Generating Functions

Section 8.4

Section Summary

- ✓Generating Functions
- ✓Counting Problems and Generating Functions
- ✓Useful Generating Functions
- ✓Solving Recurrence Relations Using Generating Functions
- ✓Proving Identities Using Generating Functions

Why should we study generating functions?

Generating functions are useful for manipulating sequences.

- ✓ to solve many kinds of counting problems

  For example, the problem of combination or permutation with constraints

- ✓ to solve the recurrence relations
- ✓ to prove combinatorial identities

Generating Functions

【Definition 1】 The generating function for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series.

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

〖Example 1〗

(1) What is the generating function for the sequence 1, 1, 1, 1, ...?

$$G(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

(2) What is the generating function for the sequence 0, 1, 2, 3, 4, 5, ...?

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

Generating Functions for Finite Sequences

The generating function for finite sequence of real numbers

$a_0, a_1, a_2, \ldots, a_n$ **is**

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

〖Example 2〗

(1) The finite sequence: 1,1,1. The generating function for this sequence is

$$G(x) = 1 + x + x^2 = \frac{1 - x^3}{1 - x}$$

(2) Let $a_k = C(m,k), k = 0,1,2,\ldots,m$ **. The generating function for this sequence is**

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \ldots + C(m,m)x^m = (1+x)^m$$

Useful Facts About Power Series

【Theorem 1】Let $f(x) = \sum_{k=0}^{\infty} \cdots$

(1) $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

(2) $\alpha \cdot f(x) = \sum_{k=0}^{\infty} \alpha \cdot a_k x^k$ $\quad \alpha \cdots$

(3) $x \cdot f'(x) = \sum_{k=0}^{\infty} k \cdot a_k x^k$

(4) $f(\alpha x) = \sum_{k=0}^{\infty} \alpha^k \cdot a_k x^k$

(5) $f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$

Proof:

$$\sum_{k=0}^{\infty} k a_k x^k = \sum_{k=0}^{\infty} a_k \cdot x \cdot k x^{k-1}$$

$$= x \sum_{k=0}^{\infty} a_k (x^k)'$$

$$= x \left(\sum_{k=0}^{\infty} a_k x^k \right)'$$

$$= x f'(x)$$

Useful facts about power series

Let $f(x) = \sum_{k=0}^{\infty} a_k x$ …

(1) $f(x) + g(x) =$ …

(2) $\alpha \cdot f(x) = \sum_{k=0}^{\infty}$ …

(3) $x \cdot f'(x) =$ …

(4) $f(\alpha x) = \sum_{k=0}^{\infty}$ …

(5) $f(x)g(x) = \sum$ …

Proof:

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2$$

$$+ \ldots + \left(\sum_{j=1}^{k} a_j b_{k-1}\right)x^k + \ldots$$

$$= (a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots)$$

$$= f(x) \cdot g(x)$$

- **Using the above properties, the generating functions of some sequence can be obtained easily.**

〖Example 3〗 (1) What is the generating function for the sequence 0,1,2,3,4,5,…?

Solution:

$$b_k = k$$

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

$$= x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2}$$

〖Example 3〗 (2) Suppose that the generating function of the sequence: $a_0, a_1, a_2, \ldots, a_n, \ldots$ is $G(x)$. What is the generating function for the sequence $b_k = \sum_{i=0}^{k} a_i$?

Solution: $a_k \leftrightarrow G(x)$,

$$c_k = 1$$

$$b_k = \sum_{i=0}^{k} a_i$$

$$= \sum_{i=0}^{k} a_i \times c_{k-i}$$

$$F(x) = G(x) \cdot \frac{1}{1-x}$$

For example:

$1,1,1,\ldots \longleftrightarrow \dfrac{1}{1-x}$

$\Downarrow$

$1,2,3,4,\ldots,k+1,\ldots \longleftrightarrow \dfrac{1}{(1-x)^2}$

〖Example 3〗 (3) What is the generating function for the sequence $a_k = k^2$ **?**

Solution:

$$a_k = 1 \longleftrightarrow \frac{1}{1-x}$$

$$a_k = k \longleftrightarrow \frac{x}{(1-x)^2}$$

$$a_k = k^2 \longleftrightarrow x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}$$

〖Example 3〗 (4) What is the generating function for the sequence $a_k = \sum_{i=1}^{k} i^2$?

