Planar Graphs

Section 10.7

Application Background

- The problem of planar graph:

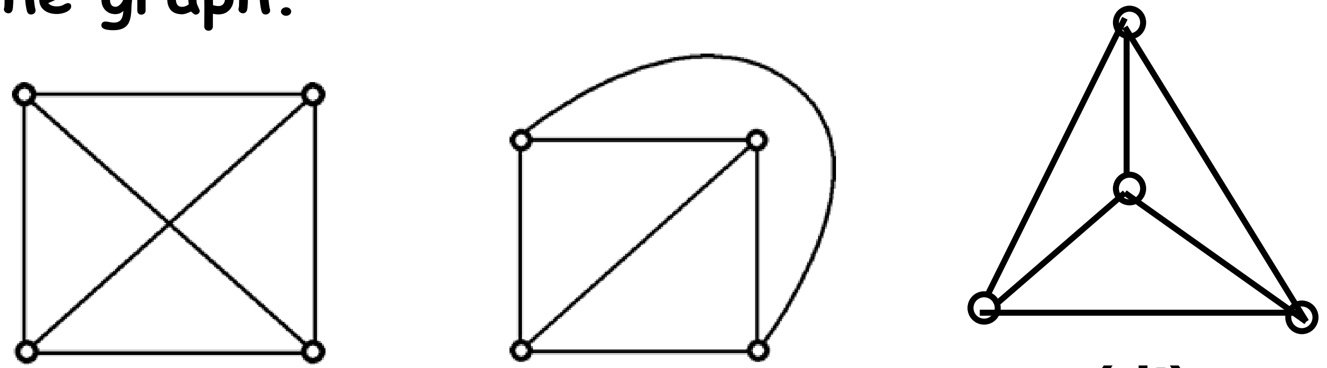
  Whether a graph can be drawn in the plane without edges crossing.

- Planarity of graphs plays an important role in the following domains:
  - ✓ The design of electronic circuits
  - ✓ The design of road networks

Definition of Planar Graph

【Definition】 A graph is called planar if it can be drawn in the plane without any edges crossing .

Such a drawing is called a planar representation of the graph.



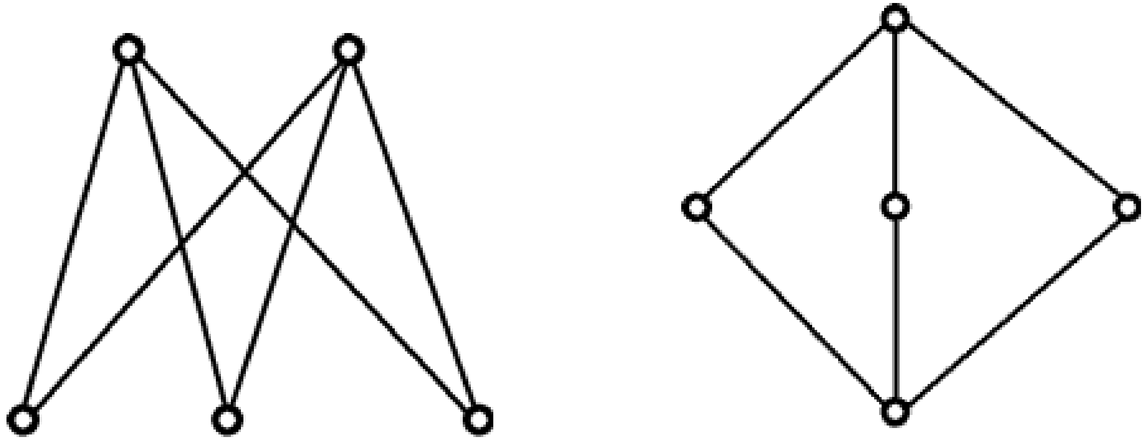
(a) (d) (d')

All of the above graphs are planar

Note:

We can prove that a graph is planar by displaying a planar representation.

〖Example 1〗 Is $K_{2,3}$ planar?

