# Research Statement

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## 1. Summary

Over the last four years, together with students and collaborators, I have enjoyed successful exploration of a web of ideas which lies in the intersection of quantum algebra, homotopy theory, and geometric combinatorics. Common themes in our publications and projects underway include: operads and operad modules, convex polytopes and related combinatorial algebras, structured categories (braided and iterated monoidal), and the superstructures those categories support. The topological center of our research is the theory of $H$-spaces, especially loop spaces and their homotopy approximations – the $A_n$-spaces. Operad actions characterize these spaces, graded Hopf algebras arise from their homology groups, and structured categories embody them as nerves. I hope to continue to contribute to the untangling of this web by uncovering the deep connections that determine its fundamental nature. Along the way there are several exciting analogies to the physical world begging to be developed into applications. Most importantly there are many potential new discoveries of mathematical reality in which beginning student investigators can participate.

1.1. **Geometric combinatorics and Hopf algebras.** Several important combinatorial graded Hopf algebras are based upon the vertices of fundamental polytopes. These include the Malvenuto-Reutenauer Hopf algebra of permutations, $MR$, and the Loday-Ronco Hopf algebra of binary trees, $LR$. F. Sottile and A. Lauve recently developed both a Hopf algebra and a Hopf module structure based on the vertices of the multiplihedra. In collaboration we have actually found several new Hopf structures on the permutohedra and multiplihedra. Some of these come from considering alternate combinatorial descriptors of the vertices, including the *marked graph tubings* developed by myself and S. Devadoss.

I have also developed a generalization of both $MR$ and $LR$ with fundamental bases the vertices of special sequences of Devadoss's graph-associahedra. The motivation for this is my discovery that the Tonks projection from the permutohedron to the associahedron can be factored through a series of graph-associahedra. This fact is simple to demonstrate; it follows from Devadoss's discovery that the complete graph-associahedron is the permutohedron while the path graph-associahedron is the Stasheff polytope. Thus by deleting edges of the complete graph one at a time, we describe a family of quotient cellular projections. Figure 1 shows one of these. The products and coproducts of the

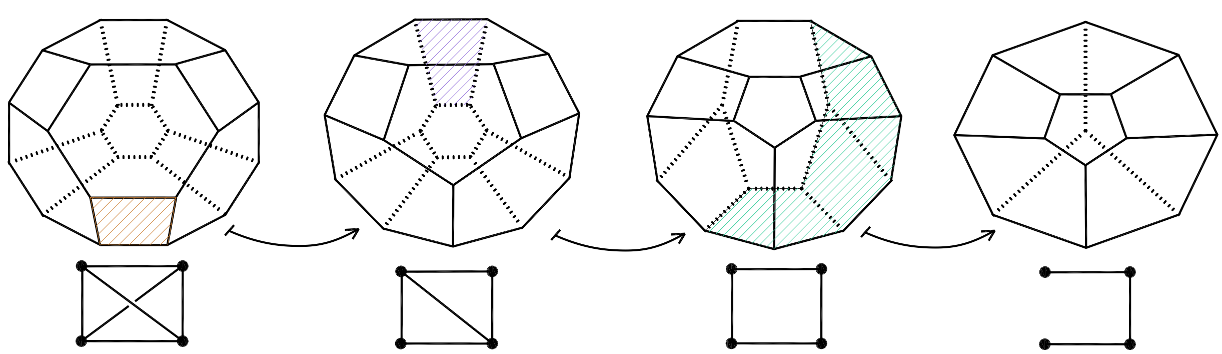


Figure 1. The cellular projection from the permutohedron to the associahedron, factored through two graph-associahedra. The third polytope is the cyclohedron. The colored facets in the first 3 pictures are collapsed by the projection.

$MR$ and $LR$ Hopf algebras are easily generalized to the vertices of the graph-associahedra, since the latter are combinatorially described by maximal collections of compatible connected subgraphs called *tubings.* Figure 2 shows a product of two vertices of the cyclohedron.

