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CONVEX POLYTOPES AND GRAFTED TREES I. STELLOHEDRA AND PTERAHEDRA.

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ABSTRACT. We answer several open questions in the affirmative. Specifically: we show that certain sets of combinatorial trees are naturally the minimal elements of face posets of convex polytopes. The polytopes that constitute our main results are well known in other contexts. They are generalized permutahedra, including graph-associahedra of the fan graphs (we call them pterahedra) and of the star graphs (stellohedra). As an aside we show that the stellahedra also appear as lifted generalized permutahedra and as cubeahedra, two guises in which they have remained until now unrecognized. Tree species considered here include ordered and unordered binary trees and ordered lists (labeled corollas). Compositions of these trees were recently introduced as bases for one-sided Hopf algebras. Thus our results show a path to extending those algebras to differential graded Hopf algebras based on the polytope faces.

1. INTRODUCTION

The mathematical operation of grafting trees, root to leaf, is the key feature for the structure of more than a few important operads and Hopf algebras. In 1998 Loday and Ronco found a Hopf algebra of plane binary trees, initiating the study of these type of structures [14]. The Loday-Ronco algebra is the image of the Malvenuto-Reutenauer Hopf algebra of permutations [15] and projects to Solomon descent algebra of Boolean subsets. The authors of [14] showed Hopf algebra maps which factor the descent map from permutations to Boolean subsets. The first factor is the Tonks projection from the vertices of the permutohedron to the vertices of the associahedron. Chapoton put this latter fact into context when he found that the Hopf algebras of vertices are subalgebras of larger ones based on the faces of the respective polytopes [6].

In 2005 and 2006 Aguiar and Sottile characterized operations in these algebras using Möbius functions [2],[1]. In ForSpr:2010 we characterize the same operations in terms of inclusions of (combinatorially equivalent) polytope faces.

In [12] we carefully defined the idea of grafting with two colors, preserving the colors after the graft in order to have two-tone, or painted, trees with various structures possible in each colored region. Here we review the definitions, adding some generality and defining poset structures on each set of painted trees. We are able to conclude that most of the coalgebras defined in [12] have underlying geometries of polytope sequences.

1.1. **Main Results.** We show that four sequences of our sets of painted trees, with their relations, are isomorphic as posets to face lattices of convex polytopes. In Theorem 3.9

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we show that forests of plane rooted trees grafted to weakly ordered trees are isomorphic to the fan-graph-associahedra, or pterahedra. In Theorem 3.10 we show that forests of corollas grafted to weakly ordered trees are isomorphic to the star-graph-associahedra, or stellohedra. In Theorem 3.11 we show that weakly ordered forests grafted to a corolla are also isomorphic to stellohedra. In Theorem 3.2 we show that weakly ordered forests grafted to weakly ordered trees are isomorphic to the permutohedra.

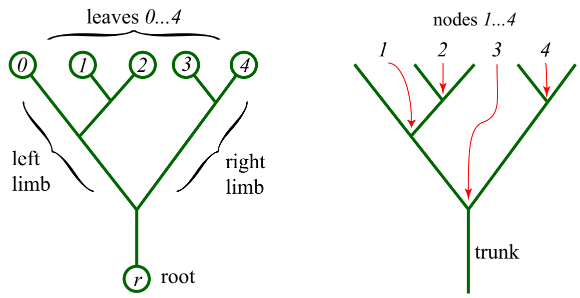
In addition, we show that the stellahedra also appear as graph-composihedra and cubeahedra.

2. Definitions

Trees are connected, acyclic graphs. We begin our discussion with rooted, plane, binary trees.

- Rooted: There is a vertex of degree one which is designated the root. The other vertices of degree one are called leaves.
- Plane: Each tree is drawn in the plane with no edges crossing. Two plane trees are only equal if they can be made into identical pictures by scaling portions of the plane, without any reflection or rotation in any dimension.
- Binary: The vertices that are not degree one are all of degree three. These vertices are called nodes.

Here is a *plane rooted binary tree*, often called a binary tree when the context is clear:



The leaves are ordered left to right as shown by the circled labels. The node ordering corresponds to the order of “gaps between leaves:” the n^{th} node is the one where a raindrop would be caught which fell just to the left of the n^{th} leaf. The branches are the edges with a leaf. Non-leafed edges are referred to as internal edges. The nodes are also partially ordered vertically; we say two nodes are comparable in this partial order if they both lie on a path to the root. The root is maximal in this partial order. In the preceding picture, we have for instance that node 3 is greater than node 1 which is greater than node 2. The set of plane rooted binary trees with n nodes and $n+1$ leaves is denoted $\mathcal{Y}_n$. The cardinality of these sets are the Catalan numbers:

$$|\mathcal{Y}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

We will also need to consider rooted plane trees with nodes of larger degree than three. An $(n+1)$-leaved rooted tree with only one node (it will have degree $n+2 \geq 3$), or, for $n = 0$, a single leaf tree with zero nodes, is called a *corolla*, denoted $\mathfrak{C}_n$. This notation for the (set of one) corolla with $n+1$ leaves is the same as used for the set of one *left comb* in [12]. In the current paper we have decided that the corollas are more easily recognized than the combs.

2.1. **Ordered and Painted binary trees.** Many variations of the idea of the binary tree have proven useful in applications to algebra and topology. First, an *ordered tree* (sometimes called *leveled*) is a binary tree that has a vertical linear ordering of the n nodes as well as horizontal. This vertical linear ordering extends the partial vertical ordering given by distances to the root. See Figure 1 for ways to draw ordered trees. This ordering allows a well-known bijection between the ordered trees with n nodes, denoted $\mathfrak{S}_n$, and the permutations on $[n]$.

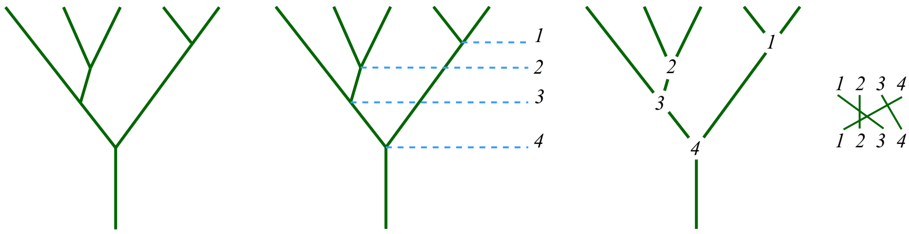
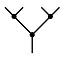
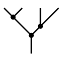
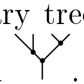


FIGURE 1. Drawing an ordered tree, in three different styles. The corresponding permutation σ is (3,2,4,1), in the notation $(\sigma(1), \sigma(2), \sigma(3), \sigma(4).)$

Thus  is a binary tree, but not an ordered tree. Instead we have two distinct ordered trees  and .

We will also consider *forests* of trees. In this paper, all forests will be a linearly ordered list of trees, drawn left to right. This linear ordering can also be seen as an ordering of all the nodes of the forest, left to right. On top of that, we can also order all the nodes of the forest vertically, giving a *vertically ordered forest*, which we often shorten to *ordered forest*. This initially gives us four sorts of forests to consider, shown in Figure 2.

Also shown in Figure 2 are three canonical, forgetful maps between the types of forests.

Definition 2.1. *We define γ to be the function that takes an ordered forest f and gives a forest of ordered trees. The output $\gamma(f)$ will have the same list of trees as f, and for a tree t in $\gamma(f)$ the vertical order of the nodes of t will respect the vertical order of the nodes in f. That is, for two nodes a, b of t we have $a \leq b$ in t iff $a \leq b$ in f.*

We define τ to be the function that takes an ordered tree and outputs the tree itself, forgetting all of the vertical ordering of nodes (except for the partial ordering based on distance from the root.) We define κ to be the function that takes a tree and gives the corolla with the same number of leaves.

Note that τ and κ are immediately both functions on forests, simply by applying them to each tree in turn. Also note that τ and κ are described in [12], but that there κ yields a left comb rather than a corolla.

Now we define larger sets of trees that generalize the binary ones. First we drop the word binary; we will consider plane rooted trees with nodes that have any degree larger than two. Then, from the non-binary vertically ordered trees we further generalize by allowing more than one node to reside at a given level. Instead of corresponding to a permutation, or total ordering, these trees will correspond to an ordered partition, or weak ordering, of their nodes.

Definition 2.2. *A* weakly ordered tree *is a plane rooted tree with a weak ordering of its nodes that respects the partial order of proximity to the root.*

Recall that this means all sets of nodes are comparable–but some are considered as tied when compared, forming a block in an ordered partition of the nodes. The linear ordering of the blocks of the partition respects the partial order of nodes given by paths to the root.

For a weakly ordered tree with $n + 1$ leaves the ordered partition of the nodes determines an ordered partition of $S = \{1, \dots, n\}$, where S is the set of "gaps between leaves," as described in [18]. (Recall that a gap between two adjacent leaves corresponds to the node where a raindrop would eventually come to rest; S is partitioned into the subsets of gaps that all correspond to nodes at a given level.) Weakly ordered trees are drawn using nodes with degree greater than two, and using numbers and dotted lines to show levels as in Figure 3. Note that an ordered tree is a (special) weakly ordered tree.

As well as forests of weakly ordered trees we also consider weakly ordered forests. This gives us three more sorts of forests to consider, shown in Figure 4. As indicated in that figure, the maps γ, τ and κ are easily extended to forests of the non-binary and/or weakly ordered trees: γ forgets the weak ordering of the forest to create a forest of weakly ordered trees, τ forgets the weak ordering, and κ forgets the partial order to create corollas.

The trees we focus on in this paper are constructed by grafting together combinations of ordered trees, binary trees, and corollas. Visually, this is accomplished by attaching

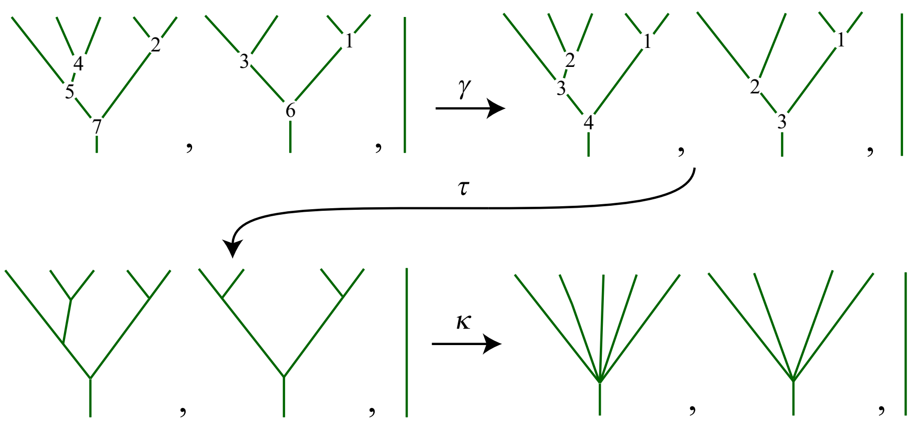


FIGURE 2. Following the arrows: a (vertically) ordered forest, a forest of ordered trees, a forest of binary trees, and a forest of corollas.

the roots of one of the above forests to the leaves of one of the above types of trees. We use two colors, which we refer to as "painted" and "unpainted." The forest is described as unpainted, and the base tree (which the forest is grafted to) is painted. At a graft the leaf is identified with the root, and in the diagram that point is drawn as a change in color (and thickness, for easy recognition) of the resulting edge. (Note that in earlier papers such as [10] our mid-edge change in color is described instead as a new node of degree two.)

See Figure 5 for an illustration of this composition. We refer to the result as a (partly) *painted tree*, regardless of the types of upper (unpainted) and lower (painted) portions. Notice that in a painted tree the original trees (before the graft) are still easily observed since the coloring creates a boundary, called the *paint-line* halfway up the edges where the graft was performed. Thus the paint line separates the painted tree into a single tree of one color and a forest of trees of another color. In Figure 6 we show all 12 ways to graft one of our types of partially ordered forest with one of our types of tree.

