1. Research Plan: Characterizing Asymmetric Loading and Efficient Flow in Networked Systems.

1.1. Nature and Importance of the Research. It is always exciting when one concept in pure mathematics can give crucial design insight into two or more kinds of physical systems. In this proposal we describe such a case. It turns out that at a practical level a structure built as a non-rigid 3d latticework is much like a communication network that uses a lattice of distributed nodes, which in turn is similar to the network of microscopic channels in a filter. In all three cases we are looking at a highly regular pattern of nodes and connections, through which must pass either force, messages, or a chemical solution. Thus the efficiency and stability of each may be compromised when certain paths in the network are overloaded.

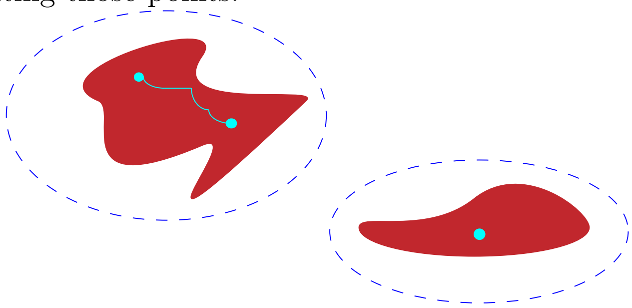
We plan to study a newly discovered mathematical phenomenon of discrete *network loading* that has implications for the designs of these networks, and possibly many more. In general we seek to gain deep understanding of how the symmetry of a regular lattice-like network gives rise to asymmetric loading at larger scales, and how selectively asymmetric design can improve performance. In this proposal we will explain with examples what these terms mean and why they are important.

Mathematically a network is modeled as a *graph* or *pseudograph*. A graph in this context refers to a set of nodes, which we will assume is a finite collection, and a set of edges each of which connects two nodes. A pseudograph allows multiple edges between a given pair of nodes as well as loops that begin and end on the same node. Often we will assign an orientation, or direction, to the edges. This will be drawn by adding an arrow to the edge between two nodes.

Classically, much study has been devoted to maximizing flow through a network, or minimizing the set of edges that must be cut to block that flow. We are interested in a complementary problem: how to avoid overstressing any subset of connections or nodes. Surprisingly, the answer is often not to simply spread the flow evenly from each node.

In applications that we will consider the nodes may represent sending and receiving agents, such as networked computers or robots, or sensors with wireless communication capacity, or cities on a map. Alternatively the nodes might be stations in an electric grid, joints in a non-rigid superstructure, or junctures of channels in a filter. In each case the edges will represent a connection, along which flows information (as in an email) or force (as in tension along a beam) or actual liquid.

Topology is about the connectivity of space. Intuitively, two points in space are connected if there is a path we can trace between them. More generally, they are connected if no two disjoint open sets cover the space while separating those points.



We will consider spaces whose points are the nodes of the graph and whose edges determine the topology. That means we will use *topological bases* such that if v and v' are the endpoints of an edge then $\{v, v'\}$ is connected as a *subspace*. Recall that a basis is a collection of open sets which must cover both the whole space (here the set of nodes) and any of its own intersections, and that a subspace topology on $\{v, v'\}$ is found by intersecting the open sets with that pair. Figure 1 shows an example of a topological basis which obeys this rule: it respects the edge-connectivity of the graph. Figure 2 shows an example of a topological basis that breaks the rule by disconnecting a pair of vertices.

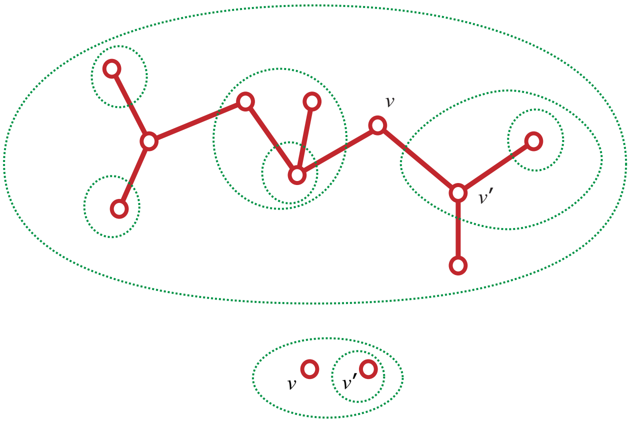


FIGURE 1. There is an edge between v and v'. The basis of open sets, represented by the dashed circled collections of nodes, gives us the subspace of v and v' shown below. They are topologically connected–no two open sets separate them.

