Advanced Counting Techniques

Chapter 8

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

Applications of Recurrence Relations

Section 8.1

Section Summary

- ✓ Applications of Recurrence Relations
  - Fibonacci Numbers
  - The Tower of Hanoi
  - Counting Problems
- ✓ Algorithms and Recurrence Relations

A Counting Problem:

The number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in *n* hours?

a_n : the number of bacteria at the end of *n* hours

$$\boxed{a_n = 2a_{n-1}}$$

$$a_0 = 5$$

A variety of counting problems can be modeled using recurrence relations.

Recurrence Relations

【Definition】 A recurrence relation for the sequence $\{a_n\}$ is an equation that express a_n in terms of one or more of the previous terms of the sequence, namely, a_0 , a_1 , a_2 , … , a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integers.

$$a_n = f(a_0, a_1, a_2, \dots, a_{n-1}) \quad n \geq n_0$$

For example,

(1) The Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$.

(2) Pascal's recursion for the binomial coefficient is a two variable recurrence equation:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

A solution of a recurrence relation is a sequence if its terms satisfy the recurrence relation.

〖Example 1〗 Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n=2,3,4,…,where $a_n = 3n$ for every nonnegative integer n.

Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution :

$a_n = 3n$ ✓

$a_n = 2^n$ ✗

$a_n = 5$ ✓

Given a recurrence relation, how many initial conditions are needed to uniquely identify the sequence?

Normally, there are many sequences which satisfy a recurrence relation. We should distinguish them by initial conditions.

The degree of a recurrence relation

$$a_n = a_{n-1} + a_{n-8}$$ ---- a recurrence relation of degree 8

For example,

(1) In the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$ we must specify a_0 and a_1 .

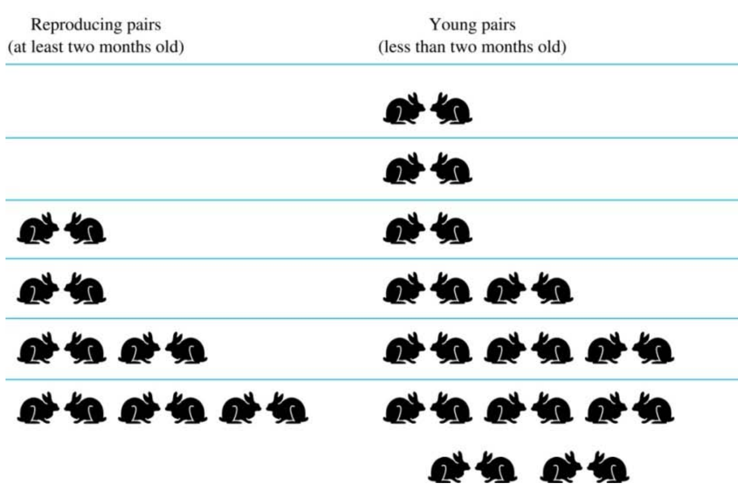
(2) In Pascal's identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

we must specify $C(1,0)$ and $C(1,1)$.

Modeling with Recurrence Relations

〖Example 1〗 ***Rabbits and the Fibonacci numbers***

A young pair of rabbits is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in the following Figure. <u>Find a recurrence relation for the number of pairs of rabbits on the island after *n* months</u>, assuming that no rabbits ever die.



