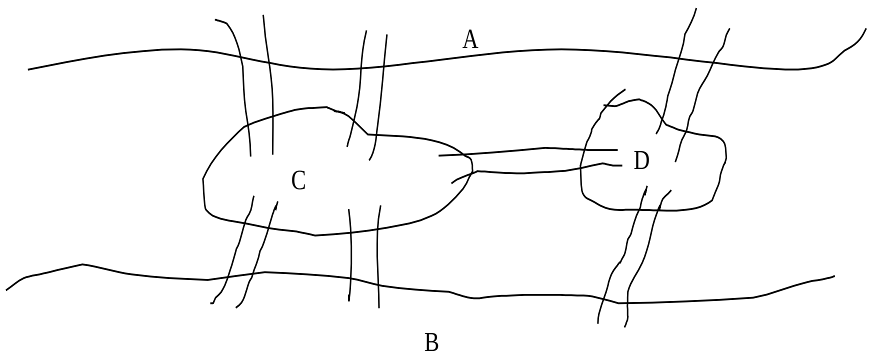
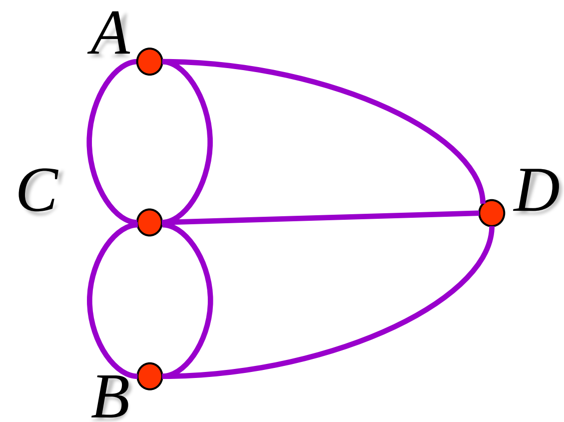
Euler and Hamilton Paths

Section 10.5

Königsberg Seven Bridge Problem

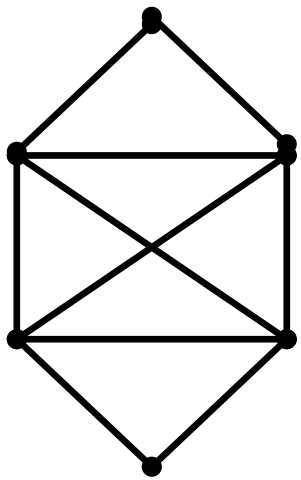




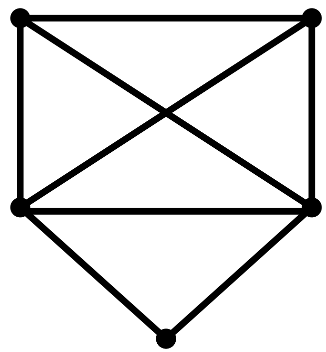
Is there a simple circuit in this multigraph that contains every edge?

Terminologies:

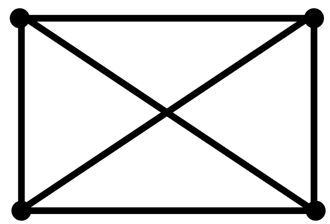
- **Euler Path:** a simple path containing every edge of G
- **Euler Circuit:** a simple circuit containing every edge of G
- **Euler Graph:** A graph contains an Euler circuit



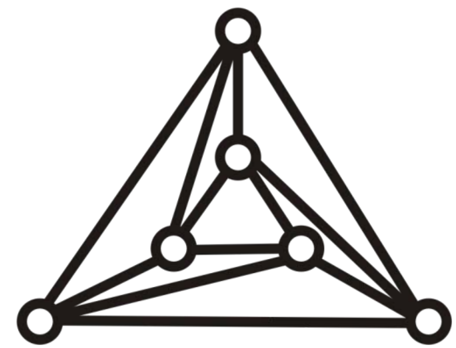
Euler graph



has Euler path



No Euler path or Euler circuit



Euler graph

Question: How to determine whether there is an Euler path or Euler circuit in a given graph?

Necessary and sufficient condition for Euler circuits and paths

【Theorem 1】 A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

Proof:

(1) Necessary condition

G has an Euler circuit $\Rightarrow$ Every vertex in V has even degree

Consider the Euler circuit starting and ending at vertex a

- **first edge of the Euler circuit contributes one to the degree of a**
- **Each time the circuit passes through a vertex it contributes two to the vertex's degree**
- **The circuit terminates where it started, contributing one to deg(a).**

(2) Sufficient condition

We will form a simple circuit that begins at an arbitrary vertex *a* of *G*.

- **Build a simple circuit $x_0=a, x_1, x_2,\ldots,x_n=a$.**
- **An Euler circuit has been constructed if all the edges have been used. otherwise,**
- **Construct a simple path in the subgraph *H* obtained from *G*.**

  Let *w* be a vertex which is the common vertex of the circuit and *H*.