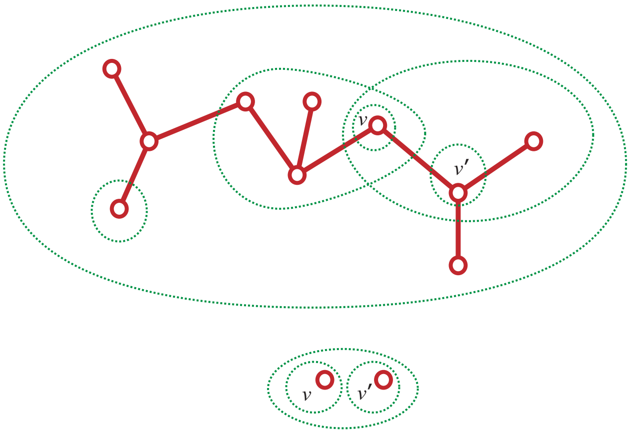


FIGURE 2. In contrast to Figure 1, this basis of open sets, represented by the dashed circled collections of nodes, gives us a subspace of v and v' in which they are topologically disconnected.

The first sort of optimization we want to focus on will apply to the problem of efficiently passing messages, either electronically or physically, from an source node to a final destination node. We will study networks that have at least initially a regular lattice (or mesh), topology, perhaps allowing connection failures to alter the topology as time progresses. In a regular lattice network there are typically many ways to travel along connections from a start node to a finish node, and there are still many choices of path even after restricting to the paths of minimal length.

For instance, Figure 3 shows two lattice network topologies in 2d. These are based respectively on the only two possible ways to tile a plane with regular polygons: the square and hexagon. These are named after the number of edges attached to each node: E_4 and E_3.

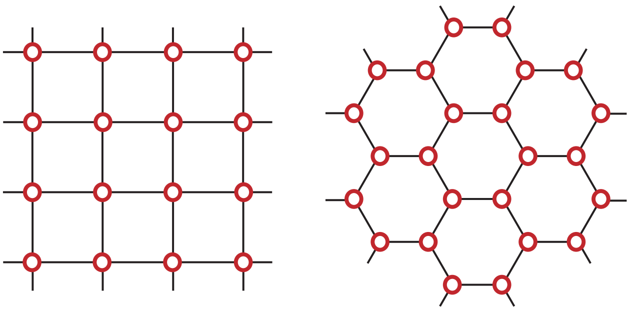


FIGURE 3. The simplest two regular lattice-based 2d networks: E_4 and E_3.

The E_4 network can be extended by allowing edges to connect diagonally positioned nodes. This is equivalently described by letting the square edges have unit length, and connecting any two nodes that are within $\sqrt{2}$. The resulting network, seen in Figure 4, is called E_8.

In [6] S. Sastry and collaborators showed that for certain two dimensional lattice networks there are precise formulas governing the load experienced by network nodes as messages are routed via shortest distance paths from a sender to a receiver. The simplest algorithm for distributing messages is for a node to equally distribute messages it receives among the nodes directly connected to it that lie on shortest paths to the eventual receiver.

Sastry et.al. begin by showing how in many lattice network topologies this simplistic approach leads to drastic overloads of some nodes and under-utilization of others. In that same paper the authors then formulated algorithms for several 2d lattices that correctly calculated the required distribution at each node in order to ensure equal loading.

For example, Figure 4 shows a lattice of nodes which are each connected to eight neighbors. The nodes labeled **S** and **D** are the source and destination respectively. The arrows show possible routes to take in order to travel along a shortest path from source to destination. We enclose the collection of shortest paths, in a rectangle in this case, and call it the *contour*. Contours come in several shapes depending on the relative locations of source and destination nodes.

First we demonstrate the round robin, or *uniform* method of disseminating messages. 99 messages are to be routed from source to destination. At each node we decide to send the received messages in equal numbers to the nodes further along shortest paths to the destination. (This can be done in real time by sending received messages to nodes "downstream" from you cyclically in sequence.) Figure 5 shows the results, where the number in a node is the (rounded up) number of messages it must process.

Notice that in Figure 5 the nodes along the long upper slanted edge end up processing many more than their share of the messages. This is obviously an inefficient algorithm for routing messages, since the overloaded nodes will be a bottleneck that puts a large lower limit on the time required to process all the messages from source to destination.