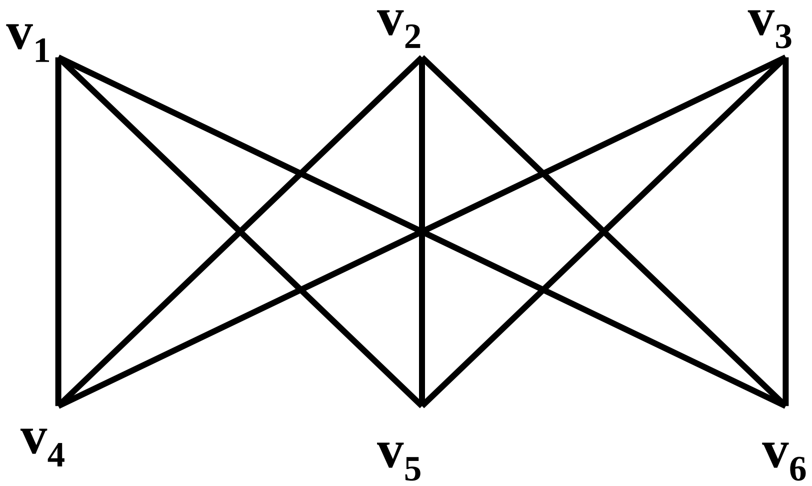


$K_{2,3}$ is planar

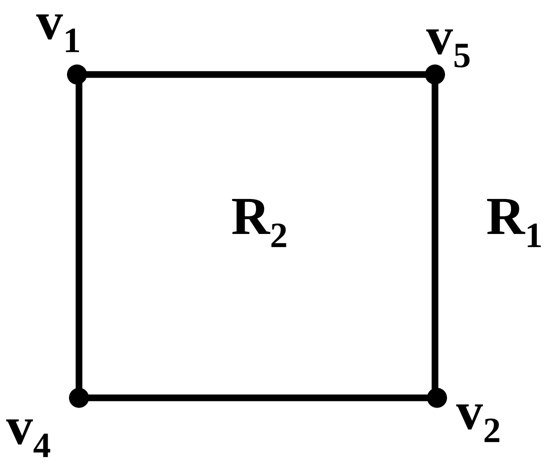
Note:

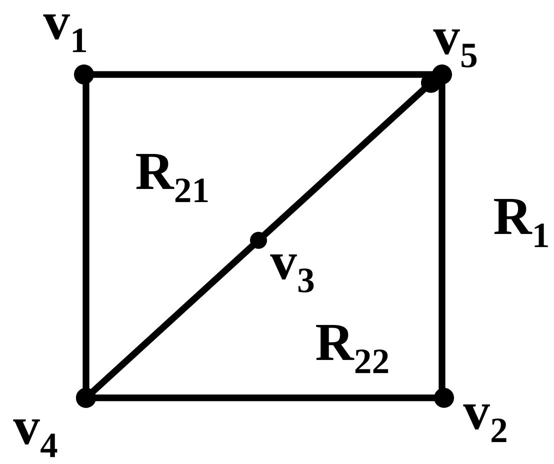
- Complete bipartite graphs $K_{2,n} (n \geq 1)$ are planar.
- Complete bipartite graphs $K_{1,n}$ are planar.

〖Example 2〗 Is $K_{3,3}$ planar?



$K_{3,3}$ is not planar





Euler's Formula

Some terminologies:

- **Region:** a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.
  - Bounded region
  - Unbounded region

  Note: There is one unbounded region in a planar graph.

- **the boundary of region**
- **the Degree of Region R (Deg(*R*)):** the number of the edges which surround R, suppose *R* is a region of a connected planar simple graph
- **adjacent regions**: two regions with a common border
- If e is not a cut edge, then it must be the **common border** of two regions

〖Example 3〗 There are 4 regions in the right graph.

