Combinatorial n-fold monoidal categories and n-fold operads

July 29, 2005

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Abstract

Operads were originally defined as $\mathcal{V}$-operads, that is, enriched in a symmetric or braided monoidal category $\mathcal{V}$. The symmetry or braiding in $\mathcal{V}$ is required in order to describe the associativity axiom the operads must obey, as well as the associativity that must be a property of the action of an operad on any of its algebras. A sequence of categorical types that filter the category of monoidal categories and monoidal functors is given by Balteanu, Fiedorowicz, Schwänzl and Vogt in [2]. These subcategories of **MonCat** have objects that are called n-fold monoidal categories. A k–fold monoidal category is n-fold monoidal for all $n \leq k$, and a symmetric monoidal category is n-fold monoidal for all n. After a review of the role of operads in loop space theory and higher categories we go over definitions of iterated monoidal categories and introduce the lower branches of an extended family tree of simple examples. Then we generalize the definition of operad by defining n-fold operads and their algebras in an iterated monoidal category. It is seen that the interchanges in an iterated monoidal category are the natural requirement for expressing operad associativity. The definition is developed from the starting point of iterated monoids in a category of collections. Since monoids are special cases of enriched categories this allows us to describe the iterated monoidal and higher dimensional categorical structure of iterated operads. We show that for $\mathcal{V}$ k-fold monoidal the structure of a $(k-n)$-fold monoidal strict n-category is possessed by the category of n-fold operads in $\mathcal{V}$. We discuss examples of these operads that live in the previously described categories. Finally we describe the algebras of n-fold $\mathcal{V}$-operads and their products.

keywords: operads, iterated monoidal categories
amsclass: 18D10; 18D20

Contents

1 Introduction

In this introductory section we will give a brief, non-chronological overview of the relationship between operads, higher category theory, and topology. This will serve to motivate the study of iterated monoidal categories and their operads that comprises the remaining sections. In the second section, in order to be self contained, we repeat the definition of the iterated monoidal categories first set down in [2]. In the third section we seek to fill a gap in the literature which currently contains few good examples of that definition Thus our first new contribution consists of a series of simple and very geometric n-fold monoidal categories based on totally ordered monoids or semigroups. By simple we mean that axioms are largely fulfilled due to relationships between max, plus, concatenation, sorting and lexicographic ordering as well as the fact that all diagrams commute when the underlying graph is merely the total order. The adjective combinatorial in the title refers to the various categories based on the nonnegative integers (hereafter refererred to as natural numbers), whose objects can be represented by Ferrer diagrams. These often exhibit products with the geometrical interpretation "combining stacks of boxes." The second new contribution is the theory of operads within, or enriched in, iterated monoidal categories. The presentation of definitions of these operads and their algebras is accompanied by examples living in the previously introduced combinatorically defined categories. Under the conviction that these n-fold operads are worthy of study in their own right, we finally spend time investigating the structure of the category they comprise. Before any of this though we motivate with some ideas about their possible applications.

Operads in a category of topological spaces are the crystallization of several approaches to the recognition problem for iterated loop spaces. Beginning with Stasheff's associahedra and Boardman and Vogt's little n-cubes, and continuing with more general A_∞, E_n and E_∞ operads described by May and others, that problem has largely been solved. [24], [8], [20] Loop spaces are characterized by admitting an operad action of the appropriate kind. More lately Batanin's approach to higher categories through internal and higher operads promises to shed further light on the combinatorics of E_n spaces. [5], [6]

Recently there has also been growing interest in the application of higher dimensional structured categories to the characterization of loop spaces. The program being advanced by many categorical homotopy theorists seeks to model the coherence laws governing homotopy types with the coherence axioms of structured n-categories. By modeling we mean a connection that will necessarily be in the form of a functorial equivalence between categories of special categories and categories of special spaces. The largest challenges currently are to find the most natural and efficient definition of (weak) n-category, and to find the right sort of connecting n-functor. The latter will almost certainly be analogous to the nerve functor on 1-categories, which preserves homotopy equivalence. In [25] Street defines the nerve of a strict n-category. Recently Duskin in [9] has worked out the description of the nerve of a bicategory.

One major recent advance is the discovery of Balteanu, Fiedorowicz, Schwänzl and Vogt in [2] that the nerve functor on categories gives a direct connection between iterated monoidal categories and iterated loop spaces. Stasheff [24] and Mac Lane [18] showed that monoidal categories are precisely analogous to 1-fold loop spaces. There is a similar connection between symmetric monoidal categories and infinite loop spaces. The first step in filling in the gap between 1 and infinity was made in [10] where it is shown that the group completion of the nerve of a braided monoidal category is a 2-fold loop space. In [2] the authors finished this process by, in their words, "pursuing an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces." The first thing they focus on is the fact that a braided category is a special case of a carefully defined 2-fold monoidal category. Based on their observation of the correspondence between loop spaces and monoidal categories, they iteratively define the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. In [2] a symmetric category is seen as a category that is n-fold monoidal for all n. The main result in that paper is that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. It is still open whether this is a complete characterization, that is, whether every n-fold loop space arises as the nerve of an n-fold category.

The connection between the n-fold monoidal categories of Fiedorowicz and the theory of higher categories is through Baez's periodic table.[1] Here Baez organizes the k-tuply monoidal n-categories, by which terminology he refers to $(n+k)$-categories that are trivial below dimension k. The triviality of lower cells allows the higher ones to compose freely, and thus these special cases of $(n+k)$-categories are viewed as n-categories with k products. Of course a k-tuply monoidal n-category is a special k-fold monoidal n-category. The specialization results from the definition(s) of n-category, all of which seem to include the axiom that the interchange transformation between two ways of composing four higher morphisms along two different lower dimensions is required to be an isomorphism. As will be mentioned in the next section the property of having iterated loop space nerves held by the k-fold categories relies on interchange transformations that are not isomorphisms. If those transformations are indeed isomorphisms then the k-fold 1-categories do reduce to the braided and symmetric 1-categories of the periodic table. Whether this continues for higher dimensions, yielding for example sylleptic monoidal 2-categories as 3-fold 2-categories with interchange isomorphisms, is yet to be determined.

A further refinement of higher categories is to require all morphisms to have inverses. These special cases are referred to as n-groupoids, and since their nerves are simpler to describe it has been long known that they model homotopy n-types. A homotopy n-type is a topological space X for which $\pi_k(X)$ is trivial for all $k > n$. Thus the homotopy n-types are classified by π_n. It has been suggested that a key requirement for the eventual accepted definition of n-category is that a k-tuply monoidal n-groupoid be associated functorially (by a nerve) to a topological space which is a homotopy n-type k-fold loop space. [1] The loop degree will be precise for $k < n+1$, but for $k > n$ the associated homotopy n-type will be an infinite loop space. This last statement is a consequence of the stabilization hypothesis , which states that there should be a left adjoint to forgetting monoidal structure that is an equivalence of $(n+k+2)$ categories between k-tuply monoidal n-categories and $(k+1)$-tuply monoidal n-categories for $k > n+1$. For the case of $n = 1$ if the interchange transformations are isomorphic then a k-fold (and k-tuply) monoidal 1-category is equivalent to a symmetric category for $k > 2$. With these facts in mind it is clear that if we wish to precisely model homotopy n-type k-fold loop spaces for $k > n$ then we need to consider k-fold as well as k-tuply monoidal n-categories. This paper is part of an embryonic effort in that direction.

Since a loop space can be efficiently described as an operad algebra, it is not surprising that there are several existing definitions of n-category that utilize operad actions. These definitions fall into two main classes: those that define an n-category as an algebra of a higher order operad, and those that achieve an inductive definition using classical operads in symmetric monoidal categories to parameterize iterated enrichment. The first class of definitions is typified by Batanin and Leinster. [4],[16] The former author defines monoidal globular categories in which interchange transformations are isomorphisms and which thus resemble free strict n-categories. Globular operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string of objects as in the case of classical operads. The binary composition in an n-category derives from the action of a certain one of these globular operads. Leinster expands this concept to describe n-categories with unbiased composition of any number of cells. The second class of definitions is typified by the works of Trimble and May. [26], [21] The former parameterizes iterated enrichment with a series of operads in $(n-1)$-Cat achieved by taking the fundamental $(n-1)$-groupoid of the kth component of the topological path composition operad E. The latter begins with an A_∞ operad in a symmetric monoidal category $\mathcal{V}$ and requires his enriched categories to be tensored over $\mathcal{V}$ so that the iterated enrichment always refers to the same original operad.

Iterated enrichment over n-fold categories is described in [11] and [12]. We would like to define n-fold operads in n-fold monoidal categories in a way that is consistent with the spirit of Batanin's globular operads, and with the eventual goal of using them to weaken enrichment over n-fold categories in a way that is in the spirit of Trimble. This program carries with it the promise of characterizing k-fold loop spaces with homotopy n-type for all n, k.

This paper comprises a naive beginning, illustrated with a very basic set of examples that we hope will help clarify the definitions. First we present the definition of n-fold monoidal category and go over a collection of related examples from semigroup theory. The examples of most visual value are from the combinatorial theory of tableau shapes. Secondly we present constructive definitions of n-fold operads and their algebras in an iterated monoidal category. We discuss examples of these that live in the previously described categories. Finally we describe the $(n-2)$-fold monoidal category of n-fold $\mathcal{V}$-operads.

A more abstract approach for future consideration would begin by finding an equivalent definition of n-fold operad in terms of monoids in a certain category. Then the full abstraction would be to find an equivalent definition in the language of Weber, where an operad lives within a monoidal pseudo algebra of a 2-monad. [27] This latter is a general notion of operad which includes as instances both classical and higher operads.

