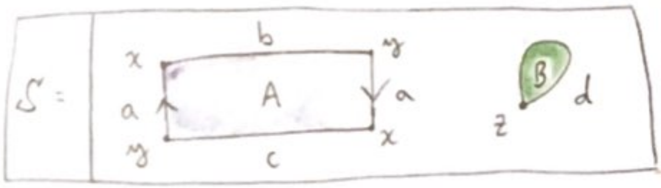
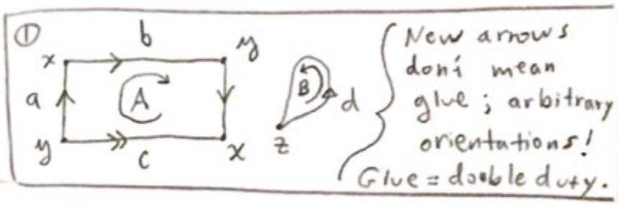
Topology Quiz

1) Find presentations for the homology groups H_0, H_1, H_2 of the Moebius strip, and a disk D^2.





② $\partial_0 x = \partial_0 y = \partial_0 z = 0$; $\partial_1 a = x - y$; $\partial_1 b = y - x$; $\partial_1 c = x - y$; $\partial_1 d = z - z = 0$

$\partial_2 A = a + b + a - c = 2a + b - c$; $\partial_2 B = d$.

③ $\ker \partial_0 = \mathrm{span}\{x, y, z\}$; $\partial_1 = \begin{array}{cc} \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \\ \begin{matrix} a & b & c & d \end{matrix} & \end{array}$ $\Rightarrow$ $\ker \partial_1 = \mathrm{span}\{a+b, c-a, d\}$

$\mathrm{Im}\, \partial_1 = \mathrm{span}\{x - y\}$

$\partial_2 = \begin{array}{cc} \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} a \\ b \\ c \\ d \end{matrix} \\ \begin{matrix} A & B \end{matrix} & \end{array}$ $\Rightarrow$ $\ker \partial_2 = \langle 0 \rangle$

$\mathrm{Im}\, \partial_2 = \mathrm{span}\{2a + b - c, d\}$.

④ $H_0 = \mathrm{span}\{x, y, z\} / \mathrm{span}\{x - y\}$

$= \langle x, y, z \mid x = y \rangle$

$= \boxed{\langle x, z \mid \emptyset \rangle} = \mathbb{Z} \oplus \mathbb{Z}$

$H_1 = \mathrm{span}\{a+b, c-a, d\} / \mathrm{span}\{2a+b-c, d\}$

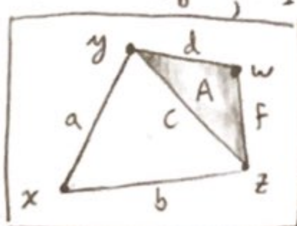
$= \langle a+b, c-a, d \mid a+b = c-a, d \rangle$

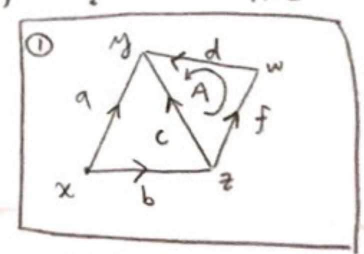
$= \boxed{\langle a+b \mid \emptyset \rangle} = \mathbb{Z}$

$H_2 = \langle 0 \rangle / \langle 0 \rangle$

$= \boxed{\langle 0 \rangle}$

2) Find presentations for H_0, H_1, H_2 of the following space S = (graph  with one triangle filled in.)





