Research Statement
Stefan Forcey

1. Summary

Recently my research with collaborators and students has focused on combinatorial algebra, with a geometrical flavor. We have been developing a diverse family of graded algebras, coalgebras, modules and comodules, often with Hopf structure, arising from sequences of convex polytopes. The key point of interest is to see how the algebraic structure reflects the combinatorial structure, and vice versa. The polytope sequences include the familiar cubes and simplices, permutohedra and associahedra, as well as newer examples like multiplihedra and composihedra. The individual polytopes often possess lattice structures; the sequences in their entirety possess operad and operad module structures, and whole families of sequences are related by cellular surjections. Immediately there are more open questions than any of us have time for! Luckily the questions are perfect for introducing students to pure math research and experimentation.

My previous research involved some of the same operads and polytope sequences, but from a category theory perspective. My interests in category theory are centered on categories with extra structure: braided, monoidal, enriched, and higher dimensional n-categories. These are exciting for many reasons. They provide the environment for comparison of operads and their algebras, and they promise new ways to model geometry and topology. The latter is where my original introduction into the subject matter began: my interest in classical topological problems of knot concordance led me to study topological quantum field theory and categorical homotopy theory. The circle is closed when we realize that the category theory can also shed light on the combinatorial Hopf algebras.

In case we need any more motivation, there are several tantalizing applications of our research. The polytope sequences we study are defined combinatorially, and those same combinatorics model physical structures and processes such as benzenoid molecules, biological clocks and Feynman diagrams.

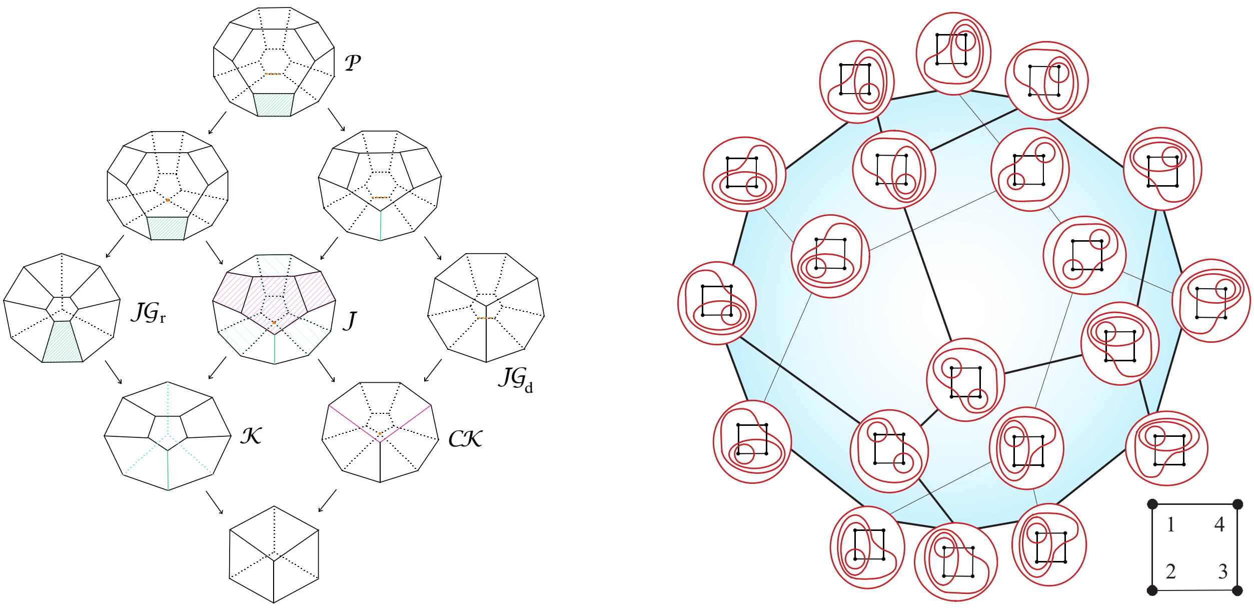

Figure 1. On the left is a commuting diagram of quotients and coverings of the multiplihedra which each underlie a new Hopf algebra. The 3 dimensional multiplihedron $\mathcal{J}$ is in the middle, and its quotients the associahedron $\mathcal{K}$ and composihedron $\mathcal{CK}$ lie beneath it. On the right is the 1-skeleton of the cyclohedron, seen as a lattice of graph tubings. Möbius inversion on this lattice gives a new basis for the corresponding noncommutative graded algebra.

