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# 320

# *Contents*

# Introduction

## Course Info

This document is based on Professor Krzysztof Klosin's Intro to Point Set Topology math class at CUNY Queens College taken Spring of 2021. The course code for this class is Math 320. Introduction to Topology 3rd edition by Bert Mendelson is used as the course textbook.

## What is a Topology?

We begin the course by asking ourselves what a topology is. In some sense topology is similar to Abstract Algebra or Algebraic Structures. In a way, topology is abstract geometry. Topology itself lays the groundwork for most of mathematics, it is considered the underlying language of geometry and analysis. It allows a mathematician to look at geometric phenomena in a much more general way.

Because it is abstract geometry, there will be pictures in this class. You should exercise caution when using pictures in topology because they are often times only there to help one understand the intuitive idea of what might be true. More commonly, they do not depict the topological reality accurately. This is because topology is so general that the space that we will discuss is impossible to draw, even impossible to imagine in any sort of geometric manner. The pictures are helpful nonetheless, and we will use them and draw them, but they should not be considered as absolute truth. For instance, in geometry a picture is everything. If you draw a triangle, you can infer a lot about what might be true about certain relationships in geometry.

The second thing about topology is that like all the languages in mathematics, it uses sets. So we will start with set theory in this course. In a sense almost everything in modern mathematics is the study of sets in the context of a specific structure. If you are familiar with abstract algebra you may know of groups, rings, and fields, and they're all sets with an additional structure. Topological spaces are also sets but with its own structure, but these structures are unique to those in abstract algebra. Later, when we study topological spaces you will see that we take the sets and put an extra structure on them with the effect of allowing a mathematician to take distances between objects in a set.

You can imagine having a set with three apples, you do not know how the apples are currently arranged. But if you have a topological space with three apples, then they will come with a certain arrangement. Topology allows you to distinguish between such things. You say, "Oh so a topology allows me to take distances between things", I respond with no, in fact topology is more general than that. There exist topological spaces with no distance. Still, there is something added to a set such that it is more than a collection of elements. That thing that is added to a set is actually called a topology. So topology is not just the name of a mathematical discipline, its also the name of the structure. We will get to this, and we will explain this as well. This course assumes you have no background in the subject matter.

# Chapter 1
# Set Theory

## The Definition of a Set

We begin this chapter by saying the beginnings of mathematics are frustrating because there is no definition of a set. We will not define what a set is. Intuitively, it is a collection of things. In mathematics, it does not have a definition. When you come out of high school you believe everything in mathematics has a definition, then you take more advanced courses in mathematics and learn otherwise. This shouldn't be surprising because definitions in mathematics are like definitions in a dictionary. In an English dictionary you define words with other words. If you do not know any words in English that dictionary will not be useful at all.

This is the position we find ourselves in with the mathematical term "set". The same thing happens in mathematics, there is no first definition. Instead mathematics tries to collect things, of which do not have a definition, and try to formulate a few. We do this for the word set. We may be able to give it some properties, but we do not define it. You ask, "why can we not define a set is a collection of things?" To do that, you first have to define what a collection is, and what things are. You end up in this endless epistemological spiral. Similarly, we do not define what an element of a set is, or what it means to belong to a set, or what equality means. We also will not define equality. We all intuitively understand what these terms mean nonetheless, and intuition is an important aspect to succeeding in mathematics. You'll find in any study of mathematics you first have to introduce fundamental things without knowing exactly what they are.

## First Axioms

We denote an element x is in the set A as so

$$x \in A$$

Observe how we denote sets with capital letters and elements with lower case letters. This is standard.

To be redundant, just because we gave a few definitions does not mean that we can now prove theorems because there is no first theorem. Perhaps we can use the properties of what we've learned so far, but again if we have no properties then there is no property that we can establish. In mathematics we have these things called *axioms*. These are things that we do not have to prove, we can take them at face value. We are not going to question them.

Sets can be defined in several ways.

1. **Axiom of Extension**
   Two sets are equal if an only if they have the same elements.

   By listing their elements within a pair of curly brackets. If the set A consisted of elements 1, 3, and the letter x, we can write

   $$A = \{1,\ 3,\ x\}$$

   If we repeat an element in such a notation the set does not change

   $$B = \{1,\ 3,\ 1,\ x\} = A = \{1,\ 3,\ x\}$$

2. **Axiom of Specification**
   **Definition:** A set A is called a *subset* of a set B if every element of A is also an element of B. We denote this with

   $$A \subset B$$

   This notation allows for A = B. The second way to define a set is as a subset of a given set whose elements satisfy some condition, we denote this with

   $$\{x \in A : \textit{condition}\}$$

   This is read, "The set where x is in the set A such that the condition is true."

**Example:**
Let A = 1, 2, 3, 4. Then,

$$B = \{x \in A : x > 1\} = \{2,3,4\}$$

We now have enough to formulate our first proposition.

**Proposition 1:** Let A and B be sets. Then A = B if and only if

$$A \subset B \textit{ and } B \subset A$$

**Proof:** Suppose that A = B. Let x be an element of A. By the axiom of extension, because A = B, we get that A and B have the same elements, so x is also an element of B. Thus A is a subset of B. Similarly if x is an element of B, the same argument gives that x is an element of A, so B is a subset of A. Conversely, suppose that A is a subset of B and B is a subset of A. Then if x is an element of A it is also an element of B. Similarly if x is an element of B it is also an element of A. Thus A and B have the same elements, and are therefore equal by the axiom of extension.

