Permutations and Combinations

Section 6.3

Section Summary

- ✓Permutations
- ✓Combinations
- ✓Combinatorial Proofs

Permutation

permutation **: an ordered arrangement of the elements of a set**

r-permutation **: an ordered arrangement of *r* elements of a set**

【Theorem 1】 The number of *r*-permutations of a set with *n* distinct elements is

$$P(n, r)=n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof: **Using the product rule.**

n choices for the first element, $(n-1)$ for the second one, $(n-2)$ for the third one…

Note :

- **$P(n,0) = 1$, since there is only one way to order zero elements.**
- **$P(n, n)=n!$**

Combinations

r-combination: **an unordered selection of *r* elements of a set**

Note: An *r*-combination is simply a subset of a set with *r* elements.

***C(n, r)*: the number of *r*-combination of a set with *n* element**

$$C(n,r)=\binom{n}{r}$$

Binomial coefficient

$$=\frac{n!}{r!(n-r)!}$$

【Theorem 2】 The number of *r*-combination of a set with *n* elements, where *n* is a positive integer and *r* is an integer with 0≤*r*≤*n*, equals

$$n(n-1)(n-2)\ldots(n-r+1)/r!$$

〖Example 1〗 **A soccer club has 8 female and 7 male members. For today's match, how many possible configurations are there?**

(1) The coach wants to have 6 female and 5 male players on the grass.

(2) The coach wants to have 11 players with at most 5 male players on the grass.

Solution:

(1) $C(8, 6) \cdot C(7, 5)$

$= 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$

$= 28\cdot 21$

$= 588$

(2) $C(8, 6)C(7, 5)+C(8, 7)C(7, 4)+C(8, 8)\,C(7, 3)$

【 Corollary 1 】 **Combination Corollary**

Let n and r be nonnegative integers with $r \le n$. Then

$$C(n, r) = C(n, n-r)$$

Proof:

(1) Using theorem 2 $$C(n,r) = \frac{n!}{r!(n-r)!}$$

(2) Using **Combinatorial Proof**

A combinatorial proof of an identity:

- **double counting proofs**

  uses counting arguments to prove that both sides of the identity count the same objects but in different ways.

- **bijective proofs**

  show that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements.
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.

Binomial Coefficients

Section 6.4

Section Summary

- ✓The Binomial Theorem
- ✓Pascal's Identity and Triangle
- ✓Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A ***binomial* expression** is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- **We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.**
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- **We have used a counting argument to show that** $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

The Binomial Theorem

【 Theorem 1 】 The Binomial Theorem

Let x and y be varaibles, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Proof:

We use combinatorial reasoning .

The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Using the Binomial Theorem

〖Example 1〗 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x\text{-}3y)^{25}$?

Solution:

We view the expression as $(2x+(-3y))^{25}$.

By the binomial theorem

$$(2x+(-3y))^{25}=\sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^{j}$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j=13$.

$$\binom{25}{13}(2)^{12}(-3)^{13}=-\frac{25!}{13!12!}2^{12}3^{13}$$

Corollaries for the Binomial Theorem

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

PASCAL'S Identity

【 Theorem 2 】 **PASCAL'S Identity**

Let n and k be positive integers with $k \le n$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof:

$$A = \{x, a_1, a_2, \ldots, a_n\}$$

the basis of *Pascal's triangle*

We construct subsets of size k from a set with $n + 1$ elements.

The total will include

- **all of the subsets from the set of size n which do not contain the element x $C(n, k)$,**

  plus

- **the subsets of size k - 1 with the element x added $C(n, k\text{-}1)$.**

Pascal's triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$,

$k = 0,1,\ldots,n.$

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

By Pascal’s identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Other Identity Involving Binomial Coefficients

【 Theorem 3 】 Vandermonde's Identity

Let m, n and r be nonnegative integer with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof:

A and B are two disjoint sets.

$|A|=m$, $|B|=n$,

$C(m+n, r)$ ---- the number of ways to pick r elements from $A \cup B$

Another way to pick r element from $A \cup B$ is to pick r-k elements from A and then k elements from B, where $0 \leq k \leq r$

【 **Corollary 4** 】 **If *n* is a nonnegative integer. Then**

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof:

We use Vandermonde's Identity with *m*=*r*=*n* to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

【 **Theorem 4** 】**Let n and r be nonnegative integer with $r \leq n$. Then**

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof:

The left-hand side counts the bit strings of length n+1 containing r+1 1s.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with r+1 ones.

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Generalized Permutations and Combinations

Section 6.5

Section Summary

- ✓Permutations with Repetition
- ✓Combinations with Repetition
- ✓Permutations with Indistinguishable Objects
- ✓Distributing Objects into Boxes

Problems:

How to solve counting problems where elements may be used more than once?