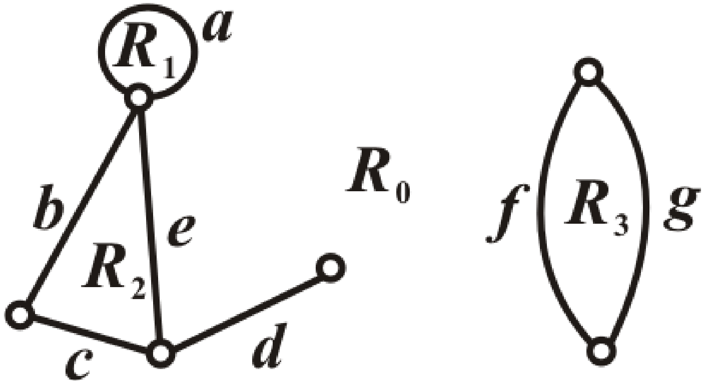
the boundary of region

R_1: *a*

R_2: *bce*

R_3: *fg*

R_0: *abcdde, fg*



$\deg(R_1) =$ 1

$\deg(R_2) =$ 3

$\deg(R_3) =$ 2

$\deg(R_0) =$ 8

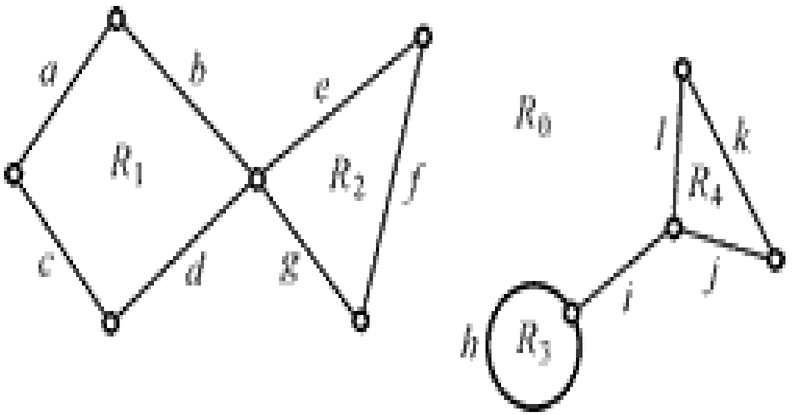
〖Example 4〗 The following graph is a planar representation of a graph.

◆There are 5 regions .

◆The boundaries of regions R_1, R_2, R_3 and R_4 are *abdc*, *efg*, *h*, *kjl.*

$\deg(R_1)=4$, $\deg(R_2)=3$,

$\deg(R_3)=1$, $\deg(R_4)=3$



◆The boundary of unbounded region R_0 is constructed by *abefgdc* and *kjihil*, $\deg(R_0)=13$。

Note: The sum of the degrees of the regions is exactly twice the number of edges in the planar graph.

$$2e=\sum_{all\ \ region\ \ R}\deg(R)$$

【 Theorem 1】 Euler's formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then r=e-v+2.

Proof:

First, we specify a planar representation of *G*. We will prove the theorem by constructing a sequence of subgraphs G_1, G_2, ⋯, $G_e = G$, successively adding an edge at each stage.

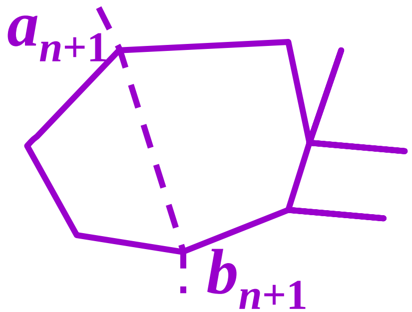
The constructing method: Arbitrarily pick one edge of *G* to obtain G_1. Obtain G_n from $G_{n\text{-}1}$ by arbitrarily adding an edge that is, incident with a vertex already in $G_{n\text{-}1}$.

Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of *G*, respectively.

(1) The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1, since $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$.

(2) Now assume that $r_n = e_n - v_n + 2$. Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}.

- **Both a_{n+1} and b_{n+1} are already in G_n.**



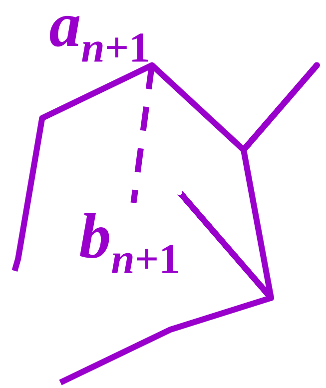
These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (and G_{n+1} is planar).

The addition of this new edge splits R into two regions.

Consequently, $r_{n+1} = r_n + 1$, $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$. Thus, $r_{n+1} = e_{n+1} - v_{n+1} + 2$.

- **One of the two vertices of the new edge is not already in G_n .**

Suppose that a_{n+1} is in G_n but that b_{n+1} is not.



Adding this new edge does not produce any new regions, since b_{n+1} must be in a region that has a_{n+1} on its boundary.

Consequently, $r_{n+1}= r_n$. Moreover, $e_{n+1}= e_n +1$ and $v_{n+1}= v_n +1$.

Hence, $r_{n+1}= e_{n+1}+1– v_{n+1}-1 +2$.

Note:

1) The Euler's formula is **necessary condition**.

2) How about unconnected simple planar graph?

