Cofree compositions of coalgebras: Trees, polytopes and indelible grafting.

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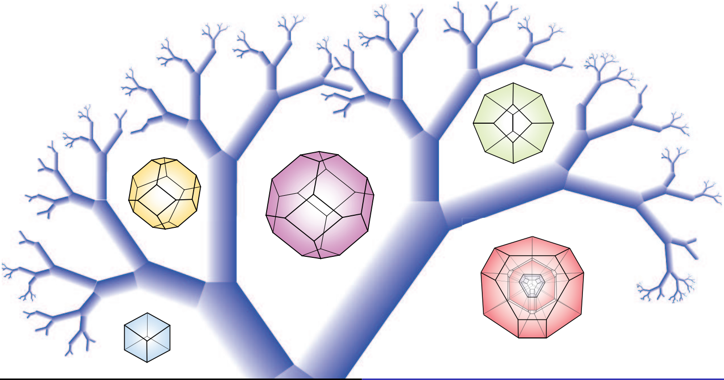
Frank Sottile, Texas A&M U.



Our slogan.

"Composing coalgebraic species preserves their niceness."

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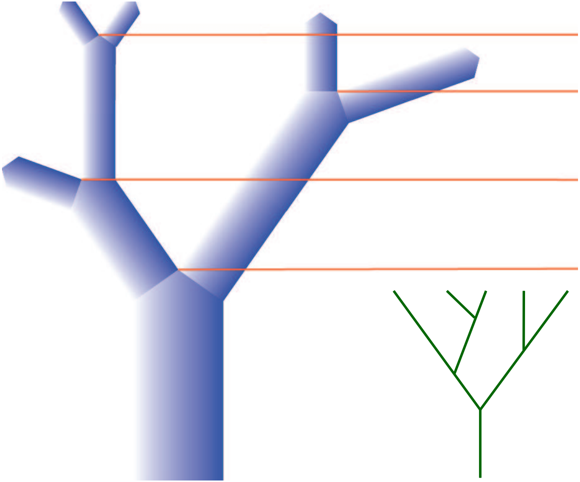
For this talk, nice properties of coalgebras will be:

1. Cofree-ness,
2. Hopf-ness,
3. Polytopal-ness.

But first, our cast of characters:

Leveled trees $\mathcal{S}$.



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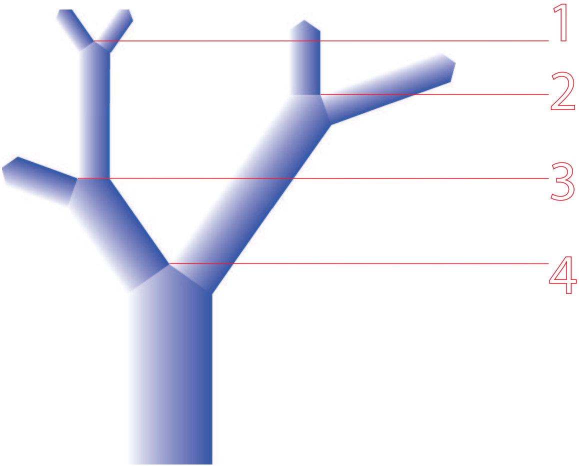
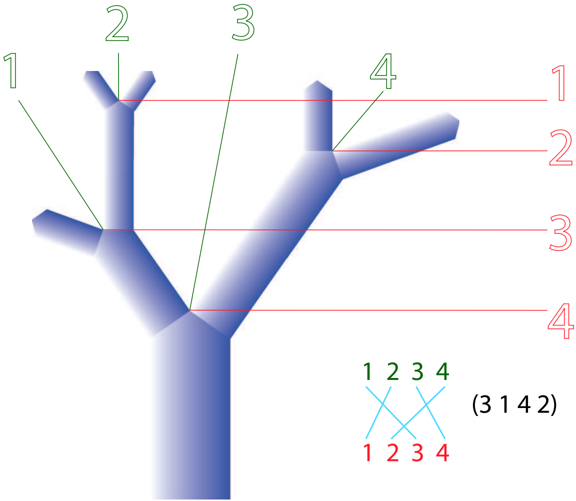
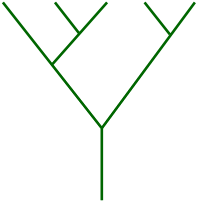


