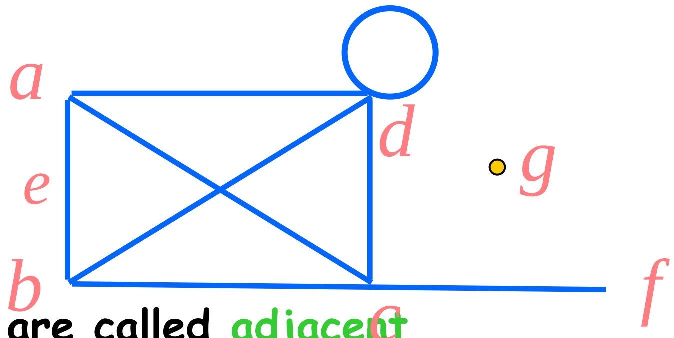
Graph Terminology and Special Types of Graphs

Section 10.2

Basic Terminology

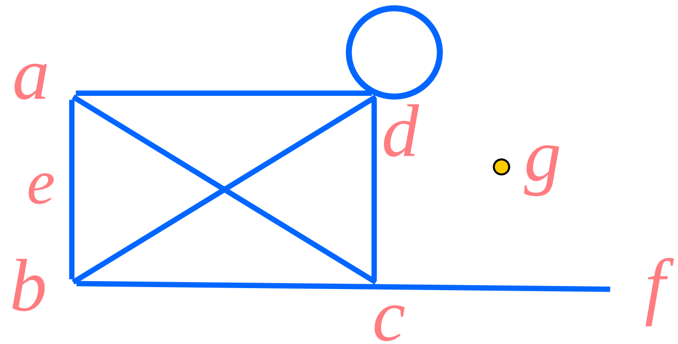
Undirected Graphs $G=(V, E)$



- Two vertices, u and v in an undirected graph G are called adjacent (or neighbors) in G, if {u, v} is an edge of G.
- An edge e connecting u and v is called incident with vertices u and v, or is said to connect u and v.
- The vertices u and v are called endpoints of edge {u, v}.
- Loop: an edge connects a vertex to itself.
- The neighborhood of v (N(v)): the set of all neighbors of a vertex v
- The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex

  Notation: deg(v)

  - If deg(v) = 0, v is called isolated.
  - If deg(v) = 1, v is called pendant.



Find the degree of all the vertices.

deg(a) =3 deg(6) = 3 deg(c) = 4 deg(d) = 5

deg(f) = 1 deg(g) = 0

TOTAL of degrees = 3 + 3 + 4 + 5 + 1 + 0 = 16

TOTAL NUMBER OF EDGES = 8

【Theorem 1】 The Handshaking Theorem

Let G = (V, E) be an undirected graph G with e edges. Then

$$\sum_{v \in V} \deg(v) = 2e$$

The sum of the degrees over all the vertices equals twice the number of edges.

Proof:

Each edge represents contributes twice to the degree count of all vertices.

Note:

This applies even if multiple edges and loops are present.

Questions:

1. The sum, over the set of people at a party, of the number of people a person has shaken hands with, is even?
2. How many edges are there in a graph with 10 vertices each of degree 6?
3. If a graph has 5 vertices, can each vertex have degree 3? 4?
   - The sum is $3 \cdot 5 = 15$ which is an odd number.

     Not possible.
   - The sum is $20 = 2\,|E|$ and $20/2 = 10$.

     May be possible.