Suppose that a planar graph G has k connected components, e edges, and v vertices. Let r be the number of regions in a planar representation of G.
Then $r=e-v+k+1$.

【Corollary 1】 If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v-6$

Proof:

Suppose that a connected planar simple graph divides the plane into r regions, the degree of each region is at least 3.

Since $2e = \sum \deg(R_i) \geq 3r$, it imply $r \leq (2/3)e$

Using Euler's formula $e-v+2=r$, we obtain

$e-v+2 \leq (2/3)e$, this shows that $e \leq 3v-6$.

Note:

- **For unconnected planar simple graph, $e \leq 3v-6$ is also holds.**

Since for a component, $e_i \leq 3v_i - 6$

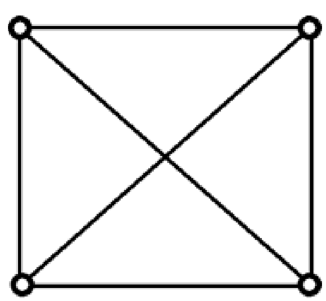
$$e = \sum e_i \leq \sum (3v_i - 6) < 3\sum v_i - 6 = 3v - 6$$

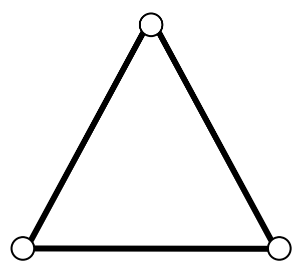
【Corollary 2】 If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v-4$.

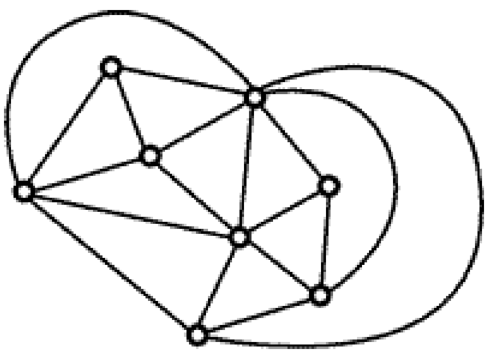
Note:

- **Generally, if every region of a planar connected graph has at least k edges, then** $e \leq \dfrac{(v-2)k}{k-2}$

- A connected planar simple graph with $e=3v-6$?


(a)


(b)


(c)

【Corollary 3】 If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof:

(1) *G* has one or two vertices

(2) G has at least three vertices

By Corollary 1 , we know that $e \le 3v-6$, so $2e \le 6v-12$

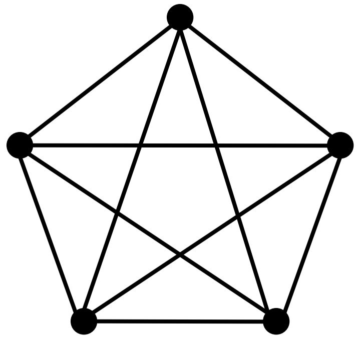
If the degree of every vertex were at least six, then

$2e \ge 6v$

〖Example 5〗 Show that k_5, $k_{3,3}$ are nonplanar.

Proof:

(1)

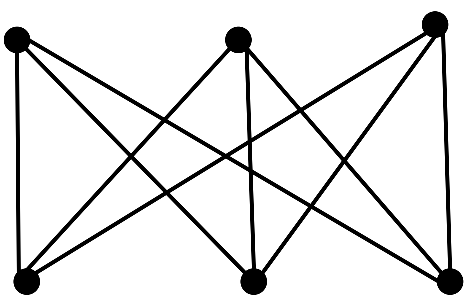


The graph k_5 has 5 vertices and 10 edges.

However, the inequality $e \leqq 3v-6$ is not satisfied for this graph since $e=10$ and $3v-6=9$.

Therefore, k_5 is not planar.

(2)



$K_{3,3}$ has 6 vertices and 9 edges.

Since $K_{3,3}$ has no circuits of length 3 (this is easy to see since it is bipartite), Corollary 3 can be used .