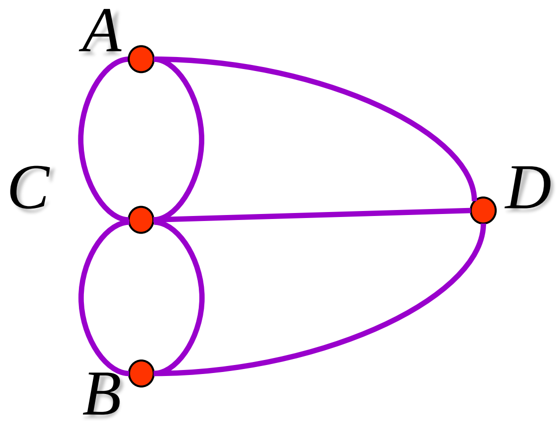
  Beginning at *w*, construct a simple path in *H*.
- **Form a circuit in G by splicing the circuit in H with the original circuit in G**
- **Continue this process until all edges have been used.**

【 Theorem 2】 A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Proof:

- **Suppose the multigraph has an Euler path from *a* to *b***
  - **The first edge of the Euler path contributes one to the degree of *a***
  - **The last edge in the Euler path contributes one to the degree of *b***
  - **The path contributes two to the degree of a vertex whenever it passes through it**
- **Suppose that a graph has exactly two vertices of odd degree, say *a* and *b*.**
  - **Add an edge {a, b}, then every vertex has even degree, so there is an Euler circuit.**
  - **The removal of the new edge produces an Euler path.**

〖Example 1〗 K*ö*nigsberg Seven Bridge Problem

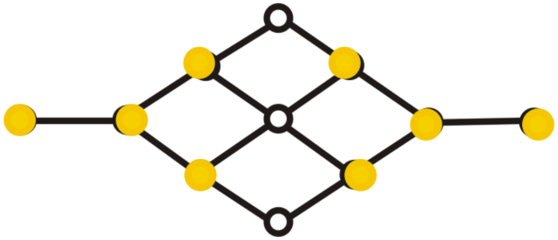


Solution:

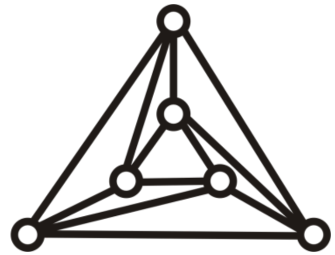
1. The graph has four vertices of odd degree. Therefore, it does not have an Euler circuit.

2. It does not have an Euler path either.

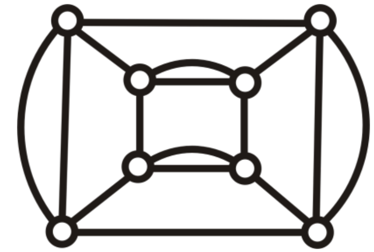
〖Example 2〗



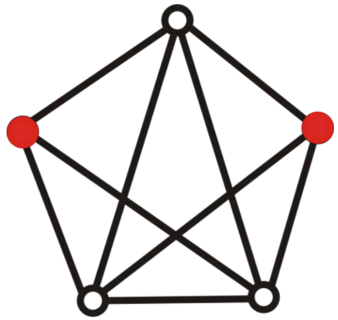
No Euler path



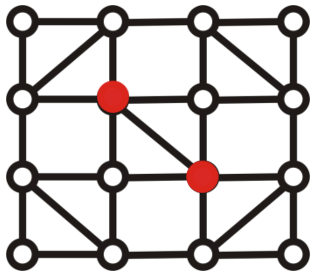
Euler graph



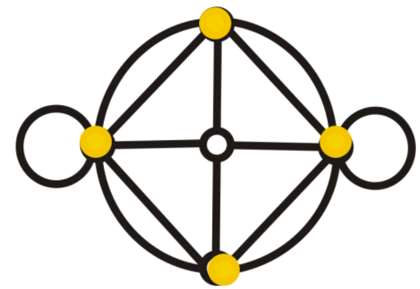
Euler graph



has Euler path, but
no Euler circuit



has Euler path, but
no Euler circuit



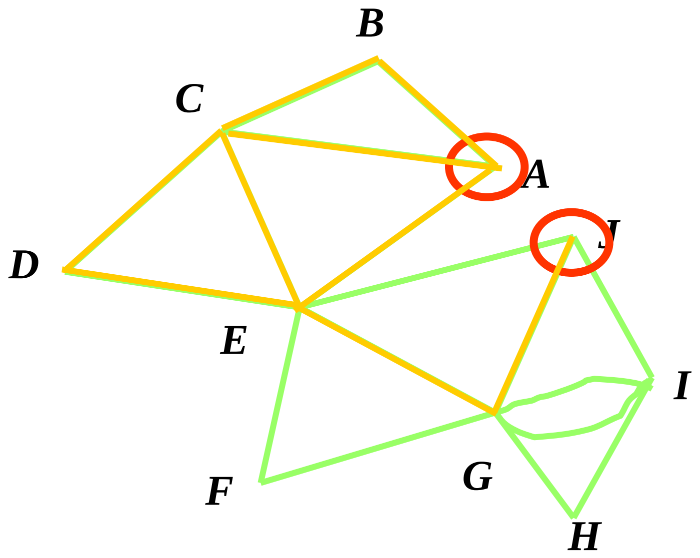
No Euler path

Question: How to find an Euler path or Euler circuit in a given graph?

ALGORITHM **Constructing Euler Circuit.**

procedure *Euler* (G: connected and all vertices of even degree)
circuit := a cirsuit in G beginning at an arbitrarily chosen
    with edges successively added to form a path that
    return to this vertex.
H := G with the edges of this circuit removed
while H has edges
begin
  subcircuit := a circuit in H beginning at a vertex in H that also
      is an endpoint of an edge of *circuit*
  H:=H with edges of *subcircuit* and all isolated vertices removed
  circuit:= *circuit* with *subcircuit* inserted at the appropriate vertex
end {circuit is an Euler circuit}

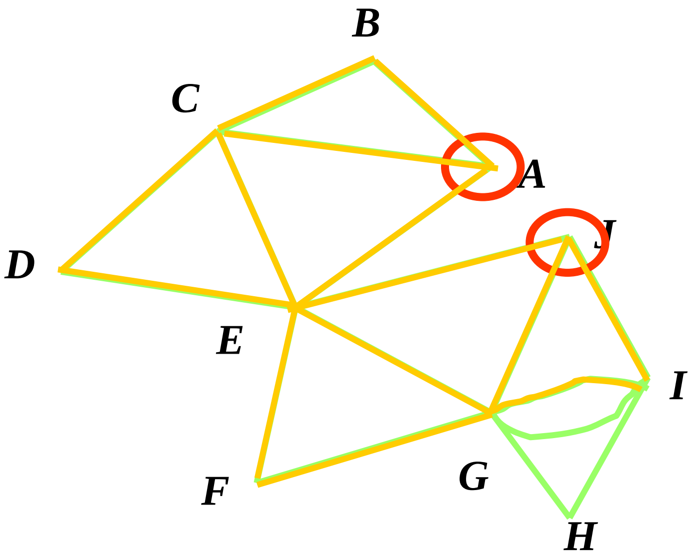
〖Example 3〗 Determine whether the following graph has an Euler path. Construct such a path if it exists.



Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

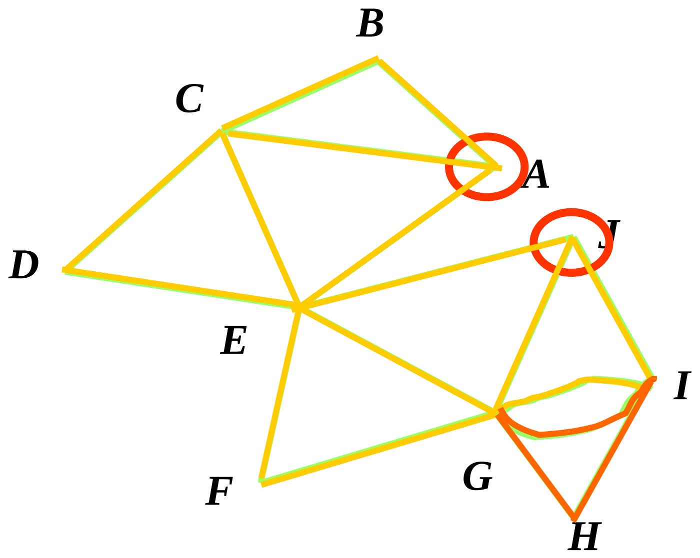
〖Example 3〗 **Determine whether the following graph has an Euler path. Construct such a path if it exists.**



Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

〖Example 3〗 Determine whether the following graph has an Euler path. Construct such a path if it exists.



The Euler path:

A,C,E,F,G,I,J,E,A,B, C,D,E,G, H,I,G,J

Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

Euler circuits and paths in directed graphs

P.704 16,17

A directed multigraph having no isolated vertices has an Euler circuit if and only if

-- the graph is weakly connected

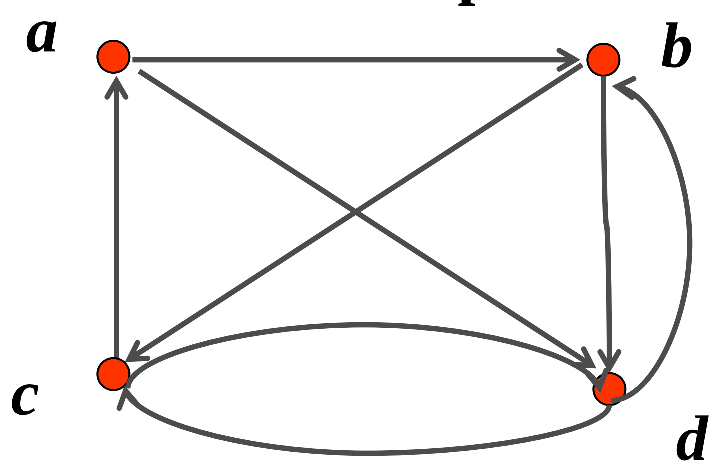
-- the in-degree and out-degree of each vertex are equal.

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if

-- the graph is weakly connected

-- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree.

〖Example 5〗 **Determine whether the directed graph has an Euler path. Construct an Euler path if it exists.**

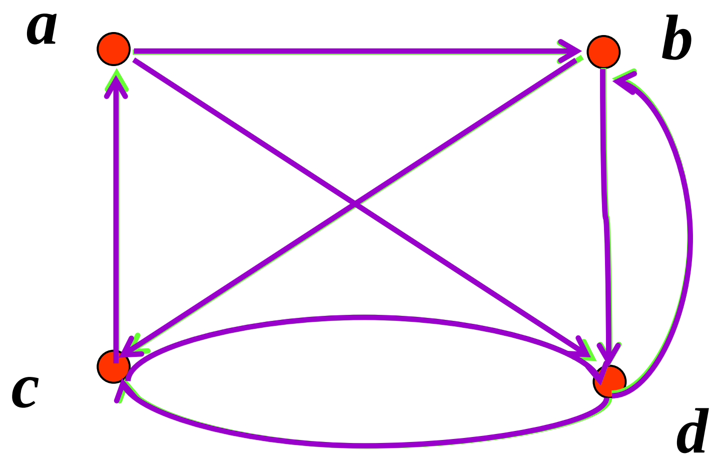


Solution:

	$\deg^-(v)$	$\deg^+(v)$
a	1	2
b	2	2
c	2	2
d	3	2

Hence, the directed graph has an Euler path.

〖Example 4〗 **Determine whether the directed graph has an Euler circuit. Construct an Euler circuit if it exists.**

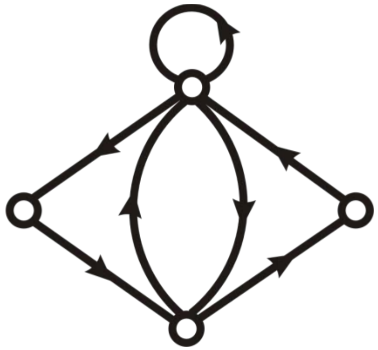


Solution:

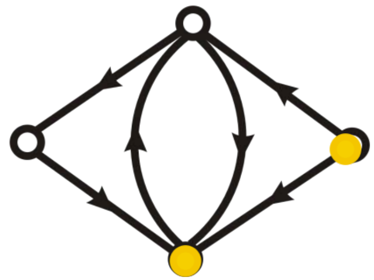
	$deg^{-}(v)$	$deg^{+}(v)$
a	1	2
b	2	2
c	2	2
d	3	2

Hence, the directed graph has an Euler path.

