Connectivity

Section 10.4

Paths

In G = (V, E), it is usually considered that starting from one vertex and terminating at another vertex by passing along some edges. This is the concept of path.

Many problems can be modeled with paths of the graph.

Definition of path in undirected graph

- **Path of length n from u to v in an undirected graph**
  - a sequence of n edges $e_1, \ldots, e_n$ for which there exists a sequence $x_0=u, x_1, \ldots, x_{n-1}, x_n=v$ such that e_i has endpoints x_{i-1}, x_i
  - When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_{n-1}, x_n$
- **Circuit**
  - if the path begins and ends with the same vertex
- The path or circuit is said to pass through the vertices $x_1, \ldots, x_{n-1}$ or traverse the edges $e_1, \ldots, e_n$
- **Simple path/circuit**
  - if it does not contain the same edge more than once

Path in directed graph

- **path of length n from u to v in a directed graph**
  - a sequence of edges $e_1, \ldots, e_n$ such that e_1 is associated with (x_0, x_1), $e_2 \ldots$
  - When there are no multiple edges in the directed graph, this path is denote by its vertex sequence $x_0, x_1, \ldots, x_{n-1}, x_n$
- **circuit or cycle**
  - if the path begins and ends with the same vertex
- **simple path/circuit**
  - if it does not contain the same edge more than once

- **Paths represent useful information in many graph models.**

Path in Acquaintanceship Graphs

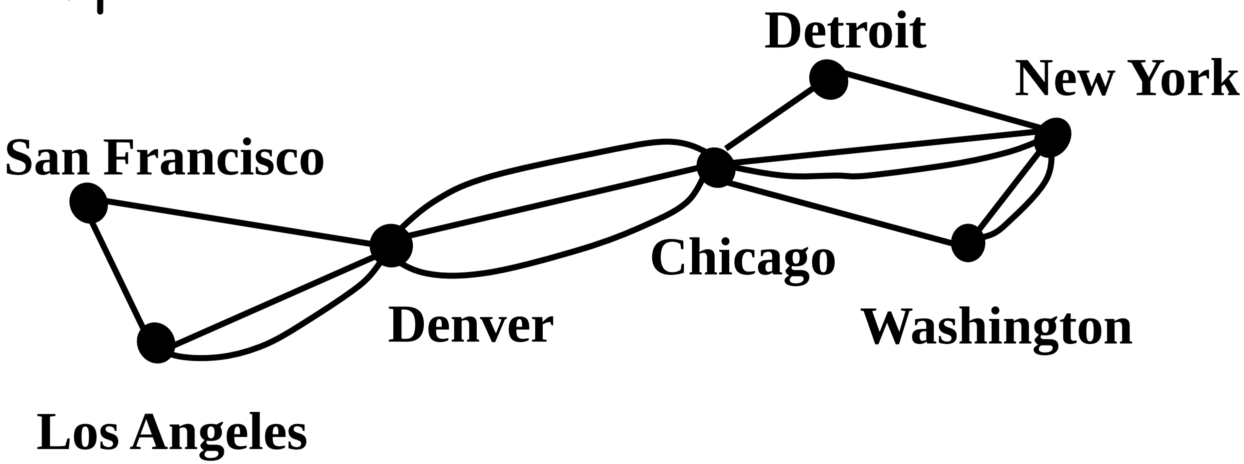
In an acquaintanceship graph there is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one other.

Many social scientists have conjectured that almost every pair of people in the world are linked by a small chain of people, perhaps containing just five or fewer people.

Six Degrees of Separation

Connectedness in undirected graphs

Example: Computer network



Question:

Can any two computers on the network communicate with each other?

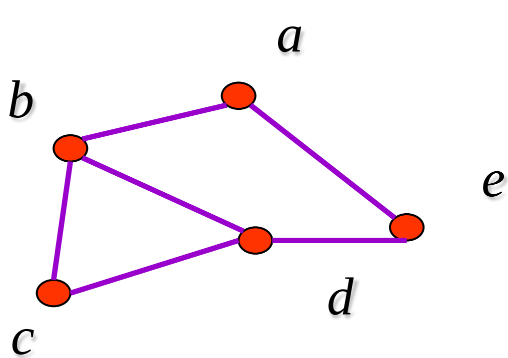
Whether there is always a path between two vertices in the graph.

