Relations

Chapter 9

Chapter Summary

- Relations and Their Properties
- n-ary Relations and Their Applications
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

- The definition of Relation
- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary relation

【Definition】A binary relation R from a set A to a set B is a subset of A×B.

Note:

- **A *binary relation R*** is a set.
- $R \subseteq A \times B$
- $R = \{(a,b) \mid a \in A, b \in B, aRb\}$

〖Example 1〗

(1) $A = \{2,3,4\}, B = \{2,3,4,5,6\}$ $\quad R = \{(x,y) \mid x \in A, y \in B, x \mid y\}$

$R = \{(2,2),(2,4),(2,6),(3,3),(3,6),(4,4)\}$

(2) Let A and B be sets, $\phi,$ $\qquad A \times B$

n-ary Relations

【Definition】 **Let $A_1, A_2, \ldots, A_n$ be sets. An *n*-ary Relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$**

The domains of relation

degree

A function *f* from a set *A* to a set *B* is a relation form *A* to *B*.

〖Example 2〗 Suppose that $A = \{1,2,3,4\}, B = \{0,1\}$.

$f: A \to B,\ f(1)=f(3)=1, f(2)=f(4)=0$

$R = \{(1,1),(3,1),(2,0),(4,0)\}$

A relation can be used to express a one to many relationship between the elements of the sets A and B.

〖Example 3〗 $R = \{(1,0),(1,1),(2,1),(3,0)\}$

Relations are a generalization of graphs of function.

Relations On A Set

【Definition】A **relation on the set *A*** is a relation form *A* to *A*.

Note:

- $R \subseteq A \times A$

〖Example 4〗

(1) Let $A = \{1,2,3,4\}, \quad R = \{(a,b) \mid a,b \in A, a \mid b\}$

(2) Suppose that *S* is a set. $R = \{(S_1, S_2) \mid S_1 \subseteq S_2, S_1, S_2 \in P(S)\}$

Question:

How many binary relations are there on a set *A* with *n* elements?

$$2^{n^2}$$

Representing Relations

The methods of representing relation:

- **list its all ordered pairs**
- **using a set build notation/specification by predicates**
- **2D table**
- **Connection matrix /zero-one matrix**
- **Directed graph/Digraph**

〖Example 5〗 $A=\{2,3,4\}, B=\{2,3,4,5,6\}$ $R=\{(x,y)\mid x\in A, y\in B, x\mid y\}$

$$R=\{(2,2),(2,4),(2,6),(3,3),(3,6),(4,4)\}$$

	2	3	4	5	6
2	×		×		×
3		×			×
4			×		

Connection Matrices

【Definition】Let R be a relation from

$$A=\{a_1,a_2,\ldots,a_m\},\quad to\quad B=\{b_1,b_2,\ldots,b_n\}$$

An $m\times n$ connection matrix $M_R=[m_{ij}]$ for R is defined by

$$m_{ij}=\begin{cases}1 & if\,(a_i,b_j)\in R,\\ 0 & if\,(a_i,b_j)\notin R.\end{cases}$$

For example,

$$A=\{2,3,4\},B=\{2,3,4,5,6\}\qquad R=\{(x,y)\mid x\in A,y\in B,x\mid y\}$$

$$\begin{array}{c c} & \begin{array}{ccccc}2 & 3 & 4 & 5 & 6\end{array} \\ \begin{array}{c}2\\3\\4\end{array} & \begin{bmatrix}1 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}\end{array}$$

Question:

How many binary relations are there on a set *A* with *n* elements?

0,1

$$\begin{bmatrix} \times & \times & \dots & \times \\ \times & \times & \dots & \times \\ \dots & & & \\ \times & \times & \dots & \times \end{bmatrix}_{n\times n}$$

By the product rule,

$$2\times 2\times \dots \times 2 = 2^{n^2}$$

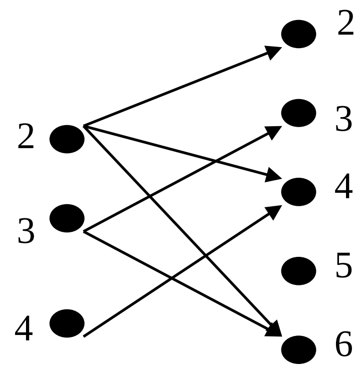
Directed graph/Digraph

【Definition】*A* **directed graph** or a **digraph**, consists of a set V of **vertices** together with a set E of ordered pairs of elements of V called **edges(or arcs)**.