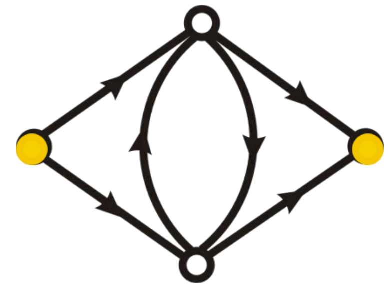
〖Example 5〗



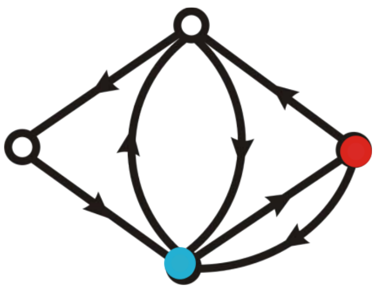
Euler graph



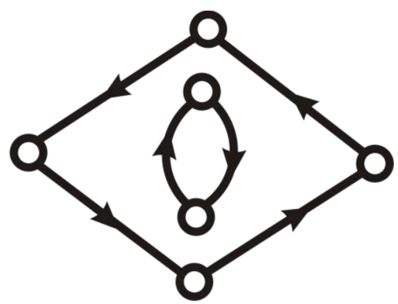
No Euler path



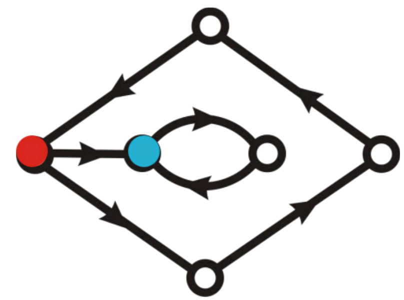
No Euler path



has Euler path, but no Euler circuit



No Euler path

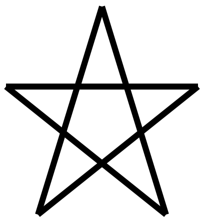
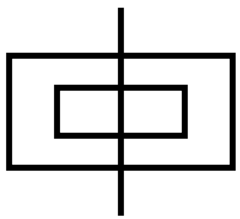
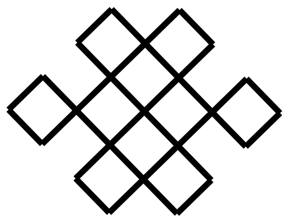
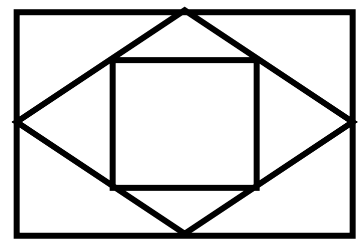
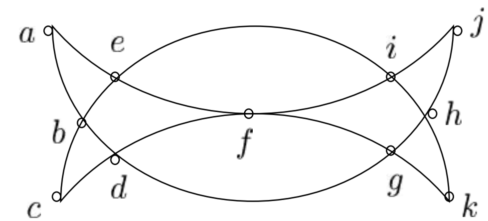


has Euler path, but no Euler circuit

Application of Euler circuits and paths

1. A type of puzzle

Draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

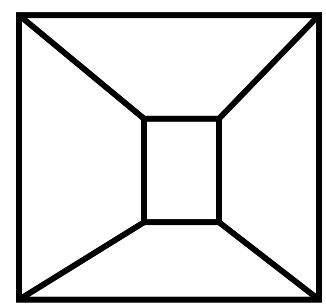






Mohammed's scimitars

✓ Now we can easily determine whether it is one-stroke drawing and how to draw.

Question:

How about the right graph?



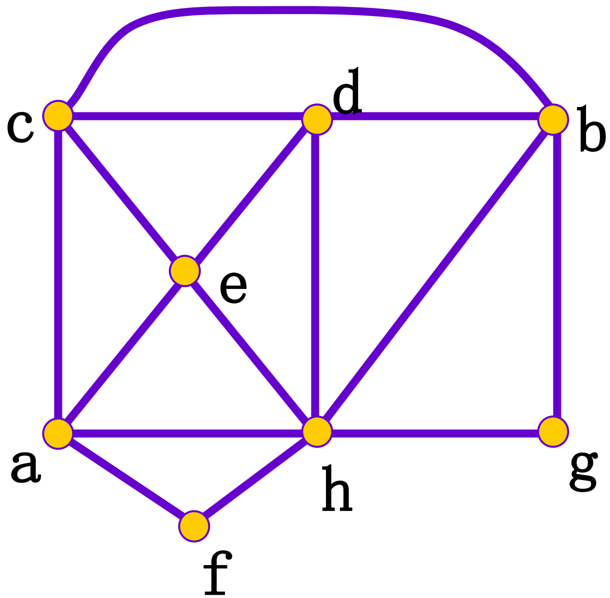
2. Route planning for the snow removal truck

The streets and intersections are represented by the following graph. The garage of the snow removal truck is located at vertex d. Show that there exists a route such that the snow removal truck can clean every street exactly once and return to the garage. Find this route.