**Example:**
Let A be a set. Then by the axiom of specification the following is also a set,

$$\{x \in A : x \neq x\}$$

This set has no elements. This also defines the empty set and denote it by Ø. Now suppose I took a different set B and define another set to be

$$\{x \in B : x \neq x\}$$

By the axiom of extension this also defines the empty set. Conclusively, given unique sets A and B,

$$\{x \in A : x \neq x\} = \{x \in B : x \neq x\}$$

In other words, there is only one empty set, the empty set. Note that sets can also be elements of other sets, or a set can be an element of another set. If A is a set, then in general, A is the set containing A as an element. It is not the same as the set A.

**Example:**
Given, ∅, { ∅ } and {{ ∅ }}, we can say that none of these sets are the same. Suppose they were the same sets, by the axiom of extension it must be the case that all sets have the same elements, but this is not the case. The first set is simply the empty set. The second set is a set containing the empty set. The third set is a set containing a set containing the empty set. Therefore, $\emptyset \neq \{\emptyset\} \neq \{\{\emptyset\}\}$.

**Example:**
For this example we will examine the Boolean value of each statement given the following three sets,

$$A = \{5, 9, 3\},\ B = \{\{5\}, \{9, 3\}\},\ C = \{5, \{9\}, \{3\}\}$$

| | | | |
|---|---|---|---|
| $5 \in A$ | True | $5 \subset B$ | False |
| $5 \subset A$ | False | $5 \subset B$ | True |
| $5 \in B$ | False | $\{3, 9\} \in C$ | False |
| $\{5\} \in B$ | True | $\{3, 9\} \subset C$ | False |
| $\{5\} \in A$ | True | $\{3, 9\} \subset B$ | False |
| $\{5\} \subset A$ | True | $\{3, 9\} \in B$ | True |

**Axiom of Power**
For any set A there exists a set P(A) whose elements are exactly all subsets of A. This set is called the *power set* of A.

**Example 1:**
Given, A = {1, 4}

$$P(A) = \{\emptyset,\ A,\ \{1\},\ \{4\}\}$$

The power set of a set A is sometimes denoted, $2^A$ The reason for this notation is the fact that if A is a finite set with n elements then the power series of A consists of $2^n$ elements.

**Example 2:**
Given,

$$A = \{\emptyset,\ \{\emptyset\}\}$$

Then the power set of A is,

$$P(A) = \{\emptyset,\ A,\ \{\emptyset\},\ \{\{\emptyset\}\}\}$$

Observe how the elements of every power set contains the empty set and the set itself. This is because the empty set and the set itself are both considered subsets of the set A by the definition of a subset.

## *Relations*

**Axiom of Unions**
Given any number of sets there exists a set whose elements are exactly those elements that belong to at least one of the sets. We call this set the union of the given sets. If the sets are, $A_1, A_2, ..., A_n$, we write $A_1 \cup A_2 \cup ... \cup A_n$ to denote their union. The union can be taken of any arbitrary number of sets, and it doesn't necessarily have to be finite. We will discuss this more later.

**Definition:** Let $A_1$, $A_2$, ..., $A_n$, be sets. The intersection of these sets is defined as the set,

$$\{x \in A_1 : x \in A_2, x \in A_3, ..., x \in A_n\}$$

and is denoted by $A_1 \cap A_2 \cap ... \cap A_n$.

**Definition:** Let S be a set and X and Y be subsets of S. Then we define X-Y to be the *compliment of Y in X*,

$$X - Y = \{x \in X : x \notin Y\}$$

Note that Y does not have to be a subset of X to begin with. X and Y can be completely unique. To put it bluntly, the compliment of a set Y in the set X is the set X such that there exist no elements in the set Y. If Y is a subset of X and X is clear from the context, we will sometimes write $Y^C$. In this sense, we can simply call X - Y the compliment of Y. That is, Y ∪ $Y^C$ = X. In some texts you will find the the compliment of Y to be denoted C(Y), of course granted that we know Y is a subset of a set X.

**Example:**
Given,

$$X = \{1, 2, 3, 4\}, Y = \{1, 3\}, Z = \{1, 4, 5\}$$

| Relation | Set |
|---|---|
| X ∪ Z | {1, 2, 3, 4, 5} |
| X ∪ $Y$ | X |
| X ∪ $Y$ ∪ Z | {1, 2, 3, 4, 5} |
| X ∩ Z | {1, 4} |
| X ∩ $Y$ | Y |
| X ∩ $Y$ ∩ Z | {1} |
| X - Y | {2, 4} |
| X - Z | {2, 3} |

## *Indexing*

In topology we sometimes have to deal with large collections of sets. Sometimes these collections may include 2 or 3 sets, but we may find that these collections include infinitely many sets, even an unaccountably infinite number of sets. This notation, X ∪ Z, is very inconvenient when specifying the union of a large number of sets. In fact, if you were to use this notation to define the union of all of the countable collections, its impossible to include those uncountable collections. So we introduce something called *indexed families* that allow you to manipulate large, and even infinitely large collections of sets. It is not necessary to know the formal definition, so we will skip that.

To use indexing families we have to use a set called the *indexing set*. This is the set where our indices will come from. The indexing set for the set $A_1$, $A_2$, ..., $A_n$ is {1, 2, ..., n}. For example, if you have a finite collection of sets that has n sets, you can use {1, 2, ..., n} for the indexing set. In general, an indexing set can be any set. From a mathematical point of view and indexing set is just a set, we call it an indexing set to remind us why we created it, that is, to index the sets in a collection of sets.

Let I be an indexing set. We say that this set indexes a collection of sets if for every element i in I we have a set $A_i$ in our collection. This gives us an indexed family of sets that we denote,

$$\{A_i : i \in I\}$$

Given such an index family of sets we can use the axiom of unions to form the union of this collection, denoted by,

$$\cup_{i \in I} A_i$$

An element belongs to this union if and only if it belongs to at least one of the sets $A_i$. Similarly, we define the intersection of such a family as so,

$$\cap_{i \in I} A_i = \{x \in A_{i_0} : x \in A_i \; for \; all \; i \in I\}$$

Here, $i_0$ is any element of I.