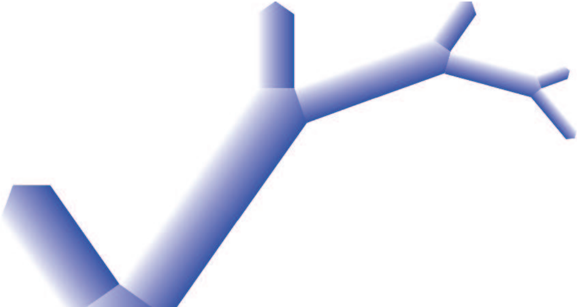
Leveled trees are permutations $\mathcal{S}_n$.

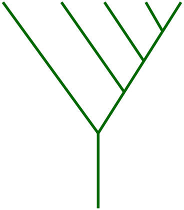


Binary trees $\mathcal{B}$.



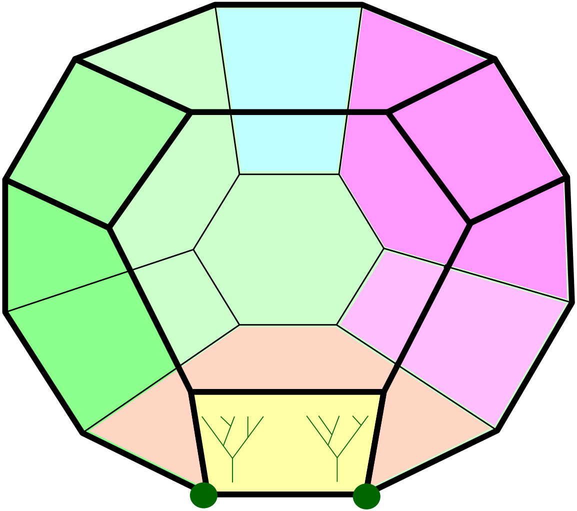
Combed binary trees $\mathcal{C}$.



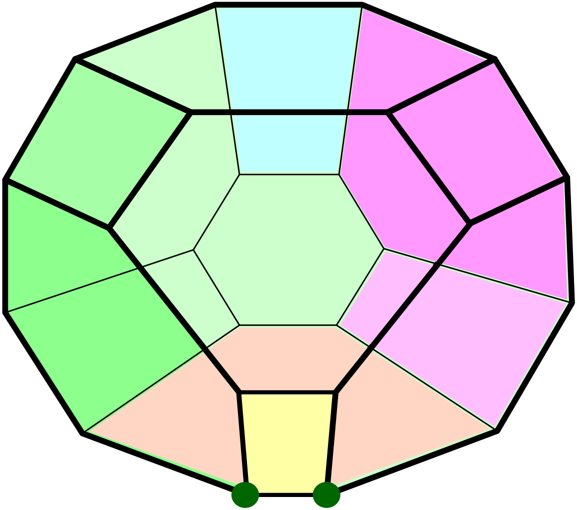


Our characters as Polytopes.

Permutohedron.



Tonks cellular projection.