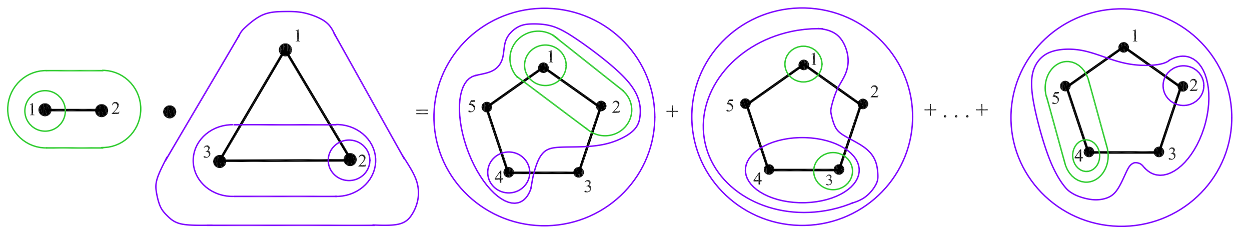



FIGURE 2. A product of cycle graph tubings (10 terms total).

The supporting results of my work in this area describe the geometric combinatorics of the polytopes in question. The first major contribution is the answering of a long-standing open question. The low dimensional multiplihedra were first described by Stasheff in 1970 in order to describe $A_n$-maps between $A_n$-spaces. The question of whether they could be realized as convex polytopes was still open three decades later. Building on Loday's famous geometric realization of the associahedra, I developed a simple algorithm that, for each choice of $q \in (0, 1)$, gives vertices in Euclidean space $\mathbb{R}^n$ whose convex hull is the multiplihedron of dimension $n$. The method that allowed the development of my algorithm was a complete description of the vertices of the multiplihedron using painted trees. This allowed as well the first enumeration of the vertices of the multiplihedron, using generating function techniques.

Metric versions of the painted trees were described by Boardman and Vogt in 1973, and a recursive definition of the multiplihedra was implicitly discovered by Iwase and Mimura in 1986. The new convex hull realization allowed connection of these early results via combinatorial equivalence to Iwase and Mimura's CW-complex, and simultaneously by homeomorphism to Boardman and Vogt's spaces.

The importance of the geometric realization of the multiplihedron was immediately apparent. First, it allowed simple generalizations to quotients of the multiplihedron corresponding to strict range and domain spaces. The former yields a new realization of the $n$-dimensional associahedron in $\mathbb{R}^n$ and the latter yields a realization of the composihedron. The composihedron are a newly discovered sequence of polytopes. I actually found them in both topological and categorical settings–where in both cases they had been misnamed or mistaken for associahedra in the early literature. In the case of strict domain and range the quotient polytopes are cubes, as shown in Figure 3.

Secondly, the geometric realization of the classic multiplihedron affords us a tool for the study of generalized multiplihedra of several kinds, a tool made necessary by the fact that these new

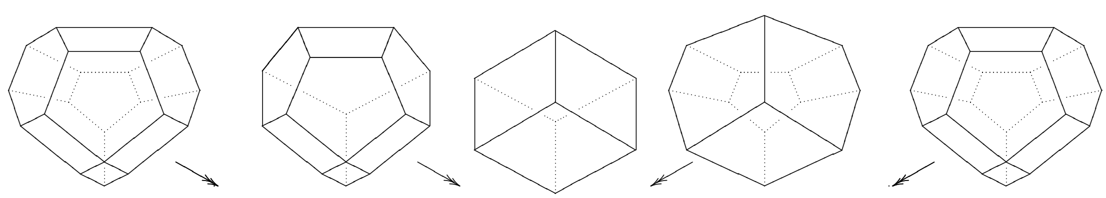


FIGURE 3. Left to right: $\mathcal{J}(4)$, $\mathcal{CK}(4)$, 3-d cube, $\mathcal{K}(5)$, and $\mathcal{J}(4)$ again, showing projection.

