CONVEX POLYTOPES FROM NESTED POSETS

SATYAN L. DEVADOSS, STEFAN FORCEY, STEPHEN REISDORF, AND PATRICK SHOWERS

Abstract. Motivated by the graph associahedron $\mathcal{K}G$, a polytope whose face poset is based on connected subgraphs of G, we consider the notion of associativity and tubes on posets. This leads to a new family of simple convex polytopes obtained by iterated truncations. These generalize graph associahedra and nestohedra, even encompassing notions of nestings on CW-complexes. However, these *poset associahedra* fall in a different category altogether than generalized permutohedra.

1. Background

1.1. Given a finite graph G, the graph associahedron $\mathcal{K}G$ is a polytope whose face poset is based on the connected subgraphs of G [4]. For special examples of graphs, $\mathcal{K}G$ becomes well-known, sometimes classical: when G is a path, a cycle, or a complete graph, $\mathcal{K}G$ results in the associahedron, cyclohedron, and permutohedron, respectively. Figure 1 shows some examples, for a graph and a pseudograph with multiple edges. These polytopes were first

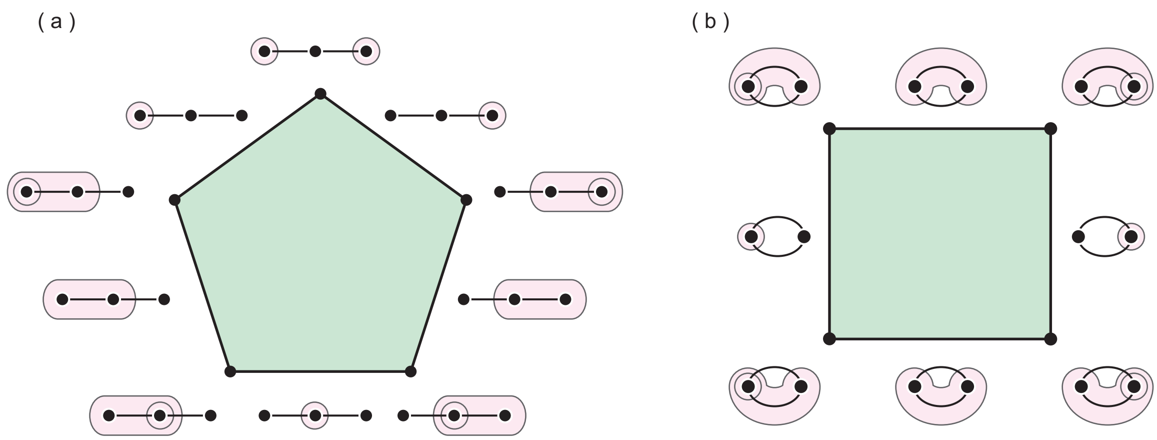


Figure 1. Graph associahedra of a path and a multi-edge.

motivated by De Concini and Procesi in their work on "wonderful" compactifications of hyperplane arrangements [6]. In particular, if the hyperplane arrangement is associated to a Coxeter system, the graph associahedron $\mathcal{K}G$ appear as tilings of these spaces, where its underlying graph G is the Coxeter graph of the system [5]. These compactified arrangements are themselves natural generalizations of the Deligne-Knudsen-Mumford compactification

2000 *Mathematics Subject Classification.* Primary 52B11, Secondary 55P48, 18D50.
Key words and phrases. poset, graph associahedron, nesting, polytope.

$\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of the real moduli space of curves [7]. From a combinatorics viewpoint, graph associahedra arise in numerous areas, ranging from Bergman complexes of oriented matroids to Heegaard Floer homology [1]. Most notably, these polytopes have emerged as graphical tests on ordinal data in biological statistics [12].

1.2. The combinatorial and geometric structures of these polytopes capture and expose the fundamental concepts of connectivity and nestings, and it is not surprising that there have been several similar notions, such as nested sets [10], nested complexes [16] and the larger class of generalized permutohedra of Postnikov [13]. However, none of these constructions capture the notion of nested sets of posets, as we do below. Indeed, our notion of the set of poset tubes is not a classical building set, but falls in a different category altogether.

In this paper, we construct a new family of convex polytopes which are extensions of nestohedra and graph associahedra via a generalization of building sets. But rather than starting with a set, we begin with a poset P. The resulting *poset associahedron* $\mathcal{K}P$, based on connected lower sets of P, covers a wide swath of existing examples from geometric combinatorics, including permutahedra, associahedra, multiplihedra, graph associahedra, nestohedra, pseudograph associahedra, and their liftings; in fact, all these types are just from two rank posets. Newly discovered are polytopes capturing associativity information of CW-complexs.

An overview of the paper is as follows: Section 2 supplies the definitions of poset associahedra along with several examples, while Section 3 provides methods of constructing them via induction. Specialization to nestohedra and permutohedra is given in Section 4, and we finish with proofs of the main theorems in Section 5.

2. Posets

2.1. We begin with some foundational definitions about posets. The reader is forewarned that definitions here might not exactly match those from earlier works. A *lower set* L is a subset of a poset P such that if $y \preceq x \in L$, then $y \in L$. The *boundary* of an element x is $\partial x := \{y \in P \mid y \prec x\}$.

Definition. Let $\mathfrak{b}_x := \{y \in P \mid \partial y = \partial x\}$ be the *bundle* of the element x. A bundle is *trivial* if $\mathfrak{b}_x = \{x\}$.

Throughout this paper, a poset will be visually represented by its Hasse diagram. Consider the example of a poset P given on the left side of Figure 2. The subset $\{1, 2, 4, 5\}$ in part (a), depicted by the highlighted region, is not a lower set since it does not include element 3. This poset is partitioned into four bundles, $\{1, 2, 3\}$, $\{4\}$, $\{5\}$, and $\{6, 7, 8\}$, with elements in a bundle having identical boundary. In particular, notice that all minimal elements of the poset are in one bundle since they share the empty set as boundary. The following is immediate:

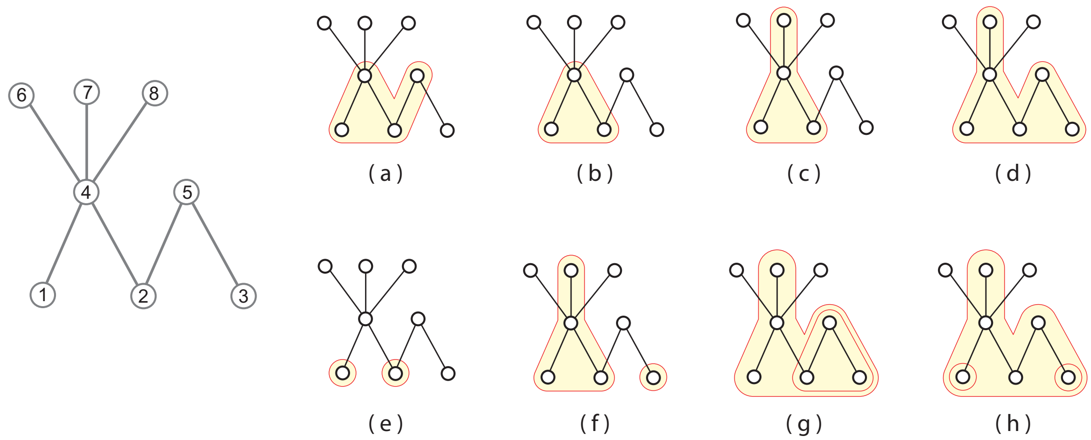


FIGURE 2. Some examples of valid and invalid tubes and tubings.