2 k-fold monoidal categories

This sort of category was developed and defined in [2]. The authors describe its structure as arising recursively from its description as a monoid in the category of $(k-1)$-fold monoidal categories. Here we present that definition (in its expanded form) altered only slightly to make visible the coherent associators as in [11]. That paper describes its structure in terms of tensor objects in the category of $(k-1)$-fold monoidal categories. This variation has the effect of making visible the associators α^i_{ABC}. It is desirable to do so for several reasons. One is that this makes easier a direct comparison with Batanin's definition of monoidal globular categories as in [4]. A monoidal globular category is a quite special case of a iterated monoidal category, with source and target maps that take objects to those in a category with one less product, and with interchanges that are isomorphisms. Another reason is that when we investigate delooping functors from the category of n-fold monoidal categories to the category of $(n-1)$-fold monoidal 2-categories, the functoriality of the 2-associator is implied by the external associativity axiom in the following definition [11]. If the associators are hidden then the full necessity of the axioms to the existence of delooping is also hidden. Finally, in this paper we will consider a category of collections in an iterated monoidal category which will be monoidal only up to natural associators. That being said, in much of the remainder of this paper we will consider examples with strict associativity, where each α is the identity, and in interest of clarity will often hide associators even in further definitions such as the expanded definition of iterated operad. One actual simplification in this defintion is that all the products are assumed to have the same unit. We note that this is easily generalized, as in the case of collections which we will consider.

Definition 1 *An n-fold monoidal category is a category $\mathcal{V}$ with the following structure.*

1. *There are n multiplications*

$$\otimes_1, \otimes_2, \dots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

each equipped with an associator α_{UVW}, a natural transformation which satisfies the pentagon equation:

$$\begin{tikzcd}
 & ((U \otimes_i V) \otimes_i W) \otimes_i X \arrow[rr, "\alpha^i_{UVW} \otimes_i 1_X"] \arrow[dl, "\alpha^i_{(U\otimes_i V)WX}"'] & & (U \otimes_i (V \otimes_i W)) \otimes_i X \arrow[dr, "\alpha^i_{U(V\otimes_i W)X}"] & \\
(U \otimes_i V) \otimes_i (W \otimes_i X) \arrow[drr, "\alpha^i_{UV(W\otimes_i X)}"] & & & & U \otimes_i ((V \otimes_i W) \otimes_i X) \arrow[dll, "1_U \otimes_i \alpha^i_{VWX}"] \\
 & & U \otimes_i (V \otimes_i (W \otimes_i X)) & &
\end{tikzcd}$$

2. *$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.*

3. *For each pair (i, j) such that $1 \leq i < j \leq n$ there is a natural transformation*

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) *Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$*

(b) *External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$*

(c) *Internal associativity condition: The following diagram commutes*

$$\begin{tikzcd}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) \arrow[r, "\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}"] \arrow[d, "\alpha^i"] & \big((U \otimes_i W) \otimes_j (V \otimes_i X)\big) \otimes_i (Y \otimes_j Z) \arrow[d, "\eta^{ij}_{(U \otimes_i W)(V \otimes_i X)YZ}"] \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) \arrow[d, "1_{U \otimes_j V} \otimes_i \eta^{ij}_{WXYZ}"] & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \arrow[d, "\alpha^i \otimes_j \alpha^i"] \\
(U \otimes_j V) \otimes_i \big((W \otimes_i Y) \otimes_j (X \otimes_i Z)\big) \arrow[r, "\eta^{ij}_{UV(W \otimes_i Y)(X \otimes_i Z)}"] & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{tikzcd}$$

(d) *External associativity condition: The following diagram commutes*

$$\begin{tikzcd}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) \arrow[r, "\eta^{ij}_{(U \otimes_j V)W(X \otimes_j Y)Z}"] \arrow[d, "\alpha^j \otimes_i \alpha^j"] & \big((U \otimes_j V) \otimes_i (X \otimes_j Y)\big) \otimes_j (W \otimes_i Z) \arrow[d, "\eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}"] \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) \arrow[d, "\eta^{ij}_{U(V \otimes_j W)X(Y \otimes_j Z)}"] & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\alpha^j"] \\
(U \otimes_i X) \otimes_j \big((V \otimes_j W) \otimes_i (Y \otimes_j Z)\big) \arrow[r, "1_{U \otimes_i X} \otimes_j \eta^{ij}_{VWYZ}"] & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{tikzcd}$$

(e) *Finally it is required for each triple (i, j, k) satisfying $1 \leq i < j < k \leq n$ that the giant hexagonal interchange diagram commutes.*

$$
\begin{tikzcd}[column sep=tiny]
 & ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}"'] \arrow[dr, "\eta^{ij}_{(A\otimes_k A')(B\otimes_k B')(C\otimes_k C')(D\otimes_k D')}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\eta^{ik}_{(A\otimes_j B)(A'\otimes_j B')(C\otimes_j D)(C'\otimes_j D')}"'] & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\eta^{jk}_{(A\otimes_i C)(A'\otimes_i C')(B\otimes_i D)(B'\otimes_i D')}"] \\
 & ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}
$$

Note that for $q > p$ we have natural transformations

$$\eta^{pq}_{AIIB} : A \otimes_p B \to A \otimes_q B \qquad \text{and} \qquad \eta^{pq}_{IABI} : A \otimes_p B \to B \otimes_q A.$$

If the authors of [2] had insisted a 2-fold monoidal category be a tensor object in the category of monoidal categories and *strictly monoidal* functors, this would be equivalent to requiring that $\eta = 1$. In view of the above, they note that this would imply $A \otimes_1 B = A \otimes_2 B = B \otimes_1 A$ and similarly for morphisms.

Joyal and Street [13] considered a similar concept to Balteanu, Fiedorowicz, Schwänzl and Vogt's idea of 2–fold monoidal category. The former pair required the natural transformation η_{ABCD} to be an isomorphism and showed that the resulting category is naturally equivalent to a braided monoidal category. As explained in [2], given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\otimes_2$, and defining the commutativity isomorphism for the remaining operation $\otimes_1$ to be the composite

$$A \otimes_1 B \xrightarrow{\eta_{IABI}} B \otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B \otimes_1 A.$$

The authors of [2] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \ldots = \otimes_n = \otimes$$

and defining

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation.

Joyal and Street [13] require that the interchange natural transformations η^{ij}_{ABCD} be isomorphisms and observed that for $n \geq 3$ the resulting sort of category is equivalent to a symmetric monoidal category. Thus as Balteanu, Fiedorowicz, Schwänzl and Vogt point out, the nerves of such categories have group completions which are infinite loop spaces rather than only n–fold loop spaces.

Because of the recursive nature of the definition of iterated monoidal category, there are multiple forgetful functors implied. Specifically, letting $n < k$, from the category of k-fold monoidal categories to the category of n-fold monoidal categories there are $\begin{pmatrix} k \\ n \end{pmatrix}$ forgetful functors which forget all but the chosen set of products.

The coherence theorem for iterated monoidal categories states that any diagram composed solely of interchange transformations commutes; i.e. if two compositions of various interchange transformations (legs of a diagram) have the same source and target then they describe the same morphism. Furthermore we can easily determine when a composition of interchanges exists between objects. Here are the necessary definitions and Theorem as given in [2].

Definition 2 *Let $\mathcal{F}_n(S)$ be the free n-fold monoidal category on the finite set S. Its objects are all finite expressions generated by the elements of S using the products $\otimes_i, i = 1..n$. By $\mathcal{M}_n(S)$ we denote the sub-category of $\mathcal{F}_n(S)$ whose objects are expressions in which each elemnt of S occurs exactly once.*

If $S \subset T$ then there is a restriction functor $\mathcal{M}_n(T) \to \mathcal{M}_n(S)$, induced by the functor $\mathcal{F}_n(T) \to \mathcal{F}_n(S)$, which sends $T - S$ to the empty expression 0.

Definition 3 *Let A be an object of $\mathcal{M}_n(S)$. For $a, b \in S$ we say that $a \otimes_i b$ in A if the restriction functor $\mathcal{M}_n(S) \to \mathcal{M}_n(a, b)$ sends A to $a \otimes_i b$.*

Theorem 1 *[2] Let A and B be objects of $\mathcal{M}_n(S)$.*

1. *Then there is at most one morphism $A \to B$.*

2. *Moreover there exists a morphism $A \to B$ if and only if:*

   for any $a, b \in S$ if $a \otimes_i b$ in A then there must be in B either $a \otimes_j b$ for some $j \geq i$ or $b \otimes_j a$ for some $j > i$.

3 Examples of iterated monoidal categories

Lemma 1 *Given a totally ordered set S with a smallest element $e \in S$, then the elements of S make up the objects of a strict monoidal category.*

The category will also be denoted S. Its morphisms are given by the ordering; there is only an arrow $a \to b$ if $a \leq b$. The product is max and the 2-sided unit is the least element e. We must check that the product is functorial since this defines monoidal structure on morphisms. Here it is so since if $a \leq b$ and $a' \leq b'$ then $\max(a, a') \leq \max(b, b')$. Also the identity is clearly preserved.

Example 1

The basic example is the nonnegative integers (natural numbers) $\mathbf{N}$ with their ordering $\leq$.

Lemma 2 *Any ordered monoid with its identity element e also its least element forms the object set of a 2-fold monoidal category.*

Proof: Morphisms are again given by the ordering. The products are given by max and the semigroup operation: $a \otimes_1 b = \max(a, b)$ and $a \otimes_2 b = ab$. The shared two-sided unit for these products is the identity element 0. The products are both strictly associative and functorial since if $a \leq b$ and $a' \leq b'$ then $aa' \leq bb'$ and $\max(a, a') \leq \max(b, b')$. The interchange natural transformations exist since $\max(ab, cd) \leq \max(a, c) \max(b, d)$. This last theorem is easily seen by checking the four possible cases: $\{c \leq a, d \leq b\}$; $\{c \leq a, b \leq d\}$; $\{a \leq c, d \leq b\}$; $\{a \leq c, b \leq d\}$; or by the quick argument that

$a \leq max(a, c)$ and $b \leq max(b, d)$ so

$ab \leq max(a, c)max(b, d)$ and similarly

$cd \leq max(a, c)max(b, d)$.

The internal and external unit and associativity conditions of Definition 1 are all satisfied due to the fact that there is only one morphism between two objects. More generally, given any ordered n-fold monoidal category with I the least object we can potentially form an $(n+1)$-fold monoidal category with morphisms ordering, and the new $\otimes_1 = \max$.

Example 2

Again we have in mind $\mathbf{N}$ with its ordering and addition.

