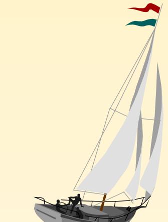
Applications of Propositional Logic

Section 1.2



Section Summary

- ✓Translating English to Propositional Logic
- ✓System Specifications
- ✓Boolean Searching
- ✓Logic Circuits
- ✓Logic Puzzles

Translating Sentences

Why ? How?

- **Steps to convert an English sentence to a statement in propositional logic**
  - **Identify atomic propositions and represent using propositional variables.**
  - **Determine appropriate logical connectives**

- "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.

If p or q then not r.

$$(p \vee q) \to \neg r$$

Another Example

How can the following sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution:

Let a, c and f represent "you can access the Internet from campus","you are a computer science major" and "you are a freshman".

This sentence can be represented as

$$a \rightarrow (c \vee \neg f)$$

System Specification

System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"It is necessary to scanned the message for viruses whenever the message was sent from an unknown system ."

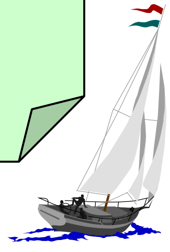
Solution:

Let *p* denote "The message is scanned for viruses "

Let *q* denote "the message was sent from an unknown system".

This specification can be represented as

$$q \rightarrow p$$



Consistent System Specifications

Definition: A list of propositions is consistent if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let p denote "The diagnostic message is stored in the buffer." Let q denote "The diagnostic message is retransmitted" The specification can be written as: $p \lor q$, $p \rightarrow q$, $\neg p$. When p is false and q is true all three statements are true. So the specification is consistent.

- **What if "The diagnostic message is not retransmitted is added."**

Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

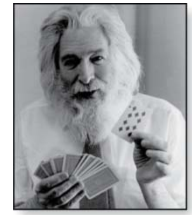
Logic Puzzles

Logic puzzle: can be solved using logical reasoning.

Solving logic puzzles is an excellent way to practice working with the rules of logic.

Computer programs designed to carry out logical reasoning often use well-known logic puzzles to illustrate their capabilities.

Example Logic Puzzle



Raymond Smullyan (Born 1919)

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution: Let *p* and *q* be the statements that A is a knight and B is a knight, respectively. So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then *p* is true. Since knights tell the truth, *q* must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Homework:

Seventh Edition:

P.22 6, 9, 24, 35

Propositional Equivalences

Section 1.3

Section Summary

- ✓Tautologies, Contradictions, and Contingencies.
- ✓Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- ✓Other logical operators
- ✓The Dual of a Compound Proposition
- ✓Functionally Complete Collection of Logical Operators
- ✓Normal Forms (covered in exercises in text)
  - DNF & Full DNF
  - CNF & Full CNF
- ✓Propositional Satisfiability
  - Sudoku Example

Tautologies, Contradictions, and Contingencies

- **A tautology is a proposition which is always true.**
  - Example: $p \lor \neg p$
- **A contradiction is a proposition which is always false.**
  - Example: $p \land \neg p$
- **A contingency is a proposition which is neither a tautology nor a contradiction, such as p**

P	¬p	p ∨¬p	p ∧¬p
T	F	T	F
F	T	T	F

Logically Equivalent

The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

Notation: $p \Leftrightarrow q$ **or** $p \equiv q$

Remark:

The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

Problem: How to determine whether two compound propositions are equivalent?

Method 1: to use a truth table

Two compound propositions *p* and *q* are equivalent if and only if the columns in a truth table giving their truth values agree.

This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	¬p	¬p ∨ q	p→ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan

1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

- **Show that $A \equiv B$ using the truth table**

  In general, 2^n rows are required if a compound proposition involves n propositions.

Key Logical Equivalences

Name	**Equivalences**	
Identity laws	$p \wedge \mathbf{T} \equiv p$	$p \vee \mathbf{F} \equiv p$
Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$	$p \wedge \mathbf{F} \equiv \mathbf{F}$
Idempotent laws	$p \vee p \equiv p$	$p \wedge p \equiv p$
Double negation law	$\neg\neg p \equiv p$	
Commutative laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	

Name	Equivalences
Negation laws	$p \vee \neg p \equiv T$, $p \wedge \neg p \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$ $\quad p \wedge (p \vee q) \equiv p$
Contrapositive law	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
Exportation law	$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$
Absurdity law	$(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$
Implication law	$p \rightarrow q \equiv \neg p \vee q$
Equivalence law	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \wedge \neg q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Using De Morgan's Laws

Example : Use De Morgan's laws to express the negation of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert".

Solution:

Miguel does not have a cellphone or he does not have a laptop computer.

Heather will not go to the concert and Steve will not go to the concert.