Lemma 1. *The elements of poset P are partitioned into equivalence classes of bundles.*

Definition. A lower set is *filled* if, whenever it contains the boundary ∂x of an element x, it also intersects the bundle $\mathfrak{b}_x$ of that element. A *tube* is a filled, connected lower set. A *tubing* T is a collection of tubes (not containing all of P) which are pairwise disjoint or pairwise nested, and for which the union of every subset of T is filled.

Figure 2(b) shows the boundary of $\{6, 7, 8\}$, which is an unfilled lower set, whereas (c) is a filled one. Note that parts (c, d) display examples of one tube. Parts (e, f) display two disjoint tubes which are not tubings, since the union of the tubes would create an unfilled lower set. Examples of tubings with two and three components are given by (g, h) respectively.

2.2. We now present our main result.

Theorem 2. *Let P be a poset with n elements partitioned into b bundles. If $\pi(P)$ is the set of tubings of P ordered by reverse containment, the* poset associahedron $\mathcal{K}P$ *is a convex polytope of dimension $n - b$ whose face poset is isomorphic to $\pi(P)$.*

This theorem follows from the construction of $\mathcal{K}P$ from truncations, described in Theorem 5 below. We now pause to illustrate several examples.

Example. Figure 3 shows the two polytopes of Figure 1, reinterpreted as tubings on posets of their underlying graphs. Both posets are *two rank*, the number of elements in any maximal chain of the poset. Part (a) has 5 elements and 3 bundles, whereas (b) has 4 elements and 2 bundles, both resulting in polygons, as given in Theorem 2.

Example. Figure 4 shows two different posets, resulting in identical poset associahedra as Figure 3. Part (a) shows a poset structure which does not even come from a CW-complex.

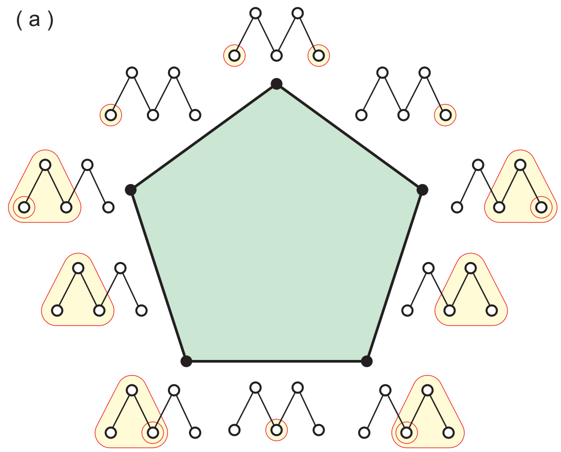


Figure 3. Poset versions of Figure 1.

Notice here that the left element of height zero cannot be a tube since it needs to be filled. Part (b) has a near identical structure to Figure 3(b). Here, the bottom-right element cannot be a tube in itself since it is unfilled. Because both posets have 5 elements and 3 bundles, the dimension of the polytopes is two.

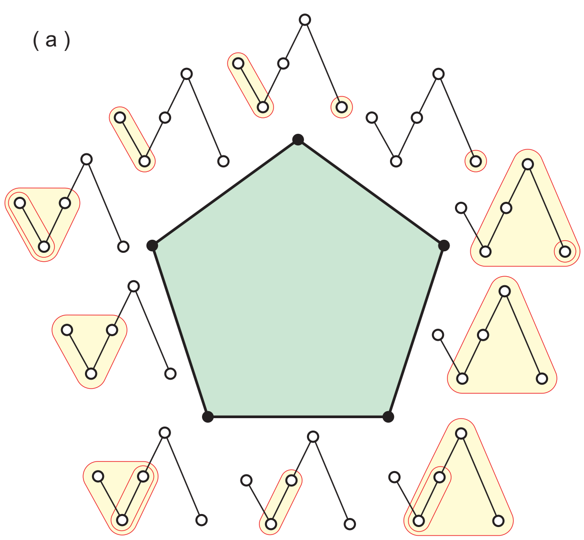


Figure 4. Alternate poset associahedra.

Example. Three examples of 3D poset associahedra are given in Figure 5. The cube in (a) can be viewed as an extension of the square in Figure 3(b), and Proposition 12 generalizes this pattern to the n-cube. A truncation of this cube results in (b), with both posets having 6 elements partitioned into 3 bundles. Part (c) shows a novel construction of the 3D associahedron K_5, with 7 elements partitioned into 4 bundles.

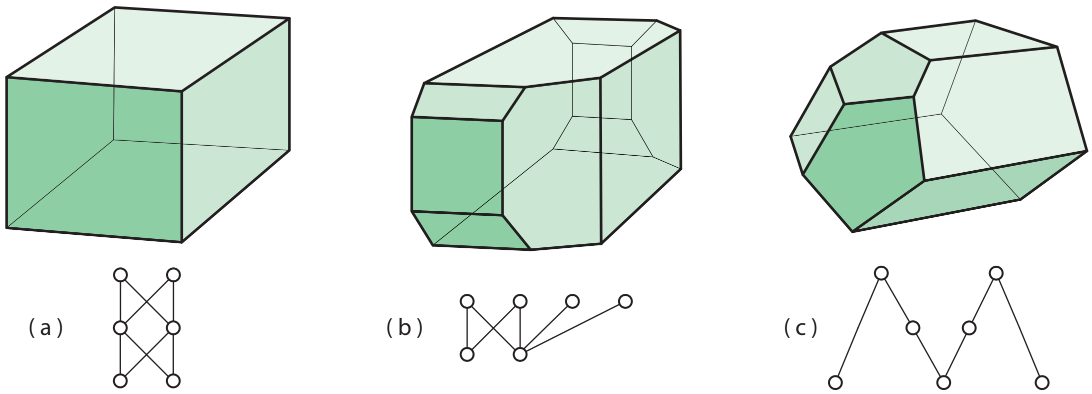


FIGURE 5. Examples of 3D poset associahedra.

2.3. The notion of being "filled" appears in different guises: For graph associahedra, a *tube* of a graph is filled because it is an induced subgraph, catalogued just by listing its set of vertices [3, Section 2]. A *tubing* is filled because its tubes are "far apart," which is equivalently described by saying that two distinct tubes in a tubing cannot have a single edge connecting them. Since a simple graph has no bundles, the following is immediate:

Proposition 3. *The graph associahedron $\mathcal{K}G$ can be obtained as a poset associahedron $\mathcal{K}P$, where P is the face poset of graph G.*

For pseudographs (having loops and multiple edges), a filled tube is a connected subgraph t where at least one edge between every pair of nodes of t is included if such edges exist [4, Section 2]. For multiple loops and edges of G, the notion of tubes on posets match perfectly with tubes on G, and the proposition above extends to the pseudograph associahedron. The first three examples of Figure 6 displays invalid tubings (all due to not being filled) and the last a valid one; the top row shows tubes on graphs whereas the bottom recasts them on posets.