【Theorem 2】 **An undirected graph has an even number of vertices of odd degree.**

Proof:

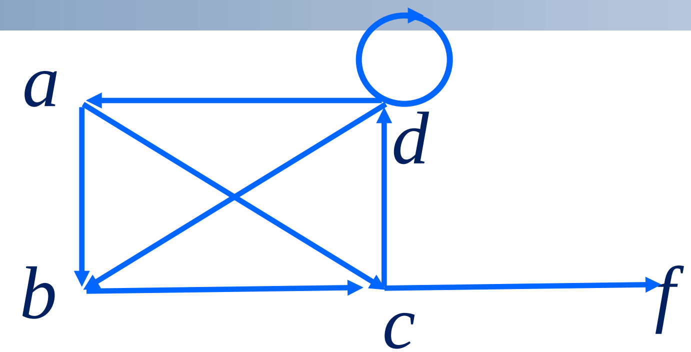
Let V_1, V_2 **be the set of vertices of even degree and the set of vertices of odd degree, respectively.**

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2m$$

Questions:

1. Is it possible to have a graph with 3 vertices each of which has degree 3?
2. Is it possible that a graph has a sequence of degrees (3,3,2,3) or (5,2,3,1,4)?
3. Show that among 9 factories,
   - It is impossible that each factory has business relation only with other three factories.
   - It is impossible that only four factories have business relation with factories with even number.
4. G is an nonempty simple graph, then there must exist vertices with same degrees.

Directed Graphs $G=(V, E)$



Let (u, v) be an edge in G. Then u is an **initial vertex** and is **adjacent to** v and v is a **terminal vertex** and is **adjacent from** u.

The **in degree** of a vertex v, denoted $deg^-(v)$ is the number of edges which terminate at v.

Similarly, the **out degree** of v, denoted $deg^+(v)$, is the number of edges which initiate at v.

underlying undirected graph

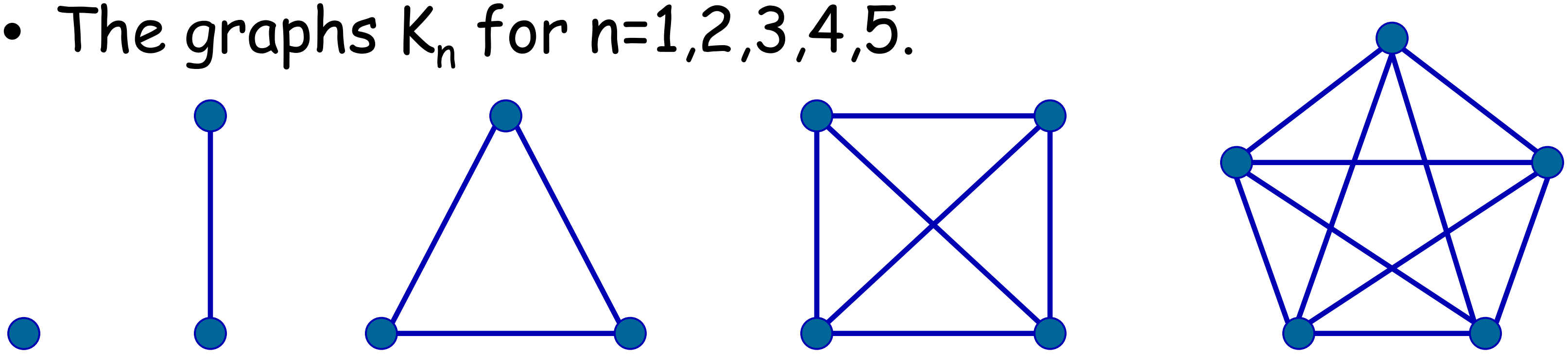
【Theorem 3】 Let G = (V, E) be a graph with direct edges. Then

$$\sum_{v\in V} d^+(v) = \sum_{v\in V} d^-(v) \quad = |E|$$

Some Special Simple Graphs

(1) Complete Graphs - K_n: the simple graph with
- n vertices
- exactly one edge between every pair of distinct vertices.

- The graphs K_n for n=1,2,3,4,5.