Constructing New Logical Equivalences

- **The logical equivalence can be used to constructing new logical equivalences**
  - **Reason: a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.**
- **We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.**
  - **To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.**

$$\begin{aligned} A &\equiv A_1 \\ &\vdots \\ A_n &\equiv B \end{aligned}$$

Equivalence Proofs

Example : Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$$ **by the De Morgan' law**

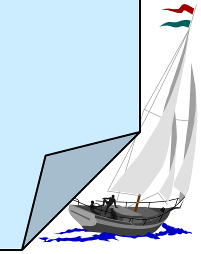
$$\equiv \neg p \wedge (\neg\neg p \vee \neg q)$$ **by the De Morgan' law**

$$\equiv \neg p \wedge (p \vee \neg q)$$ **by the double negation law**

$$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$$ **by the distributive law**

$$\equiv F \vee (\neg p \wedge \neg q)$$ **by the negation law**

$$\equiv \neg p \wedge \neg q$$ **by the identity law**



Example : Show that $((P \to Q) \to R) \to ((R \to P) \to (S \to P))$ **is a tautology.**

Solution:

$$((p \to q) \to r) \to ((r \to p) \to (s \to p))$$

$$\equiv \neg(\neg(\neg p \vee q) \vee r) \vee (\neg(\neg r \vee p) \vee (\neg s \vee p))$$

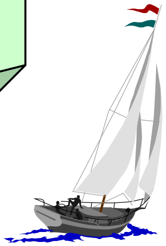
$$\equiv ((\neg p \vee q) \wedge \neg r) \vee (r \wedge \neg p) \vee (\neg s \vee p)$$

$$\equiv (\underline{\neg p \wedge \neg r}) \vee (q \wedge \neg r) \vee (\underline{r \wedge \neg p}) \vee (\neg s \vee p)$$

$$\equiv (\neg p \wedge (\underline{\neg r \vee r})) \vee (q \wedge \neg r) \vee (\neg s \vee p)$$

$$\equiv \underline{\neg p} \vee (q \wedge \neg r) \vee (\neg s \vee \underline{p})$$

$$\equiv T$$



Other logical operators

Sheffer stroke |:

$$p|q \equiv \neg(p \land q) \quad \textbf{NAND}$$

Peirce arrow ↓:

$$p \downarrow q \equiv \neg(p \lor q) \quad \textbf{NOR}$$

The Dual of a Compound Proposition

The **dual** of compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$,each $\wedge$ by $\vee$,each T by F and each F by T. The dual of *S* is denoted by *S**.

For example,

(1) $S=(p \vee \neg q) \wedge r \vee T \quad S^{*}= (p \wedge \neg q) \vee r \wedge F$

(2) $S= (p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$

$S^{*}= \neg(p \vee q) \wedge (p \wedge q)$

【Theorem】 let *s and t* are two compound propositions, $s \equiv t$ if and only if $s^{*} \equiv t^{*}$.

Functionally Complete Collection of Logical Operators

A collection of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

For example,

$\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$, $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\,|\,\}$, $\{\downarrow\}$
are all functionally complete operators.

Propositional Normal Forms

Why it is needed to introduce propositional normal forms?

Formulas can be transformed into standard forms

- **Make identification and comparison of two formulas**
- **Become more convenient to determine whether a formula is tautology, contradiction or contingency**
- **Become more convenient to find the assignments for which the formula is true or false**
- **Can simplify the formula**

There are two types of normal forms in propositional calculus:

disjunctive normal form(DNF) and conjunctive normal form(CNF)

Disjunctive normal form

A *literal* is a variable or its negation.

Conjunctions with literals as conjuncts are called *conjunctive clauses (clauses)*.

For example, $p \wedge q,\quad p \wedge \neg q,\quad \neg p \wedge q,\quad \neg p \wedge \neg q$

A formula is said to be in disjunctive normal form if it is written as a disjunction, in which all the terms are conjunctions of literals.

- Yes $(p \wedge q) \vee (p \wedge \neg q)$
- No $p \wedge (p \vee q)$

More DNF or CNF

p	DNF & CNF
$\neg p \vee q$	DNF & CNF
$\neg p \wedge q \wedge \neg r$	DNF & CNF
$\neg p \vee (q \wedge \neg r)$	DNF
$\neg p \wedge (q \vee \neg r) \wedge (\neg q \vee r)$	CNF

Existence of normal form

【 **Theorem 1** 】 **Any formula *A* is tautologically equivalent to some formula in DNF (CNF).**

Proof:

Construct the truth table for the proposition. Then an equivalent proposition is the disjunction with *n* disjuncts (where *n* is the number of rows for which the formula evaluates to T). Each disjunct has m conjuncts where *m* is the number of distinct propositional variables. Each conjunct includes the positive form of the propositional variable if the variable is assigned T in that row and the negated form if the variable is assigned F in that row. This proposition is in disjunctive normal from.

How to obtain normal form

(1) Use of the following logical equivalences to eliminate $\rightarrow, \leftrightarrow$.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

(2) Use of the following logical equivalences to eliminate $\neg$, $\vee$, $\wedge$ from the scope of $\neg$ such that any $\neg$ has only an atom as its scope.

$$\neg (p_1 \vee p_2 \vee \ldots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$$

$$\neg\neg p \equiv p$$

(3) Use of the commutative laws, the distributive laws and the associative laws to obtain normal form.

