Lecture 6

The Beta Distribution

Median

$$\hat{\theta}_{MMAE} = Med[\theta|x] = a$$

Where MMAE stands for minimum mean absolute error. Such that,

$$a = \int_{-\inf}^{a} P(\theta|x)d\theta = \frac{1}{2}$$

Using our iid $Bern(\theta)$ model and data $x =< 1, 1, 0 >$, we can compute the MMAE Bayesian point estimate:

$$\int_0^a 12\theta^2(1-\theta)d\theta = 12[\frac{\theta^3}{3} - \frac{\theta^4}{4}]_0^a$$

$$= 12(\frac{a^3}{3} - \frac{a^4}{4}) = \frac{1}{2}$$
$$\rightarrow a = 0.614$$

This is called a 'quartic equation' and has a formulaic solution. The MLE is not a bayesian estimate, whereas MAP, MMSE and MMAE are. The data $x =< 1, 1, 0 >$ was a specific case. We will now solve this generally for any data set $x =< x_1, x_2, ..., x_n >$. Also using Laplace's prior of indifference, $\Theta \sim \cup(0, 1)$.

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)} = \frac{P(x|\theta)P(\theta)}{\int_0^1 P(x|\theta)P(\theta)d\theta} \rightarrow \frac{\theta^{\Sigma_{x_i}}(1-\theta)^{n-\Sigma_{x_i}}}{\int_0^1 \theta^{\Sigma_{x_i}}(1-\theta)^{n-\Sigma_{x_i}}d\theta}$$

This integral in the denominator is a special integral and is known as the *beta function*:

$$\mathcal{B}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

The beta function has no closed form solution but can be calculated to arbitrary precision using a scientific calculator. So we get the following,

$$\rightarrow \frac{1}{\mathcal{B}(\Sigma_{x_i}, n - \Sigma_{x_i})}\theta^{\Sigma_{x_i}}(1-\theta)^{n-\Sigma_{x_i}} = Beta(\Sigma_{x_i}, n - \Sigma_{x_i})$$

We just derived that the posterior for the iid bernoulli likelihood is a beta distribution. Let's go back to probability class and examine the beta distribution...

$$\mathcal{Y} \sim Beta(\alpha, \beta) =^{PDF} \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} = p(y)$$

$$Supp[\mathcal{Y}] = (0, 1)$$

$$int_0^1 \frac{1}{B(\alpha, \beta} y^{\alpha-1}(1-y)^{\beta-1} dy = 1$$

Where,

$$\alpha \ > \ 0, \ \beta \ > \ 0$$

For instance,

$$\alpha = 0, \ \beta = 1 \rightarrow \int_0^1 \frac{1}{y} dy = \inf$$

$$E[\mathcal{Y}] = \int_0^1 yP(y)dy = \int_0^1 y \frac{1}{B(\alpha, \beta)} y^{\alpha-1}(1-y)^{\beta-1} dy$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 y^{\alpha-1+1}(1-y)^{\beta-1} = \frac{B(\alpha+1, \beta}{B(\alpha, \beta)}$$

To simplify this, we need the gamma function:

$$\gamma(\alpha) = \int_0^{\inf} t^{\alpha-1} e^{-t} dt, \alpha > 0$$

Facts:

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
2. $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$Mode[\mathcal{Y}] = argmax_{y \in (0,1)} \{ \frac{1}{B(\alpha, \beta} y^{\alpha-1}(1-y)^{\beta-1} \}$$

$$= argmax\{(\alpha-1)ln(y) + (\beta-1)ln(1-y)\}$$

$$\rightarrow y^* = \frac{\alpha-1}{\alpha+\beta-2}$$

If we take the second derivative to check if it's negative, we find it's only negative if both alpha and beta are greater than one. This formula only works if alpha and beta are greater than 1.

$Med[\mathcal{Y}]$ has no closed form expression, and this must be done with a computer. We will denote the answer to this using notation from the R programming language, 'qbeta(0.5, α, β)'.