COMBINATORIC N-FOLD CATEGORIES AND N-FOLD OPERADS

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Abstract. Operads were originally defined as $\mathcal{V}$-operads, that is, enriched in a symmetric monoidal category $\mathcal{V}$. The symmetry in $\mathcal{V}$ is required in order to describe the associativity axiom the operads must obey, as well as the associativity that must hold in the action of an operad on one of its algebras. A sequence of categorical types that filter the category of monoidal categories and monoidal functors is given by Balteanu, Fiedorowicz, Schwanzl and Vogt in [Balteanu et.al, 2003]. The subcategories of **Mon-Cat** are called n-fold monoidal categories. A k–fold monoidal category is n-fold monoidal for all $n \leq k$, and a symmetric monoidal category is n-fold monoidal for all n. We begin with definitions and simple examples of iterated monoidal categories. Then we generalize the definition of operad by defining n-fold operads and their algebras in an iterated monoidal category. We discuss examples of these that live in the previously described categories. Finally we describe the $n-2$-fold monoidal category of n-fold $\mathcal{V}$-operads.

A possible aside: n-fold operads as certain monoids. We may also want to compare n-fold operads with globular operads as defined by Michael Batanin in "Monoidal Globular Categories As a Natural Environment for the Theory of Weak n-Categories." (Advances in Math 136, 39-103, 1998). An MGC is very close to being n-fold monoidal. Interchanges are isos. Globular ("higher order") operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string as in the case of normal operads (which can be seen as parameterizing a collection of n-ary operations for natural numbers n).

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1. Introduction

2. k-fold monoidal categories

This sort of category was developed and defined in [Balteanu et.al, 2003]. The authors describe its structure as arising from its description as a monoid in the category of $(k-1)$-fold monoidal categories. Here is that definition altered only slightly to make visible the coherent associators as in [Forcey, 2004]. In that paper I describe its structure as arising from its description as a tensor object in the category of $(k-1)$-fold monoidal categories.

2.1. Definition. *An n-fold monoidal category is a category $\mathcal{V}$ with the following structure.*

1. *There are n distinct multiplications*

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

for each of which the associativity pentagon commutes

$$\begin{tikzcd}
 & ((U \otimes_i V) \otimes_i W) \otimes_i X \arrow[r, "\alpha^i_{UVW} \otimes_i 1_X"] \arrow[dl, "\alpha^i_{(U \otimes_i V)WX}"'] & (U \otimes_i (V \otimes_i W)) \otimes_i X \arrow[dr, "\alpha^i_{U(V \otimes_i W)X}"] & \\
(U \otimes_i V) \otimes_i (W \otimes_i X) \arrow[drr, "\alpha^i_{UV(W \otimes_i X)}"] & & & U \otimes_i ((V \otimes_i W) \otimes_i X) \arrow[dl, "1_U \otimes_i \alpha^i_{VWX}"] \\
 & & U \otimes_i (V \otimes_i (W \otimes_i X)) &
\end{tikzcd}$$

$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.

2. *For each pair (i, j) such that $1 \leq i < j \leq n$ there is a natural transformation*

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) *Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$*

(b) *External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$*

(c) *Internal associativity condition: The following diagram commutes*

$$\begin{tikzcd}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) \arrow[r, "\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}"] \arrow[d, "\alpha^i"] & \big((U \otimes_i W) \otimes_j (V \otimes_i X)\big) \otimes_i (Y \otimes_j Z) \arrow[d, "\eta^{ij}_{(U \otimes_i W)(V \otimes_i X)YZ}"] \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) \arrow[d, "1_{U \otimes_j V} \otimes_i \eta^{ij}_{WXYZ}"] & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \arrow[d, "\alpha^i \otimes_j \alpha^i"] \\
(U \otimes_j V) \otimes_i \big((W \otimes_i Y) \otimes_j (X \otimes_i Z)\big) \arrow[r, "\eta^{ij}_{UV(W \otimes_i Y)(X \otimes_i Z)}"] & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes

$$\begin{tikzcd}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) \arrow[r, "\eta^{ij}_{(U\otimes_j V)W(X\otimes_j Y)Z}"] \arrow[d, "\alpha^j \otimes_i \alpha^j"] & \big((U \otimes_j V) \otimes_i (X \otimes_j Y)\big) \otimes_j (W \otimes_i Z) \arrow[d, "\eta^{ij}_{UVXY} \otimes_j 1_{W\otimes_i Z}"] \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) \arrow[d, "\eta^{ij}_{U(V\otimes_j W)X(Y\otimes_j Z)}"] & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\alpha^j"] \\
(U \otimes_i X) \otimes_j \big((V \otimes_j W) \otimes_i (Y \otimes_j Z)\big) \arrow[r, "1_{U\otimes_i X} \otimes_j \eta^{ij}_{VWYZ}"] & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{tikzcd}$$

(e) Finally it is required for each triple (i, j, k) satisfying $1 \leq i < j < k \leq n$ that the giant hexagonal interchange diagram commutes.

$$\begin{tikzcd}[column sep=tiny]
& ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}"'] \arrow[dr, "\eta^{ij}_{(A\otimes_k A')(B\otimes_k B')(C\otimes_k C')(D\otimes_k D')}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\eta^{ik}_{(A\otimes_j B)(A'\otimes_j B')(C\otimes_j D)(C'\otimes_j D')}"'] & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\eta^{jk}_{(A\otimes_i C)(A'\otimes_i C')(B\otimes_i D)(B'\otimes_i D')}"] \\
& ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}$$

The authors of [Balteanu et.al, 2003] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \ldots = \otimes_n = \otimes$$

and defining (associators added by myself)

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation.