K_1 K_2 K_3 K_4 K_5

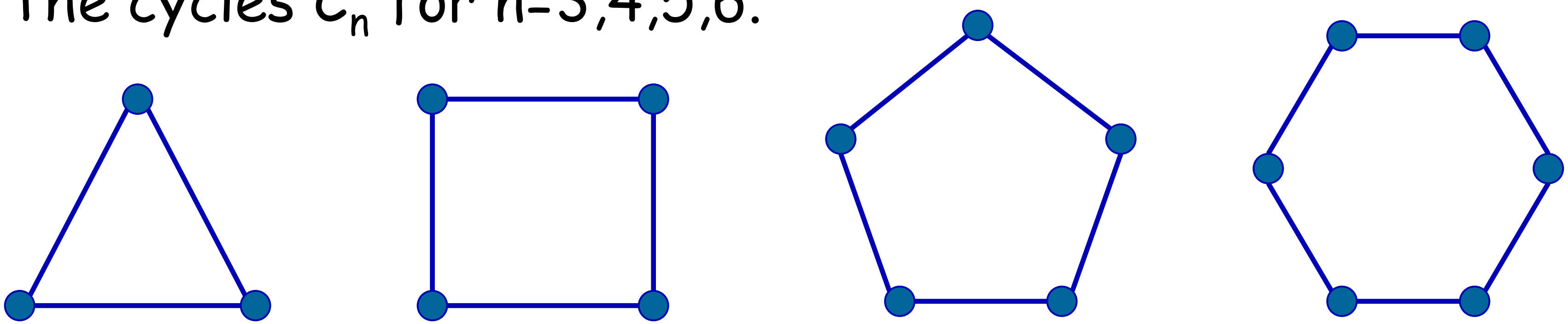
Qusetion: The number of edges in K_n ?

(2) Cycles C_n (n>2)

$$C_n=(V,E), \text{ where } V=\{v_1,v_2,\ldots,v_n\},$$

$$E=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n),(v_n,v_1)\},\ n \geq 3$$

- The cycles C_n for n=3,4,5,6.

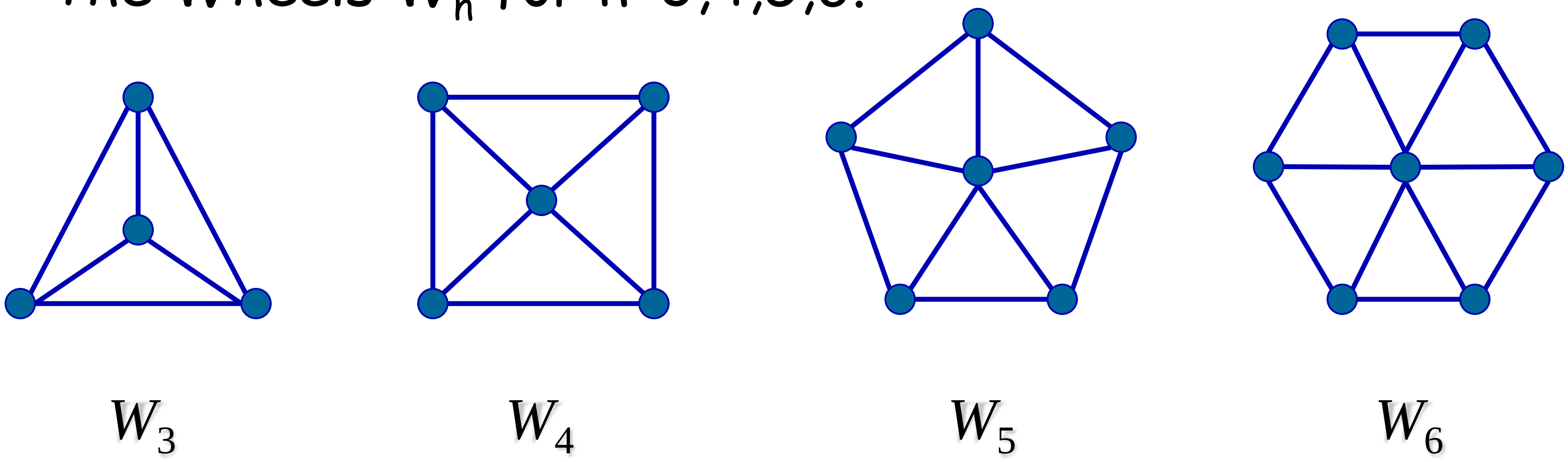


C_3 C_4 C_5 C_6

(3) Wheels W_n (n>2)

Add one additional vertex to the cycle C_n and add an edge from each vertex to the new vertex to produce W_n.

