Representing Graphs and Graph Isomorphism

Section 10.3

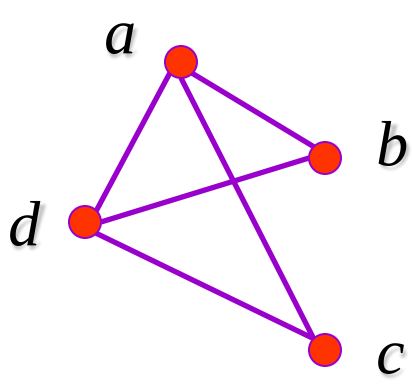
Representing Graphs

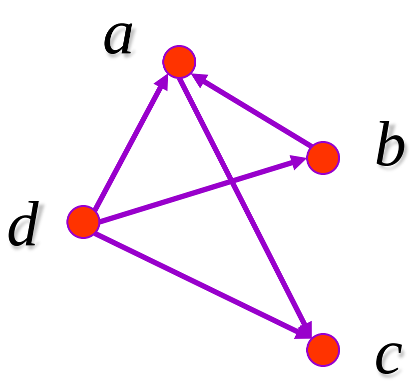
Common methods for representing graphs:

- **Adjacency lists**
- **Adjacency matrices**
- **Incidence matrices**
- **…**

Adjacency lists

- lists that specify the vertices that are adjacent to each vertex





vertex	*Adjacent vertices*
a	*b, c, d*
b	*a, d*
c	*a, d*
d	*a, b, c*

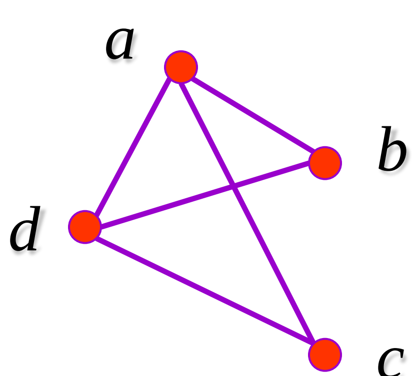
Initial vertex	*terminal vertices*
a	*c*
b	*a*
c	
d	*a, b, c*

Adjacency Matrices

A simple graph $G = (V, E)$ with n vertices $(v_1, v_2, \ldots, v_n)$ can be represented by its adjacency matrix, A, where

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,
$a_{ij} = 0$ otherwise.

Note: An adjacency matrix of a graph is based on the ordering chosen for the vertices.



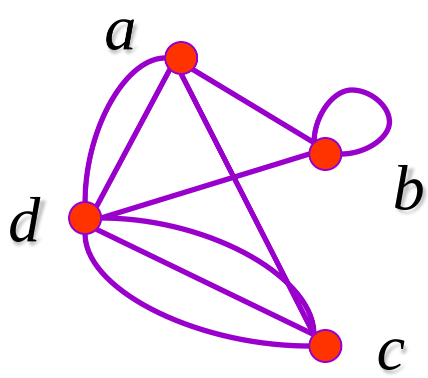
the adjacency matrix A_G based on the order of vertices ***a*, *b*, *c*, *d***

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

◆ The adjacency matrix of a multigraph or pseudograph

The (i, j)th entry of such a matrix equals the number of edges that are associated to $\{v_i, v_j\}$.



the adjacency matrix based on the order of vertices *a*, *b*, *c*, *d*

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

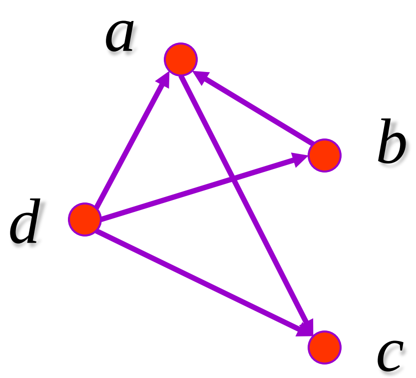
Note: **For undirected multigraph or pseudograph , adjacency matrices are symmetric.**

◆ The adjacency matrix of a directed graph

For directed graph $G = (V, E)$ with $|V| = n$, suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$, the adjacency matrix $A = [a_{ij}]$, where

$a_{ij} = 1$ if (v_i, v_j) is an edge of G,

$a_{ij} = 0$ otherwise.