Definition 2.3. *The maps γ, τ and κ are now extended to the painted trees, just by applying them to the unpainted forest and/or to the painted tree beneath. We indicate*

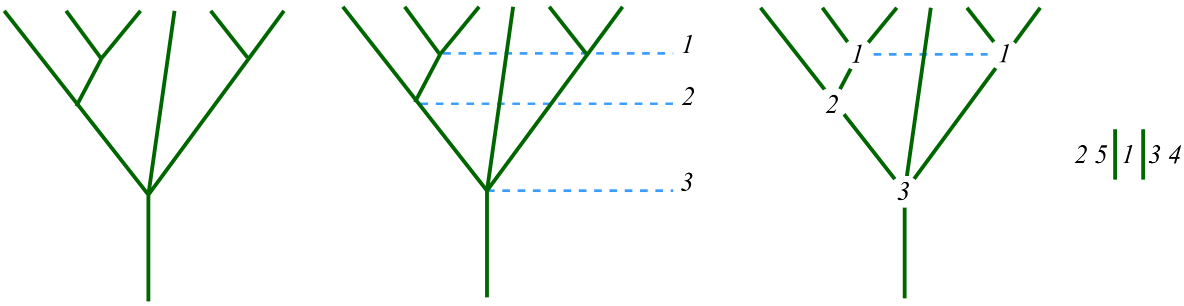


FIGURE 3. Drawing a weakly ordered tree, in three different styles. The corresponding ordered partition is $(\{2,5\},\{1\},\{3,4\})$.

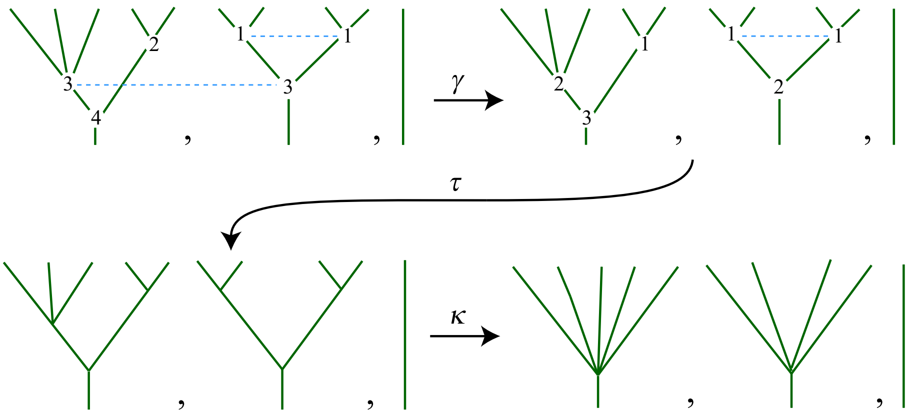


FIGURE 4. Following the arrows: a (vertically) weakly ordered forest, a forest of weakly ordered trees, a forest of plane rooted trees, and a forest of corollas. Note that the forests in Figure 2 are special cases of these.

this by writing a fraction: $\frac{f}{g}$ *for two of our three maps, or the identity map, as seen in Figure 6. That is,* $\frac{f}{g}$ *indicates applying* f *to the forest and* g *to the painted base tree, for* $f, g \in \{\gamma, \tau, \kappa, 1\}$.

2.2. **General painted trees.** Now our definition of painted trees is expanded to include any of our types of forest grafted to any of our types of tree. On top of that we will also permit a further broadening of the allowed structure of our painted trees. The paint-line, where the graft occurs, is allowed to coincide with nodes, where branching occurs. We call it a *half-painted node*. In terms of the grafting of a forest onto a tree our description depends on the type of forest. If the forest is weakly ordered, or is a forest of weakly ordered trees, then we see each half-painted node as grafting on a single tree at its least node, after removing its trunk and root. If the forest is only partially ordered (i.e. of binary trees or corollas) then we see the half-painted nodes as (possibly) several roots of several trees simultaneously grafted to a given leaf.

For these general painted trees we can again extend the "fractional" maps using γ, τ and κ. We reiterate from above how the half-painted nodes are interpreted, since that determines the input for the "numerator" map. Specifically $\frac{\gamma}{g}$ operates by taking as input for γ the weakly ordered forest of trees, one tree for each half-painted node. That is, $\frac{\gamma}{1}$ treats the half-painted nodes as being the location of a single tree that is grafted on without a trunk. This description is the same for $\frac{\tau}{g}$. In contrast however, the map $\frac{\kappa}{g}$ takes as input the forest found by listing all the unpainted trees while assuming each has a visible trunk, some of which are simultaneously grafted at the same half-painted node. Examples of these maps are shown in Figure 7, where we show 12 general painted trees that consist of one of the four general types of forest and one of the three general types of trees.

2.3. **The face poset relations.**

2.3.1. *Partial ordering of nodes and gaps.* Each of our 12 types of painted tree comes with a canonical vertical partial ordering of its nodes (branch points), produced by concatenating the orders that exist before the graft. Each new partial ordering is a refinement of the partial ordering given by distances to the root of the newly minted painted tree, and also preserves the relations that existed before the graft. We add the rules that 1) all half-painted nodes must be forced to remain at the same level, that is, incomparable to each other (or tied in a weak order); and 2) that nodes below the paint line will never surpass half painted nodes, and neither of the former will surpass

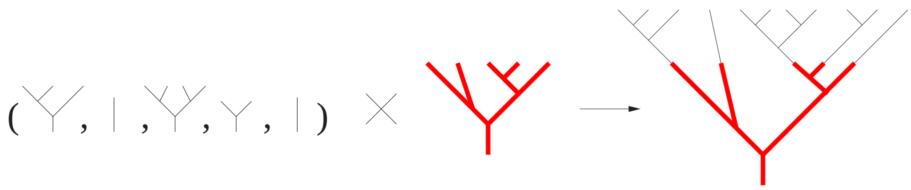


FIGURE 5. Construction of a painted tree, using a forest of binary trees and an ordered tree.

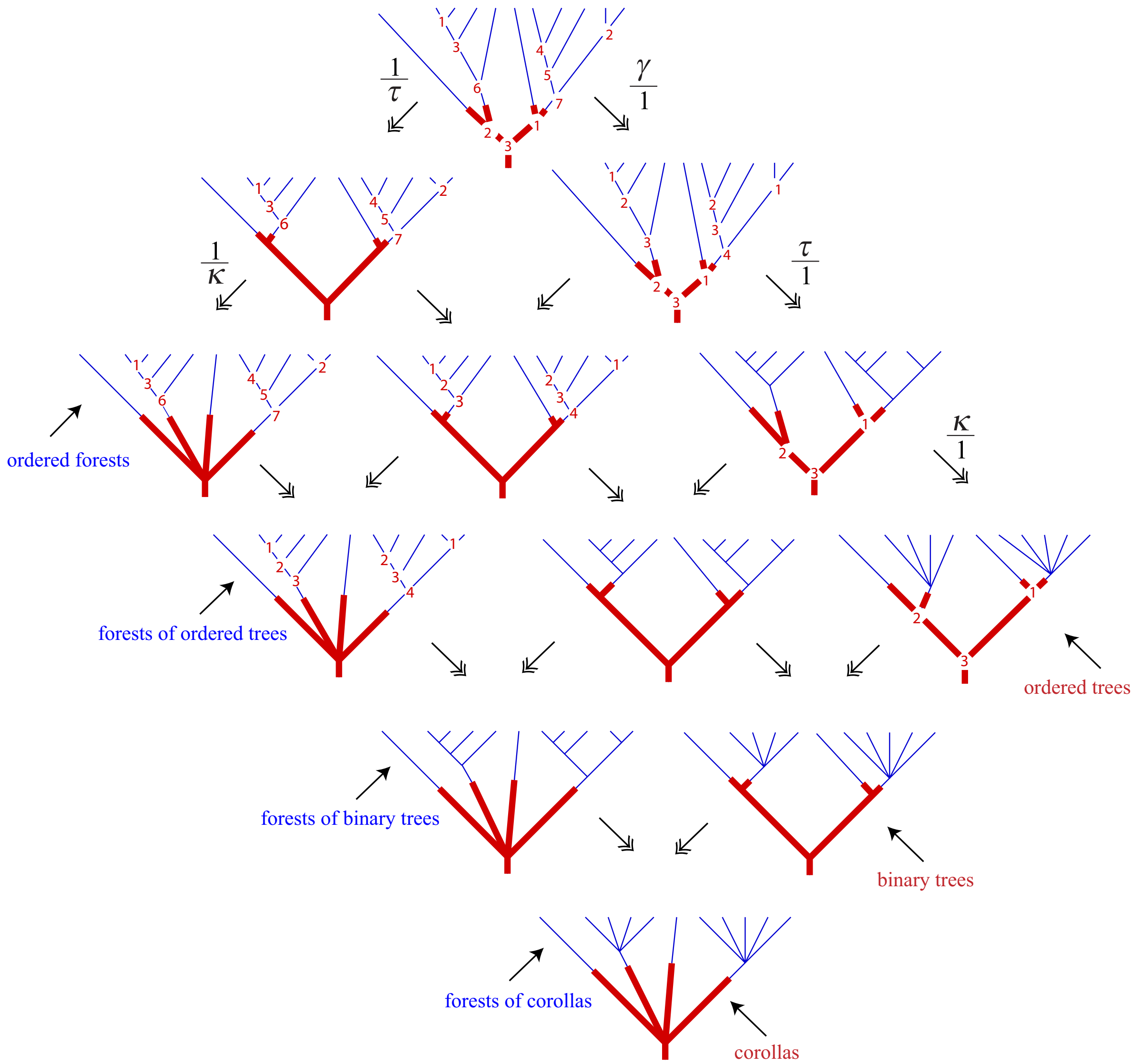


FIGURE 6. Varieties of grafted, painted trees. Each diagonal shares a type of tree on the bottom (painted) or a type of forest grafted on, as indicated by the labels. Since these trees are all binary, they correspond to vertex labels of the polytope sequences whose 3d versions are shown in Figure 9. The forgetful maps are shown with example input and output. Parallel arrows all denote the same map, except of course that the identity is context dependent.

unpainted nodes in the partial order. Furthermore, this ordering of nodes implies an ordering of the gaps between leaves of the tree. Some gaps share a node. Two gaps that share a node are considered to be incomparable in the partial order (or tied in a weak order).

Now we can define 12 separate posets whose elements are trees: one poset on each of our 12 types of painted trees shown in Figure 7. Note that the simplest painted tree

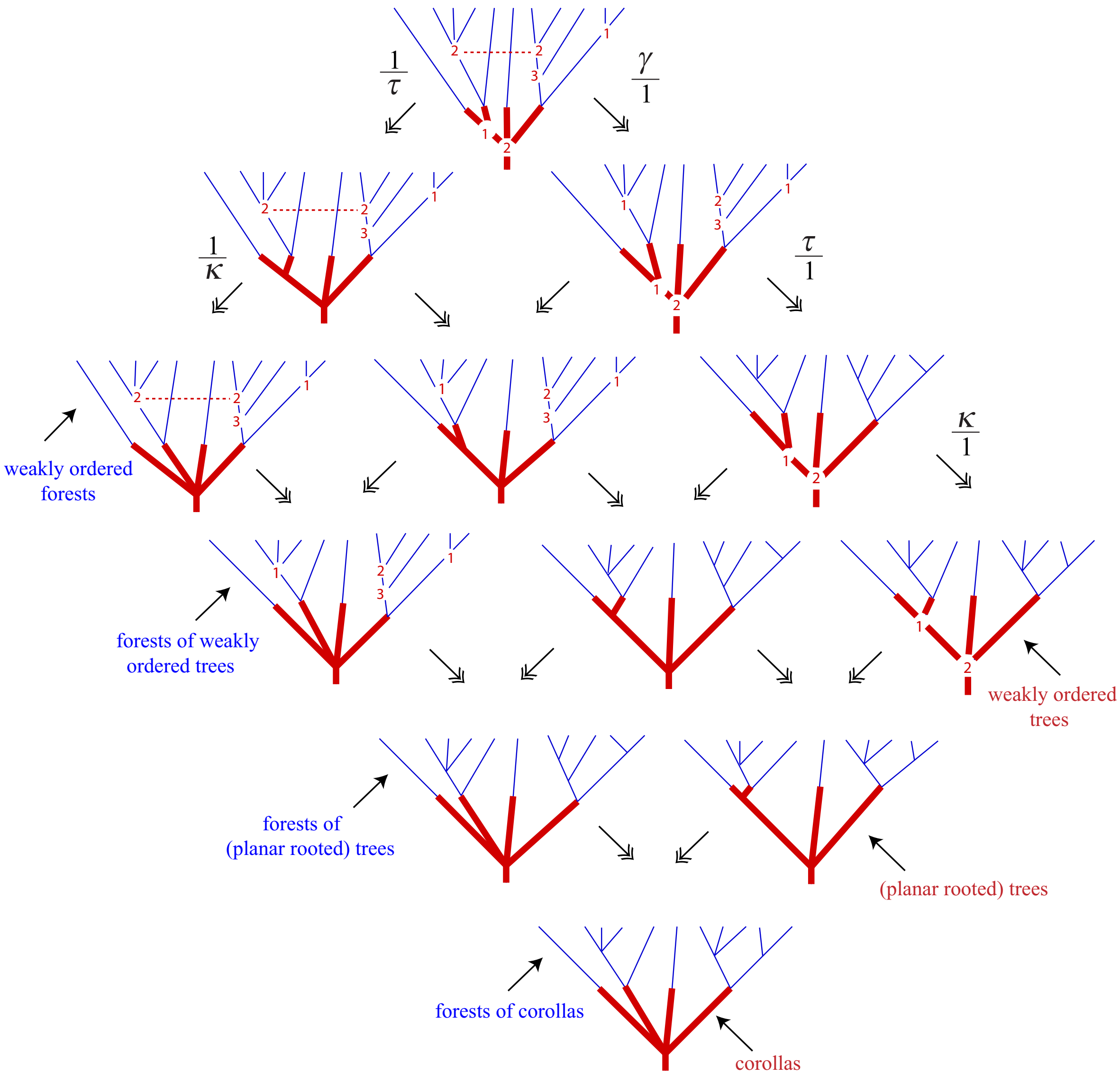


FIGURE 7. More varieties of grafted, painted trees. These correspond to face labels of the polytope sequences whose 3d versions are shown in Figure 9. Parallel arrows all denote the same map. Note that the trees in Figure 6 are special cases–vertex trees, or minimal in the face lattice–of the types illustrated in this figure.

with n leaves has one half-painted node: n single leafed unpainted trees all grafted to a painted trunk, the node coinciding with the paint line. This *half-painted corolla* can be interpreted as one of any of the 12 painted tree varieties, and it will be the unique maximal element in all 12 posets.