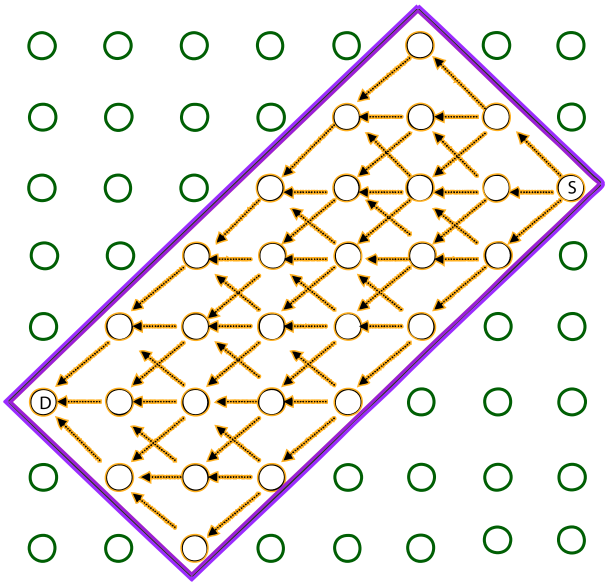


FIGURE 4. A contour in the E_8 2-dimensional network. Each path from the source (S) to the destination (D) has exactly seven steps. This network connects each node (circle) to all 8 of its nearest neighbors, but here we only show the edges (arrows) that are actually used in a shortest path. Adapted from [6].

Instead, the authors develop an algorithm that chooses how many messages to send to various nodes downstream based on whether a node finds itself in one of three regions: expansion, contraction or propagation. In Figure 6 we show the result of their new algorithm.

In the next stage of this research we want to study the analogous questions for the more realistic situations of 3d networks. This brings up the question of classification of networks.

3d networks arise from consideration of different *lattices* in 3d space. A lattice in this context is a regular array of points (our network nodes) generated by discrete translations. It is well know to those who study the mathematical theory of crystal structures

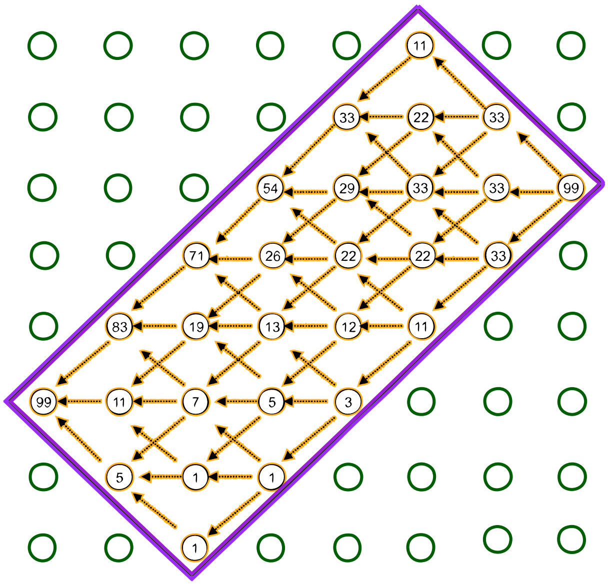


FIGURE 5. Uniform dissemination of messages showing asymmetrical loads. Note the high numbers of messages being handled by the nodes along the upper long slanted path. Adapted from [6].

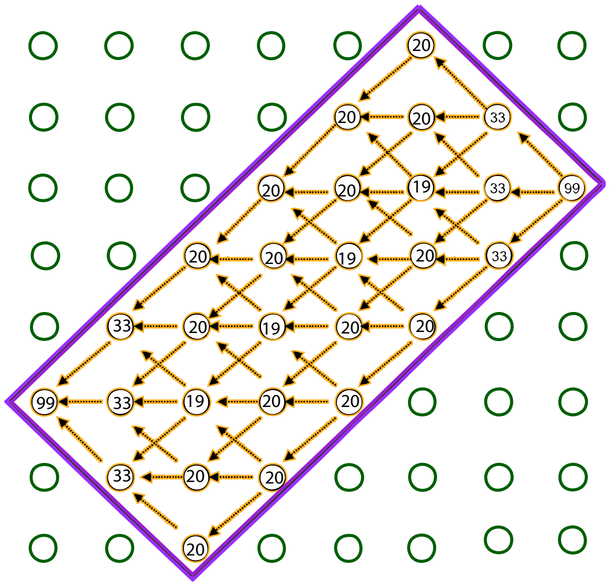


FIGURE 6. Contour guided dissemination of messages. Adapted from [6].

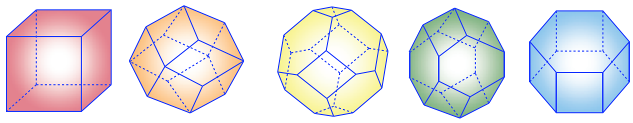


FIGURE 7. The primary parallelohedra, left to right: cube, rhombic dodecahedron, truncated octahedron, rhombo-hexagonal dodecahedron, and hexagonal prism.

that there are exactly 14 isomorphism classes of regular 3d lattices, known as the *Bravais lattices*. However, many of these give equivalent networks since we are more interested in the combinatorial properties than the geometrical. In fact if we consider the *Voronoi tessellation* of the lattice there are only five combinatorial possibilities.