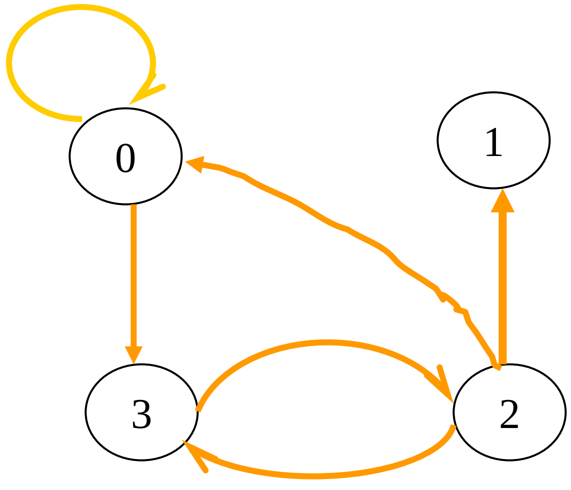
The vertices a,b is called the **initial** and **terminal** vertices of the edge (a,b).

For example,

$A=\{2,3,4\}, B=\{2,3,4,5,6\}$ $\quad R=\{(x,y) \mid x \in A, y \in B, x \mid y\}$



〖Example 6〗 $A = \{0, 1, 2, 3\}$



$$R = \{(0,0), (0,3), (2,0), (2,1), (2,3), (3,2)\}.$$

Properties of Binary Relations

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Transitive

Reflexive Relations

【Definition】A relation R on a set A is reflexive if

$$(x,x) \in R, \textit{ for every element } x \in A.$$

$$\forall x(x \in A \to (x,x) \in R)$$

Questions:

(1) What do we know about matrices representing reflexive relations?

All the elements on the main diagonal of $\mathbf{M}_R$ must be 1s.

$$M_r = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

(2) What do we know about digraphs representing reflexive relations?

There is a loop at every vertex of the directed graph.

Irreflexive Relations

【Definition】A relation R on a set A is irreflexive if

$$\forall x(x \in A \to (x,x) \notin R)$$

Questions:

(1) The connection matrix of a irreflexive relations?

(2) Digraph?

(3) Can a relation on a set be neither reflexive nor irreflexive?

Yes.

$$\begin{bmatrix} 1 & \times & \dots & \times \\ \times & 0 & \dots & \times \\ \mathrm{L} & & & \\ \times & \times & \dots & 0 \end{bmatrix}_{n\times n}$$

Symmetric Relations

【Definition】A relation R on a set A is symmetric if

$$\forall x \forall y ((x, y) \in R \rightarrow (y, x) \in R)$$

Questions:

(1) The connection matrix of a symmetric relations?

$$\begin{bmatrix} \times & 1 & \cdots & \times \\ 1 & \times & \cdots & 0 \\ \cdots & & & \\ \times & 0 & \cdots & \times \end{bmatrix}_{n\times n}$$

(2) Digraph?

If there is an arc (*x*, *y*) there must be an arc (*y*, *x*).

Antisymmetric Relations

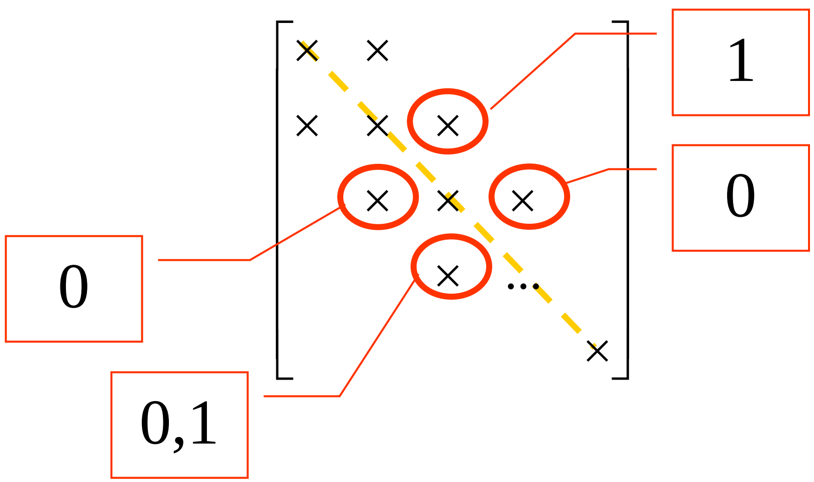
【Definition】A relation R on a set A is antisymmetric if

$$\forall x \forall y((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$$

Note:

(1) $\forall x \forall y((x, y) \in R \wedge x \neq y \rightarrow (x, y) \notin R)$

(2)



(3) If there is an arc from *x* to *y* there cannot be one from *y* to *x* if $x \neq y$.

