MULTILAYER GRAPHENE

CONTENTS

I. SCALES OF THE HAMILTONIAN

We set $\hbar=1$ throughout all the text. The UV cutoff $\mathcal{K}$ of the system, fixed by the lattice parameter $2\pi/\mathcal{K}$, provides us a natural scale to measure the momenta, such that any momentum index $\mathbf{k}$ is normalized by $\mathcal{K}$. In this way, we get the kinetic non-interacting Hamiltonian

$$H_{\text{kin}} = E_{\text{UV}} \sum_{\mathbf{k}\sigma\sigma'} \left(\frac{|\mathbf{k}|}{\mathcal{K}}\right)^m \psi^\dagger_{\mathbf{k},\sigma} \left(\hat{\mathbf{n}}_m \cdot \boldsymbol{\sigma}_{\sigma\sigma'}\right) \psi_{\mathbf{k},\sigma'}, \quad (1)$$

where $[\hat{\mathbf{n}}_m] = \big(\cos(m\phi_m)\ \sin(m\phi_m)\big)$ and E_{UV} is the numerical value of the kinetic term evaluated at the momentum of the UV cutoff, and whose shape depends on m:

- $m=1$: $E_{\text{UV}} = v\mathcal{K}$, where v is the Fermi velocity of the Dirac fermions.
- $m=2$: $E_{\text{UV}} = \mathcal{K}^2/2m_e$, where m_e is the effective mass of the electron with parabolic dispersion relation.

The potential is given by

$$V_{\mathbf{x}} = V_0 e^{-r^2/a^2} \quad (2)$$

with V_0 the intensity of the Gaussian potential and a sets the spreading of the potential in the reciprocal space. Its Fourier transform is expressed as

$$V_{\mathbf{k}} = \pi a^2 V_0 e^{-(a\mathcal{K})^2\left(\frac{k}{\mathcal{K}}\right)^2} \quad (3)$$

where $k=|\mathbf{k}|$. It is assumed that $a > 2\pi/\mathcal{K}$, i.e., the spreading of the Gaussian should be larger than the lattice parameter $2\pi/\mathcal{K}$. The effect of the potential is then described by two nondimensional quantities:

- $a_{\mathcal{K}} = a\mathcal{K}$: Determines how concentrated is the potential in the reciprocal space. It is constrained to be larger than one.
- $g_{\mathcal{K}} = \pi a_{\mathcal{K}}^2 V_0/E_{\text{UV}}$: Relates the intensity of the Gaussian potential in the reciprocal space with E_{UV}, the kinetic energy at the UV cutoff.

Lastly, the interaction Hamiltonian can be expressed as

$$H_{\text{int}} = \frac{1}{2A}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\sigma\sigma'} V_{\mathbf{q}} \psi^\dagger_{\mathbf{k}'+\mathbf{q},\sigma'}\psi^\dagger_{\mathbf{k}-\mathbf{q},\sigma}\psi_{\mathbf{k},\sigma}\psi_{\mathbf{k}',\sigma'} \quad (4)$$

$$= \frac{E_{\text{UV}}}{2A\mathcal{K}^2}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\sigma\sigma'} g_{\mathcal{K}} e^{-a_{\mathcal{K}}^2\left(\frac{k}{\mathcal{K}}\right)^2} \psi^\dagger_{\mathbf{k}'+\mathbf{q},\sigma'}\psi^\dagger_{\mathbf{k}-\mathbf{q},\sigma}\psi_{\mathbf{k},\sigma}\psi_{\mathbf{k}',\sigma'}.$$

II. RESTRICTED HILBERT SPACE FOR $\mathbf{Q}=0$ EXCITONS AND SPIN BASIS

By restricting the Hilbert space of the system to the singly-occupied sites in the momentum lattice, we can build the spin operators

$$\mathbf{s}_{\mathbf{k}} = \sum_{\sigma,\sigma'} \psi^\dagger_{\mathbf{k},\sigma}\boldsymbol{\sigma}_{\sigma\sigma'}\psi_{\mathbf{k},\sigma'} \quad (5)$$

so that we get the new Hamiltonian $\mathcal{P}H\mathcal{P}$ after projecting into the restricted Hilbert space.

