Project description:
Natural polyhedra as a foundation for high dimensional transformations.

1 Introduction

Herein is described proposed work on a set of dual problems in geometric combinatorics and higher dimensional category theory. First is the problem of finding a complete combinatorial description of two sequences of polytopes which have important universal properties. Both are related directly to Stasheff's associahedra. The immediate descendents of the associahedra are the composihedra. The naturahedra are second generation descendents of the associahedra in this line. The boundary of the i^{th} associahedron $\mathcal{K}(i)$ is topologically equivalent to the $(i-3)$ sphere. The boundary of the i^{th} composihedron $\mathcal{CK}(i)$ is topologically equivalent to $S^{(i-2)}$ and that of the i^{th} naturahedron $\mathcal{N}(i)$ to $S^{(i-1)}$. The sequences of naturahedra and composihedra will be recursively defined and the facets of each term will be constituted of instances of the preceding term as well as terms of its ancestral sequences. A constructive description will precede the complete combinatorial description. The latter will consist of two parts. The first data is the listing of the faces, or dimension $(i-1)$ polytopes that constitute the i^{th} naturahedron or composihedron. The second data is the recipe for the intersection possessed by any two faces, which determines their assembly into the full polytope. This proposal will give details of the problem, describe initial work already completed, outline the steps to completion together with the sources that will enable progress and discuss the applications of the work to physics, computer science and broader mathematics.

By way of a brief introduction here is a comparison of vertex and edge data in the underlying graphs of the associahedra, composihedra and naturahedra. The language of a free monoid on a collection of letters is convenient. Elements are words in the given alphabet. Strings of elements will be denoted by comma delimited lists. The vertexes of the n^{th} associahedron are indexed by complete bracketings of a string of n distinct letters. A complete bracketing is a parenthization of the string so that there is no ambiguity with respect to the order of the performance of a binary multiplication; this is why the associahedra are named as such and used to parameterize homotopy associative spaces. The edges are indexed by incomplete bracketings that have two possible completions. The vertexes of the n^{th} composihedron are indexed by the complete bracketings on all strings of $k \le n$ nonempty words equivalent to a given string of n letters. For two strings of words to be equivalent is for their respective entire concatenations to yield the same final word. Hence $a, b, c \sim ab, c \sim a, bc$. Here edges include both those of the evident included associahedra and those indexed by triples of words where the third is the concatenation of the first two. Finally, the n^{th} naturahedron contains the $(n-1)^{st}$ composihedron. Given a string of $n-1$ letters its vertices are indexed in two groups: the first by completely bracketed equivalent strings of $k \le n$ words based on the original string of $n-1$ letters and a single copy of the empty word (for example, without showing bracketing, $a, b, \_\,, c \quad \sim \quad ab, c \quad \sim \quad \_\,, a, bc \quad \sim \quad abc \quad \sim \dots$); the second by completely bracketed strings of $k \le n$ words equivalent to the original string of $n-1$ letters and a single central element 1 (hence $a, b, 1, c \quad \sim \quad ab, 1, c \quad \sim \quad ab, c1 \quad \sim \quad 1abc \quad \sim \dots$) The edges here include the edges of the included composihedra, those indexed by the replacement of the empty word with the central element 1 and (optionally) those indexed by the switching inside a word of the central element with an adjacent subword.

Secondly it is essential to prove the existence of parity complex structures on the composihedra and naturahedra. This refers to a decomposition of the boundary of each facet of a polytope into a disjoint union of "negative" and "positive" faces. This then allows (after some flipping from positive to negative) a comprehensive such structure on the entire polytope. From the top down perspective of the entire polytope there is a choice of two special vertices and a division of the codimension one faces of the polytope into two contiguous regions whose shared boundary contains the special vertices. The division of a given term is required to preserve the already defined divisions on the preceding terms that are included as faces. A parity complex structure allows the categorical interpretation of a polytope as giving the source and target of a higher morphism. The two halves of the polytope are pasting diagrams for the categorical codimension one morphisms. Part of the proposed project then is to provide the translation to category theory once the full description and existence of parity is in place.

The categorical problem is to use the polytopes to define two important concepts. The composihedra are used to define the concept of weak enrichment. Enriched categories usefully generalize the standard sort of category in which there are sets of maps between objects. Enrichment allows either more or less structure than that of a set to be specified of what exists between objects. In short, rather than a set of maps, for every pair of objects there is a map object in a monoidal category. One way in which the concept of enrichment is central to category theory is that the category of categories, **Cat**, is actually enriched over itself. For every pair of categories there is a category of functors and natural transformations between them. In general a category enriched over **Cat** is called a 2-category and an n-category is a category enriched over the category of $(n-1)$-categories. By categorical dimension is meant the level of this enrichment. High dimension allows the enrichment process to be weakened. Weakening in this context refers to the transition from strictly associative composition of morphisms to the existence of higher morphisms that mediate associativity. This transition allows for far more applications to other areas of algebra, topology and analysis. Weak enrichment will be shown to be parameterized by the composihedra.

High categorical dimension also allows the definition of higher levels of natural transformations. The first few levels in the n-dimensional setting are referred to as n-functors, n-natural transformations and n-modifications. Where weakness is the accepted term for describing structure in a category, laxness is the corresponding term for the exterior structure of transformations. Maps between categories are required to preserve categorical structure such as composition of morphisms. Laxifying is a transition from strict preservation to a version with higher morphisms that mediate the structure preservation. The first two examples of this are the lax enriched n-natural transformations between lax enriched n-functors. The naturahedra model precisely the series of diagrams needed to describe the lax enriched k-cells between strict enriched $(k-1)$ cells between lax enriched n-functors between two weakly enriched n-categories.

Special cases of the expected results will include the redescription of bicategory and tricategory morphisms as in [30] and [19] respectively. Potential special cases include the probicategory morphisms and tetracategory morphisms neither of which have ever been fully described. Eventually weakened unit axioms will also have to be described to complete the picture; at the first step of the research strict units will be assumed in interest of clarity. Further possible developments include the extension to lax enriched k-cells between lax enriched $(k-1)$ cells. From initial investigations it appears that these will be parameterized by a further generalization of the naturahedra, the n-naturahedra. Where the naturahedra vertices are indexed by a strings with a single copy of the empty word and a single central element the n-naturahedra have n of each.

2 Applications

Topology provides ample justification for studying higher dimensional categories. The fact that n is the number of ways of composing n-cells is quite useful in modeling cobordisms between cobordisms and homotopies between homotopies. In both of these cases however the strict version of n-categories leads to a trivial sort of model, one that loses too much information. Degenerate (weak) n-categories with only one cell in the lower dimensions allow the modeling of nontrivial embeddings of cobordisms, such as braids, ribbon tangles, and knotted surfaces. In homotopy theory the composition of homotopies is not strictly associative, it is only associative up to a higher homotopy. This situation is also familiar from the study of spaces with homotopy associative products. These include the A_n spaces and the loop spaces.

Operads in a category of topological spaces are the crystallization of several approaches to the recognition problem for iterated loop spaces. Beginning with Stasheff's associahedra and Boardman and Vogt's little n-cubes, and continuing with more general A_∞, E_n and E_∞ operads described by May and others, that problem has largely been solved. [35], [8], [33] Loop spaces are characterized by admitting an operad action of the appropriate kind. More lately Batanin's approach to higher categories through internal and higher operads promises to shed further light on the combinatorics of E_n spaces. [6], [7]

Recently there has also been growing interest in the application of higher dimensional structured categories to the characterization of loop spaces. The program being advanced by many categorical homotopy theorists seeks to model the coherence laws governing homotopy types with the coherence axioms of structured n-categories. By modeling we mean a connection that will necessarily be in the form of a functorial equivalence between categories of special categories and categories of special spaces. The largest challenges currently are to find the most natural and efficient definition of (weak) n-category, and to find the right sort of connecting n-functor. The latter will almost certainly be analogous to the nerve functor on 1-categories, which preserves homotopy equivalence. In [36] Street defines the nerve of a strict n-category. Recently Duskin in [13] has worked out the description of the nerve of a bicategory. This project hopes to shed light on both problems, chiefly by providing a framework in which to compare existing definitions.

