Cardinality of Sets

Section 2.5

Introduction

The cardinality of a finite set

- two sets have the same size or when one is bigger than the other

Extend to infinite set

- A way to measure the relative sizes of infinite sets
- Countable infinite sets and uncountable infinite sets

Applications

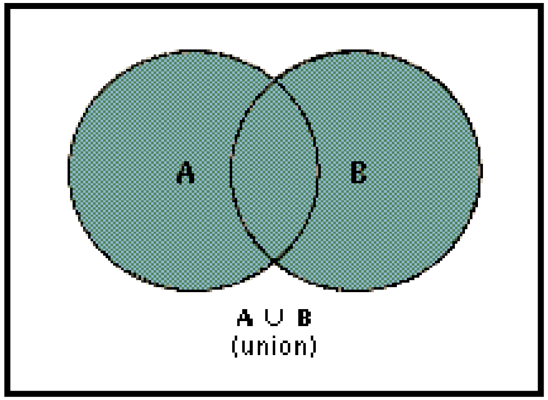
- Solve some interesting problems
- Computer science: explain why uncomputable functions exist

Review the cardinality of a finite set

- **The cardinality of a finite set was defined to be the number of distinct elements in the set.**
  - This definition tell us when two sets have the same size or when one is bigger than the other
- **The cardinality of the union of two finite sets:**
  - The principle of Inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The union of more finite sets?



For the union of three finite sets:

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$$

Proof:

Let $Q \cup R = S$

$$|P \cup S| = |P| + |S| - |P \cap S|$$

$$= |P| + |Q \cup R| - |P \cap (Q \cup R)|$$

$$= |P| + |Q \cup R| - |(P \cap Q) \cup (P \cap R)|$$

For the union of *n* finite sets:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Extend to infinite sets

Some questions:

1) How to determine the cardinality of an infinite set?
2) Is the number of positive integers double the number of positive even integers?
3) Do the set of rational numbers and the set of real numbers in (0,1) have same cardinality?
4) Do the set of real numbers in (a, b) and the set of real numbers in (0,1) have same cardinality?
5) Are all of the cardinalities of infinite sets same?

Cardinality

Definition: **The sets A and B have the same cardinality (denoted by $|A| = |B|$) iff there exists a one-to-one correspondence (bijection) from A to B.**

Note: This provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set.

〖Example 1〗 let A be the set of 26 lowercase English alphabets. B={1,2,…,26}. Then $|A| = |B|$.

〖Example 2〗 The set of natural numbers is denoted by $N = \{1,2,\ldots,n,\ldots\}$ **.** $M = \{1,2^2,\ldots,n^2,\ldots\}$**. Then $|N| = |M|$.**

〖Example 3〗 Let *A* be the set of real numbers between *a* and *b* (*a*<*b*), and *B* be the set of real numbers between 0 and 1. Show that | *A* | = | *B* |.

Proof:

Let *f* be a function form *A* to *B*.

$$\frac{x-a}{b-a}=\frac{y-0}{1-0}$$

$$y=f(x)=\frac{x-a}{b-a}$$

Then *y* is a bijection from (*a*, *b*) to (0,1).

Hence, | *A* | = | *B* |.

Cardinality

Definition: If there is a one-to-one function form A to B, the cardinality of A is less than or the same as cardinality of B ($|A| \le |B|$).

When $|A| \le |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Now we analysis infinite set…

Countable Sets

Definition: A set that is either finite or has the same cardinality as <u>the set of positive integers</u> is called **countable**.

A set that is not countable is called **uncountable**.

- **The set of real numbers R is an uncountable set.**

When an infinite set is countable (**countably infinite**) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Hilbert's Grand Hotel

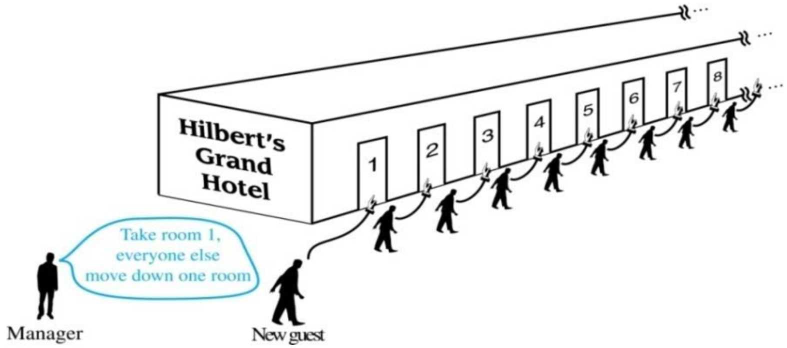
David Hilbert

- **Invented by David Hilbert**
- **A paradox that shows that something impossible with finite sets may be possible with infinite sets.**