The definition of connected and disconnected

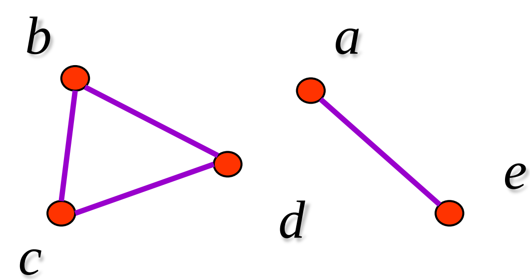
An undirected graph is connected: if there is a path between every pair of distinct vertices

An undirected graph is disconnected: the graph is not connected

Disconnect a graph: remove vertices or edges, or both, to produce a disconnected subgraph.



connected



disconnected

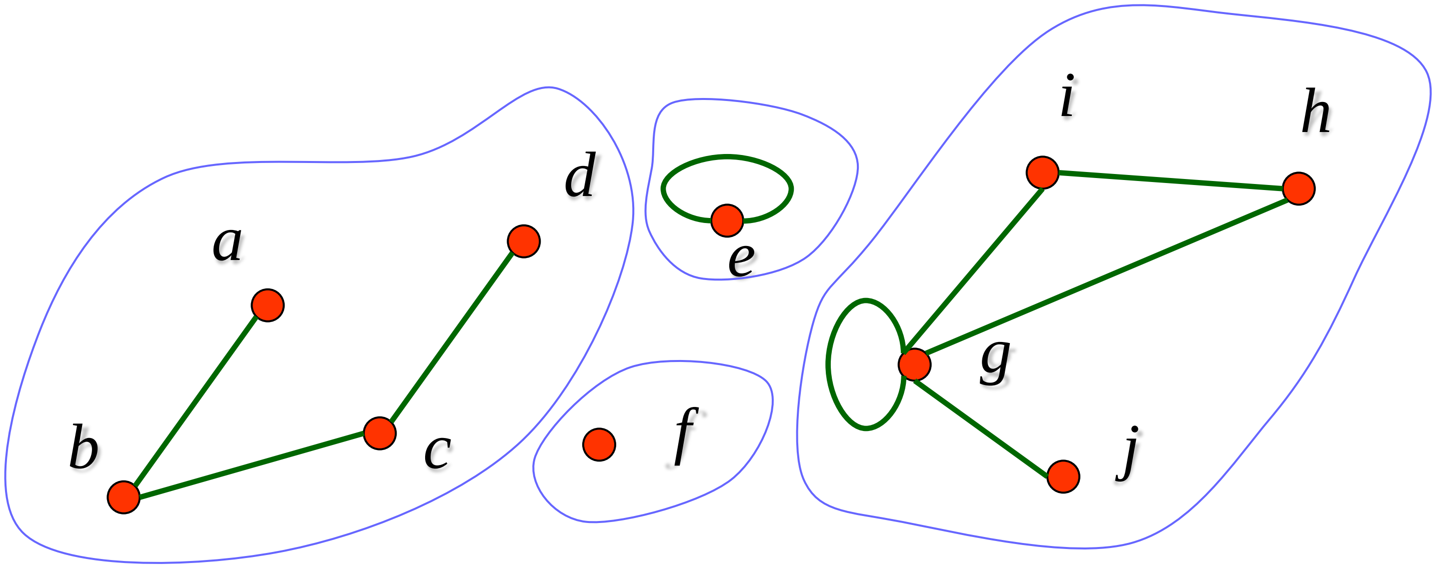
【 Theorem 1】 **There is a simple path between every pair of distinct vertices of a connected undirected graph.**

Proof:

Because the graph is connected there is a path between *u* and *v*. Throw out all redundant circuits to make the path simple.

◆ Connected Components

The maximally connected subgraphs of *G* are called the connected components or just the components.

