Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.
- Prenex normal form

Nested Quantifiers

- **Quantifiers that occur within the scope of other quantifiers**
  - often necessary to express the meaning of sentences as well as important concepts in computer science and mathematics.

Example 1 : "Every real number has an inverse" is

$\forall x \exists y (x + y = 0)$

where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

◆Nested Loops

- To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.

- To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

- **The order of quantifiers is important unless all the quantifiers are universal quantifiers or all the quantifiers are existential quantifiers**

Example 2 :

① Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then

$\forall x\, \forall y P(x,y)$ and $\forall y\, \forall x P(x,y)$ have the same truth value.

② Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then

$\forall x \exists y P(x,y)$ is true, but $\exists y\, \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 3: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\, P(x,y)$

   Answer: True

4. $\exists x \exists y\, P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 4: Let U be the real numbers,

Define $P(x,y)$: $x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\, P(x,y)$

   Answer: False

4. $\exists x \exists y\, P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

Translating Nested Quantifiers into English

Example 5: Translate the statement

$$\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Translating Nested Quantifiers into English

Example 6: Translate the statement

$$\exists x \forall y \forall z ((F(x, y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

where $F(a,b)$ means a and b are friends and the universe of discourse for x, y, and z is the set of all students in you school.

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic Involving Nested Quantifiers

Example 7 : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain(all integers), to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

   $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$

   $\forall x > 0 \forall y > 0 (x + y > 0)$ **quantifiers with restricted domains**

Translating English into Logical Expressions

Example 8: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w\, \forall a\, \exists f\ (P(w,f) \land Q(f,a))$$

Translating English into Logical Expressions

Example 9: Use quantifiers to express the statement "*Everyone has exactly one best friend.*"

Solution:

1. Let $B(x, y)$ be the statement "y is the best friend of x".
2. Let the universe of discourse for the variables be the set of all people in the world.
3. Consequently, we can translate the sentence as :

$$\forall x \, \exists y \, B(x, y) \quad \times$$

$$\forall x \exists y \forall z (B(x, y) \wedge ((z \neq y) \rightarrow \neg B(x, z)))$$

Translating English into Logical Expressions

Example 10: Use quantifiers to express the statement

"Not all of the real numbers are rational numbers.."

Solution:

1. $R(x)$: x is a real number, $Q(x)$: x is a rational number
2. The domain of x is all numbers
3. we can write it symbolically as :

$$\neg\forall x(R(x) \rightarrow Q(x))$$

$$\equiv \exists x(R(x) \wedge \neg Q(x))$$

Calculus in Logic

Example 11: Use quantifiers to express the definition of the limit of *a* real-valued function *f(x)* of a real variable *x* at a point *a* in its domain.

Solution: Recall the definition of the statement

$$\lim_{x\to a} f(x) = L$$

is "For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$."

① The domain for the variables ε and δ consists of all positive real numbers and the domain for *x* consists of all real numbers.

$$\forall\epsilon\exists\delta\forall x(0 < | x - a | < \delta \to | f(x) - L | < \epsilon)$$

② The domain for the variables ε， δ and *x* consists of all real numbers

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta) \to |f(x) - L| < \varepsilon)$$

③ The domain for the variables ε， δ and *x* consists of all numbers

$$\forall \varepsilon (R(\varepsilon) \wedge P(0,\varepsilon) \to \exists \delta (R(\delta) \wedge P(0,\delta) \wedge \forall x (R(x) \wedge P(|x - x_0|, \delta) \to P(|f(x) - L|, \varepsilon))))$$

Calculus in Logic

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① The domain for the variables ε and δ consists of all positive real numbers and the domain for *x* consists of all real numbers.

$$\forall\epsilon\exists\delta\forall x(0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon)$$

② The domain for the variables ε， δ and *x* consists of all real numbers

$$\forall\varepsilon > 0 \exists\delta > 0 \forall x(0 < |x - a| < \delta) \to |f(x) - L| < \varepsilon)$$

③ The domain for the variables ε， δ and *x* consists of all numbers

$$\forall\varepsilon(R(\varepsilon) \wedge P(0,\varepsilon) \to \exists\delta(R(\delta) \wedge P(0,\delta) \wedge \forall x(R(x) \wedge P(|x - x_0|,\delta) \to P(|f(x) - L|,\varepsilon))))$$

Questions on Translation from English

Example 12: Choose the obvious predicates and express in predicate logic.

1. *Brothers are siblings.*

$$\forall x\, \forall y\, (B(x, y) \rightarrow S(x, y))$$

2. *Siblinghood is symmetric.*

$$\forall x\, \forall y\, (S(x, y) \rightarrow S(y, x))$$

3. *Everybody loves somebody.*

$$\forall x\, \exists y\, L(x, y)$$

4. *There is someone who is loved by everyone.*

$$\exists y\, \forall x\, L(x, y)$$

5. *There is someone who loves someone.*

$$\exists x\, \exists y\, L(x, y)$$

6. *Everyone loves himself.*

$$\forall x\, L(x, x)$$

Negating Nested Quantifiers

Example 13: Recall the logical expression developed some slides back:

$$\exists w \forall a \exists f\, (P(w,f) \land Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\, (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\, (P(w,f) \land Q(f,a))$
2. $\forall w \neg \forall a \exists f\, (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\, (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Return to Calculus and Logic

Example 14: Recall the logical expression developed in the calculus example three slides back.

