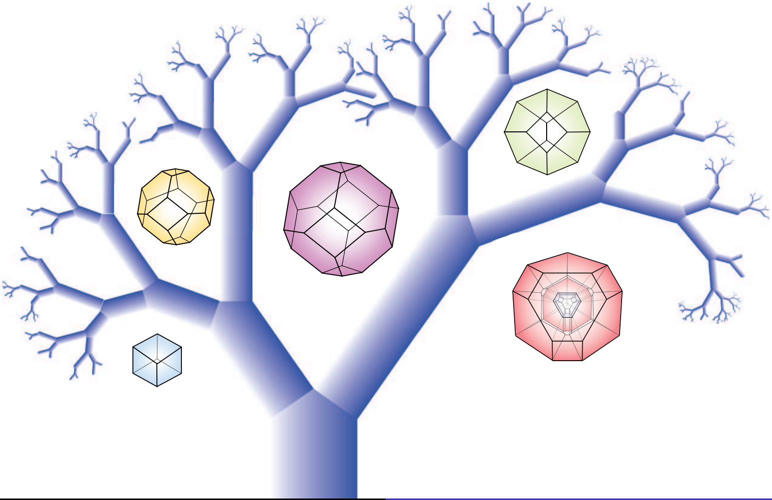
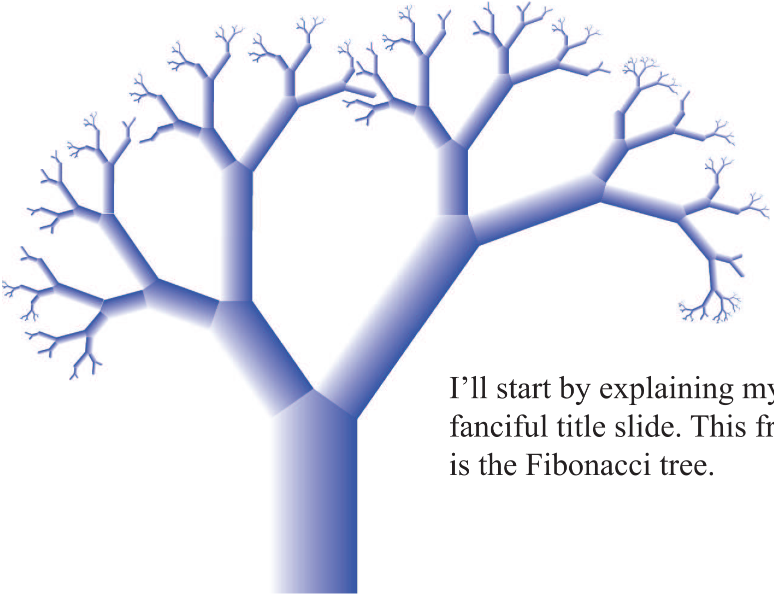
Trees and Polytopes   Stefan Forcey

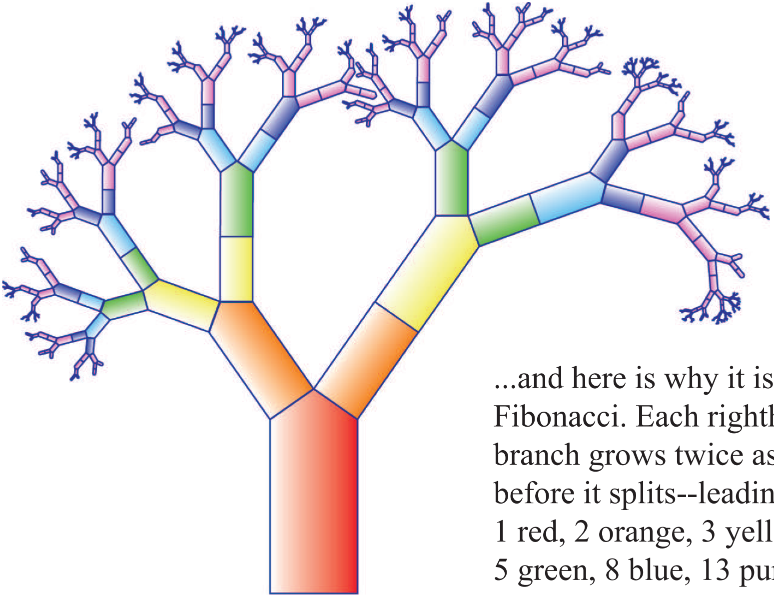


Fibonacci tree $\mathcal{F}$.