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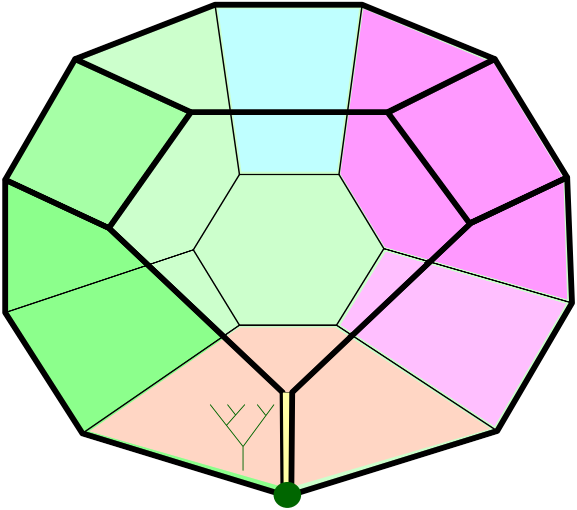

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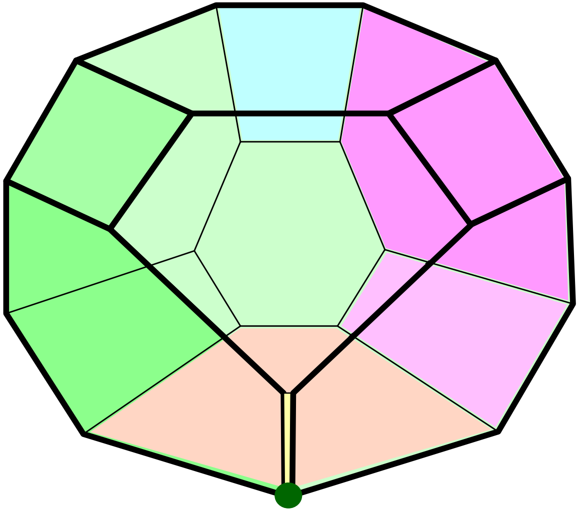

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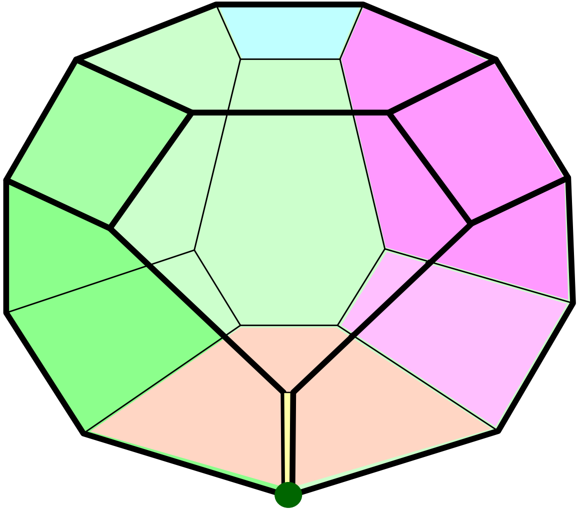

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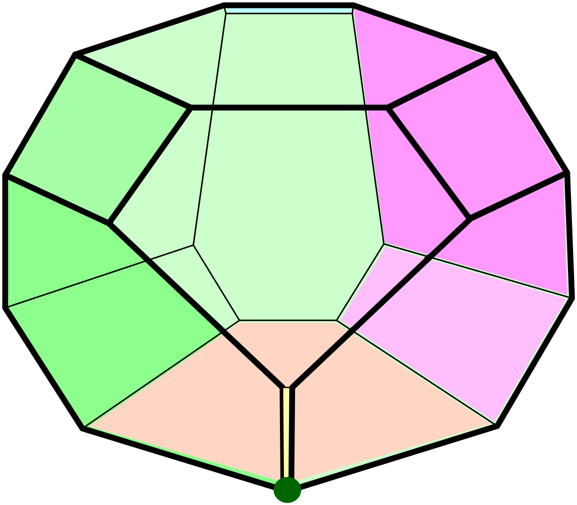

