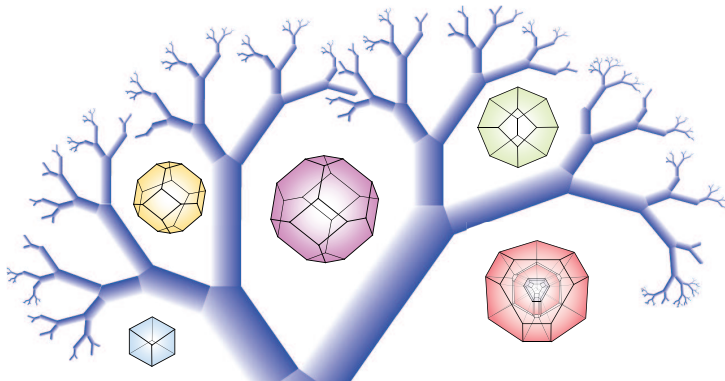


Cofree compositions of coalgebras: Trees, polytopes and indelible grafting.

Stefan Forcey, U. Akron

Aaron Lauve, Loyola U. Chicago

Frank Sottile, Texas A&M U.



“Composing coalgebraic species preserves their niceness.”

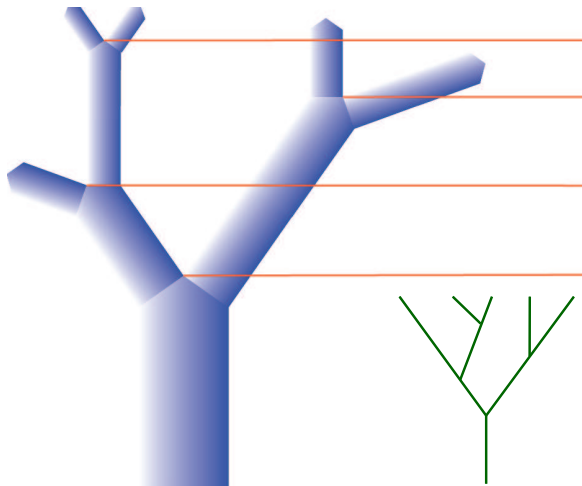
“Composing coalgebraic species preserves their niceness.”

For this talk, nice properties of coalgebras will be:

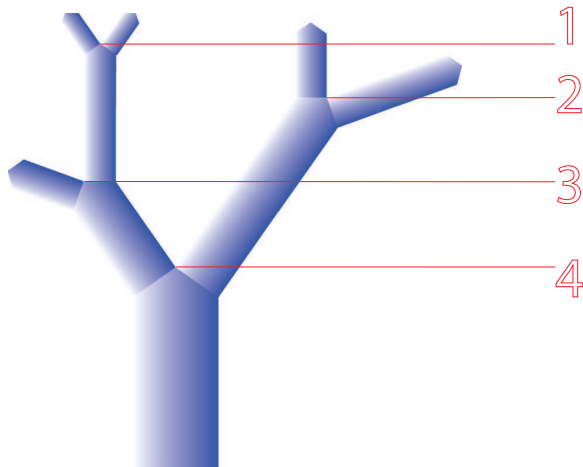
1. Cofree-ness,
2. Hopf-ness,
3. Polytopal-ness.

But first, our cast of characters:

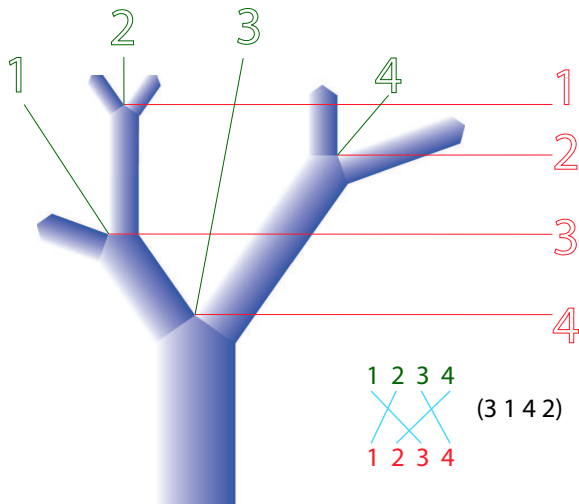
Leveled trees \mathcal{S} .



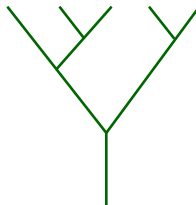
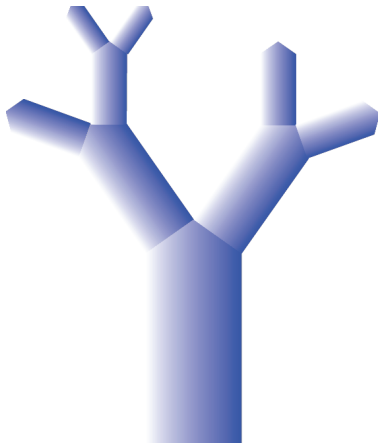
Leveled trees \mathcal{S} .



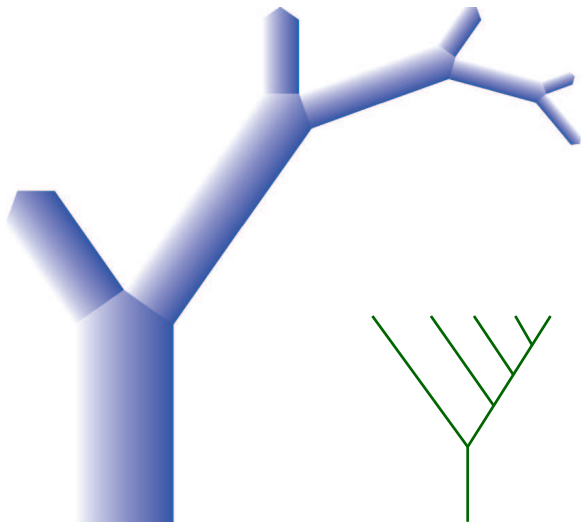
Leveled trees are permutations \mathcal{S}_n .



