ex) $T: P^4 \to P^2$

given by $T(f(x)) = f''(x)$

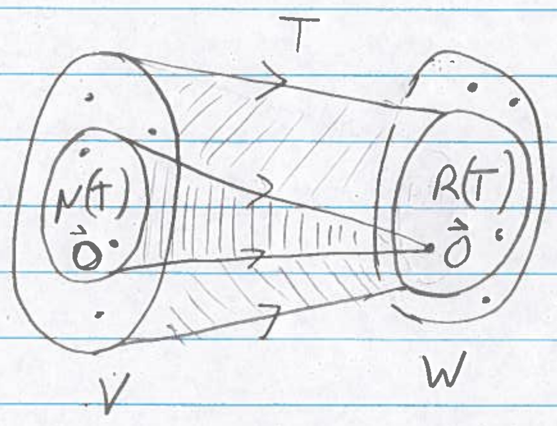
$[T]_{\mathcal{E}}^{\mathcal{E}}$ uses $\mathcal{E}_4$ for inputs: $\{1, x, x^2, x^3, x^4\}$
and $\mathcal{E}_2$ for outputs.

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \left[[0]_{\mathcal{E}} \; [0]_{\mathcal{E}} \; [2]_{\mathcal{E}} \; [6x]_{\mathcal{E}} \; [12x^2]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}, \quad 5 \times 3$$

Terminology: $T: V \to W$

- V is the domain, $\boxed{\text{dom}(T)}$
- W is the codomain, $\boxed{\text{codom}(T)}$
- Range (T) is a subspace of W, $\boxed{R(T)}$ which is all the outputs of T.
- Null Space of T, $\boxed{N(T)}$ is a subspace of V which is all the inputs that get taken to $\vec{0}$ by T.



- Null space is also known as kernel (T).

- Composition: for $T: V \to W$
and $S: W \to Y$
we make $S \circ T : V \to Y$
by $(S \circ T)(\vec{x}) = S(T(\vec{x}))$

$$V \xrightarrow[A]{T} W \xrightarrow[B]{S} Y$$

- If A represents T and B represents S (for same basis on W)
then $S \circ T$ is represented by
BA (matrix multiplication)

More terminology

- $T: V \to W$ is one-to-one (1-1) when each output has only exactly one input.
For $\vec{y} \in R(T)$ if $T(\vec{a}) = \vec{y} = T(\vec{b})$
then $\vec{a} = \vec{b}$. (T is injective)

Theorem. T is one-to-one if and only if
$$N(T) = \{\vec{0}\}.$$

Proof: Assume $N(T) = \{\vec{0}\}$.
Then if $T(\vec{a}) = T(\vec{b})$

$$\Rightarrow T(\vec{a}) - T(\vec{b}) = \vec{0}$$
$$\Rightarrow T(\vec{a} - \vec{b}) = \vec{0} \quad \text{(linearity)}$$
$$\Rightarrow \vec{a} - \vec{b} = \vec{0} \quad \text{(by assumption)}$$
$$\Rightarrow \vec{a} = \vec{b}$$

Next, Assume $N(T) \neq \{\vec{0}\}$, so $N(T) = \{\vec{0}, \vec{x}, \ldots\}$
then $T(\vec{0}) = \vec{0} = T(\vec{x})$, not 1-1. $\square$

- $T: V \to W$ is $\boxed{\text{onto}}$ (surjective) when $R(T) = W$.
- If T is 1-1 and onto, T is an $\boxed{\text{isomorphism}}$

Finding $N(T)$ and $R(T)$:

→ Same exact process as finding solution to $A\vec{x} = \vec{0}$ and $\text{col}(A)$, where $A = [T]_{\mathcal{B}}^{\mathcal{C}}$.

→ Find both: note that augment is $\vec{0}$

1) r.r. A to r.r.e.f.
   Recall: free variables are all non-pivot columns
2) write solution as a span, that's $N(T)$.
3) write $\text{col}(A)$ as a span of the original columns of A which correspond to pivots in r.r.e.f. That's $R(T)$.
4) Use bases $\mathcal{B}$ & $\mathcal{C}$ to describe $N(T)$ (using $\mathcal{B}$, the input basis) and $R(T)$ (using $\mathcal{C}$, the output basis.)

→ Note: since pivots + non-pivots = all the columns of A, we see that:

$$\dim(R(T)) + \dim(N(T)) = \dim(\text{dom}(T))$$