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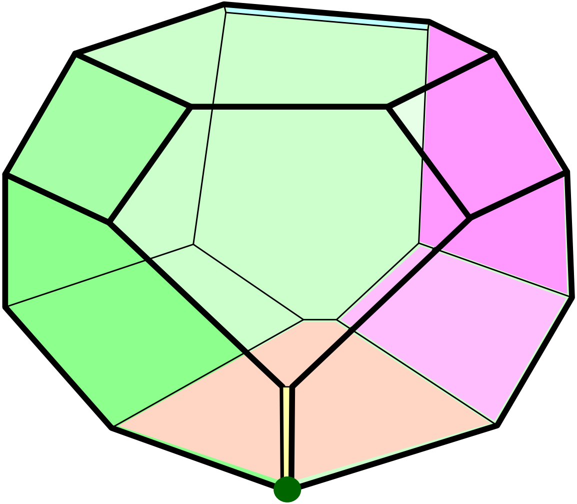

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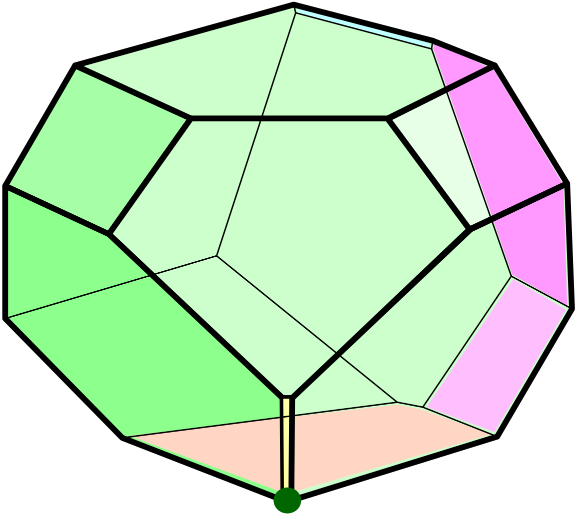

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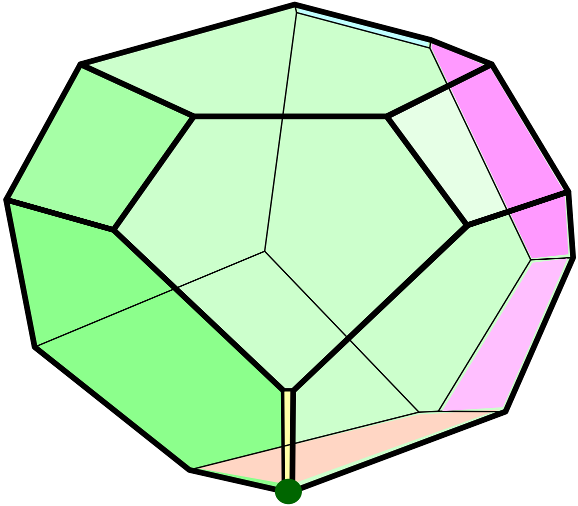

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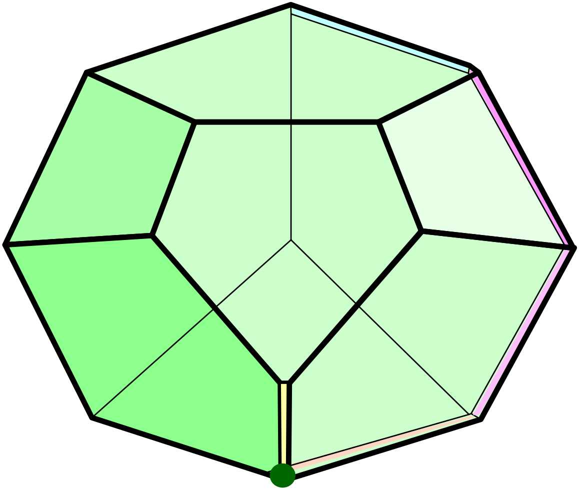
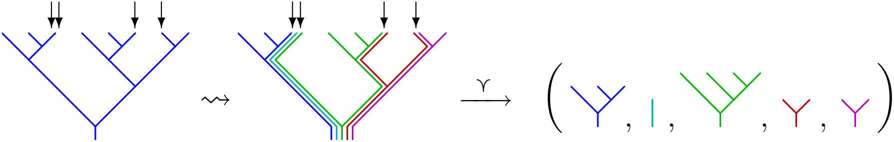


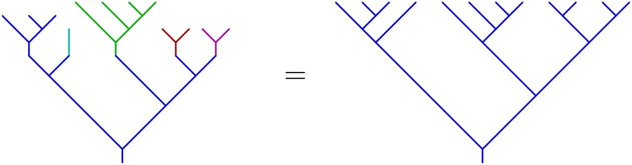
Our characters as graded Hopf algebras.