- The Wheels W_n for n=3,4,5,6.



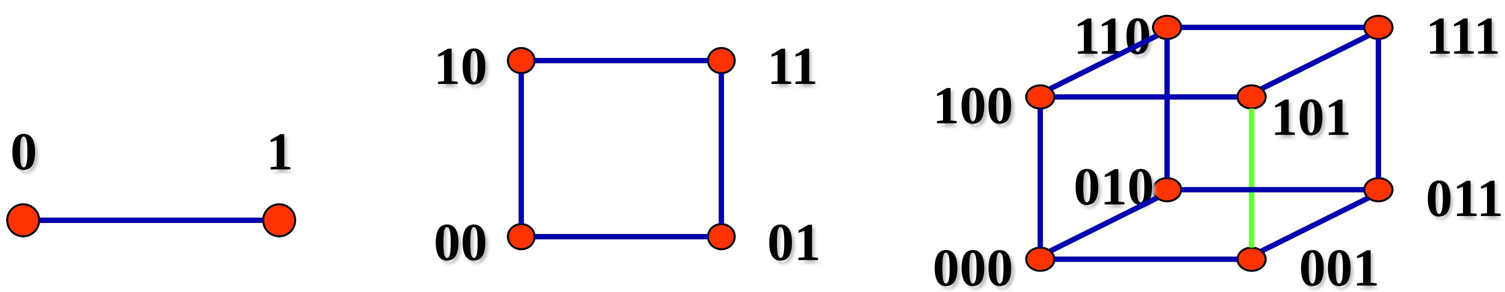
(4) n-Cubes Q_n (n>0)

Q_n =<V,E> is the graph with 2^n vertices representing bit strings of length n, where

$V = \{ v \mid v=a_1a_2\ldots a_n,\ a_i=0,1,\ i=1,2,\ldots,n\}$

$E=\{(u,v) \mid u,v \in V \wedge u$ and v differ by one bit position$\}$.

- The n-Cubes Q_n for n=1,2,3

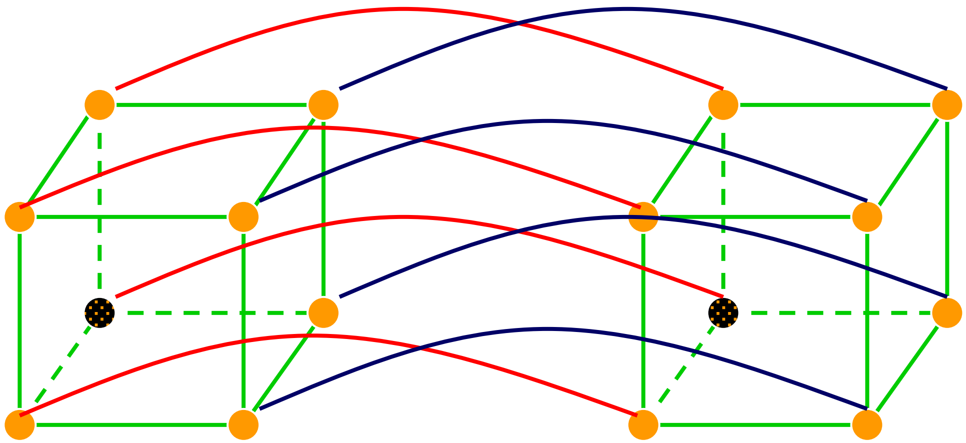


Q_1 Q_2 Q_3

Q_4 ?