**Example 1:**
Given the indexing set I as the set of rationals, I = Q, and
$A_i = \{x \in Q \ : x < i\}$. What is the union of this collection of sets?

$$\cup_{i \in I} A_i = Q$$

Simply the set of rational numbers. This is because every $A_i$ is a subset of the rationals, so the union is going to be a subset of the rational numbers. Secondly, every rational number can be found in some set $A_i$. For instance, say I wanted to find the set that contains the number 101. We can find this number in the sets $A_{102,103,...,\infty}$. This is the case for any rational number. Therefore, the union of this collection is the set of rational numbers.

**Example 2:**
Given the indexing set I is the set of rationals, I = Q, and
$A_i = \{x \in Q \ : x < i\}$. What is the intersection of this collection of sets?

$$\cap_{i \in I} A_i = \emptyset$$

**Proof by Contradiction:** Suppose x is a rational number that belongs to this intersection. However, x does not belong to $A_x$, this is because according to the conditions of every set in the collection, only numbers less than x can be in the set $A_x$. Hence, x cannot belong to the intersection.

**Example 3:**
Given, $A_n = \{1,\ 2,\ 3,\ ...,\ n\}$, n ∈ $Z_+$, $Z_+$ serves as the indexing set.

$$\cup_{n \in Z_+} A_n = Z_+$$

$$\cap_{n \in Z_+} A_n = \{1\}$$

$$\cup_{n \in Z_+} A_n = A_2 \cup A_4 \cup A_6 \cup ...,\ A_n = Z_+$$

For any positive real number a we define $A_a$ = (-a, a). If I is the set of positive real numbers,

**Example 4:**

$$\cup_{a \in I} A_a = A_2 \cup A_4 \cup A_6 \cup ...,\ A_n$$

This hints at how we can have unions of unaccountably infinite collections of sets. This notation does not work in this case because we are missing a great number of positive real numbers, such as 1.5. You will have to take the union of $A_i$ for every i ∈ $R$.

**Example 5:**

$$\cap_{n \in Z_+} A_n = \{1\}$$

We prove this by contradiction. Suppose there is a positive or negative real number B. If we divide this real number B by a real number, we should expect to get a real number. So, if we take the interval ($\frac{-|b|}{2}$, $\frac{|b|}{2}$), it is impossible for b to be in this interval. This logic works for all B in $R$.

**Example 6:**
Given,

$$B_1 = (-2,\ 2)$$

$$B_2 = (-\frac{3}{2},\ \frac{3}{2})$$

$$B_3 = (-\frac{4}{3},\ \frac{4}{3})$$

Then,

$$\cup_{i \in I} B_i = (2,\ -2)$$

$$\cap_{i \in I} B_i = [-1,\ 1]$$

**Proof:** To show that two sets are equal it is enough to show that each is a subset of the other. Suppose that a number x lies in the left-hand side. By the definition of the intersection, if you lie in the intersection of sets then x lies in $B_i$ for every $i \in I$. We want to show that $x \in [-1, 1]$. Let us prove this by contradiction. So, suppose not, such that x > 1 or x < -1. Suppose x > 1. Let i be a positive integer such that $1/i$ < x - 1. Thus, we claim that x $\notin B_i$ because $1 + \frac{1}{i} < x$. Hence,

$$x \notin \cap_{\ i \in I}\ B_i$$

So we have proved that

$$\cap_{i \in I} B_i \subset [-1,\ 1]$$

You treat the other case similarly. Now to prove the other containment, not that

$$[-1, 1] \subset (-1 - \frac{1}{i}, 1 + \frac{1}{i}) = B_i$$

This shows that

$$\cap_{i \in I} B_i = [-1,\ 1]$$

## *Relations and Functions*

An ordered pair is a sequence of two objects, where order matters. We denote an ordered pair where x is the first element and y is the second element by (x, y). Note the clash of notation with an open-ended interval. Context will help you discriminate between the two.

(x, y) = (y, x)   if and only if   x = y (x, y) = (z, v)
if and only if   x = z and y = v

**Definition:** Let A and B be non-empty sets. The set whose elements are all possible ordered pairs (a, b) with a belonging to A and b belonging to B is called the *Cartesian Product of A and B* denoted by,

$$AxB$$

**Example 1:**
Say,

$$A = \{1,\ 2,\ 3\},\ B = \{2, 7\}$$

Then,

$$AxB = \{(1,2),\ (1,7),\ (2,2),\ (2,7),\ (3,2),\ (3,7)\}$$

$$BxA = \{(2,1),\ (2,2),\ (2,3),\ (7,1),\ (7,2),\ (7,3)\}$$

Note, $A \times B \neq B \times A$ is not always true true. We can generalize the notion of a Cartesian product of two sets to the notion of a Cartesian product of n sets. The latter consists of all sequences with length n where the first coordinate in the the first set, the second coordinate in the second set, and so on, such that

$A_1 \times A_2 \times ... \times A_n = \{(a_1,\ a_2,\ ...,\ a_n) : a_1 \in A_1,\ a_2 \in A_2,\ ...,\ a_n \in A_n\}$

We can also define a 3 dimensional cube using Cartesian products, as so,

$$[-1,1] \times [-1,1] \times [-1,1]$$

**Definition:** Let A and B be two non-empty sets. Any subset of $A \times B$ is called a *relation* between A and B, also read "from A to B". Lets define the definition of a relation one more time.

Definition: Let A and B be two non-empty sets. A *relation* between A and B (or from A to B) is any subset of A cross B.

$$A \times B$$

The set A is called the *domain* of the relation and the set B is called the *codomain* of the relation. If the element (a, b) $\in$ the relation R, then we sometimes write aRb and say that a is in relation with b. The subset of B given by $\{b \in B : \exists a \in A : (a,b) \in R\}$ is called the *image* (or the *range*) of R. A relation is any subset of $A \times B$.