The Voronoi tessellation is the division of space into regions, one containing each lattice point and all the points in space closer to that lattice point than any other. The regions for a lattice are *polyhedra*, 3d shapes with polygonal facets. There are exactly 5 different 3d polyhedron that tile space under translations alone and which are Voronoi tessellations. They are known as the primary parallelohedra, and we show them in Figure 7.

We will focus first on the cube, rhombic dodecahedron, and truncated octahedron. This choice is due to the fact that they are more symmetric than the last two parallelohedra in Figure 7, and thus more likely to yield tractable loading patterns. They also appear as the Voronoi tessellations of many of the most common lattices. These include the cubic lattice, where network nodes are at the corners of cubes, body centered cubic where we also add nodes in the center of each cube, and face centered cubic lattices where nodes are at centers of square facets as well. For each type of tessellation we study, the simplest corresponding network connects nodes at the centers of polyhedrons in the tessellation which share polygonal facets. That is, an edge between nodes goes through the shared facet.

In Figure 8 we show part of a cubic tessellation next to a corresponding part of a cubic lattice. In Figure 9 we show part of a rhombic dodecahedral tessellation next to a corresponding part of a face-centered cubic lattice. In Figure 10 we show part of a truncated octahedral tessellation next to a corresponding part of a body-centered cubic lattice.

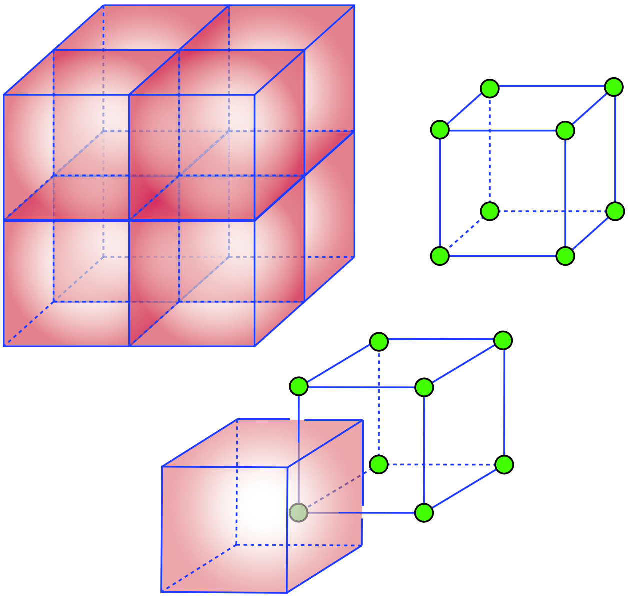


Figure 8. Eight Voronoi cells of a cubic tessellation on the left correspond to eight nodes in the cubic lattice. Underneath we show how the lattice nodes fit into the parallelahedron (cube). In this case the 6 network edges that "go through" the facets of the tessellation coincide with the edges of the cubes in the cubic lattice.

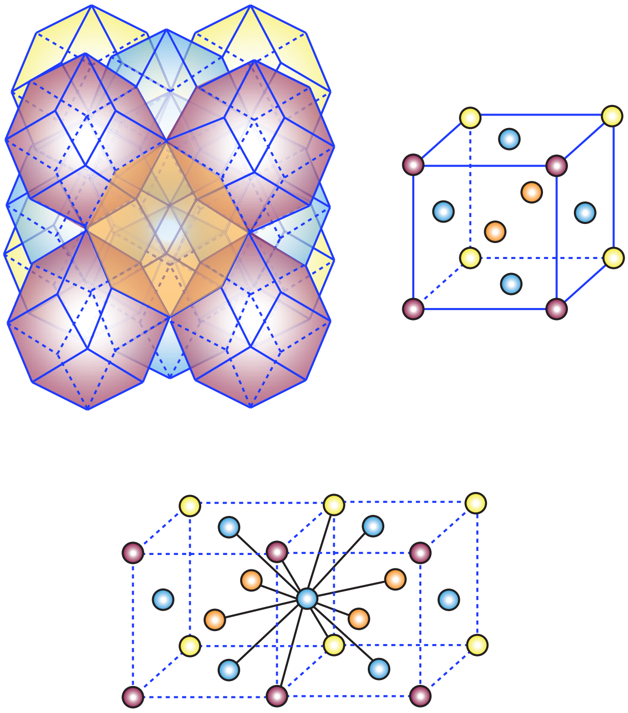


Figure 9. 14 Voronoi cells of a rhombic dodecahedral tessellation on the left correspond to 14 nodes in the face-centered cubic lattice. In this case twelve network edges "go through" facets, and those 12 edges are shown as solid lines in the lower picture.