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci numbers

Solution :

f_n : The number of pairs of rabbits after n month

We can show that $f_1, f_2, \ldots$ are the terms of the Fibonacci numbers

$$f_1 = 1$$

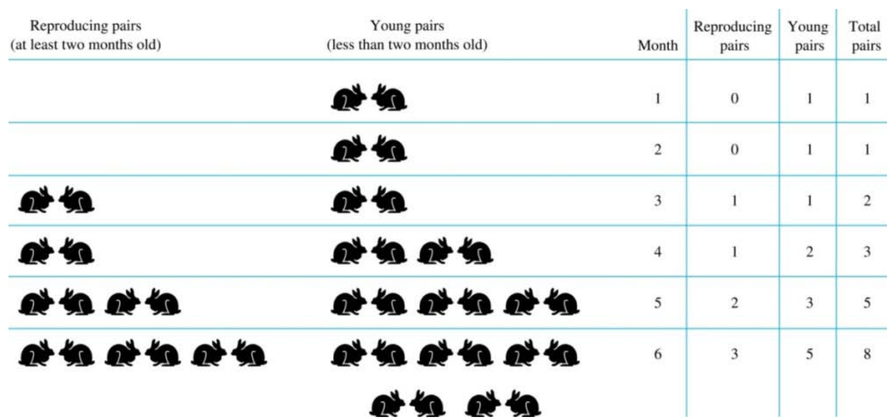
$$f_2 = 1$$

$$f_3 = 1+1 = 2$$

$$f_4 = 2+1 = 3$$

……

$$f_n = f_{n-1} + f_{n-2}$$



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

f_{n-1}: the number of previous month

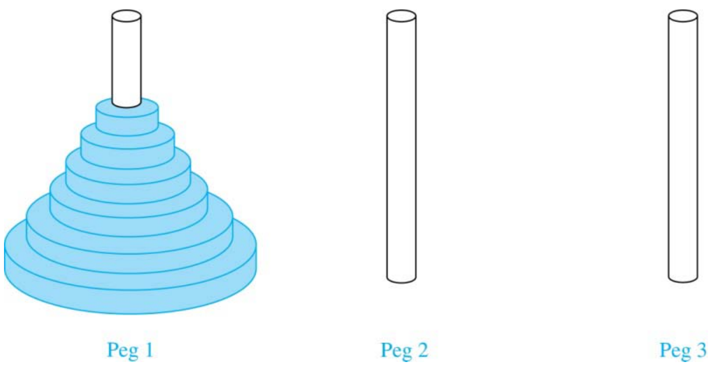
f_{n-2}: the newborn pairs

The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.



The Initial Position in the Tower of Hanoi Puzzle

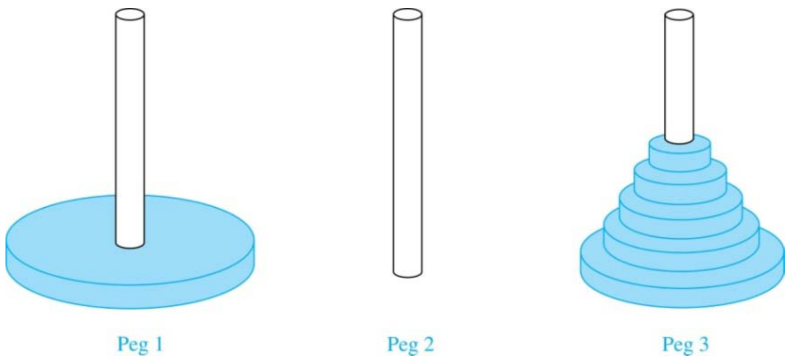
Let H_n denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. $H_n = ?$

Set up a recurrence relation for the sequence $\{H_n\}$

The Tower of Hanoi

Solution:

Let H_n denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks.



Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.

First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$\boxed{H_n = 2H_{n-1} + 1.}$$

The initial condition is $\boxed{H_1 = 1}$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi

- $\boldsymbol{H_n} =$**?**

use an **iterative approach** to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\ \ \vdots \\
&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \qquad \textit{because } H_1 = 1 \\
&= 2^n - 1 \qquad \textit{using the formula for the sum of the terms of a geometric series}
\end{aligned}
$$

- There was **a myth created with the puzzle**. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each second. When the puzzle is finished, the world will end.
  - Using this formula for the 64 gold disks of the myth,

    $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$

    seconds are needed to solve the puzzle, which is more than 500 billion years.