2. Geometric combinatorics and Hopf algebras

In broadest terms, the ideas we are focusing on trace their origin to Gian-Carlo Rota and his collaborators. A combinatorial sequence that is created by a recursive process often carries the seed of a graded bi-algebraic structure. The algebra reflects the process of building a new object from prior ones; and the coalgebra arises from deconstructing an object into its constituent components.

The discrete geometric approach we are taking is to study recursive combinatorial structures among polytopes and polytopal cones. There is a vast landscape of these shapes which have inductively defined facets. The challenge we face is to pick out from this space the sequences that give rise to interesting algebras and bialgebras.

2.1. **Background.** The historical examples of Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Q}Sym$ which we study were created using the bases of permutations and boolean subsets. Loday and Ronco realized that binary trees provided a general format for both, and used this generalization to discover the Hopf algebra $\mathcal{Y}Sym$ lying between them. Chapoton capitalized on the fact that the three graded bases could actually be described as the vertex sets of polytope sequences, and defined larger algebras on the faces of the permutohedra, associahedra and cubes.

2.2. **New approaches.** We begin with the unified description of associahedra and permutohedra as graph-associahedra developed by Carr and Devadoss. Our first new insight is to use that point of view to describe the algebra and coalgebra structures of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in completely geometric terms, using cartesian products, inclusion maps, and cellular surjections of the polytopes.

This in turn allows us to describe new algebraic structures, based on the vertex sets and on the faces of special sequences of graph associahedra, which are quotients of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. These include algebras based upon the cyclohedra (cycle graphs), the simplices (edgeless graphs), and upon several infinite sequences of polytope families which filter the historical algebras and coalgebras. Figure 2 shows a product of two vertices of the cyclohedron.