The importance of seeing the relationships between the 3d lattice networks we were already studying and the parallelohedra is that the latter show how to increase the number of connections between nodes while preserving the regularity and symmetry of the network. For example, in 2d, a simple rectangular grid network has each node in a square connected to 4 squares that share a side. It becomes more highly connected when we also connect to nodes in squares that share a corner: eight in all. Next we should look next at analogous densifications of 3d networks.

For instance, there are several possible networks based on a unit cubical lattice. First, there are three basic networks that arise from allowing nodes to communicate with others within a spherical radius of 1, $\sqrt{2}$, or $\sqrt{3}$ units. Equivalently, radius 1 gives a network in 3d we call Γ_6, where edges between nodes

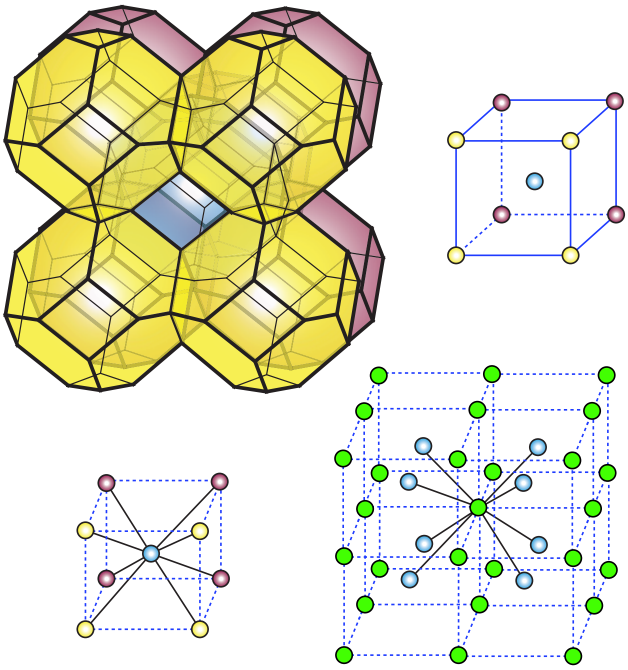


FIGURE 10. Nine Voronoi cells of a truncated octahedral tessellation on the left correspond to nine nodes in the body-centered cubic lattice. In this case we start by considering eight network edges that "go through" the hexagonal facets, and those 8 edges are shown as solid lines in the lower pictures.

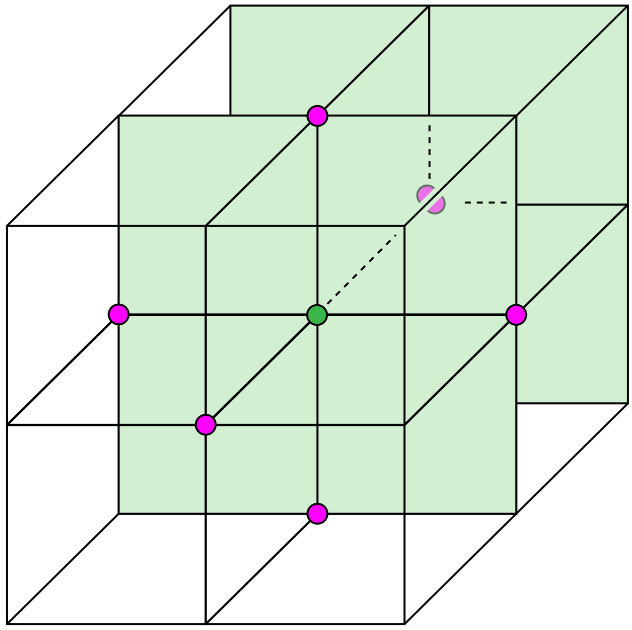


FIGURE 11. There are eight cubes stacked here, four in front and four in back. The center node is green and the six other nodes within radius 1 are also shaded.

pass through the cube facets, radius $\sqrt{2}$ gives Γ_{18} where edges also pass through the cube ridges, and radius $\sqrt{3}$ gives Γ_{26} where edges in the network pass through facets, ridges and corners. Figure 11 shows the radius 1 situation while Figures 12 and 13 show the $\sqrt{2}$ and $\sqrt{3}$ radii.

Next we can extend the rhombic dodecahedron in a way analogous to going from E_4 in 2d to E_8 in 2d. For the rhombic dodecahedron we get Γ_{12} from facets, Γ_{36} including the ridges and Γ_{50} including ridges and corners. Thus the largest amount of regular connection seen this way using a rhombic dodecahedron is 50 connections. Similar expansions are planned for the truncated octahedral networks.

1.1.1. *Background.* Dr. Sastry and students have conducted extensive searches in the current literature dealing with 3d peer-to-peer networks, their efficiency and stability. The following is an abbreviated selection from the background section of the paper we are currently in progress of preparing.