Other examples of such semigroups as in Lemma 2 are the pure braids on n strands with only right-handed crossings [15]. Notice that braid composition is a non-symmetric example. Further examples are found in the papers on semirings and idempotent mathematics, such as [17] and its references as well as on the related concept of tropical geometry, such as [23] and its references. Semirings that arise in these two areas of study usually require some translation of the lemmas we have stated thus far, since the idempotent operation is usually min and its unit ∞. Since the operation given by addition has unit 0 we have to broaden our definition of 2-fold monoidal category. Working from the principle that the second operation is the muliplication of a categorical monoid with respect to the first, the additional requirement is that the two distinct units obey each other's operations: i.e $I_1 \otimes_2 I_1 = I_1$ and $I_2 \otimes_1 I_2 = I_2$. For example, $\min(0,0) = 0$ and $\infty + \infty = \infty$.

Example 3

If S is an ordered set then by $\mathrm{Seq}(S)$ we denote the infinite sequences X_n of elements of S for which there exists a natural number $l(X)$ such that $k > l(X)$ implies $X_k = e$ and $X_{l(X)} \neq e$. Under lexicographic ordering $\mathrm{Seq}(S)$ is in turn a totally ordered set with a least element. The latter is the sequence 0 where $0_n = e$ for all n. We let $l(0) = 0$. The lexicographic order means that $A \leq B$ if either $A_k = B_k$ for all k or there is a natural number $n = n_{AB}$ such that $A_k = B_k$ for all $k < n$, and such that $A_n < B_n$.

The ordering is easily shown to be reflexive, transitive, and antisymmetric. See for instance [22] where the case of lexicographic ordering of n-tuples of natural numbers is discussed. In our case we will need to modify the proof by always making comparisons of $\max(l(A), l(B))$-tuples.

As a category $\mathrm{Seq}(S)$ is 2-fold monoidal since we can demonstrate two interchanging products: max using the lexicographic order– $A \otimes_1 B = \max(A, B)$; and concatenation of sequences. Here $(A \otimes B)_n = A_n, n \leq l(A)$ and $= B_n, n > l(A)$. Concatenation clearly preserves the ordering. This is actually the free monoid on the ordered set S.

Example 4

Letting S be the set with a single element recovers Example 2.

Lemma 3 *If we have an ordered monoid $(M, +)$ as in Lemma 2 and reconsider Seq(M) as in Example 3 then we can describe a 3-fold monoidal category Seq$(M, +)$ (with Seq(M) the image of forgetting the third product) iff the monoid operation $+$ is such that $0 < a < b$ and $c \leq d$ implies both $a + c < b + d$ and $c + a < d + b$ strictly.*

Proof: The first two products are again lexicographic max and concatenation of sequences. The third product $\otimes_3$ is pointwise application of $+$, $(A \otimes_3 B)_n = A_n + B_n$. The last condition that the monoid operation $+$ strictly respect strict ordering is necessary to guarantee that the third product both respect the lexicographic ordering and interchange corectly with concatenation. To see the former let sequences $A \leq B, C \leq D$. Note that if $A = B, C = D$ then $A \otimes_3 C = B \otimes_3 D$ and if instead (without loss of generality) $A_j < B_j$ for j such that $A_i = B_i$ and $C_i = D_i$ for $i \leq j$, then $A \otimes_3 C < B \otimes_3 D$, since $C_j \leq D_j$. To see the converse consider a case where $0 < a < b$ and $c \leq d$ but $a + c = b + d$. Then the sequences $A = (a, a), B = (b, 0), C = (c, 0), D = (d, 0)$ are such that lexicographically $A < B$ and $C \leq D$ but $A \otimes_3 C = (a+c, a) > B \otimes_3 D = (b+d, 0)$. To see the interchange $\text{concat}(A+B, C+D) \leq \text{concat}(A, C) + \text{concat}(B, D)$ notice that we can assume that $l(A) > l(B)$. Then $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$ due to the fact that if D has a first non-zero term, it will be added to an earlier term of the concatenation of A and C in the second four-way product.

Remark 1

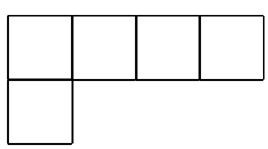
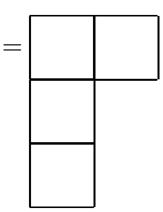
A non-example is seen if we begin with the monoid of Lemma 1, an ordered set with a least element where the product is max. Here max does not strictly preserve strict ordering, and so pointwise max does not respect lexicographic ordering. Neither do concatenation and pointwise max interchange.

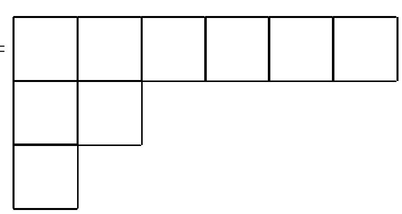
Corollary 1 *Given any ordered n-fold monoidal category C with I the least object and $\otimes_1$ the max, and whose higher products strictly respect strict ordering, we can form an $(n+1)$-fold monoidal category Seq(C).*

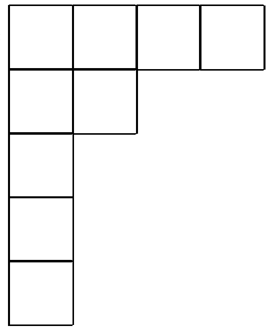
The new products of $\text{Seq}(C)$ are the lexicographic max, the concatenation, and the pointwise application of $\otimes_i$ for $i = 2 \ldots n$. The pointwise application of the original products to the sequences directly inherits the interchange properties. For instance, if $A, B, C, D \in \text{Seq}(C)$ then $(A_n \otimes_2 B_n) \otimes_1 (C_n \otimes_2 D_n) \leq (A_n \otimes_1 C_n) \otimes_2 (B_n \otimes_1 D_n)$ for all n, which certainly implies that the pointwise 4-way products are ordered lexicographically.

Example 5

Even more symmetrical structure is found in examples with a natural geometric representation which allows the use of addition in each product. One such category is that whose objects are Ferrers diagrams, by which we mean the underlying shapes or diagrams of Young tableaux. These can be presented by a decreasing sequence of nonnegative integers in two ways: the sequence that gives the heights of the columns or the sequence that gives the lengths of the rows. We let $\otimes_2$ be the product which adds the heights of columns of two tableaux, $\otimes_1$ adds the length of rows. We often refer to these as vertical and horizontal stacking respectively.

$A =$  and $B =$ 

then $A \otimes_1 B =$ 

and $A \otimes_2 B =$ 

There are several possibilities for morphisms. We can take as morphisms the totally ordered structure of the tableau diagrams given by lexicographic ordering. In interest of focusing on the stacking products though we often choose to restrict these morphisms further, and say an arrow given by ordering can only exist between similar mass objects, i.e. the two objects in question have equal sums of their respective sequences or, in reference to the pictures, an equal total number of blocks. This restriction eliminates the product described by lexicographic max. By the category of 2-d block castles we will refer to the object set of tableau diagrams, morphisms as restricted lexicographic ordering, and the two stacking products demonstrated above. We will also find occasion to relax the morphisms to include all ordering and reintroduce the lexicographic max as $\otimes_1$, and will refer to that category simply as the tableau diagrams. By previous discussion of sequences the tableau diagrams with $\otimes_1$ the lexicographic max and $\otimes_3$ the piecewise addition (thought of here as vertical stacking) form a subcategory of the 3-fold monoidal category called Seq($\mathbf{N}$,+). To see that with the additional $\otimes_2$ of horizontal stacking that this becomes a valid 3-fold monoidal category we look at that operation from another point of view. Note that the horizontal product of tableau diagrams A and C can be described as a reorganization of all the columns of both A and C into a new tableau diagram made up of those columns in descending order of height. Rather than (but equivalent to) the addition of rows, we see horizontal stacking as the concatenation of monotone decreasing sequences (of columns) followed by sorting greatest to least. We call this operation merging.

Lemma 4 *Let $(S, \leq, +)$ be an ordered semigroup and consider Seq$(S, +)$with lexicographic ordering, piecewise addition $+$ and the function of sorting denoted by s:Seq$(S, +)$ $\rightarrow$Seq$(S, +)$. Then the triangle inequality holds for two sequences: $s(A + B) \leq s(A) + s(B)$.*

Proof: Consider $s(A + B)$, where we start with the two sequences and add them piecewise before sorting. We can metamorphose this into $s(A) + s(B)$ in stages by using an algorithm to sort A and B. Note that if A and B are already sorted, the inequality becomes an equality. For our algorithm we choose parallel bubble sorting. This consists of a series of passes through the sequences comparing A_n and A_{n+1} and comparing B_n and B_{n+1} simultaneously. If the two elements of a given sequence are not already in strictly decreasing order we switch their places. We claim that switching consecutive sequence elements into order always results in a lexicographically larger sequence after adding piecewise and sorting. If both the elements of A and of B are switched, or if neither, then the result is unaltered. Therefore without loss of generality we assume that $A_n < A_{n+1}$ and that $B_{n+1} < B_n$. Then we compare the original result of sorting after adding and the same but after the switching of A_n and A_{n+1}. It is simplest to note that the new result includes $A_{n+1} + B_n$, which is larger than both $A_n + B_n$ and $A_{n+1} + B_{n+1}$. So after adding and sorting the new result is indeed larger lexicographically. Thus since each move of the parallel bubble sort results in a larger expression after first adding and then sorting, and after all the moves the result of adding and then sorting the now presorted sequences is the same as first sorting then adding, the triangle inequality follows.

Theorem 2 *The category of 2-d block castles forms a 2-fold monoidal category, and the category of tableau diagrams forms a 3-fold monoidal category.*

Proof: We show the latter statement is true, and then note that the the former statement follows since the block castles are the image of forgetting the first product on tableau diagrams and then passing to a subcategory by restricting morphisms. The products on tableau diagrams are $\otimes_1$ = lexicographic max, $\otimes_2$ = horizontal stacking and $\otimes_3$ = vertical stacking. We need to check first that horizontal stacking, or merging, is functorial with respect to morphisms (defined as the $\leq$ relations of the lexicographic ordering.) The cases where $A = B$ or $C = D$ are easy. For example let $A_k = B_k$ for all k and $C_k = D_k$ for all $k < n_{CD}$ Thus the columns from the copies of, for instance A in $A \otimes_1 C$ and $A \otimes_1 D$ fall into the same final spot under the sortings right up to the critical location, so if $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D$. Similarly, it is clear that $A \leq B$ implies $(A \otimes_1 D) \leq (B \otimes_1 D)$. Hence if $A \leq B$ and $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D \leq B \otimes_1 D$ which by transitivity gives us our desired property.