A Hopf algebra of binary trees.

Two operations on trees: splitting



and grafting:



Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n+1$ leaves, denoted $\mathcal{Y}_n$.

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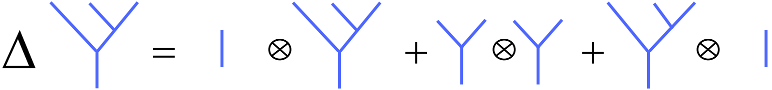

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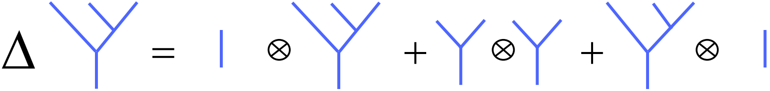
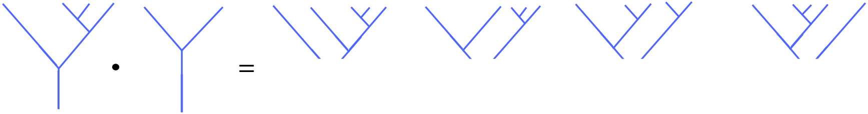


Here is how to multiply two trees:



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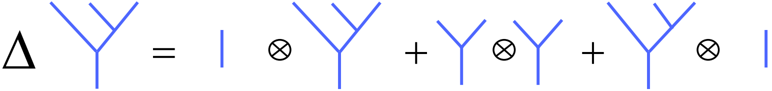

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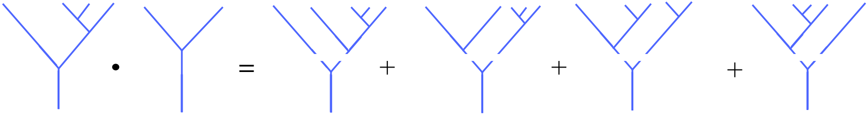

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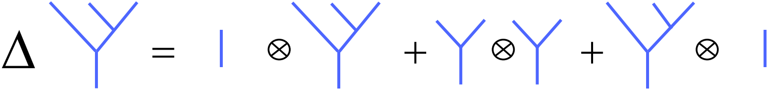

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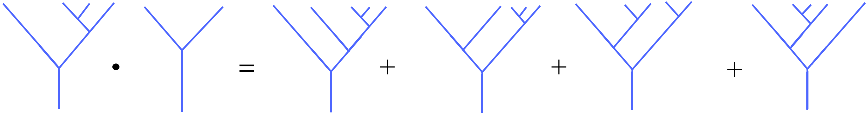
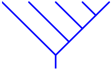


The way to comultiply in $\mathfrak{C}Sym$.

We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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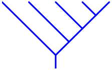
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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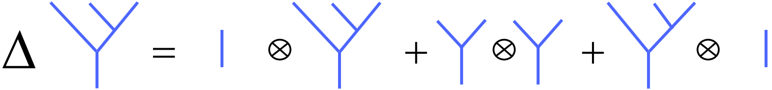

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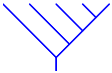
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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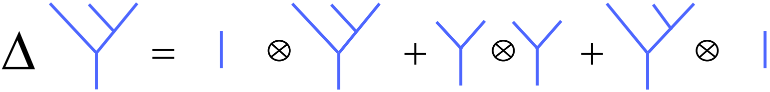

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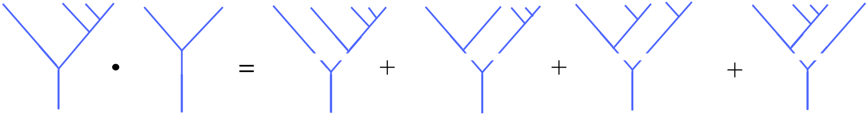

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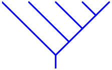
We draw basis elements of $\mathfrak{C}Sym$ as *right combs*. $x^{(4)} =$  .