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation:

- Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.
- When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n.
- This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



Hilbert's Grand Hotel can also accommodate

- ✓ A finite group of new guests
- ✓ A countably infinite number of new guests
- ✓ a countable number of new guests, all the guests on a countably infinite number of buses where each bus contains a countably infinite number of guests.

How to show that a set is countable?

- **An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).**

Reason:

A one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

How to show that a set is countable?

Example 4: Show that the set of positive even integers *E* is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then f is a bijection from N to E since f is both one-to-one and onto.

- **To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$.**
- **To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.**

Note :

- **E is countable infinite.**
- **E is a proper subset of Z^+! But $|E| = |Z^+|$.**

How to show that a set is countable?

Example 5: Show that the set of integers Z is countable.

Solution: Can list in a sequence:

$$0, -1, 1, -2, +2, \ldots$$

Or can define a bijection from N to Z:

$$f(i) = \begin{cases} 2|i| & i < 0 \\ 1 & i = 0 \\ 2i + 1 & i > 0 \end{cases}$$

f **is a bijection from Z to Z^+ .**

Hence, Z is countable infinite set.

The Positive Rational Numbers are Countable

Example 6: Show that the positive rational numbers are countable.

- $\forall x \in \mathbf{Q}^+, x = q/p,\ p, q \in \mathbf{N}$
- **Let** $S = \{ (p, q) \mid p, q \in \mathbf{N} \} = \mathbf{N} \times \mathbf{N}$.
- $$\left.\begin{array}{l} |Q^+| \le |S| \\ |S| = |N| \\ |N| \le |Q^+| \end{array}\right\} \Rightarrow |Q^+| = |N|$$

SCHRÖDER-BERNSTEIN THEOREM:

If A and B are sets with $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.

In other words, if there are one-to-one functions f form A to B and g from B to A, then there is a one-to-one correspondence between A and B.

(1) $|Q^+| \le |S|$

Suppose that $\dfrac{q}{p} \in Q^+$

$\dfrac{q}{p} \to (p, q)$ is a injective

$\therefore |Q^+| \le |S|$

(2) $|S| = |N|$

$$\begin{array}{cccccc}
\mathbf{(1,1)} & \mathbf{(1,2)} & \mathbf{(1,3)} & \dots & \mathbf{(1,\mathit{q})} & \dots \\
\mathbf{(2,1)} & \mathbf{(2,2)} & \mathbf{(2,3)} & \dots & \mathbf{(2,\mathit{q})} & \dots \\
\mathbf{(3,1)} & \mathbf{(3,2)} & \mathbf{(3,3)} & \dots & \mathbf{(3,\mathit{q})} & \dots \\
\dots & & & & & \\
\mathbf{(\mathit{p},1)} & \mathbf{(\mathit{p},2)} & \mathbf{(\mathit{p},3)} & \dots & \mathbf{(\mathit{p},\mathit{q})} & \dots \\
\dots & & & & &
\end{array}$$

$$1 + 2 + \ldots + (p + q - 2) = \frac{(p+q-2)(p+q-1)}{2}$$

$$n = \frac{1}{2}(p+q-2)(p+q-1) + p$$

(3) $|N| \leq |Q^+|$

Since $N \subseteq Q^+$

Therefore, $|N| \leq |Q^+|$

Note :

- **The set of all rational numbers Q, positive and negative, is countable infinite.**
- **The set of rational numbers and the set of natural numbers have same cardinality.**

Strings

Example 7: Show that the set of finite strings *S* over a finite alphabet *A* is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution:

Show that the strings can be listed in a sequence.

- **First list all the strings of length 0 in alphabetical order.**
- **Then all the strings of length 1 in lexicographic (as in a dictionary) order.**
- **Then all the strings of length 2 in lexicographic order.**
- **And so on.**

This implies a bijection from N to *S* and hence it is a countably infinite set.

The set of all Java programs is countable.