Use quantifiers and predicates to express that $\lim_{x \to a} f(x)$ does not exist.

1. We need to say that for all real numbers L, $\lim_{x \to a} f(x) \neq L$

2. The result from the previous example can be negated to yield:

$$\neg \forall \epsilon \exists \delta \forall x (0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon)$$

3. Now we can repeatedly apply the rules for negating quantified expressions:

$$\begin{aligned}
&\neg \forall \epsilon \exists \delta \forall x (0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon) \\
&\quad \equiv \exists \epsilon \neg \exists \delta \forall x (0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon) \\
&\quad \equiv \exists \epsilon \forall \delta \neg \forall x (0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon) \\
&\quad \equiv \exists \epsilon \forall \delta \exists x \neg (0 < |\, x - a \,| < \delta \to |\, f(x) - L \,| < \epsilon) \\
&\quad \equiv \exists \epsilon \forall \delta \exists x \cancel{\neg} (0 < |\, x - a \,| < \delta \wedge |\, f(x) - L \,| \geq \epsilon)
\end{aligned}$$

Calculus in Predicate Logic

4. Therefore, to say that $\lim_{x \to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x \to a} f(x) \neq L$ can be expressed as:

$$\forall L \exists \epsilon \forall \delta \exists x (0 < | x - a | < \delta \wedge | f(x) - L | \geq \epsilon)$$

Remember that ε and δ range over all positive real numbers and *x* over all real numbers.

5. Translating back into English we have, for every real number *L*, there is a real number ε > 0, such that for every real number δ > 0, there exists a real number *x* such that $0 < | x - a | < \delta$ and $|f(x) - L| \geq \varepsilon$.

Prenex Normal Forms

Motivation:

- simplifies the surface structure of the sentence.
- useful to automated theorem proving.

Prenex normal form: $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n B$

Where $Q_i (i = 1, \ldots, n)$ is $\forall$ or $\exists$ and the formula *B* is quantifier free.

For example,

$(1) \forall x P(x) \vee \exists x Q(x)$ ✘

$(2) \neg \forall x \forall y (P(x) \to Q(y))$ ✘

$(3) \forall x \forall y \neg (P(x) \to Q(y))$ ✓

$(4) R(x, y)$ ✓

Algorithm for prenex normal form

Any expression can be converted into prenex normal form.

How to obtain prenex normal form?

1. Eliminate all occurrences of $\rightarrow$ and $\leftrightarrow$ from the formula in question.
2. Move all negations inward such that, in the end, negation only appear as part of literals.
3. Standardize the variables a part(when necessary).
4. The prenex normal form can now be obtained by moving all quantifiers to the front of the formula.

〖Example 15〗 Convert the following formulas into prenex normal form.

$$\forall x((\exists y R(x,y) \wedge \forall y \neg S(x,y)) \to \neg(\exists y M(x,y) \wedge P))$$

Solution:

$$\forall x((\underline{\exists y R(x,y) \wedge \forall y \neg S(x,y)}) \to \neg(\underline{\exists y M(x,y) \wedge P}))$$

$$\Leftrightarrow \forall x(\neg(\exists y R(x,y) \wedge \forall y \neg S(x,y)) \vee \neg(\exists y M(x,y) \wedge P))$$

$$\Leftrightarrow \forall x((\neg\exists y R(x,y) \vee \neg\forall y \neg S(x,y)) \vee (\neg\exists y M(x,y) \vee \neg P))$$

$$\Leftrightarrow \forall x((\forall y \neg R(x,y) \vee \exists y S(x,y)) \vee (\forall y \neg M(x,y) \vee \neg P))$$

$$\Leftrightarrow \forall x(\forall y \neg R(x,y) \vee \exists z S(x,z) \vee \forall u \neg M(x,u) \vee \neg P)$$

$$\Leftrightarrow \forall x \forall y \exists z \forall u(\neg R(x,y) \vee S(x,z) \vee \neg M(x,u) \vee \neg P)$$

Homework:

1) **Seventh Edition:**

P.53 16, 24, 34, 51, 62

P.65 7(b,d,f),12(d,h ,k,n), 19, 33,38(b,d),48

2) Give the prenex normal forms of the following formulas:

1) $(\forall x)(P(x) \rightarrow (\exists y)Q(x,y))$

2) $(\forall x)(\forall y)(((\exists z)P(x,y,z) \wedge (\exists u)Q(x,u)) \rightarrow (\exists v)Q(y,v))$