Functions

Section 2.3

Section Summary

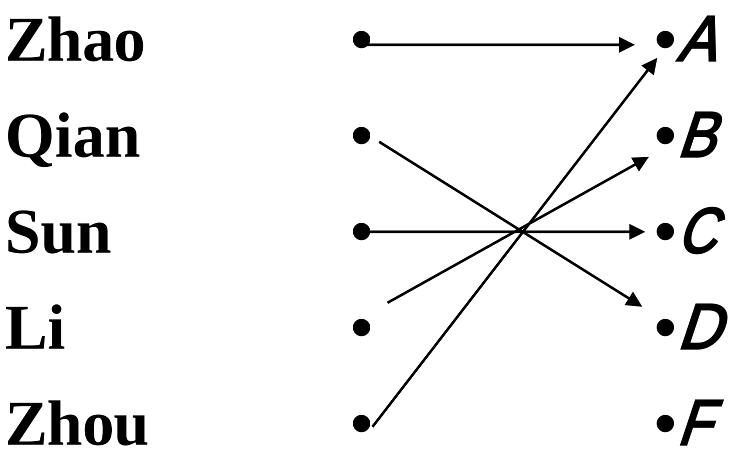
- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

Functions are sometimes called *mappings* or *transformations*.



Functions

- A function f: $A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \to \exists y[y \in B \land (x, y) \in f]]$$

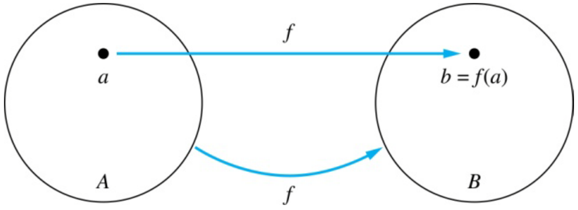
and

$$\forall x, y_1, y_2[[(x, y_1) \in f \land (x, y_2)] \to y_1 = y_2]$$

Functions

Given a function $f: A \to B$:

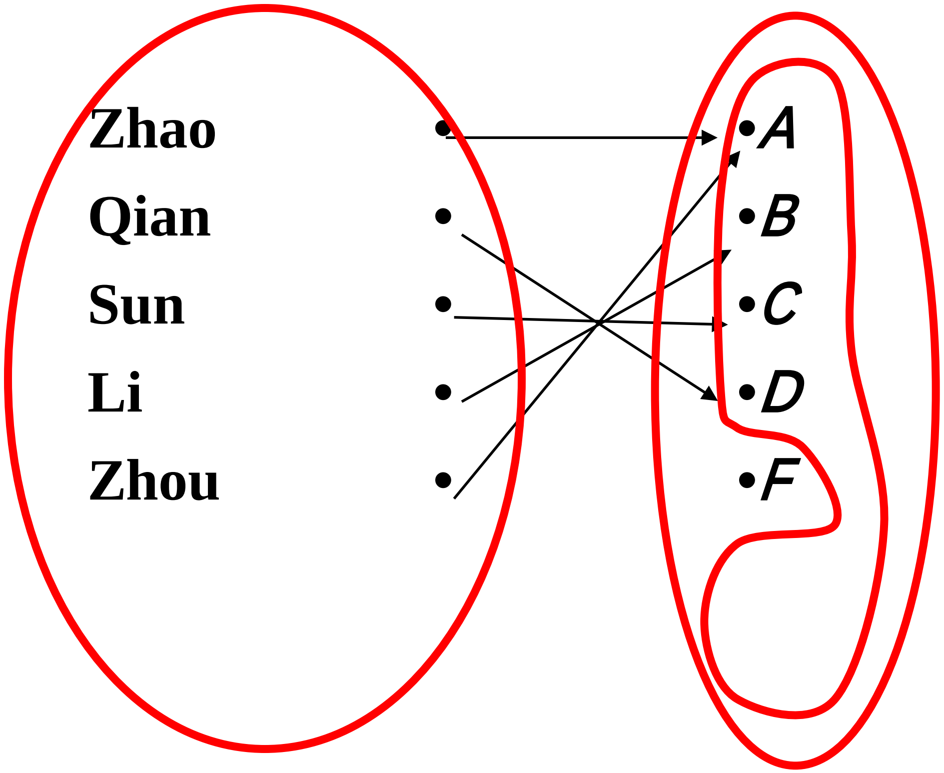
- We say *f maps A* to *B or*
  f is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.



- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b*.
- The range of *f* is the set of all images of points in **A** under *f*. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

For example,

Suppose that each student in a class is assigned a letter grade from the set $\{A,B,C,D,F\}$. Let G be the function that assigns a grade to a student.



Domain

Codomain

Range

Representing Functions

Functions may be specified in different ways:

- **An explicit statement of the assignment**

  Students and grades example.

- **A formula**

  $f(x) = x + 1$

- **A computer program**

  A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Chapter 5).

【**Definition**】 Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x)\, f_2(x)$$

【**Definition**】 Let f be a function from A to B and let S be a subset of A. The ***image*** of S is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so that

$$f(S) = \{ f(s) \mid s \in S \}$$

Remark:

$f(S)$ denotes a set, and not the value of the function f for the set S.

One-to-one Functions

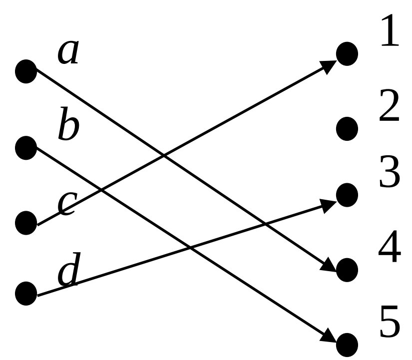
A function f is one-to-one (denoted 1-1), or injective

$$\forall a \forall b (a \in A \wedge b \in A \wedge (f(a) = f(b) \rightarrow a = b))$$

$$\forall a \forall b (a \in A \wedge b \in A \wedge (a \neq b \rightarrow f(a) \neq f(b)))$$

Note: Preimages are unique.

Example ,



Onto Functions

A function f from A to B is called onto, or surjective

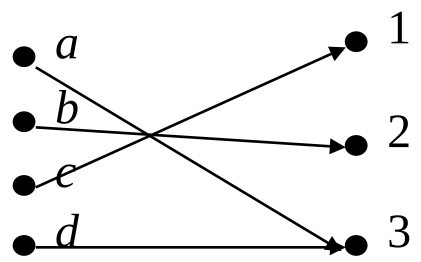
$$\forall b(b \in B \rightarrow \exists a\ (a \in A \wedge f(a) = b))$$

Note:

This means that for every *b* in *B* there must be an *a* in *A* such that *f*(*a*) = *b*.

Every *b* in *B* has a pre-image.

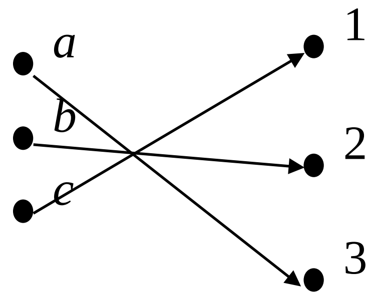
Example ,



One-to-one Correspondence Functions

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

For example,



Note:

Whenever there is a bijection from *A* to *B*, the two sets must have the same number of elements or the same *cardinality*.

That will become our *definition*, especially for infinite sets.

Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto

Example : Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example : Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

A function is

increasing	$\forall x \forall y (x < y \rightarrow f(x) \leq f(y))$
strictly increasing	$\forall x \forall y (x < y \rightarrow f(x) < f(y))$
decreasing	$\forall x \forall y (x < y \rightarrow f(x) \geq f(y))$
strictly decreasing	$\forall x \forall y (x < y \rightarrow f(x) > f(y))$

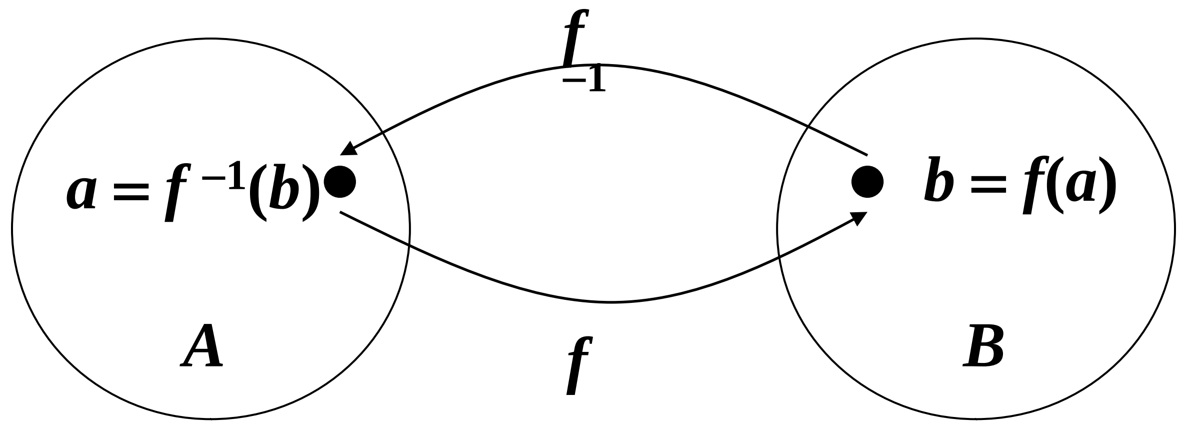
Questions:

1. A function that is either strictly increasing or strictly decreasing must be one to one?
2. The number of one to one functions form a set S to a set T?

Inverse Functions

Let f be a bijection from A to B. Then the inverse function of f, denoted f^{-1}, is the function from B to A defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b$$



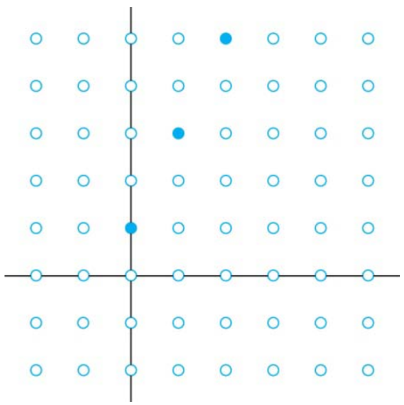
Note:

No inverse function exists unless *f* is a bijection.

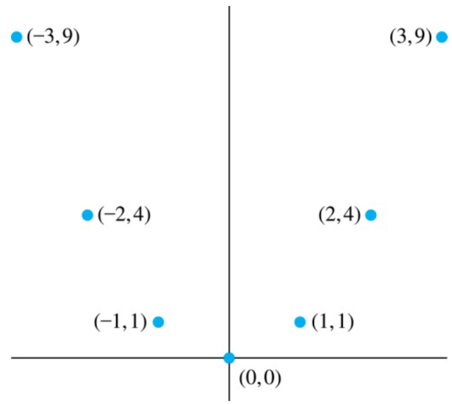
The Graphs of Functions

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs

$$\{(a, b) \mid a \in A \text{ and } f(a) = b\}.$$



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

The ***floor function*** **f (x) is the largest integer less than or equal to x.**