The fermion bilinears can be expanded in terms of the generators of the Lie algebra of the group SU(2)

$$\psi^\dagger_{\mathbf{k},\tau'}\psi_{\mathbf{k},\tau} = \frac{1}{2}\sum_{\mu=0}^{3} s^\mu_{\mathbf{k}}\sigma^\mu_{\tau\tau'} \quad (6)$$

where $\sigma^\mu_{\tau,\tau'}$ is the component $\tau\tau'$ of the matrices σ^μ give by

$$\begin{aligned} \sigma^0 &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\quad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ ,\\ \sigma^2 &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ ,\ \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ , \end{aligned} \quad (7)$$

and $s^\mu_{\mathbf{k}}$ are the coefficients of the expansion

$$s^\mu_{\mathbf{k}} = \sum_{\sigma\sigma'} \psi^\dagger_{\mathbf{k},\sigma}\sigma^\mu_{\sigma\sigma'}\psi_{\mathbf{k},\sigma'} \quad (8)$$

The singly-occupied condition is forced by the constrain

$$\begin{aligned} s^0_{\mathbf{k}} &= \sum_{\sigma\sigma'} \psi^\dagger_{\mathbf{k},\sigma}\tau^0_{\sigma\sigma'}\psi_{\mathbf{k},\sigma'} = \sum_{\sigma\sigma'} \psi^\dagger_{\mathbf{k},\sigma}\delta_{\sigma\sigma'}\psi_{\mathbf{k},\sigma'}\\ &= \sum_{\sigma\sigma'} \psi^\dagger_{\mathbf{k},\sigma}\psi_{\mathbf{k},\sigma} = \sum_{\sigma} n_{\mathbf{k},\sigma} = 1, \end{aligned} \quad (9)$$

because the subspace corresponds to singly-occupied sites in the momentum lattice.

The following steps show how the four-fermion term in the interacion Hamiltonian can be expressed under the

singly-occupied constrain. The completeness relation of the Pauli matrices

$$\sum_{\mu=0}^{3}\sigma^{\mu}_{\sigma\sigma'}\sigma^{\mu}_{\tau\tau'}=2\delta_{\sigma\tau'}\delta_{\sigma'\tau} \tag{10}$$

Trace of bilinear

$$\mathrm{tr}\left(\sigma^{\mu}\sigma^{\mu'}\right)=\sum_{\tau,\tau'}\sigma^{\mu}_{\tau\tau'}\sigma^{\mu'}_{\tau'\tau}=2\delta^{\mu\mu'} \tag{11}$$

$$\begin{aligned}\sum_{\mu=0}^{3}\sigma^{\mu}_{\tau\tau'}s^{\mu}_{\mathbf{k}}&=\sum_{\mu=0}^{3}\sum_{\sigma\sigma'}\psi^{\dagger}_{\mathbf{k},\sigma}\sigma^{\mu}_{\sigma\sigma'}\tau^{\mu}_{\tau,\tau'}\psi_{\mathbf{k},\sigma'}\\&=2\sum_{\sigma\sigma'}\psi^{\dagger}_{\mathbf{k},\sigma}\delta_{\sigma\tau'}\delta_{\sigma'\tau}\psi_{\mathbf{k},\sigma'}\\&=2\psi^{\dagger}_{\mathbf{k},\tau'}\psi_{\mathbf{k},\tau}\end{aligned} \tag{12}$$

$$\psi^{\dagger}_{\mathbf{k},\tau'}\psi_{\mathbf{k},\tau}=\frac{1}{2}\sum_{\mu=0}^{3}\tau^{\mu}_{\tau,\tau'}s^{\mu}_{\mathbf{k}} \tag{13}$$

