Linear Algebra

Chp. 1

Examples:

3, 5, -1, $\frac{1}{2}$, 1 scalar coefficients

(1)

system of equations

$$\begin{cases} 3x + 5y - z = 2 \\ x - y = 0 \\ y + \frac{1}{2} z = 1^{*} \end{cases}$$

x, y, z scalar variables

2, 0, 1 scalar constants

homogeneous linear equation

*affine linear equation

(2)

homogeneous system of linear equations

$$\begin{cases} 3x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_3 - x_2 = 0 \end{cases}$$

(Alternate scalar variables $x_1, x_2, x_3, \ldots$)

Solve: simultaneous solution (x_1, x_2, x_3)

(1) Subtract equations:

$$\begin{array}{r} x_1 + x_3 = 0 \\ -(x_1 + x_3 - x_2 = 0) \\ \hline \Rightarrow x_2 = 0 \end{array}$$

(2) Substitute back:

$3x_1 - 0 = 0 \Rightarrow x_1 = 0$ | $0 + x_3 = 0$, $x_3 = 0$

$(x_1, x_2, x_3) = (0,0,0)$ makes all 3 true.

Solving with a matrix of coefficients

→ same as combining (subtracting) equations to eliminate variables

→ Three allowed moves, to break it down:

Row Reduction Moves

① Switch 2 rows (just like reordering equations)

$R_3 \leftrightarrow R_5$

② Replace a row with a multiple of itself $R_5 \leftarrow -\tfrac{2}{3} R_5$

③ Replace a row with a combination of itself with another row.

$R_7 \leftarrow R_7 + 2R_5$

Ex:
$$\begin{aligned} 3x_1 - x_2 &= 0 \\ x_1 + x_3 &= 2 \\ x_1 + x_3 - x_2 &= 1 \end{aligned}$$

Matrix A

Rows: Equations

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

Columns: coeffs of x_1, x_2, x_3 | augmented constants

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & -1 & 0 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -3 & -3 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{array}\right]$$

$R_3 \leftarrow -\tfrac{1}{3} R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array}\right]$$

... so $\begin{cases} x_1 = 1/3 \\ x_2 = 1 \\ x_3 = 5/3 \end{cases}$

Check: these make all three original equations true

$$3\left(\tfrac{1}{3}\right) - 1 = 0 \quad \checkmark$$
$$1/3 + 5/3 = 2 \quad \checkmark$$
$$1/3 + 5/3 - 1 = 1 \quad \checkmark$$

→ **Geometric meaning and 3 types of solution.**

For any number of linear equations, with any number of variables, there can only be one of 3 possibilities
- → Zero solutions
- → One solution
- → ∞ solutions