Construct Q_{n+1} from Q_n :

- **making two copies of Q_n , prefacing the labels on the vertices with a 0 in one copy and with a 1 in the other copy**
- **adding edges connecting two vertices that have labels differing only in the first bit**

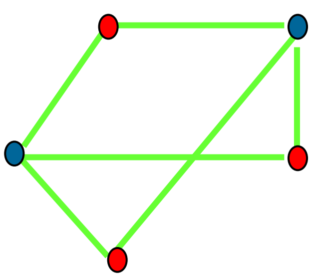


Q_4

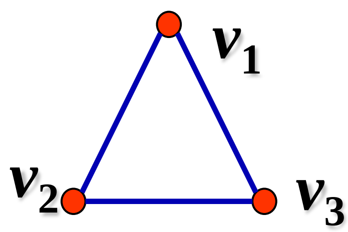
Bipartite Graphs

- **A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.**
- the pair $\{V_{1,} V_2\}$ is called a bipartition of the vertex V of G.

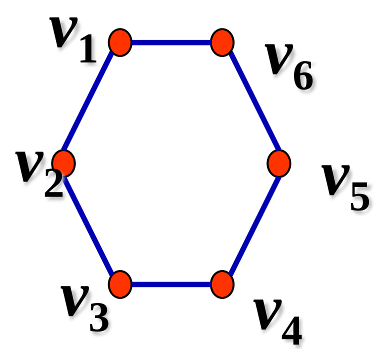
For example,



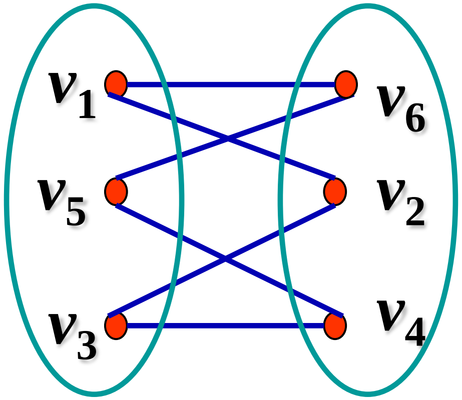
bipartite



C_3 is not bipartite

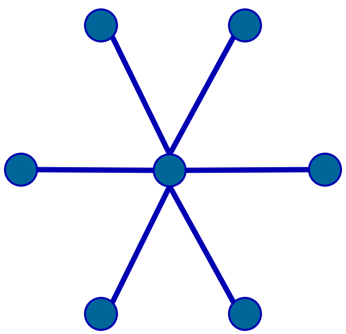


C_6 is bipartite

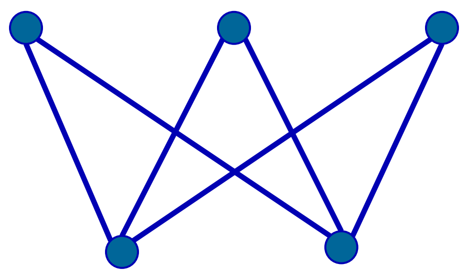


Bipartite Graphs

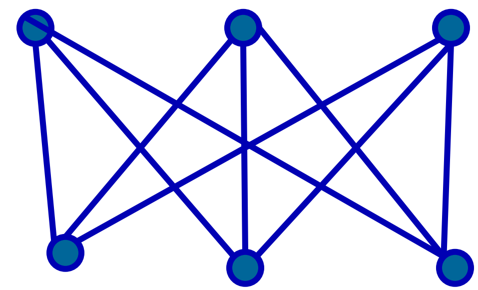
- The **complete bipartite graph** is the simple graph that has its vertex set partitioned into two subsets V_1 and V_2 with m and n vertices, respectively, and **every vertex** in V_1 is connected to **every vertex** in V_2, denoted by $K_{m,n}$, where $m = | V_1 |$ and $n = | V_2 |$.



$K_{1,n}$



$K_{3,2}$



$K_{3,3}$

【Theorem 4】 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof:

(1) **Suppose that $G=(V, E)$ is a bipartite simple graph. Then $V=V_1 \cup V_2$, where V_1, V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.**

(2) **Suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color.**

Regular graph

- A simply graph is called **regular** if every vertex of this graph has the same degree.
- A **regular graph** is called **n-regular** if every vertex in this graph has degree n.