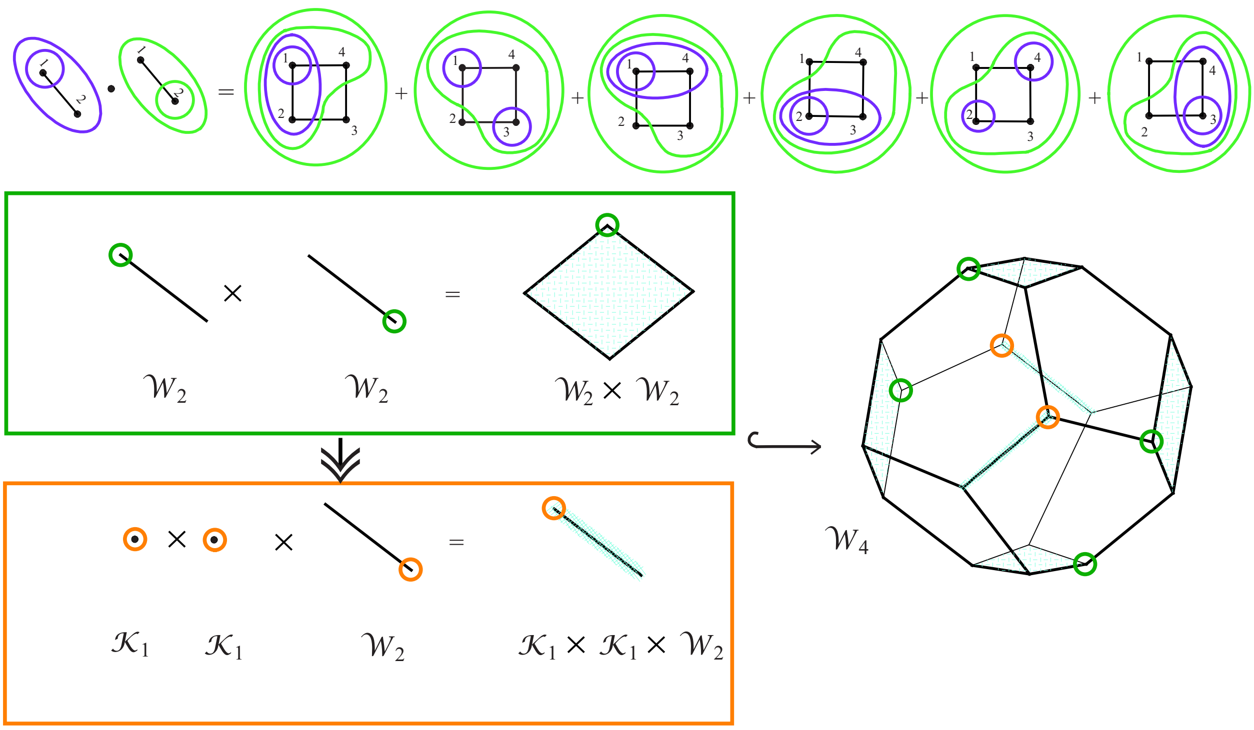

Figure 2. A product of cycle graph tubings, demonstrating its geometric character.

Collaborators Aaron Lauve, Frank Sottile and myself have recently developed algebra, Hopf algebra and Hopf module structures based on the vertices of the multiplihedra, seen as painted trees. Even more algebras arise from considering alternate combinatorial descriptors of the vertices, including the *marked graph tubings* developed by myself and S. Devadoss.

2.3. **Maps.** An original motivation for these new algebras was my discovery that the Tonks projection from the permutohedron to the associahedron can be factored through a series of graph-associahedra. This fact is simple to demonstrate; it follows from Devadoss's discovery that the complete graph-associahedron is the permutohedron while the path graph-associahedron is the Stasheff polytope. Thus by deleting edges of the complete graph one at a time, we describe a family of quotient cellular projections. Figure 3 shows one of these. Our important result is that the cellular projections give rise to graded algebra maps.