- **Reve's puzzle** (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the the minimum number of moves needed to solve this puzzle. (*see Exercises* 38-45)

〖Example 3〗Find a recurrence relation for the number of bit strings of length *n* that don't have two consecutive 0s.

Solution :

Let a_n denote the number of bit strings of length *n* that don't have two consecutive 0s.

Any bit string of length n-1 with no two consecutive 0s. | 1 → a_{n-1}

Any bit string of length n-2 with no two consecutive 0s. | 1 | 0 → a_{n-2}

Recurrence relation:

$$a_n = a_{n-1} + a_{n-2} .$$

Initial conditions: $a_1=2,\ a_2=3$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Many relationships are most easily described using recurrence relations.

Recurrence relations and complexity of algorithm

```
long int  f(int n)
{ if (n <= 1)   return 1;
  else  return f(n - 1) + f(n - 2);
}
```

$T(n)$: the running time for the method call $f(n)$

$$T(n) = T(n-1) + T(n-2) + 2$$

$$T(0) = 1$$

$$T(1) = 1$$

ALGORITHM **The Binary Search Algorithm.**

Procedure *binary search* (x: integer,

$a_1, a_2, \ldots, a_n$: increasing integers)

$i := 1$ {i is left endpoint of search interval }

$j := n$ { j is right endpoint of search interval }

while $i < j$

begin

$m := \lfloor (i + j)/2 \rfloor$

if $x > a_m$ then $i := m + 1$

else $j := m$

end

if $x = a_i$ then $location := i$

else $location := 0$

{location is the subscript of term equal to *x* ,or 0 if x is not found }

f_n*: the number of comparisons required to search for an element in a search sequence of size *n

f(n)=f(n/2)+2

f(1)=1

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- ✓ **Linear Homogeneous Recurrence Relations**
- ✓ **Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.**
- ✓ **Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.**

Linear Homogeneous Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

- ***linear:*** **the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of *n***
- ***constant coefficients*** **:the coefficients in the sum of the a_i' s are constants, independent of *n*.**
- ***degree k:*** **a_n is expressed in terms of the previous *k* terms of the sequence**
- ***homogeneous*** **: because no terms occur that are not multiples of the a_js. Otherwise *inhomogeneous* or *nonhomogeneous*.**

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the *k* initial conditions $a_0 = C_1, a_0 = C_1, \ldots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

〖Example 2〗

(1) $a_n = (1.02)a_{n-1}$

linear; constant coefficients; homogeneous; degree 1

(2) $a_n = (1.02)\, a_{n-1} + 2^{n-1}$

linear; constant coefficients; nonhomogeneous; degree 1

(3) $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

linear; constant coefficients; nonhomogeneous; degree 3

(4) $a_n = n\, a_{n-1} + n^2\, a_{n-2} + a_{n-1}\, a_{n-2}$

nonlinear; coefficients are not constants; homogeneous; degree 2

Solving Linear Homogeneous Recurrence Relation With Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

The basic approach:

***To look for solution of the form $a_n = r^n$*, where *r* is a constant.**

Solution Procedure:

(1) Put all a_i' s on the left side of the equation, everything else on the right. If nonhomogeneous, stop (for now).

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_k a_{n-k} = 0$$

(2) Substitute the solution into the equation, factor out the lowest power of *r* and eliminate it.

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \ldots - c_k r^{n-k} = 0$$

$$r^{n-k}(r^k - c_1 rk^{n-1} - c_2 r^{k-2} - \ldots - c\ )$$

Characteristic equation

(3) We obtain the equivalent equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

Characteristic root

(4) Find its *k* roots $r_1, r_2, \ldots, r_k$

These characteristic roots can be used to given an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

【Theorem 1】 Let c_1, c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2 . Then the Sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ for $n = 0,1,2,\ldots$, where α_1, α_2 are constants.