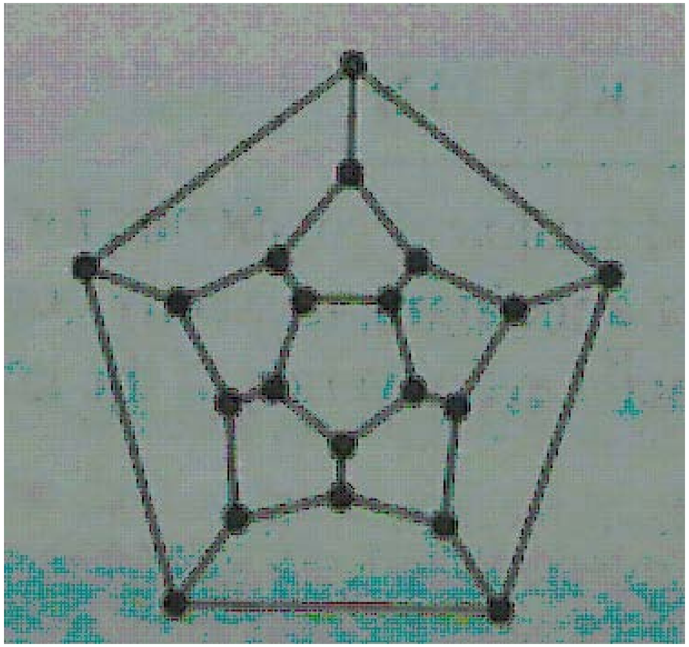
Since $e=9$ and $2v-4=8$, corollary 3 shows that $k_{3,3}$ is nonplanar.

〖Example 6〗 If G is a planar simple graph with vertices not exceeding 11, then G must exist vertices of degrees less than five.

〖Example 7〗 $K_n(n\geq7)$ is not planar.

〖Example 8〗 The construction of Dodecahedron .

Solution:



Since the degree of every vertex is 3 and the degree of every region is 5. Then

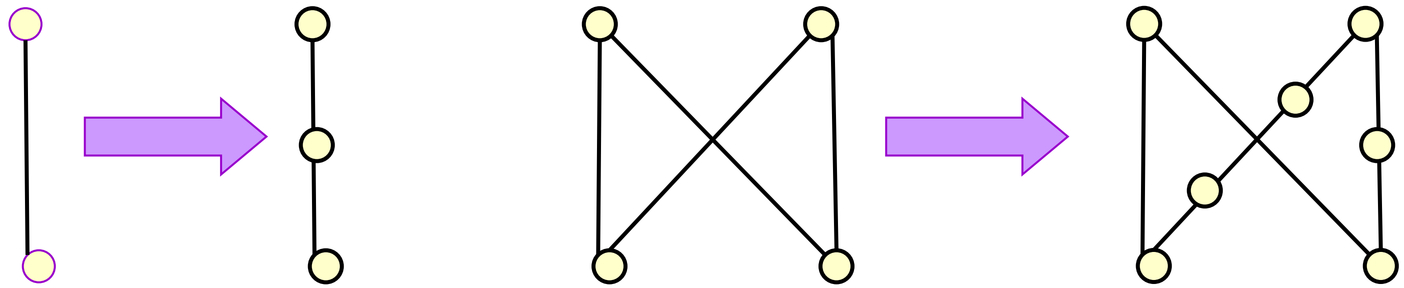
$$\begin{cases} 2e = 3v \\ 2e = 5r \\ r = e - v + 2 \end{cases}$$

It follows that *v*=20, *e*=30 and *r*=12.