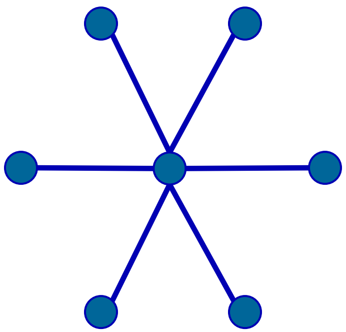
For example,

(1) k_n **is a (n-1)-regular.**

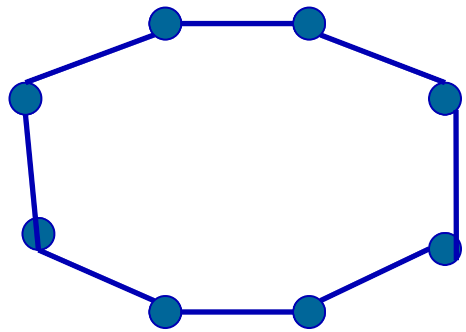
(2) For which values of m and n is $K_{m,n}$ regular?

Some applications of special types of graphs

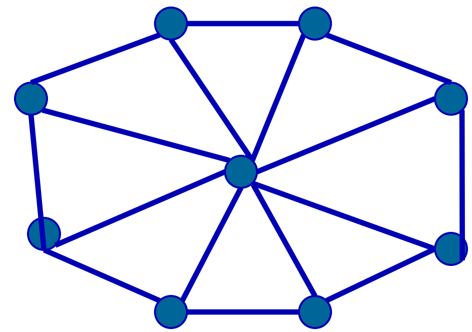
Local Area Networks.



Star topology



Ring topology



Hybrid topology

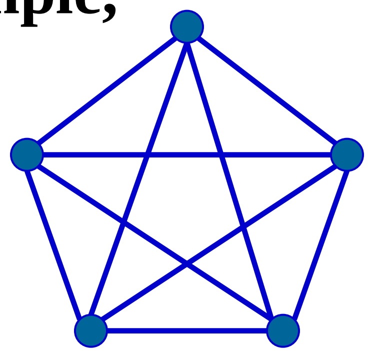
New Graphs From Old

◆Subgraph

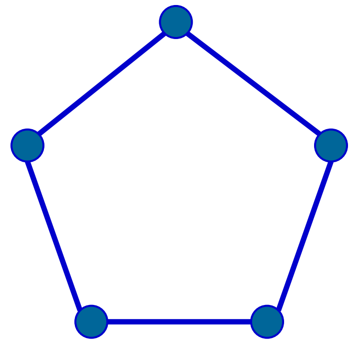
G= (V, E) , H= (W, F)

- ***H*** **is a** ***subgraph*** **of** ***G*** **if** $W \subseteq V, F \subseteq E$ **.**
- **subgraph** ***H*** **is a** ***proper subgraph*** **of** ***G if H≠G*****.**
- ***H*** **is a** ***spanning subgraph*** **of** ***G*** **if** $W = V, F \subseteq E$ **.**

For example,



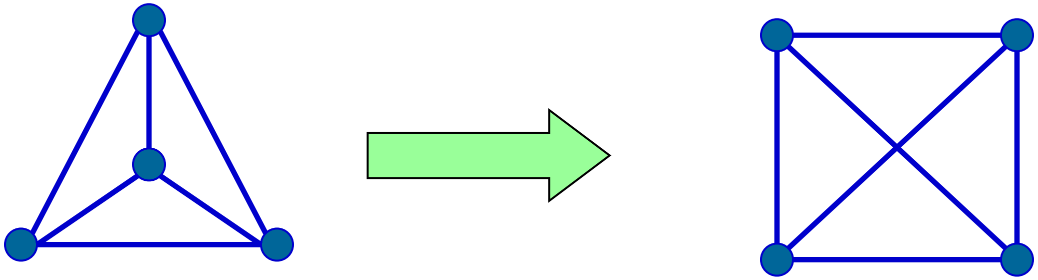
K_5



C_5 is subgraph of K_5

Question:

How many subgraphs with at least one vertex does W_3 have?