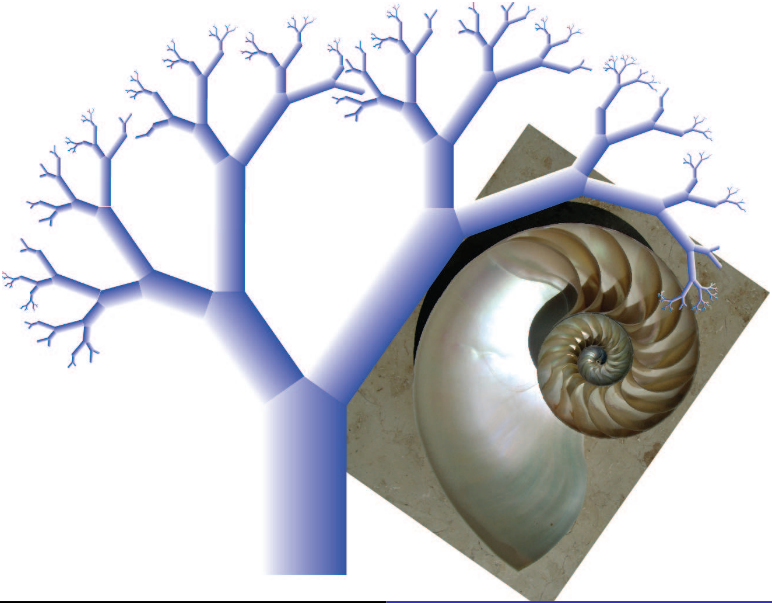
I'll start by explaining my fanciful title slide. This fractal is the Fibonacci tree.

Fibonacci tree$F$.



...and here is why it is called Fibonacci. Each righthand branch grows twice as far before it splits--leading to 1 red, 2 orange, 3 yellow 5 green, 8 blue, 13 purple...

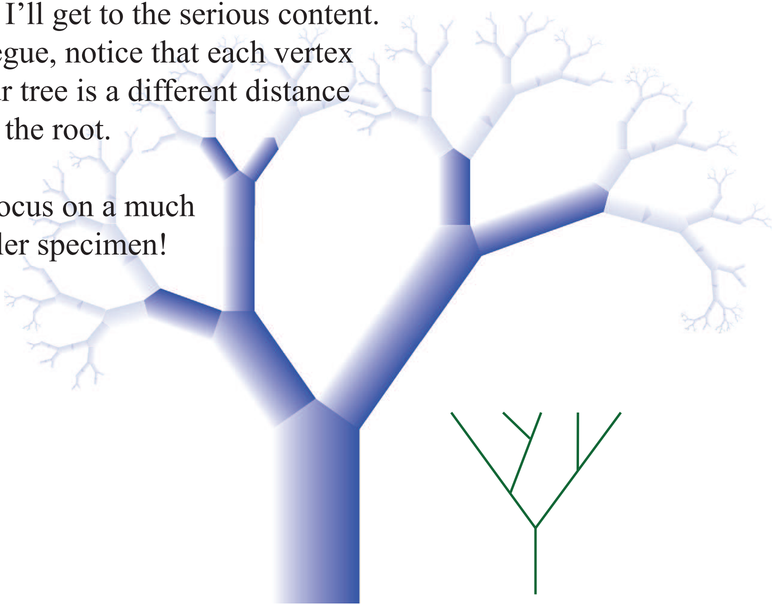
Logarithmic spiral.



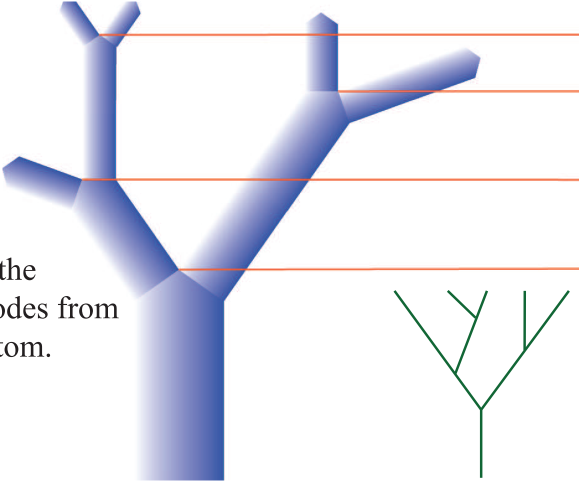
$\mathcal{F}$ is a *leveled* tree.