② $\partial_0(x) = \partial_0(y) = \partial_0(z) = \partial_0(w) = 0$; $\partial_1 a = y - x$; $\partial_1 b = z - x$; $\partial_1 c = y - z$

$\partial_1 d = y - w$; $\partial_1 f = w - z$; $\partial_2 A = f + d - c$

③ $\ker \partial_0 = \mathrm{span}\{x, y, z, w\}$; $\partial_1 = \begin{array}{cc} \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} & \begin{matrix} x \\ y \\ z \\ w \end{matrix} \\ \begin{matrix} a & b & c & d & f \end{matrix} & \end{array}$ $\Rightarrow$ $\ker \partial_1 = \mathrm{span}\{b + c - a, b + d + f - a\}$

$\mathrm{Im}\, \partial_1 = \mathrm{span}\{y - x, z - x, y - w\}$

$\partial_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathrm{Im}\, \partial_2 = \mathrm{span}\{f + d - c\}$

$H_0 = \langle x, y, z, w \mid y = x, z = x, y = w \rangle$

$= \boxed{\langle x \mid \emptyset \rangle} = \mathbb{Z}$

$H_1 = \langle b + c - a, b + d + f - a \mid f + d = c \rangle$

$= \boxed{\langle b + c - a \mid \emptyset \rangle}$

$= \mathbb{Z}$

$H_2 = \langle 0 \rangle / \langle 0 \rangle$

$\boxed{= \langle 0 \rangle}$

Find presentations of H_1, H_2 for S :

①

② $\partial_0 z = 0$; $\partial_1 p = \partial_1 q = \partial_1 r = z - z = 0$

$\partial_1 t = z - z = 0$.

$\partial_2 Q = p + q - r + p + t + r - t - q = 2p$

③ $\ker \partial_0 = \text{span}\{z\}$; $\partial_1 = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} & z \\ \begin{array}{cccc} p & q & r & t \end{array} & \end{array}$ $\Rightarrow$ $\ker \partial_1 = \text{span}\{p, q, r, t\}$

$\text{Im}\ \partial_1 = \langle 0 \rangle$

$\partial_2 = \begin{array}{cc} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{c} p \\ q \\ r \\ t \end{array} \\ Q & \end{array}$ $\Rightarrow$ $\ker \partial_2 = \langle 0 \rangle$

$\text{Im}\ \partial_2 = \text{span}\{2p\}$

④ $H_0 = \boxed{\langle z \mid \emptyset \rangle}$

$= \mathbb{Z}$

$H_1 = \langle p, q, r, t \mid 2p \rangle$

$\boxed{= \langle p, q, r, t \mid p = -p \rangle}$

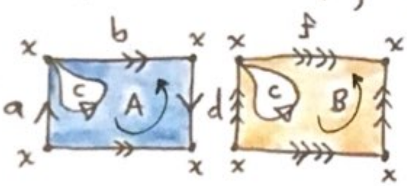
$= \mathbb{Z}/_{2\mathbb{Z}} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$H_2 = \langle 0 \rangle /_{\langle 0 \rangle}$

$\boxed{= \langle 0 \rangle}$

Find presentations of H_1, H_2 for $K^2 \# T^2$

①



② $\partial_0 x = 0$

$\partial_1 a = \partial_1 b = \partial_1 c = \partial_1 d = \partial_1 f = 0$

$\partial_2 A = b - a - b - c - a = -2a - c$

$\partial_2 B = f + d - f - c - d = -c$

③ $\ker \partial_0 = \text{span}\{x\}$

$\ker \partial_1 = \text{span}\{a, b, c, d, f\}$

$\text{Im}\ \partial_1 = \langle 0 \rangle$

$\partial_2 = \begin{array}{cc} \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ f \end{array} \\ \begin{array}{cc} A & B \end{array} & \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $\ker \partial_2 = \langle 0 \rangle$, $\text{Im}\ \partial_2 = \text{span}\{-2a - c, -c\}$

④ $H_0 = \boxed{\langle x \mid \emptyset \rangle}$

$H_1 = \langle a, b, c, d, f \mid -2a = c,\ -c = 0 \rangle$

$\boxed{= \langle a, b, d, f \mid a = -a \rangle}$

$H_2 = \langle 0 \rangle /_{\langle 0 \rangle}$

$\boxed{= \langle 0 \rangle}$