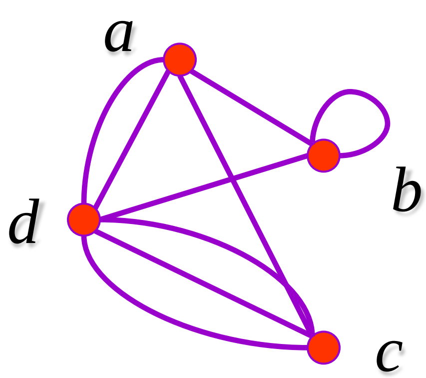


the adjacency matrix A_G based on the order of vertices a, b, c, d

$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?



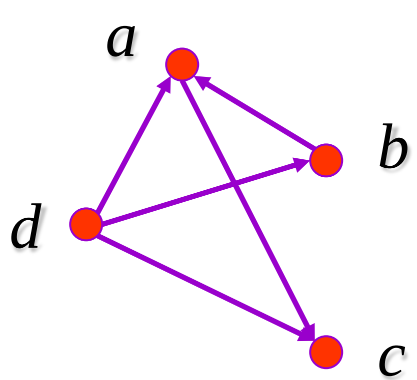
$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?



$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?

   $deg^+(v_i)$

2. What is the sum of the entries in a column of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?

   $deg^-(v_i)$

Question:

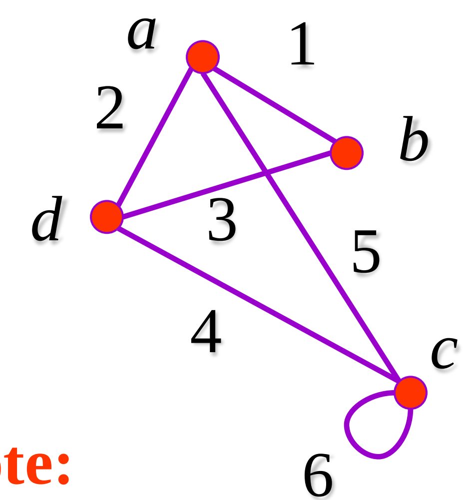
Adjacency lists or adjacency matrices ?

Incidence matrices

G = (*V*, *E*), $V=\{v_1,v_2,\ldots,v_n\}, E=\{e_1,e_2,\ldots,e_m\}.$

The incidence matrix with respect to this ordering of *V* and *E* is *n*×*m* matrix $\mathbf{M}=[m_{ij}]_{n\times m}$**, where**

$$m_{ij}=\begin{cases}1 & \text{when edge } e_j \text{ is incident with } v_i\\ 0 & \text{otherwise}\end{cases}$$



$$M=\begin{array}{c}\\ a\\ b\\ c\\ d\end{array}\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \begin{bmatrix}1&1&0&0&1&0\\1&0&1&0&0&0\\0&0&0&1&1&1\\0&1&1&1&0&0\end{bmatrix}\end{array}$$

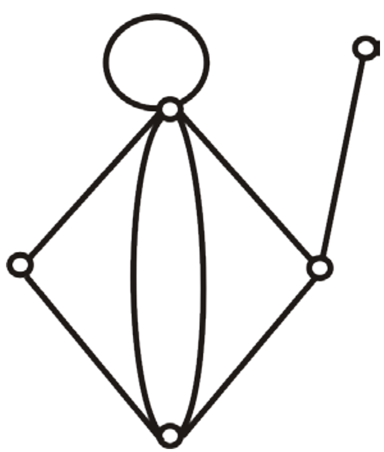
Note:

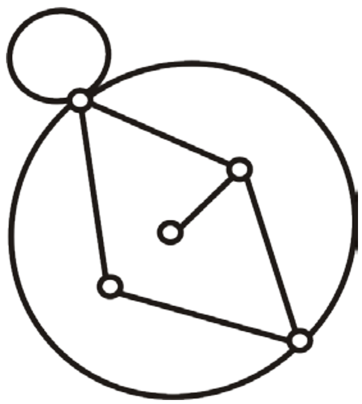
Incidence matrices of undirected graphs contain two 1s per column for edges connecting two vertices and one 1 per column for loops.