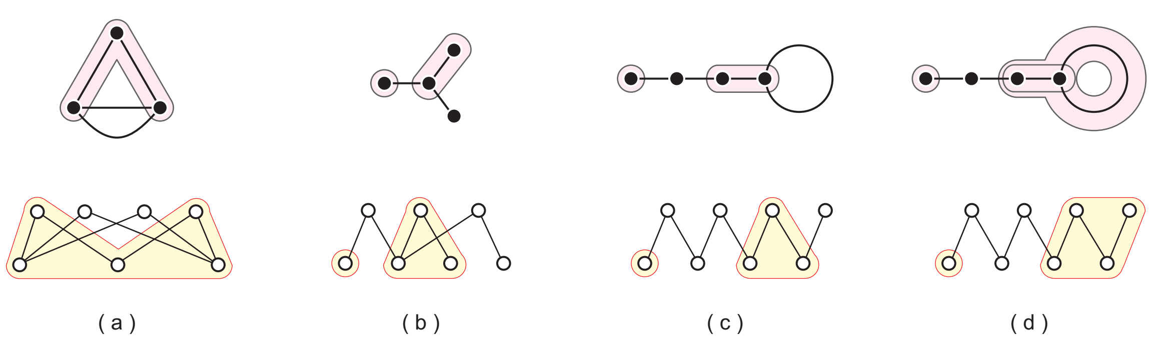


FIGURE 6. Valid and invalid tubings on graphs and posets.

Remark. For a single loop attached to a node v of G, the pseudograph associahedron defined in [4] gave a choice of choosing or ignoring the loop when v is chosen in a tube; see Figure 6(c) and (d). In this paper, however, for the sake of consistency in the notion of "filled", we always include the loop for poset associahedron. This allows us to always obtain convex polytopes, rather than unbounded polyhedral chambers of [4].

The notion of *associativity*, encapsulated by drawing tubes on graphs, has a natural generalizations to higher-dimensional complexes. In particular, for any CW-complex structure X, consider its face poset P_X. The poset associahedron $\mathcal{K}P_X$ captures the analogous information of X that the graph associahedron captures for a graph.

Example. Figure 7(a) shows a CW-complex, with three 2-cells, 1-cells, and 0-cells. Part (b) shows the poset structure of this complex, and (c) its poset associahedron.

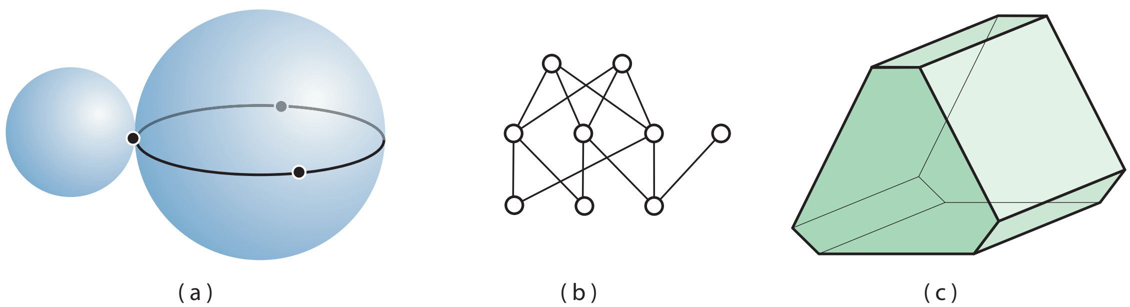


FIGURE 7. The poset associahedron of a CW-complex.

3. CONSTRUCTIONS

3.1. The poset associahedron $\mathcal{K}P$ is recursively built by a series of truncations. Because the truncation procedure is a delicate one, we present an overview here and save the details for the proof in Section 5. First, the following result allows us to consider only connected posets:

Proposition 4. *Let P be a poset with connected Hasse components $P_1, \ldots, P_m$. Then $\mathcal{K}P$ is isomorphic to $\mathcal{K}P_1 \times \cdots \times \mathcal{K}P_m \times \Delta_{m-1}$.*

Proof. Any tubing of P can be described as:

(1) a listing of tubings $T_1 \in \mathcal{K}P_1, \ \ldots, \ T_k \in \mathcal{K}P_m$, and
(2) for each component P_i either including or excluding the tube $T_i = P_i$, as long as all tubes P_i are not included.

The second part of this description is clearly isomorphic to a tubing of the edgeless graph H_m on m nodes. But from [8, Section 3], since $\mathcal{K}H_m$ is the simplex Δ_{m-1}, we are done. $\square$

Theorem 5. *The poset associahedron $\mathcal{K}P$ is constructed inductively on the number of elements of P. Choose a maximal element x of a maximal length chain of P.*

(1) *If bundle $\mathfrak{b}_x$ is trivial, truncate $\mathcal{K}(P - x)$ to obtain $\mathcal{K}P$.*
(2) *If bundle $\mathfrak{b}_x$ is nontrivial, truncate $\mathcal{K}\left(P - (\mathfrak{b}_x - x)\right) \times \Delta_{|\mathfrak{b}_x - x|}$ to obtain $\mathcal{K}P$.*

This immediately implies the combinatorial result of Theorem 2. The following is a notable consequence:

Proposition 6. *There are different ways to construct $\mathcal{K}P$, based on the possible choices of maximal elements in the recursive process.*

In certain situations, altering the underlying poset does not affect the polytope. This occurred in Figures 3(b) and 4(b), and can be presented as

Corollary 7. *Let x be a maximal element of a maximal chain of P such that $\mathfrak{b}_x$ is trivial. If ∂x is connected, then $\mathcal{K}P = \mathcal{K}(P - x)$.*

Proof. We show that t is a tube of P if and only if $t - x$ is a tube of $P - x$. If $x \notin t$, then $\partial x \notin t$, and t has the properties of a tube in both P and $P - x$, or in neither. On the other hand, if $x \in t$, then $\partial x \in t$, and so t is connected if and only if $t - x$ is connected. Extending this isomorphism of tubes to tubings preserves this containment. □