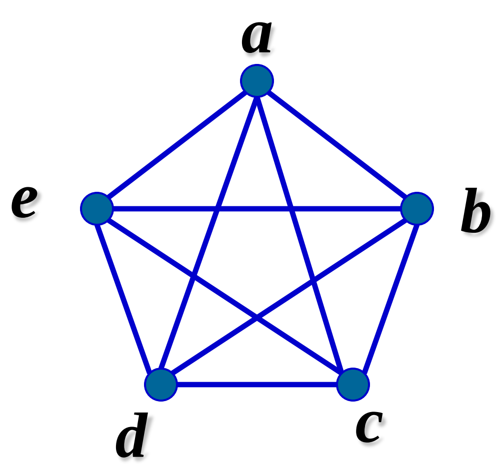


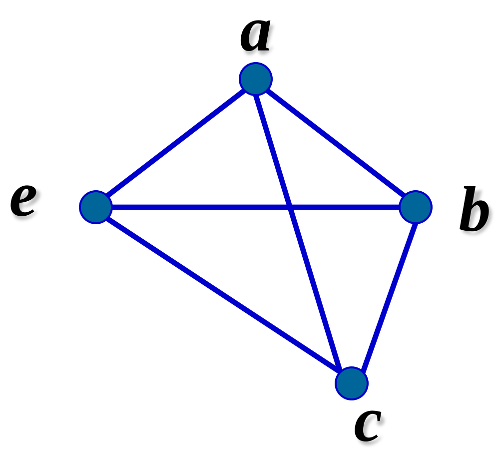
$$C(4,1)+C(4,2)\times 2+C(4,3)\times 2^3+C(4,4)\times 2^6$$

◆ Subgraph induced by a subset of V

Let G=(V,E) be a simple graph. The subgraph induced by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E iff both endpoints of this edge are in W.

For example,





- Removing edges of a graph

  $G-e = (V, E-\{e\})$

- Adding edges to a graph

  $G+e = (V, E\cup\{e\})$

- Edge contration

  Remove an edge e with endpoints u and v, merge u and v into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u and v and with the same second endpoint.

- Removing vertices from a graph

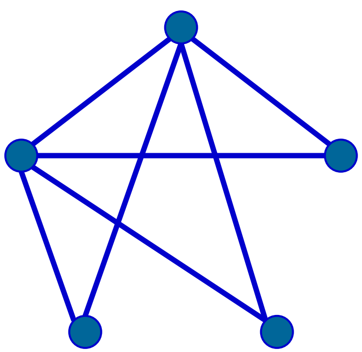
  $G-v = (V-v, E')$, where E' is the set of edges of G not incident to v

◆ Graph Union

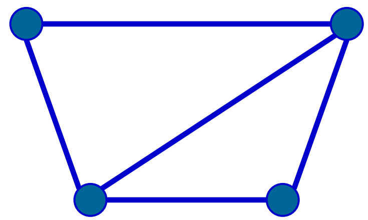
The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Notation: $G_1 \cup G_2$

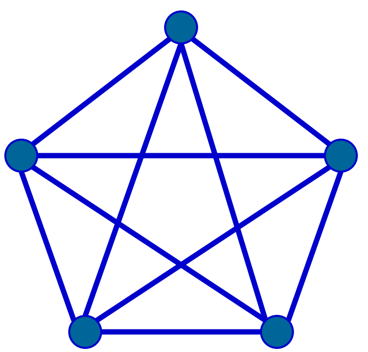
For example,



G_1



G_2



$G_1 \cup G_2 = K_5$

Homework:

Seventh Edition:

P. 665 5,21-25,41,53,60