3. Examples of iterated monoidal categories

1. The first of our examples is the 2-fold monoidal category $\hat{G}$ that is formed from any totally (pre)-ordered semigroup $\{G \leq\}$ such that the identity element $e \in G$ is less than every other element, $g \in G \Rightarrow e \leq g$. Inverses thus cannot be present. Then the objects are elements of G. Morphisms are given by the ordering; there exists exactly one morphism from an object to a greater or equal object. The products are given by max and the group operation: $a \otimes_1 b = \max(a, b)$ and $a \otimes_2 b = ab$. The shared two-sided unit for these products is the identity element e. The products are both strictly associative and functorial since if $a \leq b$ and $a' \leq b'$ then $aa' \leq bb'$ and $\max(a, a') \leq \max(b, b')$. The interchange natural tranformations exist since $\max(ab, cd) \leq \max(a, c) \max(b, d)$. This last theorem is easily seen by checking the four possible cases: $\{c \leq a, d \leq b\}$; $\{c \leq a, b \leq d\}$; $\{a \leq c, d \leq b\}$; $\{a \leq c, b \leq d\}$; or by the quick argument that

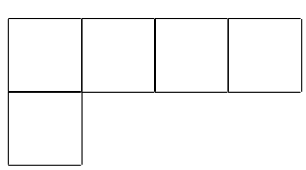
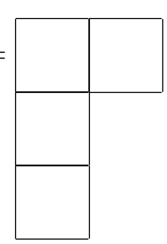
   $a \leq max(a, b)$ and $c \leq max(c, d)$ so

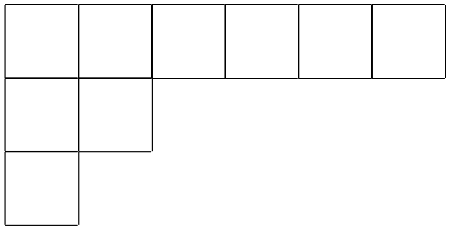
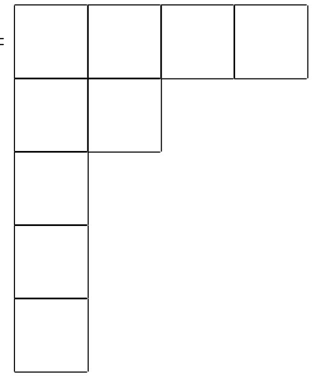
   $a + c \leq max(a, b) + max(c, d)$ and similarly

   $b + d \leq max(a, b) + max(c, d)$.

   The internal and external unit and associativity conditions of Definition 2.1 are all satisfied due to the fact that there is only one morphism between two objects. Examples of such semigroups are the nonzero integers under addition and the braids on n strands with only right-handed crossings.

2. The preorder structure is very convenient in creating examples, but the use of max requires a total ordering and is thus somewhat limiting. To find examples that are more natural we turn to geometric and combinatorial posets that allow a use of addition in each product. The first such category is that whose objects are regular graphs, by which we mean the shapes of Young tableaux. These can be presented by a decreasing sequence of nonnegative integers in two ways: the sequence that gives the heights of the columns or the sequence that gives the lengths of the rows. We let $\otimes_2$ be the product which adds the heights of columns of two tableaux, $\otimes_1$ adds the length of rows.

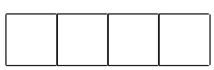
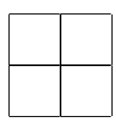
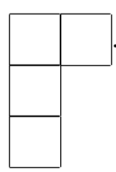
   $A =$  and $B =$ 

then $A \otimes_1 B =$  and $A \otimes_2 B =$ 

There are several possibilities for morphisms. We can create a category equivalent to the non-negative integers in the previous example by pre-ordering the tableaux by height. Here the height $h(A)$ of the tableau is the number of boxes in its leftmost column, and we say $A \leq B$ if $h(A) \leq h(B)$. Two tableaux with the same height are isomorphic objects, and the one-column stacks form both a full subcategory and a skeleton of the category. Everything works as for the previous example since $h(A \otimes_2 B) = h(A) + h(B)$ and $h(A \otimes_1 B) = max(h(A), h(B))$.

