New Hopf structures on planar binary trees

Stefan Forcey†, Aaron Lauve‡, Frank Sottile‡

†Tennesee State University, ‡Texas A&M University

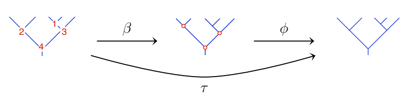
Question

Given that there are polytope quotients $\mathfrak{S}_\bullet \twoheadrightarrow \mathcal{M}_\bullet \twoheadrightarrow \mathcal{Y}_\bullet$ from the permutahedra to the multiplihedra to the associahedra, and given that the two extremes have been made into Hopf algebras,

What Hopf structures exist in the middle?

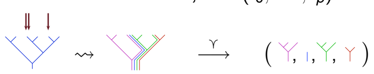
Ordered, bi-leveled, and ordinary trees

Different types of planar binary trees index the vertices of the permutahedra, the multiplihedra, and the associahedra:

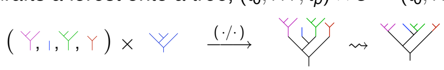


Operations on trees

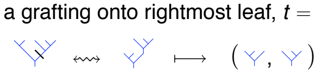
Splitting. Splits a tree into a forest, $t \xrightarrow{\curlyvee} (t_0, \ldots, t_p)$.



Grafting. Grafts a forest onto a tree, $(t_0, \ldots, t_p) \times s \mapsto (t_0, \ldots, t_p) / s$.



Pruning. Undo a grafting onto rightmost leaf, $t = r \setminus s \mapsto (r, s)$.



Hopf structures on $\mathfrak{S}_\bullet$ and $\mathcal{Y}_\bullet$

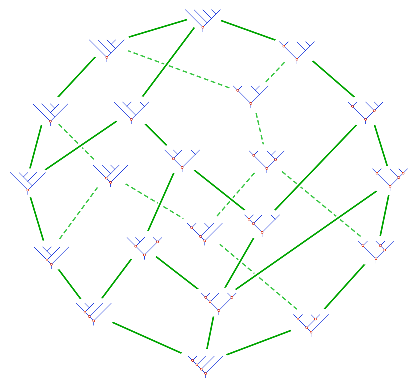
Let $\mathfrak{S}Sym := \mathrm{span}_{\mathbb{K}}\left\{F_w \mid w \in \mathfrak{S}_\bullet\right\}$ *and* $\mathcal{Y}Sym := \mathrm{span}_{\mathbb{K}}\left\{F_t \mid t \in \mathcal{Y}_\bullet\right\}$. *These spaces become Hopf algebras [1], [2] under the maps*

$$F_t \cdot F_s = \sum_{t \xrightarrow{\curlyvee} (t_0,\ldots,t_p)} F_{(t_0,\ldots,t_p)/s} \tag{1}$$

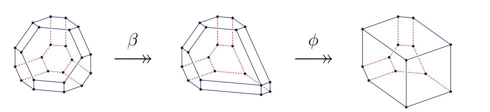
and

$$\Delta(F_t) = \sum_{t \xrightarrow{\curlyvee} (t_0,t_1)} F_{t_0} \otimes F_{t_1}. \tag{2}$$

Moreover, the map τ *induces a Hopf algebra map* $\tau : \mathfrak{S}Sym \to \mathcal{Y}Sym$.



The 1-skeleton of the ***multiplihedron*** $\mathcal{M}_4$.



Change of basis

Use Möbius inversion (in posets $\mathfrak{S}_\bullet$ and $\mathcal{Y}_\bullet$) to define a new basis, e.g.,

$$M_v := \sum_{v \le w} \mu_{\mathfrak{S}}(v, w) F_w .$$

Changing basis (from fundamental basis $\{F_\bullet\}$ *to monomial basis* $\{M_\bullet\}$*) reveals additional structure for the coproduct [1], e.g.,*

$$\Delta(M_t) = \sum_{t = r \setminus s} M_r \otimes M_s . \tag{3}$$

In particular, the trees (ordered or ordinary) with no nontrivial prunings are ***primitive elements***. (In fact, they form bases for the primitives.)

Operations on bi-leveled trees

Define the space $\mathcal{M}Sym := \mathrm{span}_{\mathbb{K}}\left\{F_b \mid b \in \mathcal{M}_\bullet\right\}$. Follow (1) and (2).

Are the splitting, grafting, and pruning operations well-defined on bi-leveled trees?

How about between ordered, bi-leveled, and ordinary trees?

Main results

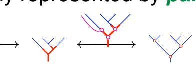
- *$\mathcal{M}Sym$ is an algebra, $\mathfrak{S}Sym$-module, and (right) $\mathcal{Y}Sym$-comodule in a natural way (in terms of operations on bi-leveled trees).*
- *The space $\mathcal{M}Sym_+$ has a natural right $\mathcal{Y}Sym$–Hopf module structure.*
- *The monomial basis reveals extra structure for coaction on $\mathcal{M}Sym_+$:*

$$\Delta(M_b) = \sum_{b = c \setminus t} M_b \otimes M_t ,$$

*where allowable prunings leave all circled nodes in c. In particular, it reveals a basis for the **coinvariants** for this Hopf module.*

Painted trees ($\mathcal{M}_\bullet$ history)

Stasheff [3] introduced $\mathcal{M}_\bullet$ in the context of *H*-spaces to catalog identities a map $f : (\mathcal{C}, \bullet) \to (\mathcal{D}, *)$ must satisfy to perserve higher homotopy associativity. These are most naturally represented by ***painted trees:***

$f(a) * (f(b \bullet c) * f(d)) \longleftrightarrow$ 

Further results

Using painted tree formulation, we can show $\mathcal{M}Sym_+$ becomes a (one-sided) Hopf algebra (and two-sided $\mathcal{Y}Sym$–Hopf module)

References

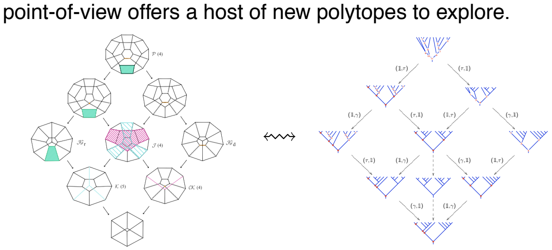
[1] M. Aguiar & F. Sottile, *Structure of the Loday-Ronco Hopf algebra of trees*, J. Algebra **295** (2006), no. 2, 473–511.

[2] J.-L. Loday & M. O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309.

[3] J. Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Mathematics., Vol. 161, Springer-Verlag, Berlin, 1970.

The future

Our point-of-view offers a host of new polytopes to explore.



http://www.math.tamu.edu/~lauve

lauve@math.tamu.edu