(4) The symmetric and antisymmetric relations are not opposites.

For example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Transitive Relations

【Definition】A relation R on a set A is transitive if whenever $\forall x \forall y \forall z((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$

Note:

(1) $\overline{(m_{ij} \wedge m_{jk})} \vee m_{ik} = 1$

Why?

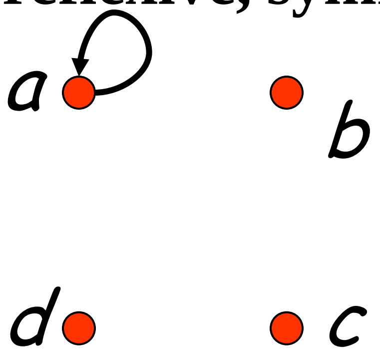
(2) If there is an arc from *x* to *y* and one from *y* to *z* then there must be one from *x* to *z*.

〖Example 7〗**Determine whether the following relations are reflexive, irreflexive, symmetric, antisymmetric and/or transitive.**

(1) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reflexive, symmetric, antisymmetric, transitive

(2)



$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

not reflexive, symmetric, antisymmetric, transitive

(3) $R_1 = \{(a,b) \mid a \mid b, a, b \in N\}$

reflexive, antisymmetric, transitive

(4) $R_2 = \{(a,b) \mid a+b = 2m, a, b, m \in N\}$

reflexive, symmetric, transitive

Question:

Symmetric, transitive $\Rightarrow$ reflexive ?

$$\left.\begin{array}{r} \left.\begin{array}{c} (a,b) \in R \\ R \quad is \quad symmetric \end{array}\right\} \Rightarrow (b,a) \in R \\ R \quad is \quad transitive \end{array}\right\} \Rightarrow (a,a) \in R$$

〖Example 8〗**Counting relations** **How many relations are there on a set with *n* elements that are**

(1) reflexive ?

(2) symmetric ?

(3) antisymmetric ?

Solution:

(1)

$$\begin{bmatrix} 1 & \times & \dots & \times \\ \times & 1 & \dots & \times \\ \dots & & & \\ \times & \times & \dots & 1 \end{bmatrix}_{n\times n}$$

0,1

$$2^{n^2-n}$$

(2)

$$\begin{bmatrix} \times & 1 & \dots & \times \\ 1 & \times & \dots & 0 \\ \dots & & & \\ \times & 0 & \dots & \times \end{bmatrix}_{n\times n}$$

$$2^n \times 2^{\frac{n^2-n}{2}} = 2^{\frac{n(n+1)}{2}}$$

Solution:

(3)

$$\begin{bmatrix} \times & 1 & & & \\ 0 & \times & 0 & & \\ & 0 & \times & 0 & \\ & & 1 & \ldots & \\ & & & & \times \end{bmatrix} \qquad 2^n \times 3^{\frac{n^2-n}{2}}$$

Questions:

- **reflexive and symmetric?**
- **transitive?**

Combining Relations

Since relations form A to B are subsets of A×B, two relations form A to B can be combined in any way two sets can be combined.

- **Set operation**
- **Composition**
- **Inverse relation**

1) $\cup, \cap, \overline{\ }, -, \oplus$

〖Example 9〗 **Let** $A=\{1,2,3,4\}, Z$ **is the set of integers,**

$$R=\{(a,b)\mid a,b\in A, \frac{a-b}{2}\in Z\}, S=\{(a,b)\mid a,b\in A, \frac{a-b}{3}\in Z, a-b>0\},$$

what are the relations $R\cup S, R\ S, \overline{R}, R-S, S\oplus R$ **?**

Solution:

$R=\{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1),(2,4),(4,2)\}$ $\quad S=\{(4,1)\}$

(1) Set operations

(2) Boolean operations/logical operations

The Boolean Sum $\quad \vee: 0\vee 0=0, 0\vee 1=1, 1\vee 0=1, 1\vee 1=1$

The Boolean product $\quad \wedge: 0\wedge 0=0, 0\wedge 1=0, 1\wedge 0=0, 1\wedge 1=1$

The complement $\quad -: \overline{0}=1, \overline{1}=0$

The logical operations of matrices:

Let $A=\{a_1,a_2,\ldots,a_n\},B=\{b_1,b_2,\ldots,b_m\},M_{R_1}=[c_{ij}],M_{R_2}=[d_{ij}]$ **, the set operations of two relations are defined by**

$$M_{R_1 \cup R_2}=[c_{ij}\vee d_{ij}]=M_{R_1}\vee M_{R_2}$$

$$M_{R_1R_2}=[c_{ij}\wedge d_{ij}]=M_{R_1}\wedge M_{R_2}$$

$$M_{\overline{R_1}}=[\overline{c}_{ij}]$$

$$M_{R_1-R_2}=M_{R_1\cap\overline{R_2}}=[c_{ij}\wedge\overline{d}_{ij}]$$

2) Composition

$$R = \{(a,b) \mid a \in A, b \in B, aRb\}, \quad S = \{(b,c) \mid b \in B, c \in C, bSc\}$$

The composite of *R* and *S*: $S \circ R$

$$S \circ R = \{(a,c) \mid a \in A \wedge c \in C \wedge \exists b(b \in B \wedge aRb \wedge bSc\}$$

Question:

How to computer *S*o*R*?

(1) Using the definition directly

(2) Using the connection matrix

〖**Example 10**〗 $A=\{a,b\}, B=\{1,2,3,4\}, C=\{5,6,7\}$

$R=\{(a,1),(a,2),(b,3)\}, S=\{(2,6),(3,7),(4,5)\}$

$S \circ R = ?\quad R \circ S = ?$

Solution:

(1) Using the definition directly

$$S \circ R=\{(a,6),(b,7)\}$$

$$R \circ S=\phi$$

Note:

$$S \circ R \neq R \circ S$$

Solution:

(2) Using the connection matrix

$$\mathbf{M}_R = [r_{ij}]_{m\times n}, \mathbf{M}_S = [s_{jk}]_{n\times l}$$

$$\mathbf{M}_{S\circ R} = \mathbf{M}_R \cdot \mathbf{M}_S = [w_{ik}]_{m\times l}, \quad w_{ik} = \bigvee_{j=1}^{n} (r_{ij} \wedge s_{jk})$$

$$A = \{a, b\}, B = \{1,2,3,4\}, C = \{5,6,7\}$$

$$R = \{(a,1), (a,2), (b,3)\}, S = \{(2,6), (3,7), (4,5)\}$$

$$\therefore \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{M}_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{S\circ R} = \mathbf{M}_R \cdot \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore S \circ R = \{(a,6), (b,7)\}$$

The Power of a relation R

【Definition】Let *R* be a relation on the set A. The powers $R^n, n = 1,2,3,\ldots$ are defined inductively by

$$R^1 = R, \quad and \quad R^{n+1} = R^n \circ R$$

〖Example 11〗 **Let *A*={0,1,2,3}. *R* is the relation on the set *A*.**

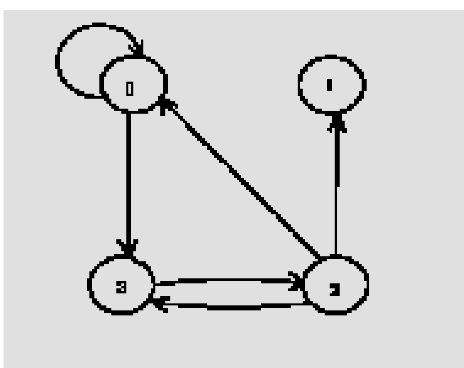
R **={(0,0),(0,3),(2,3),(3,2),(2,1),(2,0)}. R^2=?**

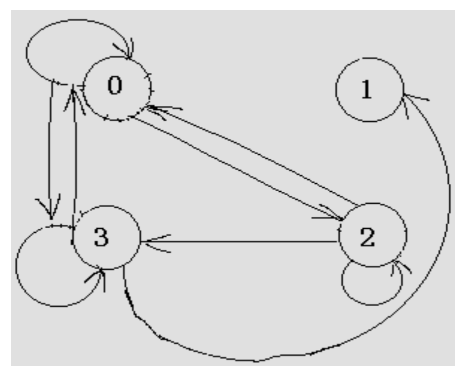
Solution:

(1) Using the definition

R^2={(0,0),(0,3),(0,2),(2,2),(3,3),(2,3),(2,0),(3,1),(3,0)}

(2) Using the digraph





(3) Using the matrix

$$\mathbf{M}_{R^2} = M_R \bullet M_R$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

