

The Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

Chapter Summary

- Sets
 - ◆ The Language of Sets
 - ◆ Set Operations
 - ◆ Set Identities
- Functions
 - ◆ Types of Functions
 - ◆ Operations on Functions
 - ◆ Computability
- Sequences and Summations
- Set Cardinality
 - ◆ Countable Sets
- Matrices



Sets

Section 2.1

Brief introduction of set theory

- ◆ Set theory is the part of mathematics devoted to the study of set
- ◆ Set theory is the foundation of modern mathematics
 - ✓ 数学的大多数分支所研究的对象或者可以看做一种特定结构的集合，或者可以通过集合定义
 - ✓ 集合概念、方法已渗透到数学的几乎所有领域，是必不可少的数学工具和表达语言



Brief introduction of set theory

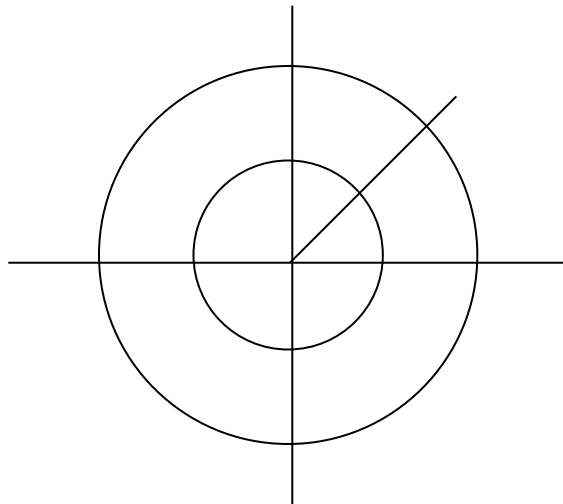
- ◆ Learning set theory in discrete mathematics
 - ♦ Much of discrete mathematics is devoted to the study of discrete structures
 - ♦ Many important discrete structures are built using sets
 - ✓ combinations
 - ✓ Relations
 - ✓ Graphs, etc.



Brief introduction of set theory

G. Cantor (1845-1918)

- German mathematician
- The founder of set theory (Naive set theory)
- Great achievement: research on **infinite sets**



1 2 3 4 ... n ...

1 2² 3² 4² ... n² ...



Brief introduction of set theory

B. Russell (1872-1970)

- ◆ English Philosopher
- ◆ His most famous work “*Principia Mathematica*” (written with A. N. Whitehead)
- ◆ Russell won the Nobel Prize for literature in 1950.
- ◆ **Russell Paradox**

In a town, the barber shaves all and only those who do not shave themselves. Does the barber shave himself?

- Let $P(x) = x \notin x$ and $B = \{x \mid x \notin x\}$
- $B \in B$ —true or false?
- $B \in B \Rightarrow B \notin B, B \notin B \Rightarrow B \in B$
- Contradiction!



Some Concepts of set theory

【**Definition**】 A **set** is an unordered collection of objects.

The objects in a set are called the **elements**, or **members**, of the set.

A set is said to **contain** its elements.

Note:

- Uppercase letters are usually used to denote sets, and lowercase letters are usually used to denote elements of sets.
- $a \in A$: a is a member of A or a is an element of A
 $a \notin A$: a is not an element of A



The description of set

One can describe a set by

1. Roster method: list all the members of a set, when this is possible (enumeration)

For example,

(1) The set of all odd positive integers less than 10.

$$\{1, 3, 5, 7, 9\}$$

(2) $S = \{a, b, c, d\}$

Brace notation with ellipses

For example,

$$S = \{ \dots, -3, -2, -1 \}$$



2. Use set builder notation (specification by predicates)

The universal set U :

-----The set contains all the objects under consideration.

$$S = \{x \mid P(x)\}$$

----- S contains all the elements from U which make the predicate P true.

For example,

$$\{x \mid x \text{ is an odd positive integers less than } 10\}$$



[[**Example 1**]] Some examples of set.