Binary trees \mathcal{B} .

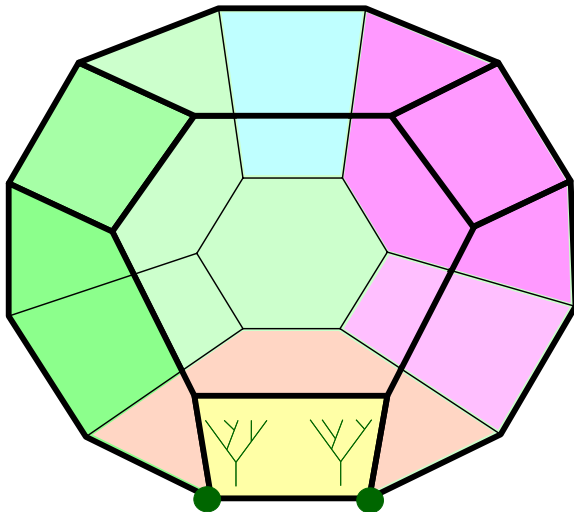


Combed binary trees \mathcal{C} .

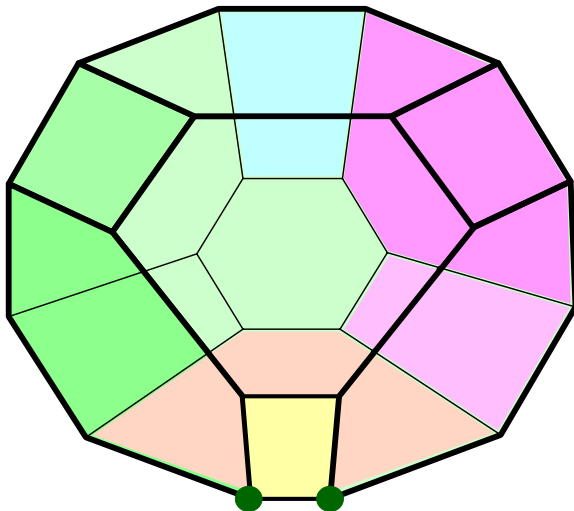


Our characters as Polytopes.

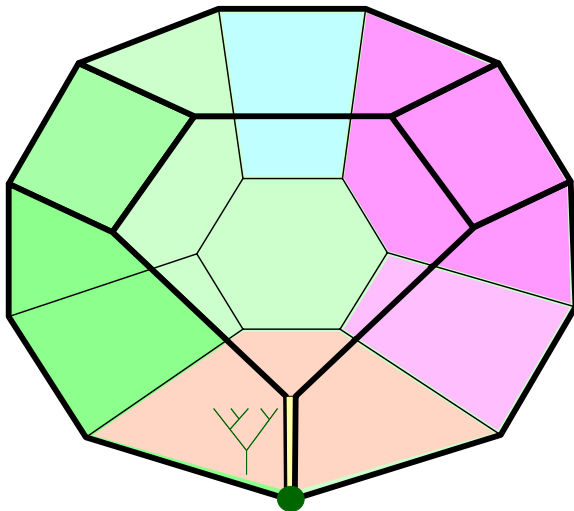
Permutohedron.



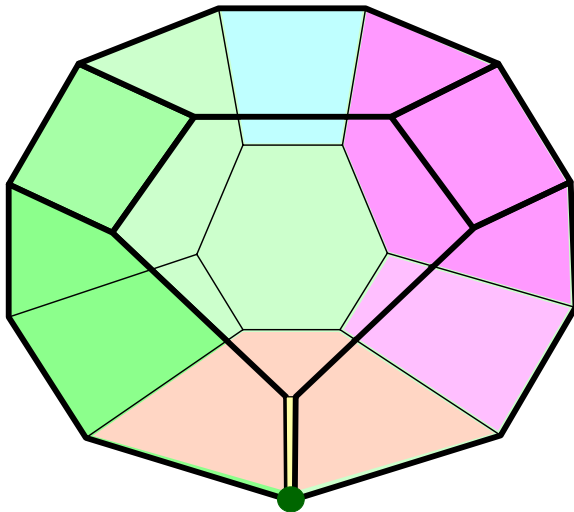
Tonks cellular projection.



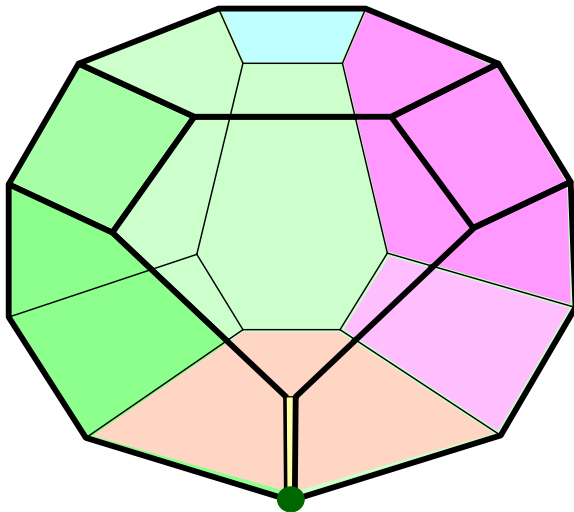
Tonks cellular projection.