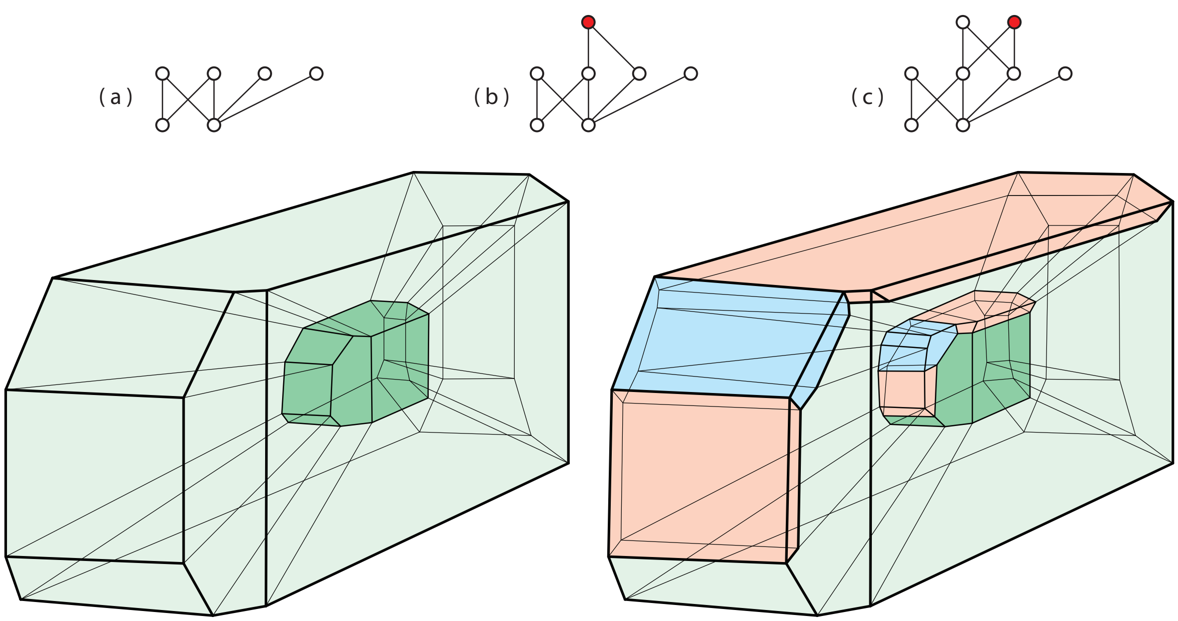


FIGURE 8. Construction of a 4D poset associahedron.

Example. A 4D case for Theorem 5 is provided in Schlegel diagram on the right side of Figure 8, where the poset steps are drawn above. Part (a) begins with the poset P_* of Figure 5(b). In Figure 8(b), by Corollary 7, adding the new maximal element to this poset does not change the structure of the polytope. Finally, part (c) shows the addition of a nontrivial bundle, now with two elements. According to Theorem 5, we first consider the 4D polytope $\mathcal{K}P_* \times \Delta_1$, the left Schlegel diagram of Figure 8. Then truncate certain faces (first the two blue chambers, then the four orange ones) to obtain the 4D poset associahedron drawn on the right.

3.2. We close this section with a corollary of Proposition 6 as it pertains to the classical associahedron K_n. Interestingly, this construction of the associahedron is novel, though examples of special cases have appeared in different parts of literature, as referenced below.

Proposition 8. *The poset associahedron of the* zigzag *poset with* $2n-1$ *elements yields the classic associahedron* K_{n+1}. *In particular, the associahedron* K_{n+1} *is obtained by truncations of codimension two faces of* $K_{p+1} \times \Delta_1 \times K_{q+1}$, *where* $n = p + q$ *and* $p, q \geq 1$.

Proof. The poset P of a path G with n nodes is the zigzag poset $2n-1$ elements; the tubings on P resulting in $\mathcal{K}P$ are in bijection with tubes on the graph G. The enumeration of the different types of truncation comes from removing a maximal element of P and using Theorem 5(b). □

Remark. The particular construction of K_{n+1} from $K_n \times \Delta_1 \times K_2 \simeq K_n \times \Delta_1$ appears in another form in the work by Saneblidze and Umble [14] on diagonals of associahedra.

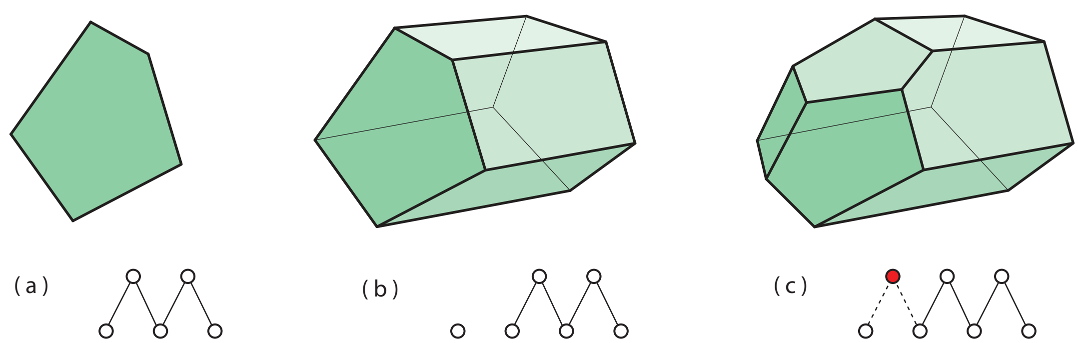


FIGURE 9. The associahedron K_5 from a pentagonal prism, $K_4 \times \Delta_1 \times K_2$.

Example. Figure 9 considers one construction of the 3D associahedron: Part (a) begins with the pentagon from Figure 3(a), and (b) adds an extra disconnected element. By Proposition 4, the result is the product with Δ_1, a pentagonal prism. Part (c) connects up the poset with a trivial bundle; two edges of the prism are truncated according to the proof of Theorem 5 to yield the associahedron. Indeed, Figure 5(c) is formed in an identical manner.

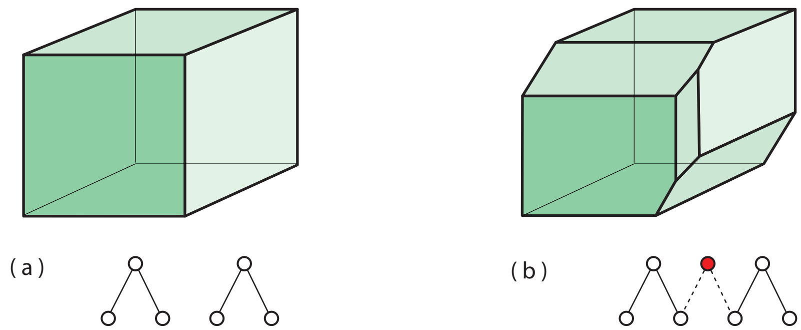


Figure 10. The associahedron K_5 from a cube, $K_3 \times \Delta_1 \times K_3$.

Example. Figure 10 considers another assembly of the 3D associahedron: Each disconnected component yields an interval, and together (by Proposition 4), the result is a cube (a), the product of three intervals. Part (b) connects up the poset with a trivial bundle, and three of its edges are truncated according to the proof of Theorem 5. This construction appears in [2], motivated by truncating codimension two faces of cubes.

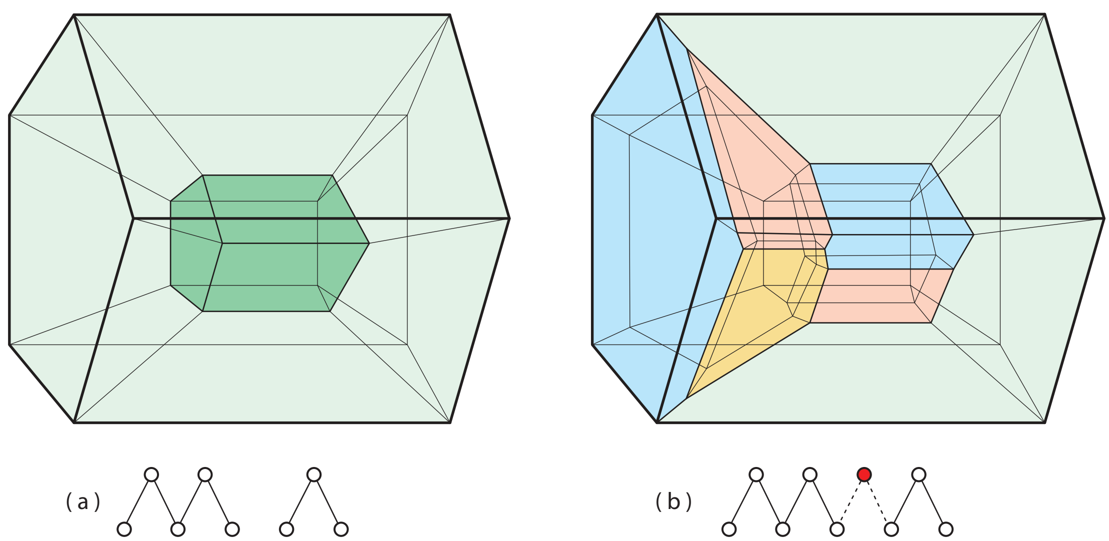


Figure 11. The associahedron K_6 from $K_4 \times \Delta_1 \times K_3$.