For example, how many strings of length n can be formed from the uppercase letters of the English alphabet?

How to solve counting problems in which some elements are indistinguishable?

For example, how many different strings can be made form the letters in MISSISSIPPI, using all the letters?

Permutations With Repetition

r-permutation with repetition:

***r*-permutations of a set with repetition allowed.**

【Theorem 1】 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof:

By the product rule.

Example:

How many strings of length n can be formed from the uppercase letters of the English alphabet?

Combination With Repetition

H_n^r

r-Combination with repetition:

combinations with repetition of elements allowed

【Theorem 2】 There are $C(n-1+r, r)$ r-combination from a set with n elements when repetition of elements is allowed.

Proof:

① The n-1 bars are used to mark off n different cells, with the ith cell contains a star for each time the ith element of the set occurs in the combination.

For example, * * | * | | * | * * *

② Each r-combination of a set with n elements when repetition is allowed can be represented by a list of n-1 bars and r stars.

③ The number of such lists is $C(n-1+r, r)$.

Combinations with Repetition

〖Example 1〗 Suppose that a cake shop provides eight different kinds of cake. There are 12 cakes in one box. How many different boxes with cake?

$$H_8^{12}$$

Question:

How many different boxes with at least one of each kind?

$$H_8^4$$

〖Example 2〗How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $x_i\,(i = 1,2,3,4)$ **is nonnegative integer?**

Solution:

Since a solution of this equation corresponds to a way of selecting 16 items from a set with four element, such that x_1 items of type one, x_2 items of type two, x_3 items of type three, x_4 items of type four are chosen.

Hence the number of solutions is

$$H_4^{16} = C\,(4\text{-}1+16,\ 16) = C\,(19,\ 16)$$

$$= C\,(19,\ 3)$$

〖Example 3〗How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $x_i (i = 1,2,3,4)$ **is nonnegative integer?**

Question:

(1) $x_i > 1, \text{ for } i = 1,2,3,4 \quad \Longrightarrow \quad \boldsymbol{x_i \geq 2}$

$$\boldsymbol{H_4^{8} = C\,(4\text{-}1+8,\ 8) = C\,(11,\ 8) = C\,(11,\ 3)}$$

(2) $x_1 + x_2 + x_3 + x_4 \leq 16$

We can introduce an auxiliary variable x_5 so that

$$\boldsymbol{x_1 + x_2 + x_3 + x_4 + x_5 = 16}$$

$$\boldsymbol{H_5^{16} = C\,(5\text{-}1+16,\ 16) = C\,(20,\ 16) = C\,(20,\ 4)}$$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

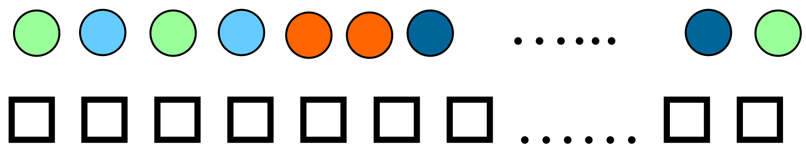
Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\dfrac{n!}{(n-r)!}$
r-combinations	No	$\dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

n-Permutation with limited repetition

$A = \{ n_1 \bullet a_1 , n_2 \bullet a_2 , \ldots, n_k \bullet a_k \}$, *where* $n_1+n_2+\ldots+n_k = n$

【Theorem 3】 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type1,…,and n_k indistinguishable objects of type k, is

$$n!/(n_1!n_2!\ldots n_k!)$$



Proof:

$$C(n,n_1)\cdot C(n-n_1,n_2)\cdot \ldots \cdot C(n-n_1-n_2-\ldots-n_{k-1},n_k)$$

$$=\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\cdots\cdot\frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!(n-n_1-n_2-\ldots-n_k)!}$$

$$=\frac{n!}{n_1!n_2!\ldots n_k!}$$

〖Example 4〗 There are 50 students in a class.

(1) How many ways to select 7 students to construct a leading group?

C(50,7)

$$= \frac{P(50,7)}{1!1!5!}$$

(2) If two students are elected as a ice monitor, then how many are there?

(3) If these 7 students are elected to have different tasks, then how many are there?

P(50,7)

〖Example 5〗

(1) How many bit strings of length 10?

2^{10}

(2) How many bit strings of length 10 are there that contain exactly two 0s, eight 1s?

$10!/(2!8!)$

〖Example 6〗 How many different strings can be made form the letters in MISSISSIPPI, using all the letters?