Next we check that our interchange transformations will always exist. η^{1j} exists by the proof of Lemma 2 for $j = 2, 3$ since the higher products both respect morphisms(ordering) and are thus ordered semigroup operations. We need to check for existence of η^{23}, i.e. we need to show that $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$. This inequality follows from Lemma 4 on the triangle inequality for sorting. To prove the new inequality we consider the special case of two sequences formed by letting A' be A followed by C and letting B' be B followed by D. By "followed by" we mean padded by zeroes so that $l(A') = \max(l(A), l(B)) + l(C)$ and $l(B') = \max(l(A), l(B)) + l(D)$. Thus piecewise addition of A' and B' results in piecewise addition of A and B, and respectively C and D. Then to our new sequences A' and B' we apply the result of Lemma 4 and have our desired result.

Remark 2

Alternatively we can create a category equivalent to the non-negative integers in Example 1 by pre-ordering the tableaux by height. Here the height $h(A)$ of the tableau is the number of boxes in its leftmost column, and we say $A \leq B$ if

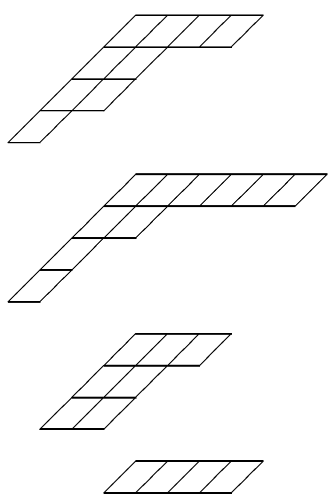
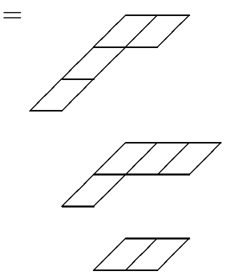
$h(A) \leq h(B)$. Two tableaux with the same height are isomorphic objects, and the one-column stacks form both a full subcategory and a skeleton of the height preordered category. Everything works as for the previous example of natural numbers since $h(A \otimes_2 B) = h(A) + h(B)$ and $h(A \otimes_1 B) = \max(h(A), h(B))$. There is also a max product. The new max with respect to the height preordering is defined as $\max(A, B) = A$ if $B \leq A$ and $= B$ otherwise. In the height preordered category this latter product is equivalent to the horizontal stacking $\otimes_1$.

Remark 3

Notice that we can start with any totally ordered semigroup $\{M, \leq, +\}$ such that the identity 0 is less than any other element and such that $0 < a < b$ and $c \leq d$ implies both $a + c < b + d$ and $c + a < d + b$ for all $a, b, c \in G$. We create a 3-fold monoidal category ModSeq$(M, +)$ with objects monotone decreasing finitely non-zero sequences of elements of M and morphisms given by the lexicographic ordering. The products are as described for the category of tableau diagrams (ModSeq($\mathbf{N}$,+)) in the previous example. The common unit is the zero sequence. The proofs we have given in the previous example for $M = \mathbf{N}$ are all still valid.

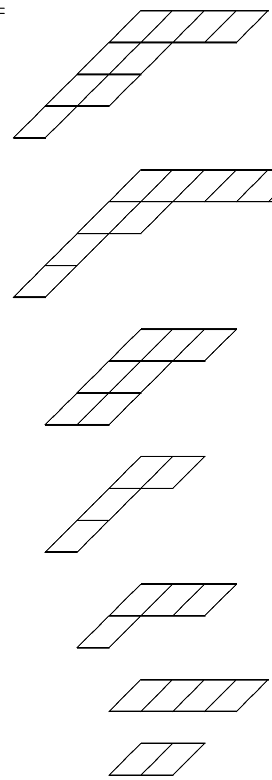
By Corollary 1 we can also consider 4-fold monoidal categories such as Seq(ModSeq(M)) and other combinations of Seq and ModSeq.

For instance if ModSeq($\mathbf{N}$,+) is our category of tableau diagrams then ModSeq(ModSeq($\mathbf{N}$,+)) has objects monotone decreasing sequences of tableau diagrams, which we can visualize along the z-axis. Here the lexicographic-lexicographic max is $\otimes_1$, lexicographic merging is $\otimes_2$, pointwise merging (pointwise horizontal or y-axis stacking) is $\otimes_3$ and pointwise-pointwise addition (pointwise x-axis stacking) is $\otimes_4$. For example, if:

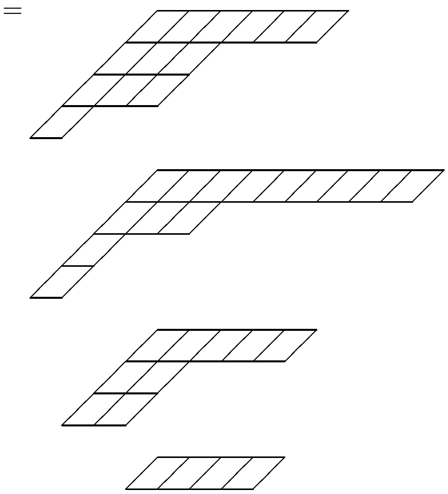
$A =$  and $B =$ 

then $A \otimes_1 B = A$ and

$A \otimes_2 B =$

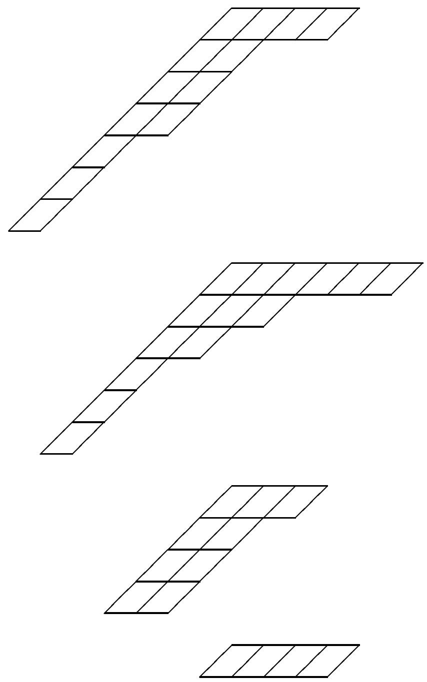


$A \otimes_3 B =$



and

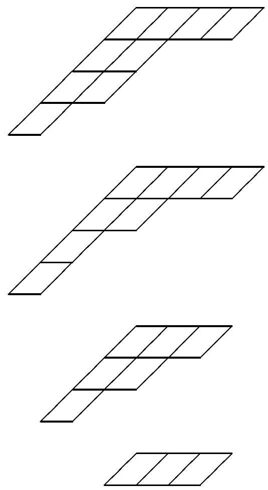
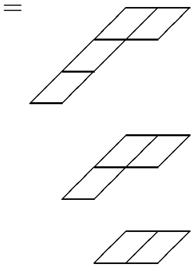
$$A \otimes_4 B =$$



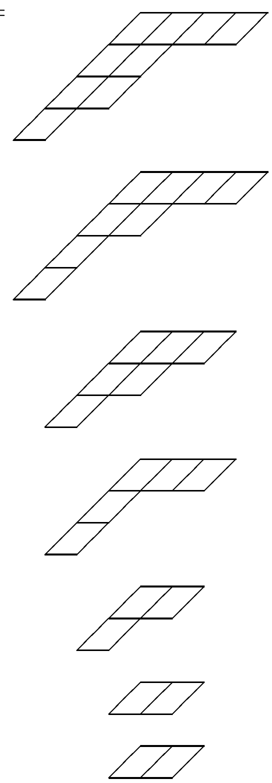
Example 6

It might be nice to retain the geometric picture of the products of tableau diagrams in terms of vertical and horizontal stacking, and stacking in other directions as dimension increases. This is not found in the just illustrated category, which relies on the merging viewpoint. The "tableau stacking" point of view is restored if we restrict to sequences of tableau diagrams that are decreasing in columns as well as rows. These are lexicographically decreasing sequences of decreasing sequences so already obey the requirement that A_{n1} is decreasing in n. We expand this to require that A_{nk} be decreasing in n for constant k, which implies decreasing in k for constant n. We can represent these objects as infinite matrices with finitely nonzero natural number entries, and with monotone decreasing columns and rows. We choose the sequence of rows to be the sequence of sequences, i.e. each row represents a tableau which we draw as being parallel to the xy plane. This choice is important because it determines the total ordering of matrices and thus the morphisms of the category. Thus y-axis stacking is horizontal concatenation (disregarding trailing zeroes) of matrices followed by sorting the new longer rows. x-axis stacking is addition of matrices. Now we define z-axis stacking as vertical concatenation of matrices followed by sorting the new long columns.

Here is a visual example of the three new products, beginning with z-axis stacking, labeled $\otimes_1$: if

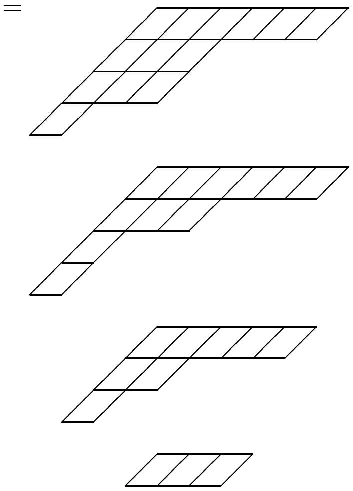
$A =$  and $B =$ 

then we let

$A \otimes_1 B =$ 

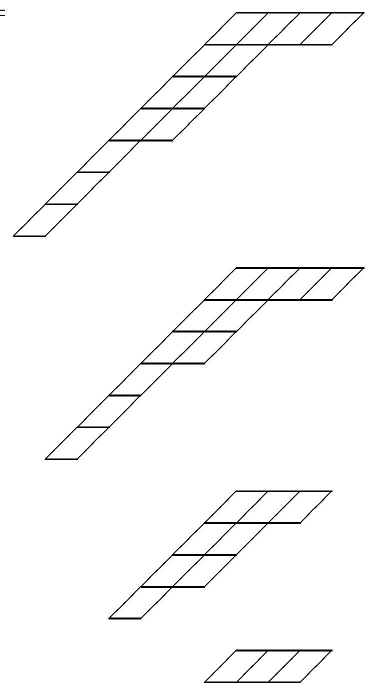
and

$A \otimes_2 B =$



and

$A \otimes_3 B =$



Note that in this restricted setting of decreasing matrices the lexicographic merging of sequences (rows) of two matrices does not preserve the total decreasing property (decreasing in rows and columns).