Example 8: Show that the set of all Java programs is countable.

Solution:

Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)

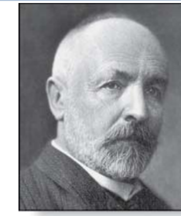
If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.

We move on to the next string.

In this way we construct an implied bijection from N to the set of Java programs. Hence, the set of Java programs is countable.

Uncountable Sets

Georg Cantor
(1845-1918)



【Theorem】The set of real numbers between 0 and 1 is uncountable.

Proof:

- use an important proof method known as the **Cantor diagonalization argument**.

$$A = \{x \mid x \in (0,1) \wedge x \in R\}$$

$$\left.\begin{array}{l}(1)\ |N| \le |A| \\ (2)\ |N| \ne |A|\end{array}\right\} \Longrightarrow |N| < |A|$$

$$(1)\ |N| \le |A|$$

$$A = \{x \mid x \in (0,1) \wedge x \in R\}$$

$$B = \{\frac{1}{n+1} \mid n \in N\}$$

$$|B| = |N| \qquad B \subseteq A$$

$$\therefore |N| \le |A|$$

(2) $|N| \neq |A|$

Let A={r_1 , r_2 , r_3 ,… , r_n ,… }
Represent each real number in the list using *its decimal expansion*.
THE LIST....

$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}. \ . \ . \ . \ .$
$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26} \ . \ . \ . \ .$
$r_3 = 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36} \ . \ . \ . \ .$
…

where $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$
Now construct the number $x = 0.x_1x_2x_3x_4x_5x_6x_7. \ . \ . \ .$

$x_i = 4$ if $d_{ii} \neq 4$
$x_i = 5$ if $d_{ii} = 4$

Then *x* is not equal to any number in the list.
Hence, no such list can exist and hence the interval (0,1) is uncountable .

【**Theorem**】**The set of real numbers is uncountable.**

Proof:

Let $f(x)=\mathrm{tg}(x)$**.**

$f(x)$ **is a bijection from** $(-\frac{\pi}{2},\frac{\pi}{2})$ **to** $R=(-\infty,+\infty)$ **.**

$\left|(-\frac{\pi}{2},\frac{\pi}{2})\right| = |(0,1)|$ $\qquad \therefore |R| = |(0,1)|$

$|\mathbf{R}| = \aleph$

It is said to have the cardinality of the continuum, c.

〖Example 9〗 Show that $|(0,1)| = |[0,1]|$ **.**

Proof:

$$A = [0,1] = \{x \mid x \in R, 0 \le x \le 1\}$$

$$B = (0,1) = \{x \mid x \in R, 0 < x < 1\}$$

$$(1) B \subseteq A \Rightarrow |B| \le |A|$$

$$(2)\ \text{Let } g(x) = \frac{1}{2}x + \frac{1}{4}, x \in [0,1]$$

Hence, *g*(*x*) is a bijection from [0,1] to [1/4,3/4].

Thus $|A| \le |B|$

Results about cardinality

1) **No infinite set has a smaller cardinality than a countable set.**

2) **The union of two countable sets is countable.**

Proof:

Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint.

Case 1: A and B are finite.

Case 2: A is infinite and B is finite.

Case 3: A and B are both countably infinite.
We can list their elements as $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$, respectively . By alternating terms of these two sequences, we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$
This means that $A \cup B$ is countably infinite.

Results about cardinality

1) No infinite set has a smaller cardinality than a countable set.

2) The union of two countable sets is countable.

3) The union of finite number of countable sets is countable.

4) The union of a countable number of countable sets is countable.

Uncomputable Function--An important application in CS

【Definition 4】A function is computable if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is uncomputable.

Question: Show that there are functions that are not computable.

- Show that the set of all computer programs in any particular programming language is countable.
- There are uncountably many different function from a paticular countably infinite set to itself.

The Continuum Hypothesis

- The cardinality of the power set of an arbitrary set has a greater cardinality than the original arbitrary set.
- The power set of Z^+ and the set of real numbers R have the same cardinality.

$$|P(Z^+)| = |R| = c$$

The Continuum Hypothesis

The continuum hypothesis (CH) asserts that there is no cardinal number a such that $\aleph_0 < a < \aleph$.

Homework:

Seventh Edition:

P.176 4, 7, 9, 29, 37, 38