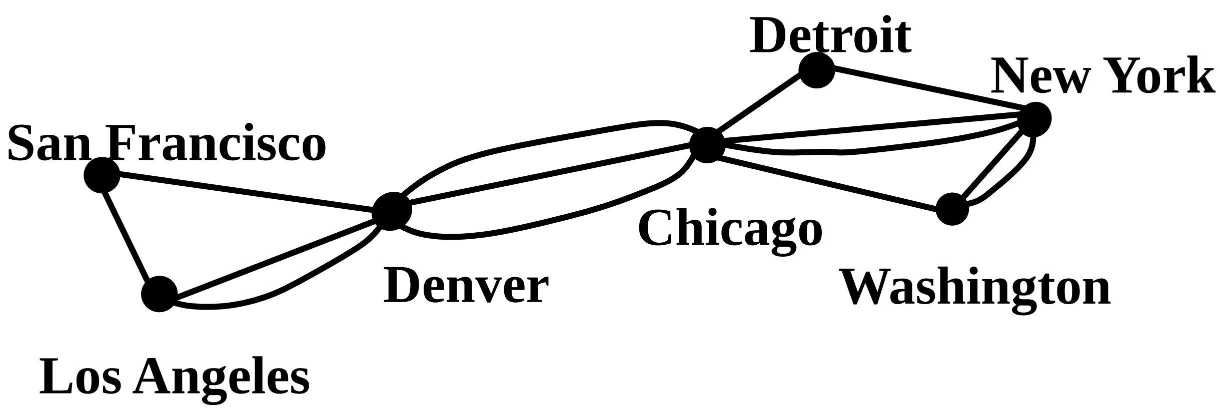


These four subgraphs are the connected components.

How connected is a graph?

Computer network:

Any two computers on the network can communicate when the graph representing this network is connected.



Another question:

How reliable this network is?

Will it still be possible for all computers to communicated after a router or a communications link fails?

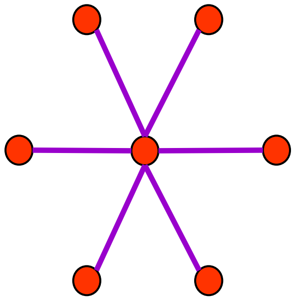
◆ cut vertex and cut edge

◆ cut vertex (or articulation point)

- if removing a vertex and all edges incident with it results in more connected components than in the original graph.

◆ cut edge or bridge

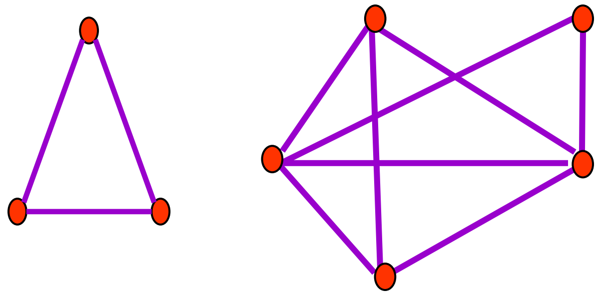
- if removing a edge creates more components.



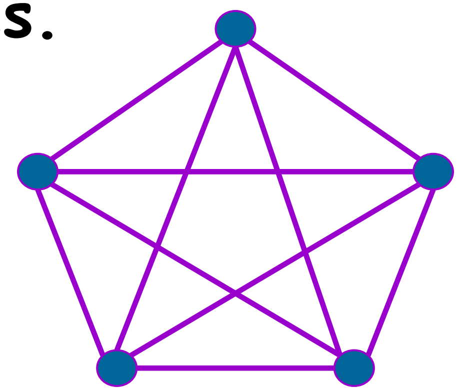
cut vertex: the center vertex

cut edges: all edges

Not all graphs have cut vertices.



no cut edges or vertices



K_n ($n \geq 3$) has no cut vertices.