The chief difficulty that arises in any naive attempt to define a weak n-category is that the legs of axiomatic diagrams that would commute in the strict case but instead form the domain and range of a higher cell are themselves not well defined pasting diagrams. Indeed all the ways of expressing the full composition of a higher dimensional pasting diagram in a weak n-category are not equal. Instead there exist coherent isomorphisms between them. To avoid a special choice of composition order one can make use of coherence theorems about the entities in which the undefined compositions take place. For instance, any diagram in a bicategory can be defined as having the value of the preimage of the composition of its image taken in the equivalent strict 2-category. The difficulty then truly arises at dimension three. In [19] Gordon, Power, and Street demonstrate that their tricategories are not all equivalent to corresponding strict versions. As examples of such pathological tricategories they offer braided monoidal categories (not symmetric and the braiding not the identity on the diagonal) as single-object, single-arrow tricategories. Conceptually then, the problem of not having an equivalent strict arena in which to paste is connected to the existence of nontrivial embeddings of codimension two, and assumed to persist as n increases. There are quite a few approaches to defining (weak) n-categories that successfully avoid this problem. The new problem then is to find ways of comparing these varied definitions and of isolating their shared axioms in a way analogous to the unifying definition of a cohomology. A first check of the validity of these definitions as is performed in [30] is that they agree with the classical definitions of category and bicategory. An application of this project is to provide an analogous series of higher checks by which full definitions of n-category may be gauged and compared. Two definitions of weak n-category could be comparatively studied by asking for how many values of n is every one of the examples of the respective definitions equivalent to a weakly enriched n-category.

The founders of category theory have stated that the reason for its conception was to make possible the study of natural transformations. Two reasons for this prioritization are immediately evident. First is the fact that there are two ways of composing natural transformations and thus they can be seen as the 2-cells between functors in the 2-category of categories, which is the model for all 2-categories. Secondly they allow the efficient definition of concepts that rely on universal properties, specifically the concepts of limits and colimits. Not only does this allow a common framework for much of algebra and analysis, but recently there have been exciting direct applications of limits to computing. At the International Joint Conference on Neural Networks (IJCNN05) in Montreal Mike Healy presented joint work with Sandia National Laboratories in research on category theory applied to neural networks. They modified standard artificial neural architecture by adding a neural representation of limit cones and using these to exert fine control over the network operation. The program was used in generating a multispectral image from satellite data, with results greatly improved over the standard result [22]. One of the reasons for carefully defining higher natural transformations is that they provide the foundation for higher limits, which one might imagine will be utilized in future iterations of this type of application.

3 Combinatorics and geometry

Stasheff has completely described the associahedra $\mathcal{K}(i)$ in [35] and [32]. The recursive definition of the n^{th} associahedron states that it is made up of the faces which correspond to a placement of exactly one set of parentheses into a (comma delimited) string of n letters. These faces each are products of smaller associahedra. The intersection of two faces corresponds to the disjoint or nested placing in the string of two sets of parentheses. There are several proofs of the fact that the associahedra are convex polytopes. One exhibits them as truncated simplices. Another utilizes the fact that the vertices of the n^{th} associahedron can be alternately indexed by full binary trees with n leaves. The number of vertices of the n^{th} associahedra is given by the n^{th} Catalan number.

Although the composihedra are newly introduced, the combinatorics of their vertices are also well known. The number of vertices in the n^{th} composihedron is the sequence that begins:

$$1,\ 2,\ 5,\ 15,\ 51,\ 188,\ 731,\ 2950,\ 12235,\ 51822,\ 223191,\ 974427,\ 4302645,\ \ldots.$$

This sequence is the binomial transform of the Catalan numbers. It can be described in several ways. The closest description, which gives an alternate way of indexing the vertices, is that the sequence gives the number of binary trees of weight n where leaves have positive integer weights. This is the non-commutative non-associative version of partitions of n. Two other combinatorial questions that this sequence answers are:

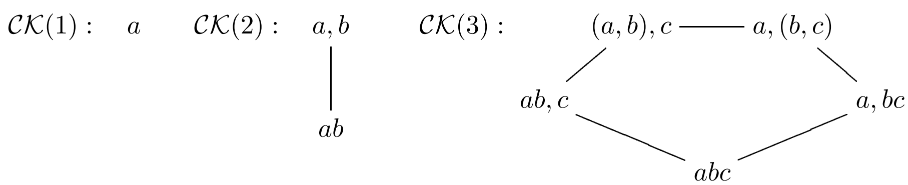
the number of Schroeder paths (i.e. consisting of steps $U = (1, 1), D = (1, -1), H = (2, 0)$ and never going below the x-axis) from $(0, 0)$ to $(2n-2, 0)$, with no peaks at even level; and the partial sums of the sequence given by the number of restricted hexagonal polyominoes with n cells [12]. Further combinatorial questions include the enumeration of the edges and other higher dimensional faces. The sequence of numbers of edges in the n^{th} composihedron begins $0,\ 1,\ 5,\ 23,\ 112,\ \ldots.$

The composihedra have not yet been described recursively, but the problem of doing so should not be difficult. Here is the beginning of the constructive definition, using strings of elements of the free monoid on an alphabet. Equivalent strings are those that give the same word when concatenated. For example $a, bc, de \quad \sim \quad a, \_\, , bcde$ since both give $abcde$ upon concatenation.

Definition 1 *(draft) Given a string of $n > 1$ letters, the n^{th} composihedra $\mathcal{CK}(i)$ is a cell complex whose vertices (0-cells) correspond to completely bracketed equivalent strings of $k \leq n$ words. Edges (1-cells) correspond to either an incomplete bracketing of a string that has as its completions the two vertices it connects, or a concatenation of two words separated only by a comma. Higher cells are automatically determined, and are instances of (products of) $\mathcal{CK}(i)$ and $\mathcal{K}(i)$ for $i < n$.*

Example 1

Here are the first few composihedra with vertices labeled by bracketed word strings. Notice how the n^{th} composihedron is based on the n^{th} associahedron. The Schlegel diagram is shown for $\mathcal{CK}(4)$.



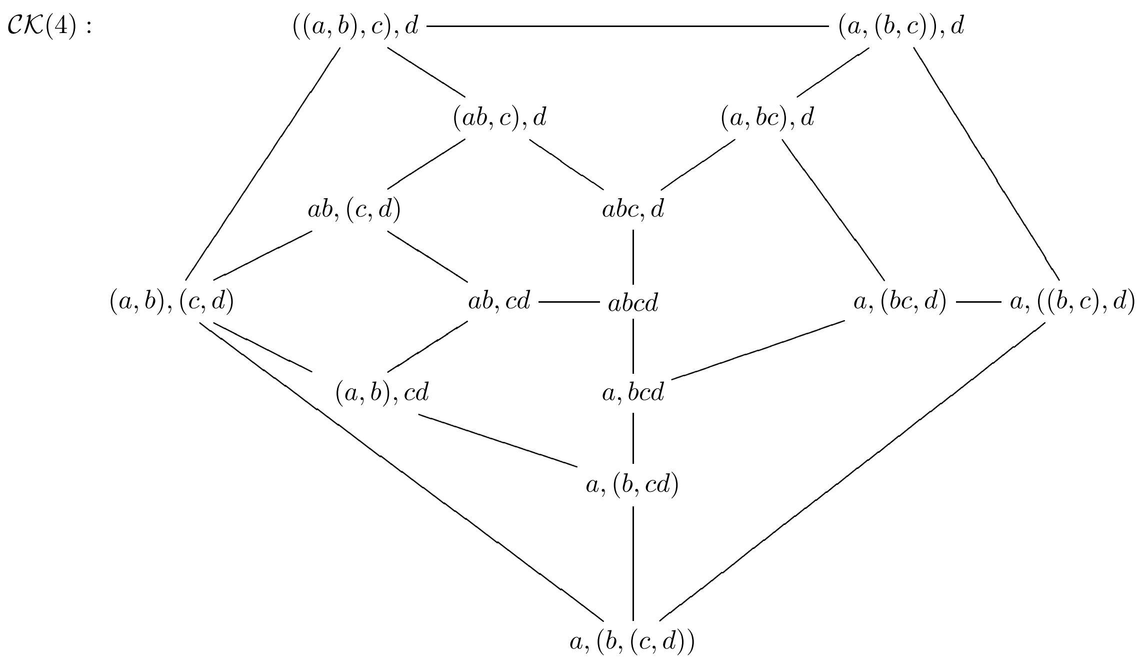


Figure 1 shows the last of the above examples, $\mathcal{CK}(4)$, drawn 3-dimensionally. The bold outlined face pentagon in the upper right is the copy of $\mathcal{K}(4)$ which appears on the outside of the above diagram.

