chp 3

Vector Spaces & Linear Transformations

$\mathbb{R}^m$, the vectors with m components, is an example of an m-dimensional vector space.

In general: a vector space over the real scalars is any set V with structures of addition and scaling; obeying: For $\vec{x}, \vec{y}, \vec{z} \in V$ and $c, d \in \mathbb{R}$

0) $\vec{x} + \vec{y} \in V$ and $c\vec{x} \in V$ closure

1) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ associative

2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ commutative

3) there exists $\vec{0} \in V$ additive
    with $\vec{x} + \vec{0} = \vec{0} + \vec{x}$ identity

4) there exists $-\vec{x} \in V$ additive
    with $\vec{x} + -\vec{x} = \vec{0}$ inverses

5) $c(d\vec{x}) = (cd)\vec{x}$ compatibility

6) $1\vec{x} = \vec{x}$ scalar identity

7) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$ distributive

8) $(c + d)\vec{x} = c\vec{x} + d\vec{x}$ distributive

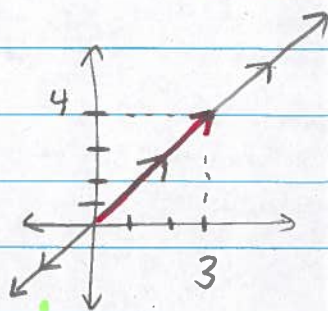ex) $\mathbb{R}^m$ any m

ex) $M^{m \times n}$ all matrices m rows, n columns

ex) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ the set of all scalings of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

that last one could be written:

$$S = \left\{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = c \begin{pmatrix} 3 \\ 4 \end{pmatrix}, c \in \mathbb{R} \right\}$$

this S is a <u>subspace</u> of $\mathbb{R}^2$

Any subset of a vector space V which is closed under addition and scaling automatically will obey 1-8, so is a subspace.

* for instance, any subspace contains $\vec{0}$

ex) $W = \left\{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} \right\}$

check: W is <u>closed</u>, so it is a <u>subspace</u>.

Also, we define the $\boxed{\text{span}}$ of a set of vectors to be the set of all lin. combs of those vectors.

so $W = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} \right\}$

and $S = \text{Span} \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$

<u>In fact</u>, <u>any</u> subspace of a (finite dimensional) vector space can be written as the span of some of its vectors.

ex) For any matrix $A_{m \times n}$ the solution to $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

- the solution contains $\vec{0}$.

We call this solution the Null Space $N(A)$.

ex) Find the null space $N(A)$ for

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

augment $\swarrow$

same as solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$\Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned} \Big\} \quad \vec{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \quad N(A) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

e) Find the null space $N(B)$

for $B = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix}$

$N(B) = \text{Span} \left\{ \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

solve $\begin{bmatrix} 1 & 2 & 1 & 0 & 1 & | & 0 \\ 0 & 3 & 6 & 0 & -3 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 3 & | & 0 \\ 0 & 1 & 2 & 0 & -1 & | & 0 \end{bmatrix}$

$\Rightarrow \quad \begin{aligned} x_1 - 3x_3 + 3x_5 &= 0 \\ x_2 + 2x_3 - x_5 &= 0 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \Big\} \quad \begin{aligned} x_1 &= 3x_3 - 3x_5 \\ x_2 &= -2x_3 + x_5 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \Big\}$

$\vec{x} = x_3 \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

ex) $S = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

is a subspace of $\mathbb{R}^2$.

But, that set of vectors is <u>lin. dep.</u>
(since $4 > 2$)
That means, some of those vectors can
be made as lin. combs. of others,
so the list is redundant: there
is a smaller list whose span is S.

Def: a $\boxed{\text{basis}}$ B of a vector space V
(or subspace) is a <u>lin. indep.</u>
set of vectors $B = \{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \}$
such that $\underline{\text{span}(B) = V}.$

To find a basis for S, row reduce
the matrix of those column vectors,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ in r.r.e.f.}$$

$\rightarrow$ find the <u>pivots</u>, and
then find the original
columns in those positions (col 1 and 3)

$\rightarrow$ Then $S = \boxed{\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}}$ for $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$
$\rightarrow$ (B is a <u>basis</u>)
$\rightarrow$ This S is also called the $\boxed{\text{column space}}$ $\underline{\text{col}(A)}$
of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$

ex) Find the column space, as span of a basis,
for $A = \begin{bmatrix} 3 & 0 & 6 & 0 & 1 & 2 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

r.r. $\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
  $\uparrow$ $\uparrow$

$\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$

Note: if you know Calc 3; $\vec{n} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

$\text{col}(A)$ is a plane: $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 6 \\ 1 & 1 & 7 \end{vmatrix} = \langle 28, -21, -1 \rangle$

and $(0,0,0)$ is in the plane
so $28x - 21y - z = 0$ is an equation of the plane.