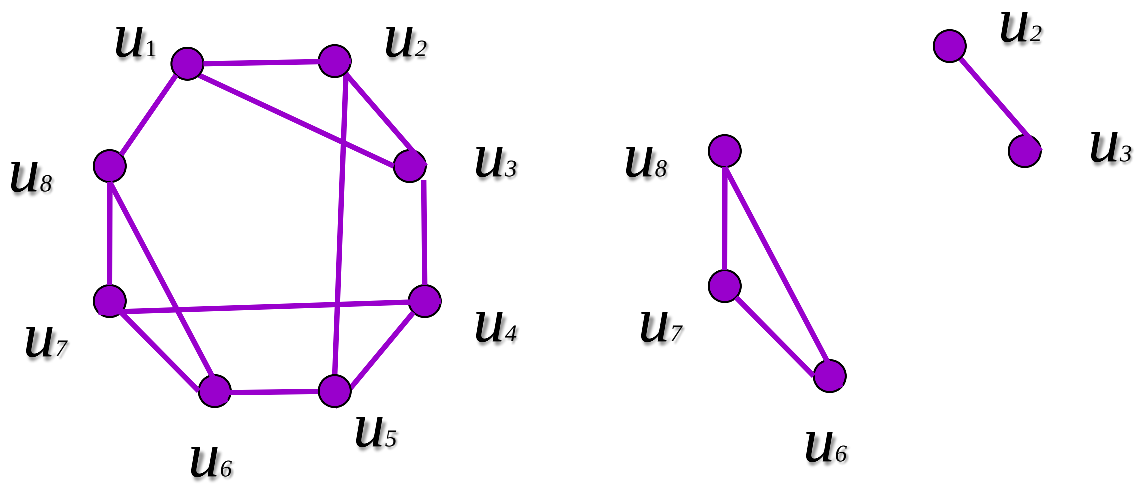
- **nonseparable graphs**
  - Connected graphs without cut vertices
  - Nonseparable graphs can be thought of as more connected than those with a cut vertex.

How to measure graph connectivity?

- based on the minimum number of vertices that can be removed to disconnect a graph.

Vertex connectivity

Vertex cut, or separating set: a subset V' of the vertex set V of G=(V,E) such that G-V' is disconnected.



$\{u_1, u_4, u_5\}$ is a vertex cut.

Note:

Every connected graph except a complete graph has a vertex cut. [Exercise 51]

Vertex connectivity $\kappa(G)$: the minimum number of vertices in a vertex cut.

Note:

① The minimum number of vertices that can be removed from G to either disconnect G or produce a graph with a single vertex.

② $\kappa(G)=0$ iff G is disconnected or $G=K_1$

③ $\kappa(G)=1$ if G is connected with cut vertices or $G=K_2$

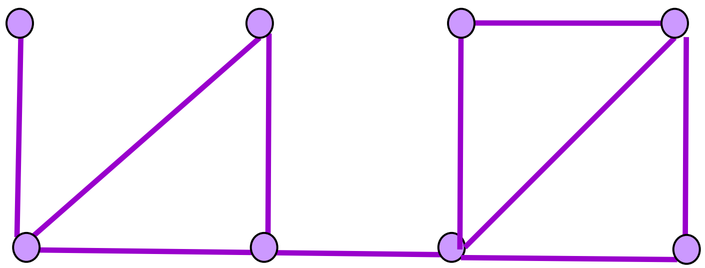
④ $\kappa(G)=n-1$ iff G is complete

$0\le\kappa(G)\le n-1$ if G has n vertices.

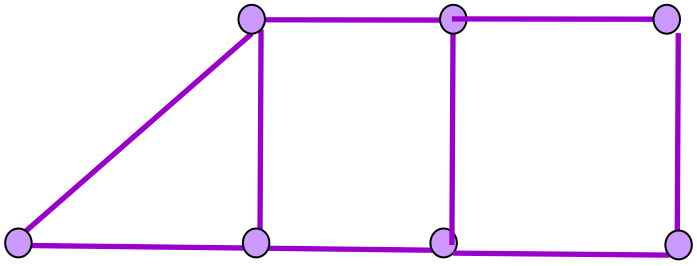
The larger $\kappa(G)$ is, the more connected we consider G to be.

A graph is **K-connected** (or **k-vertex-connected**), if $\kappa(G)\ge K$

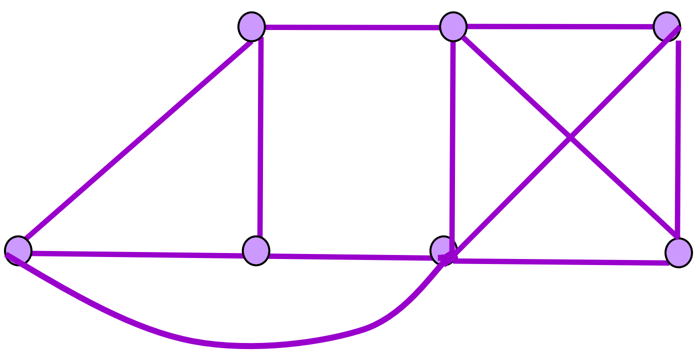
〖Example 〗Find the vertex connectivity for each of the following graphs.