Solution:

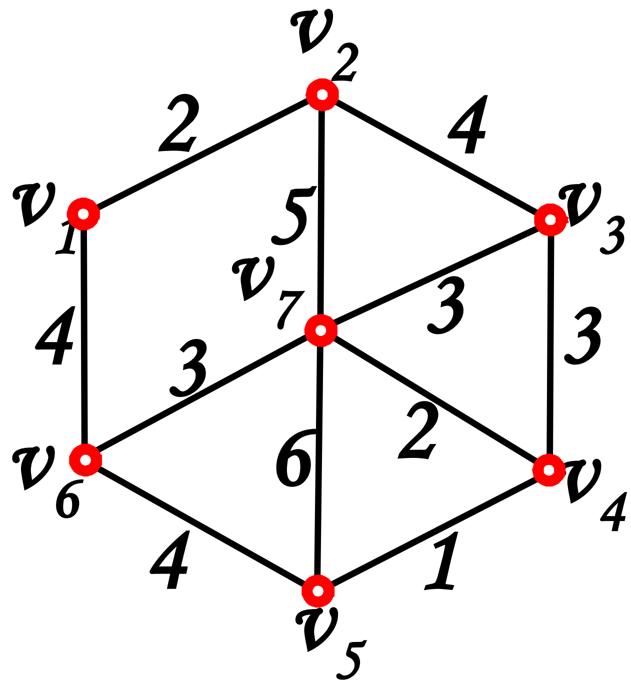
The graph is connected and each vertex has even degree, so there exists Euler circuits.

C=*dbghbceahfacdhed*



3) The Chinese postman problem

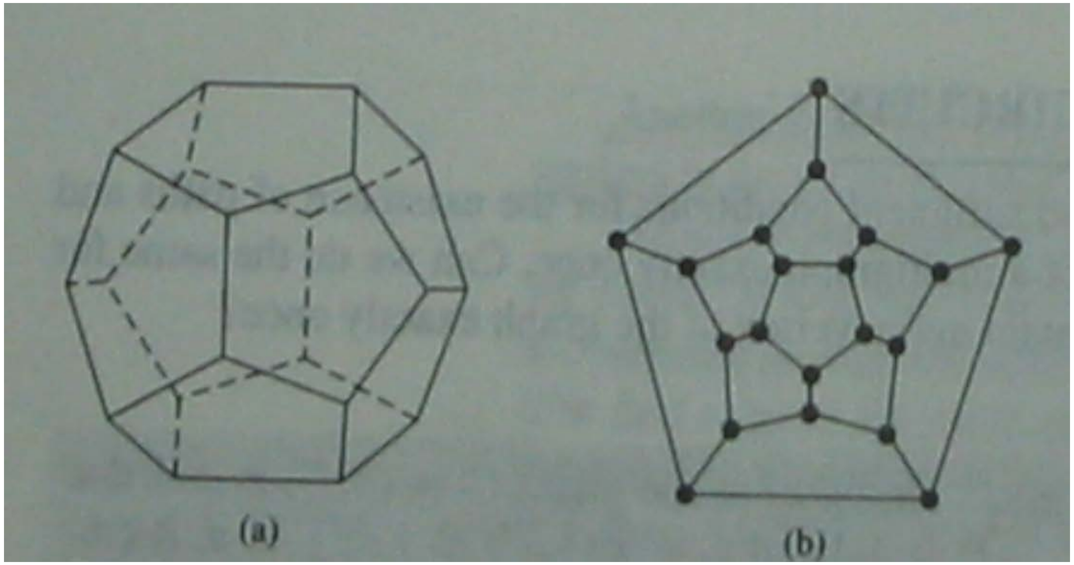
- The problem is named in honor of Guan Meigu, who posed it in 1962.



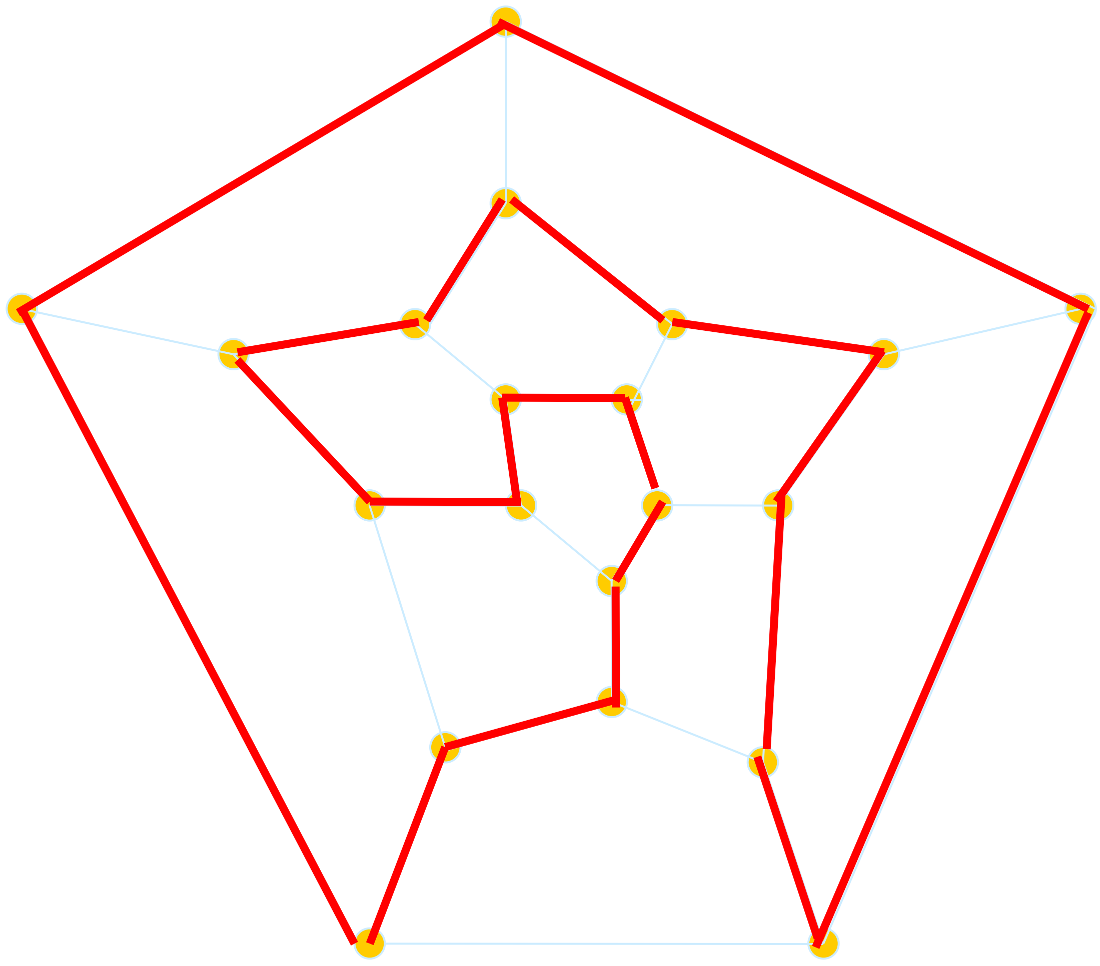
4) The other area, such as networking, molecular biology etc.