Notation: $\lfloor x \rfloor$

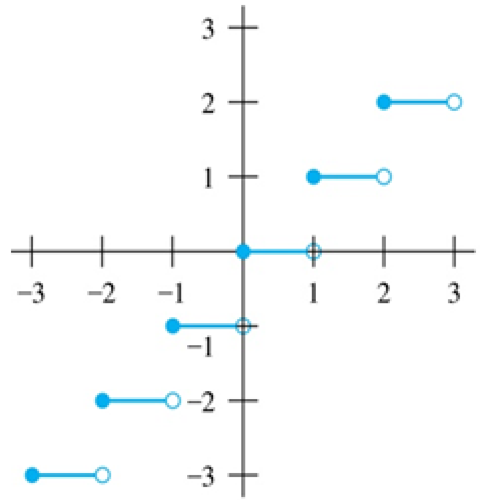
For example:

$\lfloor 0.5 \rfloor = 0$

$\lfloor 1.5 \rfloor = 1$

$\lfloor 2 \rfloor = 2$

$\lfloor -0.5 \rfloor = -1$



(a) $y = [x]$

Graph of Floor Functions

Remark:

The floor function is often also called the ***greatest integer function*****. It is often denoted by [*x*].**

Some Important Functions

The ceiling function f (x) is the smallest integer greater than or equal to x.

Notation:

$$\lceil x \rceil$$

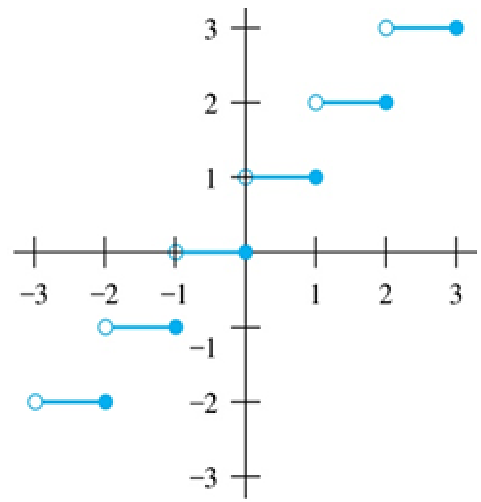
For example:

$\lceil 0.5 \rceil =$ 1

$\lceil 1.5 \rceil =$ 2

$\lceil 2 \rceil =$ 2

$\lceil -0.5 \rceil =$ 0



(b) $y = \lceil x \rceil$

Graph of Ceiling Functions

The floor and ceiling functions are useful in wide variety of application.

- **Data storage and data transmission**
- **The pigeonhole principle**

Example, Date stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

$$\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$$

Useful Properties of the Floor and Ceiling Functions

(1a) $\lfloor x \rfloor = n$ **if and only if** $n \leq x < n + 1$ **where** n **is an integer**

(1b) $\lceil x \rceil = n$ **if and only if** $n - 1 < x \leq n$ **where** n **is an integer**

(1c) $\lfloor x \rfloor = n$ **if and only if** $x - 1 < n \leq x$ **where** n **is an integer**

(1d) $\lceil x \rceil = n$ **if and only if** $x \leq n < x + 1$ **where** n **is an integer**

(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ **where** m **is an integer**

(4b) $\lceil x + m \rceil = \lceil x \rceil + m$ **where** m **is an integer**

Prove property (4a)

(4a) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ **where** m **is an integer**

Proof:

Suppose that $\lfloor x \rfloor = n$**, where n is a positive integer.**

By property (1*a*), it follows that if $n \le x < n + 1$**.**

Then $n + m \le x + m < n + m + 1$**.**

Using property (1*a*) again, we see that $\lfloor x + m \rfloor = n + m = \lfloor x \rfloor + m$**.**

Example, Prove that if *x* is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor.$$

Hint:

The floor function: **Let** $x = n + \varepsilon$**, where** $n = \lfloor x \rfloor$ **is an integer, and** ε**, the fractional part of** x **,** $0 \leq \varepsilon < 1$**.**

The ceiling function: **Let** $x = n - \varepsilon$**, where** $n = \lceil x \rceil$ **is an integer, and** $0 \leq \varepsilon < 1$**.**

Homework:

Seventh Edition： P.153 12,40, 56, 72, 76