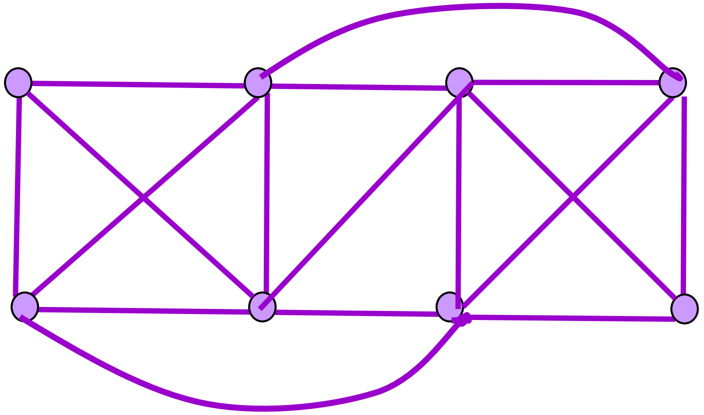
$\kappa(G)=1$



$\kappa(G)=2$



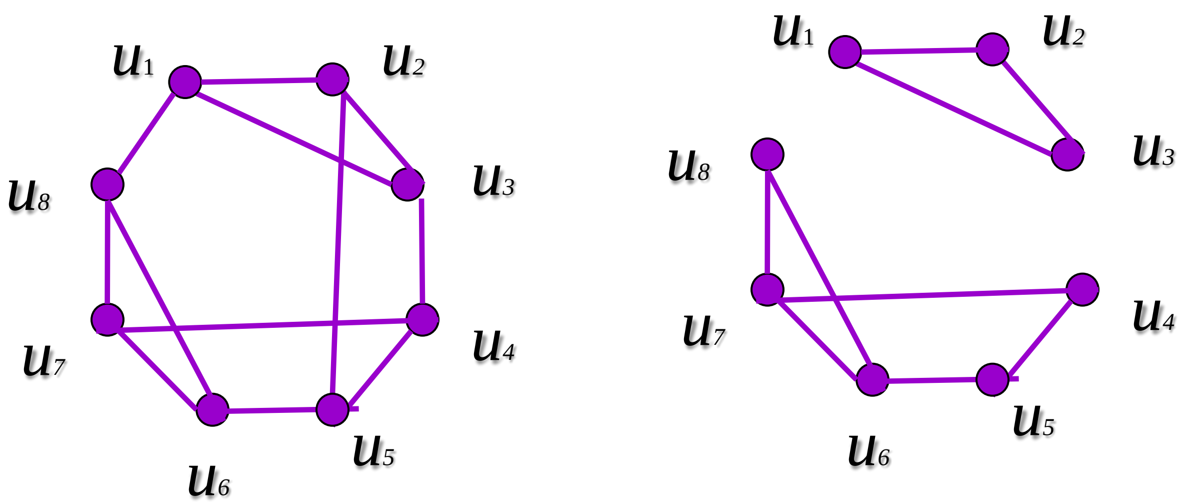
$\kappa(G)=2$



$\kappa(G)=3$

◆ Edge connectivity

edge cut: a set of edges E' is called an edge cut of G if the subgraph G-E' is disconnected.



$\{(u_1, u_8), (u_3, u_4), (u_2, u_5)\}$ is a edge cut

◆ Edge connectivity

edge connectivity $\lambda(G)$: the minimum number of edges in an edge cut of G.

Note:

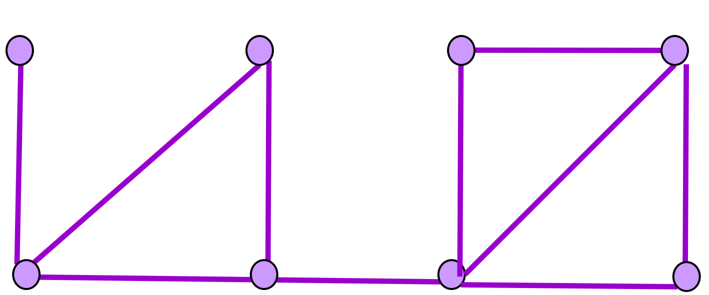
① The minimum number of edges that can be removed from G to disconnect G