In order to construct the recursive definition the investigators may consult [23], where the related sequence of polytopes called multiplihedra is described. These are constructed by considering the way in which a homomorphism ϕ respects multiplication. Starting from one of the associahedra, new edges are added to represent each relationship of the form $\phi(ab) \to \phi(a)\phi(b)$.

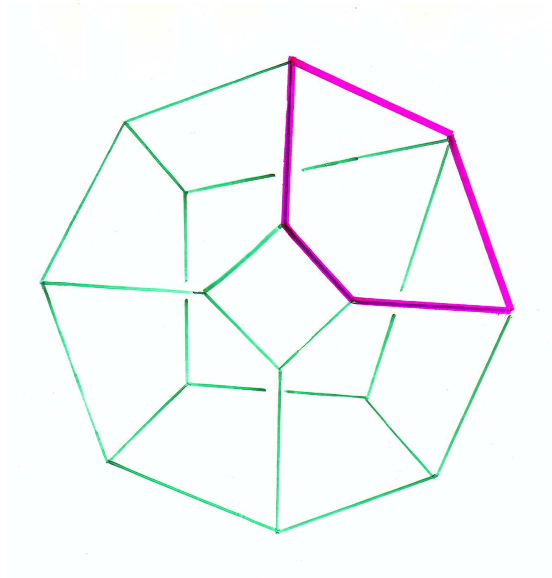


Figure 1: $\mathcal{CK}(4)$

Lemma 1 *(draft) A truncation of the polytope $\mathcal{CK}(i)$ gives us the well known polytope $\mathcal{J}(i)$ of the family that goes by the name of multiplihedra.*

The proof of this will rest on the observed pattern. The first few of the truncations needed to get from $\mathcal{CK}(i)$ to $\mathcal{J}(i)$ are shown in Figure 2.

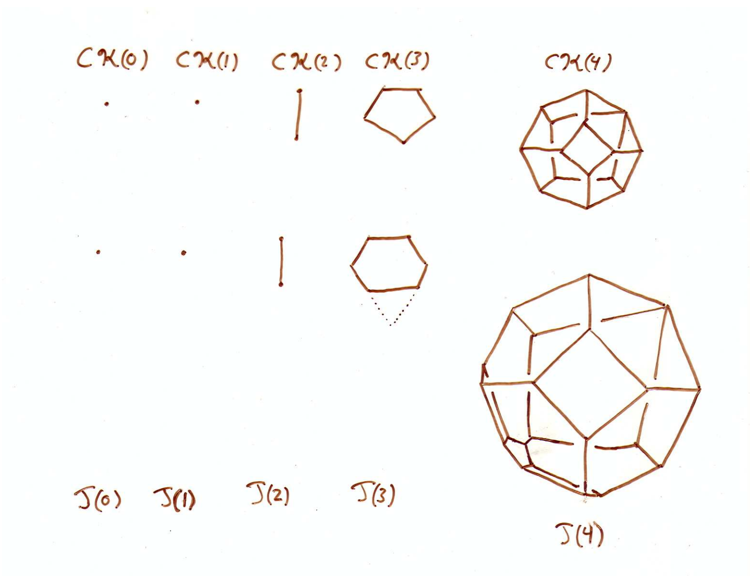


Figure 2: Multiplihedra are truncated composihedra.

Theorems presented here and labeled as "draft" are meant to demonstrate some precise goals of the project.

Theorem 1 *(draft) The n^{th} composihedron has boundary topologically equivalent to the $(n-2)$-sphere. In fact it is the boundary of a convex $(n-1)$ dimensional polytope.*

Proof sketch: The case of the polyhedron $\mathcal{CK}(4)$ follows from the Steinitz theorem which states that any simple planar 3-connected graph can be realized as a convex polyhedron. The general case will follow from a complete recursive combinatorial description, and use of the fact that the composihedra are basically a subdivided cone on the n^{th} associahedron. Thus it will depend also on the related theorem for the associahedra in [32] and [35]. There is also a great deal of knowledge about the nature of the graph of a n-dimensional polytope. For instance Balinski's theorem does state that the graph of an n-polytope is n-connected [2]. Also, every graph of an n-polytope contains a subdivision of the complete graph on $n+1$ vertices [20]. Other valuable sources include: [28], [25], [26] and [39].

Theorem 2 *(draft) The composihedra possess a parity complex structure.*

Proof sketch: This will follow from the parity complex structure on the associahedra shown by Trimble, and the parity structure on prisms and products of polytopes with parity structures shown by Street.

The strings of words in the composihedra are assumed to be made up of nonempty words, and the letters are assumed to be non-central. This suggests two ways to generalize the indexing of the vertices of the composihedra. One is to consider bracketed strings that include a certain number of instance of the empty word. The simplest next step is to allow at most one instance of the empty word. This corresponds to weighted trees of a given weight that may include at most one leaf of weight zero. The number of these for weight n form a new sequence that begins $1,\ 3,\ 10,\ 39,\ 165,\ \ldots$ The general formula for this sequence is unknown but would make a good problem for an undergraduate project. The other generalization is to consider including in the original string of letters a central generator of the monoid. It is denoted by 1. Now there are words that are considered equal, such as $ab1c = abc1$ and for the sake of simplicity in the definition these equal words are identified and written with the unit first, $1abc$. The number of bracketed strings of words based on n letters including one central letter is a sequence that begins $1,\ 3,\ 11,\ 45,\ 195,\ 873,\ \ldots$ This is the same beginning as the sequence formed by taking the binomial transform of the central binomial coefficients. To prove the conjecture that the two sequences are identical is another good undergraduate problem. Neither generalization of combinatorial indexing leads immediately to a sequence of polytopes. However, both indexes contain the examples in which the new element, respectively the empty word and the central generator, have yet to be concatenated with any neighboring words in the sequence. The number of these examples is the same for both cases, and thus by connecting the corresponding instances of these cases a polytope is formed.