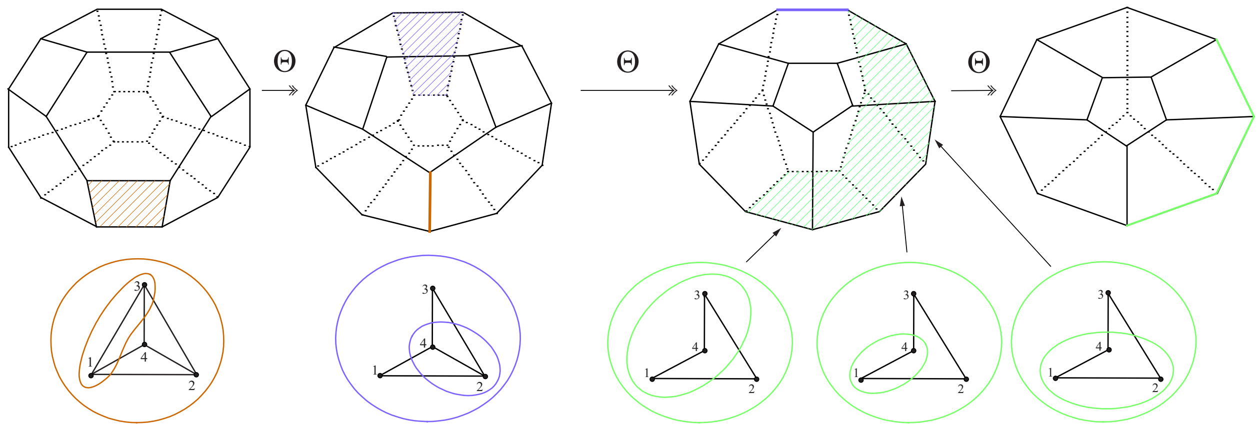

FIGURE 3. A factorization of the Tonks projection through 3 dimensional graph associahedra. The shaded facets correspond to the shown tubings, and are collapsed as indicated to respective edges. The first, third and fourth pictured polytopes are above views of $\mathcal{P}_4, \mathcal{W}_4$ and $\mathcal{K}_4$ respectively.

2.4. **Geometry.** The supporting results of my work in this area describe the geometric combinatorics of the polytopes in question. The first major contribution is the answering of a long-standing open question. The low dimensional multiplihedra were first described by Stasheff in 1970 in order to describe A_n-maps between A_n-spaces. The question of whether they could be realized as convex polytopes was still open three decades later.

Building on Loday's famous geometric realization of the associahedra, I developed a simple algorithm that, for each choice of $q \in (0,1)$, gives vertices in Euclidean space $\mathbb{R}^n$ whose convex hull is the multiplihedron of dimension n. The method that allowed the development of my algorithm was a complete description of the vertices of the multiplihedron using painted trees. This allowed as well the first enumeration of the vertices of the multiplihedron, using generating function techniques.

Metric versions of the painted trees were described by Boardman and Vogt in 1973, and a recursive definition of the multiplihedra was implicitly discovered by Iwase and Mimura in 1986. The new convex hull realization allowed connection of these early results via combinatorial equivalence to Iwase and Mimura's CW-complex, and simultaneously by homeomorphism to Boardman and Vogt's spaces.

The importance of the geometric realization of the multiplihedron was immediately apparent. First, it allowed simple generalizations to quotients of the multiplihedron corresponding to strict range and domain spaces. The former yields a new realization of the n-dimensional associahedron in $\mathbb{R}^n$ and the latter yields a realization of the composihedron. The composihedron are a newly discovered sequence of polytopes. I actually found them in both topological and categorical settings–where in both cases they had been misnamed or mistaken for associahedra in the early literature. In the case of strict domain and range the quotient polytopes are cubes, as shown in Figure 1.

Secondly, the geometric realization of the classic multiplihedron affords us a tool for the study of generalized multiplihedra of several kinds, a tool made necessary by the fact that these new polytopes are rarely simple themselves. By use of an analogous realization myself and collaborator Devadoss were able to experimentally discover the structure of the graph-multiplihedra and their quotients. Only then were we able to prove the details of the facet structure of the graph-multiplihedra. This same approach, along with new results regarding truncations of products of simplices, is being applied to my newly defined multigraph-associahedra, and their corresponding multiplihedra.