Isomorphism Of Graphs

The problem of isomorphism of graphs?

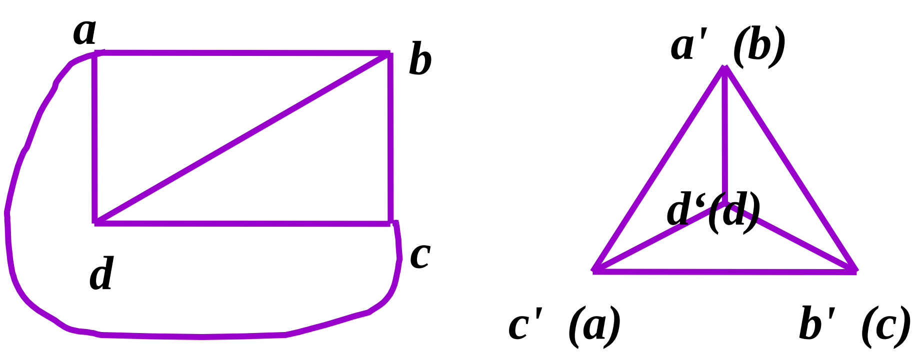
It is possible that two graphs are the same although these two graphs looks very different.





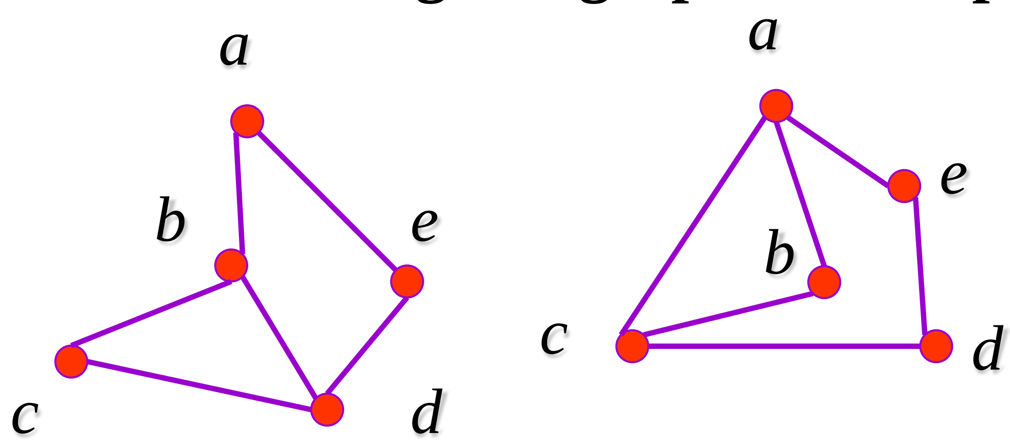
The definition of isomorphism of graphs?

- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a 1-1 and onto function f (f is called an isomorphism) from V_1 to V_2 such that for all a and b in V_1, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
- In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.



How to determine?

〖Example 〗 Are the following two graphs isomorphic?