Definition 2 *Given a string of $n-1$ letters the n^{th} naturahedron is a cell complex whose vertices are indexed by completely bracketed distinct equivalent strings of $k \leq n$ words based on the original string of $n-1$ letters together with either a single copy of the empty word or a single central element 1. In addition to edges inherited from included composihedra are those where the empty word and the unit 1 occupy the same location in the bracketed strings which are the vertices connected by that edge.*

Example 2

Here are the first few naturahedra. Notice how the n^{th} naturahedron is based on the $(n-1)^{st}$ composihedron.

$$\mathcal{N}(1): \quad \_ \; \text{———} \; 1 \qquad\qquad \mathcal{N}(2): \quad \begin{tikzcd}[column sep=small] & \_\,,a \arrow[r, dash] \arrow[dl, dash] & 1,a \arrow[dr, dash] & \\ a \arrow[dr, dash] & & & 1a \\ & a,\_ \arrow[r, dash] & a,1 \arrow[ur, dash] & \end{tikzcd}$$

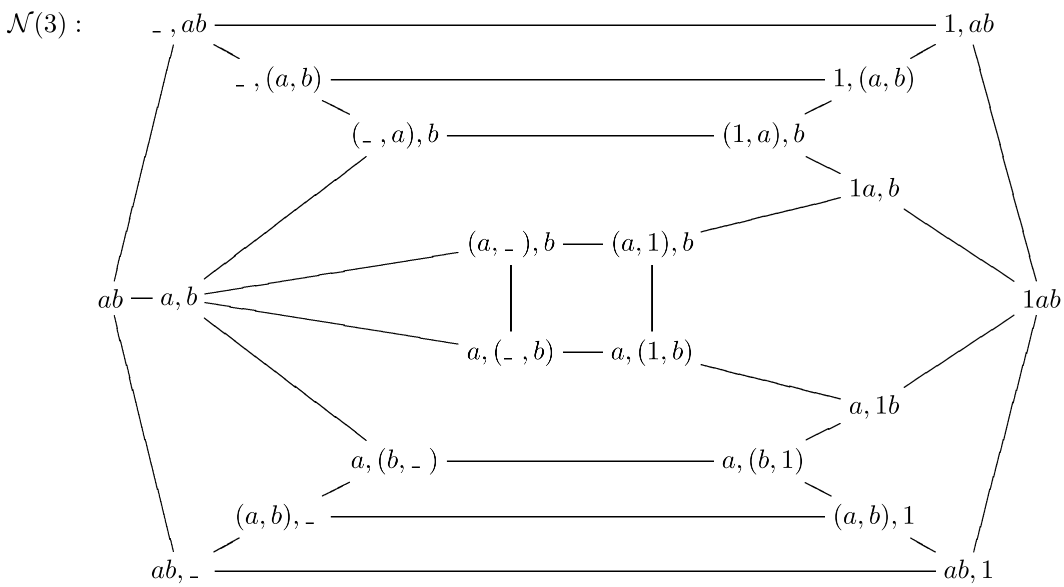


Figure 3 shows the last of the above examples, $\mathcal{N}(3)$, drawn 3-dimensionally. The bold edge in the upper center is the copy of $\mathcal{CK}(2)$ which appears at the far left of the above diagram.

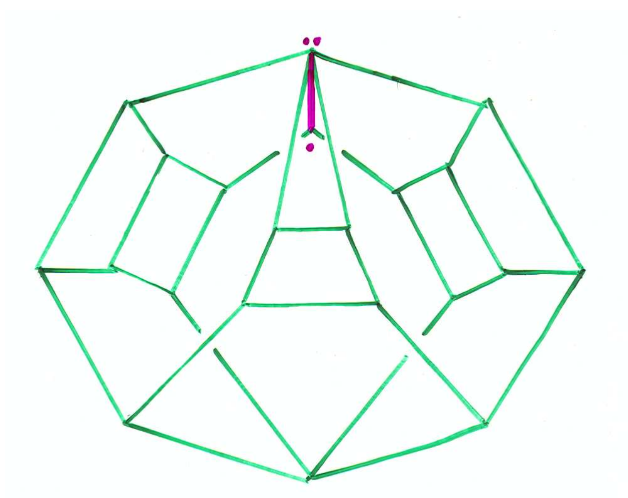


Figure 3: $\mathcal{N}(3)$

Theorem 3 *(draft) The n^{th} naturahedron has boundary topologically equivalent to the $(n-1)$-sphere. In fact it is the boundary of a convex n dimensional polytope.*

Proof sketch: The case of the polyhedron $\mathcal{N}(3)$ follows from the Steinitz theorem which states that any simple planar 3-connected graph can be realized as a convex polyhedron. The general case will follow from a complete recursive combinatorial description, and use of the fact that the naturahedra are basically the joining of two generalized copies of composihedra.

Theorem 4 *(draft) The naturahedra posess a parity complex structure.*

Proof sketch: This will follow from the previous parity structures shown for the associahedra and composihedra. Also important may be the well known theorem that any polytopal directed graph has exactly one sink and one source [38].

Further geometric combinatorial questions involve further generalizations of the composihedra and analogous polytopes based on the cousins of the associahedra. These include the permutohedra, associa-permutohedra, cyclohedra and more generally the graph associahedra of Devadoss [10].

4 Strict Enrichment: the Category of $\mathcal{V}$-n-Categories

In [17] the principal investigator defines enriched n-categories and their morphisms. Also is shown how the morphisms are composed. Here are just the definitions. The definition of a $\mathcal{V}$-n-category for $\mathcal{V}$ symmetric monoidal is just a category enriched over $\mathcal{V}$-$(n-1)$-Cat. In the following a superscript in parentheses adorning a product denotes the level of enrichment of its operands. Here is the expanded definition:

Definition 3 *A (small, strict) $\mathcal{V}$-n-category $\boldsymbol{\mathcal{U}}$ consists of*

1. *A set of objects $|\boldsymbol{\mathcal{U}}|$*
2. *For each pair of objects $A, B \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-category $\boldsymbol{\mathcal{U}}(A, B)$.*
3. *For each triple of objects $A, B, C \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-functor*
$$\boldsymbol{\mathcal{M}}_{ABC} : \boldsymbol{\mathcal{U}}(B, C) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A, B) \to \boldsymbol{\mathcal{U}}(A, C)$$
4. *For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$-$(n-1)$-functor*
$$\boldsymbol{\mathcal{J}}_A : \boldsymbol{\mathcal{I}}^{(n-1)} \to \boldsymbol{\mathcal{U}}(A, A)$$
5. *Axioms: The $\mathcal{V}$-$(n-1)$-functors that play the role of composition and identity obey commutativity of a pentagonal diagram (associativity axiom) and of two triangular diagrams (unit axioms). This amounts to saying that the functors given by the two legs of each diagram are equal. To save space "$\bullet\bullet \to \bullet$" will represent $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{U}}(B, C) \otimes \boldsymbol{\mathcal{U}}(A, B) \to \boldsymbol{\mathcal{U}}(A, C)$. Thus the first location in the pentagon stands in for*
$$(\boldsymbol{\mathcal{U}}(C, D) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(B, C)) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A, B)$$
and the others can be easily determined as range and domain of the arrows.

$$\begin{tikzcd}
 & (\bullet\bullet)\bullet \arrow[rr, "\alpha"] \arrow[dl, "\boldsymbol{\mathcal{M}}_{BCD}\otimes^{(n-1)}1"'] & & \bullet(\bullet\bullet) \arrow[dr, "1\otimes^{(n-1)}\boldsymbol{\mathcal{M}}_{ABC}"] & \\
\bullet\bullet \arrow[drr, "\boldsymbol{\mathcal{M}}_{ABD}"'] & & & & \bullet\bullet \arrow[dll, "\boldsymbol{\mathcal{M}}_{ACD}"] \\
 & & \boldsymbol{\mathcal{U}}(A, D) & &
\end{tikzcd}$$

$$\begin{tikzcd}
\boldsymbol{\mathcal{I}}^{(n-1)} \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A, B) \arrow[dr, "="] \arrow[dd, "\boldsymbol{\mathcal{J}}_B\otimes^{(n-1)}1"'] & & \boldsymbol{\mathcal{U}}(A, B) \otimes^{(n-1)} \boldsymbol{\mathcal{I}}^{(n-1)} \arrow[dl, "="'] \arrow[dd, "1\otimes^{(n-1)}\boldsymbol{\mathcal{J}}_A"] \\
 & \boldsymbol{\mathcal{U}}(A, B) & \\
\boldsymbol{\mathcal{U}}(B, B) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A, B) \arrow[ur, "\boldsymbol{\mathcal{M}}_{ABB}"] & & \boldsymbol{\mathcal{U}}(A, B) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A, A) \arrow[ul, "\boldsymbol{\mathcal{M}}_{AAB}"']
\end{tikzcd}$$

This definition requires that there be definitions of the unit $\boldsymbol{\mathcal{I}}^{(n)}$ and of $\mathcal{V}$-n-functors in place. First, from the proof of monoidal structure on $\mathcal{V}$-n-Cat, there should be a recursively defined unit $\mathcal{V}$-n-category.