- Publications and student papers:
— New Hopf structures on binary trees. (with A. Lauve and F. Sottile)
to appear, DMTCS Proceedings, 2009.
— Geometric combinatorial algebras: cyclohedron and simplex. (with D. Springfield)
submitted, 2009.
— Marked tubes and the graph multiplihedron. (with S.L. Devadoss)
Algebraic and Geometric Topology, 8(4) 2081-2108, 2008.
— Quotients of the multiplihedron as categorified associahedra.
Homotopy, Homology and Applications, vol. 10(2), 227-256, 2008.
— Convex Hull Realizations of the Multiplihedra
Topology and its Applications, 156, 326-347, 2008.
— Derriell Springfield, Masters Thesis, TSU
Algebras based upon the cyclohedron, 2009.
— Jerome Lecointre, Senior project, TSU
Polytope structure of the composihedra, 2006.
- Main results:
— New geometrical characterization of the products in $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$.
— Algebra structures on the cyclohedra and simplices.
— Graded algebra and Hopf module based on vertices of the multiplihedra.
— Discovery of new quotient of multiplihedra: the composihedra.
— Discovery of graph-multiplihedra and their quotients.
— Geometric realizations for the multiplihedra, composihedra, graph-multiplihedra, and quotients of the graph multiplihedra.
— Enumeration of vertices of multiplihedra, composihedra; recognition of the former as the Catalan transform and the latter as the binomial transform of the Catalan numbers.
— Facet structure of composihedra and graph-multiplihedra.
— Factorization of the Tonks projection through graph-associahedra.
— Factorization of Saneblidze and Umble's projection through the graph-multiplihedra (special cases demonstrated, general case conjectured.)
- Current projects and collaborations:
— Pending grant proposal: *Geometric combinatorial Hopf algebras and modules.*
— Algebraic filtrations of $\mathfrak{S}Sym$ and quotient lattices of the weak order on $\mathfrak{S}_n$.
— New families of Hopf algebras using painted trees.
(with A. Lauve, F. Sottile)
— New classes of combinatorial polytopes and polytopal cones, based on multigraphs and CW-complexes(with M. Carr and S. Devadoss.)
— New recursive sequences of polytopes, and their supported
Hopf algebras using multigraph associahedra, graph-multiplihedra.

3. Structured categories and their operads

A common generalization of symmetric, braided, and tensor categories exists – the iterated monoidal categories introduced by Fiedorowicz, Vogt, and their students in 2003. Many of my contributions in this area, with students and collaborators, are due to the insight that results about braided categories can often be extended to the realm of Fiedorowicz's n-fold monoidal categories.

Specifically we have demonstrated that the concepts of enrichment, substitution product, operads and operad algebras all make sense in a base category that is n-fold monoidal. In order to show

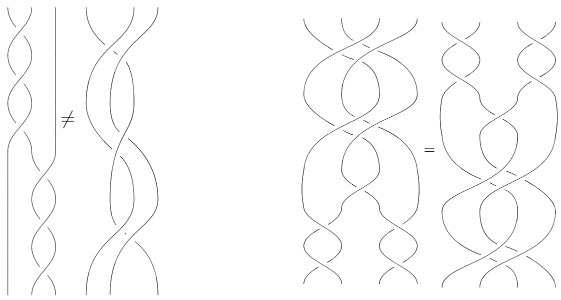

FIGURE 4. Examples of braid inequality and equality used in categorical existence proofs.

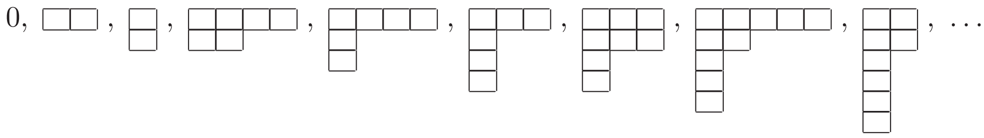

FIGURE 5. A minimal operad of Young diagrams.

the sharpness of certain theorems about operads in n-fold monoidal categories we developed a large family of simple combinatorial posets which exemplify n-fold monoidal categories. These were the first concrete examples besides the free n-fold monoidal categories originally described by Fiedorowicz. There are many open questions raised by our research – ranging from the broad question of how the 2-fold monoidal structures on species defined by M. Aguiar and S. Mahajan relate to our combinatorial 2-fold monoidal structures, to the specific question of whether certain operads of Young diagrams display chaotic growth patterns.

3.1. **Braidings.** The most recent contributions to the theory of structured categories we can claim are in regard to what sort of additional structures can be found when a braiding is present. The proofs in this area are often more instructive than the theorem statements. For example, we use the embedding of the semigroups of positive braids into the braid groups to show a family of braid inequalities (see Figure 4 for the first) which preclude existence of a second braiding based on an odd number of half twists using the original braiding in a category. Then we use the existence of opposites in the category of enriched categories over a braiding, and the existence of monoids in the free braided category with duals to classify the 4-strand braids which obey a certain *interchanging* identity. Thanks are due to Imre Tuba for helpful conversations about the latter. Also important to our classification of the 2-fold monoidal structures based on a braiding is the fact that the interchanging braids are equivalent to their own 180 degree rotations, as shown in Figure 4. Finally, both the existence of a braiding and an involution in the 2-category of enriched categories are precluded in general by showing braids of two types to be non-conjugate, via arguing that their braid closures always have different linking number.