Solution:

$$A = \{ 1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$$

$$\frac{11!}{4!4!2!}$$

Distributing objects into boxes

◆**Many counting problems can be solved by counting the ways objects can be placed in boxes.**

- **The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).**
- **The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).**

◆ Distinguishable objects and distinguishable boxes

【Theorem 5】 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are place into box i, i=1,2,…,k, equals

$$n!/(n_1!n_2!\ldots n_k!)$$

Question:

Why both the ways in theorem 3 and 5 are the same?

〖Example 8〗 **How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?**

Solution:

It is typical problem that involves distributing distinguishable objects into distinguishable boxes.

- The distinguishable objects are the 52 cards.

- The five distinguishable boxes are the hands of the four players and the rest of the deck.

Indistinguishable objects and distinguishable boxes

There are $C(n + k - 1, n - 1)$ ways to place k indistinguishable objects into n distinguishable boxes.

Proof based on one-to-one correspondence between *n*-combinations from a set with *k*-elements when repetition is allowed and the ways to place *k* indistinguishable objects into *n* distinguishable boxes.

Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

◆ Distinguishable objects and indistinguishable boxes

- ✓ **counting the ways to place *n* distinguishable objects into *k* indistinguishable boxes**

〖Example 9〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

There are 1+4+3+6=14 ways to put four different employees into three indistinguishable offices.

Solution:

We represent the four employees b

(1) All four are put into one office:

{A,B,C,D}

(2) Three are put into one office and a fourth is put into a second office: 4 ways

{{A,B,C},{D}}; {{A,B,D}, {C}}; {{A,C,D}, {B}}; {{B,C,D}, {A}}

(3) Two are put into one office and two put into a second office:3 ways

{{A,B},{C,D}}; {{A,D},{B,C}}; {{A,C},{B,D}}

(4) Two are put into one office, and one each put into the other two office:6 ways

{{A,B},{C},{D}}; {{A,C},{B},{D}}; {{A,D},{B},{C}}; {{B,C},{A},{D}};

{{B,D},{A},{C}}; {{C,D},{A},{B}}

〖Example 9〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

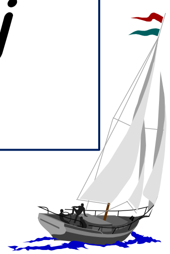
Solution:

Another way:

Look at the number of offices into which we put employees.

1) **There are 6 ways to put four different employees into three indistinguishable offices so that no office is empty.**
2) **There are 7 ways to put four different employees into two indistinguishable offices so that no office is empty.**
3) **There are 1 ways to put four different employees into one offices so that it is not empty.**

Problem: distribute n distinguishable objects into j indistinguishable boxes so that no boxes is empty



Stirling numbers of the second kind

Notation: $S(n,j)$

the number of ways to distribute *n* distinguishable objects into *j* indistinguishable boxes so that no boxes is empty

(1) $S(r,1)=S(r,r)=1 \quad (r\geq 1)$

(2) $S(r,2)=2^{r-1}-1$

(3) $S(r,r-1)=C(r,2)$

(4) $S(r+1,n)=S(r,n-1)+nS(r,n)$

Note:

1. *S(n,j)* is the number of ways to partition the set with *n* elements into *j* nonempty and disjoint subsets.

2. *S(n,j)j!* is the number of ways to distribute *n* distinguishable objects into *j* distinguishable boxes so that no boxes is empty
   - the number of onto functions from a set with *n* elements to a set with *j* elements

$$S(n,j)\,j! = \left(\sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^n\right)$$

3. the number of ways to place n distinguishable objects into k indistinguishable boxes

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \left(\left(\sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^n\right) / j!\right)$$

〖Example 9〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution:

Another way:

Look at the number of offices into which we put employees.

1) **There are 6 ways to put four different employees into three indistinguishable offices so that no office is empty.**
2) **There are 7 ways to put four different employees into two indistinguishable offices so that no office is empty.**
3) **There are 1 ways to put four different employees into one offices so that it is not empty.**

S(4,3)+S(4,2)+ S(4,1)=6+7+1=14

◆ Indistinguishable objects and indistinguishable boxes

- ✓ counting the ways to distribute indistinguishable objects into indistinguishable boxes

〖Example 10〗 **How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?**

Solution:

We can enumerate all ways to pack the books. For each ways to pack the books, we will list the number of books in the box with the largest of books, followed by the number of books in each box containing at least one book, in order of decreasing number of books in a box.

The ways we can pack the books are

6

5,1 4,2 4,1,1

3,3 3,2,1 3,1,1,1

2,2,2 2,2,1,1

- **The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.**
- **No simple closed formula exists for this number.**

Homework:

Seventh Edition:

P.413   20,28, 30

P.421   10,24,27

P. 432   10, 16, 32, 42, 48, 50, 63

Homework:

Seventh Edition:

P. 432 10, 16, 32, 42, 48, 50, 63