3d network structures are relevant for many wireless sensor applications including civil infrastructure monitoring (e.g., bridges, dams, roads, and levies), surveillance, and in the area of Network-on-Chip (NoC). A collection of regular and non-regular 3D mesh topologies was reported on in [7]. These topologies were realized as transformations of a 3d six-neighbor topology. Their analysis showed that higher degree topologies could sustain higher communication loads with a fewer number of channels.

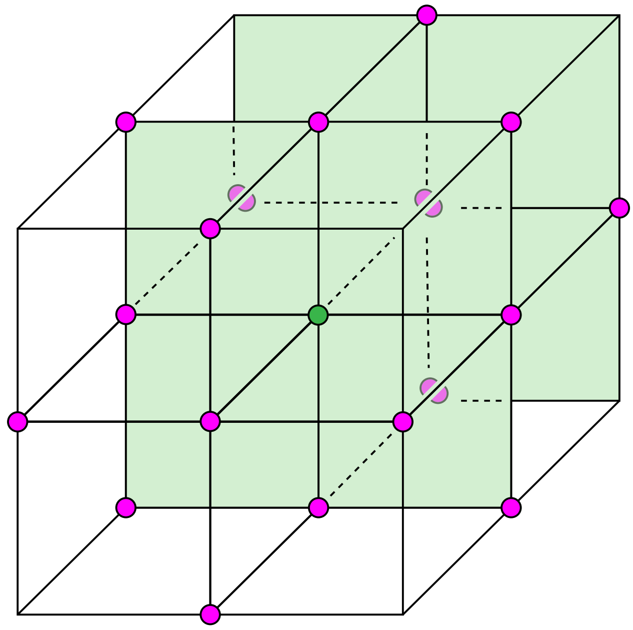


FIGURE 12. Now included are all of the 18 neighbors within $\sqrt{2}$ of the central node.

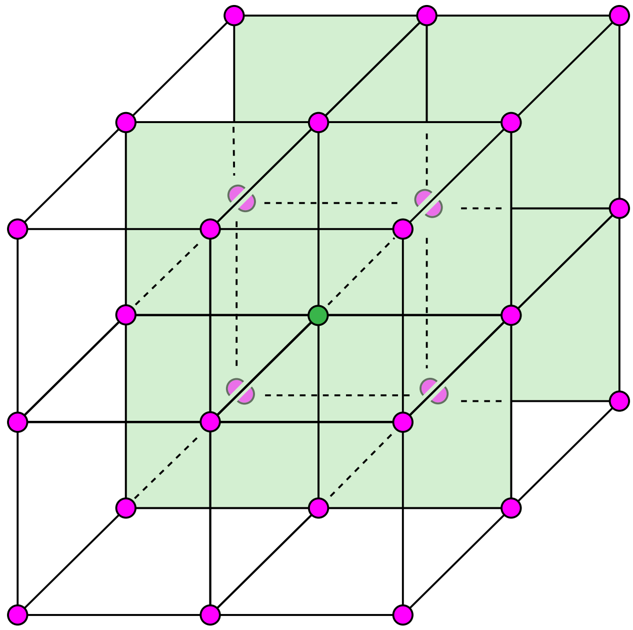


FIGURE 13. Now included are all of the 26 neighbors within $\sqrt{3}$ of the central node.

The 3d peer-to-peer topology described in [2] is derived from a planar triangular tessellation. Multi-Mesh architecture for a network of processors as reported on in [1] combines multiple 3d blocks. Each block is obtained by connecting each processor to six other processors along the usual x, y and z major axes. A novel addressing scheme and optimal routing algorithm for 2d hexagonal networks which uses a formula based on the distance formula is reported in [4]. The authors also proposed a 3d network based on two dimensional hexagonal meshes for cellular mobile computing indoor environment.

An algorithm in [5] provides load balancing in general hybrid mesh grid based on serial graph partitioning. Simplification of a 3d mesh by applying Delaunay topology constraints is achieved in [3]. Neither of these papers focus on, nor exploit, the union of all the shortest paths between a pair of nodes.

1.1.2. *Goals, impacts and expected outcomes.* Our long-term goal is to develop a complete theory (collection of useful theorems) describing contours of and loading under uniform dissemination for 3d lattice embeddings, as well as precise connections (modeling) between the discrete theorems and the behavior of continuous systems that they approximate. Then we want to actually publish explanations of how to use the theorems and the models in applications such as constructing filters, grounding electrical grids, and building smart superstructures for construction at various scales.