Now I'll get to the serious content.
To segue, notice that each vertex
of our tree is a different distance
from the root.

We focus on a much
smaller specimen!

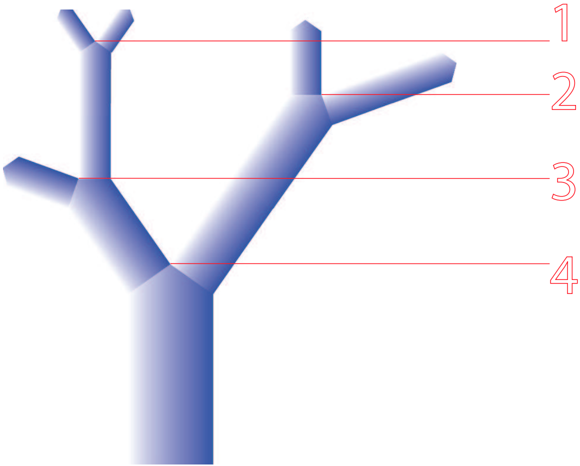


Leveled trees S.

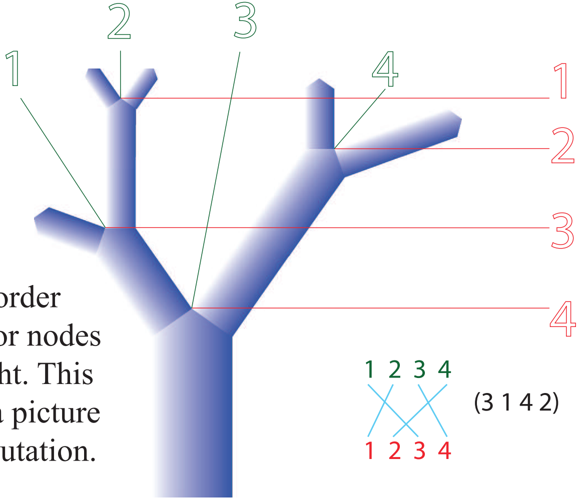


We order the interior nodes from top to bottom.

Leveled trees $\mathcal{S}$.



Leveled trees are permutations S_n.



..and we order the interior nodes left to right. This gives us a picture of a permutation.

1 2 3 4

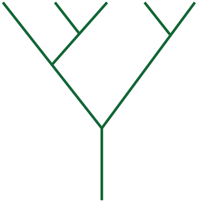
1 2 3 4

(3 1 4 2)

Q. What permutations are subtrees of $\mathcal{F}$?

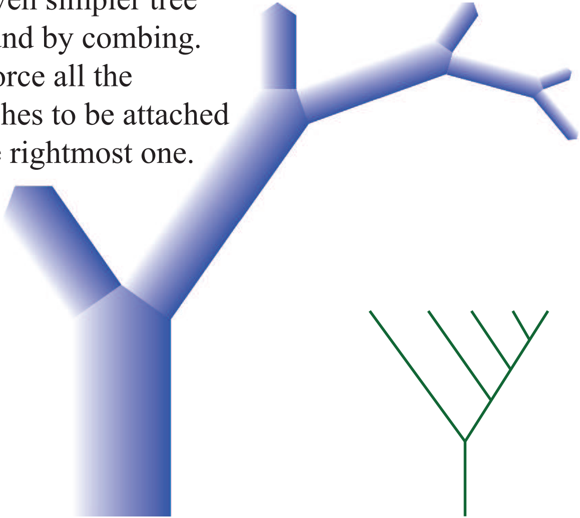
Binary trees B .

Next, a simpler tree is formed by forgetting the vertical levels. We only remember the branching structure.



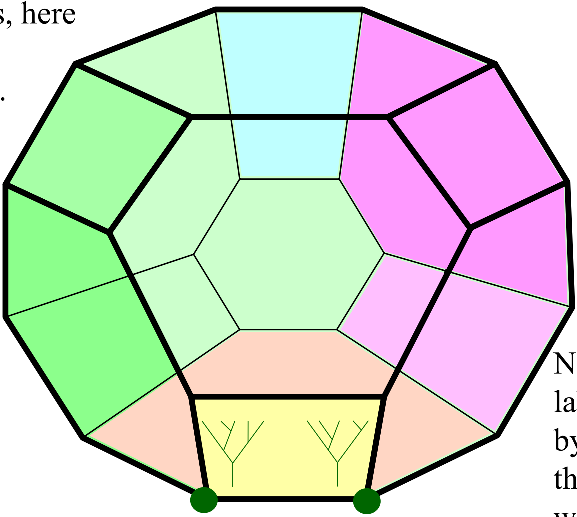
Combed binary trees C .