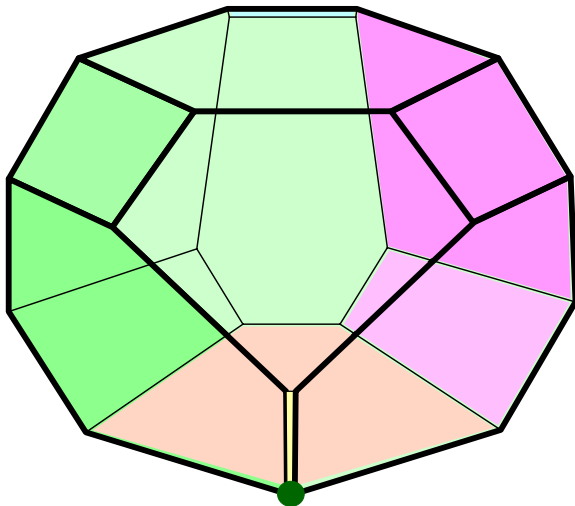
Tonks cellular projection.



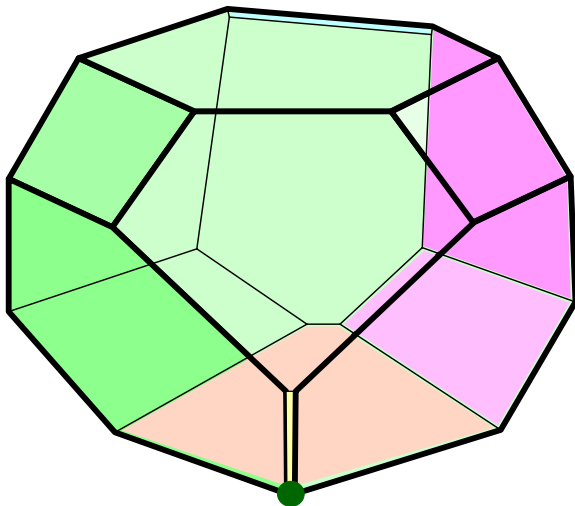
Tonks cellular projection.



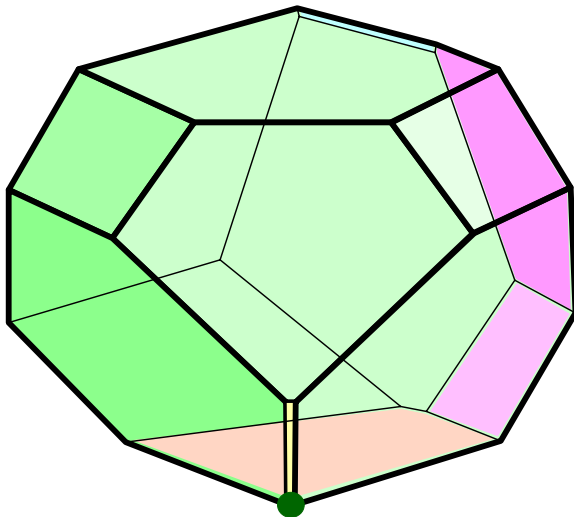
Tonks cellular projection.



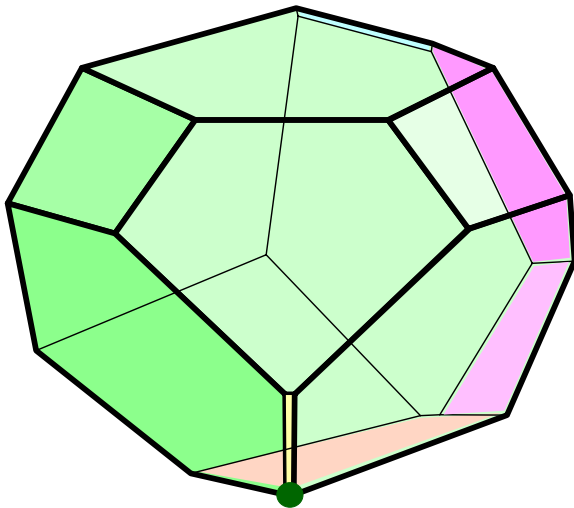
Tonks cellular projection.



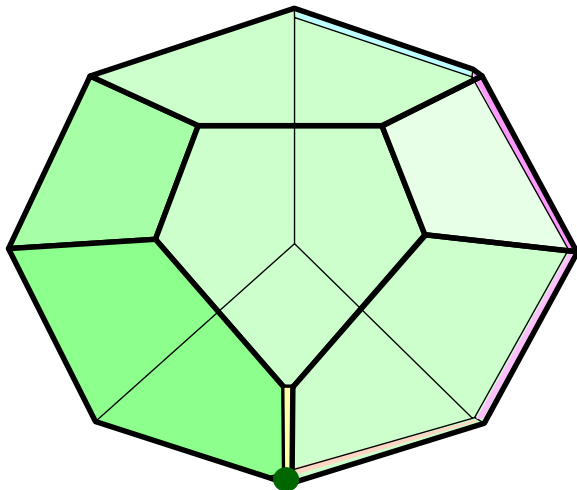
Tonks cellular projection.



Tonks cellular projection.