Solution:

$$a_k = k^2 \quad \longleftrightarrow \quad x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}$$

$$a_k = \sum_{i=1}^{k} i^2 \quad \longleftrightarrow \quad \frac{x(1+x)}{(1-x)^4}$$

〖Example 4〗 **Let** $f(x)=\dfrac{1}{1-4x^2}$ **. Find the coefficient** $a_0, a_1, a_2, \ldots, a_n, \ldots$ **in the expansion** $f(x)=\sum_{k=0}^{\infty} a_k x^k$ **.**

Solution:

$$f(x)=\frac{1}{1-4x^2} \qquad =\frac{1}{(1-2x)(1+2x)} \qquad =\frac{1}{2}\left(\frac{1}{1-2x}+\frac{1}{1+2x}\right)$$

$$2^k \qquad (-2)^k$$

$$\frac{1}{2}\left(2^k+(-2)^k\right) \quad =\begin{cases} 2^k & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

The extended binomial coefficient

Recall:

$$\binom{m}{k} = C(m,k) = \frac{m!}{k!(m-k)!}$$

Where *m*, *k* are nonnegative integers, *k*≤*m*

【Definition 2】 Let *u* be a real number and *k* a nonnegative integer. Then the extended binomial coefficient is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots(u-k+1)/k! & \text{if} \quad k > 0 \\ 1 & \text{if} \quad k = 0 \end{cases}$$

〖Example 5〗 (1)$\binom{1/2}{3} = ?$ (2)$\binom{-n}{k} = ?$

Solution:

$$(1)\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = 1/16$$

$$(2)\binom{-n}{k} = \frac{(-n)(-n-1)\ldots(-n-k+1)}{k!}$$

$$= \frac{(-1)^k n(n+1)\ldots(n+k-1)}{k!}$$

$$= (-1)^k C(n+k-1, k)$$

The extended Binomial Theorem

【Theorem 2】Let x be a real number with $|x|<1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

〖Example 6〗**Find the generating functions for**

$$(1+x)^{-n} \; and \; (1-x)^{-n}$$

where *n* is a positive integer, using the extended Binomial Theorem.

Solution:

By the extended Binomial Theorem , it follows that

$$(1+x)^{-n}$$

$$=\sum_{k=0}^{\infty}\binom{-n}{k}x^k$$

$$=\sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)x^k$$

$$(1-x)^{-n}$$

$$=\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k$$

$$=\sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)(-1)^k x^k$$

$$=\sum_{k=0}^{\infty}C(n+k-1,k)x^k$$

Some Common Used Generating Functions

Sequence	Generating function
(1) $C(n,k)$	$\sum_{k=0}^{\infty} C(n,k)x^k = (1+x)^n$
(2) $C(n,k)a^k$	$(1+ax)^n$
(3) $1,1,\ldots,1$	$1+x+x^2+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$
(4) $1,1,1,\ldots$	$\frac{1}{1-x}$
(5) a^k	$\frac{1}{1-ax}$
(6) $k+1$	$\frac{1}{(1-x)^2}$

Some Common Used Generating Functions

Sequence	Generating function
(7) $C(n+k-1,k)$	$(1-x)^{-n}$
(8) $(-1)^k C(n+k-1,k)$	$(1+x)^{-n}$
(9) $C(n+k-1,k)a^k$	$(1-ax)^{-n}$
(10) $\dfrac{1}{k!}$	e^x
(11) $\dfrac{(-1)^{k+1}}{k}$	$\ln(1+\mathrm{x})$

Counting Problems and Generating Functions

〖Example 7〗Find the number of solutions of $e_1+e_2+e_3=17$ **where** e_1,e_2,e_3 **are nonnegative integers with** $2\le e_1\le 5, 3\le e_2\le 6, and\ 4\le e_3\le 7$

Solution:

$$e_1+e_2+e_3=17$$

(1) $e_i\ge 0 \Rightarrow H_3^{17}=C(3-1+17,17)$

(2) $e_1\ge 10 \Rightarrow e_1+e_2+e_3=7(e_i\ge 0)$

$\Rightarrow H_3^{7}=C(3-1+7,7)$

(3) $2\le e_1\le 5, 3\le e_2\le 6, and\ 4\le e_3\le 7?$

The generating function for this counting problem is

$$G(x)=(x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(x^4+x^5+x^6+x^7)$$

The number of solutions is the coefficient of x^{17} in the expansion of $G(x)$.

