Linear. Final Review.

Also study all the quizzes, the two previous tests, the reviews for those tests, and homework problems!

(1) Consider the following matrices:

$$A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Find the eigenvalues and eigenspaces for each of these. Are either diagonalizable? If so, find the matrix P, and check the similarity relationship, where the diagonal matrix D is related to the original by multiplying by P and P^{-1}.

$$\det(A-\lambda I) = (3-\lambda)(0-\lambda)\big((0-\lambda)(2-\lambda)-(-1)\big)$$
$$= (3-\lambda)(-\lambda)(\lambda^2-2\lambda+1)$$
$$= (3-\lambda)(-\lambda)(\lambda-1)^2$$

need $\dim(V_1) = 2$ for diagonalizable

$$\Rightarrow \lambda = 3, 0, 1$$

$$V_3: \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned} \quad V_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$V_0: \left[\begin{array}{cccc|c} 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned} \quad V_0 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$V_1: \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 + \tfrac{1}{2}x_4 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 - x_4 &= 0 \\ x_4 &= x_4 \end{aligned} \quad V_1 = \operatorname{span}\left\{\begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Not Diagonalizable

$$\det(B-\lambda I) = (0-\lambda)^3(4-\lambda)$$

$$\Rightarrow \lambda = 0, 4$$

V_0 :

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$V_0 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

V_4 :

$$\left[\begin{array}{cccc|c} -4 & 1 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} -4 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} -4 & 0 & 0 & +3/8 & 0 \\ 0 & 1 & 0 & -3/8 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3/32 & 0 \\ 0 & 1 & 0 & -3/8 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= \tfrac{3}{32} x_4 \\ x_2 &= \tfrac{3}{8} x_4 \\ x_3 &= \tfrac{3}{4} x_4 \\ x_4 &= x_4 \end{aligned}$$

$$V_4 = \text{span}\left\{\begin{bmatrix} 3/32 \\ 3/8 \\ 3/4 \\ 1 \end{bmatrix}\right\}$$

$$\boxed{\text{Not diagonalizable}}$$

(we need the $\dim(V_0) = 3$, but $\dim(V_0) = 1$.)

2 (2) Consider the two bases for $\mathcal{P}_3$:

$$\mathcal{E} = \{1, x, x^2, x^3\},\ \mathcal{C} = \{x^3 + 3x^2 + 1, x^2 - 2, x - 7, 2\}$$

Find the eigenvalues of $T(f) = f'' + xf''$.

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is T diagonalizable? If so, find P.

If you found $[T]_{\mathcal{C}}^{\mathcal{C}}$, what would its eigenvalues be?

$$\det(T - \lambda I) = (-\lambda)^4 \Rightarrow \boxed{\lambda = 0}$$

V_0 :

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$V_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \boxed{\text{Span}\{1, x\}}$$

Note: $V_0 = N(T)$ always true.

$\boxed{\text{Not diagonalizable.}}$

The eigenvalues of $[T]_{\mathcal{C}}^{\mathcal{C}}$ are the same: $\boxed{\lambda = 0}$

Example:

Is $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ diagonalizable?

If so, find P and D such that $A = PDP^{-1}$.

Ans. $\det(A - \lambda I) = (3-\lambda)\big((3-\lambda)(-1-\lambda) - 0\big)$
$= (3-\lambda)(3-\lambda)(-1-\lambda) = (3-\lambda)^2(-1-\lambda)$

$$\boxed{\begin{array}{ll} \lambda = 3 & \text{mult} = 2 \\ \lambda = -1 & \end{array}}$$

V_3: $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$ $\quad \begin{array}{l} x_1 = x_1 \\ x_2 - 2x_3 = 0 \\ x_3 = x_3 \end{array}$

$V_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\} \Rightarrow \boxed{\text{diagonalizable}}$

V_{-1}: $\left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = x_3 \end{array}$

$V_{-1} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad A = P \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} P^{-1}$

$\lambda = 3$ (any order) $\qquad \lambda = -1$