**Example 1:**
A = {1, 2, 3}, B = {2, 4}

$$R \subset A \times B$$

$$R' = \{(1,2),(1,4),(2,4)\}$$
$$R'' = \{(3,4)\}$$

Another way to write a relation is as so,

$$R' = \{(1,2),(1,4),(2,4)\} = 1R2, 1R4, 2R4$$
$$R'' = \{(3,4)\} = 3R4$$

**Example 2:**
Given,

$$\begin{array}{ll} \text{R} \subset A \times B\text{domR = A} & codR = B \\ im(R) = \{2,\ 4\} & \\ domR' = A & codR' = B \\ im(R') = B & \\ domR'' = A & codR'' = B \\ im(R'') = 4 & \end{array}$$

**Example 3:**
Given, $A = \mathbb{Z}$, $B = \mathbb{Z}$, $R \subset A \times B$, $aRb\ iff\ a - b$ is even.

$$R = 5R3, 3R5, 6R4$$

$domR = A \ \ codR = B \ \ im(R) = B$

**Example 4:**
$A = \mathbb{Z}$, $B = \mathbb{Z}$, $R \subset A \times B$, $aRb\ iff\ a^2 = b$ is even.

$$R = 3R9, 5R25, 10R100$$

$$im(R) = the\ set\ of\ perfect\ squares$$

**Definition:** Let A be non-empty sets. Let R be a relation between A and A (we sometimes call this a relation on A). We say that:

1. R is *reflexive* if $aRa$ for all $a \in A$.
2. R is *symmetric* if whenever $aRb$ then $bRa$ for all elements $a, b \in A$.
3. R is *transitive* if for every a, b, c in A whenever aRb and bRc are relations, then aRc is also a relation

A relation is called an *equivalence* relation if the relation has these three properties. We will now look at examples of each property.

**Example 1:**
Let A be a non-empty set. aRb if a = b. This relation is reflexive because aRa for all a in A. This relation is symmetric because a=b and b=a. This relation is transitive because if a = b, and b=c, the a=c. This is therefore an equivalence relation.

**Example 2:**
Let A = {1, 2, 3} = B Let R = {(1,1),(1,2),(2,1),(1,3),(3,1)}.
This set is not reflexive because 2 and 2 do not relate. R is also symmetric because the law is not violated. For instance, 1R2 implies 2R1, which is the case. This relation is not transitive, for instance, 3R1 and 1R2 are two ordered pairs in the relation, but 3R2 is not. Although the relation is reflexive and symmetric, it is not transitive and is therefore not an equivalence relation.

**Example 3:**
Let A = 1, 2, 3 = B, R = (1, 1), (2, 2). This set is not reflexive because (3, 3) is not in the relation. This set is symmetric and transitive though because these laws are not violated.

**Definition:** Let R be an equivalence relation on a set A. Let a be an element of A. The subset of A consisting of all elements b such that aRb is called the *equivalence class* of a and is denoted by $\pi(a)$, or [a].

**Proposition:** Two equivalence classes are either equal or disjoint, such that two equivalence classes cannot just overlap.

**Example 1:**
A = {1, 2, 3} aRb if and only if a=b.

$$\pi(1) = \{1\}$$

$$\pi(2) = \{2\}$$

$$\pi(3) = \{3\}$$

We can see here that there is no overlap between the equivalence classes of different values. Would you agree that in this case the set of equivalence classes = A? No, A%R = {{1},{2},{3}}.

**Example 2:**
Let $A = \mathbb{Z}$, aRb iff $a - b$ is even. This is an equivalence relation.

$$[1] = the\ set\ of\ all\ odd\ integers.$$

$$[5] = \{2n + 5 : n \in \mathbb{Z}\} = [1]$$

$$[2] = the\ set\ of\ all\ even\ integers.$$

The set of equivalence classes of A are defined as so, A%R = {[1],[2]}.

**Definition:** Let A and B be non-empty sets, and let R be a relation between A and B. If for every element $a \in A$ there exists one and only one element $b \in B$ such that $aRb$ then the relation is called a *function*.

**Example 1:**
Let A = {1, 2, 3}.

$$R = \{(1,2),(2,2),(2,4)\}$$
$$R' = \{(1,2),(2,4),(1,4),(2,2)\}$$
$$R'' = \{(1,2),(2,4)\}$$

R is not a function because an element a of the set A can only relate to only one element in B. This is not the case. R′ is a function. R″ is not a function because not all elements of the domain relate to an element in the codomain, specifically 3.

If R is a function from A to B then instead of writing aRb, we write

$$b = R(a)$$

or,

$$R : a \to b$$

**Example 1:**
Let A = {1, 2, 3}, B = P(A) (the power set of A)

$$f : A \to B, f(a) = \{1, a\}$$

$$f(1) = \{1, 1\} = \{1\}$$

$$f(2) = \{1, 2\}$$

$$f(3) = \{1, 3\}$$

**Definition:** A function $f : A \to B$ is called *injective* if different elements of A are mapped to different elements of B. Formally,

$$f(a) = f(b) \to a = b$$

The function f is called *surjective* if its image is B. The function f is called *bijective* if it is both injective and surjective.

**Example 1:**
Let A be a non-empty set, then the function $f : A \to A$ given by $f(a) = a$ is called the *identity function* and A and is denoted by $id_A$

**Example 2:**
Let A be a subset of B. The function $f : A \to B$ given by $f(a) = a$ is called the *inclusion* of A into B.

**Example 3:**
Let A and B be sets. Let b be a fixed element of B. The function $f : A \to B$ given by $f(a) = b$ is called a *constant function* with value b.