(1) $\mathbf{N} = \{ 0,1,2,3,\dots \}$ the set of natural numbers

(2) $\mathbf{Z} = \{ x \mid x \text{ is an integer} \}$ the set of integers

(3) $\mathbf{Z}^+ = \{ 1,2,3,\dots \}$ the set of positive integers

(4) $\mathbf{Q} = \{ p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0 \}$, the set of rational numbers

(5) $\mathbf{R} = \{ x \mid x \text{ is a real number} \}$

[[**Example 2**]] The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

Remark: The concept of a datatype, or type, in computer science is the name of a set, together with a set of operations that can be performed on objects from that set.



the empty set

- the set with no elements
- The *void* set, the *null* set
- Notation: ϕ , $\{\}$

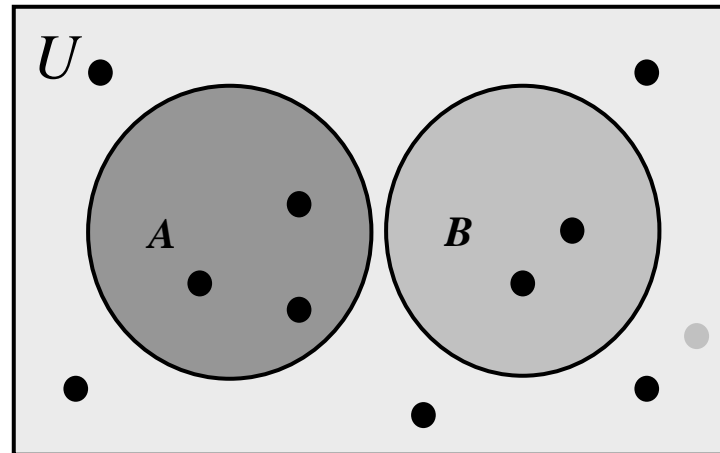
Singleton set

- A set with one element.
- Note that ϕ and $\{\phi\}$ are different.



Venn diagrams

- In Venn diagrams, the universal set U is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Some points are used to represent the particular elements of the set.



Subsets

$A \subseteq B$ -- A is a subset of the set B .

every element of A is also an element of B .

$$A \subseteq B \Leftrightarrow \forall x (x \in A \rightarrow x \in B)$$

Problems:

For any set A ,

(1) $\emptyset \subseteq A$?

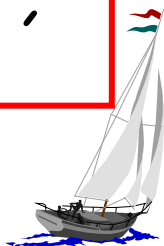
Y. The empty set is a subset of any set.

(2) $A \subseteq A$?

Y.

Showing that A is a subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B .

Showing that A is Not a subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.



Equal

$A = B$: contain exactly the same elements

$$A = B \Leftrightarrow \forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

$$\Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

Problem: ϕ is unique?

[[Example 2]] $S1 = \{a, b, c, d\}$, $S2 = \{a, d, b, c\}$, $S3 = \{b, c, a, d, d\}$.
 $S1 = S2 = S3$?

Note:

- The Order of elements does not matter.
- Repetition of elements does not matter.

Showing Two Sets are Equal To show that $A=B$, show that $A \subseteq B$ and $B \subseteq A$.



Proper subset

$$A \subset B \Leftrightarrow \forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

$$\Leftrightarrow A \subseteq B \text{ and } A \neq B$$



The Size of a Set

【Definition】 Let S be a set. If there are **exactly** n **distinct** elements in S where n is a nonnegative integer, we say that S is a **finite** set and that n is the **cardinality** of S .

Notation:

$|S|$ ---- the cardinality of S

【Definition】 A set is said to be **infinite** if it is not finite.



[[Example 3]]

(1) Let A be the set of odd positive integers less than 10.

Then $|A| = ?$

(2) $B = \{1, \{2, 3\}, \{4, 5\}, 6\}$

$|B| = 4$

(3) $C = \emptyset$

$|C| = 0$

(4) $D = \{x \in \mathbb{N} \mid x \leq 7000\}$

$|D| = 7001$

(5) $E = \{x \in \mathbb{N} \mid x \geq 7000\}$

E is infinite!

Problem: What is the cardinality of an infinite set?



Power Sets

[[**Example 4**]] Let $A = \{a, b, c\}$, find all subsets of the set A .

