# The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic

# Summary

- **Predicate Logic (First-Order Logic (FOL), Predicate Calculus)**
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
  - Translation from Predicate Logic to English
  - Translation from English to Predicate Logic

# Predicates and Quantifiers

Section 1.4

# Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic
- Logic Programming

# Propositional Logic Not Enough

- If we have:

  **"All men are mortal."**

  **"Socrates is a man."**

- Does it follow that "**Socrates is mortal?**"
- Can't be represented in propositional logic. **Need a language that talks about objects, their properties, and their relations.**

Later we'll see how to draw inferences.

# Introducing Predicate Logic

- **Predicate logic uses the following new features:**
  - **Variables:** $x, y, z$
  - **Predicates:** $P, M$
  - **Quantifiers** (*to be covered in a few slides*)
- **Propositional functions are a generalization of propositions.**
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

# Propositional Functions

- **Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain or bound by a quantifier (as we will see later).**
- **The statement *P(x)* is said to be the value of the propositional function *P* at *x*.**
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  P(-3) is false.

  P(0) is false.

  P(3) is true.
- Often the domain is denoted by $U$. So in this example $U$ is the integers.

# Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and $U$ (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  **Solution: F**

  R(3,4,7)

  **Solution: T**

  R($x$, 3, $z$)

  **Solution: Not a Proposition**

- In general, a statement involving the $n$ variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function $P$ at the $n$-tuple $(x_1, x_2, \ldots, x_n)$ and $P$ is called a $n$-ary predicate.

# Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(-1) \rightarrow P(3)$ **Solution**: T
- **Expressions with variables are not propositions** and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

# Quantifiers

- We need ***quantifiers*** to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- **The two most important quantifiers are:**

  **Universal Quantifier, "For all," symbol: $\forall$**

  **Existential Quantifier, "There exists," symbol: $\exists$**
- We write as in $\forall x\, P(x)$ (*the universal quantification of P(x))*and $\exists x\, P(x)$.

  **$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> $x$ in the *domain*.**

  **$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> $x$ in the *domain*.**
- The quantifiers are said to bind the variable $x$ in these expressions.

# Universal Quantifier

- $\forall x\ P(x)$ is read as "For all $x$, P($x$)" or "For every $x$, P($x$)"

**Examples**:

1) If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\forall x\ P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and $U$ is the positive integers, then $\forall x\ P(x)$ is true.
3) If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\forall x\ P(x)$ is false.

*Remark:*

The universal quantification of $P(x)$ create a proposition from a propositional function.

# Existential Quantifier

- **$\exists x\, P(x)$ is read as "For some $x$, P($x$)", or as "There is an $x$ such that P($x$)," or "For at least one $x$, P($x$)."**

**Examples**:

1. If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and $U$ is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\exists x\, P(x)$ is true.

# Thinking about Quantifiers

- **When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.**
- To evaluate $\forall x\, P(x)$ loop through all $x$ in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all $x$ in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an $x$ for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

# Properties of Quantifiers

- **The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain $U$.**

| Statement | When true? | When false? |
|---|---|---|
| $\forall x\, P(x)$ | $P(x)$ is true for every $x$. | There is an $x$ for which $P(x)$ is false. |
| $\exists x\, P(x)$ | There is an $x$ for which $P(x)$ is true. | $P(x)$ is false for every $x$. |

# Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If $U$ consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

# Uniqueness Quantifier

- **$\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> $x$ in the universe of discourse.**
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique $x$ such that $P(x)$."
  - "There is one and only one $x$ such that $P(x)$"
- Examples:
  1. If *P(x)* denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if *P(x)* denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique $x$ such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \wedge \forall y\, (P(y) \rightarrow y = x))$$

# Precedence of Quantifiers

- **The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.**

For example,

- $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

# Translating from English to Logic

**Example 1**: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

**Solution**:

First decide on the domain *U*.

**Solution 1**: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

**Solution 2**: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

# Translating from English to Logic

**Example 2**: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

**Solution**:

First decide on the domain $U$.

**Solution 1**: If $U$ is all students in this class, translate as

$$\exists x\, J(x)$$

**Solution 1**: But if $U$ is all people, then translate as

$\exists x\, (S(x) \wedge J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

# Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal."
- The two premises are:

$$\forall x Man(x) \rightarrow Mortal(x)$$

$$Man(Socrates)$$

- The conclusion is:

$$Mortal(Socrates)$$

Later we will show how to prove that the conclusion follows from the premises.

# Equivalences in Predicate Logic

- **Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter**
  - ✓ which predicates are substituted into these statements and
  - ✓ which domain of discourse is used for the variables in these propositional functions
- The notation $S \equiv T$ indicates that $S$ and $T$ are logically equivalent.
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

# Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in calculus" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken calculus."