② $\lambda(G)=0$ if G is disconnected or G is a graph consisting of a single vertice

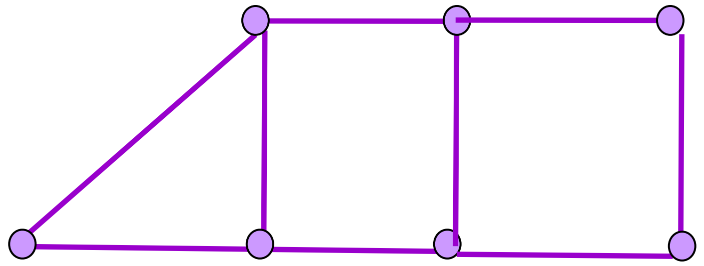
③ $\lambda(G)=n-1$ iff $G=K_n$

$0 \leq \lambda(G) \leq n-1$ if G has n vertices.

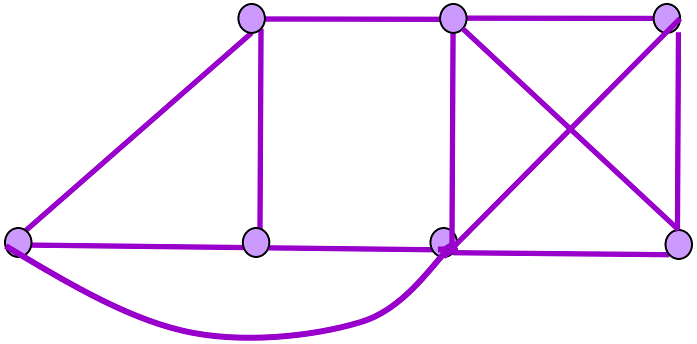
〖Example 〗Find the edge connectivity for each of the following graphs.



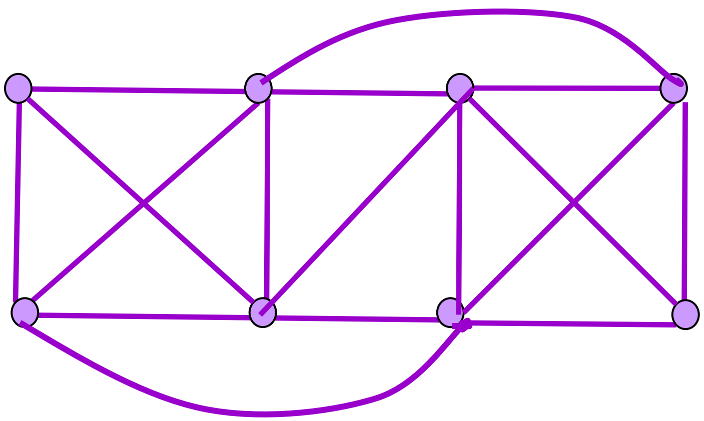
$\lambda(G)=1$



$\lambda(G)=2$



$\lambda(G)=3$



$\lambda(G)=3$

An inequality for vertex connectivity and edge connectivity

When G=(V,E) is a noncomplete connected graph with at least three vertices

$$\kappa(G) \leq \min_{v\in} \deg(v)$$

$$\lambda(G) \leq \min_{v\in} \deg(v)$$

$$\kappa(G) \leq \lambda(G) \leq \min_{v\in} \deg(v)$$

Application of vertex and edge connectivity

- **To analysis the reliability of network**

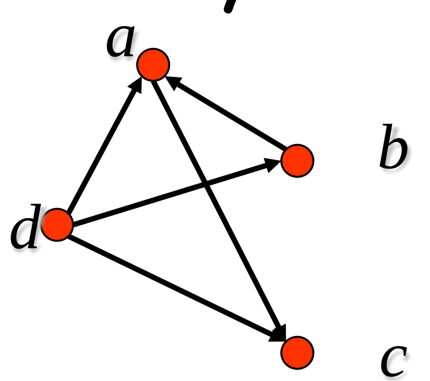
  The vertex connectivity of the graph representing network equals the minimum number of routers that disconnect the network when they are out of service.

  The edge connectivity represents the minimum number of fiber optic links that can be down to disconnect the network.