Solution:

ϕ

$\{a\}, \{b\}, \{c\}$

$\{a, b\}, \{b, c\}, \{a, c\}$

$\{a, b, c\}$

Problem: How many subsets of the set A are there?

$$C_3^0 + C_3^1 + C_3^2 + C_3^3 = 8$$

In general, for a set A with n elements, the number of subsets is

$$C_n^0 + C_n^1 + C_n^2 + \dots + C_n^n = \sum_{i=0}^n C_n^i = 2^n$$

$$(x + y)^n = \sum_{i=0}^n C_n^i x^i y^{n-i} = 2^n \quad (\text{Q } x = y = 1)$$



【Definition】 Given a set S , the **power set** of S is the set of all subsets of the set S .

Notation:

$P(S)$ ---- the power set of S .

$$P(S) = \{x \mid x \subseteq S\}$$

Note:

- (1) $|S|=n$ implies $|P(S)| = 2^n$
- (2) S is finite and so is $P(S)$.

$$(3) \left\{ \begin{array}{l} x \in P(S) \Rightarrow x \subseteq S \\ x \in S \Rightarrow \{x\} \in P(S) \\ S \in P(S) \end{array} \right.$$



【Example 5】 What is the power set of the set $\{\phi\}$ and $\{\phi, \{\phi\}\}$?

Solution:

$$(1) S = \{\phi\}$$

$$P(S) = \{\phi, \{\phi\}\}, \quad |P(S)| = 2$$

$$(2) S = \{\phi, \{\phi\}\}$$

$$P(S) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}, \quad |P(S)| = 4$$



[[**Example 6**]] Show that $P(A) \in P(B) \Rightarrow A \in B$?

Proof:

$$P(A) \in P(B) \Rightarrow P(A) \subseteq B$$

$$A \in P(A) \subseteq B \Rightarrow A \in B$$

Problem:

$$A \subseteq B \Rightarrow P(A) \subseteq P(B) ?$$



Cartesian Products

【Definition】 The ordered n -tuple (a_1, a_2, \dots, a_n) is the ordered collection that has a_1 as its first element, a_2 as its second element, ... , and a_n as its n th element.

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n) \Leftrightarrow a_i = b_i \ (i=1, 2, \dots, n)$$

In particular, 2-tuples are called **ordered pairs**.

Note:

If $x \neq y$, then $(x, y) \neq (y, x)$.

$$(x, y) = (u, v) \Rightarrow x = u \text{ and } y = v$$

The **Cartesian product** of A and B : $A \times B = \{(a, b) \mid a \in A, b \in B\}$

The **Cartesian product** of A_1, A_2, \dots, A_n :

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \dots, n\}.$$



[[**Example 7**]] $A = \{a, b\}, B = \{0, 1, 2\}$, $A \times B = ?$ $B \times A = ?$

solution:

$$A \times B = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$$

$$B \times A = \{(0, a), (1, a), (2, a), (0, b), (1, b), (2, b)\}$$

Note:

- ❖ If $|A|=m, |B|=n$, Then $|A \times B| = |B \times A| = mn$
- ❖ $A \times B \neq B \times A$
- ❖ $A \times \phi = \phi \times A = \phi$
- ❖ $(x, y) \in A \times B \Rightarrow x \in A \text{ and } y \in B; (x, y) \notin A \times B \Rightarrow x \notin A \text{ or } y \notin B$



Using Set Notation with Quantifiers

Restrict the domain of a quantified statement explicitly by making use of a particular notation.

$$\forall x \in S (P(x)) : \quad \forall x (x \in S \rightarrow p(x))$$

$$\exists x \in S (P(x)) : \quad \exists x (x \in S \wedge p(x))$$

Problems:

$$\forall x \in \mathbf{R} (x^2 \geq 0) \quad ?$$

$$\exists x \in \mathbf{Z} (x^2 = 1) \quad ?$$



Truth Sets of Quantifiers

Given a predicate P , and a domain D .

The **truth set** of P to be the set of elements x in D for which $p(x)$ is true. namely,

The **truth set** of P : $P = \{x / p(x)\}$

Problems:

$\forall x P(x)$ is true over the domain U ?