**Definition:** Restriction
Let A, B be sets and let C be a subset of A. Let $f : A \to B$ be a function. The function from C to B denoted by $f|_C C \to B$ given by $f|_C(c) = f(c)$ is called the *restriction* of f to C.

Just to review, let A and B be non-empty sets. Recall that a relation from A to B is any subset of $A \times B$, where A is called the domain, and B is called the codomain of the relation. The image of $\mathcal{R}$ is the subset of B consisting of all elements b in B such that there exists a in A with $(a.b) \in \mathcal{R}$. A relation $\mathcal{R}$ is called a function if for every element a in A there is no more than one element b in B such that $(a.b) \in \mathcal{R}$. If $\mathcal{R}$ is a relation that is also a function we use a more traditional notation for this, namely to indicate the domain, the codomain and the relation R, we write $\mathcal{R} : A \to B$, read, R is a function from A to B. Also if $(a.b) \in R$, we usually write $b = \mathcal{R}(a)$. The image of a function is simply the image of the relation that the function is. But we can rewrite what this means in the more traditional notation, namely, the image of a function,

$$f : A \rightarrow B$$

is the subset of B consisting of all elements b in B such that there exists an element a of A with $f(a) = b$.

**Definition:** Let $f : A \rightarrow B$ be a function, and let $X$ be a subset of A. Then the *image* of $X$ under $f$ denoted by $f(X)$ is the subset of B consisting of all elements b in B for which there exists an element $a$ in $X$ such that $f(a) = b$.

**Example:**
Given,

$$f : \mathcal{R} \rightarrow \mathcal{R}, f(x) = x^2 + 1$$

, then,

$$f([0,1]) = [1,2]$$

this is because the given function is an increasing function on the interval, where $f(0) = 1$ and $f(1) = 2$, and all values $f((0,1))$ may be found between 1 and 2. In the following two problems we use the same logic, but instead of dealing with an uncountably infinite interval, we deal with sets.

$$f(\{1,2\}) = \{2,5\}$$

$$f(\{-1,1\}) = \{2\}$$

**Definition:** Let $f : A \rightarrow B$ be a function, and let $Y$ be a subset of B. Then the *inverse image* of $Y$ under $f$, denoted by

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}$$

The notation looks decievingly similar to the inverse function, but this is not that. Not every function has an inverse function. In order for $f : X \rightarrow Y$ to have an inverse function, f must be one-to-one and onto. However, for ANY function, the inverse image of ANY subset of the target is defined. Unfortunately, the notation for inverse function is part of the notation for inverse image, so that you have to determine from context which meaning is meant.

**Example 1:**
Given,

$$f : \{1,2,3\}\{4,5,6\}$$

$$f(1) = 4$$

$$f(2) = 7$$

$$f(3) = 4$$

$$f^{-1}(\{7\}) = \{2\}$$

$$f^{-1}(\{4\}) = \{1, 3\}$$

$$f^{-1}(\{4, 5\}) = \{2\}$$

$$f^{-1}(\{5\}) = \{\emptyset\}$$

$$f^{-1}(\{\emptyset\}) = \{\emptyset\}$$

$$f^{-1}(\{4, 5, 7\}) = \{1, 2, 3\}$$

$$f(f^{-1}(\{5, 7\})) = f(\{2\}) = \{7\}$$

It is not generally the case that $f(f^{-1}(Y) = Y$, as we see with the final example above.

**Example 2:**
Given,

$$g : \mathcal{R}\mathcal{R}$$

$g(x) = 2$ for all $x1$, where $g(1) = 0$.

$$g^{-1}(\{2\} = \mathbf{R} - 1$$

$$g^{-1}(\{1\} = \emptyset$$

If Y contains a subset of R which contains neither 2 nor 0 then $g^{-1}(Y) = \emptyset$

$$g^{-1}(\{[-1, 1]\} = \{1\}$$

$$g^{-1}([-2, 2)\} = \{1\}$$

$$g^{-1}([-2, 2]\} = \mathbf{R}$$

**Proposition:** Let $f : A \to B$ be a bijection. Then there exists a function $g : B \to A$ such that $g(b) = a$ where a is the unique element of A such that $f(a) = b$. we call $g$ the inverse function of f and denote it by $f^{-1}$.

**Example 1:**
Given, $f : \mathbf{R} \to \mathbf{R}$, what is the inverse function of $f(x) = e^x$? Well, $f$ is not bijective, because it is not surjective, so $f$ does not have an inverse function.

**Example 2:**
Given, $f : \mathbf{R} \to \mathbf{R}_+$, what is the inverse function of $f(x) = e^x$? The function $g$ is bijective, so it does have an inverse function, $g^{-1}(x) = ln(x)$

**Example 3:**
Given, $f : \mathbf{R} \to \mathbf{R}$, what is the inverse function of $f(x) = sin(x)$? This is not a bijective function. So we restrict $f$ to the set $[\frac{-\pi}{2}, \frac{\pi}{2}]$. Given this restriction, the function $f$ does have an inverse function that we usually denote by

$$sin^{-1} : [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$$

**Definition:** Lry $f : AB$ and $g : BC$ be functions. The the function $g(f(x)) : AC$ is called the *composition* pf g with f. This only works if the domain of the second function is the domain of the first function.

# Chapter 2
# Defining a Topology

## Topological Spaces

**Definition:** Let $X$ be a non-empty set. Let $T$ be a subset of the power set of X (i.e., $T$ consists of some subsets of X). We demand that $T$ satisfies all of the following conditions.