Hamilton's paths and Circuits

- Hamilton's puzzle (1856)



The object of the puzzle:
start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city.



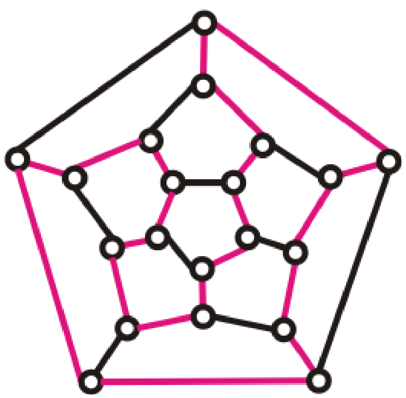
Terminologies:

- **Hamilton path:** a path which visits every vertex in *G* exactly once
- **Hamilton circuit** (or **Hamilton cycle**): a cycle which visits every vertex exactly once, **except for the first vertex**, which is also visited at the end of the cycle.
- **Hamilton graph:** a connected graph *G* has a Hamilton circuit

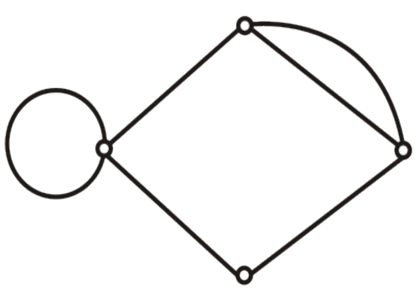
Note: The definition applies both to undirected as well as directed graphs of all types.

Question:

1) H path is a simple path?
2) H path and Euler path.

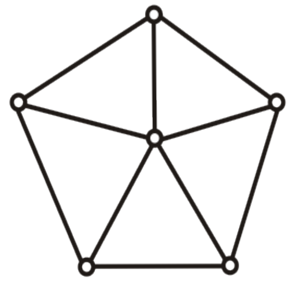


〖Example 6〗



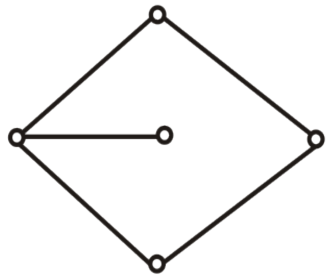
(1)

Hamilton graph



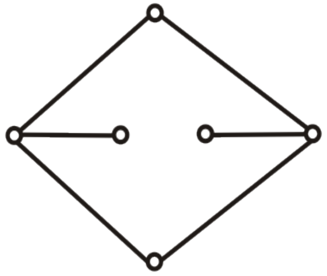
(2)

Hamilton graph



(3)

has H path, but no H circuit



(4)

No H path, no H circuit

Conditions for the existence of Hamilton path and Hamilton circuit

- No useful necessary and sufficient conditions for the existence of Hamilton circuit.
- A few sufficient conditions have been found.

【Theorem 3】 DIRAC'THEOREM

If G is a simple graph with n vertices with $n>=3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton path.

【Theorem 4】ORE'THEOREM

If G is a simple graph with n vertices with $n>=3$ such that $deg(u)+deg(v) >= n$ for every pair of nonadjacent vertices u and v in G , then G has a Hamilton circuit.

- Certain properties can be used to show that a graph has no Hamilton circuit.

  For instance, a graph with a vertex of degree one cannot have a Hamilton circuit.

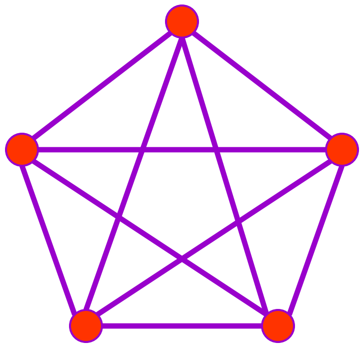
- If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit.

- When a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit , can be removed from consideration.

〖Example 6〗 Show that K_n has a Hamilton circuit whenever n>=3 ?

Proof:

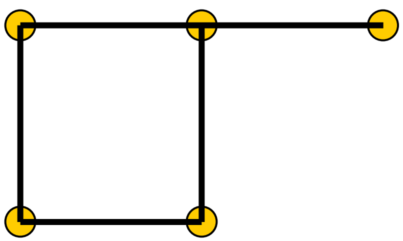
We can form a Hamilton circuit in K_n beginning at any vertex.



Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once.

This is possible since there are edges in K_n between any two vertices.