$\exists x P(x)$ is true over the domain U ?



Set Operations

Section 2.2

Section Summary

- Set Operations
 - ◆ Union
 - ◆ Intersection
 - ◆ Complementation
 - ◆ Difference
 - ◆ Symmetric difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables



Two, or more, sets can be combined in many different ways:

- Union
- Intersection
- Difference
- Symmetric difference



Union

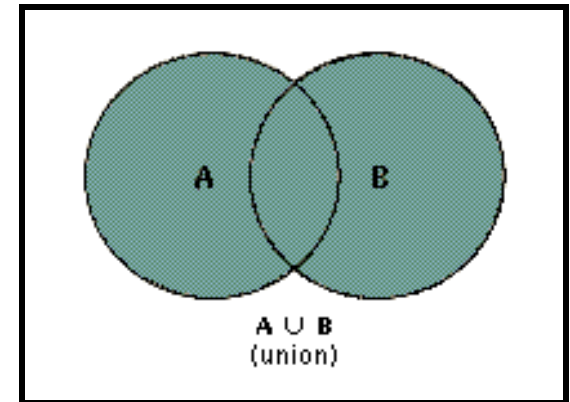
$A \cup B$: The union of the sets A and B

the set that contains those elements that are either in A or in B , or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Useful rules:

- (1) $A \subseteq A \cup B, \quad B \subseteq A \cup B$
- (2) $A \subseteq C, B \subseteq C \Rightarrow A \cup B \subseteq C$
- (3) $|A \cup B| \leq |A| + |B|$
- (4) $A \cup B = B \Leftrightarrow A \subseteq B$



Intersection

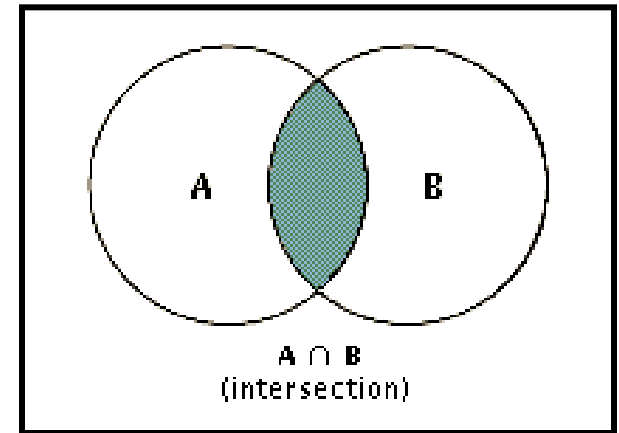
$A \cap B$: The intersection of sets A and B
the set containing those elements in both A and B .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Note: Two sets are called **disjoint** if their intersection is the empty set, namely $A \cap B = \emptyset$

Useful rules:

- (1) $A \cap B \subseteq A, \quad A \cap B \subseteq B$
- (2) $C \subseteq A, C \subseteq B \Rightarrow C \subseteq A \cap B$
- (3) $|A \cap B| \leq |A|, |A \cap B| \leq |B|$
- (4) $A \cap B = A \Leftrightarrow A \subseteq B$

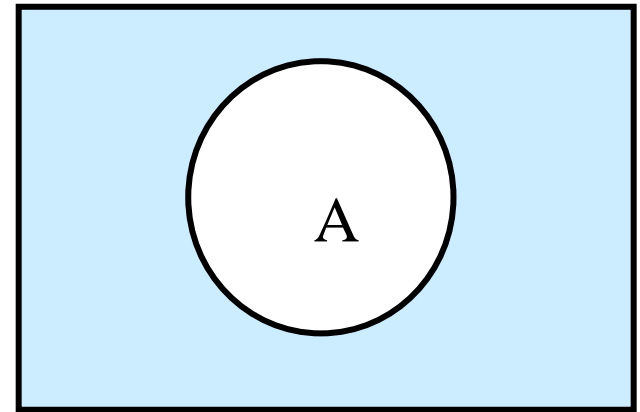


Complement

Let U be universal set. The complement of the set A denoted by \bar{A} , is the complement of A with respect to U , namely, $U - A$.