  Symbolically $\neg\forall x\, J(x)$ **and** $\exists x\, \neg J(x)$ **are equivalent**

# Negating Quantified Expressions (cont)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x J(x)$ **and** $\forall x \neg J(x)$ **are equivalent**

# De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

**TABLE 2** De Morgan's Laws for Quantifiers.

| *Negation* | *Equivalent Statement* | *When Is Negation True?* | *When False?* |
|---|---|---|---|
| $\neg\exists x P(x)$ | $\forall x \neg P(x)$ | For every $x$, $P(x)$ is false. | There is an $x$ for which $P(x)$ is true. |
| $\neg\forall x P(x)$ | $\exists x \neg P(x)$ | There is an $x$ for which $P(x)$ is false. | $P(x)$ is true for every $x$. |

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

# More Logical Equivalences

$$\forall x(A(x) \wedge B(x)) \equiv \forall x A(x) \wedge \forall x B(x)$$
$$\exists x(A(x) \vee B(x)) \equiv \exists x A(x) \vee \exists x B(x)$$

Note:

$$\forall x(A(x) \vee B(x)) \not\Leftrightarrow \forall x A(x) \vee \forall x B(x)$$
$$\exists x(A(x) \wedge B(x)) \not\Leftrightarrow \exists x A(x) \wedge \exists x B(x)$$

$$\exists x(A(x) \wedge B(x)) \Rightarrow \exists x A(x) \wedge \exists x B(x)$$
$$\forall x A(x) \vee \forall x B(x) \Rightarrow \forall x(A(x) \vee B(x))$$

# More Logical Equivalences

$x$ is not occurring in $P$.

(1) $\forall x A(x) \vee P \equiv \forall x (A(x) \vee P)$

(2) $\forall x A(x) \wedge P \equiv \forall x (A(x) \wedge P)$

(3) $\exists x A(x) \vee P \equiv \exists x (A(x) \vee P)$

(4) $\exists x A(x) \wedge P \equiv \exists x (A(x) \wedge P)$

# More Logical Equivalences

$x$ is not occurring in $P$ and $B$.

(1) $\forall x A(x) \vee P \equiv \forall x (A(x) \vee P)$

(2) $\forall x A(x) \wedge P \equiv \forall x (A(x) \wedge P)$

(3) $\exists x A(x) \vee P \equiv \exists x (A(x) \vee P)$

(4) $\exists x A(x) \wedge P \equiv \exists x (A(x) \wedge P)$

(5) $\forall x (B \to A(x)) \equiv B \to \forall x A(x)$

*Proof:*

$$\forall x (B \to A(x)) \equiv \forall x (\neg B \vee A(x))$$
$$\equiv \neg B \vee \forall x A(x)$$
$$\equiv B \to \forall x A(x)$$

# Translation from English to Logic

**Examples**:

① "Some student in this class has visited Mexico."

**Solution**:

Let $M(x)$ denote "$x$ has visited Mexico" and $S(x)$ denote "$x$ is a student in this class," and $U$ be all people.

$$\exists x \ (S(x) \land M(x))$$

② "Every student in this class has visited Canada or Mexico."

**Solution**: Add $C(x)$ denoting "$x$ has visited Canada."

$$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

# Some Fun with Translating from English into Logical Expressions

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

**Translate "Everything is a lion"**

*Solution:* $\forall x\, L(x)$

# Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

“**Nothing is a mammal.**”

*Solution:* $\neg\exists x\, M(x)$

What is this equivalent to?

*Solution:* $\forall x\, \neg M(x)$

# Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

**"All lions are mammals."**

*Solution:* $\forall x\, (L(x) \rightarrow M(x))$

# Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

**"Some mammals are carnivorous animals."**

*Solution:* $\exists x\, (M(x) \land C(x))$

# Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

**"No mammal is a carnivorous animal."**

*Solution:* $\neg\exists x\,(M(x) \land C(x))$

What is this equivalent to?

*Solution:* $\forall x\,(\neg M(x) \lor \neg C(x))$

$\equiv \forall x\,(M(x) \rightarrow \neg C(x))$

# Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: $x$ is a lion

  $M(x)$: $x$ is a mammal

  $C(x)$: x is a carnivorous animal

**"If any lion is a mammal then it is also a carnivorous animal."**

*Solution:* $\forall x\ ((L(x) \land M(x)) \rightarrow C(x))$

# System Specification Example

- Predicate logic can be used for representing system specification also.
- For example, translate into predicate logic:
  - "Every mail message larger than one megabyte will be compressed."
  - "If a user is active, at least one network link will be available."
- Decide on predicates and domains (left implicit here) for the variables:
  - Let $L(m, y)$ be "Mail message $m$ is larger than $y$ megabytes."
  - Let $C(m)$ denote "Mail message $m$ will be compressed."
  - Let $A(u)$ represent "User $u$ is active."
  - Let $S(n, x)$ represent "Network link $n$ is state $x$.
- Now we have:

$$\forall m(L(m,1) \to C(m))$$

$$\exists u\, A(u) \to \exists n\, S(n, available)$$

$$\boldsymbol{\forall u\,(A(u) \to \exists n\, S(n, available))?}$$

# Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

An argument

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

The first two are called *premises* and the third is called the *conclusion*.

One way to translate these statements to predicate logic:

Let $p(x)$, $q(x)$, and $r(x)$ be the propositional functions "$x$ is a lion," "$x$ is fierce," and "$x$ drinks coffee," respectively. Domain of $x$: All creatures.

1. $\forall x\, (p(x) \rightarrow q(x))$
2. $\exists x\, (p(x) \wedge \neg r(x))$
3. $\exists x\, (q(x) \wedge \neg r(x))$

Later we will see how to prove that the conclusion follows from the premises.

**Homework:**

**Seventh Edition:**

**P.53 16, 24, 34, 51, 62**