〖Example 7〗 **Determine whether there is a Hamilton circuit in the following graphs.**



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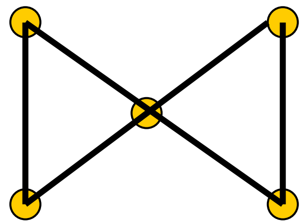

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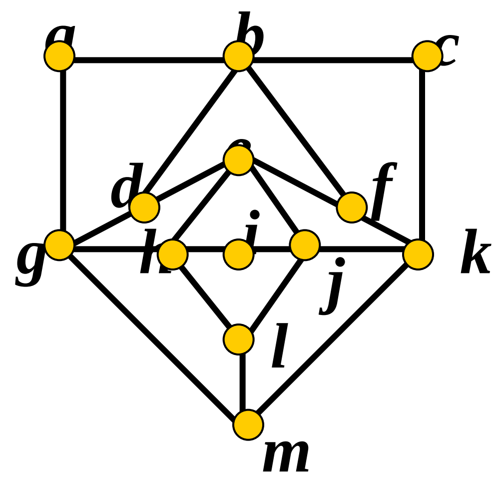

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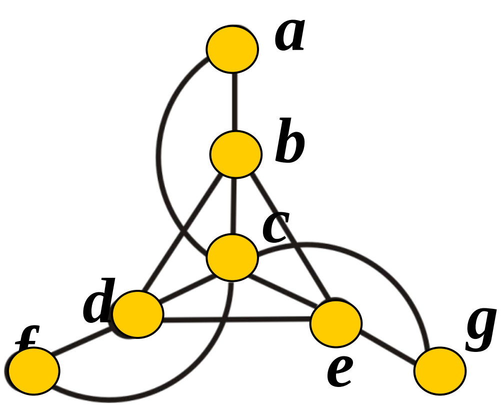


✗

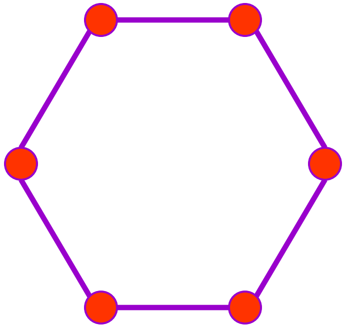
Another important necessary condition

- For any nonempty subset S of set V, the number of connected components in G-S <=|S|.

Note:

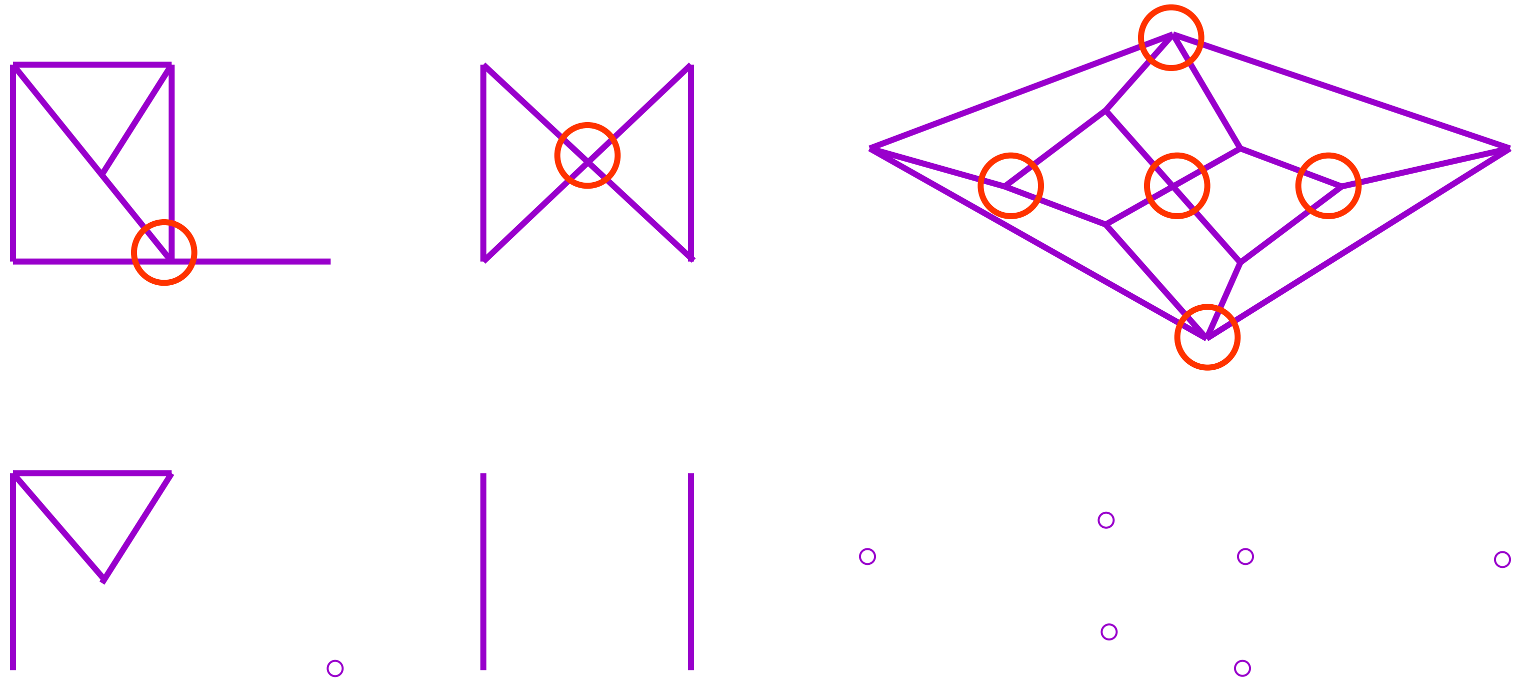
(1) ***G-S*** **is a subgraph of *G***

(2) Suppose that *C* is a H circuit of *G*. For any nonempty subset *S* of set *V*, the number of connected components in *C*-*S* <=|*S*|.