(The complement of A is sometimes denoted by A^c .)

$$\bar{A} = \{x \mid x \notin A, x \in U\} \text{ or } \{x \mid \neg(x \in A)\}$$



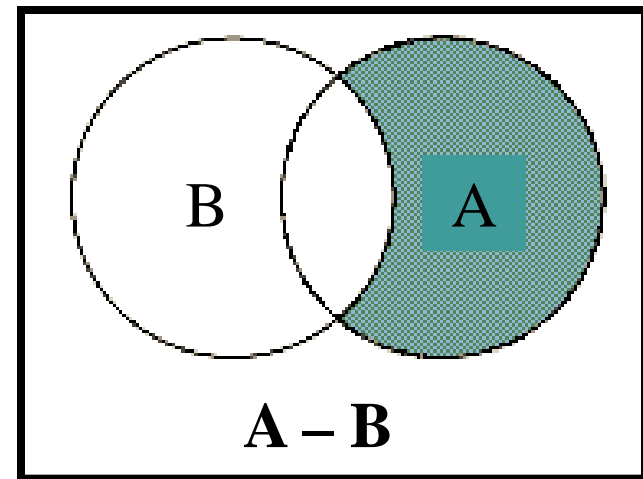
Difference

$A - B$: The difference of A and B

the set containing those elements that are in A but not in B .

Note: The difference of A and B is also called the complement of B with respect to A .

$$\begin{aligned} A - B &= \{x \mid x \in A \wedge x \notin B\} \\ &= A \cap \overline{B} \end{aligned}$$

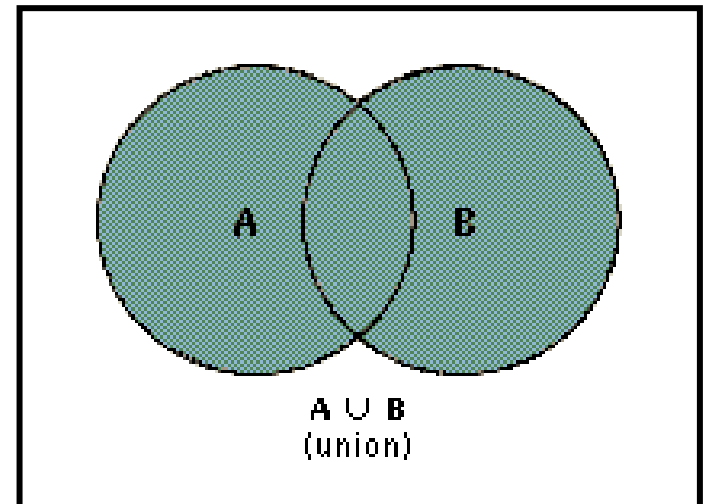


The Cardinality of a Union of Two Infinite Sets

- The principle of Inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

counts each elements that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice



The Cardinality of a Union of Two Infinite Sets

- The principle of inclusion-exclusion is an important techniques used in enumeration.
- **Example:** Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- Note:
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.



Symmetric difference

$A \oplus B$: The symmetric difference of A and B

$$A \oplus B = (A \cup B) - (A \cap B)$$

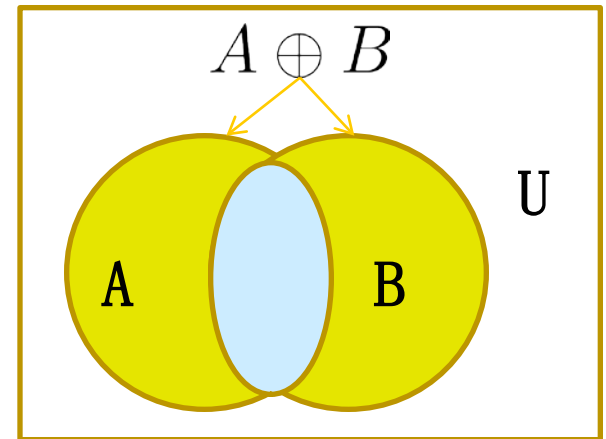
Example:

$$U = \{0,1,2,3,4,5,6,7,8,9,10\}$$

$$A = \{1,2,3,4,5\} \quad B = \{4,5,6,7,8\}$$

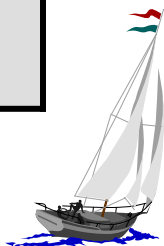
What is:

$$\text{Solution: } \{1,2,3,6,7,8\}$$



Set Identities

Set Identities	
Identity	Name
$A \cup \emptyset = A, A \cap U = A$	Identity laws
$A \cup U = U, A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A, A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A, A \cap B = B \cap A$	Commutative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws



Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.