For the short term, in the summer of 2013, we want to publish several initial results and develop one or two external funding proposals. One opportunity under serious consideration is the Integrated NSF Support Promoting Interdisciplinary Research and Education (INSPIRE) grant program. Its stated purpose is to support "bold interdisciplinary projects in all NSF-supported areas of science, engineering, and education research." The first step is a letter of intent, after which the program invites full applications based on whether they promise to "address some of the most complicated and pressing scientific problems that lie at the intersection of traditional disciplines." We think that the multiple levels of new theory and new applications, from peer-to-peer networks to filters to bio-structures , fit this bill.

Simultaneously there are also more specialized possibilities, including the NSF Department of Mathematical Sciences Combinatorics grant program, to which we can apply with a focus on the theoretical part of our overall research program. Either way (one large grant or several small ones) we have high hopes that this new collaboration will be the foundation of a long-term productive research group. The ideas are well-suited to student involvement, and the proposals we write will be heavily focused on integrating graduate and undergraduates, and securing them financial support.

The initial results we are aiming for include description of the contours for the basic 3d lattices. To prepare the proposal we need both these theorems on contours and a good set of experimental results for the loading under uniform dissemination.

1.2. **Procedures.** The first method for developing our theory is heavy experimentation, on paper and using simple programs to examine the kinds of contours and loading that occur in Γ_6, Γ_{18}, Γ_{26}, Γ_{12}, Γ_{36} and Γ_{50}.

Figures 14 and 15 show an example of finding the contour, or collection of shortest paths, in a cubic network, specifically Γ_{18} (the one allowing connections in a radius of $\sqrt{2}$ of the central node, as seen in Figure 12.)

Next we plan to continue experiments that test the results of uniform dissemination on contours in 3d. This will be facilitated by computer calculations. In fact, we also have the use of 3d modeling software that will allow the contours of various source and destination arrangements to be visualized.

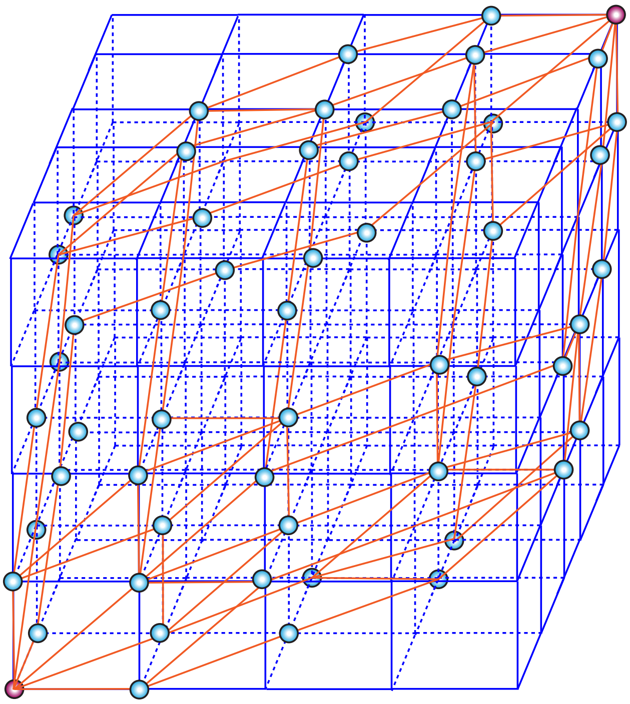


FIGURE 14. Finding the contour, or collection of shortest paths, in the cubic network Γ_{18}. Here the source and destination are, respectively, at the upper back right and lower left front of a $4 \times 5 \times 5$ cubic lattice.

1.3. **Timeline.** The two grant programs we are interested in initially are accepting proposals both in 2013 and 2014. Optimistically we are hoping to have grant proposals ready in this year's cycle. Figure 16 shows the deadlines for the year 2013, as well as our goal for an initial publication in mid-summer.

The NSF Inspire grant requires a letter of intent (LOI) which is due on March 29. The program directors reply by the end of the next month with an invitation to submit a full proposal, which is due at the end of May. Here is the url for the INSPIRE program: http://www.nsf.gov/pubs/2013/nsf13518/nsf13518.htm

The NSF DMS Combinaorics program does not require an LOI, and accepts proposals on the first Tuesday of October. Here is the