Definition 4 *The unit object in $\mathcal{V}$-n-Cat is the $\mathcal{V}$-n-category $\boldsymbol{\mathcal{I}}^{(n)}$ with one object $\mathbf{0}$ and with $\boldsymbol{\mathcal{I}}^{(n)}(\mathbf{0}, \mathbf{0}) = \boldsymbol{\mathcal{I}}^{(n-1)}$, where $\boldsymbol{\mathcal{I}}^{(n-1)}$ is the unit object in $\mathcal{V}$-$(n-1)$-Cat. Of course let $\boldsymbol{\mathcal{I}}^{(0)}$ be I in $\mathcal{V}$. Also $\boldsymbol{\mathcal{M}}_{000} = \boldsymbol{\mathcal{J}}_0 = 1_{\boldsymbol{\mathcal{I}}^{(n)}}$.*

Now the functors can be defined:

Definition 5 *For two $\mathcal{V}$-n-categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a $\mathcal{V}$-n-functor $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$-$(n-1)$-functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U,U') \to \boldsymbol{\mathcal{W}}(TU,TU')$. These latter obey commutativity of the usual diagrams. The range and domain of the arrows should be clear.*

1. *For $U, U', U'' \in |\boldsymbol{\mathcal{U}}|$*

$$\begin{tikzcd}
\bullet \arrow[r, "\boldsymbol{\mathcal{M}}_{UU'U''}"] \arrow[d, "T_{U'U''} \otimes^{(n-1)} T_{UU'}"] & \bullet \arrow[d, "T_{UU''}"] \\
\bullet \arrow[r, "\boldsymbol{\mathcal{M}}_{(TU)(TU')(TU'')}"'] & \bullet
\end{tikzcd}$$

2.

$$\begin{tikzcd}
& \bullet \arrow[dd, "T_{UU}"] \\
\boldsymbol{\mathcal{I}}^{(n-1)} \arrow[ur, "\boldsymbol{\mathcal{J}}_U"] \arrow[dr, "\boldsymbol{\mathcal{J}}_{TU}"'] & \\
& \bullet
\end{tikzcd}$$

Here a $\mathcal{V}$-0-functor is just a morphism in $\mathcal{V}$.

The 1-cells just defined play a special role in the definition of a general k-cell for $k \geq 2$.

Definition 6 *A $\mathcal{V}$-n:k-cell α between $(k-1)$-cells ψ^{k-1} and ϕ^{k-1}, written*

$$\alpha : \psi^{k-1} \to \phi^{k-1} : \psi^{k-2} \to \phi^{k-2} : ... : \psi^2 \to \phi^2 : F \to G : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$$

where F and G are $\mathcal{V}$-n-functors and where the superscripts denote cell dimension, is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$-$((n-k)+1)$-functor

$$\alpha_U : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU,GU)(\psi^2_U\mathbf{0}, \phi^2_U\mathbf{0})...(\psi^{k-1}_U\mathbf{0}, \phi^{k-1}_U\mathbf{0})$$

in such a way that the following diagram commutes. Note that the final (curved) equal sign is implied recursively by the diagram for the $(k-1)$-cells.

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\boldsymbol{\mathcal{W}}(FU',GU')(\psi^2_{U'}\mathbf{0}, \phi^2_{U'}\mathbf{0})...(\psi^{k-1}_{U'}\mathbf{0}, \phi^{k-1}_{U'}\mathbf{0}) \\ \otimes^{((n-k)+1)}\boldsymbol{\mathcal{W}}(FU,FU')(F(x_2),F(y_2))...(F(x_{k-1}),F(y_{k-1}))\end{array} \arrow[ddr, "\boldsymbol{\mathcal{M}}"] & \\
\boldsymbol{\mathcal{I}}^{((n-k)+1)} \otimes^{((n-k)+1)} \boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1}) \arrow[ur, "\alpha_{U'} \otimes^{((n-k)+1)} F"] & & \\
& & \boldsymbol{\mathcal{W}}(FU,GU')(\psi^2_{U'}\mathbf{0}F(x_2), \phi^2_{U'}\mathbf{0}F(y_2))...(\psi^{k-1}_{U'}\mathbf{0}F(x_{k-1}), \phi^{k-1}_{U'}\mathbf{0}F(y_{k-1})) \arrow[dd, equal, bend left] \\
\boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1}) \arrow[uu, "="] \arrow[dd, "="'] & & \\
& & \boldsymbol{\mathcal{W}}(FU,GU')(G(x_2)\psi^2_U\mathbf{0}, G(y_2)\phi^2_U\mathbf{0})...(G(x_{k-1})\psi^{k-1}_U\mathbf{0}, G(y_{k-1})\phi^{k-1}_U\mathbf{0}) \\
\boldsymbol{\mathcal{U}}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1}) \otimes^{((n-k)+1)} \boldsymbol{\mathcal{I}}^{((n-k)+1)} \arrow[dr, "G \otimes^{((n-k)+1)} \alpha_U"'] & & \\
& \begin{array}{c}\boldsymbol{\mathcal{W}}(GU,GU')(G(x_2),G(y_2))...(G(x_{k-1}),G(y_{k-1})) \\ \otimes^{((n-k)+1)}\boldsymbol{\mathcal{W}}(FU,GU)(\psi^2_U\mathbf{0}, \phi^2_U\mathbf{0})...(\psi^{k-1}_U\mathbf{0}, \phi^{k-1}_U\mathbf{0})\end{array} \arrow[uur, "\boldsymbol{\mathcal{M}}"'] &
\end{tikzcd}$$

Theorem 5 *$\mathcal{V}$-n-categories, $\mathcal{V}$-n-functors, and $\mathcal{V}$-n:k-cells for $k = 2 \ldots n+1$ together have the structure of an $(n+1)$-category.*

The proof is in [17]. That reference details the various compositions of the enriched cells described above and verifies the associativity and interchange laws in each case.

The definition of a category enriched over $\mathcal{V}$-$(n-1)$-Cat for $\mathcal{V}$ k-fold monoidal is simply stated by describing the process as enriching over $\mathcal{V}$-$(n-1)$-Cat with the first of its $k-n$ ordered products. These definitions reduce to the definitions of strict n-categories and their morphisms when $\mathcal{V}$ is **Set**. They reduce to the definition of an enriched category over $\mathcal{V}$, enriched functors and enriched natural transformations when $n = 1$. Thus the application of both reductions leads to an ordinary category.

5 Weak Enrichment

The definition of 1-weak $\mathcal{V}$-n-Cat is based on an arbitrary symmetric (or k-fold) monoidal category $\mathcal{V}$. The means of the construction is to define a (1-weak) $\mathcal{V}$-n-category as being weakly enriched over $\mathcal{V}$-$(n-1)$-Cat. The latter is a strict n-category by the above theorem. In a general weak n-category only the composition of top dimensional cells along next highest dimension cells is an operation that is associative. In this construction all the higher cells will compose associatively except when composing along 0-cells. Thus this is a study of weak n-category theory restricted to horizontal compositions. In the following the theorems of Section 2 are assumed to hold, since this is the second stage of the proposed project.

In an enriched category $\mathcal{A}$ (over $\mathcal{V}$) the role of composition is taken over by special morphisms in the monoidal category $\mathcal{V}$. A string of these hom-objects (such as the string of length two in the domain of the composition morphism $M : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$) will be called composable if they can be reduced to a single hom-object by repeated uses of M. Of course the parenthization of the original string matters. Keep in mind then also the associator α, used to get from one parenthization to another. For a string of length n one can draw the associahedron $\mathcal{K}(n)$ and put the various parenthizations at the vertexes, and the associators on the edges.