- Publications and student papers:
  - — Operads in iterated monoidal categories (with J. Siehler, E. Seth Sowers)
    Journal of Homotopy and Related Structures 2, 1-43, 2007.
  - — Classification of braids which give rise to interchange (with F. Humes)
    Algebraic and Geometric Topology 7, 1233-1274, 2007.
  - — Govina M. Eyum, Masters Thesis, TSU
    Products of Young diagrams in a 2-fold monoidal category, 2007.
  - — E. Seth Sowers, Masters Thesis, TSU
    Operads in 2-fold monoidal categories, 2006.
  - — Felita N.C. Humes, Masters Thesis, TSU
    Iterated monoidal categories based on a braiding, 2006.
  - — Ahmad Kheder, Senior project, TSU
    Investigating minimal recursive growth., 2007.

- Main results:
  - — Complete classification of 2-fold structures based on a braiding: the classification is by the four-strand braids $(\sigma_2\sigma_1\sigma_3\sigma_2)^{\pm n}\sigma_2^{\pm 1}(\sigma_1\sigma_3)^{\mp n}$.
  - — Discovery of 2-fold categories of sequences and Young diagrams.
  - — Discovery of n-fold categories of n-dimensional Young diagrams.
  - — Definition of n-fold operads in a k-fold monoidal category.
  - — We prove that the category of n-fold operads in a k-fold monoidal category is itself a $(k-n)$-fold monoidal, strict 2-category, and show that n-fold operads are automatically $(n-1)$-fold operads.
  - — Characterization of operads of Young diagrams for simple generators.
- Current projects and collaborations:
  - — Proof of the 2-fold structure of higher multiplication of Young diagrams. (with G. Eyum.)
  - — Classification of operads of natural numbers. Given a finite generating sequence, what is the formula for the terms in the operad it generates? (with A. Kheder).
  - — Classification of operads of Young diagrams for mutiple generators. Investigating chaotic growth for the operad generated by the sequence $0, \square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}$. See Figure 5.

4. Categorical homotopy theory

I first came across the associahedron and multiplihedron in their incarnations as commuting pasting diagrams in the theory of bicategories and tricategories. Closely connected is the fact that actions of these polytope families serve to characterize A_n-spaces and their maps.

4.1. **Background.** The definition of enriched category generalizes the usual definition of category by replacing the hom–sets of morphisms between each two objects by hom–objects in some monoidal category $\mathcal{V}$. The collection of enriched categories over a given monoidal category, with enriched functors and natural transformations, is the 2-category known as $\mathcal{V}$-Cat. Joyal and Street showed $\mathcal{V}$-Cat to inherit structure from $\mathcal{V}$: if $\mathcal{V}$ is symmetric then so is $\mathcal{V}$-Cat, if $\mathcal{V}$ is merely braided then $\mathcal{V}$-Cat is merely monoidal.

4.2. **Iterative Enrichment.** My early contributions to enrichment theory are related to this inheritance of properties. I defined enrichment over a very general type of monoidal category with extra structure; a k-fold monoidal, or iterated monoidal category. I then proved that enrichment decrements the number of available interchanging tensor products. Next I recursively defined higher dimensional enrichment. $\mathcal{V}$-n-Cat is the collection of categories enriched over $\mathcal{V}$-$(n-1)$-Cat. I defined the enriched higher morphisms for these objects and showed that iterated enrichment increments the categorical dimension.