1. $\emptyset, X \in T$
2. If $A_i \in T$ for all i in some indexing set I, then $\cup_{i \in I} A_i \in T$ (i.e., arbitrary unions of sets in T must be in T).
3. if $A_1, A_2, ..., A_n \in T$, then $A_1 \cap A_2 \cap ... \cap A_n \in T$ (i.e., finite intersections of sets in T must be in T).

The elements of $X$ are called *points*. The elements of $T$ are called *open sets* or open subsets, of $X$. The set $T$ is called a *topology* on $X$. The pair $(X, T)$ is called a *topological space*.

**Example 1:** $X = \{x\}$, we take a set with only one element. Let us see all the possible topologies we can put on X.

$$T = \{\emptyset, X\} = P(X)$$

Sure, the power set of X are in T. But we have to satisfy the other axioms. The union of the empty set and X is X, and X is already part of T, so that condition is satisfied, the same logic can be applied to the condition regarding intersections. This T is the only possible topology on this one-element set. There is only one topological space with this one point.

**Example 2:** Given, $X = \{a, b\}$, and $a \neq b$, is $T$ a topology? Yes, this is a topology on X called the *trivial topology*. So, $(X, T)$ is an example of a topological space whose points are a and b.

$T' = \{\emptyset, X, \{a\}\}$ is also a topology on $X$ as well. So, $(X, T')$ is an example of a topological space whose points are a and b different from $(X, T)$. Such a two-point space with topology that includes one of the singletons but not the other is called the *Sierpinski space*.

**Definition:** Trivial Topology
If $(X, T)$ is a topological space, then $T$ is called a *trivial topology*.

**Definition:** Discrete Topology
Let X be a non-empty set. Set T to be the power set of X. Then T is also a topology called the *discrete topology*.

If $\mathcal{X}$ is a two element set, also called a two point space, $\{x, y\}$, there are four topologies, a trivial topology, the discrete topology, the topology that contains one of the singletons but not the other, which gives rise to the Sierpinski space.

Let $X = \{x, y, z\}$ be a three element set.

$$T_1 = \{\emptyset, X, \{x\}$$

$$T_2 = \{\emptyset, X, \{x\}, \{y\}\}$$

$T_1$ is a topology, all conditions are met. All the possible unions and intersections are present. But $T_2$ is not a topology because the union of $\{x\}$ and {y} are not in the topology.

To reiterate, Let $\mathcal{X}$ be a not-empty set, and let $T$ be a collection of subsets of X such that T contains the empty set and $\mathcal{X}$ such that T is closed under arbitrary unions and finite intersections. This $T$ is called a topology on $\mathcal{X}$, its elements are called opens sets, and the pair ($\mathcal{X}$, $T$) is called a topological space.

**Definition:** Standard Topology
Let $\mathcal{X} = \mathbb{R}$, let $T$ consist of the empty set, $\mathcal{X}$ and all subset of $\mathcal{X}$ which are unions of open intervals. Recall that an open interval is a subset of $\mathbb{R}$ denoted $(a, b)$ where $a < b$ defined by $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. So, any open interval is an open set in $T$. Therefore,

$$(1, 2) \cup (3, 5) \in T$$

$$(1, 2) \cup (2, 3) \in T$$

$$A = \cup_{n \in \mathbb{Z}} (n, n+1) \in T$$

???

**Proposition 1:** $T$ as above is a topology on $\mathbb{R}$ called the *standard topology*. Note that the union of two intervals need not be an open interval.

If $\{A_i\}_{i\in I}$ is a collection of open sets, i.e. elements of $T$, then each of the $A_i$'s is a union of open intervals. So, the union of the $A_i$'s is also a union of open intervals, so it belongs in $T$. So $T$ is closed under arbitrary unions. By definition, $T$ also contains the empty set and $\mathcal{X}$. Therefore,

$$\mathbb{R} = \cup_{n\in\mathbb{Z}}(n, n+2)$$

so R can also be written as a union of open intervals, but we don't nee it because we include it in $T$ by definition of $T$.

**Proof:** The intersection of two open intervals of the form $(a, b)$ with $a < b$ and $(c, d)$ with $c < d$ is either the empty set or an open interval,

$$(a, b) \cap (c, d) = \{^{\emptyset\ if\ b \leq c}_{(c,b)\ if\ b > c}$$

So, it remains to verify that T is closed under finite intersections, i.e. that if we intersect a finite collection of sets, each of which is a union of open intervals, then we get a union of open intervals. To do this, let us prove an auxiliary lemma first.

**Lemma:** Let $I$, $J$ be indexing sets and let $\{A_{i,j}\}_{i\in I, j\in J}$ be a collection of sets. Then,

$$\cup_{i\in I}(\cap_{j\in J}A_{i,j}) = \cap_{j\in J}(\cup_{i\in I}A_{i,j})$$

**Proof of Lemma:** We have that x is an element of the left-hand side if and only if there exists such an $i_0 \in I$ such that $x \in \cap_{j\in J}A_{i_0,j}$ if and only if there exists an $i_0 \in I$ such that for every $j \in J$ we have $x \in A_{i_0,j}$ is and only if for every $j \in J$ we have that $x \in \cup_{i\in I}A_{i,j}$ if and only if $x \in \cap_{j\in J}(\cup_{i\in I}A_{i,j}$.