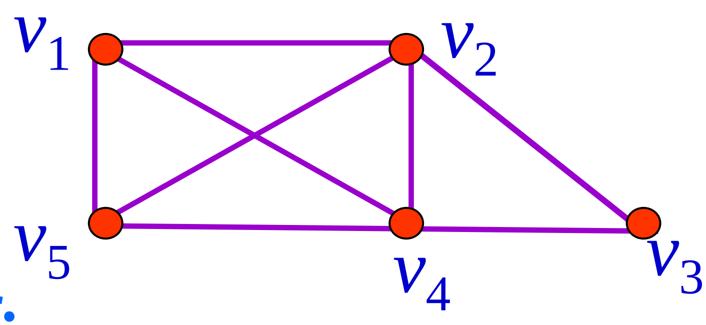
Solution:

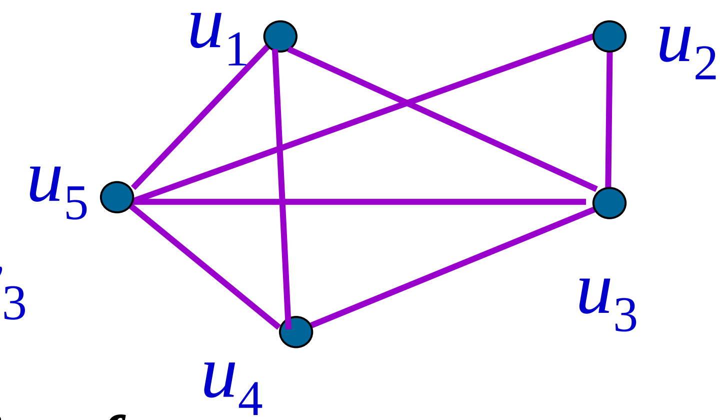
They are isomorphic, because they can be arranged to look identical.

You can see this if in the right graph you move vertex *b* to the left of the edge {*a*, *c*}. Then the isomorphism *f* from the left to the right graph is:

$f(a) = e,\ \ f(b) = a,$
$f(c) = b,\ \ f(d) = c,\ \ f(e) = d.$

〖Example 〗 **Show that the following two graphs are isomorphic.**





Proof:

- **Try to find an isomorphism *f***
- **Show that *f* preserves adjacency relation**

- The adjacency matrix of a graph *G* is the same as the adjacency matrix of another graph *H*, when rows and columns are labeled to correspond to the images under *f* of the vertices in *G*

$$A_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ u_4 \\ u_5 \\ u_2 \\ u_3 \\ u_1 \end{array} \begin{array}{c} \begin{array}{ccccc} u_4 & u_5 & u_2 & u_3 & u_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

- It is usually difficult to find an isomorphism *f* since there are *n*! possible 1-1 correspondence between the two vertex sets with *n* vertices.
- some properties (called invariants) in the graphs may be used *to* show that they are not isomorphic.

Graph invariant

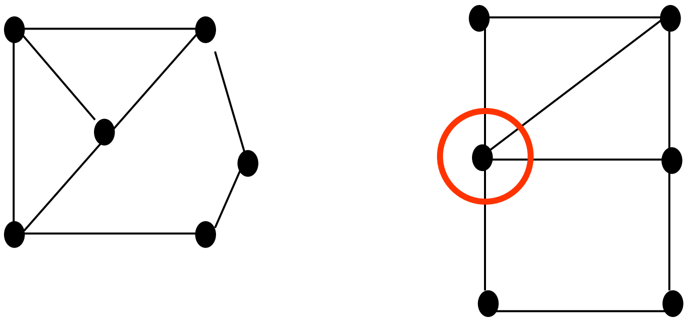
-- A property preserved by isomorphism of graphs.

Important invariants in isomorphic graphs:

- the number of vertices
- the number of edges
- the degrees of corresponding vertices
- if one is bipartite, the other must be
- if one is complete, the other must be
- if one is a wheel, the other must be

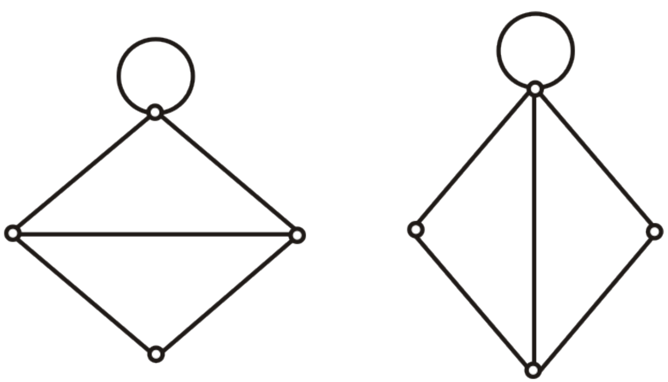
etc.