polytopes are rarely simple themselves. By use of an analogous realization myself and collaborator Devadoss were able to experimentally discover the structure of the graph-multiplihedra and their quotients. Only then were we able to prove the details of the facet structure of the graph-multiplihedra. This same approach, along with new results regarding certain Minkowski sums, is being applied to the generalized associahedra of Fomin and Zelevinsky and the generalized permutohedra of Postnikov. The former specialize to the associahedron and the cyclohedron via Dynkin diagrams, the latter via the graph-associahedra as special examples of nestohedra. We hope to develop new multiplihedra based upon all these, and then to extend the theory of combinatorial Hopf modules to the new examples. I have also defined a new family of associahedra, the multigraph-associahedra, and corresponding multiplihedra.

- Publications and student papers:

— Marked tubes and the graph multiplihedron. (with S.L. Devadoss)
to appear in *Algebraic and Geometric Topology*, 2008.

— Quotients of the multiplihedron as categorified associahedra.
to appear in *Homotopy, Homology and Applications*, 2008.

— Convex Hull Realizations of the Multiplihedra
to appear in *Topology and its Applications*, 2008.

— Jerome Lecointre, Senior project, TSU
*Polytope structure of the composihedra*, 2006.

- Main results:

— Discovery of new quotient of multiplihedra: the composihedra.

— Discovery of graph-multiplihedra and their quotients.

— Geometric realizations for the multiplihedra, composihedra, graph-multiplihedra,
and quotients of the graph multiplihedra.

— Enumeration of vertices of multiplihedra, composihedra; recognition of the former
as the Catalan transform and the latter as the binomial transform of the Catalan numbers.

— Facet structure of composihedra and graph-multiplihedra.

— Factorization of the Tonks projection through graph-associahedra.

— Factorization of Saneblidze and Umble's projection through the graph-multiplihedra.
Special cases of above factorizations proven, general case conjectured.

- Current projects and collaborations:

— Pending grant proposal: *Geometric combinatorial Hopf algebras and modules.*

— Hopf algebras and modules using composihedra and multiplihedra.
(with A. Lauve, F. Sottile [Texas A&M])

— Hopf algebras and modules using graph-associahedra, graph-multiplihedra.
(with S. Devadoss [Williams College], A. Lauve, F. Sottile.)

— New polytopes and Hopf structures:
multigraph-multiplihedra, Dynkin diagram multiplihedra, nested complex multiplihedra.
(with S. Devadoss, N. Reading [North Carolina.] )

1.2. **Structured categories and their operads.** A common generalization of symmetric, braided, and tensor categories exists – the iterated monoidal categories introduced by Fiedorowicz, Vogt, and their students in 2003. Many of my contributions in this area, with students and collaborators, are due to the insight that results about braided categories can often be extended to the realm of Fiedorowicz's $n$-fold monoidal categories. Specifically we have demonstrated that the concepts of enrichment, substitution product, operads and operad algebras all make sense in a base category that is $n$-fold monoidal. In order to show the sharpness of certain theorems about operads in $n$-fold monoidal categories we developed a large family of simple combinatorial posets which exemplify $n$-fold monoidal categories. These were the first concrete examples besides the free $n$-fold monoidal

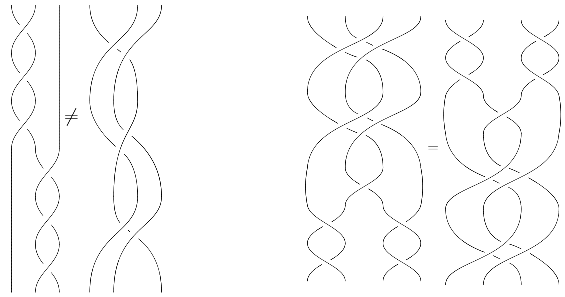


FIGURE 4. Examples of braid inequality and equality used in categorical existence proofs.