$$\begin{aligned}&\sum_{\tau,\tau'}\psi^{\dagger}_{\mathbf{k},\tau}\psi_{\mathbf{k},\tau'}\psi^{\dagger}_{\mathbf{k}',\sigma}\psi_{\mathbf{k}',\sigma'}=\\&=\sum_{\tau,\tau'}\left(\frac{1}{2}\sum_{\mu=0}^{3}\sigma^{\mu}_{\tau'\tau}s^{\mu}_{\mathbf{k}}\right)\left(\frac{1}{2}\sum_{\mu'=0}^{3}\sigma^{\mu'}_{\tau\tau'}s^{\mu'}_{\mathbf{k}'}\right)\\&=\frac{1}{4}\sum_{\mu,\mu'}s^{\mu}_{\mathbf{k}}s^{\mu'}_{\mathbf{k}'}\sum_{\tau,\tau'}\sigma^{\mu}_{\tau'\tau}\sigma^{\mu'}_{\tau\tau'}\\&=\frac{1}{4}\sum_{\mu,\mu'}s^{\mu}_{\mathbf{k}}s^{\mu'}_{\mathbf{k}'}\mathrm{tr}\left(\sigma^{\mu}\sigma^{\mu'}\right)\\&=\frac{1}{2}\sum_{\mu,\mu'}s^{\mu}_{\mathbf{k}}s^{\mu'}_{\mathbf{k}'}\delta^{\mu\mu'}=\frac{1}{2}\sum_{\mu,\mu'}s^{\mu}_{\mathbf{k}}s^{\mu}_{\mathbf{k}'}=\frac{1+\mathbf{s}_{\mathbf{k}}\cdot\mathbf{s}_{\mathbf{k}'}}{2}\end{aligned} \tag{14}$$

$$\begin{aligned}&\mathcal{P}H\mathcal{P}=\\&E_{\mathrm{UV}}\sum_{\mathbf{k}}\left(\frac{|\mathbf{k}|}{\mathcal{K}}\right)^{m}\hat{\mathbf{k}}_{m}\cdot\mathbf{s}_{\mathbf{k}}-\sum_{\mathbf{k}_1\neq\mathbf{k}_2}\frac{V_{\mathbf{k}_1-\mathbf{k}_2}}{4A}\mathbf{s}_{\mathbf{k}_1}\cdot\mathbf{s}_{\mathbf{k}_2},\end{aligned} \tag{15}$$

where the first term represents a m-folded spin vortex and the second term is an effective ferromagnetic exchange.

III. EXPANSION ABOUT THE NON-INTERACTING GROUND STATE

A. Band basis and interaction matrix

In this section, we provide details of the derivation of Eqs. (29) to (31) starting from Eq. (1). We begin describing the transfomation from pseudo-spin basis onto band basis. In the band basis $s=\{+,-\}$ the kinetic term is:

$$\begin{aligned}&\psi^{\dagger}_{\mathbf{k}\sigma}\left(\hat{\mathbf{k}}_m\cdot\boldsymbol{\sigma}_{\sigma\sigma'}\right)\psi_{\mathbf{k}\sigma'}=\\&\quad=e^{-im\phi}\psi^{\dagger}_{\mathbf{k}\uparrow}\psi_{\mathbf{k}\downarrow}+e^{+im\phi}\psi^{\dagger}_{\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}.\end{aligned} \tag{16}$$

Band and pseudospin basis are related by:

$$\begin{aligned}\psi_{\mathbf{k}\sigma}&=\sum_{s}\langle\sigma|\mathbf{k}s\rangle\,\psi_{\mathbf{k}s},\\\begin{pmatrix}\psi_{\mathbf{k}\uparrow}\\\psi_{\mathbf{k}\downarrow}\end{pmatrix}&=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-im\phi/2}&e^{-im\phi/2}\\e^{+im\phi/2}&-e^{+im\phi/2}\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{k}+}\\\psi_{\mathbf{k}-}\end{pmatrix}.\end{aligned} \tag{17}$$

Fermion bilinears transform as

$$\sum_{\sigma}\psi^{\dagger}_{\mathbf{k}_1\sigma}\psi_{\mathbf{k}_2\sigma}=\psi^{\dagger}_{\mathbf{k}_1 s_1}\sum_{s_1 s_2}\langle\mathbf{k}_1 s_1|\mathbf{k}_2 s_2\rangle\,\psi^{\dagger}_{\mathbf{k}_1 s_1}\psi_{\mathbf{k}_2 s_2}, \tag{18}$$

where

$$\langle\mathbf{k}_1 s_1|\mathbf{k}_2 s_2\rangle=\begin{pmatrix}\cos{}^{m\phi_{12}}\!/_{2}&i\sin{}^{m\phi_{12}}\!/_{2}\\i\sin{}^{m\phi_{12}}\!/_{2}&\cos{}^{m\phi_{12}}\!/_{2}\end{pmatrix}, \tag{19}$$

where ϕ_i is the polar angle of $\mathbf{k}_i$, and $\phi_{12}=\phi_1-\phi_2$. Therefore, the Hamiltonian in Eq. (1) of the Main text in the band basis is expressed as follows:

$$\begin{aligned}\mathcal{P}H\mathcal{P}&=\sum_{\mathbf{k}}E^{m}_{\mathbf{k}}\left(\psi^{\dagger}_{\mathbf{k}+}\psi_{\mathbf{k}+}-\psi^{\dagger}_{\mathbf{k}-}\psi_{\mathbf{k}-}\right)\\&-\sum_{\mathbf{k}_1\neq\mathbf{k}_2}\sum_{s_1 s_2}(T^{m}_{\mathbf{k}_1\mathbf{k}_2})^{s_1 s_2}\psi^{\dagger}_{\mathbf{k}_1 s_1}\psi_{\mathbf{k}_1\bar{s}_1}\psi^{\dagger}_{\mathbf{k}_2\bar{s}_2}\psi_{\mathbf{k}_2 s_2},\end{aligned} \tag{20}$$

where $E^{m}_{\mathbf{k}}=E_{\mathrm{UV}}(|\mathbf{k}|/\mathcal{K})^{m}+\Sigma^{m}_{\mathbf{k}}$ is the effective dispersion relation of the electrons, in which $\Sigma^{m}_{\mathbf{k}}$ is the self-energy

$$\Sigma^{m}_{\mathbf{k}}=\frac{1}{2A}\sum_{\mathbf{p}}V_{\mathbf{k}-\mathbf{p}}\cos(m\phi_{\mathbf{kp}}). \tag{21}$$

and $(T^{m})^{s_1 s_2}_{\mathbf{k}_1\mathbf{k}_2}$ is the interaction matrix in the band basis, given by:

$$T^{m}_{\mathbf{k}_1\mathbf{k}_2}=\frac{V_{\mathbf{k_1-k_2}}}{4A}\begin{pmatrix}1+\cos(m\phi_{12})&1-\cos(m\phi_{12})\\1-\cos(m\phi_{12})&1+\cos(m\phi_{12})\end{pmatrix}. \tag{22}$$

B. Holstein-Primakoff expansion

We select the following spin basis

$$\mathbf{s}_{\mathbf{k}}=-s^{z}_{\mathbf{k}}\hat{\mathbf{k}}_m+s^{x}_{\mathbf{k}}\hat{\mathbf{z}}+s^{y}_{\mathbf{k}}\hat{\boldsymbol{\phi}}_m, \tag{23}$$

which diagonalizes the kinetic term, and where $\hat{\mathbf{z}}$ is the normal axis to the layer and $\hat{\boldsymbol{\phi}}_m=\hat{\mathbf{z}}\times\hat{\mathbf{k}}_m=\big(-\sin(m\phi_m)\,\cos(m\phi_m)\big)$, and the exchange term is expanded as

$$\begin{aligned}\hat{\mathbf{s}}_{\mathbf{k}_1}\cdot\hat{\mathbf{s}}_{\mathbf{k}_2}&=\left(-\hat{s}^{z}_{\mathbf{k}_1}\hat{\mathbf{k}_1}+\hat{s}^{x}_{\mathbf{k}}\hat{\mathbf{z}}+\hat{s}^{y}_{\mathbf{k}}\hat{\boldsymbol{\varphi}}\right)\cdot\left(-\hat{s}^{z}_{\mathbf{k}'}\hat{\mathbf{k}'}+\hat{s}^{x}_{\mathbf{k}'}\hat{\mathbf{z}}+\hat{s}^{y}_{\mathbf{k}'}\hat{\boldsymbol{\varphi}}'\right)\\&=(\hat{s}^{z}_{\mathbf{k}}\hat{s}^{z}_{\mathbf{k}'}+\hat{s}^{y}_{\mathbf{k}}\hat{s}^{y}_{\mathbf{k}'})\cos\phi_{\mathbf{kk}'}+\hat{s}^{x}_{\mathbf{k}}\hat{s}^{x}_{\mathbf{k}'}\\&+\;(\hat{s}^{y}_{\mathbf{k}}\hat{s}^{z}_{\mathbf{k}'}-\hat{s}^{z}_{\mathbf{k}}\hat{s}^{y}_{\mathbf{k}'})\sin\phi_{\mathbf{kk}'}\end{aligned} \tag{24}$$