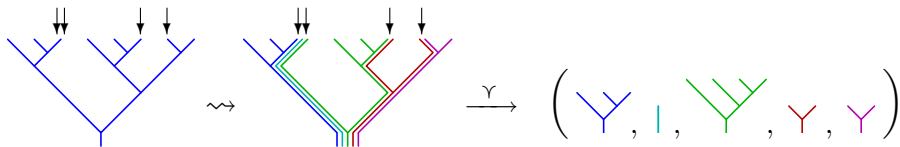
Tonks cellular projection.



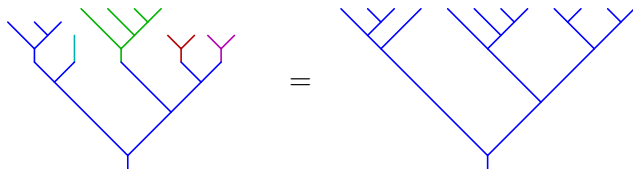
Our characters as graded Hopf algebras.

A Hopf algebra of binary trees.

Two operations on trees: splitting



and grafting:



Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \text{ (rooted tree with 2 children) } = | \otimes \text{ (rooted tree with 2 children) } + \text{ (rooted tree with 1 child) } \otimes \text{ (rooted tree with 1 child) } + \text{ (rooted tree with 2 children) } \otimes |$$

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}\text{Sym}$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \left(\text{Y-shape} \right) = \left| \right| \otimes \left(\text{Y-shape} \right) + \left(\text{Y-shape} \right) \otimes \left(\text{Y-shape} \right) + \left(\text{Y-shape} \right) \otimes \left| \right|$$

Here is how to multiply two trees:

$$\left(\text{Y-shape} \right) \cdot \left(\text{Y-shape} \right) = \left(\text{Y-shape} \right) + \left(\text{Y-shape} \right) + \left(\text{Y-shape} \right)$$

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}\text{Sym}$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \text{ (tree with 2 interior nodes) } = | \otimes \text{ (tree with 2 interior nodes) } + \text{ (tree with 1 interior node) } \otimes \text{ (tree with 1 interior node) } + \text{ (tree with 2 interior nodes) } \otimes |$$

Here is how to multiply two trees:

$$\text{ (tree with 2 interior nodes) } \cdot \text{ (tree with 1 interior node) } = \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) }$$

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The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

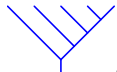
$$\Delta \text{ (tree with 2 interior nodes) } = | \otimes \text{ (tree with 2 interior nodes) } + \text{ (tree with 1 interior node) } \otimes \text{ (tree with 1 interior node) } + \text{ (tree with 2 interior nodes) } \otimes |$$

Here is how to multiply two trees:


$$\text{ (tree with 2 interior nodes) } \cdot \text{ (tree with 1 interior node) } = \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) }$$

The way to multiply in \mathfrak{CSym} .

We draw basis elements of \mathfrak{CSym} as *right combs*. $x^{(4)} =$



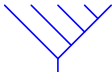
The way to multiply in \mathfrak{CSym} .

We draw basis elements of \mathfrak{CSym} as *right combs*. $x^{(4)} =$  .

Here is the coproduct:

$$\Delta \text{ (right comb with 3 teeth)} = | \otimes \text{ (right comb with 2 teeth)} + \text{ (right comb with 1 tooth)} \otimes \text{ (right comb with 1 tooth)} + \text{ (right comb with 3 teeth)} \otimes |$$

The way to multiply in \mathfrak{CSym} .

We draw basis elements of \mathfrak{CSym} as *right combs*. $x^{(4)} =$  .


Here is the coproduct:

$$\Delta \text{ (right comb with 3 teeth)} = \text{ (vertical line)} \otimes \text{ (right comb with 3 teeth)} + \text{ (right comb with 2 teeth)} \otimes \text{ (right comb with 1 tooth)} + \text{ (right comb with 1 tooth)} \otimes \text{ (vertical line)}$$

Here is how to multiply two combs:

$$\text{ (right comb with 3 teeth)} \cdot \text{ (right comb with 1 tooth)} = \text{ (right comb with 4 teeth)} + \text{ (right comb with 3 teeth)} + \text{ (right comb with 2 teeth)} + \text{ (right comb with 1 tooth)}$$

The way to multiply in \mathfrak{CSym} .

We draw basis elements of \mathfrak{CSym} as *right combs*. $x^{(4)} =$  .

Here is the coproduct:

$$\Delta \text{ (right comb with 3 teeth)} = \text{vertical line} \otimes \text{right comb with 3 teeth} + \text{right comb with 2 teeth} \otimes \text{right comb with 1 tooth} + \text{right comb with 1 tooth} \otimes \text{vertical line}$$

Here is how to multiply two combs:

$$\text{right comb with 3 teeth} \cdot \text{right comb with 2 teeth} = \text{right comb with 5 teeth} + \text{right comb with 4 teeth} + \text{right comb with 3 teeth} + \text{right comb with 2 teeth}$$

Species.

A *species* is an endofunctor of Finite Sets with bijections.

- *Example:* The species \mathcal{L} of lists takes a set to linear orders of that set.

$$\mathcal{L}(\{a, d, h\}) = \{ a < d < h, a < h < d, h < a < d, h < d < a, d < a < h, d < h < a \}$$

- *Example:* The species \mathcal{B} of binary trees takes a set to trees with labeled leaves.

$$\mathcal{B}(\{a, d, h\}) = \{ \begin{array}{c} a \quad d \quad h \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \begin{array}{c} a \quad h \quad d \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \dots, \begin{array}{c} a \quad d \quad h \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \begin{array}{c} a \quad h \quad d \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \dots \}$$

Species composition.