Definition 2.4. Given two painted trees s and t that are of the same painted type (i.e. they share the same types of tree and forest, below and above the paint line) we define the *painted growth preorder*, where :

$$s \prec t$$

if $s = t$ or if s is formed from t using a series of pairs $(a, b)_i$ of the following two moves, each pair performed in the following order:

a) "growing" internal edges of t : introducing new internal edges or increasing the length of some internal edges (either painted or unpainted). This is precisely described as a possible refinement of the vertical partial order of gaps between leaves, by adding relations to the partial order between previously incomparable elements. Relations may not be deleted by the growth (nor ties formed in a weak order), but if the growing of painted edges occurs at a collection of half-painted nodes in t then the partial order may be preserved rather than strictly refined. Note that internal edges can grow where there was no internal edge before, such as at a half-painted node or any node that had degree larger than three. Note also that the rules for painted trees must be obeyed by the growing process– for instance an unpainted edge cannot grow from a completely painted node (and vice versa), and if some painted edges are grown from a half-painted node then all the edges possible must be formed, to allow the paint line to be drawn horizontally.

b) ...followed by "throwing away," or forgetting, any superfluous structure introduced by the edge growing. This is described precisely by taking the tree that results from growing edges, and applying to it the forgetful map (from the set of $\gamma, \tau, \kappa, 1$ and their fractions and compositions) that is needed to ensure that the result is in the original type of the painted tree t. E.g. if the original type had weakly ordered forests grafted to weakly ordered trees, we only apply the identity. However if t originally was a forest of weakly ordered trees grafted to a weakly ordered tree we should apply $\frac{\gamma}{1}$.

For examples of (non-covering) relations in the 12 posets see the trees in respective locations of figure 6 and Figure 7: the latter are all greater than the former in the same positions. Several more covering relations for some of our 12 classes of general painted trees are shown in Figure 8.

3. Bijections

The painted growth relation is reflexive and transitive by construction, for all 12 types. We conjecture also that in all 12 of cases the the painted growth preorder is in fact a poset, and moreover we conjecture that all the posets thus defined are realized as the face posets of sequences of convex polytopes. Four of the cases have been proven in previous work. These four appear as the highlighted diamond in Figure 7. The polytope sequences are the cubes, associahedra, composihedra and multiplihedra.

In this section four more sequences of our sets of painted trees, with their relations, will be shown to be isomorphic as posets to face lattices of convex polytopes. These four are the species whose structure types are: a forest of plane rooted trees grafted to a weakly ordered tree (pterahedra), a forest of corollas grafted to a weakly ordered tree (stellohedra), a weakly ordered forest grafted to a corolla (stellohedra again), and a weakly ordered forest grafted to a weakly ordered tree (permutohedra). There remain four cases in Figure 7 that we leave as a conjecture. Some of our proofs and corollaries will use the concept of tubings, which we review next.

3.1. **Tubes, tubings and marked tubings.**

Definition 3.1. Let G be a finite connected simple graph. A *tube* is a set of nodes of G whose induced graph is a connected subgraph of G. We will often refer to the induced graph itself as the tube. Two tubes u and v may interact on the graph as follows:

(1) Tubes are *nested* if $u \subset v$.

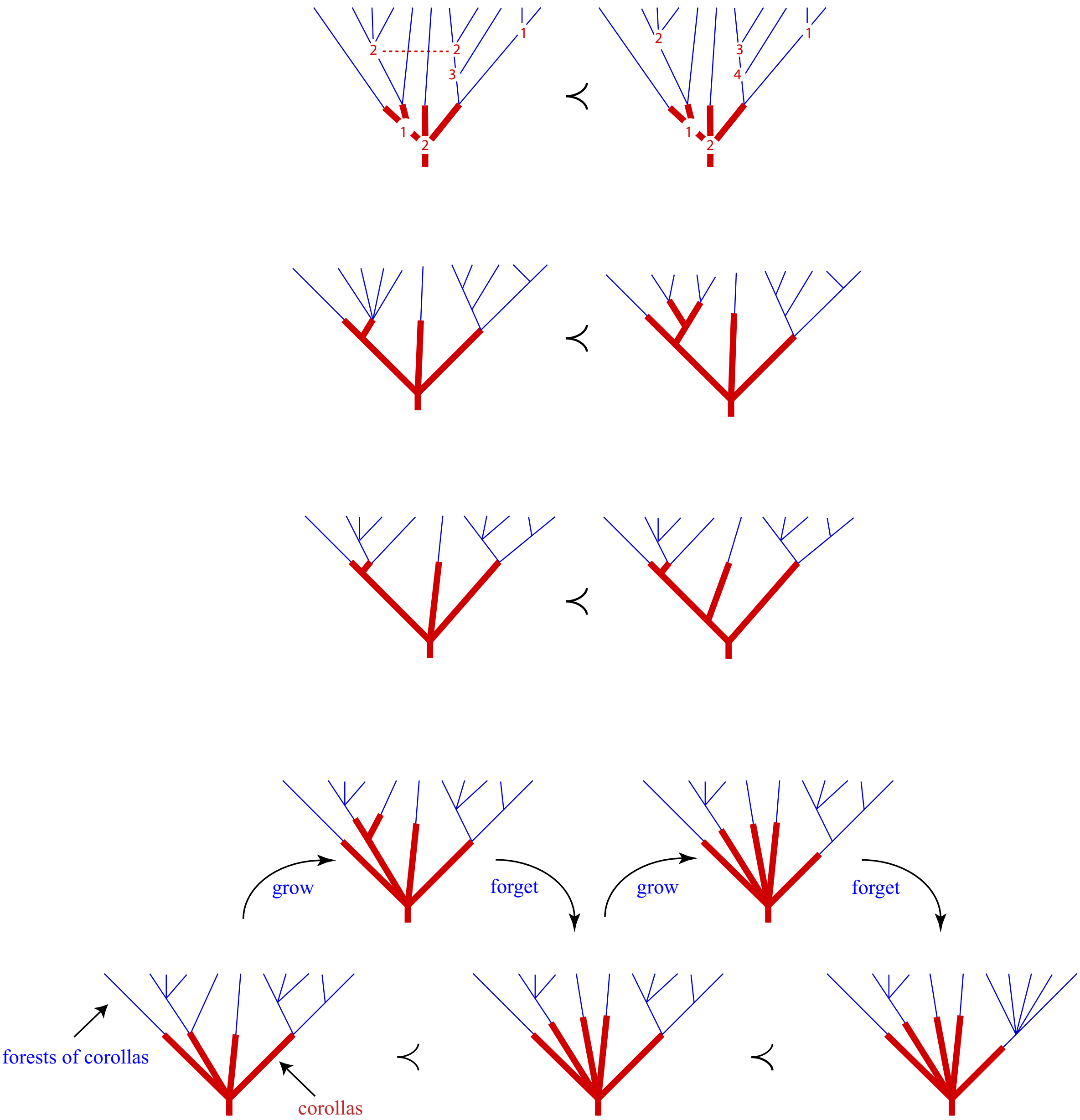


FIGURE 8. Some covering relations. In the first (top) we are looking at weakly ordered forests grafted to a weakly ordered trees, so growing an edge is a covering relation. In the next two relations we are looking at rooted trees above and below the graft–again no forgetting is needed. At the bottom for both covering relations the forgetful map is κ.

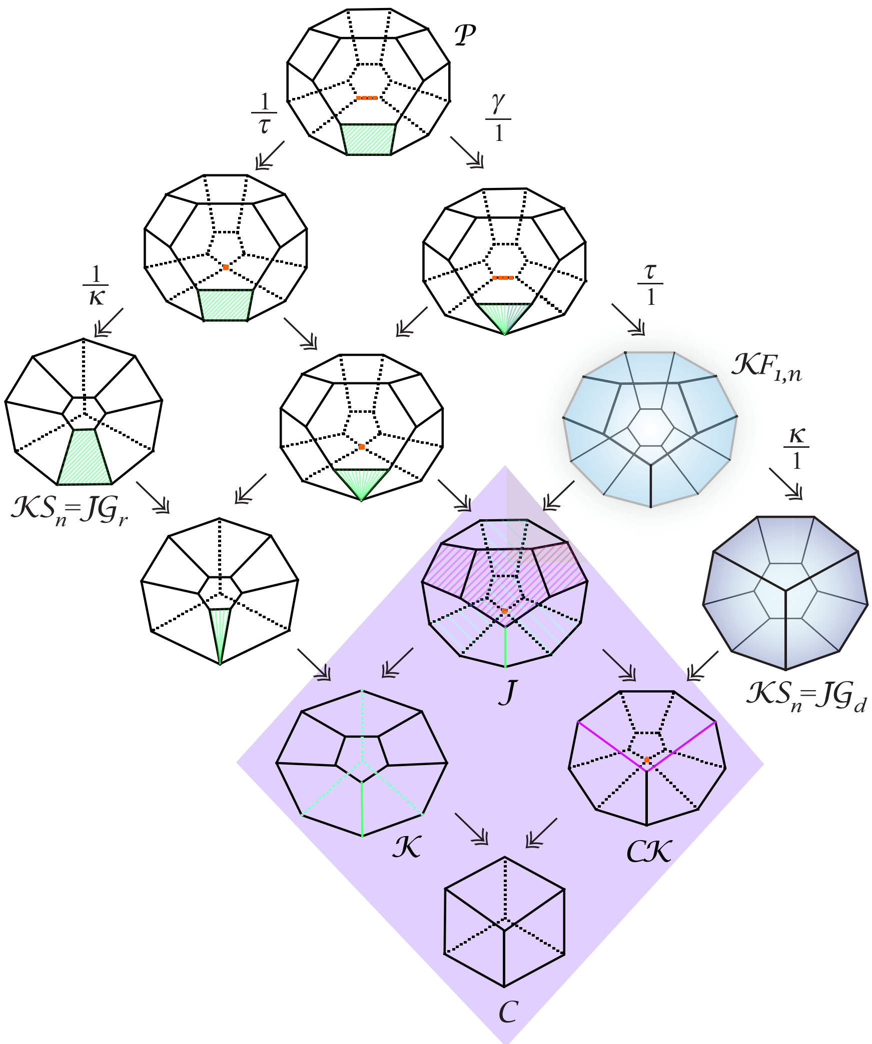


FIGURE 9. The 3d terms of some new and old polytope sequences. The four in the shaded diamond are the cube, associahedron $\mathcal{K}$, multiplihedron $\mathcal{J}$ and composihedron $\mathcal{CK}$. The other two shaded are the pterahedron $\mathcal{P}_t = \mathcal{K}F_{1,n}$ (fan graph associahedron) and the stellahedron $\mathcal{K}S$. The topmost is the permutohedron and the others are conjectured to be polytopes (clearly they are in three dimensions–the conjecture is about all dimensions.) Each of these corresponds the type of tree shown in Figure 6, in the corresponding position.

(2) Tubes are *far apart* if $u \cup v$ is not a tube in G, that is, the induced subgraph of the union is not connected, (equivalently none of the nodes of u are adjacent to a node of v).

Tubes are *compatible* if they are either nested or far apart. We call G itself the *universal tube.* A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible; moreover, we force every tubing of G to contain (by default) its universal tube. By the term *k-tubing* we refer to a tubing made up of k tubes, for $k \in \{1, \ldots, n\}$.