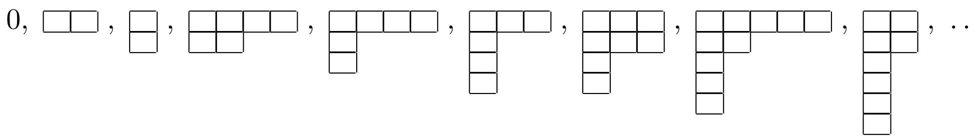


FIGURE 5. A minimal operad of Young diagrams.

categories originally described by Fiedorowicz. There are many open questions raised by our research – ranging from the broad question of how the 2-fold monoidal structures on species defined by M. Aguiar and S. Mahajan relate to our combinatorial 2-fold monoidal structures, to the specific question of whether certain operads of Young diagrams display chaotic growth patterns.

The most recent contributions to the theory of structured categories we can claim are in regard to what sort of additional structures can be found when a braiding is present. The proofs in this area are often more instructive than the theorem statements. For example, we use the embedding of the semigroups of positive braids into the braid groups to show a family of braid inequalities (see Figure 4 for the first) which preclude existence of a second braiding based on an odd number of half twists using the original braiding in a category. Then we use the existence of opposites in the category of enriched categories over a braiding, and the existence of monoids in the free braided category with duals to classify the 4-strand braids which obey a certain *interchanging* identity. Thanks are due to Imre Tuba for helpful conversations about the latter. Also important to our classification of the 2-fold monoidal structures based on a braiding is the fact that the interchanging braids are equivalent to their own 180 degree rotations, as shown in Figure 4. Finally, both the existence of a braiding and an involution in the 2-category of enriched categories are precluded in general by showing braids of two types to be non-conjugate, via arguing that their braid closures always have different linking number.

- Publications and student papers:

— Operads in iterated monoidal categories (with J. Siehler, E. Seth Sowers)
*Journal of Homotopy and Related Structures* 2, 1-43, 2007.

— Classification of braids which give rise to interchange (with F. Humes)
*Algebraic and Geometric Topology* 7, 1233-1274, 2007.

— Govina M. Eyum, Masters Thesis, TSU
*Products of Young diagrams in a 2-fold monoidal category*, 2007.

— E. Seth Sowers, Masters Thesis, TSU
*Operads in 2-fold monoidal categories*, 2006.

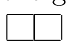
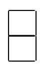
— Felita N.C. Humes, Masters Thesis, TSU
*Iterated monoidal categories based on a braiding*, 2006.

— Ahmad Kheder, Senior project, TSU
*Investigating minimal recursive growth.*, 2007.

- Main results:

— Complete classification of 2-fold structures based on a braiding: the classification is by the four-strand braids $(\sigma_2\sigma_1\sigma_3\sigma_2)^{\pm n}\sigma_2^{\pm 1}(\sigma_1\sigma_3)^{\mp n}$.
— Discovery of 2-fold categories of sequences and Young diagrams.
— Discovery of $n$-fold categories of $n$-dimensional Young diagrams.
— Definition of $n$-fold operads in a $k$-fold monoidal category.
— We prove that the category of $n$-fold operads in a $k$-fold monoidal category is itself a $(k-n)$-fold monoidal, strict 2-category, and show that $n$-fold operads are automatically $(n-1)$-fold operads.
— Characterization of operads of Young diagrams for simple generators.

- Current projects and collaborations:

— Proof of the 2-fold structure of higher multiplication of Young diagrams. (with G. Eyum.)
— Classification of operads of natural numbers. Given a finite generating sequence, what is the formula for the terms in the operad it generates? (with A. Kheder).
— Classification of operads of Young diagrams for mutiple generators. Is the growth actually chaotic for the operad generated by the sequence 0,  ,  ? See Figure 5.

1.3. **Categorical homotopy theory.** I first came across the associahedron and multiplihedron in their incarnations as commuting pasting diagrams in the theory of bicategories and tricategories. Closely connected is the fact that actions of these polytope families serve to characterize $A_n$-spaces and their maps.