The homotopy theoretical implications of my results consist of an analogy between the process of creating the classifying space of a topological monoid and the the process of creating the higher category of enriched categories over a categorical monoid. Balteanu, Fiedorowicz, Schwänzl, and Vogt show a direct correspondence between k-fold monoidal categories and k-fold loop spaces through the categorical nerve. When we find the loop space of a topological space, we see that 1 dimensional paths in the original are now points in the derived space. Delooping is the inverse of this process, and it yields the classifying space of a loop space which has one less multiplication. Thus an enrichment functor acting on these categories has precisely the expected domain and codomain for a categorical delooping.

4.3. **Composihedra.** More recently my research in this area revolves around the relationship of the composihedra to enrichment. The first few polytopes in our new sequence correspond to cocyle coherence conditions in the definition of enriched bicategories. It was incorrectly believed before my clarification that these cocycle conditions had the combinatorial form of the associahedra, just as it was incorrectly assumed that A_n-maps from a topological monoid to an A_n-space were likewise governed by associahedra. With this new understanding in hand, it should be possible to start with the data of such a map, and create from it the data and structure of an enriched bicategory.

There are two projects for the future that are supported by this work. One is to make rigorous the implication that enriched bicategories may be exemplified by certain maps of topological monoids. It could be hoped that if this endeavor is successful that A_∞ categories and their maps might also be amenable to the same approach, yielding more interesting examples of enriched bicategories. A second is more philosophical. The facts that the composihedra are used for defining enriched categories and bicategories, and that they form an operad bimodule (left-module over the associahedra and right-module over the associative operad) lead us to propose that enriching over a weak n-category should in general be accomplished by use of operad bimodules as well.

- Publications and student papers:
  - Enrichment over iterated monoidal categories
    Algebraic and Geometric Topology 4, 95-119, 2004.
  - Vertically iterated classical enrichment
    Theory and Applications of Categories 12, 299-325, 2004.
  - Ph.D. Dissertation, Virgina Tech, 2004.
    Loop Spaces and Higher-Dimensional Iterated Enrichment.
- Main results:
  - For $\mathcal{V}$ k-fold monoidal $\mathcal{V}$-Cat is a $(k-1)$-fold monoidal 2-category.
  - For $\mathcal{V}$ k-fold monoidal $\mathcal{V}$-n-Cat is a $(k-n)$-fold monoidal strict $(n+1)$-category.
  - The composihedra: their role in enriched bicategory axioms,
    pseudomonoid axioms, and A_n-map axioms
- Current projects and collaborations:
  - Operad bimodule characterization of enrichment over weak n-categories.
  - Describing the nerve of an n-operad using n-dendroidal sets.
    (with I. Moerdijk and I. Weiss, by correspondence).

5. Application: Polyhexes.

In this section I'll mention an interesting convergence which has arisen in my studies. In communication with Emeric Deutsch, I have begun to investigate the *polyhexes* which consist of arrangements of a number of hexagons which share at least one side with another in the group. These arrangements look very familiar to an organic chemist, since they are the pictures of *polycyclic benzenoid hydrocarbons*. This name refers to the way that carbon often occurs in a molecule as a hexagonal ring of six atoms. One of these rings alone is the molecule benzene, C_6H_6.

There has been much recent research into the enumeration of hydrocarbons. It is still unknown how to calculate the number of possible hydrocarbons of a given size. Enumeration of hydrocarbons is closely related to special polyhexes, such as the tree-like ones with a chosen root edge. It turns out that the n^{th} composihedron has the same number of vertices as the number of all the rooted tree-like polyhexes with up to n cells. See Figure 6 for examples. If collections of molecules could be arranged meaningfully around facets of a polytope then there might be revealed interesting insights into the relational properties of those molecules. This knowledge should also accelerate the processes of building and searching digital libraries of molecules.