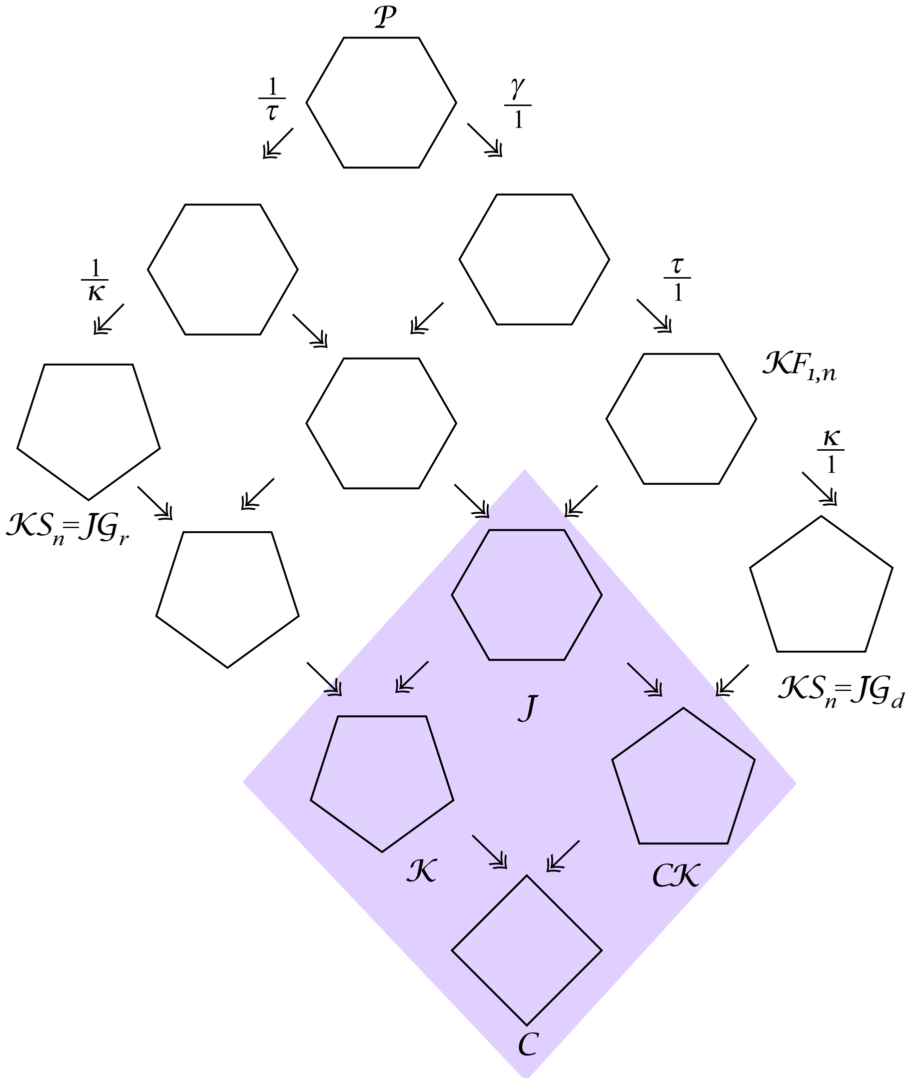


FIGURE 10. These are the 2d terms in the same sequences as in Figure 9.

When G is a disconnected graph with connected components $G_1, \ldots, G_k$, an additional condition is needed: If u_i is the tube of G whose induced graph is G_i, then any tubing of G cannot contain all of the tubes $\{u_1, \ldots, u_k\}$. However, the universal tube is still included despite being disconnected. Parts (a)-(c) of Figure 12 from [7] show examples of allowable tubings, whereas (d)-(f) depict the forbidden ones.

Theorem 3.2. [5, Section 3] *For a graph G with n nodes, the* graph associahedron $\mathcal{K}G$ *is a simple, convex polytope of dimension $n-1$ whose face poset is isomorphic to the set of tubings of G, ordered by the relationship $U \prec U'$ if U is obtained from U' by adding tubes.*

The vertices of the graph associahedron are the n-tubings of G. Faces of dimension k are indexed by $(n-k)$-tubings of G. In fact, the barycentric subdivision of $\mathcal{K}G$ is precisely the geometric realization of the described poset of tubings. Many of the face

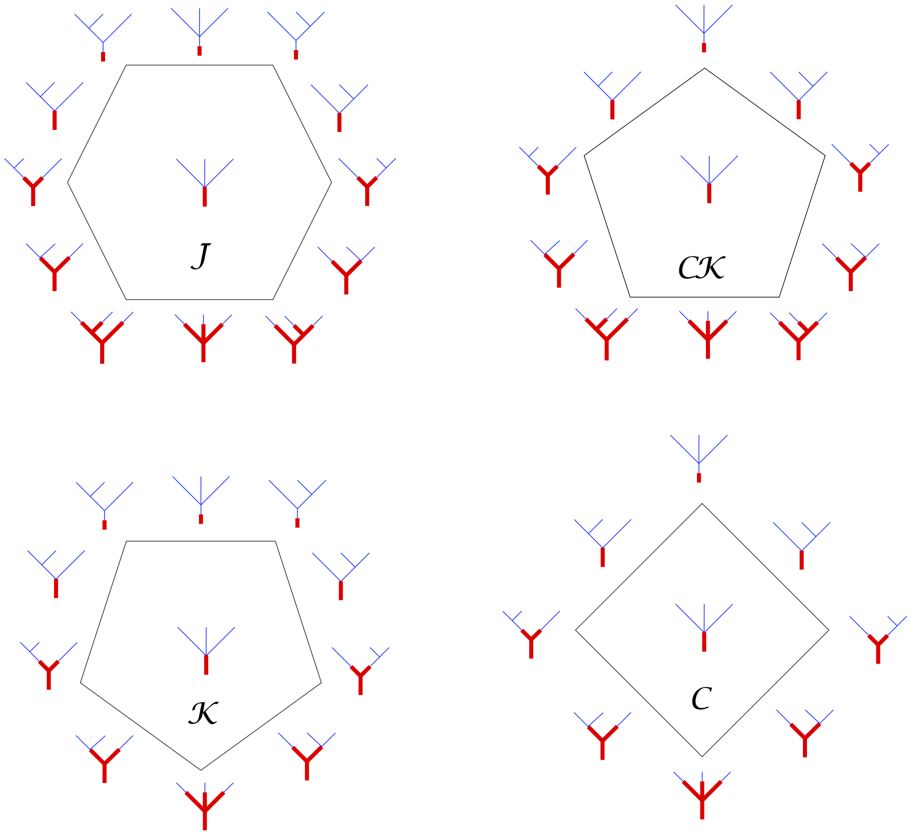


FIGURE 11. These are the 2d terms with their faces labeled. The same labels are used in 2d no matter where the shape occurs in figure 10.

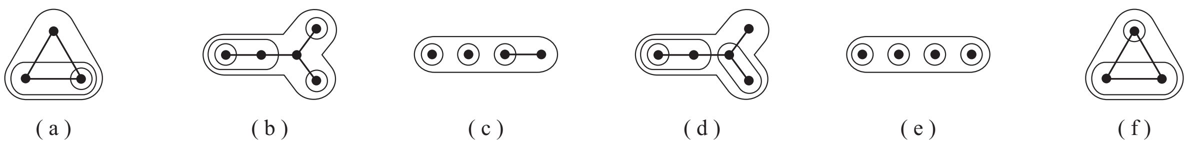


FIGURE 12. (a)-(c) Allowable tubings and (d)-(f) forbidden tubings, figure from [7].

vectors of graph associahedra for path-like graphs have been found, as shown in [16]. This source also contains the face vectors for the cyclohedra.

To describe the face structure of the graph associahedra we need a definition from [5, Section 2].

Definition 3.3. *For graph G and a collection of nodes t, construct a new graph $G^*(t)$ called the* reconnected complement*: If V is the set of nodes of G, then $V - t$ is the set of nodes of $G^*(t)$. There is an edge between nodes a and b in $G^*(t)$ if $\{a, b\} \cup t'$ is connected in G for some $t' \subseteq t$.*

Example 3.4. *Figure 13 illustrates some examples of graphs along with their reconnected complements.*

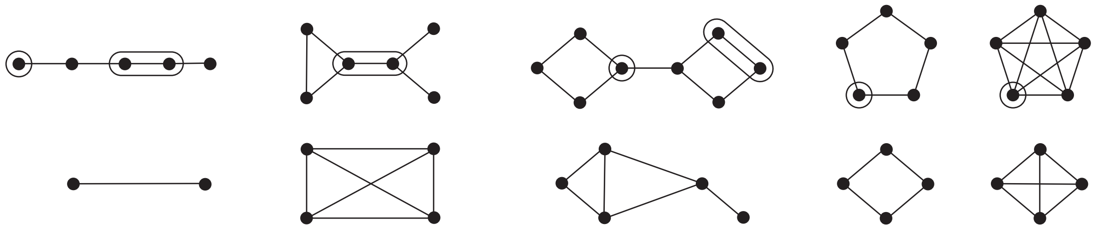


FIGURE 13. Examples of tubes and their reconnected complements.

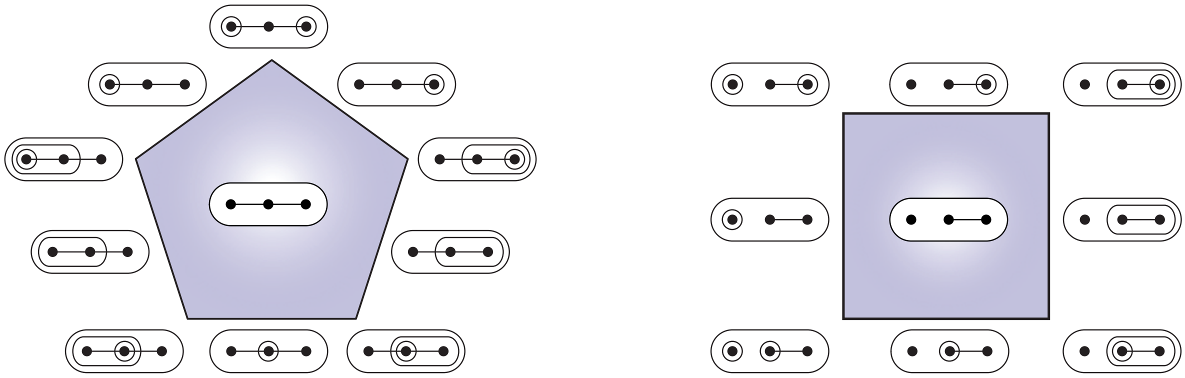


FIGURE 14. Graph associahedra of a path and a disconnected graph. The 3-cube is found as the graph-associahedron of two disjoint edges on four nodes, but no simple graph yields the 4-cube.

A lifting of a generalized permutahedron, and a nestohedron in particular, is a way to get a new generalized permutahedron of one greater dimension from a given example, using a factor of $q \in [0,1]$ to produce new vertices from some of the old ones. This procedure was first seen in the proof that Stasheff's multiplihedra complexes are actually realized as convex polytopes–answering a long-standing open question [10].

Soon afterwards the lifting procedure was applied to the graph associahedra–well-known examples of nestohedra first described by Carr and Devadoss. We completed an initial study of the resulting polytopes, dubbed *graph multiplihedra*, published as [7].

This application raised the question of a general definition of lifting using q. At the time it was also unknown whether the results of lifting, then just the multiplihedra and the graph-multiplihedra, were themselves generalized permutahedra. These questions were both answered in the recent paper of Ardila and Doker [3]. They defined nesto-multiplihedra and showed that they were generalized permutohedra of one dimension higher in each case.

Combinatorially, lifting of a graph associahedron occurs when the notion of a tube is extended to include markings. The following definitions are from [7].

Definition 3.5. A *marked tube* of P is a tube with one of three possible markings:

(1) a *thin* tube given by a solid line,
(2) a *thick* tube given by a double line, and
(3) a *broken* tube given by fragmented pieces.

Marked tubes u and v are *compatible* if

(1) they form a tubing and
(2) if $u \subset v$ where v is not thick, then u must be thin.

A *marked tubing* of P is a tubing of pairwise compatible marked tubes of P.

A partial order is now given on marked tubings of a graph G. This poset structure is then used to construct the *graph multiplihedron* below.

Definition 3.6. The collection of marked tubings on a graph G can be given the structure of a poset. A marked tubing $U \prec U'$ if U is obtained from U' by a combination of the following four moves. Figure 15 provides the appropriate illustrations, with the top row depicting U' and the bottom row U.