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

where the union is over partitions of U into any number of nonempty disjoint parts.

$$\pi = \{U_1, U_2, \dots, U_n\} \text{ such that } U_1 \sqcup \dots \sqcup U_n = U.$$

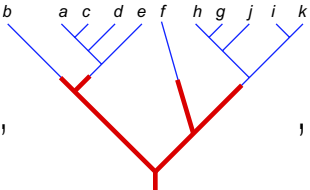
Familiar(?): also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

$$\gamma : \mathcal{F} \circ \mathcal{F} \rightarrow \mathcal{F}$$

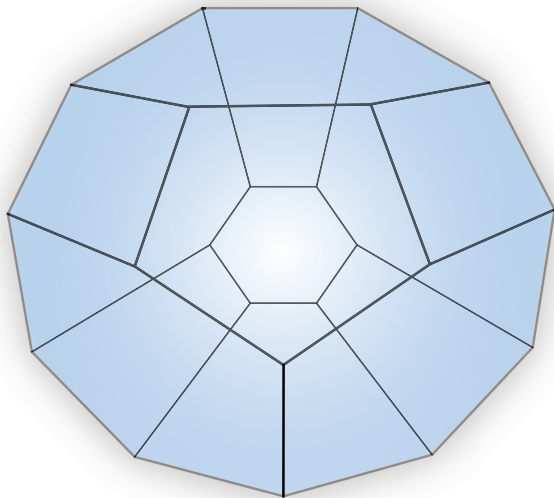
Leveled tree of trees: indelible grafting.

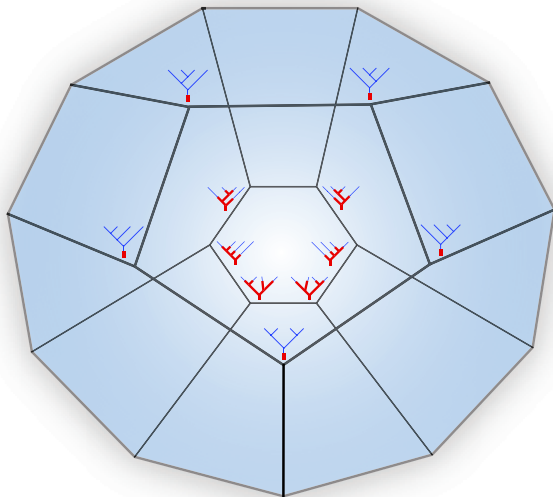
Example:

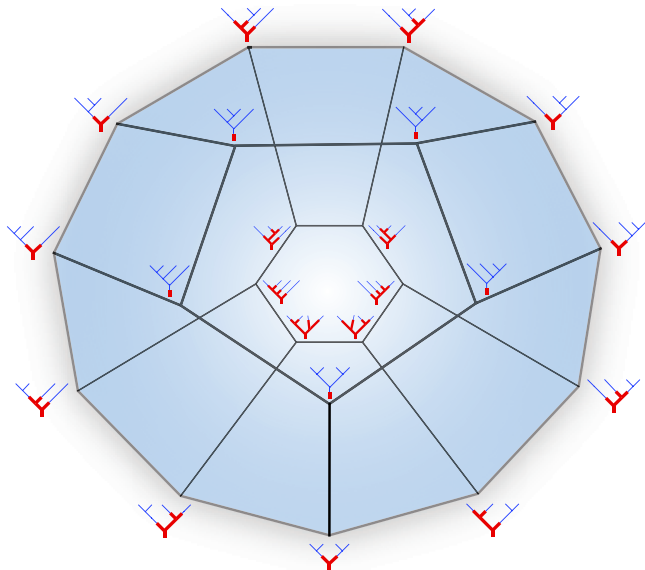
$$(S \circ \mathcal{B})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

$$\left\{ \begin{array}{c} b \quad a \quad c \quad d \quad e \quad f \quad h \quad g \quad j \quad i \quad k \\ \text{[Tree Diagram]} \end{array} \right\} = \left\{ \frac{\text{trees}}{\text{leveled tree}} \right\}$$