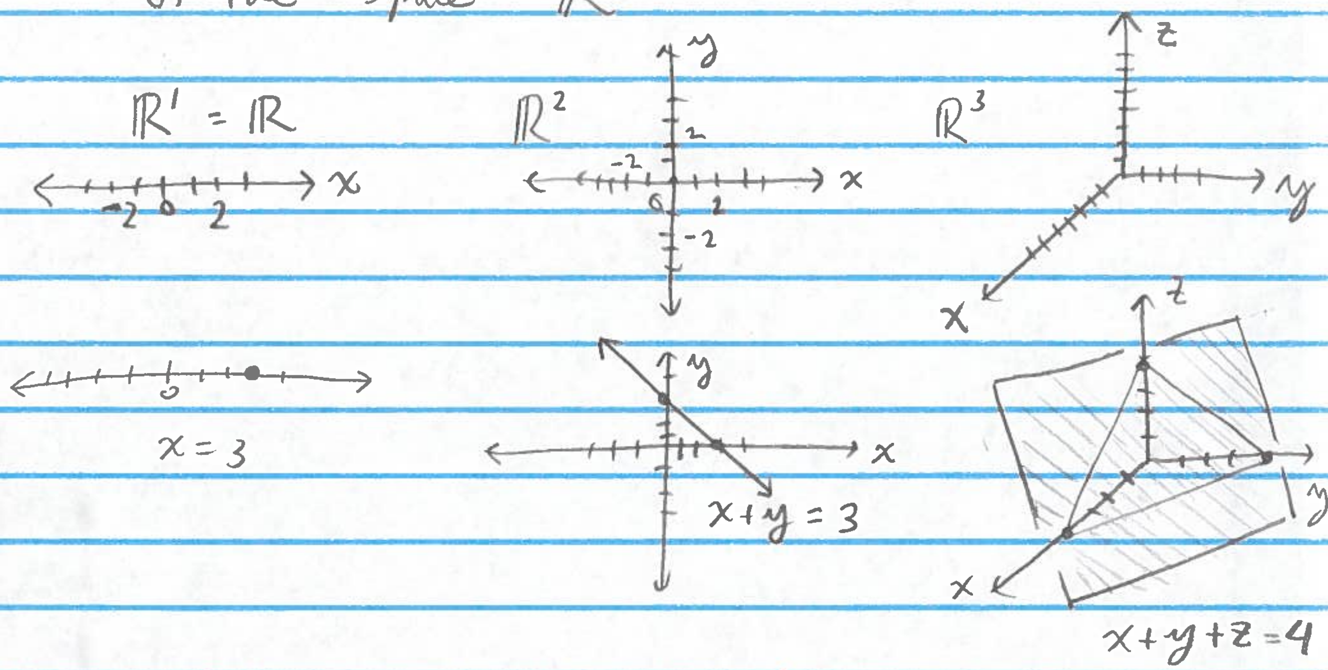
Why?

1) dimension of a space is a counting number 0, 1, 2, 3, 4, 5, 6, ... which describes a collection of points. It tells:
   - → the number of independent, free decisions for perpendicular motions
   - → the number of real numbers needed to describe a single point location in that space

degrees of freedom

coordinates, # axes R

2) Each single (affine) linear equation describes the points in a hyperplane of the space $\mathbb{R}^d$

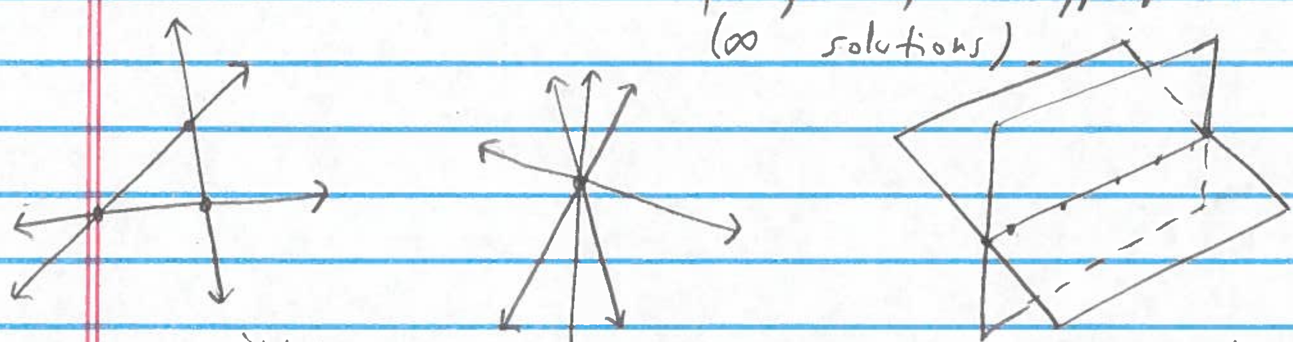


3) Several hyperplanes in $\mathbb{R}^d$ can either:

→ not share a common point of intersection (0 solutions)

→ share exactly one common point (needs at least d hyperplanes but no guarantee)

→ OR, all intersect in a lower-dimensional plane, line, or hyperplane (∞ solutions)



no point in common - no solution

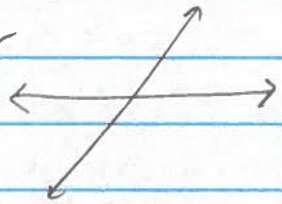
one point solution

line of solution points → ∞ solutions

Goal # 1 : be able to look at equations & know what the picture is, and vice versa : look at the picture and know things about the equations.

