The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

Summary

- Valid Arguments and Rules of Inference
- Proof Methods
- Proof Strategies

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Revisiting the Socrates Example

- We have the two **premises**:
  - “All men are mortal.”
  - “Socrates is a man.”
- And the **conclusion**:
  - “Socrates is mortal.”

1. **How to express?**
2. **How do we get the conclusion from the premises?**

Arguments

- **An argument in propositional logic is a sequence of propositions. All but the final proposition are called premises. The last statement is the conclusion.**
- We can express the premises (above the line) and the conclusion (below the line) as an **argument**.
- **An argument form in propositional logic : a sequence of compound proposition involving propositional variables.**

$$\begin{array}{c} p_1 \\ p_2 \\ \cdots \\ p_n \\ \hline \therefore q \end{array}$$

$$\boldsymbol{p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q}$$

Express The Socrates Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an **argument**:

$$\begin{array}{rl} & \forall x(Man(x) \rightarrow Mortal(x)) \\ & Man(Socrates) \\ \hline \therefore & Mortal(Socrates) \end{array}$$

Valid Arguments & Argument Form

- The argument is valid if the premises imply the conclusion.
- An argument form is valid if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

$p_1 \land p_2 \land \ldots \land p_n \to q$ is a tautology.

- How to prove an argument is valid?
- How to construct valid arguments?

Rules of Inference

- The rules of inference are the essential building block in the construction of valid arguments.
  - **Propositional Logic**

    Inference Rules
  - **Predicate Logic**

    Inference rules for propos al logic plus additional inference rules to handle va and quantifiers.

✓ **Modus Ponens**
✓ **Modus Tollens**
✓ **Hypothetical Syllogism**
✓ **Disjunctive Syllogism**
✓ **Addition**
✓ ……

Modus Ponens

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Example:

Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

Corresponding Tautology:

$$(\neg p \land (p \to q)) \to \neg q$$

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Example:

Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

Corresponding Tautology:

$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Example:

Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

Corresponding Tautology:

$(\neg p \wedge (p \vee q)) \rightarrow q$

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Example:

Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

Corresponding Tautology:

$p \rightarrow (p \vee q)$

$$\frac{p}{\therefore\ p \vee q}$$

Example:

Let p be “I will study discrete math.”
Let q be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit Las Vegas.”

Simplification

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

$$\frac{p \wedge q}{\therefore q}$$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

Corresponding Tautology:

$((p) \wedge (q)) \rightarrow (p \wedge q)$

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Example:

Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Example:

Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature

Resolution plays an important role in AI and is used in Prolog.

Build Valid Arguments

To prove an argument is valid or the conclusion follows logically from the hypotheses:

- Assume the hypotheses are true.
- Use the rules of inference and logical equivalences to determine that the conclusion is true.

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Another example of Valid Arguments

Example 2:

- With these hypotheses:

  "It is not sunny this afternoon and it is colder than yesterday."

  "We will go swimming only if it is sunny."

  "If we do not go swimming, then we will take a canoe trip."

  "If we take a canoe trip, then we will be home by sunset."

- Using the inference rules, construct a valid argument for the conclusion:

  "We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon." r : "We will go swimming." t : "We will be home by sunset."

   q : "It is colder than yesterday." s : "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

Continued on next slide →

Another example of Valid Arguments

3. Construct the Valid Argument

Hypotheses: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$
Conclusion: t

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

More complex example

Example 3, Show that $\neg w$ logically follows from the hypotheses

$$(w \vee r) \to v, v \to (c \vee s), s \to u, \neg c \wedge \neg u$$

solution:

	Step	**Reason**
1.	$(w \vee r) \to v$	**Hypothesis**
2.	$v \to (c \vee s)$	**Hypothesis**
3.	$(w \vee r) \to (c \vee s)$	**Hypothetical syllogism using (1) and (2)**
4.	$\neg c \wedge \neg u$	**Hypothesis**
5.	$\neg u$	**Simplification using (4)**
6.	$s \to u$	**Hypothesis**
7.	$\neg s$	**Modus tollens using (5) and (6)**
8.	$\neg c$	**Simplification using (4)**
9.	$\neg s \wedge \neg c$	**Conjunction using (7) and (8)**
10.	$\neg (c \vee s)$	**(9) and De Morgan**
11.	$\neg (w \vee r)$	**Modus tollens using (3) and (10)**
12.	$\neg w \wedge \neg r$	**(10) and De morgan**
13.	$\neg w$	**Simplification using (12)**

Valid Arguments

If the conclusion is given in form $p \to q$, we can convert the original problem to

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge p \Rightarrow q$$

This method is based on

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge p) \to q \equiv (p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to (p \to q)$$

Valid Arguments

Example 4, Show that $r \to s$ logically follows from the hypotheses

$$p \to (q \to s), \neg r \vee p, q$$

solution:

Step	Reason
1. $\neg r \vee p$	**Hypothesis**
2. r	**Additional hypothesis**
3. p	**Disjunctive syllogism using step 1 and 2**
4. $p \to (q \to s)$	**Hypothesis**
5 $q \to s$	**Modus ponens using steps 3 and 4**
6. q	**Hypothesis**
7. s	**Modus ponens using steps 5 and 6**
8. $r \to s$	

Resolution

Resolution can be used to reason automatically or build automatic theorem proving system.