〖Example 7〗 Determine whether the given pair of graphs is isomorphic?

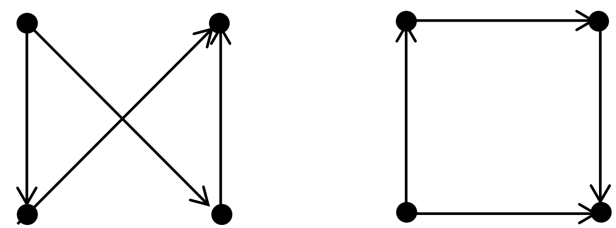


✗

✗



✗



✗

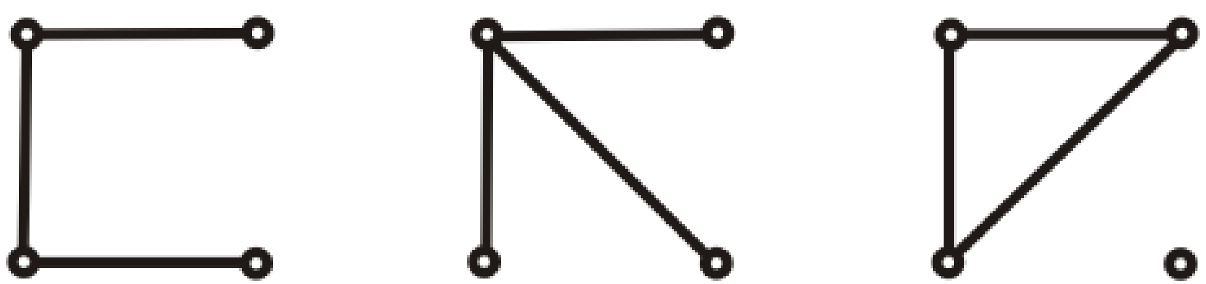
〖Example 〗 **Draw all nonisomorphic undirected simple graph with four vertices and three edges.**

Solution:

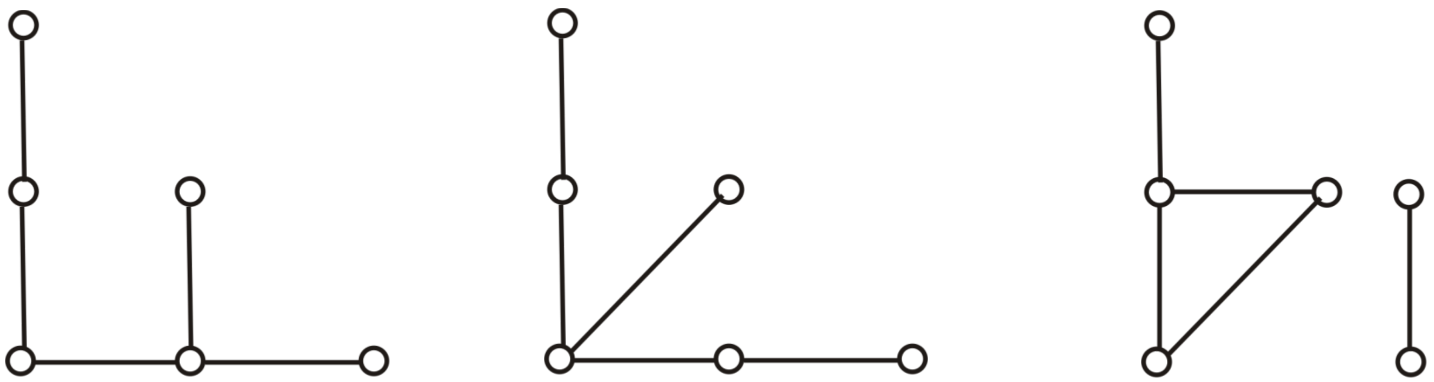
By the handshaking theorem, the sum of the degrees over four vertices is 6.