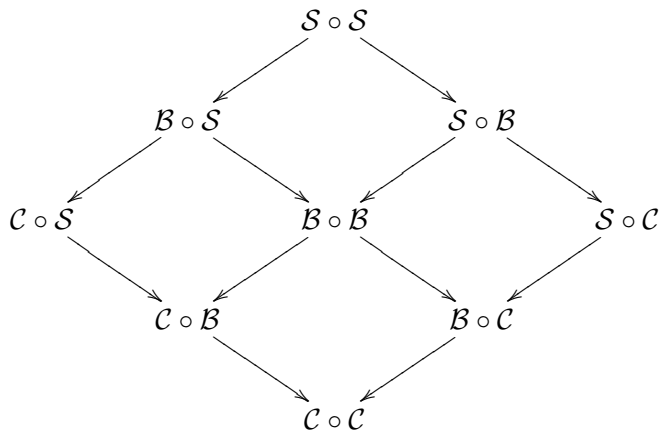
Example $\mathcal{S} \circ \mathcal{B}$



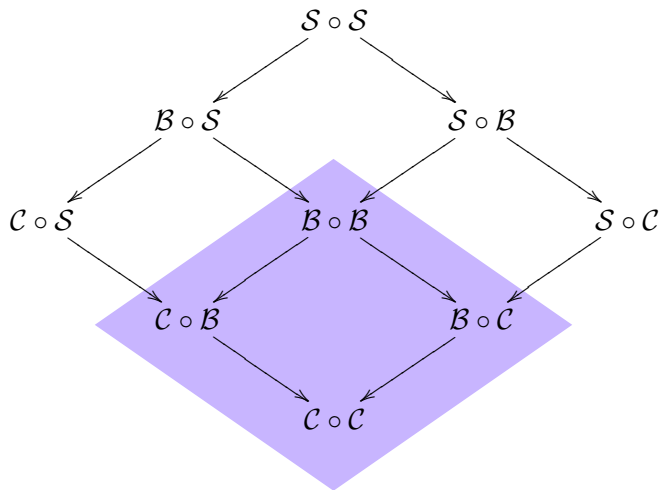




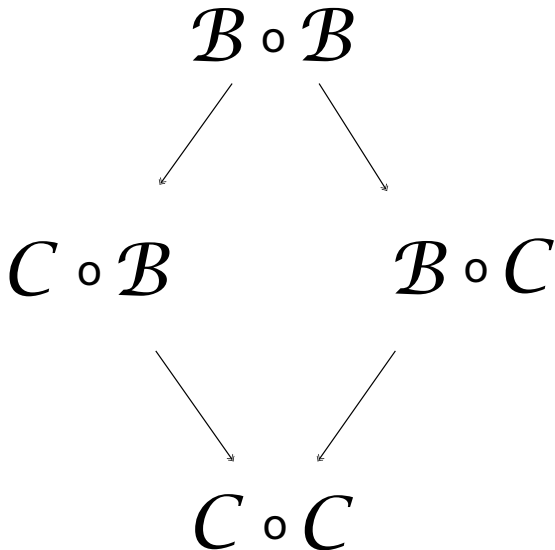
Composing species of trees.



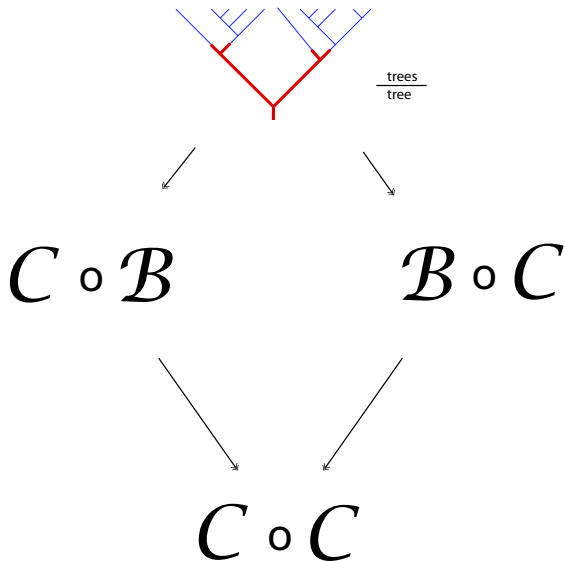
Composing species of trees.