When enriching over $\mathcal{V}$-n-Cat, composition morphisms $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$ are enriched n-functors. The pentagon they are usually required to satisfy exactly for each triple of objects can instead be filled in with an (invertible) enriched n-natural transformation $\boldsymbol{\mathcal{M}}_2$.

$$\begin{tikzcd}
& (\bullet\bullet)\bullet \arrow[rr, "\alpha"] \arrow[dl, "\boldsymbol{\mathcal{M}}_{BCD}\otimes^{(n-1)}1"'] & & \bullet(\bullet\bullet) \arrow[dr, "1\otimes^{(n-1)}\boldsymbol{\mathcal{M}}_{ABC}"] & \\
\bullet\bullet \arrow[rrrr, phantom, "\overset{\boldsymbol{\mathcal{M}}_2}{\Longrightarrow}"] \arrow[drr, "\boldsymbol{\mathcal{M}}_{ABD}"'] & & & & \bullet\bullet \arrow[dll, "\boldsymbol{\mathcal{M}}_{ACD}"] \\
& & \boldsymbol{\mathcal{U}}(A,D) & &
\end{tikzcd}$$

For each quadruple of objects $\boldsymbol{\mathcal{M}}_2$ is required to participate in a higher law. Draw the pentagonal diagram of $\mathcal{K}(4)$ with vertices labeled by parenthesized strings of composable hom-categories, and compose each vertice by use of $\boldsymbol{\mathcal{M}}$ to a common final hom-category. What this does is to subdivide $\mathcal{K}(4)$ into two-cells that are again filled with instances of $\boldsymbol{\mathcal{M}}_2$. Figure 4 shows this division.

There are exactly two 2-dimensional paths that make up the front and back of the polyhedron. The division between the two is along the line of the quickest path to the center from the upper left, as well as the path from the upper left to the center which goes through the lowest location. Between the two there should now exist an enriched modification $\boldsymbol{\mathcal{M}}_3$. Figure 5 illustrates this latter 3-cell.

The series of enriched k-cells $\boldsymbol{\mathcal{M}}_k$ fill in polytopal diagrams that are in fact the composihedra: the boundary of $\mathcal{CK}(i) = s(\boldsymbol{\mathcal{M}}_{i-1}) \sqcup t(\boldsymbol{\mathcal{M}}_{i-1})$ where s and t denote the source and target of the morphism in question.

In general for each n-tuple of objects there exists an invertible $(n-2)$ -cell $\boldsymbol{\mathcal{M}}_{(n-2)}$, until at last for each set of $n+3$ objects $\boldsymbol{\mathcal{M}}_{n+1}$ is an identity morphism – i.e. the last diagram involving $\boldsymbol{\mathcal{M}}_n$, in the form of $\mathcal{CK}(n+2)$, is required to commute.

For two 1-weak $\mathcal{V}$-n-categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a lax $\mathcal{V}$-*n-functor* $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$-$(n-1)$-functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U,U') \to \boldsymbol{\mathcal{W}}(TU,TU')$. Then for each set of $k > 2$ objects there

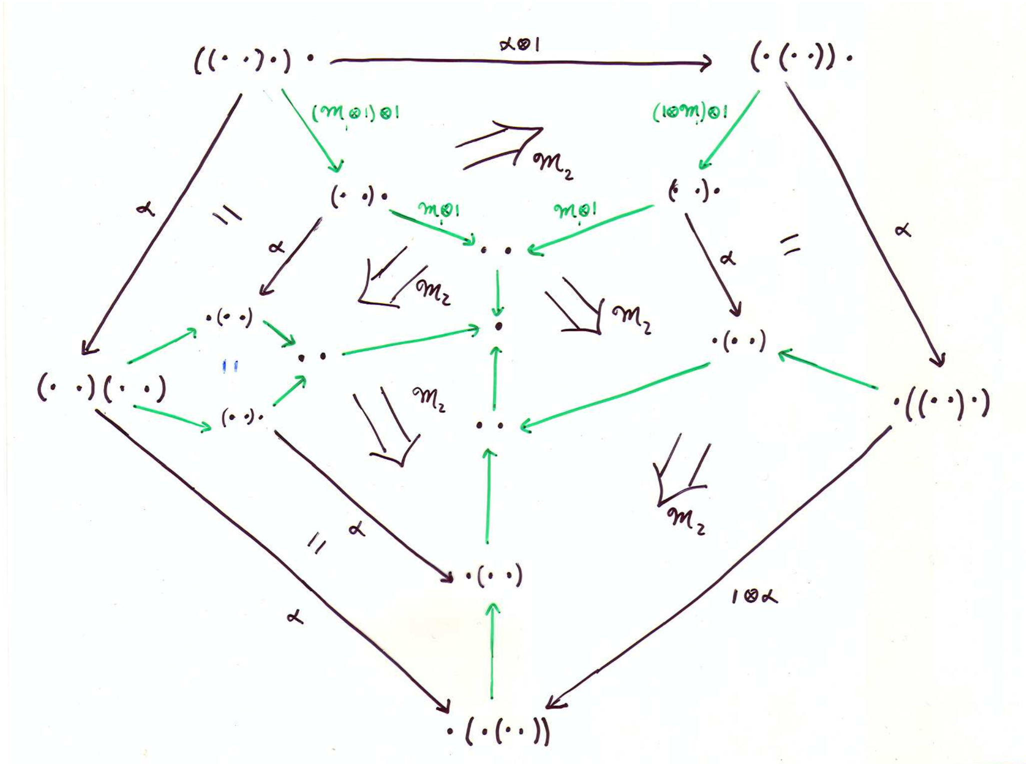


Figure 4: Source and target of $\boldsymbol{\mathcal{M}}_3$.

is an enriched k-cell ϕ_k that fills in a polytope diagram made by taking a right prism of the polytope $\mathcal{CK}(k)$. Since the $T_{UU'}$ are enriched functors the square they are usually required to satisfy exactly can instead be filled in with an (invertible) enriched n-natural transformation ϕ_2.

$$\begin{tikzcd}
\bullet \arrow[r, "\boldsymbol{\mathcal{M}}_{UU'U''}"] \arrow[d, "T_{U'U''}\otimes^{(n-1)}T_{UU'}"'] \arrow[dr, phantom, "\Nearrow\phi_2"] & \bullet \arrow[d, "T_{UU''}"] \\
\bullet \arrow[r, "\boldsymbol{\mathcal{M}}_{(TU)(TU')(TU'')}"'] & \bullet
\end{tikzcd}$$

The 2-cell ϕ_2 is required to participate in the following: draw the pentagonal prism of $\mathcal{CK}(3)$, faces filled in with $\boldsymbol{\mathcal{M}}_2$ on both pentagonal ends and with ϕ_2 on the sides. Then the prism itself is filled in with the enriched modification ϕ_3. This process continues, and ϕ_{n+1} is the identity. Figure 6 shows the shape of the domain and range of ϕ_3.

All strict higher morphisms have a shared form of their axiomatic commuting diagram, as seen above in Definition 6. Thus only a single new sequence of morphisms is required to describe lax enriched k-cells between strict enriched $(k-1)$ cells between lax enriched n-functors. For this reason the naturahedra are described as having the "universal" property of playing the same role at each categorical dimension. Here is a draft of the theorem to be proven.

Theorem 6 *(draft) The sources and targets of the mediating morphisms of a higher lax enriched transformation are together given by the boundary of the appropriate naturahedron.*

The proof will use a translation from the combinatorial description to a categorical one.