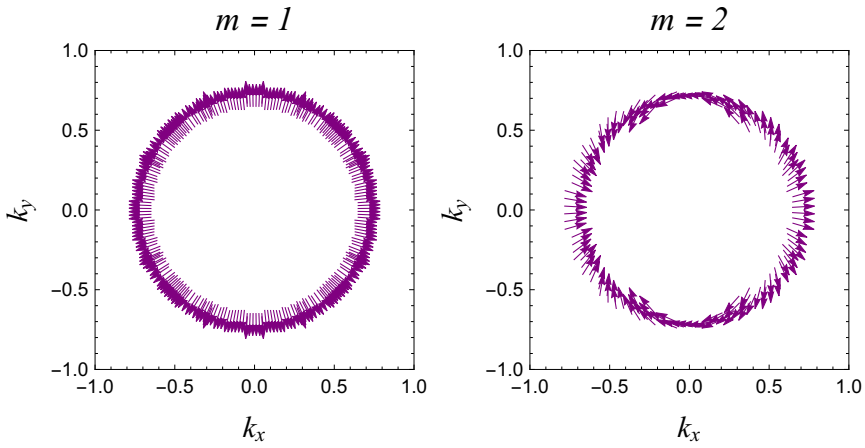

FIG. 1. Polar distribution of the vector $[\hat{\mathbf{n}}_m] = \big(\cos(m\phi_m)\ \sin(m\phi_m)\big)$ used to describe the classical ground state of the system.

with $\cos\phi_{\mathbf{k}\mathbf{k}'} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' = \hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}}'$. On this basis, the Hamiltonian can be expanded in a bosonic representation by means of the Holstein-Primakoff (HP) transformations ($S = 1/2$):

$$\begin{aligned} s^z_{\mathbf{k}} &= 2\left(S - b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) = 1 - 2b^\dagger_{\mathbf{k}} b_{\mathbf{k}}, \\ s^x_{\mathbf{k}} &\approx \sqrt{2S}\left(b_{\mathbf{k}} + b^\dagger_{\mathbf{k}}\right) = b_{\mathbf{k}} + b^\dagger_{\mathbf{k}}, \\ is^y_{\mathbf{k}} &\approx \sqrt{2S}\left(b_{\mathbf{k}} - b^\dagger_{\mathbf{k}}\right) = b_{\mathbf{k}} - b^\dagger_{\mathbf{k}}. \end{aligned} \tag{25}$$

The term corresponding to the exchange coupling in Eq. (15) can be transformed into pairing and hopping terms of bosons up to bilinears:

$$\begin{aligned} \mathbf{s}_{\mathbf{k}}\cdot\mathbf{s}_{\mathbf{k}'} \approx{}& \left(1 + b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + b^\dagger_{\mathbf{k}'} b_{\mathbf{k}'}\right)\cos\phi_{\mathbf{k}\mathbf{k}'} \\ &+ \left(b^\dagger_{\mathbf{k}} b_{\mathbf{k}'} + b_{\mathbf{k}} b^\dagger_{\mathbf{k}'}\right)(1 + \cos\phi_{\mathbf{k}\mathbf{k}'}) \\ &+ \left(b^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}'} + b_{\mathbf{k}} b_{\mathbf{k}'}\right)(1 - \cos\phi_{\mathbf{k}\mathbf{k}'}) \\ &+ i\left(b^\dagger_{\mathbf{k}} - b^\dagger_{\mathbf{k}'} - b_{\mathbf{k}} + b_{\mathbf{k}'}\right)\sin\phi_{\mathbf{k}\mathbf{k}'}. \end{aligned} \tag{26}$$

The resulting bosonic Hamiltonian after applying the HP transformations is:

$$\begin{aligned} H_{HP} = {}& \sum_{\mathbf{k}} 2E_{\mathrm{UV}}\left(\frac{|\mathbf{k}|}{\mathcal{K}}\right)^m b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \sum_{\mathbf{k}\neq\mathbf{p}} 2\frac{V_{\mathbf{k}-\mathbf{p}}}{2A} b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\cos(m\phi_{\mathbf{k}\mathbf{p}}) \\ &+ \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{V_{\mathbf{k}-\mathbf{k}'}}{4A}\left(1 + \cos(m\phi_{\mathbf{k}_1\mathbf{k}_2})\right)\left(b^\dagger_{\mathbf{k}_1} b_{\mathbf{k}_2} + b_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2}\right) + \\ &+ \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{V_{\mathbf{k}-\mathbf{k}'}}{4A}\left(1 - \cos(m\phi_{\mathbf{k}_1\mathbf{k}_2})\right)\left(b^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2} + b_{\mathbf{k}_1} b_{\mathbf{k}_2}\right). \end{aligned} \tag{27}$$