(1) *Resolving markings*: A broken tube becomes either a thin tube (15a) or a thick tube (15b).
(2) *Adding thin tubes*: A thin tube is added inside either a thin tube (15c) or broken tube (15d).
(3) *Adding thick tubes*: A thick tube is added inside a thick tube (15e).
(4) *Adding broken tubes*: A collection of compatible broken tubes $\{u_1, \ldots, u_n\}$ is added simultaneously inside a broken tube v only when $u_i \Subset v$ *and* v becomes a thick tube; two examples are given in (15f) and (15g).

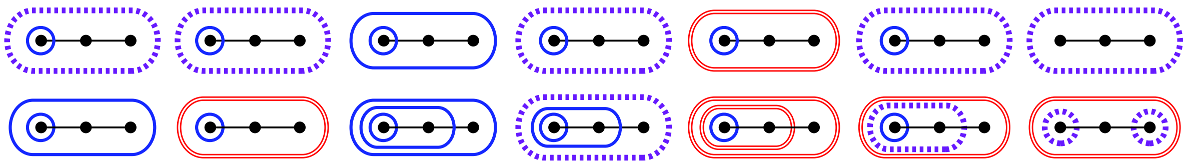


FIGURE 15. The top row are the tubings and bottom row their refinements.

Here is the key theorem from [7]

Theorem 3.7. *For a graph G with n nodes, the* graph multiplihedron $\mathcal{J}G$ *is a convex polytope of dimension n whose face poset is isomorphic to the set of marked tubings of G with the poset structure given above.*

There are two important quotient polytopes mentioned in [7]: $\mathcal{J}G_d$ and $\mathcal{J}G_r$.

In terms of tubings, the face poset of $\mathcal{J}G_d$ is isomorphic to the poset $\mathcal{J}G$ modulo the equivalence relation on marked tubings generated by identifying any two tubings $U \sim V$ such that $U \prec V$ in $\mathcal{J}G$ precisely by the addition of a thin tube inside another thin tube, as in Figure 15(c).

In terms of tubings, the face poset of $\mathcal{J}G_r$ is isomorphic to the poset $\mathcal{J}G$ modulo the equivalence relation on marked tubings generated by identifying any two tubings $U \sim V$ such that $U \prec V$ in $\mathcal{J}G$ precisely by the addition of a thick tube, as in Figure 15(e).

The four well-known examples of polytopes from Figure 9 can be seen as tubing posets, as pointed out in [7]. The multiplihedra $\mathcal{J} = \mathcal{J}P$ have face posets equivalent to the marked tubes on path graphs P. The composihedra are the domain quotients of these: $\mathcal{J}P_d$; and the associahedra are the range quotients of these: $\mathcal{J}P_r$. The cubes show up as the result of taking both quotients simultaneously.

3.2. **Permutohedra.** First we prove that the poset of painted trees made by grafting a weakly ordered forest to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with n leaves this polytope is the permutohedron $\mathcal{P}_n$. It is well known (see [14]) that the permutohedron has faces indexed by the weak orders, which in turn may be represented by weakly ordered trees. The face poset is the partial ordering of these trees by refinements.

Theorem 3.8. *There is an isomorphism φ from the poset of $(n+1)$-leaved weakly ordered trees to the painted growth preorder of n-leaved weakly ordered forests grafted to weakly ordered trees.*

Proof. The isomorphism and its inverse are described as switching between the paint line and an extra branch. Given a weakly ordered tree t, we find $\varphi(t)$ by adding a paint line at the level of left-most node of t, and then deleting the left-most branch of t. This process clearly describes a set isomorphism, since the inverse is easy to perform. Given a weakly ordered forest grafted to a weakly ordered tree, painted to make the graft visible, we attach a new branch to the left of the tree, precisely at the level of the paint line. Then the paint is forgotten and the unpainted result is a weakly ordered tree.

Next we argue that the isomorphism just described respects the poset structures. If $a \prec b$ for two weakly ordered trees, we have that the weak ordering of the nodes of a is a refinement of the weak ordering for b. We can visualize this refinement as the growing of some internal edges of a to break ties between nodes that were at the same level. If the refinement involves breaking a tie that does not include the left-most node, then the same growing produces the same relation between the painted tree images $\varphi(a)$ and $\varphi(b)$. If the growing does break a tie involving the left-most node, then the image of b may differ from that of a only in that the set of nodes of $\varphi(a)$ which coincide with the paint line will be a subset of those in $\varphi(b)$. This can be seen as a growing of some painted edges at some half-painted nodes. $\square$

This theorem immediately implies that the poset of n-leaved weakly ordered forests grafted to weakly ordered trees is isomorphic to the face poset of the n-dimensional permutohedron. That is because the poset of $(n+1)$-leaved weakly ordered trees is well known to represent the face poset of the permutohedron (via seeing each tree as a weak order of $[n]$, that is, an ordered partition.)

A corollary, from [7], is that the poset of n-leaved weakly ordered forests grafted to weakly ordered trees is isomorphic to the face poset of the n-dimensional graph multiplihedron of the complete graph.

3.3. **Pterahedra.** Now we prove that the poset of painted trees made by grafting a forest of plane rooted trees to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is the graph-associahedron $\mathcal{K}G$ where G is the fan graph $F_{1,n}$.

Recall that the fan graph $F_{1,n}$ is defined as follows: we use the set $\{0, 1, 2, \ldots, n\}$ as the set of nodes. Edges are $\{i, i+1\}$ for $i = 1, \ldots, n-1$, together with $\{0, i\}$ for $i = 1, \ldots, n$.

Theorem 3.9. *The poset of tubings on the fan graph $F_{1,n}$ is isomorphic to the poset of n-leaved forests of plane trees grafted to weakly ordered trees.*

Proof. We first note that any tubing T of the fan graph includes a unique smallest tube t_0 which contains node 0. All other tubes of T are either contained in t_0 or contain t_0, since the node 0 is adjacent to all other nodes. The tubes contained in t_0 form a tubing of a graph which is a (possibly) disconnected set of path graphs. The tubes containing t_0 form a tubing on the reconnected complement of t_0, which is the complete graph on the nodes not in t_0.

There is an isomorphism between the poset of weakly ordered trees and the tubings on the complete graph: this is pictured in Figures 16 and 17 from [13]. Also in those pictures is shown the isomorphism between plane trees and tubings on the path graph.

Thus the bijection from the tubing on the fan graph to the tree is formed of the bijection from tubings on a complete graph to weakly ordered trees, together with the bijection from tubings on a path graph to plane trees. The tube t_0 plays the same role as the paint line in the corresponding tree. The nodes $1, \ldots, n$ of the fan graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. The tubing outside of t_0 maps to the painted weakly ordered tree, the tubings inside t_0 map to the unpainted trees, and nodes that are inside t_0 but not inside any smaller tube determine the gaps-between leaves that coincide with the paint line. Examples are seen in Figure 18.

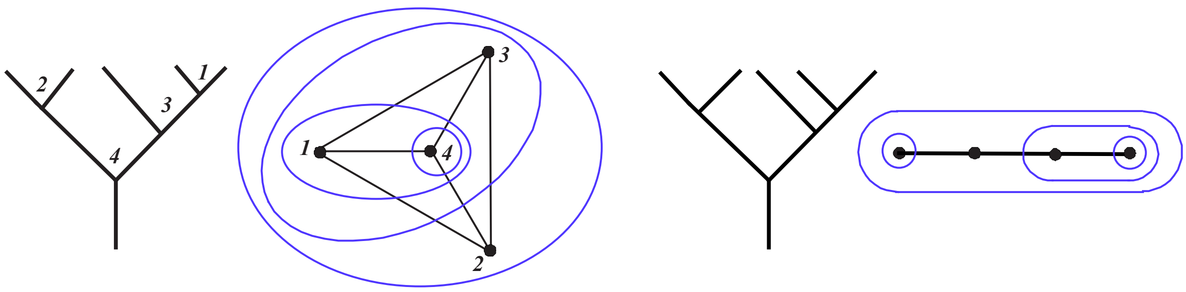


FIGURE 16. The permutation $\sigma = (2431) \in S_4$ pictured as an ordered tree and as a tubing of the complete graph; An unordered binary tree, and its corresponding tubing.

The fact that this bijection preserves the ordering follows easily from the definitions. Just note that adding a tube to a tubing of the fan graph corresponds to growing an internal edge in the tree. Adding a tube far outside of t_0 corresponds to growing an edge in the painted base. Adding a tube just outside t_0 (so that it becomes the new t_0) corresponds to growing painted edge(s) from a half-painted node. Adding a tube just inside t_0 corresponds to growing unpainted edge(s) from a half-painted node. Adding a tube well inside of t_0 corresponds to growing an edge in the unpainted forest. $\square$

The isomorphism in 3d is shown pictorially in Figure 3.3.

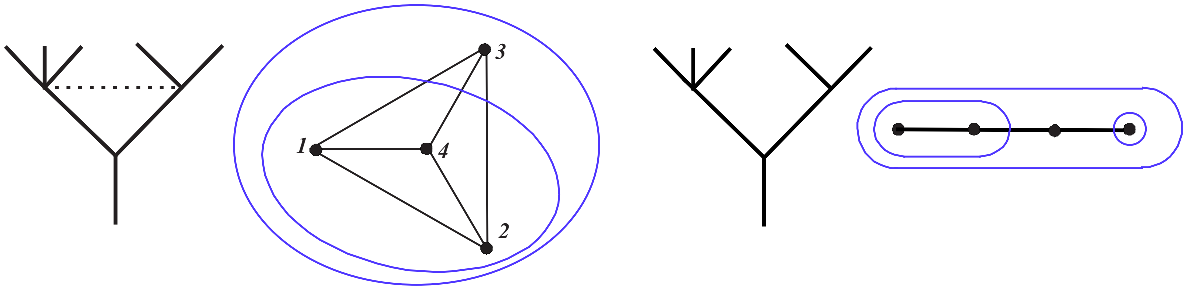


FIGURE 17. The ordered partition $(\{1,2,4\},\{3\})$ pictured as a leveled tree and as a tubing of the complete graph; the underlying tree, and its corresponding tubing.

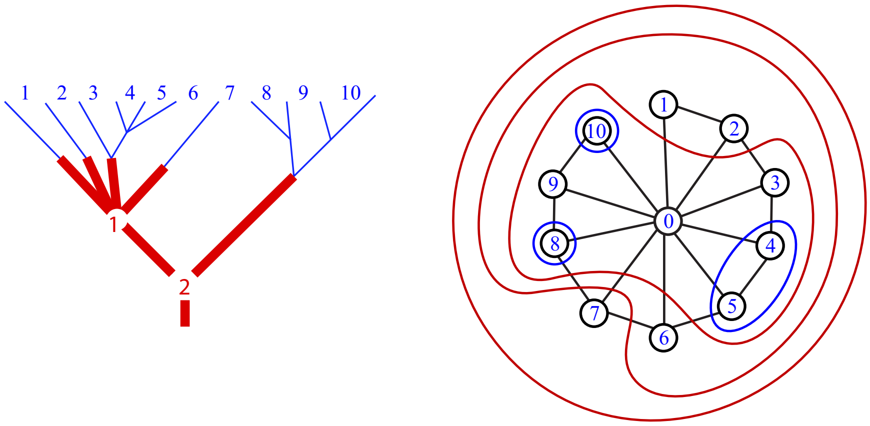


FIGURE 18. Example of the bijection in Theorem 3.9.

3.4. **Enumeration.** As well as uncovering the equivalence between the pterahedra and the fan-graph associahedra, we found some new counting formulas for the vertices and facets of the pterahedra.

First the vertices of the pterahedra, which are forests of binary trees grafted to an ordered tree. If there are k nodes in the ordered, painted portion of a tree, then there are:

- $k!$ ways to make the ordered portion of this tree with k nodes,
- $k+1$ leaves of the ordered portion of this tree, and
- $n-k$ remaining nodes to be distributed among the $k+1$ binary trees that will go on the leveled/painted leaves.