A weak *$\mathcal{V}$-n:k-cell* τ between $(k-1)$-cells ψ^{k-1} and ϕ^{k-1} is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$-$((n-k)+1)$-functor

$$\tau_U : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU, GU)(\psi_U^2\mathbf{0}, \phi_U^2\mathbf{0})...(\psi_U^{k-1}\mathbf{0}, \phi_U^{k-1}\mathbf{0}).$$

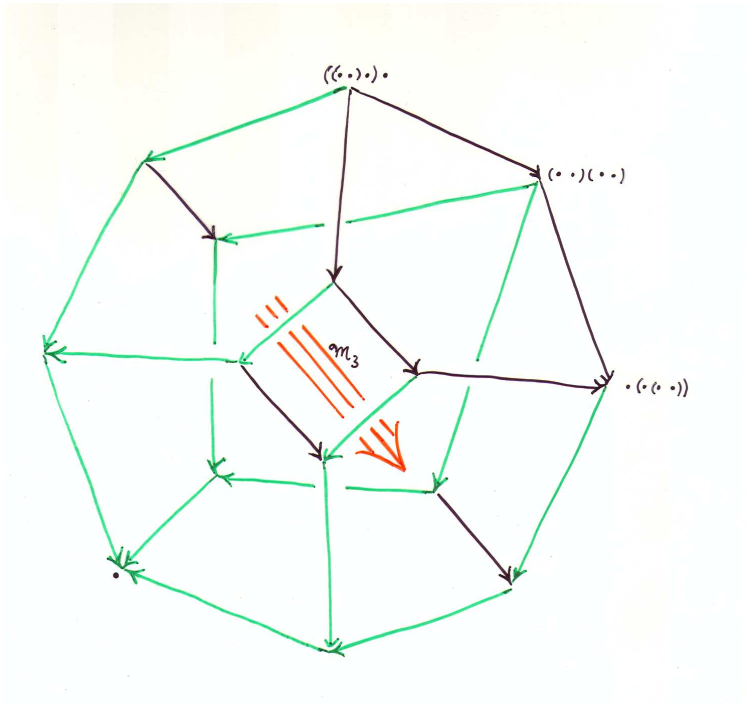


Figure 5: $\boldsymbol{\mathcal{M}}_3$

Rather than a commuting diagram there is a sequence of higher enriched cells that mediate the commutativity. The source and target of these form the boundaries of naturahedra. The enriched k-cell that fills the polytope diagram described by $\mathcal{N}(k)$ will be called τ_k. This process is illustrated here for the first few steps. Here is shown the domain and range for τ_2. The nodes have been abbreviated and the two equal ending locations have been collapsed into one; please refer to the previous section for their expansion.

$$
\begin{tikzcd}
 & & \boldsymbol{\mathcal{W}}(FU',GU') \otimes^{((n-k)+1)} \boldsymbol{\mathcal{W}}(FU,FU') \arrow[ddr, "\boldsymbol{\mathcal{M}}"] & \\
\boldsymbol{\mathcal{I}}^{((n-k)+1)} \otimes^{((n-k)+1)} \boldsymbol{\mathcal{U}}(U,U') \arrow[urr, "\alpha_{U'} \otimes^{((n-k)+1)} F"] & & & \\
\boldsymbol{\mathcal{U}}(U,U') \arrow[u, "="] \arrow[d, "="'] \arrow[rrr, phantom, "\Uparrow \tau_2"] & & & \boldsymbol{\mathcal{W}}(FU,GU') \\
\boldsymbol{\mathcal{U}}(U,U') \otimes^{((n-k)+1)} \boldsymbol{\mathcal{I}}^{((n-k)+1)} \arrow[drr, "G \otimes^{((n-k)+1)} \alpha_U"'] & & & \\
 & & \boldsymbol{\mathcal{W}}(GU,GU') \otimes^{((n-k)+1)} \boldsymbol{\mathcal{W}}(FU,GU) \arrow[uur, "\boldsymbol{\mathcal{M}}"'] &
\end{tikzcd}
$$

Now τ_2 is required to obey a commuting diagram of its own, or only to obey it up to a further higher morphism called τ_3. For brevity the superscripts showing dimension are omitted. The outermost hexagon is precisely the above diagram for τ_2. The bold arrows demarcate the division of source and target.

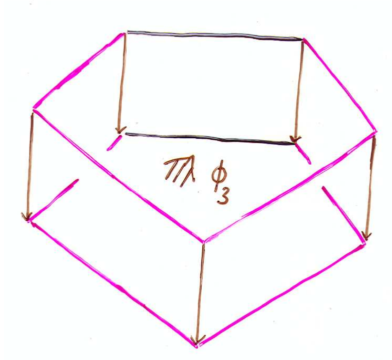


Figure 6: ϕ_3

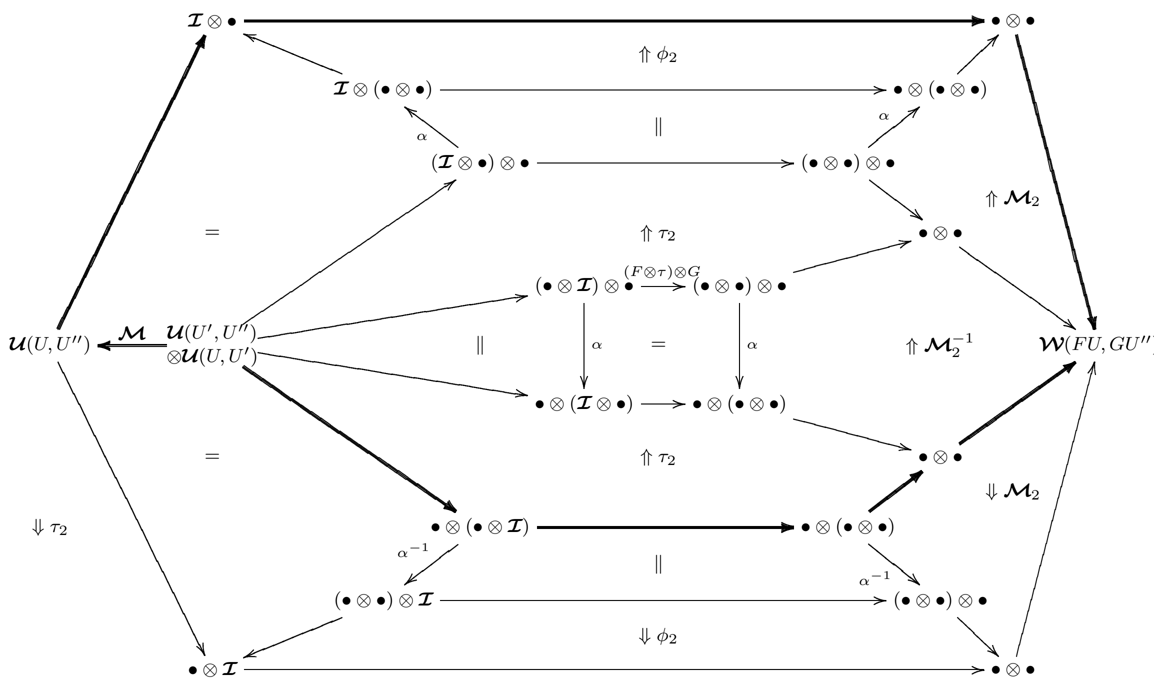


Figure 7 shows a 3-dimensional view of τ_3.

Once again this process will continue until at the top dimension $s(\tau_n) \sqcup t(\tau_n)$ is the boundary of $\mathcal{N}(n)$. Then τ_n will be required to obey a commuting diagram which has the form of $\mathcal{N}(n+1)$.

There are several secondary questions that immediately arise. The first regards the structure of the lax enriched morphisms just defined. It is likely that one of two things is true: either they form a strict n-category or a weakly enriched n-category. This needs to be investigated, especially since there is an unlikely third possibility that they conform to the axioms of one of the definitions of weak n-category. Another question concerns the correct way in which to strengthen lax transformations into equivalencies, in order to talk about equivalent weak enriched categories. Finally, of course, is the question of a theory of weak enriched limits. This will draw from the established theories of enriched (filtered) limits and lax limits (bilimits.)