〖Example 8〗Use generating functions to find the number of *r*-combinations from a set with *n* elements when repetition of elements is allowed.

Solution:

Since there are *n* elements in the set, each can be selected zero times, one times and so on. It follows that

$$G(x)=(1+x+x^2+x^3+\ldots)^n=\left(\frac{1}{1-x}\right)^n=\frac{1}{(1-x)^n}$$

the number of *r*-combinations from a set with *n* elements when repetition of elements is allowed, is the coefficient a_r of x^r in the expansion of $G(x)$. Since

$$\frac{1}{(1-x)^n}=\sum_{k=0}^{\infty}C(n+k-1,k)x^k$$

Then the coefficient a_r equals $C(n+r-1,r)$

〖Example 9〗**Suppose that there are 2*r* red balls, 2*r* blue balls, and 2*r* white balls. How many ways to select 3*r* balls from these balls?**

Solution:

How to find a_{3r} ?

$$G(x)=(1+x+x^2+\dots+x^{2r})^3$$

The coefficient a_{3r} of x^{3r} in the expansion of *G*(*x*) is the solution of this problem.

$$G(x)=(1+x+x^2+\dots+x^{2r})^3=\left(\frac{1-x^{2r+1}}{1-x}\right)^3=\frac{1-3x^{2r+1}+3x^{4r+2}-x^{6r+3}}{(1-x)^3}$$

$$F(x)=\frac{1}{(1-x)^3}=(1+x+x^2+\dots)^3$$

The coefficient of term x^i in *F*(*x*) is $H_3^i=C_{3+i-1}^i=C_{i+2}^i$

$$2r+1+y=3r \qquad \therefore y=r-1$$

The coefficient of term x^{r-1} in *F*(*x*) is C_{r+1}^{r-1}

$$\therefore a_{3r}=C_{3r+2}^{3r}-3C_{r+1}^{r-1}$$

〖Example 10〗Determine the number of ways to insert tokens worth \$1,\$2 and \$5 into a vending machine to pay for an item that costs *r* dollars in both the case when the order in which the tokens are inserted does not matter and when the order does matter.

Solution:

(1) The order in which the tokens are inserted does not matter

$$G(x) = (1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^5 + x^{10} + x^{15} + \ldots)$$

The coefficient of x^r in the expansion of *G(x)* is the solution of this problem.

(2) The order in which the tokens are inserted does matter

- **The number of ways to insert exactly *n* tokens to produce a total of *r*\$ is the coefficient of x^r in** $(x + x^2 + x^5)^n$
- **Since any number of tokens may be inserted, the number of ways to produce *r*\$ using \$1,\$2 and \$5 tokens, is the coefficient of x^r in**

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \ldots = \frac{1}{1 - (x + x^2 + x^5)}$$

Use Generating Function To Solve Recurrence Relations

The Method:

(1) Use the recurrence relation find the generating function of this sequence.

(2) $G(x) \Rightarrow a_n$

〖Example 11〗 Use generating functions to solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6$ **with initial conditions** $a_0 = 20, a_1 = 60$ **.**