(3) the number of connected components in *G*-*S* <= the number of connected components in *C*-*S*

〖Example 8〗 Does the following graphs have a Hamilton circuit?



Applications of Hamilton circuit

Hamilton path or circuit can be used to solve many practical problems also.

For example,

1) Find a path or circuit that visits each road intersection in a city, or each node in a communication network exactly once.

2) The famous Traveling Salesperson Problem (TSP)

3) ……

〖Example 9〗 There are seven people denoted by A, B, C, D, E, F, G. Suppose that the following facts are known.

A--English (A can speak English.)

B--English, Chinese

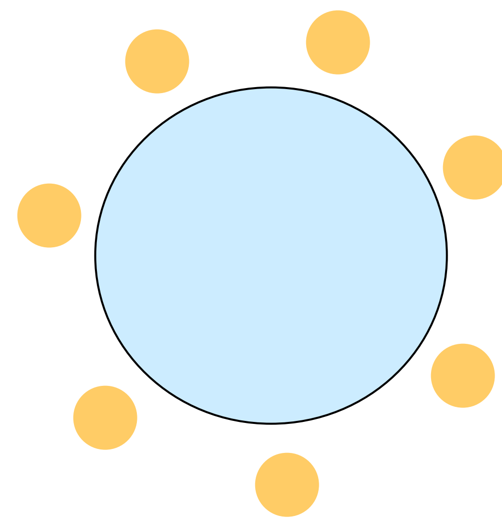
C--English, Italian, Russian

D--Japanese, Chinese

E--German, Italia

F--French, Japanese, Russian

G--French, German

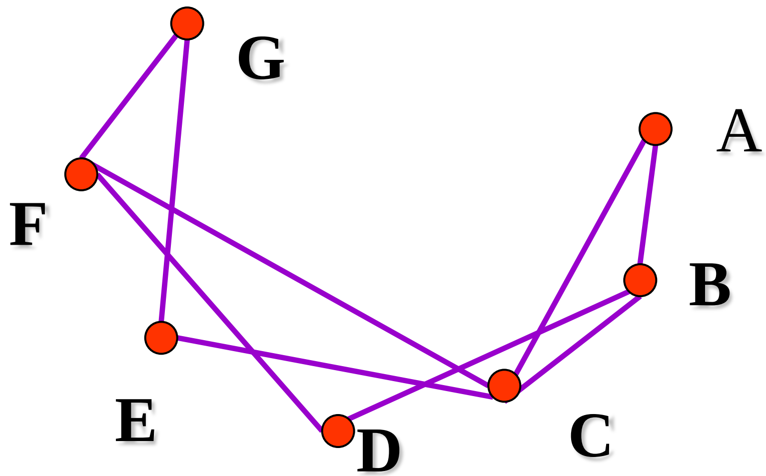


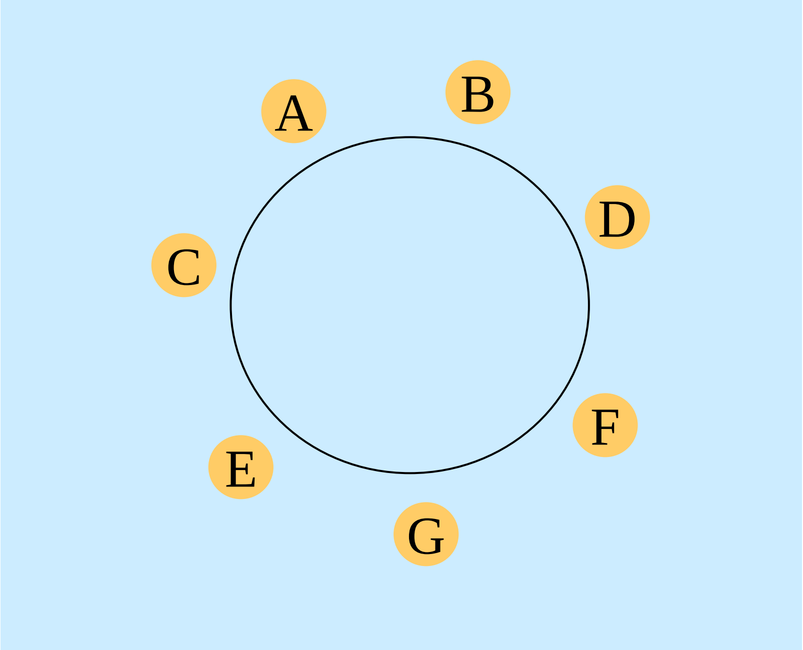
How to arrange seat for the round desk such that the seven people can talk each other?

Solution:

(1) Construct graph

***V*={A,B,C,D,E,F,G}, *E*={(u,v)|u,v can speak at least one common language.}**





(2) If there is a H circuit, then we can arrange seat for the round desk such that the seven people can talk each other.

H circuit: A,B,D,F,G,E,C,A

〖Example 10〗 **Seven examination must be arranged in a week. Each day has one examination. The examinations are in charged by the same teacher cannot be arranged in the adjacent two days. One teacher is in charge at most four examinations. Show that the arrangement is possible.**

Proof:

(1) Construct graph

***V*: seven examination, *E*={(u,v)|u,v are examinations are not in charged by the same teacher .}**

(2) If there is a H path, then the arrangement is possible.

Since one teacher is in charge at most four examinations, then every vertex in the graph has at least three adjacent vertices. It follows that the sum of the degrees of a pair vertices is at least 6.

Homework:

Seventh Edition:

P. 703 4, 6, 31, 34, 38, 41