KURATOWSKI'S THEOREM

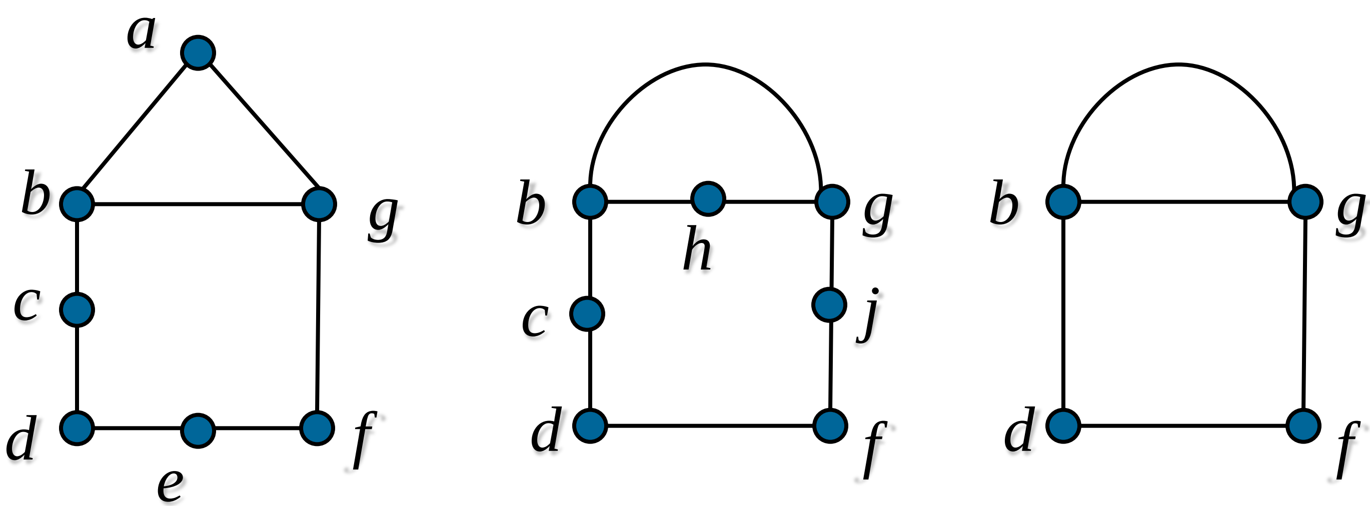
Terminologies:

- **Elementary subdivision:** If a graph is planar, so will be any graph obtained by removing an edge {u, v} and adding a new vertex w together with edges {u,w} and {w,v}.

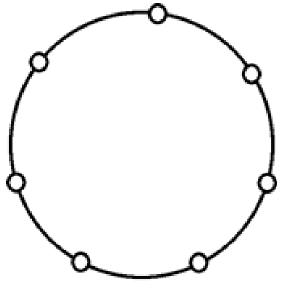


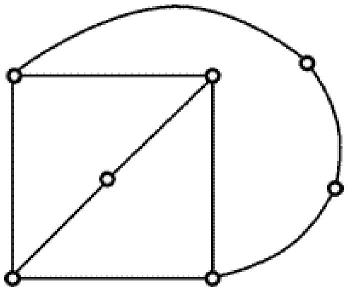
- **Homeomorphic:** the graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

Examples of homeomorphic graphs



These three graphs are homeomorphic


(a)


(b)

(a) is homeomorphic to K_3

(b) is homeomorphic to K_4

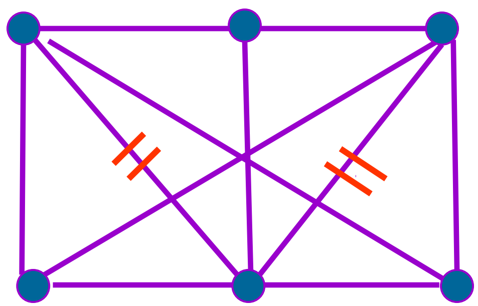
【Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

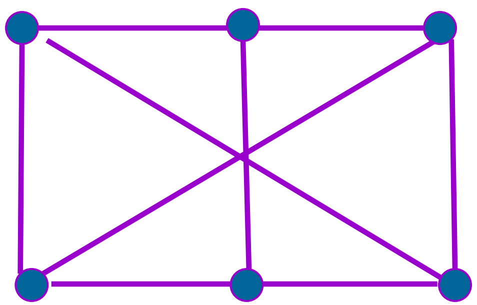
Proof:

- ✓ **It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is nonplanar.**
- ✓ **Every nonplanar graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5**

〖Example 9〗 **Determine whether the following graphs are planar.**

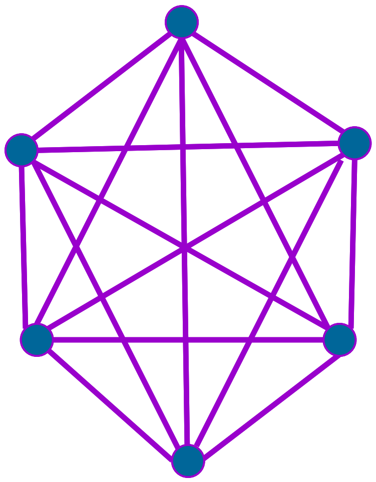
(1)





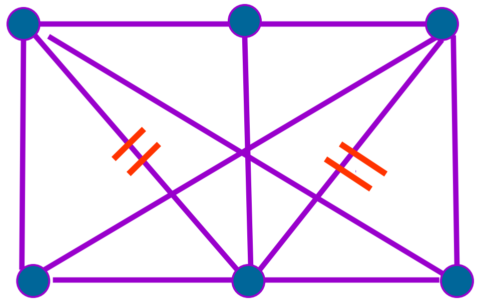
This graph is not planar.