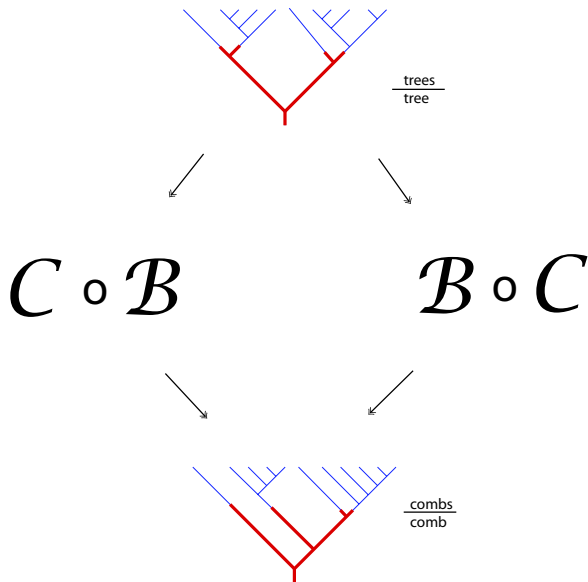
A small commuting diamond



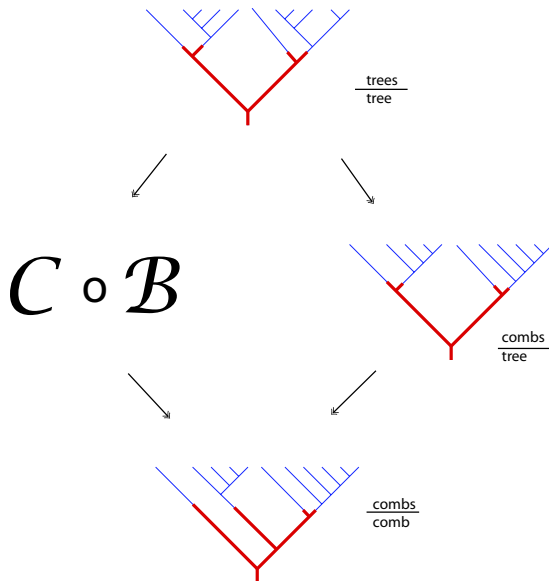
A small commuting diamond



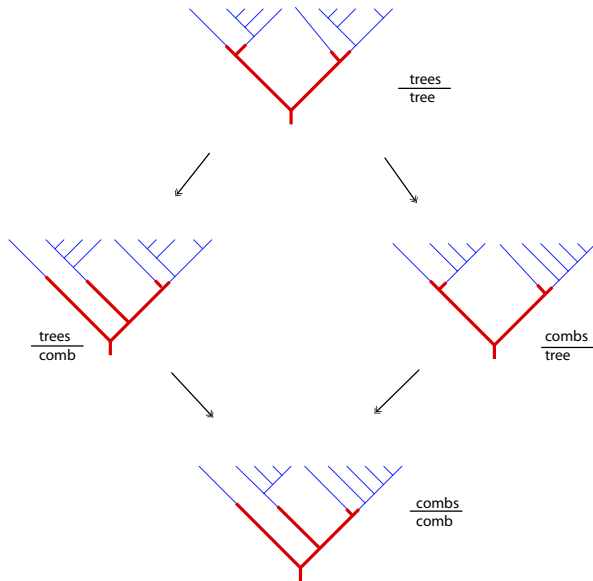
A small commuting diamond



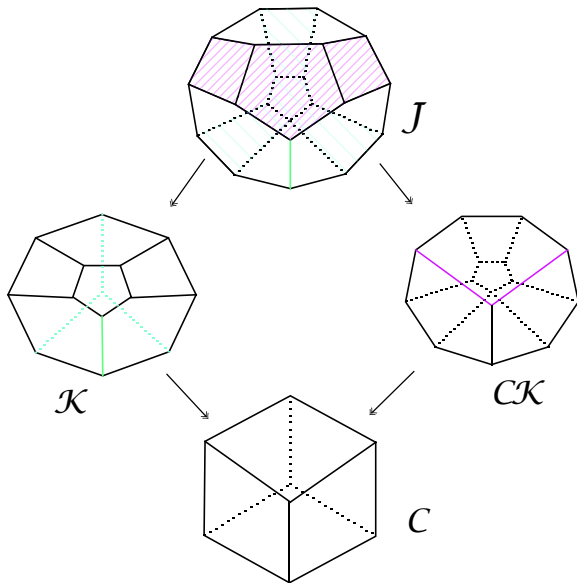
A small commuting diamond



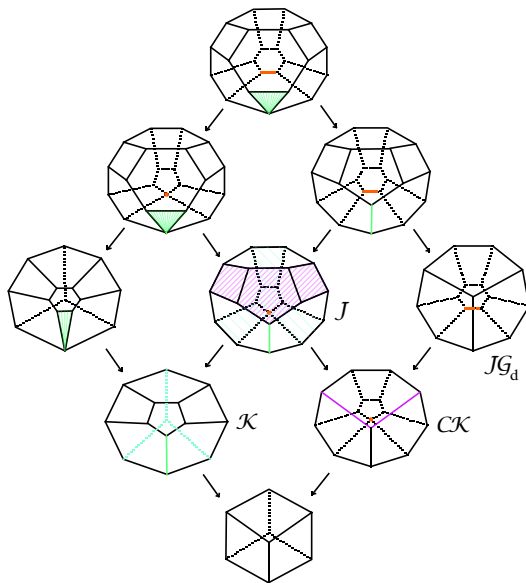
A small commuting diamond



A small commuting diamond



More polytopes.



Composition of coalgebras

Given two graded coalgebras we combine them in a way reminiscent of species composition.

Let \mathcal{C} and \mathcal{D} be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

$$\mathcal{D} \circ \mathcal{C} := \bigoplus_{n \geq 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes (n+1)}. \quad (1)$$

Examples

The motivating example is when \mathcal{C} and \mathcal{D} are spaces of rooted trees. Then \circ may be interpreted as some rule for grafting $n+1$ trees from \mathcal{C} onto the leaves of a tree in \mathcal{D}_n .

Example

Suppose $\mathcal{C} = \mathcal{D} = \mathcal{YSym}$ and consider some $(c_0, \dots, c_n) \times d \in (\mathcal{Y}^{n+1}) \times \mathcal{Y}_n$. Then defining \circ by grafting with color coding:



One sided Hopf algebras from compositions.

If our composition contains one of the Hopf operads, we get lots of free extra structure: a one-sided Hopf algebra, a Hopf module and a comodule algebra.

Examples from trees.

Here is an example of the coproduct in $\mathcal{YSym} \circ \mathcal{YSym}$:

$$\Delta \left(\text{Diagram 1} \right) = \left(\text{Diagram 2} \otimes \text{Diagram 3} \right) + \left(\text{Diagram 4} \otimes \text{Diagram 5} \right) + \left(\text{Diagram 6} \otimes \text{Diagram 7} \right) + \left(\text{Diagram 8} \otimes \text{Diagram 9} \right)$$

The diagram shows the coproduct Δ applied to a tree with 4 internal nodes (red) and 6 external nodes (blue). The result is a sum of four tensor products of trees. In each tensor product, one of the original tree's internal nodes is isolated as a single vertical red line, while the rest of the tree structure remains. The four terms correspond to isolating each of the four internal nodes in turn.

Here is an example of the product in $\mathcal{YSym} \circ \mathcal{YSym}$:

$$\left(\text{Diagram 1} \right) \cdot \left(\text{Diagram 2} \right) = \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6}$$

The diagram shows the product \cdot applied to the same tree from the coproduct example and a single vertical red line (representing a tree with one internal node). The result is a sum of four trees, each with 5 internal nodes (red) and 6 external nodes (blue). Each term in the sum represents a different way of inserting the single internal node from the second tree into the structure of the first tree.