Thus the number of vertices of the pterahedron, labeled by trees with n nodes, is:

$$v(n) = \sum_{k=0}^{n} [\ k! \sum_{\substack{\gamma_0+\ldots+\gamma_k \\ =n-k}} (\prod_{i=0}^{k} C_{\gamma_i})\ ].$$

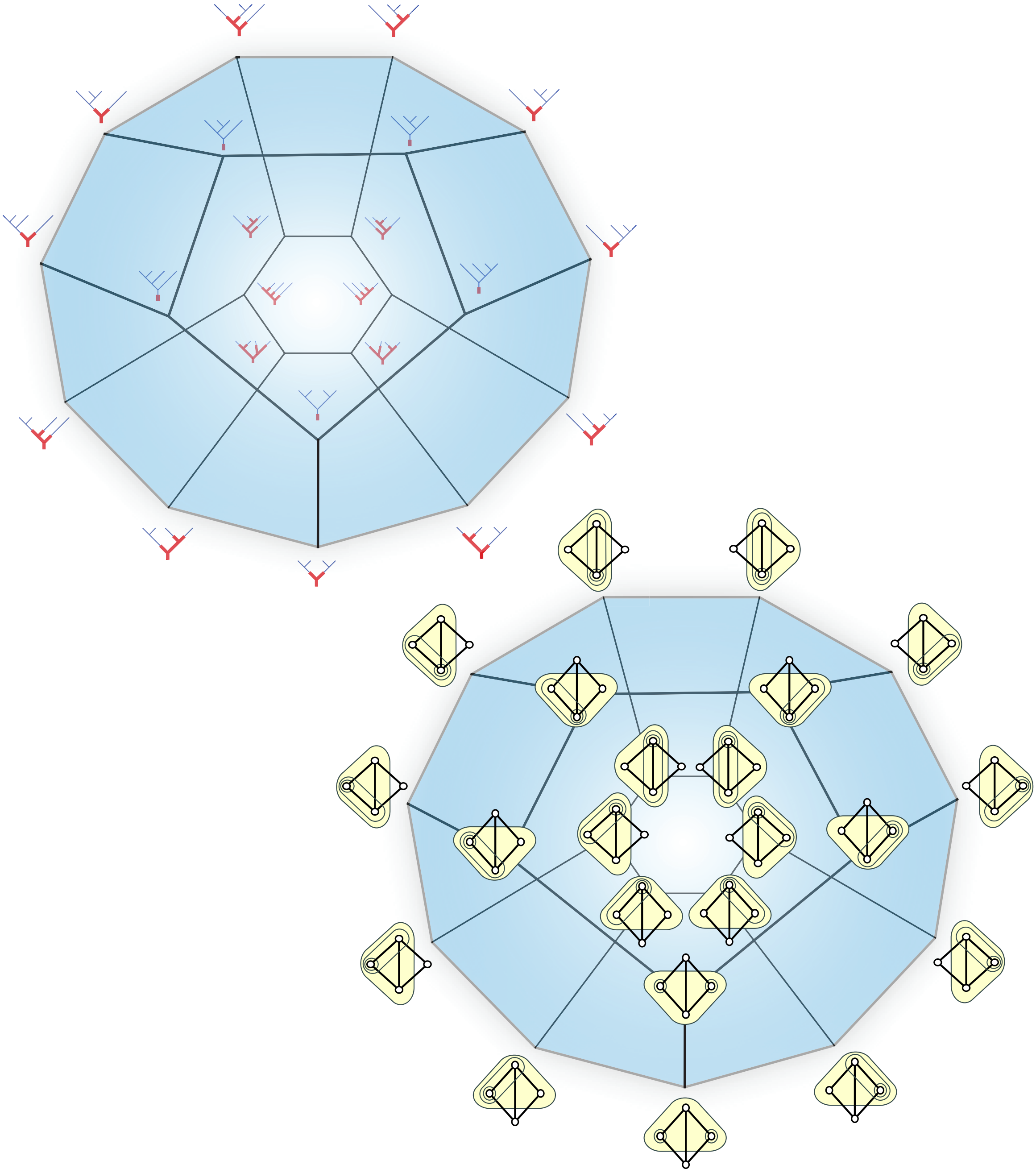


FIGURE 19. Two pictures of the pterahedron.

where C_j is the jth Catalan number. As an example, the number of trees with $n = 4$ is:

$$\begin{aligned}
&0!\ [C_4]\\
&+1!\ [C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3]\\
&+2!\ [C_2C_0C_0 + C_1C_1C_0 + C_1C_0C_1 + C_0C_2C_0 + C_0C_1C_1 + C_0C_0C_2]\\
&+3!\ [C_1C_0C_0C_0 + C_0C_1C_0C_0 + C_0C_0C_1C_0 + C_0C_0C_0C_1]\\
&+4!\ [C_0C_0C_0C_0C_0]\\
&= 1(14) + 1(14) + 2(9) + 6(4) + 24(1)\\
&= 14 + 14 + 18 + 24 + 24\\
&= 94
\end{aligned}$$

We have computed the cardinalities for $n = 0$ to 9 and they are shown in table 1.

TABLE 1. The number $v(n)$ vertices of the pterahedra, $n = 0$ to 9

n	$v(n)$	n	$v(n)$
0	1	5	464
1	2	6	2652
2	6	7	17,562
3	22	8	133,934
4	94	9	1,162,504

There does not seem to be an entry for this sequence in the OEIS.

Examination of the computations of $v(n)$ leads to an interesting discovery. If we strip off the factorial factors in $v(n)$ and build a triangle of just the sums of C_{γ_i} products, it appears we are building the Catalan triangle.

1									
1	1								
2	2	1							
5	5	3	1						
14	14	9	4	1					
42	42	28	14	5	1				
132	132	90	48	20	6	1			
429	429	297	165	75	27	7	1		
1430	1430	1001	572	275	110	35	8	1	
4862	4862	3432	2002	1001	429	154	44	9	1

For example,

$$v(4) = 94 = 0!(\mathbf{14}) + 1!(\mathbf{14}) + 2!(\mathbf{9}) + 3!(\mathbf{4}) + 4!(\mathbf{1})$$

leads to the values of the $n = 4$ row in the triangle above. In fact, Zoque [20] states that the entries of the Catalan triangle, often called *ballot numbers*, count "the number of ordered forests with m binary trees and with total number of ℓ internal vertices" where m and ℓ are indices into the triangle. These forests describe exactly the sets of binary trees we are grafting onto the leaf edges of individual leveled trees, which are counted by the sums of C_{γ_i} products. Thus we know that the ballot numbers are equivalent to the sums of C_{γ_i} products and can be used in the calculation of $v(n)$ for all values of n.

The formula for the entries of the Catalan triangle [19] leads to a simpler formula for $v(n)$, namely

$$v(n) = \sum_{k=0}^{n} k! \tfrac{(2n-k)!(k+1)}{(n-k)!(n+1)!},$$

Lastly, by considering the Catalan triangle as a matrix as in [4], we can say that the sequence of cardinalities $v(n)$ for all n is the Catalan transform of the factorials $(n-1)!$.

This means that the ordinary generating function for $v(n)$ is:

$$\sum_{k=1}^{\infty}(k-1)!\left(\frac{1-\sqrt{1-4x}}{2}\right)^{k}.$$

Also, it will be helpful to have a formula for the number of facets for the fan graphs, $F_{m,n}$. Recall $F_{m,n}$ is defined to be the graph join of $\bar{K}_m$ the empty graph on m nodes, and P_n the path graph on n nodes. Thus, $F_{m,n}$ has $m+n$ vertices, n of which comprise a subgraph isomorphic to the path graph on m nodes, the other n vertices connected to each of these m. Thus, $F_{m,n}$ has $m-1+mn = m(n+1)-1$ edges.

Now, counting tubes in this case is again a matter of counting subsets of vertices whose induced subgraph is connected. The structure of $F_{m,n}$ makes it useful to let V_m denote those vertices coming from the empty graph of m nodes, and likewise V_n those from the path graph on n nodes.

It is clear that some tubes are simply tubes of the path graph P_n, hence there are at least $\frac{n(n+1)}{2}-1$ tubes. We must not forget that V_n is itself now a tube since it is a proper subset of nodes of $F_{m,n}$.These tubes include every subset of V_n that is a valid tube of $F_{m,n}$.

It is simple to see that the only subsets of V_m that are valid tubes are precisely the singletons, since no pair of vertices in V_m are connected by an edge. Thus $F_{m,n}$ has at least $\frac{n(n+1)}{2}+m$ tubes.

The remaining possibility for tubes must include at least one node from V_m as well as at least one node from V_n. This produces all (possibly improper) tubes, since any subset of $V = V_m \cup V_n$ satisfying this criterion is connected. It is straightforward to see that there are exactly $(2^m-1)(2^n-1)$ tubes arising in this fashion. Now, however, we must subtract 1 from the above since we have allowed ourselves to count V as a tube, although it is not proper.

Hence we count

$$\frac{n(n+1)}{2}+(2^m-1)(2^n-1)+m-1$$

as the number of tubes of $F_{m,n}$ and the number of facets of the corresponding graph associahedron. For the pterahedra, where $m=1$, the formula becomes:

$$\frac{n(n+1)}{2}+2^n-1.$$

Interestingly, this is the same number of facets as possessed by the multiplihedron $\mathcal{J}(n)$, as seen in [10], where we enumerate the facets by describing their associated trees.

3.5. **Stellohedra.** Now we prove that the poset of painted trees made by grafting a forest of corollas to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is the graph-associahedron $\mathcal{K}G$ where G is the star graph S_n.

Recall that the star graph S_n is defined as follows: we use the set $\{0,1,2,\dots,n\}$ as the set of nodes. Edges are $\{0,i\}$ for $i=1,\dots,n$.

Theorem 3.10. *The poset of tubings on the star graph S_n is isomorphic to the poset of n-leaved forests of corollas grafted to weakly ordered trees.*

Proof. We first note that any tubing T of the star graph includes a unique smallest tube t_0 which contains node 0. All other tubes of T are either contained in t_0 or contain t_0, since the node 0 is adjacent to all other nodes. The tubes contained in t_0 form a tubing of an edgeless graph. The tubes containing t_0 form a tubing on the reconnected complement of t_0, which is the complete graph on the nodes not in t_0.

Thus the bijection from the tubing on the star graph to the tree is formed of the bijection from tubings on a complete graph to weakly ordered trees, together with the bijection from tubings on an edgeless graph to subsets of $[n]$. The tube t_0 plays the same role as the paint line in the corresponding tree. The nodes $1, \ldots, n$ of the star graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. The tubing outside of t_0 maps to the painted weakly ordered tree, the tubing inside t_0 maps to a subset of $[n]$ which is the subset of the gaps-between-leaves that end in nodes of unpainted corollas, and nodes that are inside t_0 but not inside any smaller tube determine the gaps-between-leaves that coincide with the paint line. Examples are seen in Figure 20.

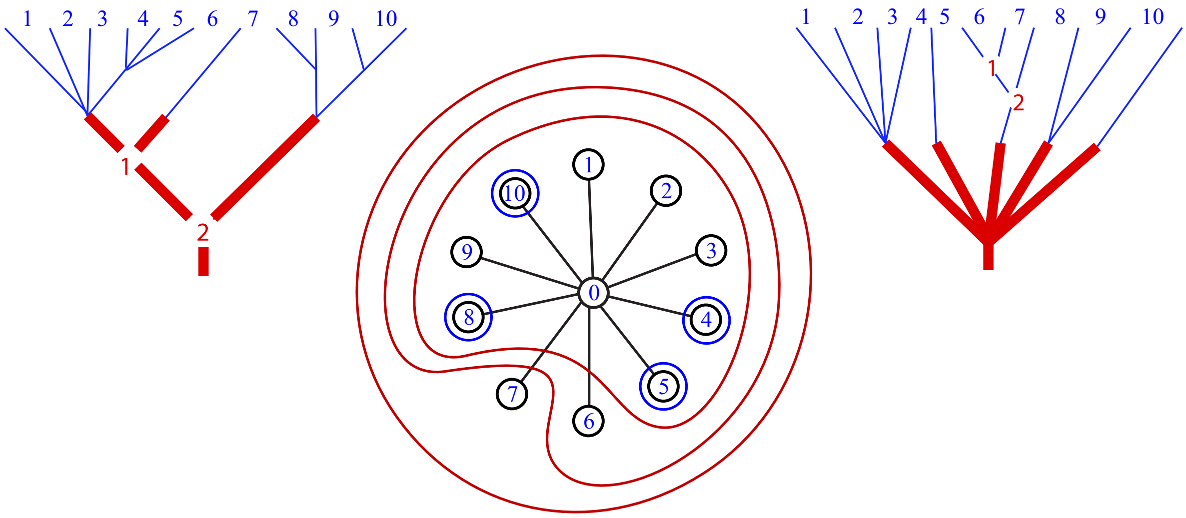


FIGURE 20. A tubing on the star graph: its bijective image in the corollas over weakly ordered trees on the left and in the weakly ordered forest over corollas on the right.

The fact that this bijection preserves the ordering again follows easily from the definitions. □

The isomorphism in 3d is shown pictorially in Figure 21.

Moreover, we show that the poset of painted trees made by grafting a weakly ordered forest to a base corolla is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is again the graph-associahedron $\mathcal{K}G$ where G is the star graph S_n.