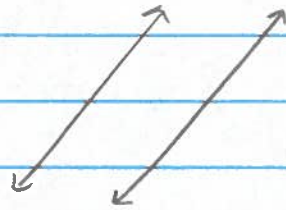
Two lines in $\mathbb{R}^2$:

either



crossing = different slopes

OR



... 2 ways!

parallel = same slope

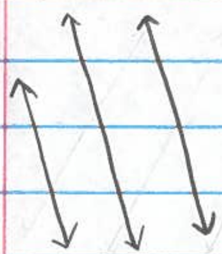
ex:

$y = \frac{2}{3}x + 1$, $y = \frac{2}{3}x - 5$

$3y - 2x = 3$, $3y - 2x = -15$

Slope = coefficients

Three lines in $\mathbb{R}^2$:



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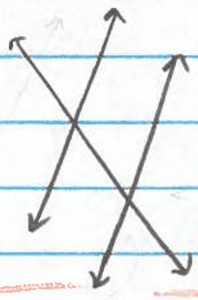

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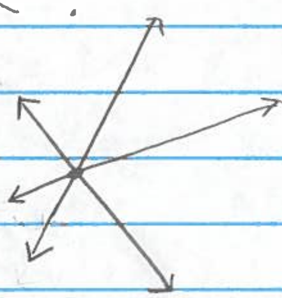

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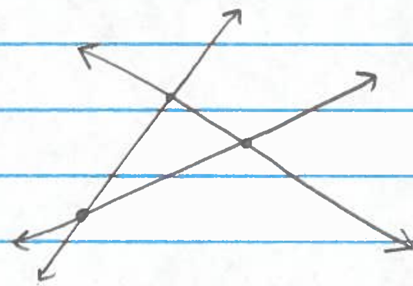


... 4 ways!

Four lines: 9 different pictures!

Five lines : 47

Six lines : 791

Seven lines : 37,830

Eight lines : 4,134,940

Nine lines : Unknown

In general n lines ?

→ open research question.

Back to solution method: matrix $A_{m \times n}$ has • m rows
• n columns

→ Recall, from a system of (affine) linear equations we write a matrix (augmented) of scalar coefficients and solve using $\boxed{\text{row reduction moves.}}$

→ Two matrices are $\boxed{\text{row equivalent,}}$ $A \sim B$, when you get from A to B by row reduction moves.

→ a $\boxed{\text{pivot}}$ in a matrix B is a "1" in a row of B with • all "0"s before it, in its row
and • all "0"s above and below, in its column

→ The $\boxed{\text{row reduced echelon form}}$ of A (r.r.e.f.) is a matrix $B \sim A$ where each row of B is either all 0's or has a pivot 1 and the pivots in earlier (higher) rows are in earlier (further left) columns, and "0" rows are at bottom.

ex:

$$B = \left[\begin{array}{cccccccccccc} 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 5 & -2 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

lots of numbers after (but not above or below) pivots

3 pivots, and one row all "0"

→ If $A \sim B$ in r.r.e.f, a $\boxed{\text{pivot column of } A}$ is a column of A where that column in B has a pivot

→ a system is solved when its matrix A of coefficients is put in r.r.e.f. B (the moves are also done on the augmented column of constants, but that column doesn't have to be in r.r.e.f.)

Then the r.r.e.f. B is returned to equations as follows:

- each column corresponds to an original variable x, y, z or $x_1, x_2, x_3, x_4, \ldots$ (except the augment column, which is constants).
- each pivot in B is a determined variable of the solution: it will be on the left of an equation.
- each non-pivot column of B is a free variable, it can be any real number.

ex:

$$\begin{array}{c|c} \overbrace{\hphantom{0\ 1\ 0\ 0\ -2\ 0\ 5}}^{B} & \text{(augment)} \end{array}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 0 & 0 & -2 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$$\begin{array}{ccccccc} x_1 & \boxed{x_2} & x_3 & \boxed{x_4} & x_5 & x_6 & x_7 \end{array}$$

pivots

$x_1 = x_1$ (free!)