Example. In a similar vein, Figure 11 obtains the 4D associahedron K_6 (right side) from truncating five codimension two faces of $K_4 \times \Delta_1 \times K_3$ (left side). The order of truncation is important: first the two blue faces, then two orange faces, and finally one yellow face.

4. Family of Associahedra

4.1. The $(n-1)$-dimensional permutahedron $\mathcal{P}_n$ is the convex hull of the points formed by the action of a finite reflection group on an arbitrary point in Euclidean space. The classic example is the convex hull of all permutations of the coordinates of the Euclidean point $(1, 2, \ldots, n)$. Changing edge lengths while preserving their directions results in the *generalized permutohedron*, as defined by Postnikov [13]. An important subclass of these is the *nestohedra* [16]: Nestohedra have the feature that each of their faces corresponds to a specific combinatorial set, and the intersection of two faces corresponds to the union of the two sets. For a given set S, each nestohedron $N(B)$ is based upon a given building set B, whose elements are known as tubes, where B must contain all the singletons of S and must also contain the union of any two tubes whose intersection is nonempty.

Proposition 9. *All nestohedra can be obtained as poset associahedra, with posets of two ranks, with no bundles of size greater than one.*

Proof. Given a building set B of a set S, we describe a ranked poset P_B, with exactly two ranks, whose tubes are in bijection with B and whose tubings are in bijection with the nested sets of B. The poset P_B has minimal elements given by set S, and has maximal elements (each a trivial bundle) given by set B, each having boundary exactly the minimal elements that it contains. The tubes of P_B are all the connected lower sets of P_B generated by a single maximal element of P_B (since each such lower set is automatically filled). And if a subset T of tubes is not filled, then T must be the boundary of a maximal element in P_B, and thus the collection of minimal elements in T make up an element of B. The ordering of nested sets by reverse inclusion corresponds to the ordering of tubings by reverse inclusion, so the nestohedron $N(B)$ is isomorphic to our polytope $\mathcal{K}P_B$. □

Note that P_B is one of many posets whose polytope is $N(B)$; many more posets (with at most two ranks) can be found. Start with set S and create any number of new maximal elements, each of which covers some of S, where each maximal element is a trivial bundle. The set of tubes of such a poset P will yield a building set B on the set of minimal elements of P, due to the definition of a tube as a connected filled lower set. This amounts to choosing a subset of the power set of S (a *hypergraph* on S), and thus the process of building $\mathcal{K}P$ for such a poset is akin to constructing the hypergraph polytope [9].

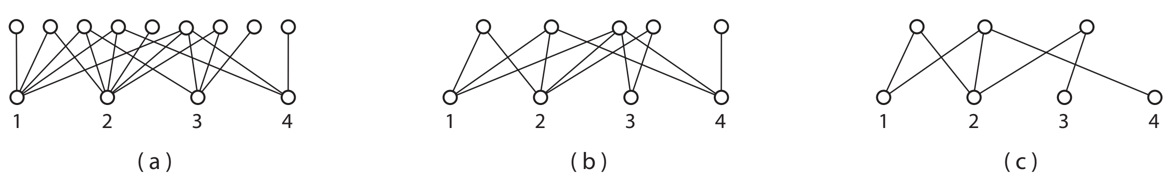


FIGURE 12. Examples of different posets resulting in identical poset associahedra.

Example. Given set $S = \{1, 2, 3, 4\}$ and building set $B = \{\{1\}, \{2\}, \{3\}, \{4\}, \{12\}, \{23\}, \{234\}, \{124\}, S\}$, Figure 12(a) shows the poset P_B constructed in the proof of Proposition 9. Moreover, all the posets in this figure result in identical poset associahedra.

4.2. Although poset associahedra contain nestohedra, they are a different class than generalized permutohedra. For instance, Figure 5(b) shows a 3D polytope which has an octagonal face, something not possible for generalized permutohedra. Figure 13 below shows this octagon in detail. Similarly, the 4D example in Figure 8 is not a nestohedron as well.

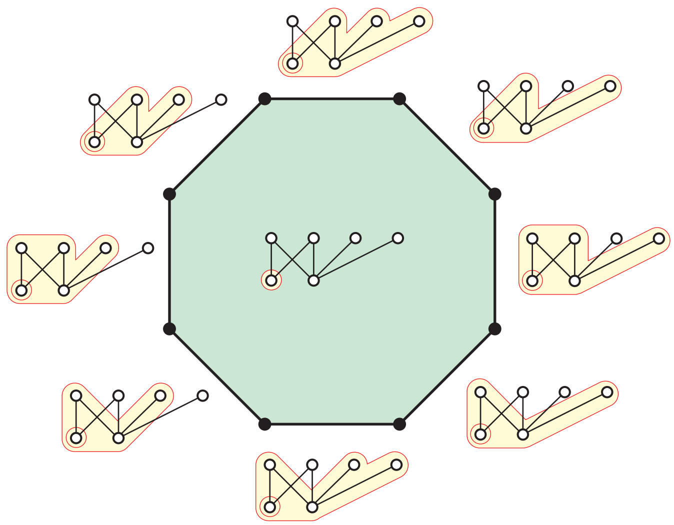


FIGURE 13. Octagonal face of the polyhedron in Figure 5(b).

All graph associahedra and nestohedra are obtained from two rank posets, and it is natural to ask whether all poset associahedra can be obtained from some two rank posets. For instance, the three rank poset of Figure 4(a) yields the same $\mathcal{K}P$ as the two rank poset of Figure 3(a). The following shows that higher ranks indeed hold deeper structure.

Theorem 10. *There exists poset associahedra $\mathcal{K}P$ which cannot be found as $\mathcal{K}P'$, for any poset P' of two rank.*

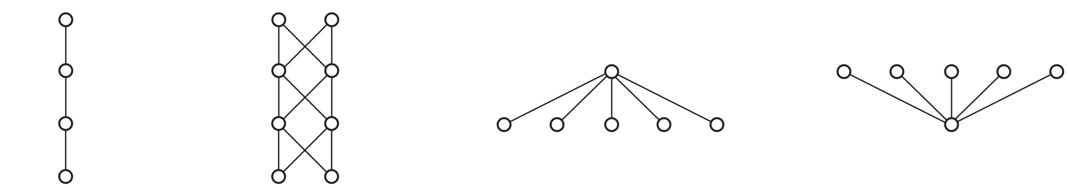
Proof. Consider the 4D example in Figure 8, a three rank poset with f-vector $(68, 136, 88, 20)$. Since this is not a nestohedron, Proposition 9 shows that at least one nontrivial bundle is needed when restricting to two ranks. Using computer calculations, we enumerated the f-vectors of all 4D poset associahedra for posets with two ranks and at least one nontrivial bundle. For each polytope with a matching f-vector (for around 500 posets), we verified using the SAGE package that it was not equivalent to the one in Figure 8. $\square$

We close with some special examples.