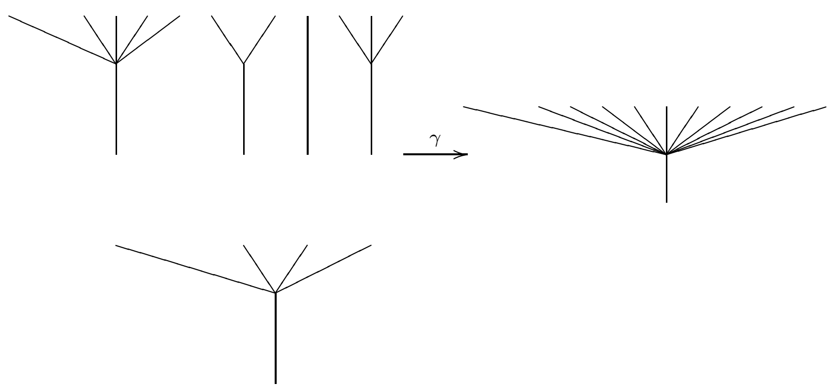
These three products just shown preserve the total sum of the entries in both matrices, and do interact via interchanges to form the structure of a 3-fold category. Renumbered they are: $\otimes_1$ (z-axis stacking) is the vertical concatenation of matrices followed by sorting the new longer columns, $\otimes_2$ (y-axis stacking) is horizontal concatenation of matrices followed by sorting the new longer rows and $\otimes_3$ (x-axis stacking) is the addition of matrices. By 3-d block castles we refer to the category that has these 3 products and morphisms given by the ordering restricted to similar mass relations.

Theorem 3 *3-d block castles and their associated products possess the sructure of a 3-fold monoidal category.*

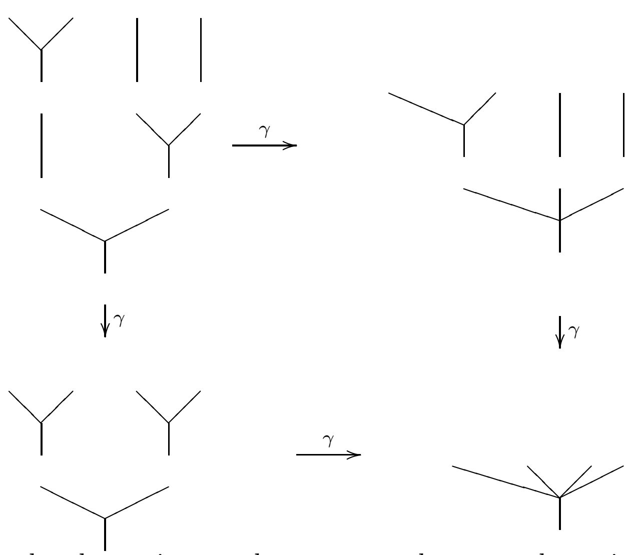
Proof: We already have existence of η^{23} by the previous argument about pointwise application of two interchanging products. To show existence of η^{13} : $(A \otimes_3 B) \otimes_1 (C \otimes_3 D) \to (A \otimes_1 C) \otimes_3 (B \otimes_1 D)$ we need to check that sorting each of the columns of two pairs of vertically concatenated matrices before pointwise adding gives a larger lexicographic result with respect to rows than adding first and then sorting columns. This follows from Lemma 4, applied to each pair of sequences which are the n^{th} columns in the two new matrices formed by vertically concatenating A and C and respectively B and D, padded with zeroes so that adding the new matrices results in adding A and B and respectively C and D. From the lemma then we have that $(A \otimes_1 C) \otimes_3 (B \otimes_1 D)$ gives a result whose n^{th} column is lexicographically greater than or equal to the n^{th} column of $(A \otimes_3 B) \otimes_1 (C \otimes_3 D)$. This implies that either the pairs of respective columns are each equal sequences or that there is some least row i and column j such that all the pairs of columns are identical in rows less than i and that the two rows i are identical in columns less than j, but that the i, j position in $(A \otimes_3 B) \otimes_1 (C \otimes_3 D)$ is less than the corresponding position in $(A \otimes_1 C) \otimes_3 (B \otimes_1 D)$. Thus the existence of the required inequality is shown. The existence of η^{12} is clear since we are ordering the matrices by giving precedence to the rows. Thus sorting columns first and then rows is guaranteed to give something larger lexicographically than sorting horizontally first.

4 n-fold operads

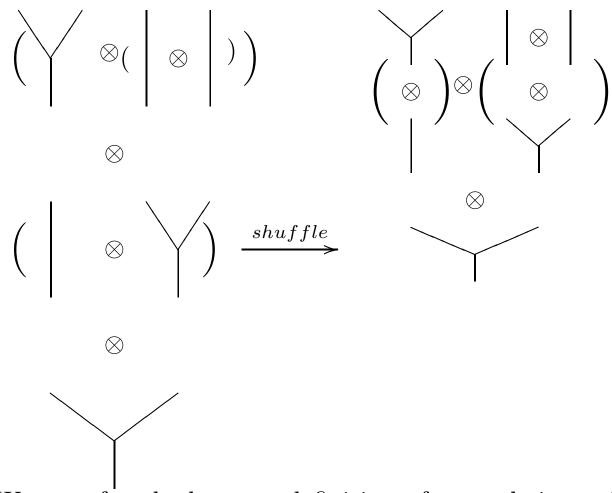
The two principle components of an operad are a collection, historically a sequence, of objects in a monoidal category and a family of composition maps. Operads are often described as paramaterizations of n-ary operations. Peter May's original definition of operad in a symmetric (or braided) monoidal category [20] has a composition γ that takes the tensor product of the nth object (n-ary operation) and n others (of various arity) to a resultant that sums the arities of those others. The nth object or n-ary operation is often pictured as a tree with n leaves, and the composition appears like this:



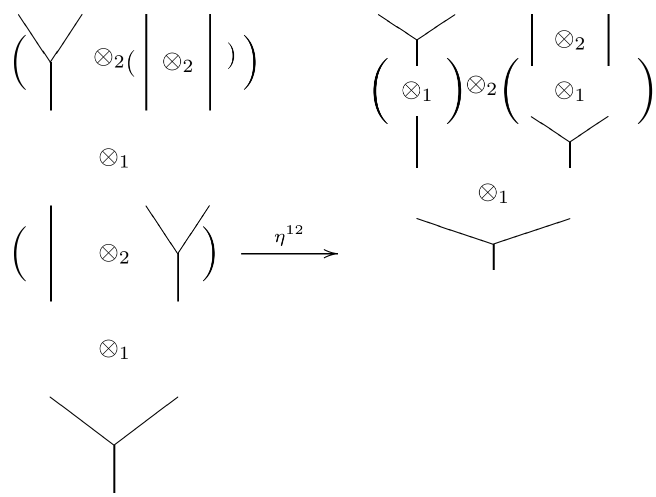
By requiring this composition to be associative we mean that it obeys this sort of pictured commuting diagram:



In the above pictures the tensor products are shown just by juxtaposition, but now we would like to think about the products more explicitly. If the monoidal category is not strict, then there is actually required another leg of the diagram, where the tensoring is reconfigured so that the composition can operate in an alternate order. Here is how that rearranging looks in a symmetric (braided) category, where the shuffling is accomplished by use of the symmetry (braiding):



We now foreshadow our definition of operads in an iterated monoidal category with the same picture as above but using two tensor products, $\otimes_1$ and $\otimes_2$. It now becomes clear that the true nature of the shuffle is in fact that of an interchange transformation.



To see this just focus on the actual domain and range of η^{12} which are the upper two levels of trees in the pictures, with the tensor product $(|\otimes_2|)$ considered as a single object.

Now we are ready to give the technical definitions. We begin with the definition of 2-fold operad in an n-fold monoidal category, as in the above picture, and then show how it generalizes the case of operad in a braided category.

Let $\mathcal{V}$ be an n-fold monoidal category as defined in Section 2.

Definition 4 *A 2-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$*

$$\gamma^{12} : \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s = \sum_{u=1}^{s} j_u$ and let $h_s = \sum_{u=1+g_{s-1}}^{g_s} i_u$.*

   The η^{12} labelling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the 12 interchange.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s)\right) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t)\right) \arrow[r, "\gamma^{12}\otimes_1 \mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes_1 \eta^{12}"'] & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t)\right) \arrow[d, "\gamma^{12}"] \\
& \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{u=1}^{j_s} {}_2\mathcal{C}(i_{u+g_{s-1}})\right)\right) \arrow[r, "\mathrm{id}\otimes_1(\otimes_2^k \gamma^{12})"'] & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(h_s)\right) \arrow[u, "\gamma^{12}"']
\end{tikzcd}$$

2. *Respect of units is required just as in the symmetric case. The following unit diagrams commute.*

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 (\otimes_2^k I) \arrow[r, equal] \arrow[d, "1\otimes_1(\otimes_2^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_1 (\otimes_2^k \mathcal{C}(1)) \arrow[ur, "\gamma^{12}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_1 \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J}\otimes_1 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_1 \mathcal{C}(k) \arrow[ur, "\gamma^{12}"] &
\end{tikzcd}$$

Note that operads in a braided monoidal category are examples of 2-fold operads. This is true based on the arguments of Joyal and Street [13], who showed that braided categories arise as 2-fold monoidal categories where the interchanges are isomorphisms. Also note that given such a perspective on a braided category, the two products are equivalent and the use of the braiding to shuffle in the operad associativity requirement can be rewritten as the use of the interchange.