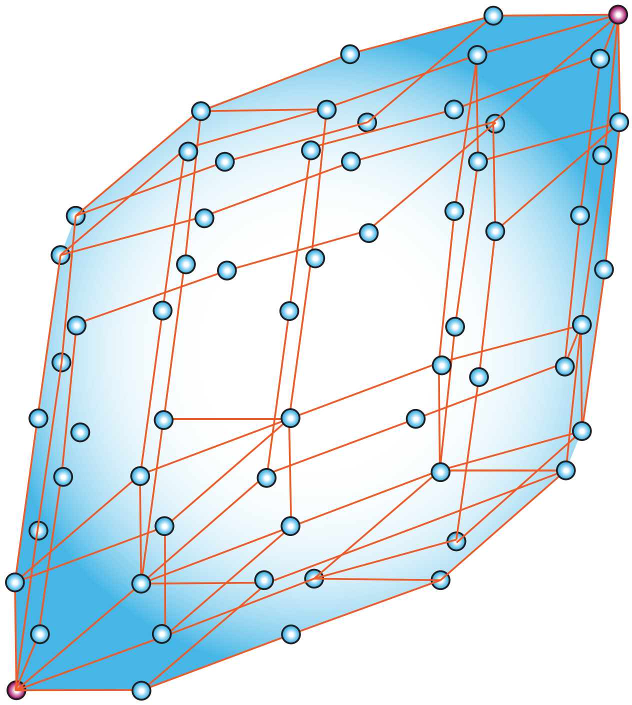


FIGURE 15. This is the shape of the contour we just found in Figure 14. Not all paths are shown, but each path is of length seven steps.

url for the Combinatorics program: http://www.nsf.gov/funding/pgm_summ.jsp?pims_id=503570&org=DMS

References

1. N. Afroz, B. Sinha, R. Islam, and S. Bandyopadhyay, *A new network topology with multiple three-dimensional meshes.*, IWDC'04 Proceedings of the 6th international conference on Distributed Computing (2004), 379–384.
2. J. Carle and J. Myoupo, *Topological properties and optimal routing algorithms for three dimensional hexagonal networks.*, IEEE International Conference on Image Processing (2006), 1893–1896.
3. I. Celasu, S. Eroksuz, R. Siddiqui, and A. Tekalp, *Hierarchical representation and coding of 3d mesh geometry.*, International Conference/Exhibition on High Per- formance Computing in the Asia-Pacific Region **1** (2000), 116–121.
4. C. Decayeux and D. Seme, *3d hexagonal network: modeling, topological properties, addressing scheme, and optimal routing algorithm.*, IEEE Transactions on Parallel and Distributed Systems **16(9)** (2005), 875–884.
5. C. Kavouklis and Y. Kallinderis., *Parallel adaptation of general three-dimensional hybrid meshes.*, Journal of Computational Physics **229** (2010), 3454–3473.
6. K.K. Mamidisetty, M. Duan, S. Sastry, and P.S. Sastry, *Multipath dissemination in regular mesh topologies.*, IEEE Transactions on Parallel and Distributed Systems **20(8)** (2009), 188–1201.
7. J. Nguyen, J. Pezaris, G. Pratt, and S. Ward, *Three-dimensional network topologies.*, Parallel Computer Routing and Communication, Lecture Notes in Computer Science, vol. vol. 853, Springer, Berlin / Heidelberg, 1994.

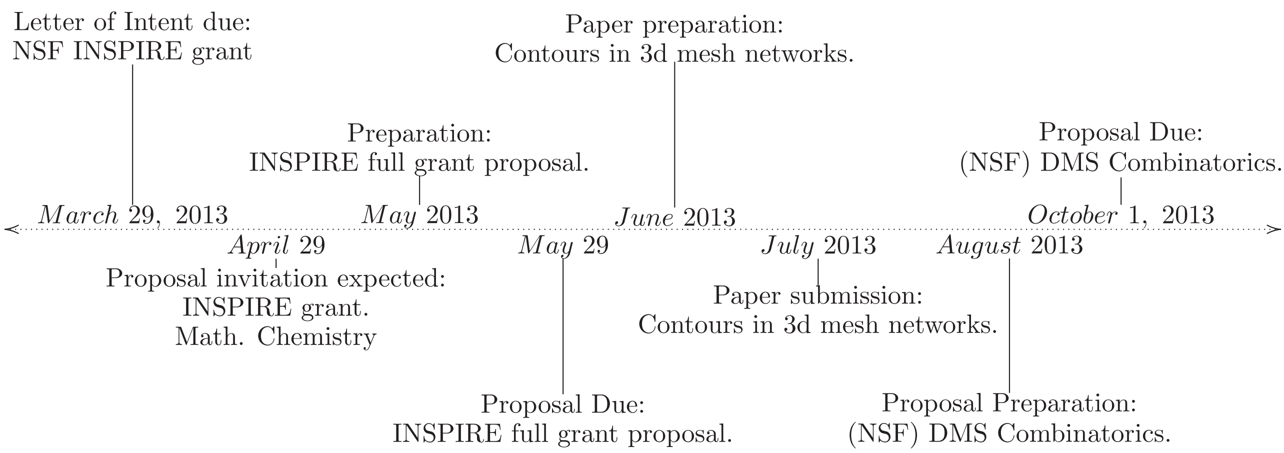


FIGURE 16. Timeline.