We will now apply the lemma to show that T is closed under arbitrary unions. Let $U_1, U_2, ..., U_n \in T$. If any $U_i = \emptyset$ then the intersection of them is empty, which belongs to $T$ and is therefore satisfies the conditions of a topology. Hence, we may assume that all $U_i$'s are non empty. Because each of the $U_j$'s is a union of open intervals, we can write,

$$\cup_1 = \cup_{i\in I}(a_{i1}, b_{i1}),\ a_{i1} < b_{i1}$$

$$\cup_2 = \cup_{i\in I}(a_{i2}, b_{i2}),\ a_{i2} < b_{i2}$$

$$...$$

$$\cup_n = \cup_{i\in I}(a_{in}, b_{in}),\ a_{in} < b_{in}$$

We let $J$ = {1, 2, ..., n}.

$$\cap_{j\in J}\cup_{ij} = \cap_{j\in J} \cup_{i\in I_j} (a_{ij}, b_{i,j})$$

By adding empty sets if necessary, we may assume that $I_1 = I_2 = ... = I_n = I$. Then we use the lemma to write the right hand side as,

$$\cap_{i\in I} \cup_{j\in J} (a_{ij}, b_{i,j})$$

So we see that this union is a union of open intervals, as desired, so is in $T$. End of proof.

How do we visualize a topology? A topological space is a pair consisting of a set $\mathcal{X}$ and a topology $T$. The role of the topology is to give a sense of how the elements of $\mathcal{X}$ are geometrically arranged, i.e., how they are spaced out. We can think of the set X as the set consisting of houses and an open set is a neighborhood in which the house is located. Topology is not the same as introducing the notion of distance, it is more general. It makes sense even in spaces where no distance can be introduced. Given, $\mathcal{X} = \{x, y\}$, where T is a discrete topology, we can think of x and y in a common neighborhood, which is the set x, but they exist in their own neighborhoods as well, $\{x\}$ and $\{y\}$. Any topological space with a discrete topology can be thought of as taking the set X and completely separating its points from other points.

Recall that a topological space is a non-empty set X together with a collection of subsets (called open sets) such that this collection (which is called a topology) contains the empty set, X, and is closed under arbitrary unions and finite intersections.

**Definition:** Neighborhood Let X be a topological space. Let x be a point in X. We say that a subset N of X is a *neighborhood* of x if there exists an open set U contains x such that $U \subset N$. Note that the neighborhood of x need not be open as long as it contains an open set that in turn contains x.

**Definition:** Closed subset Let X be a topological space. Any subset of X whose complement is open is called a *closed subset* of X. Note that a set could potentially be both open and closed.

**Example 1:** Let X={1,2} be a Sierpinski space with {x} being open. Then the closed sets of X are {y}=X-x, X=X-∅, and ∅ = X-X. Note that in a Sierpinski space only one point can be open.

**Example 2:** Let X=ℝ with the standard topology. Examples of closed set include the empty set, ∅, and there are infinitely more.

$$[a, b] = X - ((-\infty, a) \cup (b, \infty))$$

$$\{a\} = X - ((-\infty, a) \cup (a, \infty))$$

$$\{a, b\} = X - ((-\infty, a) \cup (a, b) \cup (b, \infty))$$

$$[0, 1] \cup [3, 4] = ((-\infty, 0) \cup (1, 3) \cup (4, \infty))$$

What about the following,

$$A = \cup_{n \in \mathcal{Z}, n>1}[\frac{1}{n}, 1] = (0, 1]$$

$$X - A = (-\infty, 0] \cup (1, \infty)$$

Suppose $(-\infty, 0]$ is the union of open intervals. Then 0 must lie in at least on of these intervals. Suppose then that $0 \in (a, b)$. We must have $a < 0 < b$. Then,

$$\frac{b}{2} \in (a, b) \subset (-\infty, 0]$$

This is a contradiction because $\frac{b}{2}$ is positive. Thus, A is not a closed set.

**Proposition:** Let S be a set. Let $\{A_i\}_{i \in I}$ be an indexed family of sets and suppose that each $A_i$ is a subset of S. Then,

$$S - \cup_{i \in I} A_i = \cap_{i \in I}(S - A_i)$$

$$S - \cap_{i \in I} A_i = \cup_{i \in I}(S - A_i)$$

**Proof:** Now let $\{A_i\}_{i \in I}$ be an indexed family of closed sets. Then for each i in I we know that the complement of $A_i$, such that the set $X - A_i$ is open. We know by the property of topology that,

$$\cup_{i \in I}(X - A_i) = \ open$$

Such that,

$$= X - \cap_{i \in I} A_i$$

This tells us that the complement of the intersection of an arbitrary number of closed sets is open. Hence, arbitrary intersections of closed sets is closed. In the same way we show that finite unions of closed sets are closed.

**Question:** Are the integers as a subset of $\mathbb{R}$ with the standard topology closed? Note the integers are a union of closed sets, namely singletons, each containing a different integer, but this does not mean that the integers are closed because unions of an infinite collection of closed sets need not be closed.

$$\mathbb{Z} = \mathbb{R} - \cup_{n \in \mathbb{Z}}(n, n+1)$$

Now consider $X = \mathbb{R}$ with the finite complement topology. This means that the open sets are the empty set and those subsets of X which are finite. In other words, we can say that the closed subsets of X are X and finite subsets of X.
Given, [1,2], this interval is not closed in this topology X, nor is it open.
Given, {1,2}, this interval is closed in this topology X, and is also closed in the standard topology.

**Example:** Consider the set of real numbers with the following topologies:

1. Trivial

2. Discrete

3. Standard

4. Finite complement

(1,2) is open in discrete, standard, and closed in discrete.

**Definition:** Hausdorff Let X be a topological space. Then X is called *Hausdorff* if given any two distinct points x and y in X there exist a neighborhood U of x and V of y such that their intersection is empty.

$$U \cap V = \emptyset$$

## *Closure, Interior, and Boundary*

**Definition:** Closure
Let X be a topological space and let A be a subset of X. We say that a point x of X lies in the *closure* of A in X if for each neighborhood $U_x$ of x one has $U_x \cap A \neq \emptyset$. The set of all points in X that lie in the closure of A in X is called the *closure of A in X*. We denote it by $\overline{A}$.

**Lemma 1:** If A is a subset of a topological space and F is a closed subset of X that contains A, then $\overline{A} \subset F$, that is any closed set that contains A also contains the closure of A.