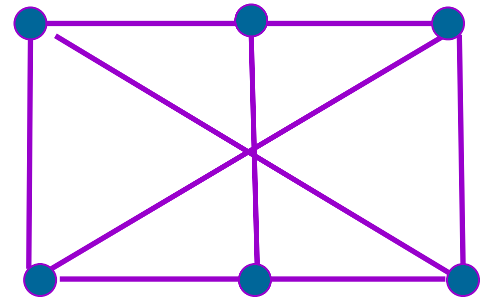
(2)



〖Example 9〗 **Determine whether the following graphs are planar.**

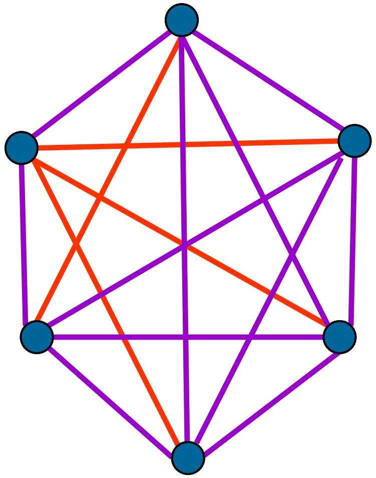
(1)

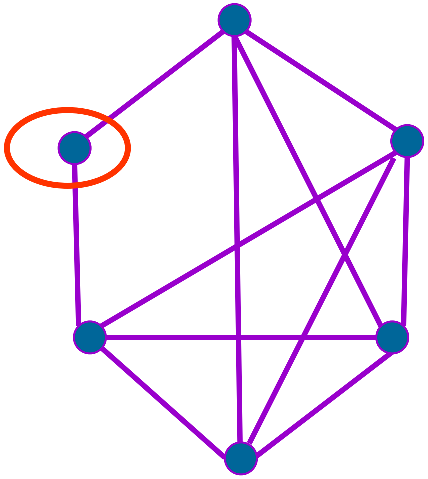




This graph is not planar.

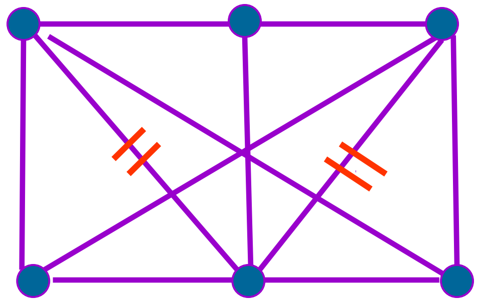
(2)

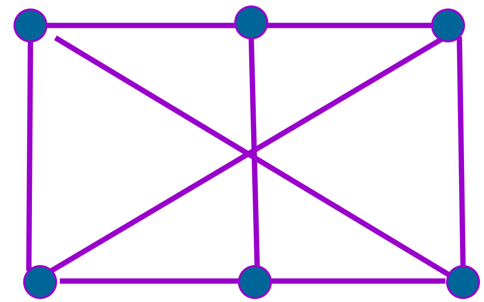




〖Example 9〗 Determine whether the following graphs are planar.

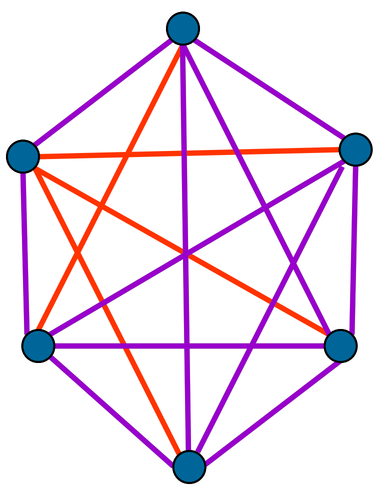
(1)

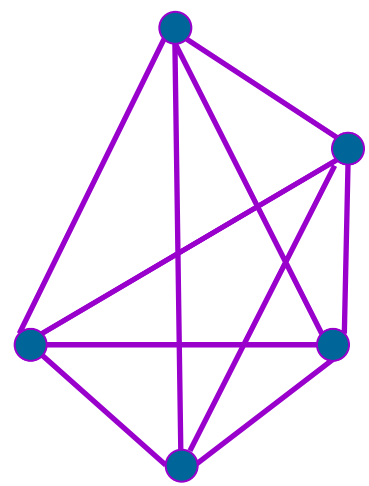




This graph is not planar.