It is immediately clear that we can define operads using more than just the first two products in an n-fold monoidal category. The best way of going about this is to use the theory of monoids, (and more generally enriched categories), in iterated monoidal categories. Operads in a symmetric (braided) monoidal category with coproducts are often efficiently defined as the monoids of a category of collections. For a braided category $\mathcal{V}$ the objects of $Col(\mathcal{V})$ are functors from the category of natural numbers to $\mathcal{V}$. In other words the data for a collection $\mathcal{C}$ is a sequence of objects $\mathcal{C}(j)$ just as for an operad. Morphisms in $Col(\mathcal{V})$ are natural transformations. The tensor product in $Col(\mathcal{V})$ is given by

$$(\mathcal{B} \otimes \mathcal{C})(j) = \coprod_{k \geq 0, j_1 + \ldots + j_k = j} \mathcal{B}(k) \otimes (\mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k))$$

where $j_i \geq 0$. This product is associative by use of the symmetry or braiding. The unit is the collection $(\emptyset, I, \emptyset, \ldots)$ where $\emptyset$ is an initial object in $(\mathcal{V})$. Here we can observe how the interchange transformations generalize braiding and for $(\mathcal{V})$ an n-fold monoidal category $(n \geq 2)$ define the objects and morphisms of $Col(\mathcal{V})$ in precisely the same way, but define the product to be

$$(\mathcal{B} \otimes^{12} \mathcal{C})(j) = \coprod_{k \geq 0, j_1 + \ldots + j_k = j} \mathcal{B}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k))$$

Associativity is seen by inspection of the two products $(\mathcal{B} \otimes^{12} \mathcal{C}) \otimes^{12} \mathcal{D}$ and $\mathcal{B} \otimes^{12} (\mathcal{C} \otimes^{12} \mathcal{D})$.

In the braided case mentioned above, the two coproducts in question are seen to be composed of the same terms up to a braiding between them. Here the terms of the two coproducts are related by instances of the interchange transformation η^{12} from the term in $((\mathcal{B} \otimes^{12} \mathcal{C}) \otimes^{12} \mathcal{D})(j)$ to the corresponding term in $(\mathcal{B} \otimes^{12} (\mathcal{C} \otimes^{12} \mathcal{D}))(j)$. For example upon expansion of the two three-fold products we see that in the coproduct which is $((\mathcal{B} \otimes^{12} \mathcal{C}) \otimes^{12} \mathcal{D})(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_2 \mathcal{C}(1)) \otimes_1 (\mathcal{D}(1) \otimes_2 \mathcal{D}(1))$$

while in $(\mathcal{B} \otimes^{12} (\mathcal{C} \otimes^{12} \mathcal{D}))(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)) \otimes_2 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)).$$

The coherence theorem of iterated monoidal categories guarantees the commutativity of the pentagon equation. Now we have a condensed way of defining 2-fold operads.

Theorem 4 *2-fold operads in n-fold monoidal $\mathcal{V}$ are monoids in $Col(\mathcal{V})$.*

Proof: A monoid in $Col(\mathcal{V})$ is an object $\mathcal{C}$ in $Col(\mathcal{V})$ with multiplication and unit morphisms. Since morphisms of $Col(\mathcal{V})$ are natural transformations the multiplication and unit consist of families of maps in $\mathcal{V}$ indexed by the natural numbers, with source and target exactly as required for operad composition and unit. The operad axioms are equivalent to the asociativity and unit requirements of monoids.

This brings us back to the question of defining operads in n-fold monoidal $(\mathcal{V})$ using the higher products and interchanges. This idea will correspond to alternate products in $Col(\mathcal{V})$ denoted by $\otimes^{pq}$. These are defined just as for the first case $\otimes^{12}$ above. Associators are as described above for the first product, using $\eta^{pq)}$ for the associator $\alpha : \mathcal{A} \otimes^{pq} (\mathcal{B} \otimes^{pq} \mathcal{C}) \to (\mathcal{A} \otimes^{pq} \mathcal{B}) \otimes^{pq} \mathcal{C}$. The unit for each is the collection $(\emptyset, I, \emptyset, \ldots)$ where $\emptyset$ is an initial object in $(\mathcal{V})$. Notice that these product do not interchange; i.e they are not functorial with respect to each other. Notice that the associators in these categories of collections are not isomorphisms unless we are considering the special cases of braiding or symmetry.

Now we will focus on the products $\otimes^{(m-1)m}$ since these will be seen to suffice for defining all operad compositions. Before defining m-fold operads as monoids with respect to $\otimes^{(m-1)m}$, we note that there is also fibrewise monoidal structure. This will be important in the description of the monoidal structure of the category of operads. In fact, we have the following

Theorem 5 *If $\mathcal{V}$ is n-fold monoidal and has coproducts then for $n \geq m \geq 2$ $Col(\mathcal{V})$ can be given the structure of an $(n-m+1)$-fold monoidal category, denoted $Col_m(\mathcal{V})$.*

Proof: The first tensor product is $\hat{\otimes}_1 = \otimes^{(m-1)m}$ and the others are the higher fibrewise products starting with fibrewise $\otimes_{m+1}$. Thus the products of $Col_m(\mathcal{V})$ are as follows:

$$(\mathcal{B}\hat{\otimes}_1\mathcal{C})(j) = \coprod_{k\geq 0, j_1+\ldots+j_k=j} \mathcal{B}(k) \otimes_m -1(\mathcal{C}(j_1) \otimes_m \ldots \otimes_m \mathcal{C}(j_k))$$

$$(\mathcal{B}\hat{\otimes}_2\mathcal{C})(j) = \mathcal{B}(j)\otimes_{m+1}\mathcal{C}(j)$$

$\vdots$

$$(\mathcal{B}\hat{\otimes}_{n-m+1}\mathcal{C})(j) = \mathcal{B}(j)\otimes_n\mathcal{C}(j)$$

The unit for $\hat{\otimes}_1$ is $\mathcal{I} = (\emptyset, I, \emptyset, \ldots)$ and the unit for all the other products is $\mathcal{M} = (I, I, \ldots)$. In [3] Batanin points out that the fibrewise products are monoidal functors with respect to the collection product $\otimes^{(m-1)m}$. Here we add that the fibrewise products interchange with the $\otimes^{(m-1)m}$ by inspection of their terms, using the interchangers of $\mathcal{V}$. It is also not hard to check the unit conditions that are required for the fibrewise products to be the multiplication for a monoid in the category of monoidal categories. The extra requirement of the two sorts of unit is that $\mathcal{M}\hat{\otimes}_1\mathcal{M} = \mathcal{M}$ and that $\mathcal{I}\hat{\otimes}_k\mathcal{I} = \mathcal{I}$ for $k > 1$. These equations do indeed hold. Thus the first product together with any of the fiberwise products are those of a 2-fold monoidal category. For the products $\hat{\otimes}_2$ and higher the associators and interchange transformations are fiberwise and the axioms hold since they hold for each fiber.

This result is quite useful for describing n-fold operads and their higher-categorical structure, especially when coupled with two other facts. The first is that monoids are equivalently defined as single object enriched categories, and the second is the following result from [11] and [12].

Theorem 6 *For $\mathcal{V}$ n-fold monoidal the category of enriched categories over $(\mathcal{V}, \otimes_1, \alpha^1, I)$ is an $(n-1)$-fold monoidal 2-category.*

For our purposes we translate the theorem about enriched categories into its single object corollary about the category of monoids in $\mathcal{V}$, Mon$\mathcal{V}$.

Corollary 2 *For $\mathcal{V}$ n-fold monoidal the category of monoids in $\mathcal{V}$ is an $(n-1)$-fold monoidal 2-category.*

Proof: The product of enriched categories always has as its object set the cartesian product of the object sets of its components. Thus one object enriched categories have products with one object as well.

Definition 5 *Now for $\mathcal{V}$ n-fold monoidal we define m-fold operads as monoids in the category of collections $Col_m(\mathcal{V})$ for $n \geq m > 2$.*

Note that a further possible avenue of investigation would be to study operads in the category of operads, intruducing new collections of operads at each iteration.

To translate the corollary about monoids to one about operads we only need to recall that we have decided to denote by 2-fold operads the monoids in $Col(\mathcal{V})$ with respect to the first product using $\otimes_1$ and $\otimes_2$ in the coproduct.

Corollary 3 1. *For $\mathcal{V}$ n-fold monoidal the category of 2-fold operads in $\mathcal{V}$, $Oper_2\mathcal{V}$, is an $(n-2)$-fold monoidal 2-category.*

2. *This process continues, i.e. the category of m-fold operads, $Oper_m\mathcal{V}$, is an $(n-m)$-fold monoidal 2-category.*

Proof: Rather than starting with monoids in n-fold monoidal $\mathcal{V}$ as in the previous corollary we are actually beginning with monoids in $(n-m+1)$-fold monoidal $Col_m(\mathcal{V})$. Note that in [11] the products in $\mathcal{V}$ are assumed to have a common unit. To generalize to our situation here, where the unit for the first product in the category of colletions is distinct, we need to add slightly to the definitions in [11]. When enriching (or more specifically taking monoids) we are doing so with respect to the first available product. Thus the unit morphism for enriched categories has its domain the unit for that first product, I. However the unit enriched category $\boldsymbol{\mathcal{I}}$ has one object, denoted 0, and $\boldsymbol{\mathcal{I}}(0,0) = \mathcal{M}$.

This theorem justifies our focus on the first m products of $\mathcal{V}$ as opposed to any subset of the n products. It is due to the way in which this focus allows us to describe the resulting structure on the category of m-fold operads. Of course, we can use the forgetful functors mentioned above to pass from n-fold monoidal $\mathcal{V}$ to any of the subcollections of products. The m-fold operads do behave as expected, retaining all but the structure that depends on the forgotten products.

This will be seen more clearly upon inspection of the unpacked definition to follow. In short, an m-fold operad is also an $(m-1)$-fold operad.