To construct proofs in propositional logic
    -- using resolution as the only rule of inference,
    -- the hypotheses and the conclusion must be expressed as clauses.

Resolution

Example 5, Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

solution:

$$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$$

$$r \rightarrow s \equiv \neg r \vee s$$

$$(p \vee r) \wedge (\neg r \vee s) \Rightarrow p \vee s$$

An Interesting Example

一名公安人员审查一件盗窃案，已知的事实如下：

A或**B**盗窃了**x**；若**A**盗窃了**x**，则作案时间不能发生在午夜前；若**B**证词正确，则在午夜时屋里灯光未灭；若**B**证词不正确，则作案时间发生在午夜前；午夜时屋里灯光灭了. **B**盗窃了**x**。

设：p: **A**盗窃了**x**；q:**B**盗窃了**x**；r:作案时间发生在午夜前；

s：**B**证词正确；t: 在午夜时屋里灯光灭了.

前提：$p \vee q$, $p \rightarrow \neg r$, $s \rightarrow \neg t$, $\neg s \rightarrow r$, t

结论：q

An Interesting Example

前提：$p \vee q$, $p \rightarrow \neg r$, $s \rightarrow \neg t$, $\neg s \rightarrow r$, t

结论：q

	Step	Reason
1.	t	Hypothesis
2.	$s \rightarrow \neg t$	Hypothesis
3.	$\neg s$	Modus tollens using (1) and (2)
4.	$\neg s \rightarrow r$	Hypothesis
5.	r	Modus ponens using (3) and (4)
6.	$p \rightarrow \neg r$	Hypothesis
7.	$\neg p$	Modus tollens using (5) and (6)
8.	$p \vee q$	Hypothesis
9.	q	Disjunctive syllogism using (7) and (8)

Fallacies

1) The Fallacy of affirming the conclusion

Method:

Reasoning based on $((p \to q) \land q) \to p$

Example, Let p: $n \equiv 1 \pmod 3$; and q: $n^2 \equiv 1 \pmod 3$.

The implication $p \to q$ is true. If q is true, so that $n^2 \equiv 1 \pmod 3$, does it follow that p is true, namely, that $n \equiv 1 \pmod 3$?

solution:

It would be incorrect to conclude that p is true.

Fallacies

2) The Fallacy of denying the hypothesis

Method:

Reasoning based on $((p \to q) \wedge \neg p) \to \neg q$

Example , Is it correct to conclude that

$$n^2 \not\equiv 1 (\bmod 3) \quad if \quad n \not\equiv 1 (\bmod 3)$$,

using the implication: if $n \equiv 1 (\bmod 3)$, then $n^2 \equiv 1 (\bmod 3)$?

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

"John Smith has two legs"

is a consequence of the premises:

"Every man has two legs." "John Smith is a man."

Solution: Let $M(x)$ denote "x is a man" and $L(x)$ "x has two legs" and let John Smith be a member of the domain.

Valid Argument:

Step	Reason
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Returning to the Socrates Example

$$\begin{array}{ll} & \forall x(Man(x) \rightarrow Mortal(x)) \\ & Man(Socrates) \\ \hline \therefore & Mortal(Socrates) \end{array}$$

Valid Argument

Step	**Reason**
1. $\forall x(Man(x) \rightarrow Mortal(x))$	Premise
2. $Man(Socrates) \rightarrow Mortal(Socrates)$	UI from (4)
3. $Man(Socrates)$	Premise
4. $Mortal(Socrates)$	MP from (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Continued on next slide →

Using Rules of Inference

Valid Argument:

$$\begin{array}{l} \exists x(C(x) \wedge \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore \ \exists x(P(x) \wedge \neg B(x)) \end{array}$$

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ P(a), \text{ where } a \text{ is a particular} \\ \quad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.

Universal Modus Tollens

Universal Modus Tollens combines universal instantiation and modus tollens into one rule.

$\forall x(P(x) \rightarrow Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore\ \neg P(a)$

Example, Determine whether the following argument is valid.

(1)	$\forall x \exists y G(x, y)$	Premise
(2)	$\exists y G(a, y)$	UI using Step 1
(3)	$G(a, c)$	EI using Step 2
(4)	$\forall x G(x, c)$	UG using Step 3
(5)	$\exists y \forall x G(x, y)$	EG using Step 3

Homework:

Seventh Edition: P.78 6, 12, 29, 31,35