Proposition 11. *Let $\mathfrak{b}_x$ be a bundle with n elements of poset P such that all of its elements are maximal. If $\partial x = P - \mathfrak{b}_x$, then $\mathcal{K}P = \mathcal{K}(P - \mathfrak{b}_x) \times \mathcal{P}_n$.*

Proof. A tubing $U \in \mathcal{K}P$ containing no tubes that intersect $\mathfrak{b}_x$ can be viewed as a tubing in $\mathcal{K}(P - \mathfrak{b}_x)$; call it $\alpha(U)$. Let $V \in \mathcal{K}P$ be a tubing with only tubes that intersect $\mathfrak{b}_x$, where a tube in V is a lower set generated by a subset of $\mathfrak{b}_x$, and compatibility of these tubes is equivalent to the subsets being nested.

Let Γ_x be the complete graph with n nodes, labeled by elements of $\mathfrak{b}_x$. Let $\beta(U)$ be the tubing on Γ_x such that if $t \subset \mathfrak{b}_x$ generates a tube in U, t is a tube in $\beta(U)$. Any tubing $T \in \mathcal{K}P$ can be written as a tubing U and a tubing V, where the map $T \to (\alpha(U), \beta(V))$ preserves compatibility and is bijective. The proof follows since the graph associahedron of a complete graph of n nodes is the permutohedron $\mathcal{P}_n$. $\square$



(a) *point* (b) *cube* (c) *simplex* (d) *permutohedron*

FIGURE 14. Examples of simple posets.

Corollary 12. *Consider Figure 14. If (a) P is a chain, (b) a cross-stack of n rank, (c) a one rank element with $n+1$ boundary elements, or (d) a bundle with n maximal elements, then $\mathcal{K}P$ is a point, an n-cube, an n-simplex, or $\mathcal{P}_n$, respectively.*

Proof. The first follows from Corollary 7, and the rest from Proposition 11. $\square$

5. PROOFS

5.1. The proof of Theorem 5 is now given, which immediately results in Theorem 2. We proceed by induction on the size of the connected poset P: First, an explicit construction of the polytope $\mathcal{K}P$ is provided, as outlined in Theorem 5, based on the truncation of a smaller polytope (using the induction hypothesis). And second, a poset isomorphism is created between the newly constructed $\mathcal{K}P$ and tubings of P, establishing the result.

Throughout this proof, we let x be a maximum element in a maximal chain of P. We first consider the case when $\mathfrak{b}_x$ is trivial, and begin with a definition.

Definition. For trivial $\mathfrak{b}_x$, if there exists a pairwise disjoint tubing T of $P - x$ such that

$$\mathrm{fill}_x(T) \ := \ \{x\} \ \cup \ \{p \in t \mid t \in T\}$$

is a tube of P, call $\text{fill}_x(T)$ the *x-fill* of T; Figure 15 gives some examples.

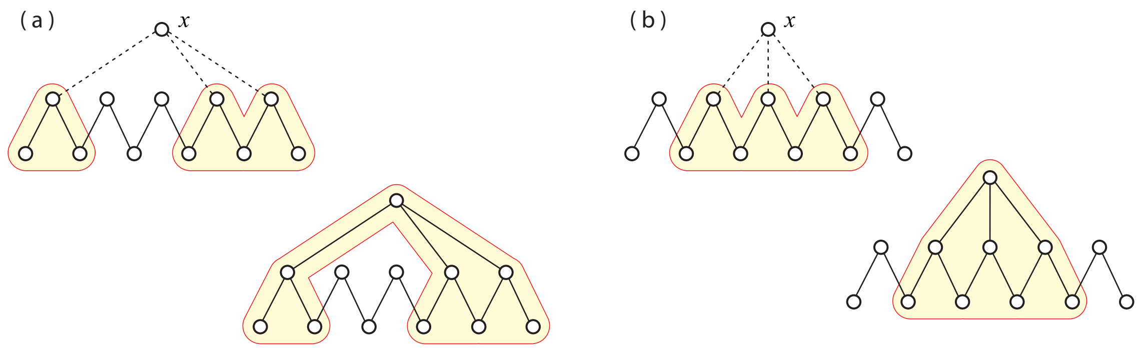


FIGURE 15. Some tubings of $P - x$ (top row) and their x-fills (bottom row).

Truncation algorithm: By induction, the theorem holds for $\mathcal{K}(P - x)$; we refer to this object both as a polytope and as the poset of tubings on $P - x$. Construct $\mathcal{K}P$ by truncating the faces T of $\mathcal{K}(P - x)$ such that $\text{fill}_x(T)$ are tubes of P. The resulting new facets inherit the labelings of $\text{fill}_x(T)$, while retaining the old labels for any facets not truncated; the label of each face is the set of labels of its adjacent facets. We perform this truncation iteratively, based on *decreasing* size of the tubes $\text{fill}_x(T)$; for a tie, the order of truncation is arbitrary.

Proposition 13. *For trivial $\mathfrak{b}_x$, there exists a poset isomorphism φ from $\mathcal{K}P$ to the simple polytope found by truncating $\mathcal{K}(P - x)$ as described above.*

Proof. It is straightforward that labeling tubes of the facets of our truncated polytope are in bijection with the tubes of P. We show that φ is a bijection of tubings by checking that a collection T of tubes $\{t\}$ of P is a tubing if and only if its corresponding facets $\{f(t)\}$ in $\mathcal{K}P$ intersect at a face. This simultaneously shows that φ preserves the ordering of tubings.

Forward direction: We show if T is not a tubing, then $\{f(t) \,|\, t \in T\}$ has empty intersection. If not a tubing, there is either a subset S of T that is not filled, or there is a pair of nonnested but intersecting tubes in T. First consider the former, when S is unfilled: If x is the only element whose absence causes S to be unfilled, our truncations will have effectively separated the facets by removing their intersection; otherwise, the conclusion follows by the inductive assumption on $\mathcal{K}(P - x)$.

Now consider when T contains intersecting, nonnested tubes, say t_1 and t_2 of P. The faces $f(t_1)$ and $f(t_2)$ have empty intersection when neither tubes contain x (due to inductive assumption on $\mathcal{K}(P - x)$) or when only one tube t_1 contains x (since $f(t_1)$ was formed by truncating a face which was not contained in $f(t_2)$). When both tubes contain x, then there must exist tube t_* of P containing their union. Now let $t_1 = \text{fill}_x(T_1)$ and $t_2 = \text{fill}_x(T_2)$ for disjoint tubings T_1 and T_2 of $P - x$. Note that we are only concerned if $T_1 \cup T_2$ is a tubing of $P - x$, implying the faces $f(t_1)$ and $f(t_2)$ originally intersected in $\mathcal{K}(P - x)$. Let $t_* = \text{fill}_x(T_*)$ for a tubing T_* of disjoint tubes, each containing some of ∂x. Since $t_1, t_2 \subset t_*$, the truncation of $T_* \subset T_1 \cup T_2$ will have occurred before either of the latter, and will have involved the truncation of the face labeled by $T_1 \cup T_2$, ensuring that future facets associated to the truncations of $f(t_1)$ and $f(t_2)$ will not intersect.