We note that since a symmetric monoidal category is n-fold monoidal for all n, then operads in a symmetric monoidal category are n-fold monoidal for all n as well. More generally, if $n \geq 3$ and the interchanges are isomorphisms, then by the Eckmann Hilton argument the products collapse into one and the result is a a symmetric monoidal category, and so operads in it are again n-fold monoidal for all n. Here we are always discussing ordinary "non-symmetric," ("non-braided") operads. The possible faithful actions of symmetry or braid groups can be considered after the definition, which we leave for a later paper. We do point out that the proper direction in which to expand this work is seen in Weber's paper [27]. He generalizes by making a distinction between the binary and k-ary products in the domain of the composition map $\gamma : \mathcal{C}(k)\otimes(\mathcal{C}(j_1)\otimes\ldots\otimes\mathcal{C}(j_k)) \to \mathcal{C}(j)$. The binary tensor product is seen formally as a pseudo-monoid structure and the k-ary product as a pseudo-algebra structure for a 2-monad which can contain the information needed to describe actions of braid or symmetry groups. The two structures are defined using strong monoidal morphisms, and so the products coincide and give rise to the braiding which is used to describe the associativity of composition. To encompass the definitions in this paper we would move to operads in lax-monoidal pseudo algebras, where instead of pseudo monoids and strong monoidal morphisms in a pseudo algebra we would consider the same picture but with lax monoidal morphisms.

The fact that monoids are single object enriched categories also leads to an efficient expanded definition of m-fold operads in an n-fold monoidal category. Let $\mathcal{V}$ be an n-fold monoidal category as defined in Section 2.

Definition 6 *For $2 \leq m \leq n$ an m-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$*

$$\gamma^{pq} : \mathcal{C}(k) \otimes_p (\mathcal{C}(j_1) \otimes_q \ldots \otimes_q \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $m \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum\limits_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $m \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum\limits_{s=1}^{k} j_s = j$ and $\sum\limits_{t=1}^{j} i_t = i$. Let $g_s = \sum\limits_{u=1}^{s} j_u$ and let $h_s = \sum\limits_{u=1+g_{s-1}}^{g_s} i_u$.*

   The η^{pq} labelling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the pq interchange.

$$
\begin{tikzcd}
\mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k} {}_q\mathcal{C}(j_s) \right) \otimes_p \left(\bigotimes_{t=1}^{j} {}_q\mathcal{C}(i_t) \right) \arrow[r, "\gamma^{pq} \otimes_p \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_p \eta^{pq}"'] & \mathcal{C}(j) \otimes_p \left(\bigotimes_{t=1}^{j} {}_q\mathcal{C}(i_t) \right) \arrow[d, "\gamma^{pq}"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k} {}_q\mathcal{C}(j_s) \otimes_p \left(\bigotimes_{u=1}^{j_s} {}_q\mathcal{C}(i_{u+g_{s-1}}) \right) \right) \arrow[r, "\mathrm{id} \otimes_p (\otimes_q^k \gamma^{pq})"'] & \mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k} {}_q\mathcal{C}(h_s) \right) \arrow[u, "\gamma^{pq}"']
\end{tikzcd}
$$

2. *Respect of units is required just as in the symmetric case. The following unit diagrams commute for all $m \geq q > p \geq 1$.*

$$
\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\otimes_q^k I) \arrow[r, equal] \arrow[d, "1 \otimes_p (\otimes_q^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_p (\otimes_q^k \mathcal{C}(1)) \arrow[ur, "\gamma^{pq}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_p \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_p 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_p \mathcal{C}(k) \arrow[ur, "\gamma^{pq}"] &
\end{tikzcd}
$$

Theorem 7 *The description of m-fold operad in Definition 6 is equivalent to that given in Definition 5.*

Proof: If a collection has an operad composition $\gamma^{q,q+1}$ using $\otimes_q$ and $\otimes_{(q+1)}$ then it automatically has an operad composition for any pair of products $\otimes_p$ and $\otimes_s$ for $p < s$, $p \leq q$ and $s \leq q+1$. This latter statement follows from the fact that for $q > p$ we have natural transformations $\eta^{pq}_{AIIB} : A \otimes_p B \to A \otimes_q B$ Thus if we have $\gamma^{p,p+1}$ then we can form $\gamma^{qs} = \gamma^{p(p+1)} \circ (\eta^{pq} \circ (1 \otimes_q \eta^{s(p+1)})$. The new $\gamma^{p(p+1)}$ is associative based on the old $\gamma's$ associativity, the naturality of η, and the coherence of η. Thus our claims above that generally operads are preserved as such by the forgetful functors mentioned in Section 2 and specifically that an m-fold operad is also an $(m-1)$-fold operad.

It is also worth while to expand the definition of the tensor products of m-fold operads that is implicit in their depiction as monoids in the category of collections in an n-fold monoidal category. Here is the expanded version of the definition:

Definition 7 *Let $\mathcal{C}$,$\mathcal{D}$ be m-fold operads. For $1 \leq i \leq (n-m)$ and using a $\otimes'_k$ to denote the product of two m-fold operads, we define that product to be given by:*

$$(\mathcal{C} \otimes'_i \mathcal{D})(j) = \mathcal{C}(j) \otimes_{i+m} \mathcal{D}(j).$$

We note that the new γ is in terms of the two old ones, for $m \geq q > p \geq 1$:

$$\gamma^{pq}_{\mathcal{C}\otimes'_i\mathcal{D}} = (\gamma^{pq}_{\mathcal{C}} \otimes_{i+m} \gamma^{pq}_{\mathcal{D}}) \circ \eta^{p(i+m)} \circ (1 \otimes_p \eta^{q(i+m)})$$

where the subscripts denote the n-fold operad the γ belongs to and the η's actually stand for any of the equivalent maps which factor into them. Note that this expansion also makes clear from a more direct viewpoint why it is that the monoidalness, or number of products of m-fold operads, must decrease by the same number m. From the condensed version this is expected due to the iterated enrichment. From the expanded view this is necessary to define the new composition since in order for the product to be closed, γ for the i^{th} product utilizes an interchange with superscript $i+m$. Thus for the product to exist i can only be allowed to be as large as $n-m$.

Now for some examples of operads in the categories from section 3. To have an operad with both the unit axioms and an element $\mathcal{C}(0)$ we will need to have an initial object included in the categories based on ordered semigroups and sequences. This we will denote by $\emptyset$ and define all products involving the object $\emptyset$ as an operand to be equal to $\emptyset$. In all the examples the composition is associative since it is based upon ordering, so all we need check for is the existence of that composition.

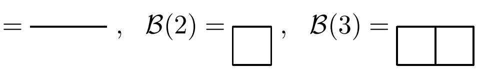
Example 7

Of course $\mathcal{C}(j) = \emptyset$ and $\mathcal{C}(j) = 0$ for all j are trivially operads, where 0 is the monoidal unit. First we look at the simplest interesting examples: 2-fold operads in an ordered semigroup such as $\mathbf{N}$, where $\otimes_1$ is max and $\otimes_2$ is $+$. A 2-fold operad here is a sequence $\mathcal{C}(j)$ of natural numbers which has the property that for any $j_1 \ldots j_k$, $max(\mathcal{C}(k), \sum \mathcal{C}(j_i)) \leq \mathcal{C}(\sum j_i)$ and for which $\mathcal{C}(1) = 0$. This translates into saying that for any two whole numbers x, y we have that $\mathcal{C}(x+y) \geq \mathcal{C}(x) + \mathcal{C}(y)$ and that $\mathcal{C}(1) = 0$. The latter condition both satisfies the unit axioms and makes it redundant to also insist that the sequence be monotone increasing. Some "minimal" examples are the sequences $(0,1,1,2,2,3,3,\ldots)$; $(0,0,1,1,1,2,2,2,3,3,3,\ldots)$; $(0,2,2,4,4,6,6,\ldots)$ and $(0,0,2,2,2,4,4,4,6,6,6,\ldots)$. It is clear that later terms in a sequence are limited by their predecessors.

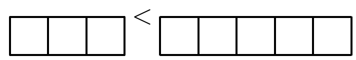
Example 8

Consider the 3-fold monoidal category Seq($\mathbf{N}$,+) of lexicographically ordered finitely nonzero sequences of the natural numbers (here we use $\mathbf{N}$ considered as an example of an ordered monoid), with products $\otimes_1$ the lexicographic max , $\otimes_2$ the concatenation and $\otimes_3$ the pointwise addition. An example of a 2-fold operad in Seq($\mathbf{N}$,+) that is not a 3-fold operad is the following:

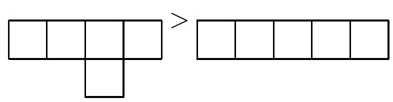
Let $\mathcal{B}(0) = \emptyset$ and let $\mathcal{B}(j)_i = 1$ for $i < j$, 0 otherwise. We can picture these as follows:

$\mathcal{B}(1) =$  , $\mathcal{B}(2) =$, $\mathcal{B}(3) =$, $\ldots$

This is a 2-fold operad, with respect to the lexicographic max and concatenation. For instance the instance of composition $\gamma^{12} : \mathcal{B}(3) \otimes_1 (\mathcal{B}(2) \otimes_2 \mathcal{B}(1) \otimes_2 \mathcal{B}(3)) \to \mathcal{B}(6)$ appears as the relation:



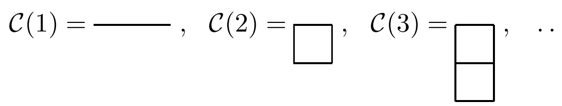
However, the relation



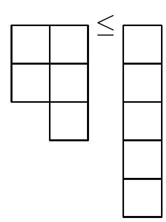
shows that $\gamma^{23} : \mathcal{B}(3) \otimes_2 (\mathcal{B}(1) \otimes_3 \mathcal{B}(3) \otimes_3 \mathcal{B}(2)) \to \mathcal{B}(6)$ does not exist.