Example : Convert the following formula into conjunctive and disjunctive normal forms.

$$\neg(p \vee q) \leftrightarrow (p \wedge q)$$

solution:

$$\neg(p \vee q) \leftrightarrow (p \wedge q)$$

$$\equiv (\neg(p \vee q) \to (p \wedge q)) \wedge ((p \wedge q) \to \neg(p \vee q))$$

$$\equiv ((p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$$

$$\equiv ((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\neg p \vee \neg q) \vee (\neg p \wedge \neg q))$$

$$\equiv (p \vee q) \wedge (\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q)$$

$$\equiv (p \vee q) \wedge (\neg p \vee \neg q) *$$

$$\equiv ((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q)$$

$$\equiv (p \wedge \neg p) \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg q) **$$

$$\equiv (q \wedge \neg p) \vee (p \wedge \neg q) ***$$

Example : Find the assignments of *p* and *q* for which the following formula is true.

$$(p \to q) \to p$$

solution:

$$(p \to q) \to p$$

$$\equiv \neg(\neg p \lor q) \lor p$$

$$\equiv (p \land \neg q) \lor p$$

$$\equiv p$$

The assignments of *p* and *q* for which the formula is true:

p	***q***
T	**T**
T	**F**

Conjunctive normal form

- A compound proposition is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions.
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by eliminating implications, moving negation inwards and using the distributive and associative laws.
- Important in resolution theorem proving used in artificial Intelligence (AI).
- A compound proposition can be put in conjunctive normal form through repeated application of the logical equivalences covered earlier.

Full disjunctive normal form

1. Minterm & Maxterm

A *minterm* is a conjunctive of literals in which each variable is represented exactly once.

For example,

If a formula has the variables p, q, r, then $p\wedge\neg q\wedge r$ is a minterm, but $p\wedge\neg q$ and $p\wedge\neg p\wedge r$ are not.

Question:

If a formula has n variables, how many minterms are there?

2. Full disjunctive normal form

If a formula is expressed as a disjunction of minterms, it is said to be in *full disjunctive normal form*.

For example,

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

3. How to obtain full disjunctive normal form

Any formula *A* is tautologically equivalent to a formula in full disjunctive normal form.

First, obtain disjunctive normal form, then use of negation law and distributive laws to obtain full disjunctive forms

$$A \equiv A \wedge (q \vee \neg q) \equiv (A \wedge q) \vee (A \wedge \neg q)$$

Example : Convert the following formula into full disjunctive normal form.

$$(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$$

solution:

$$(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$$

$$\equiv (p \wedge q \wedge (r \vee \neg r)) \vee (\neg p \wedge (q \vee \neg q) \wedge r) \vee ((p \vee \neg p) \wedge q \wedge r)$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r)$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

From the above expression, we can see

(1) The formula is true for the following four assignments

p	***q***	***r***
T	**T**	**T**
T	**T**	**F**
F	**T**	**T**
F	**F**	**T**

(2) The formula is a contingency.

4. Full disjunctive normal form from truth tables

p	*q*	*r*	*f*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

$$f \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

Why?

p	q	$\neg p \wedge \neg q$	$\neg p \wedge q$	$p \wedge \neg q$	$p \wedge q$
T	T	F	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	F	F	F

(1) Each minterm is true for exactly one assignment.

(2) If A and B are two distinct minterms, then $A \wedge B \equiv$F.

(3) A disjunction of minterms is true only if at least one of its constituents minterms is true.

4. Full disjunctive normal form from truth tables

p	*q*	*r*	*f*
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

$$f \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Example : Find the full disjunctive normal form for *f* given by the table.

p	q	r	f
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$$f \equiv (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

Example : Find the full conjunctive normal form for *f* given by the table.

p	***q***	***r***	***f***
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

Solution:

(1) $f \equiv (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

(2) Find the full disjunctive normal form of $\neg f$

$$\neg f \equiv (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

$$f \equiv (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r)$$

Question:

Use the full disjunctive normal form,

- **how to determine whether two formulas are logical equivalent?**
- **how to determine whether a given formula is tautology, contradiction or contingency?**
- **how to find the assignments for which a given formula is True?**

Homework:

(1) Seventh edition: P.34 10(d), 24, 30, 40, 51, 59

(2)Give the simplest DNF and CNF of the following formulas:

1) $((P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)) \wedge R$

2) $(P \wedge (Q \wedge S)) \vee (\neg P \wedge (Q \wedge S))$

(3) Give the full DNF of the following formulas, Find the assignments of p , q and r for which the formula is true.

1) $(\neg R \wedge (Q \rightarrow P)) \rightarrow (P \rightarrow (Q \vee R))$

2) $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$