Proof:

- **Show that if r_1, r_2 are the roots of the characteristic equation, and α_1, α_2 are constant, then the sequence $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ is a solution of the recurrence relation**

$$r_1^{\,2} = c_1 r_1 + c_2$$

$$r_2^{\,2} = c_1 r_2 + c_2$$

$$c_1a_{n-1}+c_2a_{n-2}=c_1(\alpha_1r_1^{\,n-1}+\alpha_2r_2^{\,n-1})+c_2(\alpha_1r_1^{\,n-2}+\alpha_2r_2^{\,n-2})$$

$$=\alpha_1r_1^{\,n-2}(c_1r_1+c_2)+\alpha_2r_2^{\,n-2}(c_1r_2+c_2)$$

$$=\alpha_1r_1^{\,n}+\alpha_2r_2^{\,n}$$

$$=a_n$$

- **Show that if** $\{a_n\}$ **is a solution, then** $a_n=\alpha_1r_1^{\,n}+\alpha_2r_2^{\,n}$ **for some constant** α_1,α_2 **.**

Suppose that $\{a_n\}$ **is a solution, and the initial condition** $a_0=C_0, a_1=C_1$ **hold.**

$$a_0=C_0=\alpha_1+\alpha_2$$

$$a_1=C_1=\alpha_1r_1+\alpha_2r_2$$

$$\alpha_1=\frac{C_1-C_0r_2}{r_1-r_2}$$

$$\alpha_2=\frac{C_0r_1-C_1}{r_1-r_2}$$

Using Theorem 1

〖Example 2〗 What is the solution of the recurrence relation

$$a_{n+2} = 3a_{n+1}, a_0 = 4$$

Solution:

(1) The Characteristic equation of the recurrence relation is
***r* – 3=0 .**

(2) Find the root of the characteristic equation: $r_1 = 3$

(3) Compute the general solution: $a_n = c3^n$

(4) Find the constants based on the initial conditions:

$$a_0 = c3^n = 4$$

(5) Produce the specific solution: $a_n = 4 \cdot 3^n$

An Explicit Formula for the Fibonacci Numbers

〖Example 3〗**Find an explicit formula for the fibonacci numbers.**

Solution:

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \quad f_1 = 1$$

(1) Determine the characteristic equation: $r^2 - r - 1 = 0$

(2) Find its roots: $r_1 = \dfrac{1+\sqrt{5}}{2}, r_2 = \dfrac{1-\sqrt{5}}{2}$

(3) Compute the general solution:

$$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \alpha_1, \alpha_2 \text{ are constant.}$$

(4) Dtermine α_1, α_2**:** $\alpha_1 = \dfrac{1}{\sqrt{5}}, \alpha_2 = -\dfrac{1}{\sqrt{5}}$

Consequently, the fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

【Theorem 2】 **Let c_1, c_2 be real numbers with $c_2 \neq 0$.**
Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.
A sequence $\{a_n\}$ is a solution of the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if
$a_n = \alpha_1 r_0^{\ n} + \alpha_2 n r_0^{\ n}$ *for* $n = 0,1,2,\ldots$, where α_1, α_2 are constants.

Proof:

Omitted.

Using Theorem 2

〖Example 4〗 $a_n = 6a_{n-1} - 9a_{n-2}, a_0 = a_1 = 1$

Solution:

(1) Recurrence system : $a_n - 6a_{n-1} + 9a_{n-2} = 0$

(2) Determine the characteristic equation: $(b-3)^2 = 0$

(3) Find its roots : $b_1 = b_2 = 3$

(4) Compute the general solution: $a_n = (\alpha_1 + \alpha_2 n)3^n$

(5) Solve for coefficients : $\begin{cases} a_0 = 1 = \alpha_1 \\ a_1 = 1 = (\alpha_1 + \alpha_2) \cdot 3 \end{cases}$

Consequently, $a_n = (1 - \frac{2}{3}n)3^n$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