Also $\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} x = 3t + s \\ y = 4t + s \\ z = 7s \end{cases} \right\}$

is a parameterization
of that plane.

Def. The dimension of a vector space, $\dim(V)$ of V (or subspace S) is the number of vectors in any basis of V (or S).

ex) $\mathbb{R}^n$ has dimension n.
The standard basis for $\mathbb{R}^n$ is called $\mathcal{E} = \mathcal{E}(\mathbb{R}^n) = \{ \vec{e}_1, \vec{e}_2, ..., \vec{e}_n \}$ where $\vec{e}_i =$ all zero components except one "1" in the ith component.

$$\mathcal{E} \text{ for } \mathbb{R}^4 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

* $\mathcal{E}$ is ordered!

Note: $\mathbb{R}^n$ has many other bases (∞).

2-out-of-3 rule: if $\dim(V) = n$

set of $\boxed{n}$ vectors in V

set of vectors that $\boxed{spans}$ V

set of $\boxed{lin.\,indep.}$ vectors in V

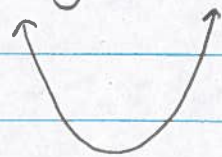" any 2 of these implies the third!

- n lin. indep. vectors $\Rightarrow$ spans
- n vectors which span $\Rightarrow$ lin. indep.
- lin. indep. and spans $\Rightarrow$ exactly n vectors

A new vector space P^n
is the set of all polynomials with
degree at most n.

ex) P^2 = all the polynomials with degree ≤ 2.
              such as: • x^2
                       • $3x^2 + 2$
                       • $\frac{1}{2}x^2 - 3x + 7$
                       • 5
                       • $x - 1$
                       • 0 always $= 0$, all x

P^2 is a vector space : obeys all 8 axioms.

$\rightarrow$ $3(3x^2 + 2) + (x-1) = 9x^2 + x + 5 \in P^2$
$\rightarrow$ $3x^2 + 2 + 0 = 3x^2 + 2$

ex) Is the set $\{x^2 + 3, x^2\}$
    lin. indep?
    Means : if $C_1(x^2 + 3) + C_2 x^2 = 0$
    then is $C_1 = C_2 = 0$ the only solution?
Solve:
    Expand $C_1 x^2 + C_1 3 + C_2 x^2 = 0$
    $\Rightarrow (C_1 + C_2)x^2 + C_1 3 = 0$
But this must be true for all x-values, including $x=0$!
    $\Rightarrow$ $C_1 3 = 0$
    $\Rightarrow$ $\boxed{C_1 = 0}$
    $\Rightarrow (0 + C_2)x^2 + 0(3) = 0$
    $\Rightarrow$ $C_2 x^2 = 0$ true for $x=1$!
    $\Rightarrow$ $\boxed{C_2 = 0}$ $\Rightarrow$ $\boxed{\text{lin. indep.}}$

Also, the set $\mathcal{E} = \{1, x, x^2\} \subseteq P^2$
is lin. indep. Plus, it spans P^2!
(any deg 2 or less polynomial looks like
$f(x) = c + bx + ax^2$, written $[f(x)]_\mathcal{E} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$
"column vector for $\mathcal{E}$"

So it is a basis for P^2, and it
has 3 items (vectors), so
P^2 has dimension $= 3$.

In general P^n has $\dim(P^n) = n+1$
and standard basis $\mathcal{E}$ (P^n)
This $\mathcal{E}$ is also ordered:
$\mathcal{E} = \{1, x, x^2, x^3, \ldots, x^n\}$.

An alternate basis for P^3 is

$$B = \{5, \; x^3+1, \; x^2, \; x+x^2\}$$

From now on, all our bases will have specific ordering.
Find $f(x) = 4x^3 + 2x$ as a lin. comb.
of B.