The definition of enriched category generalizes the usual definition of category by replacing the hom–sets of morphisms between each two objects by hom–objects in some monoidal category $\mathcal{V}$. The collection of enriched categories over a given monoidal category, with enriched functors and natural transformations, is the 2-category known as $\mathcal{V}$-Cat. Joyal and Street showed $\mathcal{V}$-Cat to inherit structure from $\mathcal{V}$: if $\mathcal{V}$ is symmetric then so is $\mathcal{V}$-Cat, if $\mathcal{V}$ is merely braided then $\mathcal{V}$-Cat is merely monoidal.

My early contributions to enrichment theory are related to this inheritance of properties. I defined enrichment over a very general type of monoidal category with extra structure; a $k$-fold monoidal, or iterated monoidal category. I then proved that enrichment decrements the number of available interchanging tensor products. Next I recursively defined higher dimensional enrichment. $\mathcal{V}$-$n$-Cat is the collection of categories enriched over $\mathcal{V}$-$(n-1)$-Cat. I defined the enriched higher morphisms for these objects and showed that iterated enrichment increments the categorical dimension.

The homotopy theoretical implications of my results consist of an analogy between the process of creating the classifying space of a topological monoid and the the process of creating the higher category of enriched categories over a categorical monoid. Balteanu, Fiedorowicz, Schwänzl, and Vogt show a direct correspondence between $k$-fold monoidal categories and $k$-fold loop spaces through the categorical nerve. When we find the loop space of a topological space, we see that 1 dimensional paths in the original are now points in the derived space. Delooping is the inverse of this process, and it yields the classifying space of a loop space which has one less multiplication. Thus an enrichment functor acting on these categories has precisely the expected domain and codomain for a categorical delooping.

More recently my research in this area revolves around the relationship of the composihedra to enrichment. The first few polytopes in our new sequence correspond to cocyle coherence conditions in the definition of enriched bicategories. It was incorrectly believed before my clarification that these cocycle conditions had the combinatorial form of the associahedra, just as it was incorrectly assumed that $A_n$-maps from a topological monoid to an $A_n$-space were likewise governed by associahedra. With this new understanding in hand, it should be possible to start with the data of such a map, and create from it the data and structure of an enriched bicategory.

There are two projects for the future that are supported by this work. One is to make rigorous the implication that enriched bicategories may be exemplified by certain maps of topological monoids. It could be hoped that if this endeavor is successful that $A_\infty$ categories and their maps might also be amenable to the same approach, yielding more interesting examples of enriched bicategories. A second is more philosophical. The facts that the composihedra are used for defining enriched categories and bicategories, and that they form an operad bimodule (left-module over the associahedra and right-module over the associative operad) lead us to propose that enriching over a weak $n$-category should in general be accomplished by use of operad bimodules as well.

- Publications and student papers:

— Enrichment over iterated monoidal categories
*Algebraic and Geometric Topology* 4, 95-119, 2004.

— Vertically iterated classical enrichment
*Theory and Applications of Categories* 12, 299-325, 2004.

— Ph.D. Dissertation, Virgina Tech, 2004.
*Loop Spaces and Higher-Dimensional Iterated Enrichment.*

- Main results:

— For $\mathcal{V}$ $k$-fold monoidal $\mathcal{V}$-Cat is a $(k-1)$-fold monoidal 2-category.

— For $\mathcal{V}$ $k$-fold monoidal $\mathcal{V}$-$n$-Cat is a $(k-n)$-fold monoidal strict $(n+1)$-category.

— The composihedra: their role in enriched bicategory axioms, pseudomonoid axioms, and $A_n$-map axioms

- Current projects and collaborations:

— Operad bimodule characterization of enrichment over weak $n$-categories.