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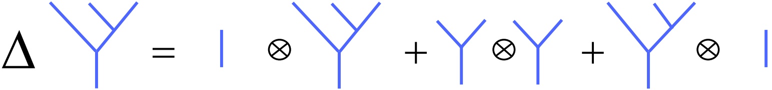

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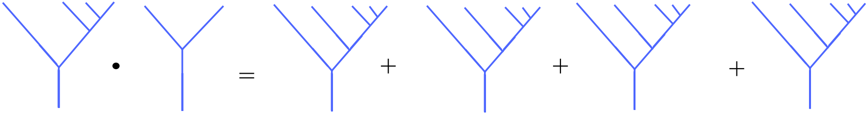


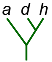
Species.

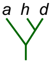
A *species* is an endofunctor of Finite Sets with bijections.

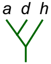
- *Example*: The species $\mathcal{L}$ of lists takes a set to linear orders of that set.

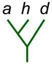
$$\mathcal{L}(\{a, d, h\}) = \{\ {\scriptstyle a<d<h,\ a<h<d,\ h<a<d,\ h<d<a,\ d<a<h,\ d<h<a}\ \}$$

- *Example*: The species $\mathcal{B}$ of binary trees takes a set to trees with labeled leaves.

$\mathcal{B}(\{a, d, h\}) = \{$ 

, 

, ... , 

, 

, ...$\}$

Species composition.

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

where the union is over partitions of U into any number of nonempty disjoint parts.

$$\pi = \{U_1, U_2, \ldots, U_n\} \text{ such that } U_1 \sqcup \cdots \sqcup U_n = U.$$

Familiar(?): also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

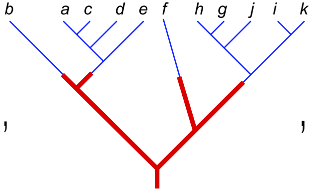
$$\gamma : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$$

Leveled tree of trees: indelible grafting.

Example:

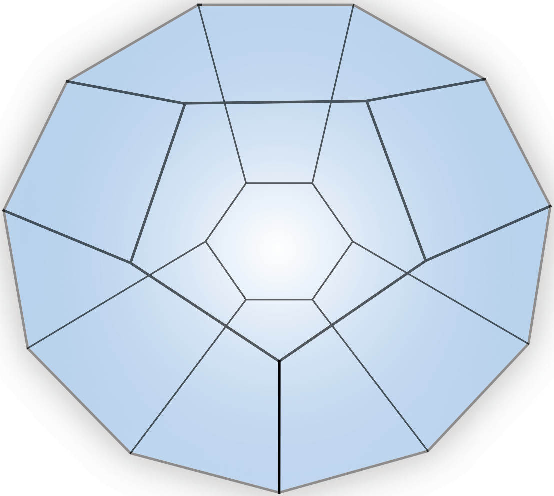
$$(\mathcal{S} \circ \mathcal{B})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