An even simpler tree
is found by combing.
We force all the
branches to be attached
to the rightmost one.



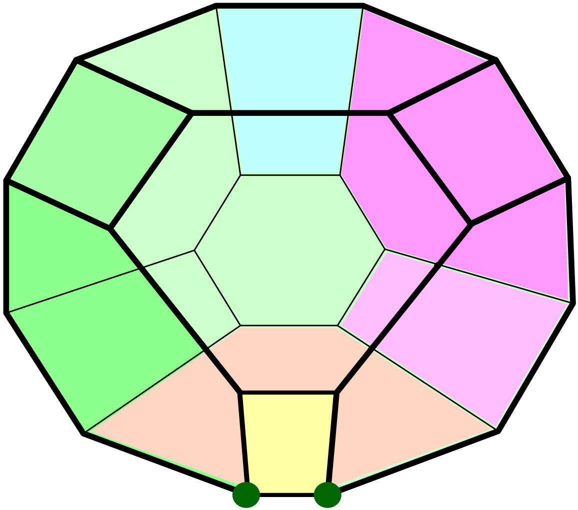
Permutohedron.

This is the 3d permutohedron. Its vertices may be labeled with permutations, here shown as leveled trees.



Notice the two labels only differ by the levels of their nodes. Now we identify them by forgetting.

Tonks cellular projection.



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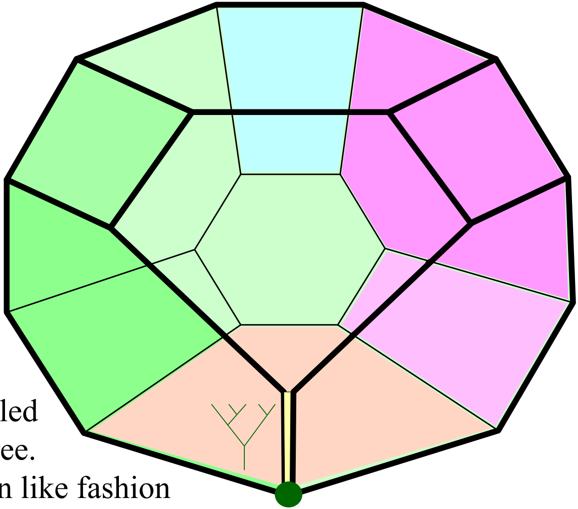
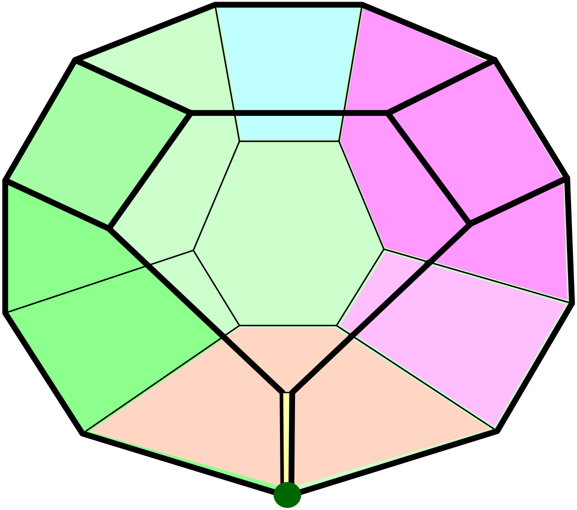


Now one
vertex is labeled
by a binary tree.
We proceed in like fashion
collapsing 5 facets in total.

Tonks cellular projection.