Theorem 3.11. *The poset of tubings on the star graph S_n is isomorphic to the poset of n-leaved forests of corollas grafted to weakly ordered trees.*

Proof. We again note that any tubing T of the star graph includes a unique smallest tube t_0 which contains node 0.

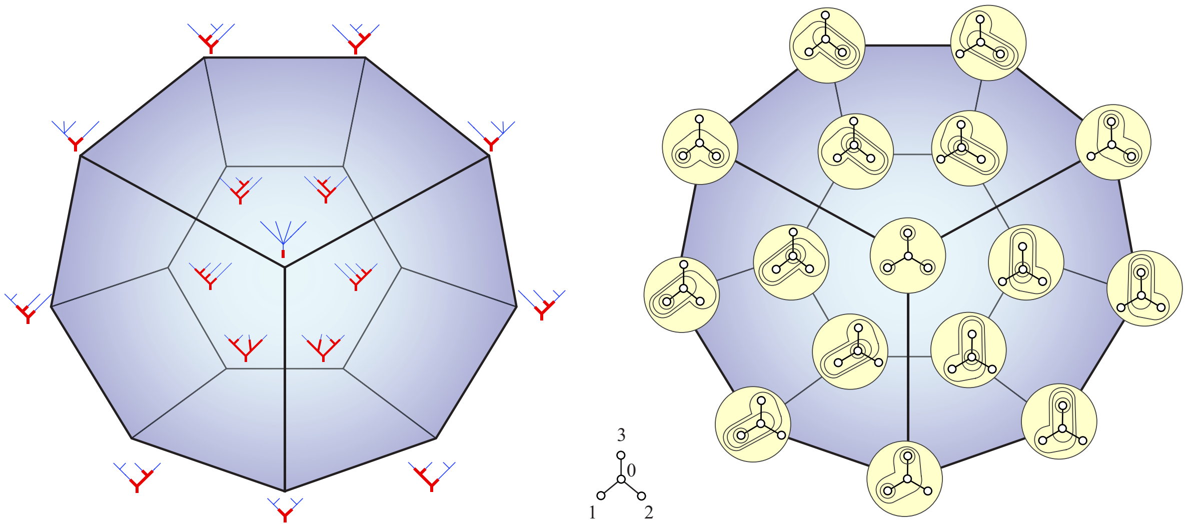


FIGURE 21. Two pictures of the stellohedron.

Again the bijection from the tubing on the star graph to the tree is formed of the bijection from tubings on a complete graph to weakly ordered trees, together with the bijection from tubings on an edgeless graph to subsets of $[n]$. The tube t_0 again plays the same role as the paint line in the corresponding tree. The nodes $1, \dots, n$ of the star graph correspond to the gaps-between-leaves $1, \dots, n$ of the tree. The tubing outside of t_0 maps to the unpainted forest of weakly ordered trees, the tubing inside t_0 maps to a subset of $[n]$ which is the subset of the gaps-between-leaves that end in nodes of the painted corolla, and nodes that are inside t_0 but not inside any smaller tube determine the gaps-between-leaves that coincide with the paint line. Examples are seen in Figure 20.

The fact that this bijection preserves the ordering again follows easily from the definitions. □

Finally we mention that the stellohedra can also be seen as the domain and range quotients $\mathcal{J}G_d$ and $\mathcal{J}G_r$ where G is the complete graph.

Theorem 3.12. *The graph-cubeahedron for a complete graph $\mathcal{K}_n$ is combinatorially equivalent to the stellohedron for the star-graph S_{n+1}. Likewise the graph-composihedron for a complete graph $\mathcal{K}_n$ is combinatorially equivalent to the stellohedron for the star-graph S_{n+1}.*

Proof. We can most easily see the isomorphism by using the stellohedra just found in Theorems 3.10 and 3.11; that is by showing isomorphisms to painted trees.

Any marked tubing T of the complete graph is entirely nested, so includes a unique smallest thick tube t_0, or no thick tube at all.

First, we show a bijection from the graph-cubeahedron to the forests of corollas grafted to weakly ordered trees. A broken tube t_0 plays the same role as the half-painted nodes in the corresponding tree. The nodes $1, \dots, n$ of the complete graph correspond to the gaps-between-leaves $1, \dots, n$ of the tree. Any nodes outside of all the thin or broken tubes map to the weakly ordered base tree, the nodes inside the largest thin tube map to the gaps-between-leaves that end in nodes of the unpainted corollas, and nodes that are inside a broken tube but not inside any thin tube determine the gaps-between-leaves that coincide with the paint line. An example of the bijection is seen in Figure 22.

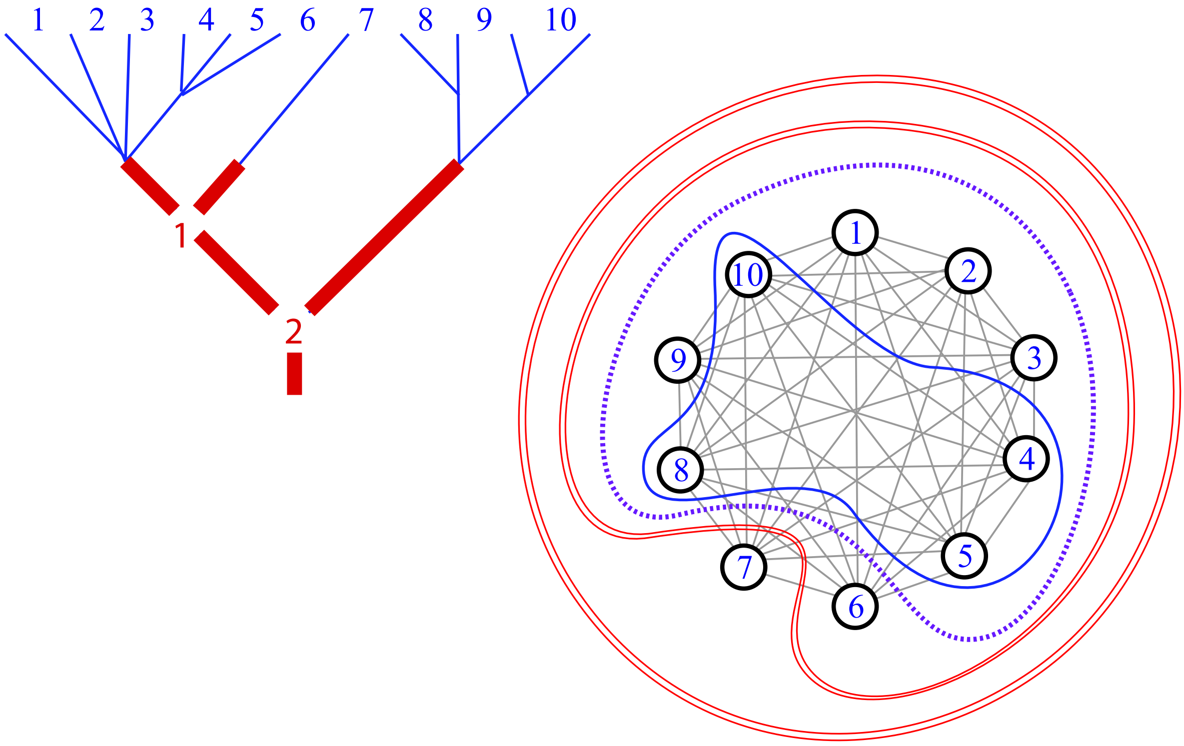


FIGURE 22. A marked tubing on the complete graph, representing an element of the complete-graph-cubahedron (no structure is shown inside the thin tube) and its bijective image in the corollas over weakly ordered trees.

3d examples are seen in Figure 24.

Secondly we show a bijection from the graph-composihedron to the weakly ordered forests grafted to corollas. The nodes $1, \ldots, n$ of the complete graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. Any nodes outside of t_0 map to the painted corolla base tree, the tubing inside a largest thin tube maps to the gaps-between-leaves that end in nodes of the unpainted weakly ordered forest, and nodes that are inside a broken tube but not inside any thin tube determine the gaps-between-leaves that coincide with the paint line. An example of the bijection is seen in Figure 23.

3d examples are seen in Figure 25.

The fact that these bijections preserve the ordering follows easily from the definitions. Note that the relations are simpler than in general for marked tubes since the tubings must all be completely nested. Adding or resolving a tube (and forgetting tubing structure that is ignored by the quotient from the marked tubing to the cubahedron

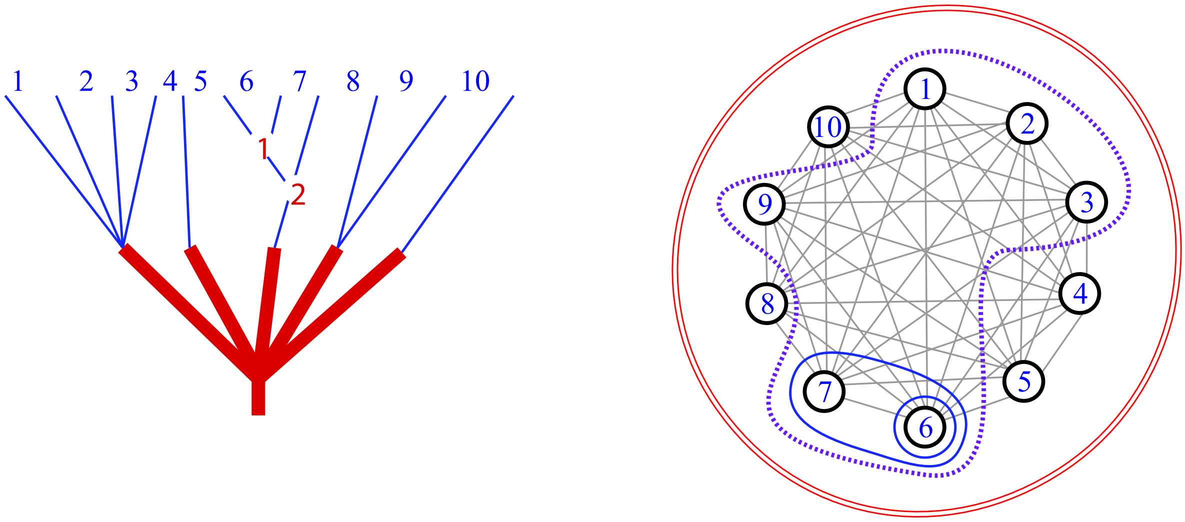


FIGURE 23. A marked tubing on the complete graph, representing an element of the complete-graph-composihedron (no structure is shown outside the broken tube) and its bijective image in the weakly ordered forest over corollas. Note the composite bijection from the cubahedra to the composihedra of the complete graph, as stated in [7].

or composihedron) corresponds to growing edges and forgetting the corresponding tree structure. □

Finally, for reference we point out that another description of the cubeahedra is found in [8], where it is described as *design-tubings* on the complete graph. In Figure 26 we show the correspondence between labels of vertices: range-equivalence classes of marked tubings and design tubings.

3.6. **Enumeration.** The number of vertices of the stellahedron is worked out in several places, including [11], where the formula is given:

$$v(n) \;=\; \sum_{k=0}^{n} k!\binom{n}{k} \;=\; \sum_{k=0}^{n} n!/k!\,,$$

which is sequence A000522 in the OEIS [17]. This is the binomial transform of the factorials.

Now for the facets. We will use the following, possibly well-known

Theorem 3.13. *The bipartite graph associahedron $\mathcal{K}K_{m,n}$ has $2^{m+n}+(m+n)-(2^m+2^n)$ facets.*

To see this, we will count subsets of nodes which give valid tubes. We will over-count and then correct. Let $K_{m,n} = (V_1 \cup V_2, E)$ where $|V_1| = m$ and $|V_2| = n$. Note that the only subsets S of nodes which do not give valid tubes are S such that $S \subseteq V_1$ (or V_2) with $|S| > 1$. These are simply empty graphs with $|S| > 1$ nodes, and do not constitute valid tubes. For the moment, let $S \subseteq V_1$ where $|V_1| = m$.