Solution:

$$a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6 \qquad \times x^n$$

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 4^n x^n + 6\sum_{n=2}^{\infty} x^n$$

$$G(x) - a_0 - a_1 x \qquad 2x\sum_{n=1}^{\infty} a_n x^n \qquad 3x^2\sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-4x} - 1 - 4x \qquad 6\left(\frac{1}{1-x} - 1 - x\right)$$

$$2x(G(x) - a_0) \qquad 3x^2 G(x)$$

$$(1-2x-3x^2)G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)}$$

$$G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)(1+x)(1-3x)}$$

$$=\frac{16/5}{1-4x}+\frac{-3/2}{1-x}+\frac{31/20}{1+x}+\frac{67/4}{1-3x}$$

$$\frac{16}{5}\times 4^n \quad -\frac{3}{2}\times 1^n \quad \frac{31}{20}\times(-1)^n \quad \frac{67}{4}\times 3^n$$

$$a_n=\frac{16}{5}\times 4^n-\frac{2}{3}+\frac{31}{20}\times(-1)^n+\frac{67}{4}\times 3^n$$

Proving Identities Via Generating Functions

〖Example 12〗 **Use generating function to prove Pascal's identity** $C(n,r)=C(n-1,r)+C(n-1,r-1)$ **when *n* and *r* are positive integers with** $r<n$ **.**

Proof:

$$G(x)=(1+x)^n=\sum_{r=0}^{n}C(n,r)x^r$$

$$(1+x)^n=(1+x)(1+x)^{n-1}=(1+x)^{n-1}+x(1+x)^{n-1}$$

$$\sum_{r=0}^{n}C(n,r)x^r=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=0}^{n-1}C(n-1,r)x^{r+1}$$

$$=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n}C(n-1,r-1)x^r$$

$$1+\sum_{r=1}^{n-1}C(n,r)x^r+x^n$$

$$=1+\sum_{r=1}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n-1}C(n-1,r-1)x^r+x^n$$

$$\sum_{r=1}^{n-1}\underline{C(n,r)}x^r=\sum_{r=1}^{n-1}[\underline{C(n-1,r)+C(n-1,r-1)}]x^r$$

Inclusion-Exclusion and Its Application

Section 8.5-8.6

Recall:

- **The principle of Inclusion-exclusion**

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- **For the union of three finite sets:**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- **For the union of *n* finite sets:**

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

How to prove?
An element in the union is counted exactly once by the right-hand side of the equation.

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

Suppose that *a* is an element of exactly *r* of the sets $A_1, A_2, \ldots, A_n$ **where** $1 \le r \le n$ **.**

This element is counted $C(r,1)$ **times by** $\sum_{i=1}^{n} |A_i|$ **.**

This element is counted $C(r,2)$ **times by** $\sum_{1 \le i < j \le n} |A_i \cap A_j|$ **.**