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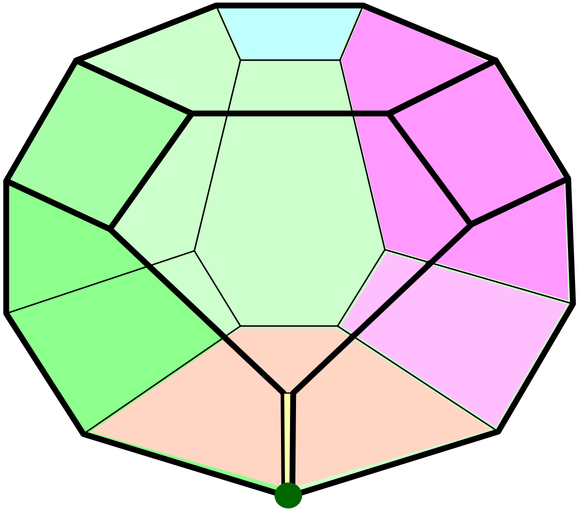

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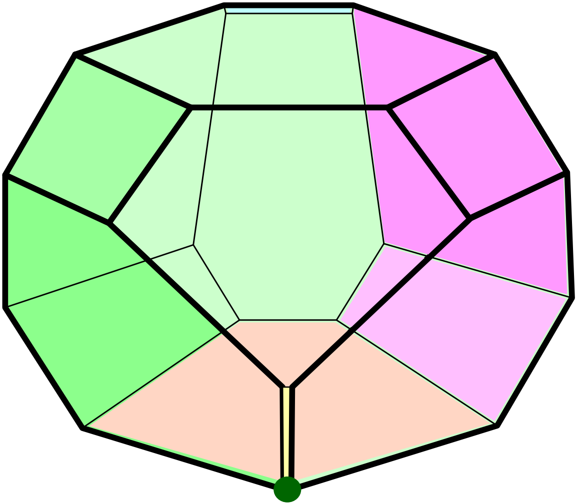

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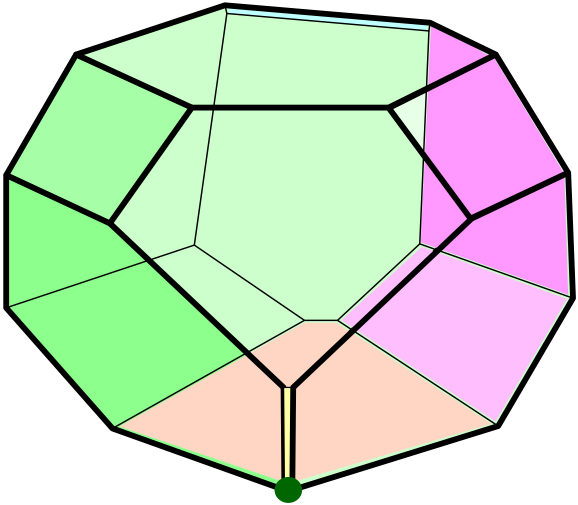

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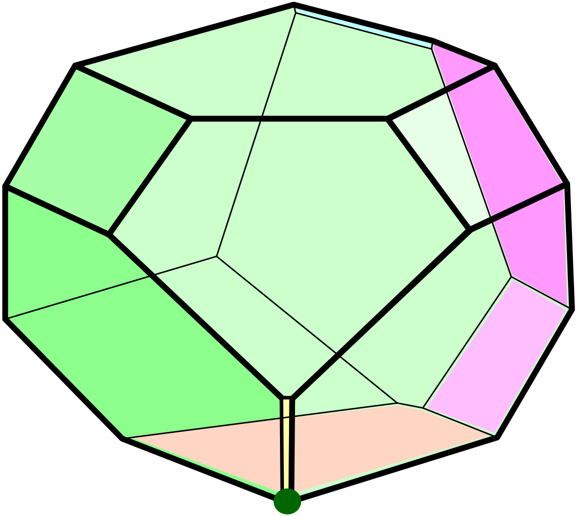

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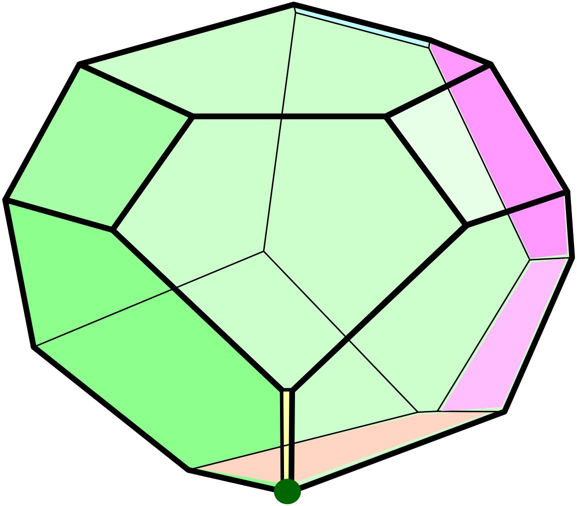