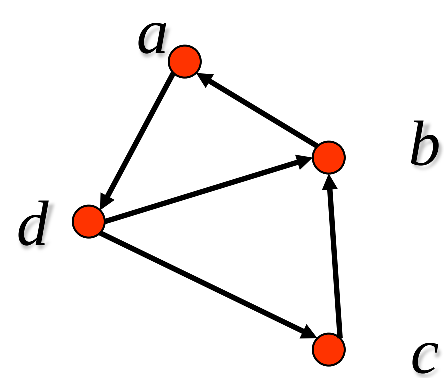
Connectedness in directed graphs

- **strongly connected**
  - if there is a path from *a* to *b* and from *b* to *a* for all vertices *a* and *b* in the graph.
- **weakly connected**
  - if the underlying undirected graph is connected.

Note: By the definition, any strongly connected directed graph is also weakly connected.



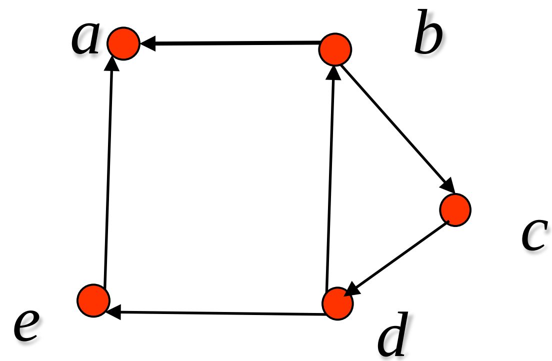
Weakly connected



Strongly connected

□strong components of a directed graph

For directed graph, the maximal strongly connected subgraphs are called the strongly connected components or just the strong components.



Three strong components: a; e; the subgraph consisting of vertices b, c, and d and edges (b, c), (c, d),(d, b)

Problem:

1. **How to determine whether a given directed graph is strongly connected or weakly connected ?**
2. **How to find the strongly connected components in a directed graph ?**

Paths and Isomorphism

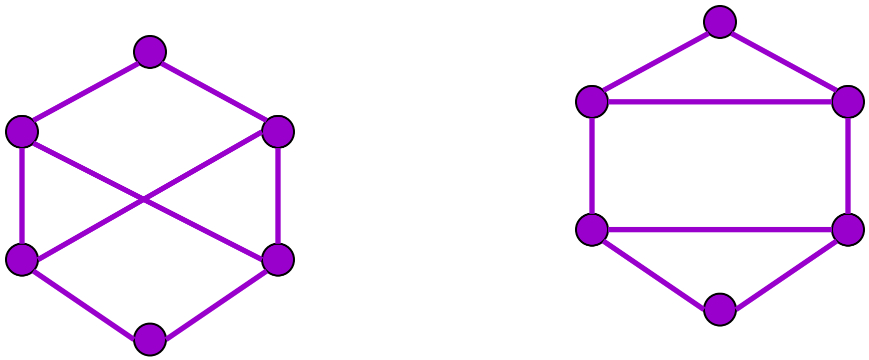
Idea:

(1) Some other graph invariants involving path

- ✓ Two graphs are isomorphic only if they have simple circuits of the same length.
- ✓ Two graphs are isomorphic only if they contain paths that go through vertices so that the corresponding vertices in the two graphs have the same degree.

(2) We can also use paths to find mapping that are potential isomorphisms.