Proof of First De Morgan Law

[[Example 1]] Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$?

solution:

(1) $A = B$ iff $A \subseteq B$ and $B \subseteq A$

$$\diamond \overline{(A \cup B)} \subseteq \bar{A} \cap \bar{B}$$

Suppose that $x \in \overline{(A \cup B)}$.

It follows that $x \notin A \cup B$.

This implies that $x \notin A$ and $x \notin B$.

Hence, $x \in \bar{A}$ and $x \in \bar{B}$.

Thus $x \in \bar{A} \cap \bar{B}$.

$$\diamond \bar{A} \cap \bar{B} \subseteq \overline{(A \cup B)}$$



Set-Builder Notation: First De Morgan Law

【Example 1】 Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$?

solution:

$$\begin{aligned}\overline{(A \cup B)} &= \{x \mid x \notin A \cup B\} \\ &= \{x \mid \neg(x \in A \cup B)\} \\ &= \{x \mid \neg(x \in A \vee x \in B)\} \\ &= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} \\ &= \{x \mid x \notin A \wedge x \notin B\} \\ &= \{x \mid x \in \bar{A} \wedge x \in \bar{B}\} \\ &= \{x \mid x \in \bar{A} \cap \bar{B}\} \\ &= \bar{A} \cap \bar{B}\end{aligned}$$



Membership Table

Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity.

[[Example 2]] Let A, B , and C be sets.
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

solution:

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0



【Example 3】 Show that $(A - B) \cup B = A \cup B$?

solution:

$$(A - B) \cup B = (A \cap \overline{B}) \cup B$$

$$= (A \cup B) \cap (\overline{B} \cup B)$$

$$= (A \cup B) \cap U$$

$$= A \cup B$$



【Example 4】 Simplify the following set.

$$((A \cup B \cup C) \cap (A \cup B) - ((A \cup (B - C)) \cap A))$$

solution:

$$((A \cup B \cup C) \cap (A \cup B) - ((A \cup (B - C)) \cap A))$$

$$= (A \cup B) - A$$

$$= (A \cup B) \cap \bar{A}$$

$$= (A \cap \bar{A}) \cup (B \cap \bar{A})$$

$$= \phi \cup (B \cap \bar{A})$$

$$= B - A$$



Generalized Unions and Intersections

- Let A_1, A_2, \dots, A_n be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

Contains those elements that are members of at least one set in the collection

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

Contains those elements that are members of all the set in the collection

These are **well defined**, since union and intersection are associative

Example, For $i = 1, 2, \dots$, let $A_i = \{i, i + 1, i + 2, \dots\}$. Then,

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{1, 2, 3, \dots\}$$

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{n, n + 1, n + 2, \dots\} = A_n$$



Computer Representation of Set

Using bit strings to represent sets.

(1) Specify an arbitrary ordering of the elements of U ,

for instance a_1, a_2, \dots, a_n

(2) Represent a subset A of U with the bit string of length n

Using bit strings to represent sets, it is easy to find complements of sets and union, intersection, and differences of sets.



[[**Example 5**]] Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 7, 9\}$. Use bit strings to find the difference of A and B .

solution:

(1) The bit string for the set A : 11 1110 000

The bit string for the set B : 10 1010 101

(2) $A - B = A \cap \overline{B}$

Questions:

Write programs.

Gives subsets A and B of a set with n elements, use bit strings to find \overline{A} , $A \cup B$, $A \cap B$, $A - B$, $A \oplus B$



Homework:

Seventh Edition: P.120 11,18,22,24,38
P.136 24, 40, 48, 61