**Proof:** Suppose that a point x of X does not belong to F. Then x belongs to the complement of F, X-F, which is also open because its complement is a closed set, therefore,

$$A \subset F \rightarrow X - F \subset X - A$$

Hence, $(X - F) \cap A = \emptyset$. Since X-F is a neighborhood of x we get that $x \notin \overline{A}$. If x were in the closure of A, then every neighborhood of x would intersect A, and here we have a neighborhood of X-F of x does not intersect A. So we have shown that $X - F \subset X - \overline{A}$ which implies $\overline{A} \subset F$.

**Example:** Suppose we take $X = \mathbb{R}$ with the standard topology and we want to find the closure of the set [0,1). If x lies in [0,1). Let $U_x$ be a neighborhood of x. Therefore, $U_x \cap [0, 1) \neq \emptyset$ because x lies in this intersection. Since x lies in this interval, we can say that x lies in the closure of the interval. Hence, the interval is a subset of the closure, $[0,1) \subset \overline{[0,1)}$. By lemma 1, we know that the closure $\overline{[0,1)} \in [0,1]$. We now know that the closure can either include the point 1 or not include the point 1, [0,1) or [0,1].

**Lemma 2:** If A is a subset of a topological space X and x is a point in X that does not belong to the closure of A in X, then x does not belong to some closed set F that contains A.

**Proof:** If $x \notin \overline{A}$ then there is an open neighborhood U (because every neighborhood contains an open neighborhood) such that $U \cap A = \emptyset$. Let F be the complement of U. Then F is closed because U is open, and $F = X - U$ and therefore $A \in F$. However, $x \in U$, and therefore, $x \notin F$.

**Corollary 1:** Let A be a subset of a topological space X. Then, $\overline{A} = \cap_{A \subset F, F \textit{ is closed}} F$, that is, the closure of A is the intersection of all closed sets that contain A, that is, the closure of A is the smallest closed set that contains A.

**Proof:** By Lemma 1 we know that the closure of A is contained in every closed set that contains A, so the closure of A is contained in the intersection of all closed sets that contain A. Conversely, if x is a point in every some closed set F that contains A such that it is a point in the intersection of all such sets, then x belongs to the closure by the contrapositive of Lemma 2, so we get the opposite inclusion.

**Corollary 2:** If A is a subset of a topologival space X then $A = \overline{A}$ if and only if A is closed.

To return to our example,

$$\overline{[0,1)} = [0,1), [0,1]$$

$[0,1)$ is not closed because its complement is $(-\infty,0) \cup [1,\infty)$. So by Corallary 2, the closure of [0,1) is [0,1].

Recall that if A is a subset of a topological space X, then a point x $\in$ X is said to be in the closure of A if every neighborhood of x has a non-empty intersection with A. The closure of A is denoted by $\overline{A}$. We proved that the closure of A is the intersection of all closed subsets of X which contain A. We also showed that A is always contained in its closure and if A is closed then it follows from the definition of the result we proved that if A is closed then it is equal to its closure. In fact, we have the following proposition.

**Proposition 1:** A subset A of a topological space X is closed if and only if A = $\overline{A}$.

**Proof:** If A is closed then as discussed above it is equal to its closure. Conversely, suppose that A is equal to its closure. This means that $A = \cap_{A \subset F}$ where F is closed. But the intersection of closed sets is closed, therefore we get that A is closed.

**Definition:** Interior
Let A be a subset of a topological space. A point x ∈ X is said to be in the interior of A if A is a neighborhood of x. We will denote that interior by Int(A).

**Easy Facts about Interior:**

1. If A is a subset of a topological space and U is an open set contained in A, then $U \subset Int(A)$. Indeed, if x is a point in U, then A is a neighborhood of x since U is open and U is contained in A. So, x is in the interior of A.

2. If A is a subset of a topological space and x is a point in Int(A) then x belongs to some open set U contained in A. Indeed, if x belongs to the interior of A, then by definition A is a neighborhood of A, so there is some open set U contained in A that contains x.

**Proposition 2:** If A is a subset of a topological space X then Int(A) is the union of all open sets contained in A, i.e., $Int(A) = \cup_{U \subset A, U = open} U$. Let A be a subset of a topological space X. Then the closure of A is the intersection of all closed sets that contain A, and the interior of A is the union of all open sets contained in A. A set is equal to its closure if and only if it is closed, and a set is equal to its interior if and only if it is open.

To see the latter property of the interior, not that if A is open then $B \subset A$ for any open set B contained in A, therefore $\cup_{B \subset A, B = Open} B = A$. In other words, Int(A) is always a subset of A for any A, and if A is open then A is part of the union, so the union is equal to A.

Conversely, if A is equal to the interior, then we have expressed A as a union of open sets, which forces A to be open, because the union of open sets is always open. Every set is a subset of its closure, and every set contains its interior as a subset. Note that the closure is always closed because it is the union of closed sets, and the interior is open because it is the union of open sets.

So for every set we have the following,

$$Int(A) \subset A \subset \overline{A}$$

**Example 1:**
$Int(A) = A =$ closure of A if and only if A is both open and closed. So, in particular, the interior and the closure of X are all equal and the same for the empty set.

**Example 2:**
Let X=R with the standard topology. Let $A = (0, 1)$. $Int(A) = (0, 1)$, therefore, the smallest closed set that contains (0,1), i.e. the intersection of all closed sets that contain (0,1), is as so, $\overline{A} = [0, 1]$. Therefore, $\overline{(0, 1)} \subset [0, 1]$.

So, the options for the closure are:

1. (0,1)

2. 0,1) (0,1

0,1

Remember, that the closure is always closed because it is the union of closed sets. (0,1) is not closed because its complement is not open.