Method: in terms of standard basis $\mathcal{E}$ for P^3
$[f]_\mathcal{E} = f = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}$, $B = \left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

so solve
(r.r.)
$\left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{array} \right] \sim \left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$

$f = -4/5 (5) + 4(x^3+1) - 2(x^2) + 2(x+x^2)$. $\quad [f]_B = \begin{pmatrix} -4/5 \\ 4 \\ -2 \\ 2 \end{pmatrix}$

Ex) Find $x^3 + 5x + 2 = f(x)$

in the basis $B = \{1, x-3, \frac{(x-3)^2}{2}, \frac{(x-3)^3}{6}\}$

That is, find $[f(x)]_B$, the col. vector

representation of f, in basis B.

$B = \{1, x-3, \frac{1}{2}x^2 - 3x + \frac{9}{2}, \frac{1}{6}x^3 - \frac{9x^2}{6} + \frac{27x}{6} - \frac{27}{6}\}$

$$\begin{bmatrix} 1 & -3 & 9/2 & -9/2 & | & 2 \\ 0 & 1 & -3 & 9/2 & | & 5 \\ 0 & 0 & 1/2 & -3/2 & | & 0 \\ 0 & 0 & 0 & 1/6 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9/2 & 9 & | & 17 \\ 0 & 1 & -3 & 9/2 & | & 5 \\ 0 & 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & 1 & | & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & | & 17 \\ 0 & 1 & 0 & -9/2 & | & 5 \\ 0 & 0 & 1 & 0 & | & 18 \\ 0 & 0 & 0 & 1 & | & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & | & 44 \\ 0 & 1 & 0 & 0 & | & 32 \\ 0 & 0 & 1 & 0 & | & 18 \\ 0 & 0 & 0 & 1 & | & 6 \end{bmatrix}$$

so $[f(x)]_B = \begin{pmatrix} 44 \\ 32 \\ 18 \\ 6 \end{pmatrix}$, $f(x) = 44 + 32(x-3) + 18\frac{(x-3)^2}{2} + 6\frac{(x-3)^3}{6}$

If you know calc II, that's the Taylor series

for $f(x)$ at $x_0 = 3$,

$f(3) = 44$, $f'(3) = 3(3)^2 + 5 = 32$, $f''(3) = 18$, $f'''(3) = 6$

Ex) in $\mathbb{R}^2$, find $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in the

basis $B = \{\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$

$$\begin{bmatrix} -2 & 1 & | & 3 \\ 1 & 0 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 2 \\ -2 & 1 & | & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 7 \end{bmatrix}$$

so $[\begin{pmatrix} 3 \\ 2 \end{pmatrix}]_B = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$; $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

Change of basis

For a given basis B, the matrix to row reduce is always the same, only the augment changes.

Note : row reduction move on A gives the same result as:

(same r.r. move on I)$\cdot A$

ex: $A = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix}$ $\xrightarrow{R2 \leftarrow R2 + 2R3}$ $\begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix}$ ✓

So if we row reduce I with all the same moves, just like for finding A^{-1}, we'll get a matrix that can do those moves (via multiplication) on any vector. It will be a change-of-basis matrix from $\mathcal{E}$ to B, we call it $[I]_{\mathcal{E}}^{B}$.

ex)
$B = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

row reduce $\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array} \right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array} \right]$ and $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ is the c.o.b.

$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{E}} = [\vec{x}]_{B}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = [I]_{\mathcal{E}}^{B}$

For any two bases B and C
we can find $[I]_B^C$. $\quad\left(\begin{array}{l} B = \{\vec{b}_1, \ldots, \vec{b}_n\} \\ C = \{\vec{c}_1, \ldots, \vec{c}_n\} \end{array}\right)$

so $\quad [I]_B^C \, [\vec{x}]_B = [\vec{x}]_C$

by $\quad [I]_B^C = \left[\, [\vec{b}_1]_C \ \ [\vec{b}_2]_C \ \ \ldots \ [\vec{b}_n]_C \,\right]$

$\underbrace{\qquad\qquad\qquad\qquad}$ columns are the basis vectors of B, written as col. vectors in C.

for our example $\rangle \quad [I]_\mathcal{E}^B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \left[\, \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_B \ \ \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_B \,\right]$

Note: $[I]_B^C$ is always square, $n \times n$.

$[I]_B^C$ is always invertible, and

$$\left([I]_B^C\right)^{-1} = [I]_\mathcal{E}^B$$

example $\rangle \quad [I]_B^\mathcal{E} = \left([I]_\mathcal{E}^B\right)^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \left[\, \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]_\mathcal{E} \ \ \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_\mathcal{E} \,\right]$