〖Example 4〗 **Are these two graphs isomorphic?**

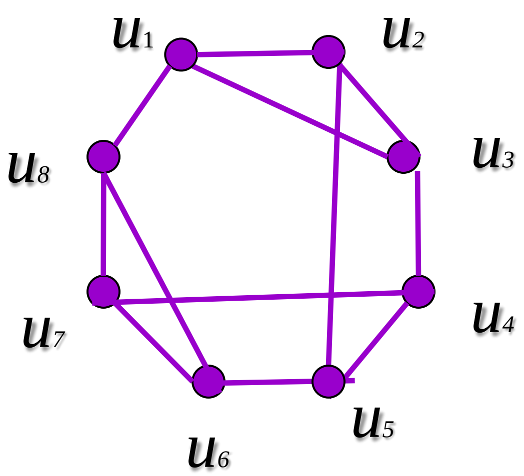


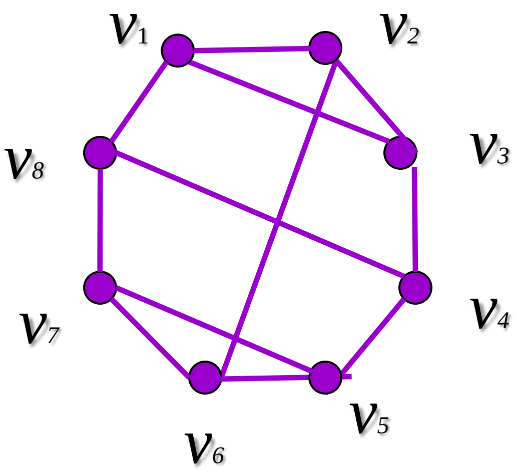
Solution:

These two graphs are not isomorphic.

Because the right graph contains circuits of length 3, while the left graph does not.

〖Example 〗 Find an isomorphism between the following graphs.





$u_1 \leftrightarrow v_2$ $u_4 \leftrightarrow v_4$

$u_8 \leftrightarrow v_6$ $u_5 \leftrightarrow v_8$

$u_2 \leftrightarrow v_1$ $u_6 \leftrightarrow v_7$

$u_3 \leftrightarrow v_3$ $u_7 \leftrightarrow v_5$

Counting paths between vertices

【Theorem 2】 The **number of different paths of length r** from v_i to v_j is equal to the $(i, j)th$ entry of A^r, where A is the adjacency matrix representing the graph consisting of vertices $v_1, v_2, \ldots v_n$.

Note: **This is the standard power of *A*, not the Boolean product.**

Proof:

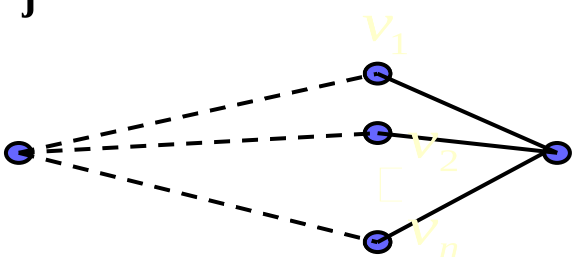
Let $A = (a_{ij})_{n \times n}$

(1) r=1.

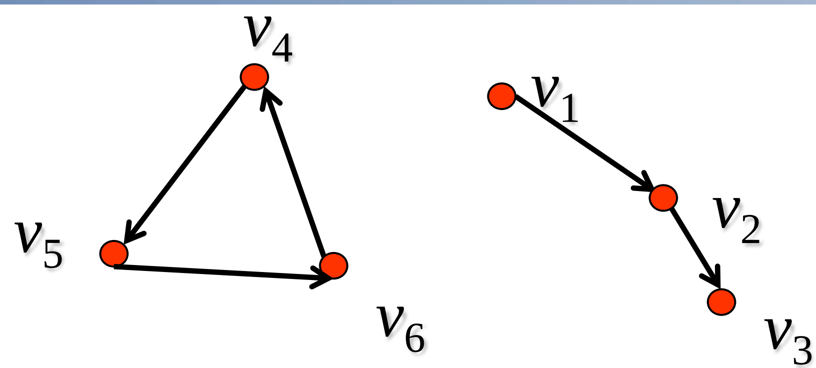
(2) Assuming that the $(i. j)$th entry of A^r is the number of different paths of length r from v_i to v_j .

$$A^{r+1} = A^r \cdot A = (d_{ij})_{n \times n}$$

$$d_{ij} = c_{i1}a_{1j} + c_{i2}a_{2j} + \cdots + c_{in}a_{nj} = \sum_{k=1}^{n} c_{ik}a_{kj}$$



〖Example 2〗



(1) How many paths of length 2 are there from v_5 to v_4?

a_{54} in A^2; 1

(2) The number of paths not exceeding 6 are there from v_4 to v_5?

a_{45} in $A+A^2+A^3+A^4+A^5+A^6$; 2

(3) The number of circuits starting at vertex v_5 whose length is not exceeding 6?

a_{55} in $A+A^2+A^3+A^4+A^5+A^6$; 2

Question: How to find the length of the shortest path from v and w in a graph?

Homework:

Seventh Edition:

P. 689 26e), 28, 29, 62