Example 9

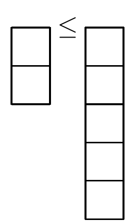
Next we give an example of a 3-fold operad in Seq($\mathbf{N}$,+). Let $\mathcal{C}(0) = \emptyset$ and let $\mathcal{C}(j) = (j-1, 0 \ldots)$. We can picture these as follows:



First we note that the operad $\mathcal{C}$ just given is a 3-fold operad since we have that the $\gamma^{23} : \mathcal{C}(k) \otimes_2 (\mathcal{C}(j_i) \otimes_3 \ldots \otimes_3 \mathcal{C}(j_k)) \to \mathcal{C}(j)$ exists. For instance $\gamma^{23} : \mathcal{C}(3) \otimes_2 (\mathcal{C}(1) \otimes_3 \mathcal{C}(3) \otimes_3 \mathcal{C}(2)) \to \mathcal{C}(6)$ appears as the relation

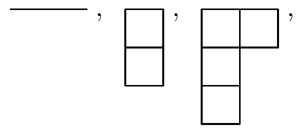


Then we remark that as expected the composition $\gamma^{12} : \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_i) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$ also exists. For instance $\gamma^{12} : \mathcal{C}(3) \otimes_1 (\mathcal{C}(1) \otimes_2 \mathcal{C}(2) \otimes_2 \mathcal{C}(3)) \to \mathcal{C}(6)$ appears as the relation



Example 10

Now we consider some products of these operads. We expect $\mathcal{B}\otimes'\mathcal{C}$ given by $(\mathcal{B}\otimes'\mathcal{C})(j) = \mathcal{B}(j) \otimes_3 \mathcal{C}(j)$ to be a 2-fold operad and it is. It appears thus:

$\emptyset$,  , ...

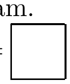
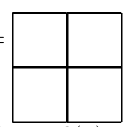
We demonstrate the tightness of the existence of products of operads by pointing our that $(\mathcal{B}\otimes\mathcal{C})(j) = \mathcal{B}(j) \otimes_2 \mathcal{C}(j)$ does not form an operad.

Example 11

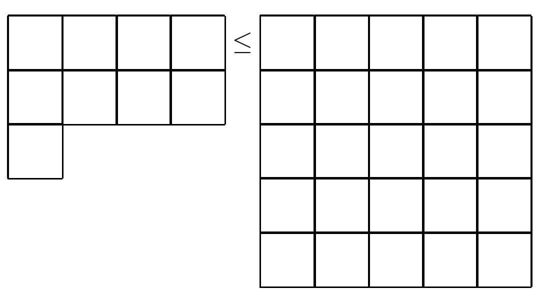
One large family of operads in the category ModSeq($\mathbf{N}$,+) is that of natural number indexed collections of tableau diagrams $\mathcal{C}(n)$, $n \in\mathbf{N}$, such that $C(0) = \emptyset$ and for $n \geq 1$ $h(\mathcal{C}(n)) = f(n)$ where $f : \mathbf{N}\to\mathbf{N}$ is a function such that $f(1) = 0$ and $f(i+j) > f(i) + f(j)$. These conditions are not necessary, but they are sufficient since the first implies that $\mathcal{C}(1) = 0$ which shows that the unit conditions are satisfied; and the second implies that the maps γ exist. We see existence of γ^{12} since for $j_i > 0$, $h(\mathcal{C}(k)\otimes_1(\mathcal{C}(j_1)\otimes_2 ... \otimes_2\mathcal{C}(j_k))) = \max(f(k), \max(f(j_i))) \leq f(j)$. We have existence of γ^{13} and γ^{23} since $\max(f(k), \sum f(j_i)) \leq f(j)$.

Examples of f include $(x-1)P(x)$ where P is a polynomial with coefficients in $\mathbf{N}$. This is easy to show since then P will be monotone increasing for $x \geq 1$ and thus $(i+j-1)P(i+j) = (i-1)P(i+j)+jP(i+j) \geq (i-1)P(i)+jP(j)-P(j)$. By this argument we can also use any $f = (x-1)g(x)$ where $g : \mathbf{N}\to\mathbf{N}$ is monotone increasing for $x \geq 1$.

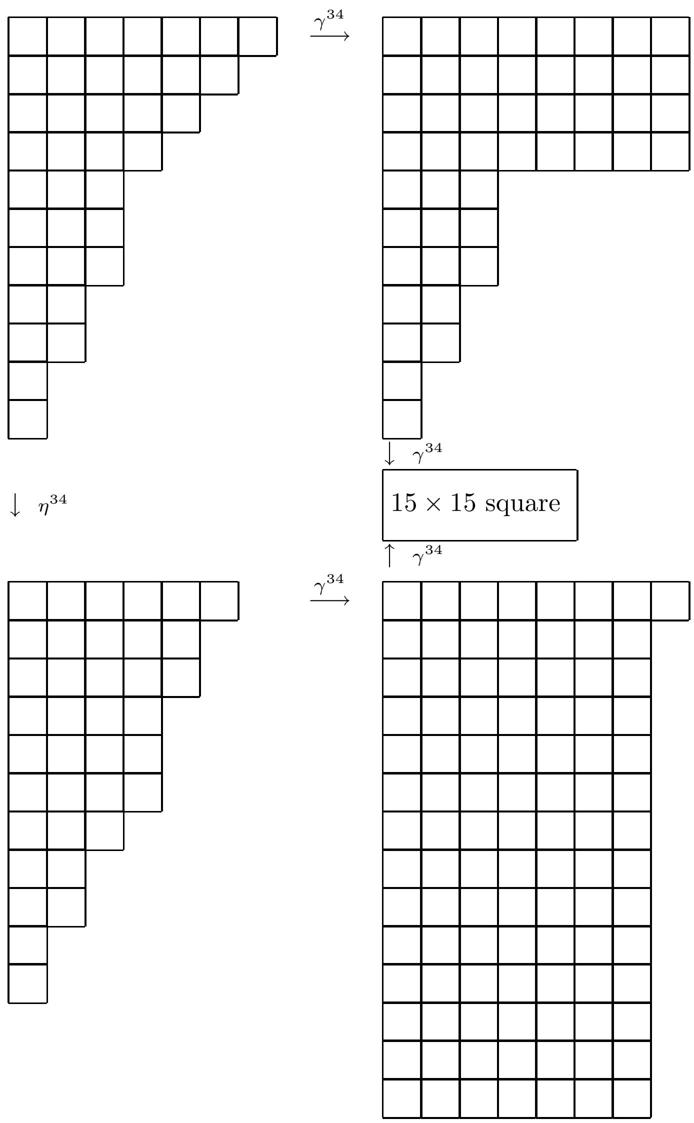
For a specific example with a handy picture that also illustrates again the nontrivial use of the interchange η we simply let $f = x - 1$. Then we have to actually describe the elements of ModSeq($\mathbf{N}$) that make up the operad. One nice choice is the operad $\mathcal{C}$ where $\mathcal{C}(n) = \{n-1, n-1, ..., n-1\}$, the $(n-1)\times(n-1)$ square tableau diagram.

$\mathcal{C}(1) = 0$, $\mathcal{C}(2) =$  , $\mathcal{C}(3) =$  ...

For instance $\gamma^{23} : \mathcal{C}(3) \otimes_2 (\mathcal{C}(1) \otimes_3 \mathcal{C}(3) \otimes_3 \mathcal{C}(2)) \to \mathcal{C}(6)$ appears as the relation



An instance of the associativity diagram with upper left position $\mathcal{C}(2)\otimes_2(\mathcal{C}(3)\otimes_3 \mathcal{C}(2)) \otimes_2 (\mathcal{C}(2) \otimes_3 \mathcal{C}(2) \otimes_3 \mathcal{C}(4) \otimes_3 \mathcal{C}(5) \otimes_3 \mathcal{C}(3))$ is as follows:



We conclude with a description of the concepts of n-fold operad algebra and of the tensor products of operad algebras.

Definition 8 *Let $\mathcal{C}$ be an n-fold operad in $\mathcal{V}$. A $\mathcal{C}$-algebra is an object $A \in \mathcal{V}$ and maps*

$$\theta^{pq} : \mathcal{C}(j) \otimes_p (\otimes_q^j A) \to A$$

for $n \geq q > p \geq 1$, $j \geq 0$.

1. *Associativity: The following diagram is required to commute for all $n \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$, and where $\sum\limits_{s=1}^{k} j_s = j$.*

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\mathcal{C}(j_1) \otimes_q \ldots \otimes_q \mathcal{C}(j_k)) \otimes_p (\otimes_q^j A) \arrow[r, "\gamma^{pq} \otimes_p \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_p \eta^{pq}"'] & \mathcal{C}(j) \otimes_p (\otimes_q^j A) \arrow[d, "\theta^{pq}"'] \\
 & A \\
\mathcal{C}(k) \otimes_p ((\mathcal{C}(j_1) \otimes_p (\otimes_q^{j_1} A)) \otimes_q \ldots \otimes_q (\mathcal{C}(j_k) \otimes_p (\otimes_q^{j_k} A))) \arrow[r, "\mathrm{id} \otimes_p (\otimes_q^k \theta^{pq})"] & \mathcal{C}(k) \otimes_p (\otimes_q^k A) \arrow[u, "\theta^{pq}"']
\end{tikzcd}$$

2. *Units: The following diagram is required to commute for all* $n \geq q > p \geq 1$.

$$\begin{tikzcd}
I \otimes_p A \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_p 1"'] & A \\
\mathcal{C}(1) \otimes_p A \arrow[ur, "\theta^{pq}"] &
\end{tikzcd}$$

Example 12

Of course the initial object is always an algebra for every operad, and every object is an algebra for the initial operad.

Let $\mathcal{C}$ and $\mathcal{D}$ be m-fold operads in an n-fold monoidal category. If A is an algebra of $\mathcal{C}$ and B is an algebra of $\mathcal{D}$ then $A \otimes_{i+m} B$ is an algebra for $\mathcal{C} \otimes_i' \mathcal{D}$.

That the product of n-fold operad algebras is itself an n-fold operad algebra is easy to verify once we note that the new θ is in terms of the two old ones:

$$\theta^{pq}_{A \otimes_{i+m} B} = (\theta^{pq}_A \otimes_{i+m} \theta^{pq}_B) \circ \eta^{p(i+m)} \circ (1 \otimes_p \eta^{q(i+m)})$$

Maps of operad algebras are straightforward to describe–they are required to preserve structure; that is to commute with θ.

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