The first line contains the kinetic and self-energy terms. The second line can be viewed as boson hopping terms in the momentum lattice. The third line can be viewed as pairing terms which change the number of bosons. Lastly, by using the Bogoliubov basis given by

$$B^\dagger_{\mathbf{k}} = \begin{pmatrix} b^\dagger_{\mathbf{k}} & b_{\mathbf{k}} \end{pmatrix}, \tag{28}$$

the Hamiltonian can be expressed as

$$H_{HP} = \sum_{\mathbf{k}_1,\mathbf{k}_2} B^\dagger_{\mathbf{k}_1} H_{\mathbf{k}_1\mathbf{k}_2} B_{\mathbf{k}_2}, \tag{29}$$

with $B^\dagger_{\mathbf{k}} = \begin{pmatrix} b^\dagger_{\mathbf{k}} & b_{\mathbf{k}} \end{pmatrix}$, and

$$H_{\mathbf{k}_1\mathbf{k}_2} = \delta_{\mathbf{k}_1\mathbf{k}_2}\begin{pmatrix} 2E^m_{\mathbf{k}_1} & 0 \\ 0 & -2E^m_{\mathbf{k}_1} \end{pmatrix} - T^m_{\mathbf{k}_1\mathbf{k}_2}, \tag{30}$$

with $E_{\mathbf{k}} = v|\mathbf{k}| + \Sigma_{\mathbf{k}}$, $\Sigma_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}-\mathbf{k}'}\cos\cos(m\phi_{\mathbf{k}\mathbf{k}'})/2A$ is the Hartree-Fock self-energy, $T_{\mathbf{k}\mathbf{k}'}$ is

$$T_{\mathbf{k}\mathbf{k}'} = \frac{V_{\mathbf{k}-\mathbf{k}'}}{4A}\begin{pmatrix} 1+\cos(m\phi_{\mathbf{k}\mathbf{k}'}) & 1-\cos(m\phi_{\mathbf{k}\mathbf{k}'}) \\ 1-\cos(m\phi_{\mathbf{k}\mathbf{k}'}) & 1+\cos(m\phi_{\mathbf{k}\mathbf{k}'}) \end{pmatrix}. \tag{31}$$

IV. HARTREE-FOCK SELF-ENERGY IN MULTILAYER GRAPHENE

The explicit Hartree-Fock self-energy found is

$$\begin{aligned} \Sigma^m(\mathbf{k}) &= \int \frac{d^2\mathbf{k}'}{(2\pi)^2} V(\mathbf{k}-\mathbf{k}')\cos(m\phi_{\mathbf{k}\mathbf{k}'}) \\ &= V_0\int \frac{d^2\mathbf{k}'}{(2\pi)^2}\left(e^{-a^2|\mathbf{k}_1-\mathbf{k}_2|^2}\right)\cos(m\phi_{\mathbf{k}\mathbf{k}'}). \end{aligned} \tag{32}$$

We did numerical integration for $V_0 = 1$, $a = 1$ and $\mathcal{K} = 1$, shown in the Fig. 2. The main feature are the asymptotic limits for large and small momentum, and the slight shift to the left with the increase of the number of layers.

V. ANGULAR MOMENTUM CHANNELS AND PARAMETRIZATION OF THE RADIAL COORDINATE

In order to exploit the emergent rotational invariance in the thermodynamic limit, we use the following polar

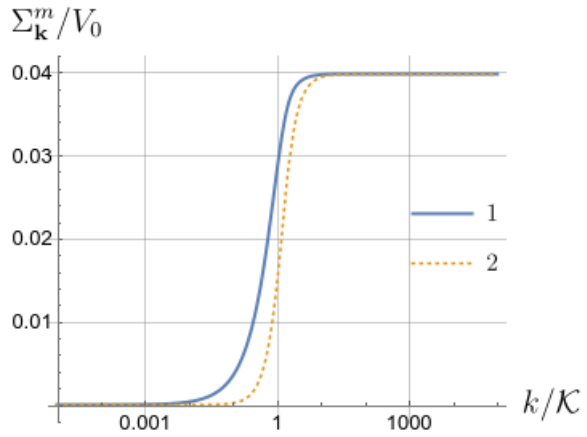

FIG. 2. Hartree-Fock self-energies of the Dirac fermions for monolayer and bilayer graphene with the Gaussian potential ($V_0 = 1$ and $a = 1$).