【 Theorem 】 The relation *R* on a set *A* is transitive if and only if $R^n \subseteq R, \quad for \quad n = 1,2,3,\ldots$

Proof:

(1) $R^n \subseteq R, \quad for \quad n = 1,2,3,\ldots$ $\Rightarrow$ ***R* is transitive**

$(a,b) \in R, \quad (b,c) \in R \quad \Rightarrow \quad (a,c) \in R$

$(a,c) \in R^2 \subseteq R$

(2) *R* is transitive $\Rightarrow$ $R^n \subseteq R, \quad for \quad n = 1,2,3,\ldots$

- **Inductive base** $n = 1, \qquad R \subseteq R$
- **Inductive step** $R^n \subseteq R \quad \Rightarrow \quad R^{n+1} \subseteq R$

$$\left.\begin{array}{l} (a,b) \in R^{n+1} \\ R^{n+1} = R^n \circ R \end{array}\right\} \Rightarrow \left.\begin{array}{l} (a,x) \in R, (x,b) \in R^n \subseteq R \\ R \text{ is transitive} \end{array}\right\} \Rightarrow (a,b) \in R$$

〖Example 12〗 Let *R* be a symmetric relation on the set *A*. Show that R^n is symmetric for all positive integers *n*.

Proof:

- **Inductive base**

  ***n*=1, *R* be a symmetric**

- **Inductive step**

  R^n is symmetric $\Rightarrow$ R^{n+1} is symmetric

$$(a,c)\in R^{n+1} \Rightarrow (c,a)\in R^{n+1}$$

$$\left.\begin{array}{l}(a,b)\in R,(b,c)\in R^n\\ R,R^n \quad are \quad symmetric\end{array}\right\} \Rightarrow (b,a)\in R,(c,b)\in R^n$$

$$\Rightarrow (c,a)\in R \circ R^n = R^{n+1}$$

3) Inverse relation

$$R = \{(a,b) \mid a \in A, b \in B, aRb\}$$

The *inverse relation* form *B* to *A*: $R^{-1}(R^c)$

$$\{(b,a) \mid (a,b) \in R, a \in A, b \in B\}$$

Question:

How to get R^{-1} **?**

(1) Using the definition directly

For example, $R = \{(a,b) \mid a \mid b, a,b \in Z^+\}$

$$R^{-1} = \{(a,b) \mid b \mid a, a,b \in Z^+\}$$

(2) Reverse all the arcs in the digraph representation of *R*

(3) Take the transpose $M_R{}^T$ of the connection matrix M_R of *R*.

4) The properties of relation operations

Suppose that *R*, *S* are the relations from *A* to *B*, *T* is the relation from *B* to *C*, *P* is the relation from *C* to *D*, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof:

$$\forall (x, y) \in (R \cup S)^{-1}$$

$$\Leftrightarrow (y, x) \in R \cup S$$

$$\Leftrightarrow (y, x) \in R \quad or \quad (y, x) \in S$$

$$\Leftrightarrow (x, y) \in R^{-1} \quad or \quad (x, y) \in S^{-1}$$

$$\Leftrightarrow (x, y) \in R^{-1} \cup S^{-1}$$

4) The properties of relation operations

Suppose that *R*, *S* are the relations from *A* to *B*, *T* is the relation from *B* to *C*, *P* is the relation from *C* to *D*, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(2) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(3) $(\overline{R})^{-1} = \overline{R^{-1}}$

(4) $(R - S)^{-1} = R^{-1} - S^{-1}$

(5) $(A \times B)^{-1} = B \times A$

Proof:

$$\forall (x, y) \in (A \times B)^{-1}$$

$$\Leftrightarrow \quad (y, x) \in A \times B$$

$$\Leftrightarrow \quad (x, y) \in B \times A$$

4) The properties of relation operations

Suppose that *R*, *S* are the relations from *A* to *B*, *T* is the relation from *B* to *C*, *P* is the relation from *C* to *D*, then

(1) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(2) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(3) $(\overline{R})^{-1} = \overline{R^{-1}}$

(4) $(R - S)^{-1} = R^{-1} - S^{-1}$

(5) $(A \times B)^{-1} = B \times A$

(6) $\overline{R} = A \times B - R$

(7) $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$

(8) $(R \circ T) \circ P = R \circ (T \circ P)$

(9) $(R \cup S) \circ T = R \circ T \cup S \circ T$

Homework:

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