An alternative, parallel avenue of research is into an operad theoretic way in which to define the weak enrichment. One can define an operad of $\mathcal{V}$-n-categories (in $\mathcal{V}$-n-Cat) which is denoted by $\overline{\mathcal{K}}$. This notation is due to the fact that $\overline{\mathcal{K}}(j)$ is defined to be the $\mathcal{V}$-n-category generated by the (directed) Stasheff associahedra $\mathcal{K}(j)$. Vertices are objects, and given two vertices there is a hom-$\mathcal{V}$-$(n-1)$-category of paths along edges. In $\mathcal{V}$-n-Cat the $n-3$ dimensional faces (including degeneracies) of $\mathcal{K}(j)$ will correspond to hom-objects in $\mathcal{V}$ and here it is required that those hom-objects be isomorphic to the unit $I \in \mathcal{V}$. The composition for this categorical operad is inclusion just as for the operad of associahedra. For the case in which $\mathcal{V}$ is k-fold

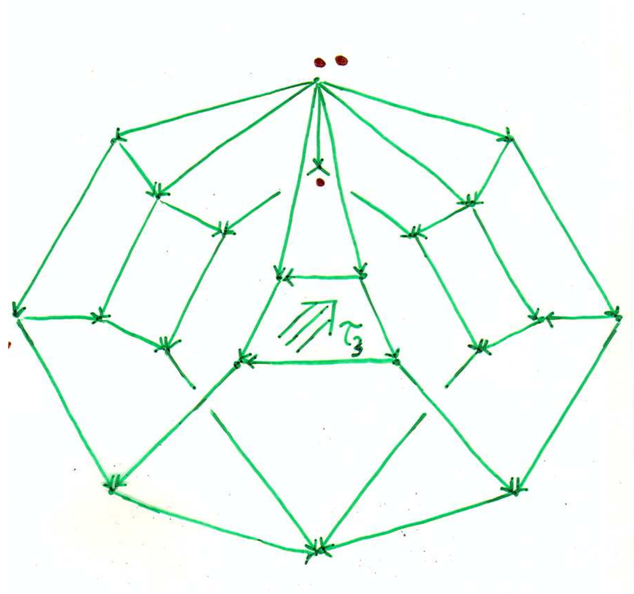


Figure 7: τ_3

monoidal rather than symmetric, the definition of an n-fold operad in an iterated monoidal category is given in [18]. Then requiring that there be an operad action on composable strings of hom-categories as in:

$$\overline{\mathcal{K}}(j) \otimes^{(n)} \boldsymbol{\mathcal{U}}(X_{j-1}, X_j) \otimes^{(n)} \cdots \otimes^{(n)} \boldsymbol{\mathcal{U}}(X_0, X_1) \to \boldsymbol{\mathcal{U}}(X_0, X_j)$$

is equivalent to the previous description of filled polytope diagrams using the composihedra.

Since a loop space can be efficiently described as an operad algebra, it is not surprising that there are several existing definitions of n-category that utilize operad actions. These definitions fall into two main classes: those that define an n-category as an algebra of a higher order operad, and those that achieve an inductive definition using classical operads in symmetric monoidal categories to parameterize iterated enrichment. The first class of definitions is typified by Batanin and Leinster. [5],[29] The former author defines monoidal globular categories in which interchange transformations are isomorphisms and which thus resemble free strict n-categories. Globular operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string of objects as in the case of classical operads. The binary composition in an n-category derives from the action of a certain one of these globular operads. Leinster expands this concept to describe n-categories with unbiased composition of any number of cells. The second class of definitions is typified by the works of Trimble and May. [37], [34] The former parameterizes iterated enrichment with a series of operads in $(n-1)$-Cat achieved by taking the fundamental $(n-1)$-groupoid of the kth component of the topological path composition operad E. The latter begins with an A_∞ operad in a symmetric monoidal category $\mathcal{V}$ and requires his enriched categories to be tensored over $\mathcal{V}$ so that the iterated enrichment always refers to the same original operad.

The operadic description of weak enrichment should facilitate further comparisons – such as determining whether 1-weak $\mathcal{V}$-n-Cat is a subcategory of May's category of n-categories, and for $\mathcal{V} = \mathbf{Set}$ whether 1-weak $\mathcal{V}$-n-Cat is a subcategory of Trimble's n-categories. Also closely related is Batanin's definition of weak n-category based on n-operads. It may be that his ideas include the concept of weak enrichment as an enriched version or special case.

6 Broader Impact: Research at an HBCU

The roles of researcher and instructor in mathematics are often seen to be at odds, one suffering when the other is focused upon. This is an unfortunate perception, since in actuality the quality and the motivational power of teaching at the university level is directly proportional to the instructor's involvement in leading edge research. The teacher/researcher is the link for the student between an esoteric world of developing science and the more familiar sphere of the classroom. Not only does the research activity of the professor keep his or her teaching relevant by forcing him to stay abreast of recent developments, but glimpses of the new results and unanswered questions he encounters energize his students with a larger view of their studies than afforded by the more mundane practice problems in their homework.

In particular the research into geometric combinatorics and high dimensional category theory discussed here will be performed largely at Tennessee State University, a historically black university with a large proportion of minority students. As well as helping to enrich the classroom instruction of the principal investigator this research project will further the participation of African Americans in mathematical research in several specific ways. The mathematics department at TSU requires a thesis from both its undergraduate senior mathematics majors and its masters degree candidates. Faculty advise students on their theses, and so active research projects such as the one being proposed are invaluable as sources of research topics for the degree candidates. The student benefits from having the experience of participating in new research and helping to develop new results and in having an adviser active in the field they are choosing to study. The funds for support of undergraduates will both allow them to focus on their research and to at the same time perform tasks for the principal investigator that will facilitate the overall project. In addition, TSU is working towards the establishment of a Doctorate program in mathematics. Steps in that direction include the hiring of additional faculty actively involved in research, and the procurement of research grants to help continue that activity. As of this date there is only one PhD in mathematics offered by any of the historically black colleges and universities in the U.S., at Howard University. There is much to be gained from increasing this number in terms of broadening the participation of severely underrepresented ethnic groups in the mathematical community. The benefits to society of encouraging contributions to scientific research from all ethnic groups should be self evident. Whenever, for whatever historical or economic reasons, there is an underrepresented segment of society in an area of scientific endeavor, it means that inevitably valuable sources of talent and creativity are remaining untapped. The incalculable rewards for correcting this state of affairs are truly mutual.

On top of simply increasing the number of minority researchers in mathematics, the proposed project has as a partial goal the strengthening of ties between Tennessee State University and its counterparts in the region and wider academic community. There is already a scheduled seminar covering this project at the neighboring Vanderbilt University. Also in place is a certain amount of networking at an even longer range. In regard to this project, the principal investigator has been in communication with experts in the subject matter such as Jim Stasheff currently at Pennsylvania State University and J.Peter May at the University of Chicago. Both have proffered invitations to travel and lecture on the subject, the invitation from the latter resulting in presentation of the work in progress at the workshop on "n–categories: Foundations and Applications" organized by John Baez and Dr. May in June 2004.

It is an unfortunate fact that though categories, functors and natural transformations were discovered here, the U.S. seriously lags behind Great Britain, Australia and Canada among others in its development of category theory and applications to computer science. However, a remedy to the situation is not to be found in isolation, and so the principal investigator plans to continue communications with experts abroad. This includes the group of category theorists led by Max Kelly and Ross Street in Australia, where the principal investigator was invited to speak in July 2005. They constitute an important link to the work of Trimble, which intersects with the proposed work at the level of dimension 4.

As results are finalized the principal investigator as well as potential student and faculty collaborators plan to disseminate the information through several venues. Articles will be prepared and submitted to appropriate scholarly journals, such as Theory and Applications of Categories and Algebraic Topology, in both of which the principal investigator has published previously. Even before acceptance of journal articles, however, the material will be made available through preprint servers such as the arxiv and the Hopf server, as well as through conference and seminar presentations. Thus the answers to important mathematical questions discussed above will be easily available to those who have interest in the subject, its practical applications and potential further research.