Backward direction: Using finite induction on the truncated tubes in P, we show if T is a tubing, then $\{f(t) \mid t \in T\}$ has nonempty intersection. Note that the tubes $\{t_i\}$ in T containing x must be nested $t_1 \supset \cdots \supset t_m$ since they intersect at least in x. Let S_i be the set of disjoint tubes of $P - x$ such that $t_i = \text{fill}_x(S_i)$. Thus, associated to a tubing T of P, there is a tubing T_0 of $P - x$, made of all the tubes of T that do not contain x, together with all the tubes in sets S_i.

Our argument considers a series of m truncations, proceeding from the list of tubes in T_0 to the list of tubes in T. By the construction of T_0, restrict attention to the truncations that form facets whose labels are in T. At each step, we show that truncation allows the tubes in the new intermediate list to label facets that intersect. The base case follows from the induction hypothesis (on the number of elements in our poset) that facets labeled by the tubes of T_0 do intersect at a unique face of the simple polytope $\mathcal{K}(P - x)$.

Recursively define an intermediate set of tubes called T_k, corresponding to having performed truncations to create the facets labeled by t_1 through t_k, where

$$T_k \;:=\; (T_{k-1} - S_k) \;\cup\; (S_k \;\cap\; (T \;\cup\; \bigcup_{i>k} S_i)) \;\cup\; (\{t_k\} \cap T).$$

Indeed, after all m truncations, we will have transformed T_0 to become T, by adding to the list of tubes in T_0 all the new tubes t_i in T and subtracting all the tubes which are not in T.

By induction, assume that after truncating to create facets $\{t_1, \ldots, t_{k-1}\}$, the facets labeled by the tubes in T_{k-1} do indeed intersect. We use this assumption to show that truncating to create the facet $t_k = \text{fill}_x(S_k)$ will preserve the property of intersection: To obtain T_k from T_{k-1}, add t_k but crucially also remove at least one tube; specifically, we claim that at least one tube in S_k will be removed.

Since truncation occurs in decreasing order of containment, the tubes $\{t_{k+1}, \ldots, t_m\}$ are all sequentially and properly contained inside of t_k. And since any tube of $T \cap S_k$ must be contained in the smallest of the tubes t_i, if there is one or more $t_i \subset t_k$, then the tubes in S_k cannot all be found again among the tubes in the sets $\{S_{k+1}, \ldots, S_m\}$, nor in $T \cap S_k$ itself. Finally, remove some of S_k when t_k is the smallest tube in T containing x, since T (being filled) cannot contain S_k.

Therefore, since $T_k \cap T_{k-1} = T_k - \{t_k\}$ does not contain S_k, truncating face f (whose containing facets are labeled by S_k) will not separate the facets labeled by the tubes of $T_k - \{t_k\}$. However, f does intersect the face where the facets labeled by $T_k - \{t_k\}$ intersect, so their intersection will further intersect the new facet labeled by t_k. Thus, the facets labeled by the tubes in T_k will have a nonempty intersection. $\square$

5.2. We now consider the second case in Theorem 2, when $\mathfrak{b}_x$ is nontrivial.

Truncation algorithm: Construct the new polytope $\mathcal{K}P$ by truncating certain faces of

$$\mathcal{K}P_* \;:=\; \mathcal{K}(P - (\mathfrak{b}_x - x)) \times \Delta_{|\mathfrak{b}_x - x|}.$$

Begin by labeling the vertices of $\Delta_{|\mathfrak{b}_x - x|}$ with the elements of $\mathfrak{b}_x$, and its faces by the subset of vertex labels which they contain. The faces of $\mathcal{K}P_*$ get labeled by the pairing (T, B), for the corresponding tubing T of $\mathcal{K}(P - (\mathfrak{b}_x - x))$ and subset B of $\mathfrak{b}_x$. Now truncate the labeled faces (T, B) with $T = \{t\}$ and either $x \in t$ or $B = \mathfrak{b}_x$. The first kind, where $x \in t$, gets relabeled by the tube $t - x + B$ and the second kind, where $B = \mathfrak{b}_x$, gets relabeled by just t. After truncations, let the label of each face be the set of labels of its adjacent facets. We perform this truncation iteratively, based on *increasing* size of tubes t.

Proposition 14. *For nontrivial $\mathfrak{b}_x$, there exists a poset isomorphism φ from $\mathcal{K}P$ to the simple polytope found by truncating $\mathcal{K}P_*$ as described above.*

Proof. It is straightforward that labeling tubes of the facets of our truncated polytope are in bijection with the tubes of P. We show that φ is a bijection of tubings by checking that a collection T of tubes t of P is a tubing if and only if its corresponding facets $f(t)$ in $\mathcal{K}P$ intersect at a face. This simultaneously shows that φ preserves the ordering of tubings.

Forward direction: We show if T is not a tubing, then $\{f(t) \,|\, t \in T\}$ has empty intersection. If not a tubing, there is either a subset S of T that is not filled, or there is a pair of nonnested but intersecting tubes in T. In the case of the former, when S is unfilled, we see implied a further subset S' that was unfilled in the poset $P - (\mathfrak{b}_x - x)$, where S' consists of the tubes of S with one modification: Replace any portion of $\mathfrak{b}_x$ in those tubes with x. Since, by induction, the facets labeled by tubes of S' have no common intersection in $\mathcal{K}(P - (\mathfrak{b}_x - x))$, and since the product of polytopes preserves this fact, then the faces of the product bearing labels from S' do not have a common intersection.

Now consider when T contains intersecting, nonnested tubes, say t_1 and t_2 of P. Replace any portion of $\mathfrak{b}_x$ contained in them with x, resulting in $t'_1 := t_1 - (\mathfrak{b}_x - x)$ and $t'_2 := t_2 - (\mathfrak{b}_x - x)$. If these tubes are still intersecting but nonnested, their facets in $\mathcal{K}(P - (\mathfrak{b}_x - x))$ had no intersection, and this property will be passed along to our new polytope. But if the tubes t'_1 and t'_2 are nested or equal, then both t_1 and t_2 contained some of $\mathfrak{b}_x$, and we must further consider the intersection

$$t_1 \;\cap\; t_2 \;\cap\; \mathfrak{b}_x\,. \tag{5.1}$$

If this is empty, then t_1 and t_2 are tubes created by truncating faces of the product polytope $\mathcal{K}P_*$, which in turn corresponded to faces of $\Delta_{|\mathfrak{b}_x - x|}$ which did not intersect. Again the non-intersection is inherited by $\mathcal{K}(P - (\mathfrak{b}_x - x))$.