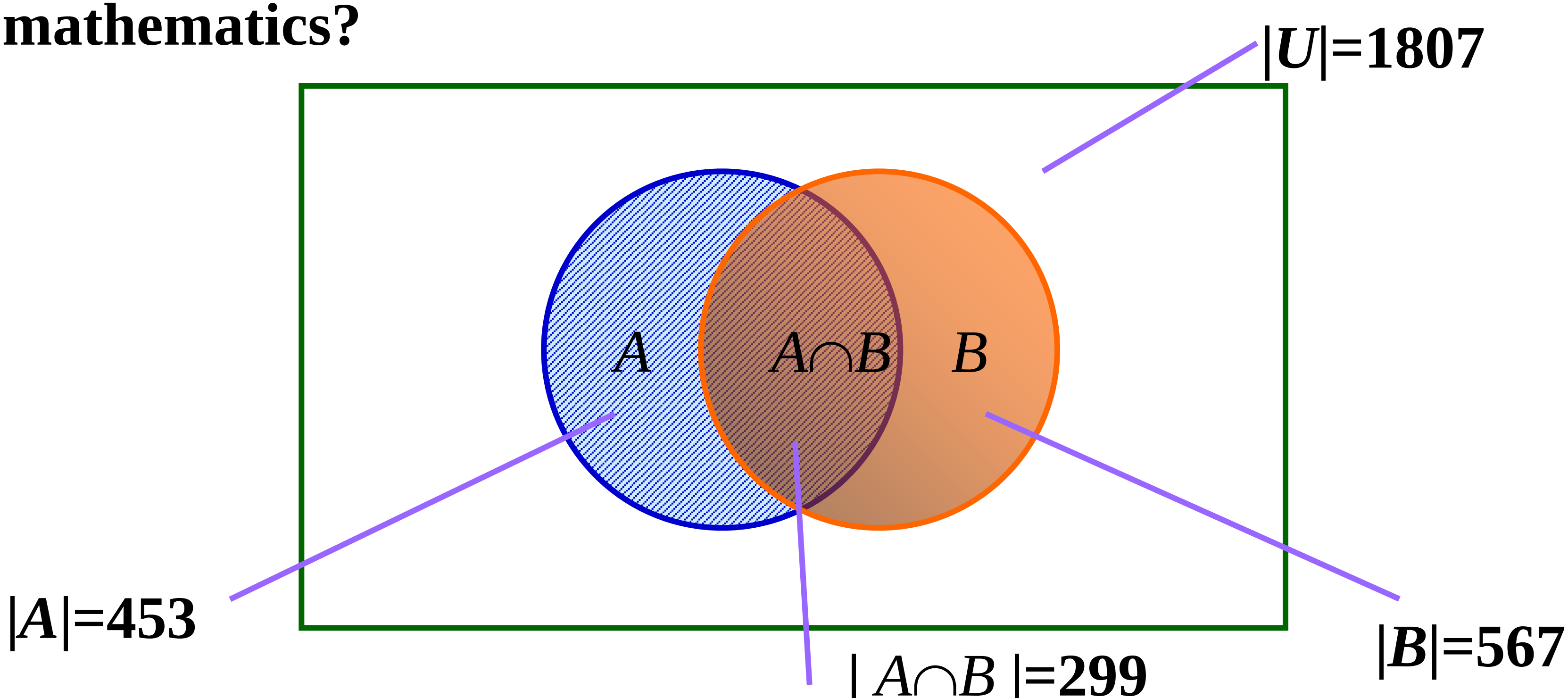
…

Thus, it is counted exactly

$$C(r,1) - C(r,2) + C(r,3) - \ldots + (-1)^{r-1} C(r,r) \ \mathbf{=}\ \mathbf{1}$$

Why ? **Since** $\mathbf{(-1+1)^r=0}$

〖Example 1〗 Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?



〖Example 2〗How many positive integers not exceeding 1000 that are not divisible by 5, 6 or 8?

Solution:

***U*: the set of positive integers not exceeding 1000**

***A*: the set of positive integers not exceeding 1000 that are divisible by 5,**

***B*: the set of positive integers not exceeding 1000 that are divisible by 6,**

***C*: the set of positive integers not exceeding 1000 that are divisible by 8.**

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 1000 - (\left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{6}\right\rfloor + \left\lfloor\frac{1000}{8}\right\rfloor - \left\lfloor\frac{1000}{5\times 6}\right\rfloor - \left\lfloor\frac{1000}{6\times 4}\right\rfloor - \left\lfloor\frac{1000}{5\times 8}\right\rfloor + \left\lfloor\frac{1000}{5\times 6\times 4}\right\rfloor)$$

$$= 600$$

〖Example 3〗How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?

Solution:

***U*: the set of permutations of the 26 letters**

***A*: the set of permutations of the 26 letters containing *fish*,**

***B*: the set of permutations of the 26 letters containing *rat*,**

***C*: the set of permutations of the 26 letters containing *bird*.**

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 26! - (23! + 24! + 23! - 21! - 0 - 0 - 0)$$

An alternative form of inclusion-exclusion

✓to solve problems that ask for the number of elements in a set that have none of *n* properties.

$$P_1, P_2, \ldots, P_n$$

Let A_i be the subset containing the elements that have property P_i .

$N(P_1P_2\ldots P_k)$ **:The number of elements with all properties $P_1, P_2, \ldots, P_k$.**

It follows that

$$N(P_1P_2\ldots P_k) = \left|A_1 \cap A_2 \cap \ldots \cap A_k\right|$$

$N(P_1'P_2'\ldots P_n')$ **:The number of elements with none of the properties $P_1, P_2, \ldots, P_n$.**

From the inclusion-exclusion principle, we see that

$$N(P_1'P_2'\ldots P_n') = N - \left|A_1 \cup A_2 \ldots \cup A_n\right| = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \ldots + (-1)^n N(P_1P_2\ldots P_n)$$

〖Example 4〗How many solutions does $x_1 + x_2 + x_3 = 13$ have, where x_i are nonnegative integers with $x_i < 6, i = 1,2,3$?