3. Alternatively for the morphisms we can introduce a partial ordering in which each subset of tableau shapes described as being the shapes which possess a total number n of blocks is a poset. In fact, tableau shapes of different total number of blocks will not be comparable. We will number the rows $1, 2, \ldots$ starting at the top and the columns starting at the left. The height of column i is denoted $h(i)$. Within a subset of block number n tableau shapes we say A is less than B if there exists a series of what we will describe as decreasing knight moves (k, i) that transform A into B. A decreasing knight move subtracts a box from a column (k) in A which is higher than its following column, i.e. $h(k+1) < h(k)$, and adds a box to an earlier column $(i \leq k)$ which is less high than its own preceding column, i.e. $h(i) < h(i-1)$. Equivalently such a move subtracts a box from the row $p = h(k)$ which is longer than its following row, and adds it to a later row $(q = h(i) + 1)$ which is shorter than its own preceding row.

   For example:

    $\leq$  $\leq$ .

   This definition of order forms a poset of tableau shapes of a given number of blocks. The relation is reflexive since we can remove a box and then add a box to the recently vacated position to see that $A \leq A$. It is antisymmetric since if in transforming A to B we increase the height of a given column, then the only way to decrease that column to its original height is to increase the height of another column to its left. Therefore since the shape has a first column there is no relation from B to

A unless $A = B$. The relation is transitive since we can combine the two series of moves implied by $A \leq B$ and $B \leq C$ to get one long series of moves that implies $A \leq C$. Note that when we consider the morphisms of the category of tableau shapes to be just the ordering on shapes we ignore the multiple ways that one shape may be transformed to another by decreasing knight moves. for instance the series (3,2),(2,1) decribes the same relation as (3,1). The order the series is given in is often important due to the need for each move to be allowable, i.e. to result in a valid tableau shape.

The ordering is preserved by our vertical and horizontal stacking, so we have tensor products of morphisms. Specifically if $A \leq B$ and $C \leq D$ then $A \otimes_i C \leq B \otimes_i D$. This is seen for vertical tensoring by noting that if $\{(i_n, k_n)\}_{n=1}^{p}$ is a series of moves transforming A to B and $\{(i_m, k_m)\}_{m=p+1}^{q} : C \to D$ then $\{(i_j, k_j)\}_{j=1}^{p+q}$ is a series of moves from $A \otimes_2 C$ to $B \otimes_2 D$. The argument is the same for horizontal tensoring, requiring only that we describe the individual moves as being on rows with the equivalent description given above.

Also of course identity morphisms are equality and composition is transitivity of the ordering. Thus functoriality for the tensor products is trivial, based on the fact that there is at most one morphism between two objects.

The key thing to prove is that our η will always exist, i.e. that $(A \otimes_2 B) \otimes_1 (C \otimes_2 D)$ is always less than $(A \otimes_1 C) \otimes_2 (B \otimes_2 D)$. This follows from considering what differences in where blocks from the operands end up in the two products. The blocks from A will end up in the top left of both products, in their original configuration. In both products the blocks from C will end up at their original vertical position. In the second product a given block from C will be potentially further left than in the first product, since in the first product blocks from B may end up horizontally between those from A and those from C. In both products the blocks from B will end up at their original horizontal position. In the second product a given block from B will end up potentially further down than in the first product because in the second product blocks from C may interpose. In the second product a given block from D will end up ?

4. n-fold operads

Let $\mathcal{V}$ be an n-fold monoidal category as defined in the last section.

4.1. Definition. *An operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, a right action by the symmetric group Σ_j on $\mathcal{C}(j)$ for each j and composition maps in $\mathcal{V}$*

$$\gamma : \mathcal{C}(k) \otimes_1 \mathcal{C}(j_1) \otimes_2 \ldots \otimes_2 \mathcal{C}(j_k) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $m_s = \sum_{n=1}^{s} j_n$ and let $h_s = \sum_{n=1+g_{s-1}}^{g_s} i_n$.*

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s)\right) \otimes_1 \left(\bigotimes_{r=1}^{j} {}_2\mathcal{C}(i_r)\right) \arrow[r, "\gamma\otimes_1\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes_1\eta^{12}"'] & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{r=1}^{j} {}_2\mathcal{C}(i_r)\right) \arrow[d, "\gamma"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{q=1}^{j_s} {}_2\mathcal{C}(i_{q+g_{s-1}})\right)\right) \arrow[r, "\mathrm{id}\otimes_1(\otimes_2^k\gamma)"'] & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(h_s)\right) \arrow[u, "\gamma"']
\end{tikzcd}$$

Equivariance and respect of units are required just as in the symmetric case.

(figure: associativity of composition of n-fold operads)

A similar translation defines algebras over an n-fold operad. The product of two operads is given by

$$(\mathcal{C} \otimes_i^{(1)} \mathcal{D})(j) = \mathcal{C}(j) \otimes_{i+2} \mathcal{D}(j).$$

If A is an algebra of $\mathcal{C}$ and B is an algebra of $\mathcal{D}$ then, for example, $A \otimes_3 B$ is an algebra for $\mathcal{C} \otimes_1^{(1)} \mathcal{D}$.

For an example of an operad in a 2-fold monoidal category we turn to the category of tableaux shapes with preorder given by height of the shape.

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