(2)

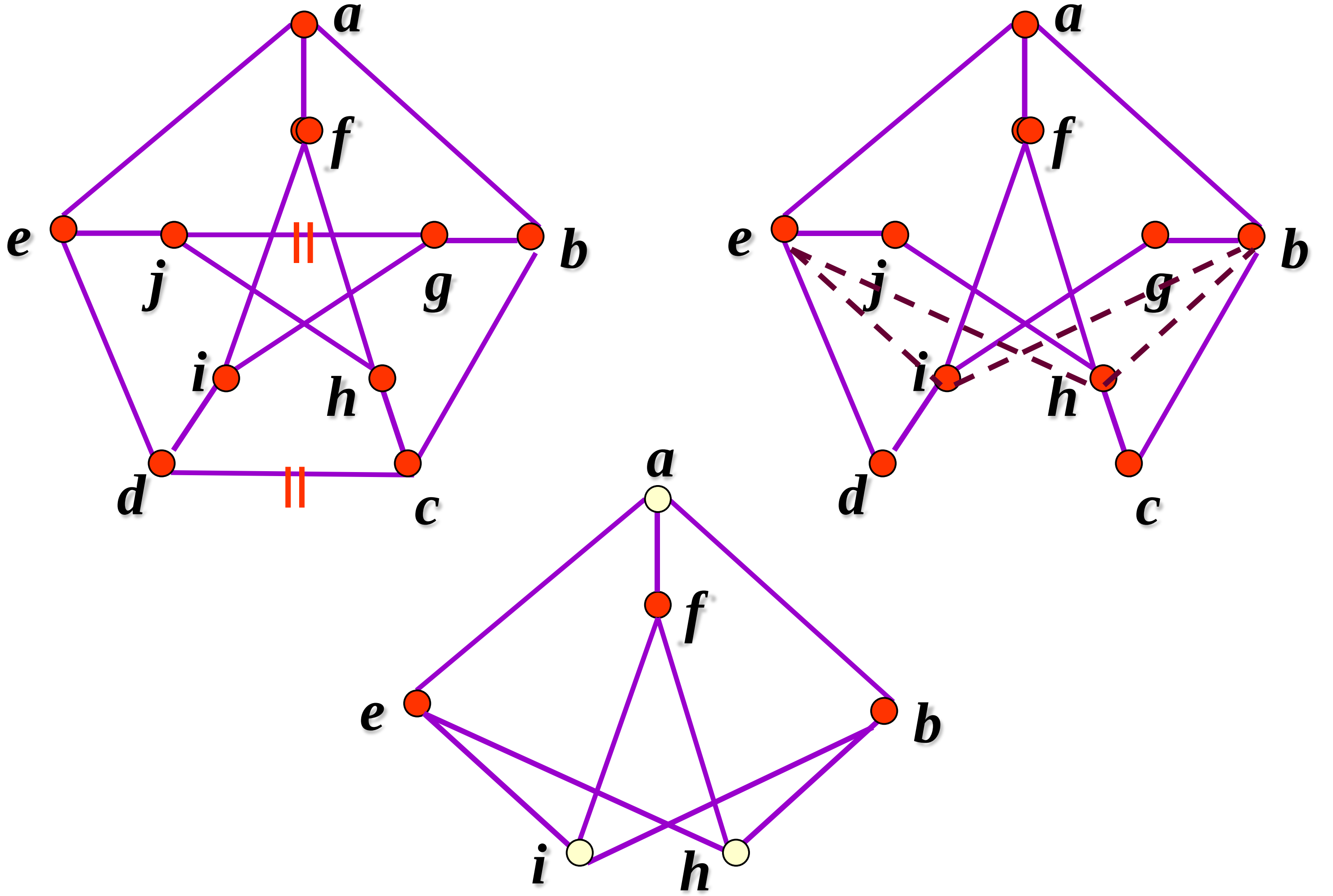




K_6 is not planar.

〖Example 9〗 **Determine whether the following graphs are planar.**

(3)



The Petersen graph is not planar.

Homework:

Seventh Edition:

P. 725 1, 7, 20, 22, 23, 25