Solution:

Let a solution has property P_1 is $x_1 \ge 6$, property P_2 is $x_2 \ge 6$, property P_3 is $x_3 \ge 6$.

The number of solutions is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

$C(3-1+13,13)$

$N(P_i) = C(3-1+7,7)$

$N(P_iP_j) = C(3-1+1,1)$

$N(P_1P_2P_3) = 0$

The Sieve of Eratoshenes

〖Example 5〗 Find the number of primes not exceeding a specified positive integer.

Take 100 for example.

Solution:

✧ **A composite integer is divisible by a prime not exceeding its square root.**

- **Composite integer not exceeding 100 must have a prime factor not exceeding 10.**
- **Since the only primes less than 10 are 2,3,5,7, the primes not exceeding 100 are these four primes and the positive integers greater than 1 and not exceeding 100 that are divisible by none of 2,3,5,7.**

P_1: the property that an integer is divisible by 2

P_2: the property that an integer is divisible by 3

P_3: the property that an integer is divisible by 5

P_4: the property that an integer is divisible by 7

The number of primes not exceeding positive integer 100 is

$$4 + N(P_1'P_2'P_3'P_4')$$

$$= 4 + N - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4)$$
$$+ N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4)$$

$$= 25$$

99

$\lfloor 100/2 \rfloor$

$\lfloor 100/(2\times 3) \rfloor$

$\lfloor 100/(2\times 3 \times 5) \rfloor$

$\lfloor 100/(2\times 3 \times 5 \times 7) \rfloor$

The sieve of Eratosthenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The number of onto functions

Theorem 1: Let *m* and *n* be positive integers with $m \geq n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with *m* elements to a set with *n* elements.

Proof:

$$A = \{a_1, a_2, \ldots, a_m\} \qquad B = \{b_1, b_2, \ldots, b_n\}$$

Let P_i be the property that b_i is not in the range of the function, respectively.

Note that a function is onto if and only if it has none of the properties $P_i (i = 1,2,\ldots,n)$.

By the principle of inclusion-exclusion, it follows that the number of onto functions is

$$N(P_1'P_2'\ldots P_n') = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \ldots + (-1)^n N(P_1P_2\ldots P_n)$$

n^m

$C(n,1)(n-1)^m$

$C(n,2)(n-2)^m$

$(-1)^n N(P_1P_2\ldots P_n) = 0$

Problem:

S(m,n) **: the number of ways to distribute m distinguishable objects into *n* indistinguishable boxes so that no boxes is empty**

the number of ways to partition the set with *m* elements into *n* nonempty and disjoint subsets.

S(m,n) n!**: the number of onto functions from a set with *m* elements to a set with *n* elements**

Application:

- **Assign *m* different jobs to *n* different employees if every employee is assigned at least one job.**
- **Distribute *m* different toys to *n* different children such that each child gets at least one toy.**

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example:

The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}]$$

Proof:

Let a permutation have property P_i if it fixes element *i*.

The number of derangements is the number of permutation having none of the properties P_i for i=1, 2, …, n, namely

$$D_n = N(P_1'P_2'\ldots P_n')$$

$$= N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \ldots + (-1)^n N(P_1P_2\ldots P_n)$$

$$= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots + (-1)^n \times C(n,n)(n-n)!$$

$$= n! - \frac{n!}{1!(n-1)!}\times(n-1)! + \frac{n!}{2!(n-2)!}\times(n-2)! - \frac{n!}{3!(n-3)!}\times(n-3)! + \ldots + (-1)^n \frac{n!}{n!(n-n)!}\times(n-n)!$$

$$= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!})$$

Homework:

Seventh Edition:

P. 549 6,16, 24, 30, 34,49

P. 557 7, 12

P. 564   6, 11, 16