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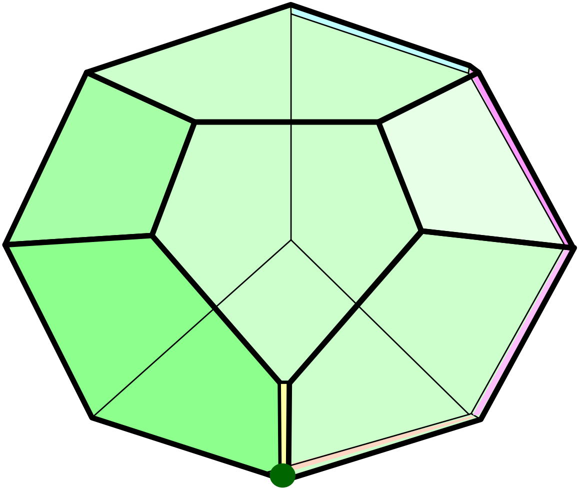
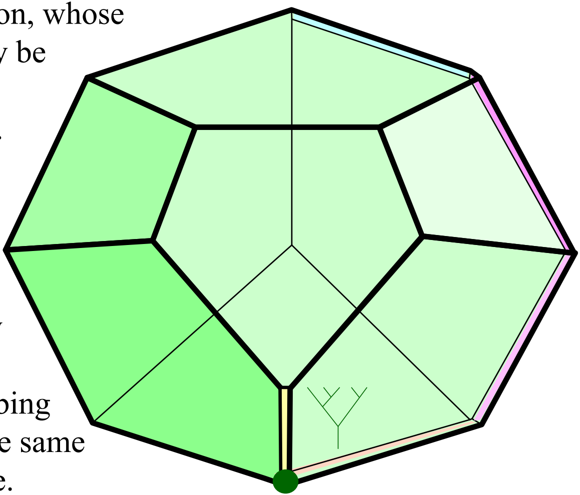


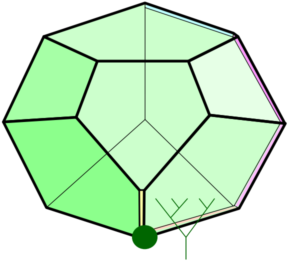
Tonks cellular projection.

The end result is the associahedron, whose vertices may be labeled by binary trees.

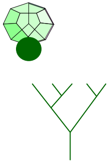


Next we identify any two trees whose combing produces the same combed tree.

Projection...

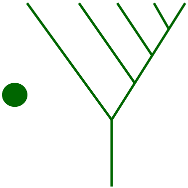


Projection...



Projection...

...resulting in a very simple polytope indeed.



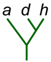
Species.

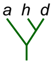
A *species* is a functor from Finite Sets to Finite Sets.

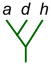
- *Example*: The species $\mathcal{L}$ of Lists takes a set to linear orders of that set.

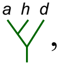
$$\mathcal{L}(\{a, d, h\}) = \{\ a<d<h,\ a<h<d,\ h<a<d,\ h<d<a,\ d<a<h,\ d<h<a\ \}$$

- *Example*: The species Gr of graphs takes a set to the collection of graphs with nodes labeled by that set.
- *Example*: The species $\mathcal{B}$ of binary trees takes a set to trees with labeled leaves.

$\mathcal{B}(\{a, d, h\}) = \{$ 

, 

, ... , 

, 

, ...$\}$

Species composition.

For a finite set U let $\mathsf{P}(U)$ = the set of partitions of U.

$$\mathsf{P}(U) = \{\{U_1, U_2, \ldots, U_n\} \mid U_1 \sqcup \cdots \sqcup U_n = U\}$$

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi \in \mathsf{P}(U)} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

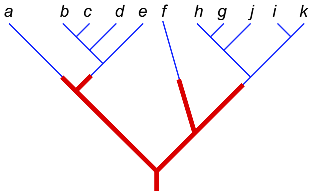
Familiar: also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

$$\gamma : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$$

Leveled tree of trees: indelible grafting.