【Theorem 3】 Let $c_1, c_2, \ldots, c_k$ **be real numbers. Suppose that the characteristic equation** $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ **has *k* distinct roots** $r_1, r_2, \ldots, r_k$ **. Then a sequence** $\{a_n\}$ **is a solution of the recurrence relation** $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$
if and only if $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n + \ldots + \alpha_k r_k{}^n$
for $n = 0, 1, 2, \ldots$ **where** $\alpha_1, \alpha_2, \ldots, \alpha_k$ **are constants.**

The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ **are found by enforcing the initial conditions**

The General Case with Repeated Roots Allowed

【Theorem 4】 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has ***t*** **distinct roots** $r_1, r_2, \ldots, r_t$ **with multiplicities** $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ *for* $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$ Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n +$$

$$(\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \ldots +$$

$$(\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \ldots$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t, 0 \leq j \leq m_i - 1$.

Linear Nonhomogeneous Recurrence Relation With Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

Where c_i(i=1,2,…,k) is real numbers, F(n) is a function not identically zero depending only on n.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

the associated *homogeneous* recurrence relation

Note:

Solution to nonhomogeneous case is sum of solution to associated homogeneous recurrence system and a particular solution to the nonhomogeneous case.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

【Theorem 5】 Let $\{a_n^{(p)}\}$ **be a *particular solution* of the nonhomogeneous linear recurrence relation with constant coefficients**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

Then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ **, where** $\{a_n^{(h)}\}$ **is a solution of the associated homogeneous recurrence relation.**

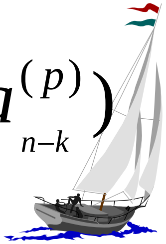
$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

Proof: $$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n)$$

Suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n)$$

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \ldots + c_k(b_{n-k} - a_{n-k}^{(p)})$$



【 Theorem 6】 Assume a linear nonhomogeneous recurrence equation with constant coefficients with the nonlinear part *F(n)* of the form

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$$

If *s* is not a root of the characteristic equation of the associated homogeneous recurrence equation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

If *s* is a root of multiplicity *m* , a particular solutions is of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n$$

〖Example 5〗 $a_n = 6a_{n-1} - 9a_{n-2} + F(n),$

$where F(n) = 3^n, n3^n, n^2 2^n, and (n^2+1)\cdot 3^n$

Solution:

(1) The general solution of the associated homogeneous recurrence equation : $a_n = (\alpha_1 + \alpha_2 n)\cdot 3^n$

(2) A particular solution of the form:

$$p_0 n^2 3^n \; if F(n) = 3^n$$

$$n^2(p_1 n + p_0)\cdot 3^n \; if F(n) = n3^n$$

$$(p_2 n^2 + p_1 n + p_0)\cdot 2^n \; if F(n) = n^2 2^n$$

$$n^2(p_2 n^2 + p_1 n + p_0)\cdot 3^n \; if F(n) = (n^2+1)3^n$$

〖Example 6〗 **Let a_n be the sum of the first n positive integers. Note that a_n satisfies the recurrence relation $a_n = a_{n-1} + n$. Find The explicit formula** of a_n

Solution:

(1) The general solution of the associated homogeneous recurrence equation : $a_n^{(h)} = c \cdot (1)^n = c$

(2) A particular solution of the form:

$$n(p_1 n + p_0) = p_1 n^2 + p_0 n$$

(3) Find p_1, p_0:

$$p_1 n^2 + p_0 n = p_1 (n-1)^2 + p_0 (n-1) + n$$

$$n(2p_1 - 1) + (p_0 - p_1) = 0$$

$$p_0 = p_1 = 1/2,$$

(4) Find c: using initial condition $a_1 = 1$

Homework:

Seventh Edition:

P. 511 8, 10, 12, 26, 32, (33-37)

P. 524 2, 4(g), 20, 30, 32, 36