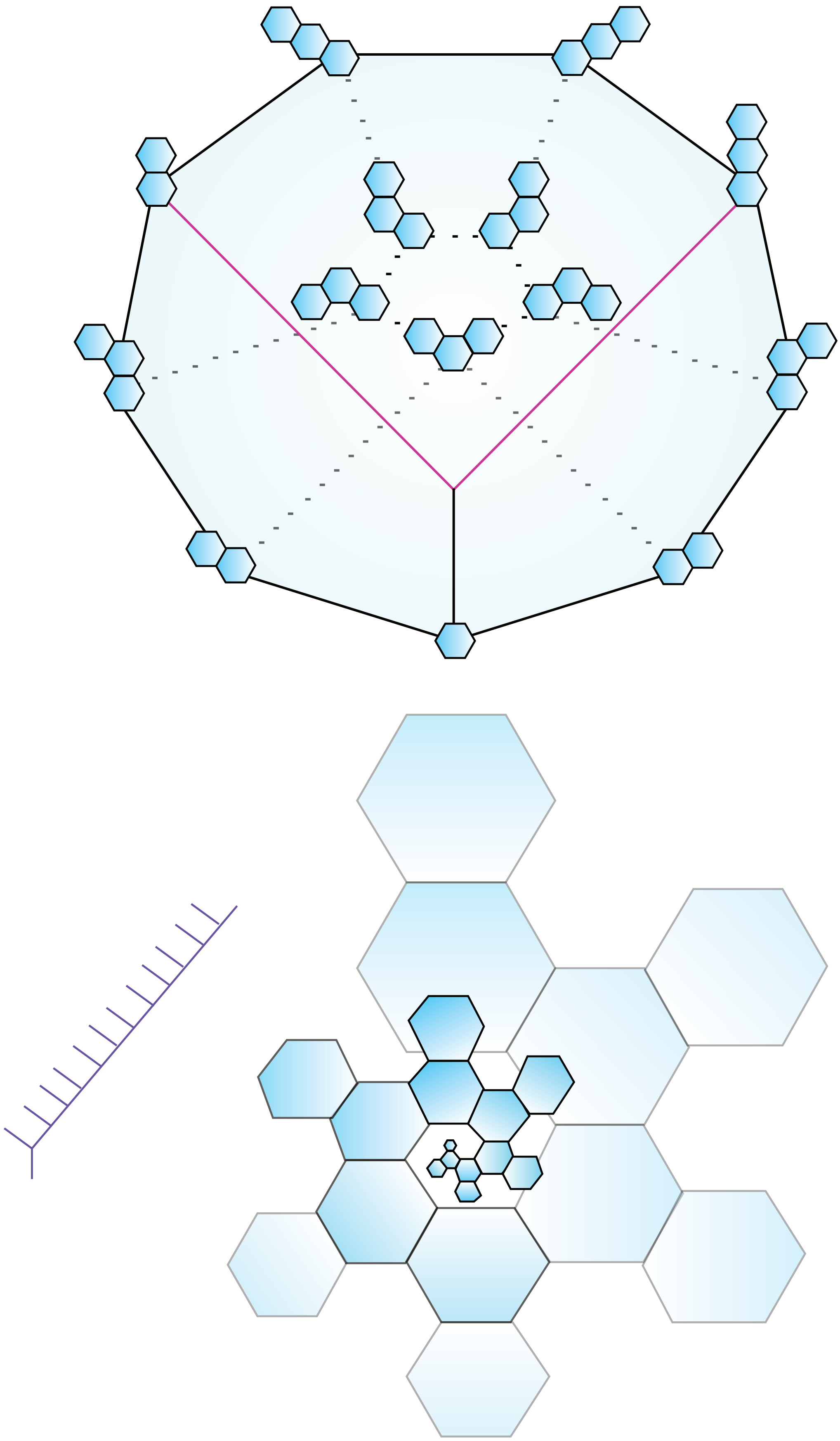

Figure 6. At the top are all the rooted polyhexes of size up to 3 cells, arranged on the composihedron CK(4). Just below them is a binary hex-tree and its associated polyhex.