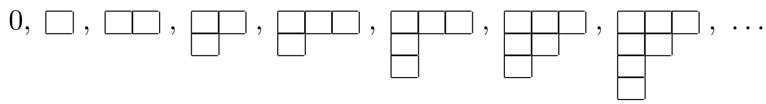
— Describing the nerve of an $n$-operad using $n$-dendroidal sets.
(with I. Moerdijk and I. Weiss, by correspondence).

1.4. **Applications: Polyhexes, crystals and Feynman diagrams.** In this section I'll simply list several analogies which have arisen in our studies between mathematical objects about which we know a good deal and some chemical, geometrical and physical phenomena that we hope to model.

First, we plan to study the polyhexes, which consist of arrangements of a number of hexagons which share at least one side with another in the group. These arrangements look very familiar to an organic chemist, since they are the pictures of *polycyclic benzenoid hydrocarbons.* It is still unknown how to calculate the number of possible hydrocarbons of a given size. Enumeration of hydrocarbons is closely related to special polyhexes, such as the tree-like ones with a chosen *root edge*. It turns out that the $n^{th}$ composihedron has the same number of vertices as the number of all the rooted tree-like polyhexes with up to $n$ cells. If collections of molecules could be arranged around facets of a polytope then there might be revealed interesting insights into the properties of those molecules and their relationships. This knowledge should also accelerate the computer processes of building and searching libraries of molecules.

Second, we plan to study the growth of crystals. In 2003 Ferreiro, Douglas, Warren and Karim observed and measured crystals growing in a film. They noticed that at certain temperatures the usual regular increase in size of the crystal became a pulsating, rhythmic growth. The researchers made several guesses at the cause of this pulsation but the state of knowledge about the driving forces here is incomplete. Upon even more detailed inspections the crystal investigators were able to pinpoint another oscillation. The first growth variable which they measured was the radial length from the center of the crystal to the tip of a main arm of the crystal. Now they extended their approach to the thickness of an arm of the crystal, measuring the length from the main arm axis to the tip of one of its sub-branches. Interestingly, this growth measurement oscillated with the same period as the radius, but was perfectly out of phase with the radius. In other words, the radius and the arm thickness take turns growing, one after the other.

Here are the first few terms of the operad of Young diagrams generated by $B = \square$.



Notice that the growth of the first column is periodic–it grows by a single box at every other step. The growth of the number of boxes in the remaining columns all to the right of the first one is also periodic, but precisely out of phase with the first column's growth. This matches the growth pattern of a crystal.

Third, we plan to study the Connes-Kreimer Hopf algebras of Feynman diagrams. Renormalization refers to a family of algorithms that generate counterterms to deal with divergences arising from loops in the Feynman diagrams. Divergences can be nested, disjoint or overlapping, but the overlapping divergences can be resolved into nested or disjoint. The operations of product, coproduct and antipode in the Hopf algebra of Feynman diagrams rely on insertion of and restriction to subdiagrams. The residue after extraction of a subgraph also appears in the formula as a reconnected complement. Here is the formula for coproduct applied to a Feynman diagram $\Gamma$ with subdiagrams $\gamma$, including the empty and trivial subdiagrams:

$$\Delta\Gamma = \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

Here $\Gamma/\gamma$ is the diagram achieved by removing $\gamma$ and then reconnecting the vertices of the graph that were connected through $\gamma$.

We have noticed analogous features in geometric combinatorics. For example, in the theory of graph-associahedra, given a graph $G$ on $n+1$ nodes an $n$-dimensional convex polytope $K_G$ is constructed. Lower dimensional faces correspond to collections of subgraphs for which each pair is either nested or disjoint. The geometry of faces is described by restricting the polytope construction to subgraphs and to their reconnected complements. The formula for the facet of a graph-associahedron corresponding to a connected subgraph $t$ of a graph $G$ is $K_t \times K_{G/t}$ where $G/t$ is again the reconnected complement. Each facet corresponds to a connected subgraph, so we can write:

$$K_G = \bigcup_{t \subset G} K_t \times K_{G/t}.$$