The maximal degree is 3, and the number of vertices with odd degrees must even. So there are 3 possible sequence of degrees:

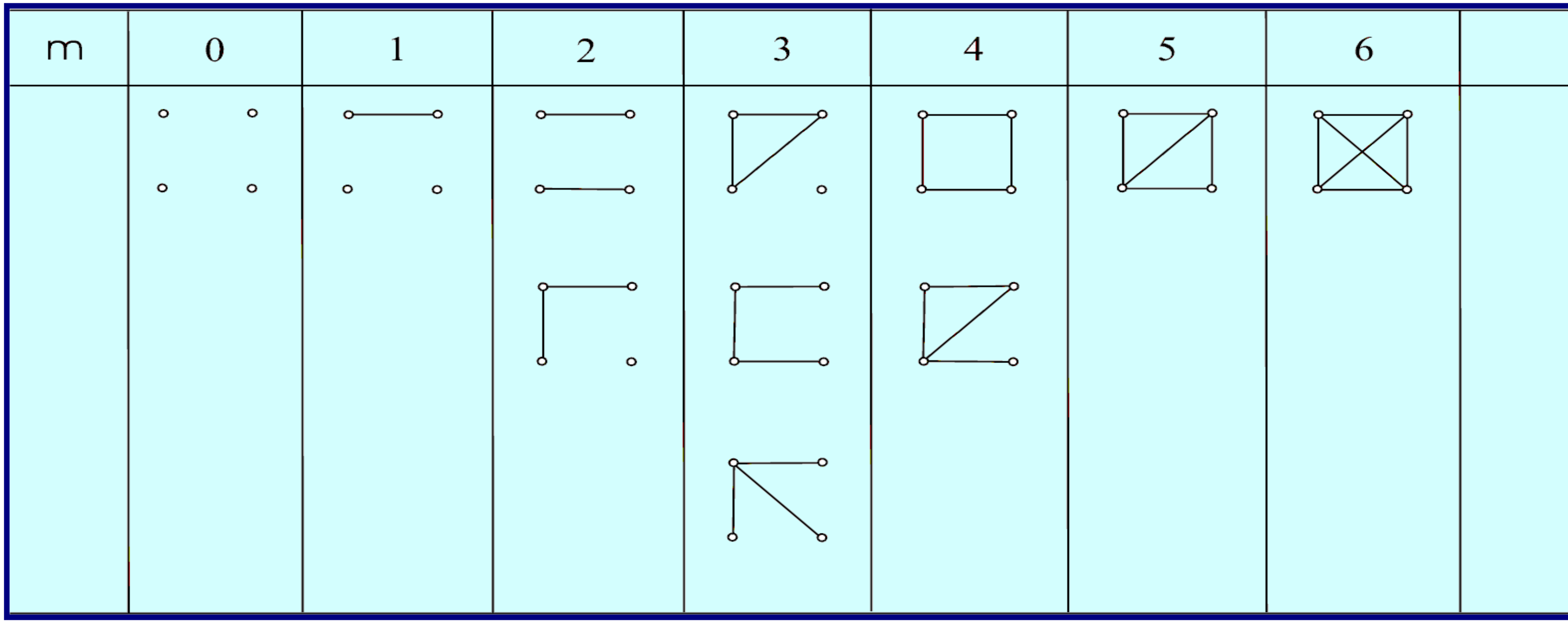
(1) 1,1,2,2; (2)1,1,1,3; (3)0,2,2,2.



〖Example 〗 **Draw 3 nonisomorphic undirected simple graph with the sequence of degrees 1,1,1,2,2,3.**



〖Example 〗 Draw all nonisomorphic spanning subgraphs of K_4.



Problem: The application of graph isomorphisms?

Homework:

Seventh Edition:

P. 675 8, 15, 17, 34-37