→ $x_2 - 2x_5 + 5x_7 = 3$

$x_3 = x_3$ (free!)

→ $x_4 + x_5 + 3x_7 = 1/4$

$x_5 = x_5$ (free!)

$x_6 = x_6$ (free!)

$x_7 = x_7$ (free!)

↑ Five free variables = 5 dimensional solution

Next we solve the non-free equations, one for each pivot.

$$\boxed{\begin{aligned} x_1 &= x_1 \\ x_2 &= 3+2x_5-5x_7 \\ x_3 &= x_3 \\ x_4 &= 1/4 - x_5 - 3x_7 \\ x_5 &= x_5 \\ x_6 &= x_6 \\ x_7 &= x_7 \end{aligned}}$$

This is the final general solution. There are ∞ solution points since choosing any values for the free variables gives a specific solution.

Specific solution example:

$x_1 = 0 \leftarrow$ pick any!

$x_2 = ? \leftarrow$ find: $3+2(-2)-5(0) = -1$

$x_3 = 1 \leftarrow$ pick any!

$x_4 = ? \leftarrow$ find: $\frac{1}{4} - (-2) - 3(0) = \frac{9}{4}$

$x_5 = -2 \leftarrow$ pick any!

$x_6 = 3 \leftarrow$ pick any!

$x_7 = 0 \leftarrow$ pick any!

$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= -1 \\ x_3 &= 1 \\ x_4 &= 9/4 \\ x_5 &= -2 \\ x_6 &= 3 \\ x_7 &= 0 \end{aligned}}$$

→ Other possibilities:

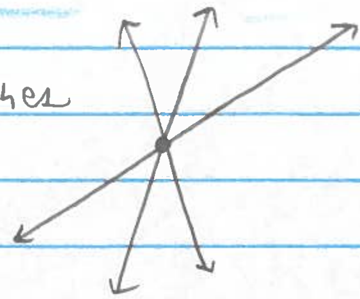
- only one unique solution: when every column is a pivot column, and any row in B of "0"s ends in an augment of 0 in that row.
- Zero solutions: when there is a row of "0"s in B but the augment column is not 0 in that row.

ex:

$$\left[\begin{array}{ccccccc|c} \multicolumn{7}{c}{\overbrace{\hphantom{0\;1\;0\;0\;0\;2\;0}}^{B}} & \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right] \rightarrow 0 = 5 \quad \boxed{\text{no solution}}$$

So now we know some facts to conclude:

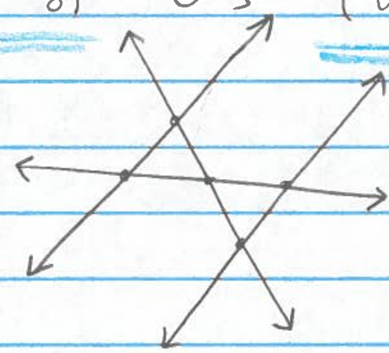
This set of lines in $\mathbb{R}^2$



has only one solution (x,y) So...

...it has a matrix $A_{3\times 2}$, (3 equations, 2 variables)
3 ↑ rows, 2 ↑ columns
(with a extra augmented column)
and together they row reduce to r.r.e.f. B,
(with augment),
that has 2 pivots (both columns)
and a row of "0"s (with 0 in augment).

This set of lines in $\mathbb{R}^2$



has no solutions!

... So it has a matrix $A_{4\times 2}$ (4 equations, 2 vars)
(with an extra augment column)
which row reduces to r.r.e.f. B,
(with augment)
that has at least one row of "0"s, with a nonzero entry in the augment of that row.

... And, it does have 2 pivots. Why? Just pick two crossing lines to be two rows. One solution!