Finally if (5.1) is nonempty, then it is straightforward to see that the facets labeled by t_1 and t_2 result from truncating faces that originally do intersect in $\mathcal{K}P_*$. Here, there is a third, prior truncation (of a face f) of the product polytope that contains the intersection of the faces that are truncated to become t_1 and t_2. Indeed, face f gives rise to the facet labeled by the tube $t_1 \cap t_2$, and thus was labeled in the product by the smaller of t'_1 and t'_2, paired with (5.1). Therefore, it is truncated first and effectively separates the others.

Backward direction: Using finite induction on the truncated tubes $\{t_1, \ldots, t_m\}$ in P, we show if T is a tubing, then $\{f(t) \mid t \in T\}$ has nonempty intersection. Our argument proceeds by constructing a series of m truncations, showing at each step that the tubes do indeed label facets that intersect. For a tubing T, create the set of pairs

$$T_0 \;=\; \{\, (t_*, B) \mid t_* \text{ is a tube of } P - (\mathfrak{b}_x - x),\; B \subset \mathfrak{b}_x \} \,,$$

where

$$t_* \;=\; \begin{cases} (t - (\mathfrak{b}_x - x),\; t \cap \mathfrak{b}_x) & \text{if } \partial x \subset t \\ (t, \mathfrak{b}_x) & \text{otherwise.} \end{cases}$$

This set gives a list of faces of $\mathcal{K}P_*$, whose intersection is nonempty, providing the base case for truncation: Since T is a tubing, the tubes of T which contain ∂x are all nested, and by our construction of T_0, the set of second elements in the pairs are subsets of $\mathfrak{b}_x$ having a common intersection.

Recursively define an intermediate set of labels T_k, formed by performing truncations to create facets labeled by $t_1, \ldots, t_k$. If t_k is not in T, let T_k be T_{k-1}. Otherwise, T_k is formed by discarding from T_{k-1} the pair $(t_k - (\mathfrak{b}_x - x),\, t_k \cap \mathfrak{b}_x)$ and replacing it with t_k itself. This corresponds to labeling the new facet with the new tube t_k. We need to show that faces labeled by elements of T_k still have a nonempty intersection after the truncation of face t_k. If t_k does not contain ∂x, then it labels a facet and its truncation does not change the polytope. Otherwise, t_k is either (1) contained in or (2) intersects (but is not nested with) some tube $\{t_{k+1}, \ldots, t_m\}$.

In the latter case (2), where t_k intersects such a tube, then $t_k \notin T$, and we argue that the truncated face does not contain the intersection of the facets labeled by T_k. This follows because t_k either inherits an empty intersection with the faces represented by T_0 (from one of the two polytopes in the cross product), or is separated from the intersection of the faces represented by T_{k-1} by an earlier truncation (from the proof in the forward direction).

Now consider the former case (1): If $(t_k \cap \mathfrak{b}_x) = (t_{k+1} \cap \mathfrak{b}_x)$, let F_* be the facet labeled by the pair $(t_k, \mathfrak{b}_x)$. Here, before any truncation, F_* originally contains faces labeled by (t_k, B), for $B \subset \mathfrak{b}_x$. On the other hand, if $(t_k \cap \mathfrak{b}_x) \subset (t_{k+1} \cap \mathfrak{b}_x)$, let F_* be the facet labeled by the pair $(P - (\mathfrak{b}_x - x), \mathfrak{b}_x - z)$, for some $z \in (t_{k+1} \cap \mathfrak{b}_x) - (t_k \cap \mathfrak{b}_x)$. In this case, this facet contains faces labeled by (t, B), where $t \subseteq P - (\mathfrak{b}_x - x)$ and $B \subseteq \mathfrak{b}_x - z$.

In either case, F_* is chosen to contain the truncated face at step k and no other face scheduled to be truncated afterwards. Moreover, F_* is chosen such that it will eventually be labeled by a tube t that intersects but is not nested with $\{t_{k+1}, \ldots, t_m\}$. Therefore, since we are dealing with simple polytopes, F_* also cannot contain any of the facets represented by the elements of T_k, both those labeled by tubes not containing ∂x, and those created by earlier truncations.

Finally, note that the truncated face t_k was assumed to intersect the common intersection of all the other faces represented by T_{k-1}, and so the facet created still does. Therefore, the faces represented by T_k have a common intersection, and thus the tubing T will be represented by facets with a common intersection in $\mathcal{K}P$. □

References

1. J. Bloom. A link surgery spectral sequence in monopole Floer homology, *Advances in Mathematics* **226** (2011) 3216–3281.
2. V. Buchstaber and V. Volodin. Combinatorial 2-truncated cubes and applications, in *Associahedra, Tamari Lattices and Related Structures*, Progess in Mathematics **299** (2013) 161–186.
3. M. Carr and S. Devadoss. Coxeter complexes and graph associahedra, *Topology and its Applications* **153** (2006) 2155–2168.
4. M. Carr, S. Devadoss, S. Forcey. Pseudograph associahedra, *Journal of Combinatorial Theory, Series A* **118** (2011) 2035–2055.
5. M. Davis, T. Januszkiewicz, R. Scott. Fundamental groups of blow-ups, *Advances in Mathematics* **177** (2003) 115–179.
6. C. De Concini and C. Procesi. Wonderful models of subspace arrangements, *Selecta Mathematica* **1** (1995) 459–494.
7. S. Devadoss. Tessellations of moduli spaces and the mosaic operad, in *Homotopy Invariant Algebraic Structures*, Contemporary Mathematics **239** (1999) 91–114.
8. S. Devadoss. A realization of graph associahedra, *Discrete Mathematics* **309** (2009) 271–276.
9. K. Došen and Z. Petrić. Hypergraph polytopes, *Topology and its Applications* **158** (2011) 1405–1444.
10. E. Feichtner and B. Sturmfels. Matroid polytopes, nested sets and Bergman fans, *Portugaliae Mathematica* **62** (2005) 437–468.
11. S. Forcey and D. Springfield. Geometric combinatorial algebras: cyclohedron and simplex, *Journal of Algebraic Combinatorics* **32** (2010) 597–627.
12. J. Morton, L. Pachter, A. Shiu, B. Sturmfels, O. Wienand. Convex rank tests and semigraphoids, *SIAM Journal on Discrete Mathematics* **23** (2009) 1117–1134.
13. A. Postnikov. Permutohedra, associahedra, and beyond, *International Mathematics Research Notices* **6** (2009) 1026–1106.

14. S. Saneblidze and R. Umble. Diagonals on the permutahedra, multiplihedra and associahedra, *Journal Homology, Homotopy and Applications* **6** (2004) 363–411.

15. J. Stasheff. Homotopy associativity of H-spaces, *Transactions of the American Mathematical Society* **108** (1963) 275–292.

16. A. Zelevinsky. Nested complexes and their polyhedral realizations, *Pure and Applied Mathematics Quarterly* **2** (2006) 655–671.

S. Devadoss: Williams College, Williamstown, MA 01267
E-mail address: satyan.devadoss@williams.edu

S. Forcey: University of Akron, OH 44325
E-mail address: sf34@uakron.edu

S. Reisdorf: University of Akron, OH 44325
E-mail address: stephenreisdorf@gmail.com

P. Showers: University of Akron, OH 44325
E-mail address: pjs36@zips.uakron.edu