Combs of combs

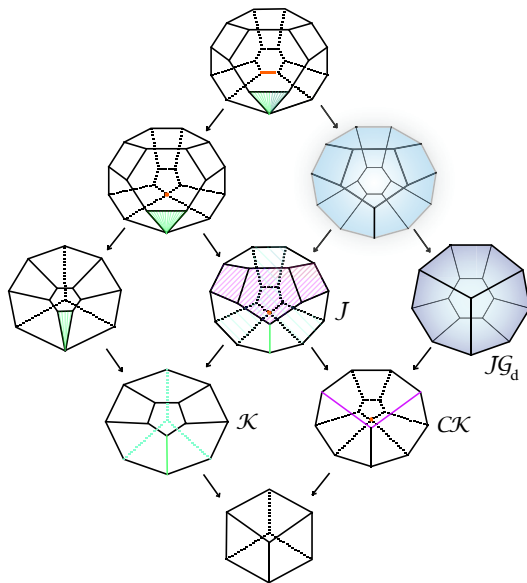
The coproduct is the usual splitting of trees:

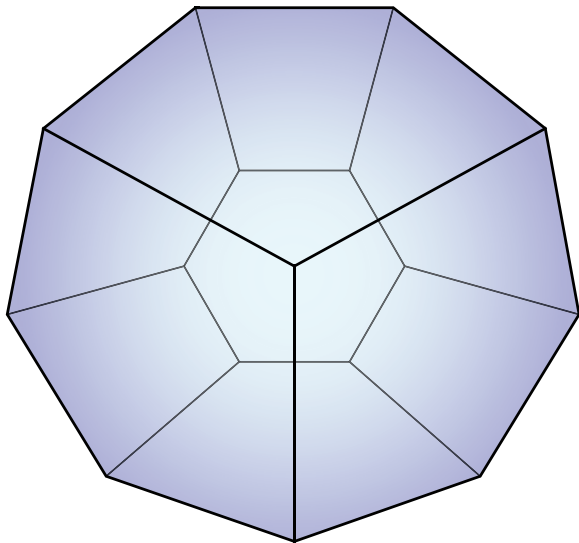
$$\begin{aligned}
 \Delta F_{13} &= \\
 \Delta \text{ (tree with root 1, children 2,3) } &= \text{ (root 1) } \otimes \text{ (tree with root 2, children 3,4) } + \text{ (tree with root 2, children 3,4) } \otimes \text{ (root 1) } \\
 &\quad + \text{ (tree with root 1, children 2,3) } \otimes \text{ (root 4) } + \text{ (tree with root 1, children 2,3) } \otimes \text{ (root 5) } \\
 &= F_1 \otimes F_{13} + F_{11} \otimes F_3 + F_{12} \otimes F_2 + F_{13} \otimes F_1
 \end{aligned}$$

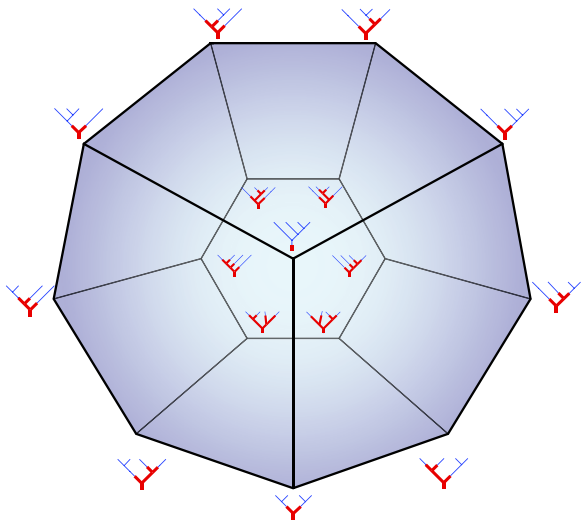
Here is the product:

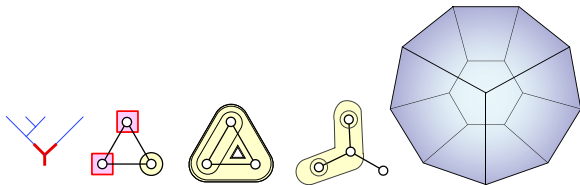
$$\begin{aligned}
 F_{13} \bullet F_2 &= \text{ (tree with root 1, children 2,3) } \bullet \text{ (root 4) } = \text{ (tree with root 1, children 2,3,4) } + \text{ (tree with root 1, children 2,3,4) } \\
 &\quad + \text{ (tree with root 1, children 2,3,4) } + \text{ (tree with root 1, children 2,3,4) } \\
 &= F_{113} + F_{113} + F_{122} + F_{131}
 \end{aligned}$$

More polytopes.









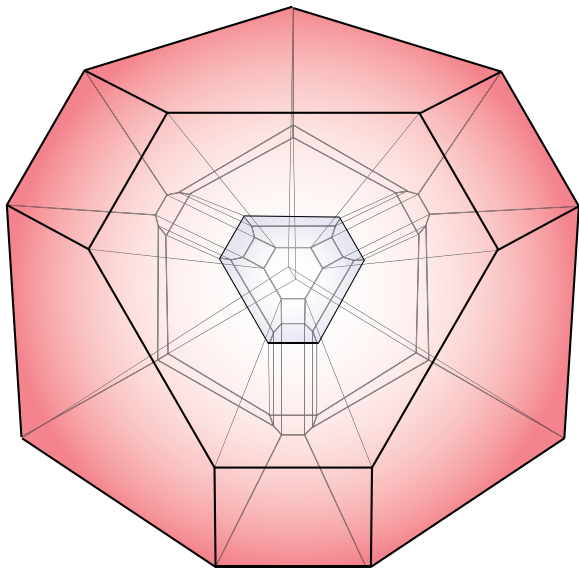
This polytope has been seen before!

Stellohedron = Complete-graph-cubeahedron

Number of vertices =

$$\sum_{k=0}^n \frac{n!}{k!}$$

Thanks!



Thanks!

Questions and comments?