Example:

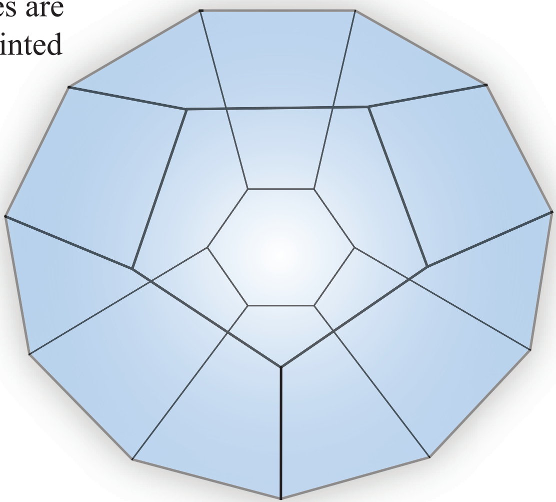
$$(\mathcal{S} \circ \mathcal{B})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

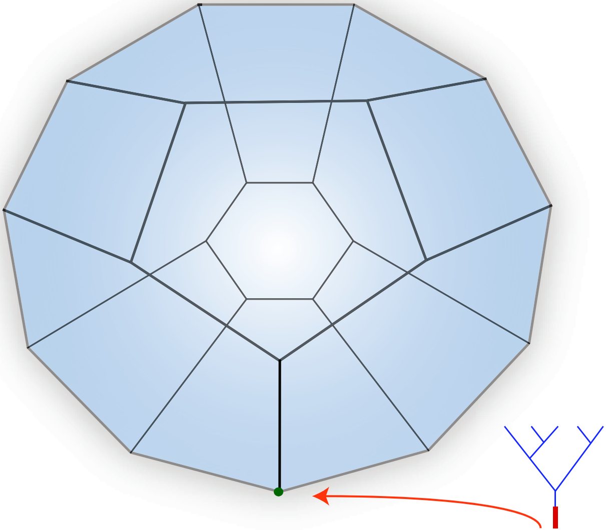


$$\frac{\text{trees}}{\text{leveled tree}}$$

$S \circ \mathcal{B}$

Here is the polytope whose vertices are labeled by painted trees with levels on the bottom, and binary trees grafted on at the top.

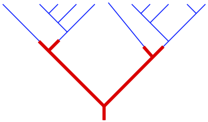




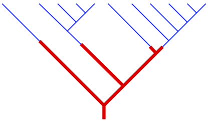
Composing species of tree.

$$
\begin{tikzcd}
 & & \mathcal{S}\circ\mathcal{S} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{B}\circ\mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{S}\circ\mathcal{B} \arrow[dl] \arrow[dr] & \\
\mathcal{C}\circ\mathcal{S} \arrow[dr] & & \mathcal{B}\circ\mathcal{B} \arrow[dl] \arrow[dr] & & \mathcal{S}\circ\mathcal{C} \arrow[dl] \\
 & \mathcal{C}\circ\mathcal{B} \arrow[dr] & & \mathcal{B}\circ\mathcal{C} \arrow[dl] & \\
 & & \mathcal{C}\circ\mathcal{C} & &
\end{tikzcd}
$$