parametrization $\mathbf{z} = (k, \phi)$:

$$k_m = \frac{\mathcal{K}}{\sqrt{2}} \tan^2(n\Delta\theta), \quad \phi_l = n\Delta\phi, \tag{33}$$

where (k_n, ϕ_l) are the polar coordinates of a given site in the polar momentum lattice, $\mathcal{K}$ is the UV momentum scale, $\Delta\theta = (\pi/2)/(N+1)$, $\Delta\phi = 2\pi/(2L+1)$, and $l = 0, ..., 2L$, $n = 1, ..., N$.

Because the Hamiltonian matrix $H_{\mathbf{k}\mathbf{k}'}$ that enters into the Hamiltonina H_{HP} in Eq. (29) of the main text only depends on the difference between the polar angles $\phi - \phi'$ we have conservation of the angular momentum ℓ of the bosons. Consequently, we perform Fourier transforms on the polar angle ϕ_l for the fields B^l_m and the matrix $H^{ll'}_{nn'}$

$$\begin{aligned} B^l_m &= \frac{1}{\sqrt{2L+1}} \sum_{\ell=-L}^{L} e^{-i\ell\phi_l} B^\ell_m, \\ H^{ll'}_{nn'} &= \sum_{\ell=-L}^{L} e^{-i\ell(\phi_l - \phi_{l'})} H^\ell_{mm'}, \end{aligned} \tag{34}$$

such that the total Bogoliubov Hamiltonian decomposes into a sum of decoupled angular mommentum channels, as follows:

$$H_{HP} = \sum_\ell H^\ell_{HP} = \sum_{nn'\ell} B^{\ell\dagger}_n H^\ell_{nn'} B^\ell_{n'}. \tag{35}$$

Therefore the problem reduces to a set of bosons moving in an effective one dimensional radial space of N sites for each angular momentum channel which in general needs to be solved numerically.

In that way, each dataset obtained from diagonalizing a $N \times N$ matrix is labelled by the intensity and spreading of the Gaussian potential V_0 and a, respectively, the number of layers m, the angular momentum ℓ and the number of sites in the radial coordinate N.

VI. BILAYER GRAPHENE IN THE CHANNEL $\ell = 0$

We begin with the sub-block $\ell = 0$ of the Hamiltonian

$$H^0_{HP} = \sum_{mm'0} B^{0\dagger}_m H^0_{mm'} B^0_{m'}. \tag{36}$$

with $V_0 = 1$, $a = 1$, $\mathcal{K} = 1$ and $m = 2$. The self-energy employed to calculate the diagonal elements is shown in §IV.

The main result we are interested in is the eigenvalue whose imaginary part is the greatest. We diagonalized H^0_{HP} for sizes $N = 5$, 10, 20, 50, 100, 200, 500 and 1000, in which we obtained only one complex eigenvalue, whose imaginary part is depicted in the Fig. 3. Below $N = 5$ there is no imaginary eigenvalue.

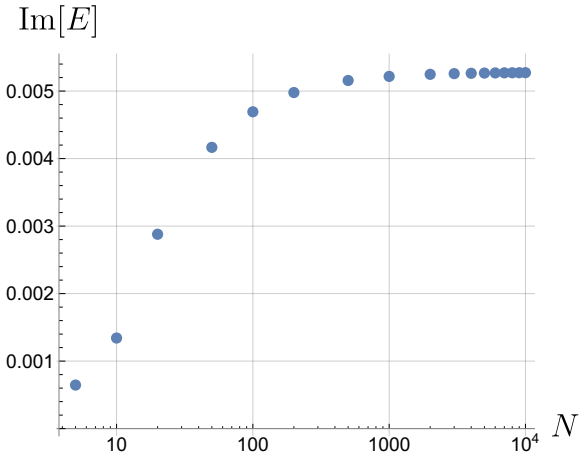

FIG. 3. Magnitude of the imaginary part of the only complex eigenvalue vs. number of sites N along radial coordinate k.