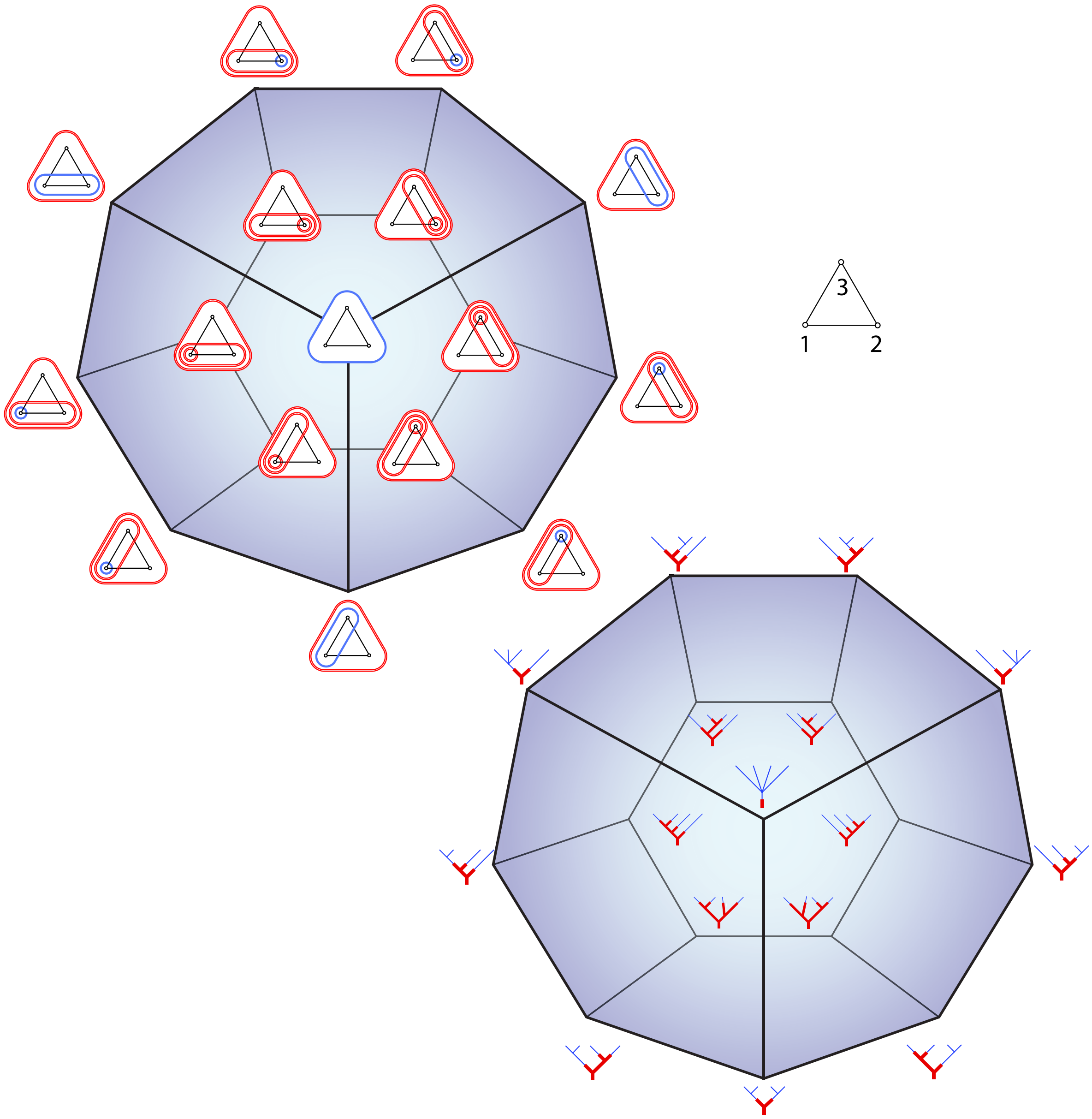


FIGURE 24. Another stellahedra bijection: when G is the complete graph then $\mathcal{J}G_r$ (the graph cubeahedron) is the stellohedron.

Let $M = \#\{S \subseteq V_1 : |S| > 1\}$. Now

$$M = \sum_{2}^{k} \binom{m}{k} = 2^m - (m+1)$$

and by the above, there are $2^n - (n+1)$ "bad subsets" that can be chosen from V_2 for a total of $2^m + 2^n - (m+n-2)$ bad subsets of $V = V_1 \cup V_2$.

It follows that we may choose any of $2^{m+n} - 2$ proper, nonempty subsets of V, and subtract off the bad choices for the total number of tubes. Thus, $K_{m,n}$ has

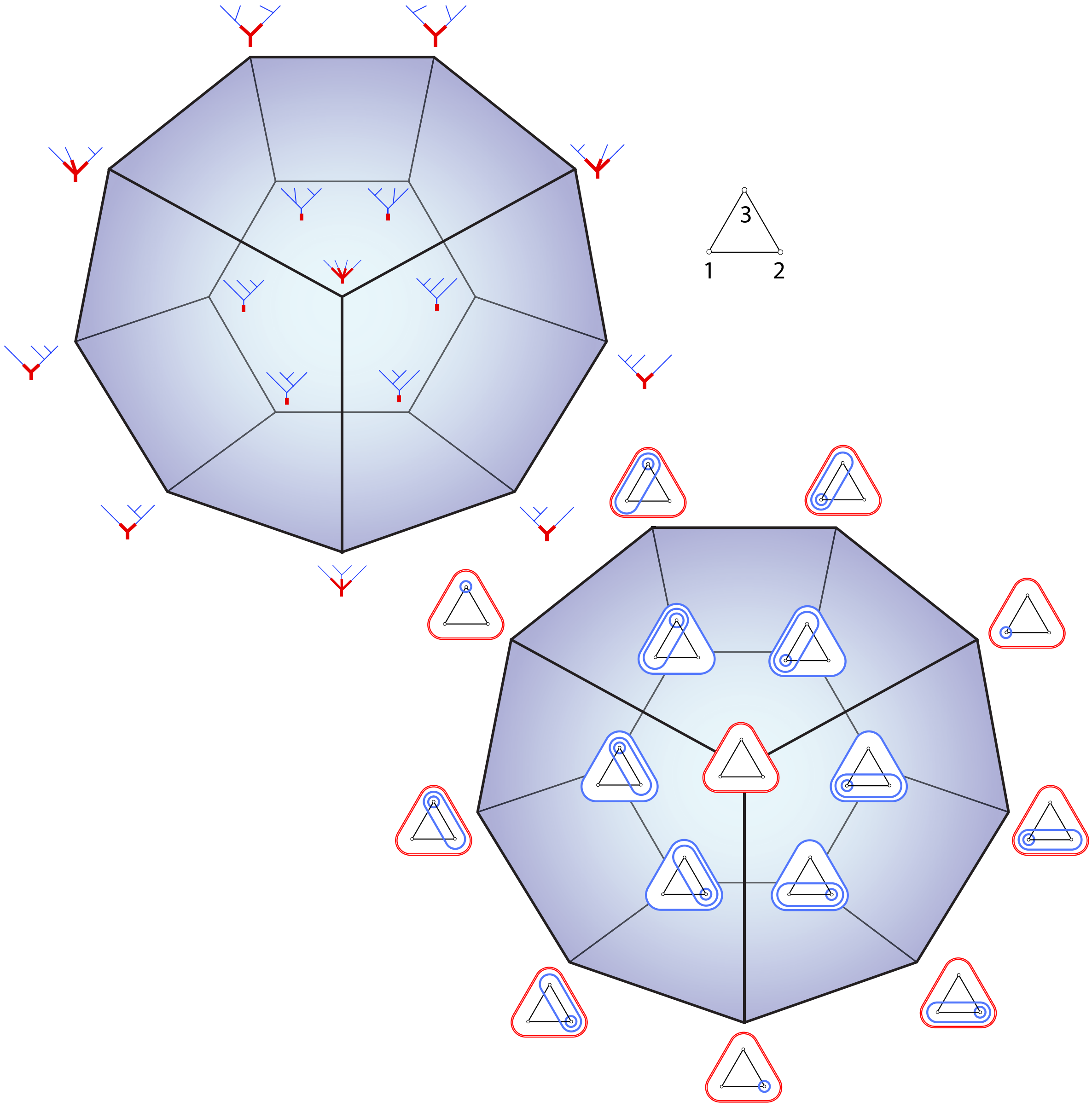


FIGURE 25. Another stellahedra bijection: the composition of ordered forests with corollas, seen in bijection with the graph composihedron.

$$\begin{aligned} 2^{m+n} - 2 - (2^m + 2^n - (m+n-2)) &= 2^{m+n} - 2 - (2^m + 2^n) + m + n + 2 \\ &= 2^{m+n} + (m+n) - (2^m + 2^n) \end{aligned}$$

tubes.

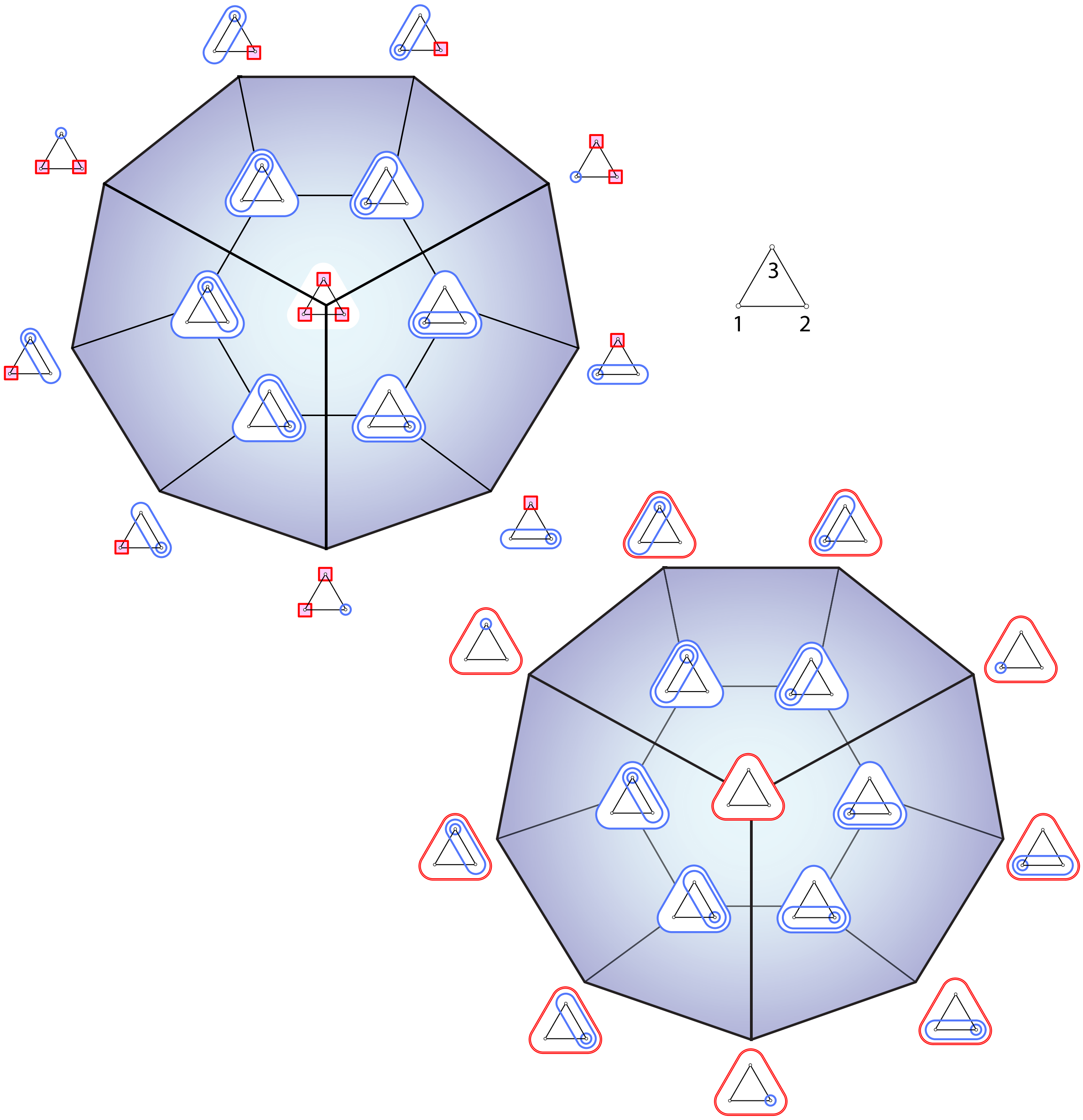


FIGURE 26. Another stellahedra bijection: the graph cubeahedron indexed by design tubings.

Note that, for $K_{1,n-1}$, we see that star graphs on n nodes have

$$\begin{aligned} 2^n + n - (2^{n-1} + 2) &= 2^n - 2^{n-1} + n - 2 \\ &= 2^{n-1}(2-1) + n - 2 \\ &= 2^{n-1} + n - 2 \end{aligned}$$

tubes.

4. Acknowledgements

The second author would like to thank the AMS and the Mathematical Sciences Program of the National Security Agency for supporting this research through grant H98230-14-0121.[1] My specific position on the NSA is published in [9]. Suffice it to say here that the second author appreciates NSA funding for open research and education, but encourages reformers of the NSA who are working to ensure that protections of civil liberties keep pace with intelligence capabilities.

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