A small commuting diamond

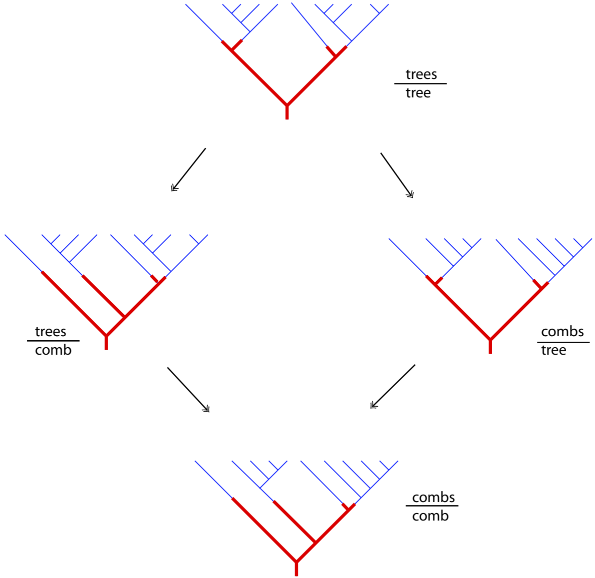


$\dfrac{\text{trees}}{\text{tree}}$

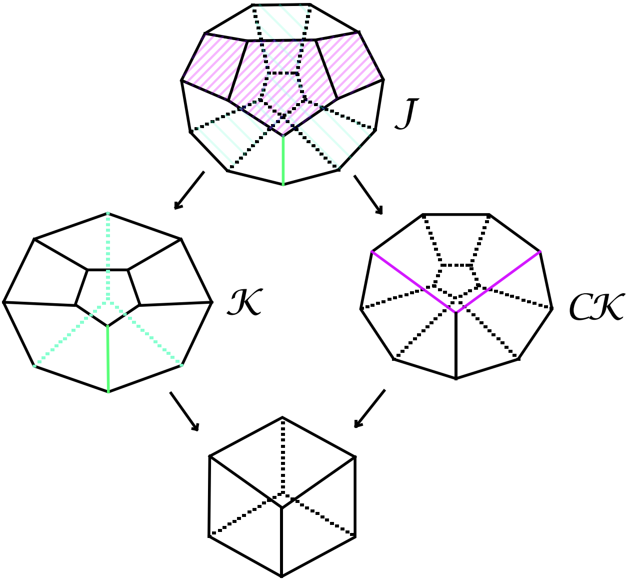


$\dfrac{\text{combs}}{\text{comb}}$

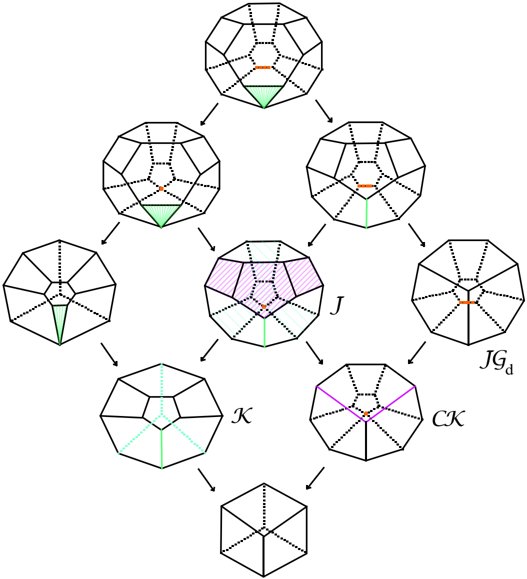
A small commuting diamond



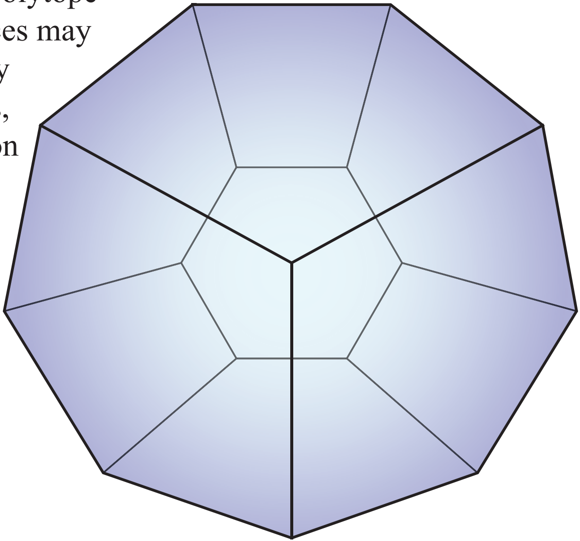
polytopes.



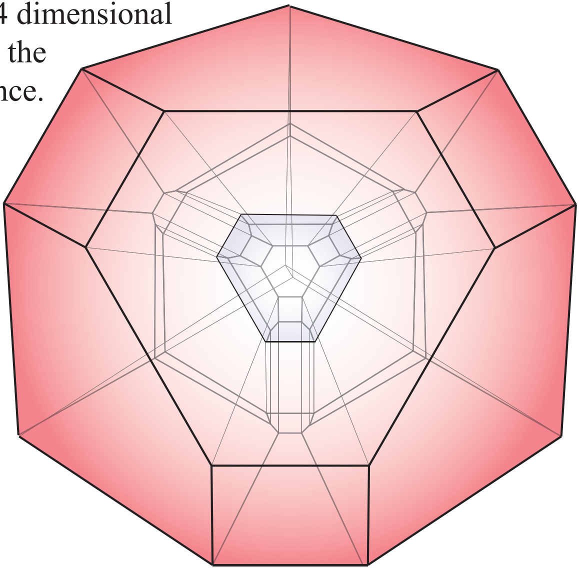
More polytopes.



Here is the polytope whose vertices may be labeled by painted trees, with levels on the bottom and combed trees grafted to the top.



Here is the 4 dimensional specimen in the same sequence.



The trees here have 4 internal nodes.