# When the Weather App Shuts Down

A tale of mathematical resilience following
2048's Y2K Part 2

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## 1 Gameplan

We define the "best" temperature to be the expected value of all temperatures. To compute this, we first define the probability density functions $f_A$, $f_B$, $f_C$ for Aaron, Brian, and Cat's temperatures selected. Using the definition of the normal distribution:

$$f_A(x) = \frac{1}{a\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-T}{a})^2}$$

$$f_B(x) = \frac{1}{b\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-T}{b})^2}$$

$$f_C(x) = \frac{1}{c\sqrt{2\pi}} e^{\frac{-1}{2}(\frac{x-T}{c})^2}$$

Now, in order to compute the expected value, we define $P(T)$ to be the probability distribution function for the event of Alex guessing $X$, Brian guessing $Y$, and Cat guessing $Z$ given that the temperature is actually $T$. We define $P(T) = f_A(X) \cdot f_B(Y) \cdot f_C(Z)$ to account for this. Then, the expected value is $\int_{-\infty}^{\infty} T \cdot \frac{P(T)}{S} dT$ where $S = \int_{-\infty}^{\infty} P(T) dT$ is divided by to ensure the probabilities used in our expected value add up to 1.

## 2 Computation Time

We wish to compute $\frac{\int_{-\infty}^{\infty} T \cdot P(T) dT}{\int_{-\infty}^{\infty} P(T) dT}$. Note that by expanding the multiplication, there exist constants $c_1$, $c_2$, and $c_3$ such that $P(T) = c_1 \cdot e^{c_2 T} \cdot e^{-c_3 T^2}$. We can compute that $c_2 = \frac{X}{a^2} + \frac{Y}{b^2} + \frac{Z}{c^2}$

and $c_3 = \frac{1}{2}(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})$. Thus the expected value is

$$\frac{\int_{-\infty}^{\infty} T \cdot c_1 \cdot e^{c_2 T} \cdot e^{-c_3 T^2} dT}{\int_{-\infty}^{\infty} c_1 \cdot e^{c_2 T} \cdot e^{-c_3 T^2} dT}$$

$$= \frac{\int_{-\infty}^{\infty} T \cdot e^{c_2 T} \cdot e^{-c_3 T^2} dT}{\int_{-\infty}^{\infty} e^{c_2 T} \cdot e^{-c_3 T^2} dT}$$

$$= \frac{\int_{-\infty}^{\infty} T \cdot e^{-c_3(T^2 - \frac{c_2}{c_3}T)} dT}{\int_{-\infty}^{\infty} e^{-c_3(T^2 - \frac{c_2}{c_3}T)} dT}$$

By completing the square and then canceling out the constant coefficient created gives that the expected value is also equal to

$$= \frac{\int_{-\infty}^{\infty} T \cdot e^{-c_3(T - \frac{c_2}{2c_3})^2 + \frac{c_2^2}{4c_3}} dT}{\int_{-\infty}^{\infty} e^{-c_3(T - \frac{c_2}{2c_3})^2 + \frac{c_2^2}{4c_3})} dT}$$

$$= \frac{\int_{-\infty}^{\infty} T \cdot e^{-c_3(T - \frac{c_2}{2c_3})^2} dT}{\int_{-\infty}^{\infty} e^{-c_3(T - \frac{c_2}{2c_3})^2} dT}$$

To evaluate this ratio, let $u = T - \frac{c_2}{2c_3}$. Then $du = dT$ and the bounds of the integrals remain the same. So, this quantity is equal to $\frac{\int_{-\infty}^{\infty} (u + \frac{c_2}{2c_3}) e^{-c_3 u^2} du}{\int_{-\infty}^{\infty} e^{-c_3 u^2} du}$.

Note that for any positive $k$, we can evaluate $\int_{-\infty}^{\infty} e^{-kx^2} dx$ by letting $u = \sqrt{k}x$, $du = \sqrt{k}dx$. Then, this integral is equal to $\int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{k}}$ which, using the well-known fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, is equal to $\sqrt{\frac{\pi}{k}}$. Since, $c_3 = \frac{1}{2}(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}) > 0$ as $a, b, c \neq 0$, we can apply this fact. Simplifying the expected value using this information gives $\frac{\int_{-\infty}^{\infty} u e^{-c_3 u^2} du + \frac{c_2}{2c_3}\sqrt{\frac{\pi}{c_3}}}{\sqrt{\frac{\pi}{c_3}}}$.

We now claim that $\int_{-\infty}^{\infty} u e^{-c_3 u^2} du = 0$. This is true because the function $xe^{-c_3 x^2}$ is an odd function, so the integral of this function from $-\infty$ to $\infty$ is 0. This means that the expected values is just $\frac{c_2}{2c_3}$ which, using our previously computed values of $c_2$ and $c_3$ means that the expected value of our temperature is $\frac{\frac{X}{a^2} + \frac{Y}{b^2} + \frac{Z}{c^2}}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$.

# 3 Other Ideas of Best

Another way we might define "best" might be the most likely temperature, meaning the temperature $T$ that maximized $P(T)$. To find this $T$, we wish to find where $P'(T) = 0$. Since $P(T) = c_1 \cdot e^{c_2 T} \cdot e^{-c_3 T^2}$ for constants $c_1$, $c_2$, and $c_3$ , $P'(T) = c_1 \cdot e^{c_2 T} \cdot e^{-c_3 T^2} \cdot (-2c_3 T + c_2)$.

Furthermore, since $c_1 = \frac{1}{abc(\sqrt{2\pi})^3}e^k$ for some constant $k$, both $c_1$ and $e^{c_2T-c_3T^2}$ are always greater than 0. This means that, in order for $P'(T)$ to be equal to 0, $-2c_3T+c_2$ must be 0, which occurs at $T=\frac{c_2}{2c_3}$. Furthermore, we can compute that $P''(T) = c_1(e^{c_2T-c_3T^2}(-2c_3 + (-2c_3T+c_2)^2)$ which, as $c_3>0$, is clearly less than 0 at $T=\frac{c_2}{2c_3}$. This, in addition to the fact that $P(T)$ is continuous and goes to 0 as $T$ goes to $\pm\infty$, mean that $T=\frac{c_2}{2c_3}$ maximizes $P(T)$. We previously computed that $c_2 = \frac{X}{a^2}+\frac{Y}{b^2}+\frac{Z}{c^2}$ and $c_3 = \frac{1}{2}(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})$. Therefore, $P(T)$ is maximized at $T=\frac{c_2}{2c_3} = \frac{\frac{X}{a^2}+\frac{Y}{b^2}+\frac{Z}{c^2}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$. This just so happens to be the same value we computed under our different definition of best, so I think we can feel pretty good about this guess for the temperature.