$\Big\{ \ldots,$

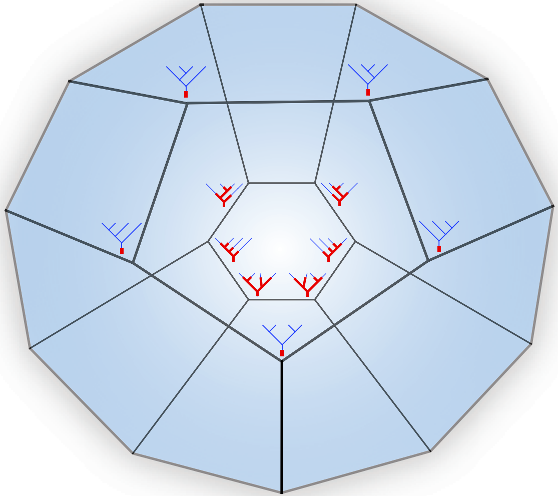


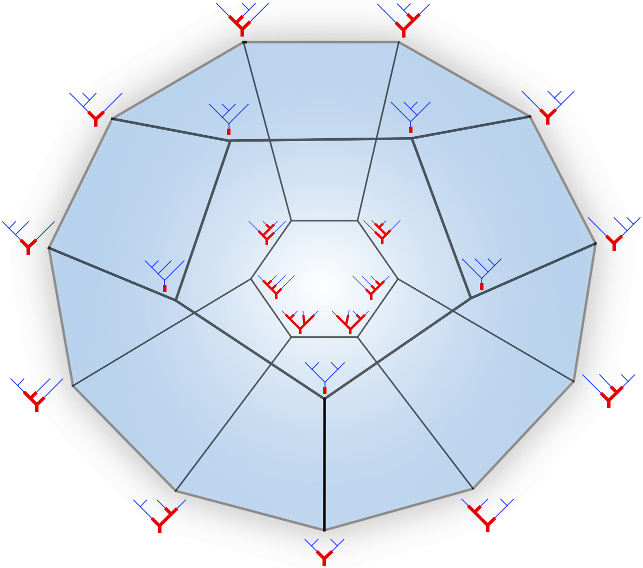
$, \ldots \Big\} = \left\{ \dfrac{\text{trees}}{\text{leveled tree}} \right\}$

Example $\mathcal{S} \circ \mathcal{B}$



$\mathcal{S} \circ \mathcal{B}$





Composing species of trees.

$$
\begin{tikzcd}[column sep=small]
& & \mathcal{S}\circ\mathcal{S} \arrow[dl] \arrow[dr] & & \\
& \mathcal{B}\circ\mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{S}\circ\mathcal{B} \arrow[dl] \arrow[dr] & \\
\mathcal{C}\circ\mathcal{S} \arrow[dr] & & \mathcal{B}\circ\mathcal{B} \arrow[dl] \arrow[dr] & & \mathcal{S}\circ\mathcal{C} \arrow[dl] \\
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 & \mathcal{C}\circ\mathcal{B} \arrow[dr] & & \mathcal{B}\circ\mathcal{C} \arrow[dl] & \\
 & & \mathcal{C}\circ\mathcal{C} & &
\end{tikzcd}
$$

A small commuting diamond

$$\begin{tikzcd}[column sep=small] & \mathcal{B} \circ \mathcal{B} \arrow[dl] \arrow[dr] & \\ \mathcal{C} \circ \mathcal{B} \arrow[dr] & & \mathcal{B} \circ \mathcal{C} \arrow[dl] \\ & \mathcal{C} \circ \mathcal{C} & \end{tikzcd}$$

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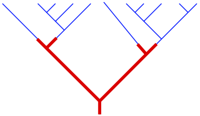

$\frac{\text{trees}}{\text{tree}}$

$$\begin{tikzcd} & {} \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathcal{C}\circ\mathcal{B} \arrow[dr, two heads] & & \mathcal{B}\circ\mathcal{C} \arrow[dl, two heads] \\ & \mathcal{C}\circ\mathcal{C} & \end{tikzcd}$$

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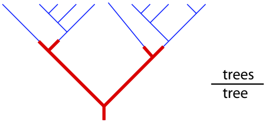
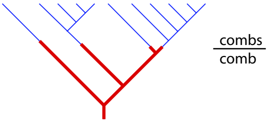


$$\mathcal{C} \circ \mathcal{B} \qquad\qquad \mathcal{B} \circ \mathcal{C}$$



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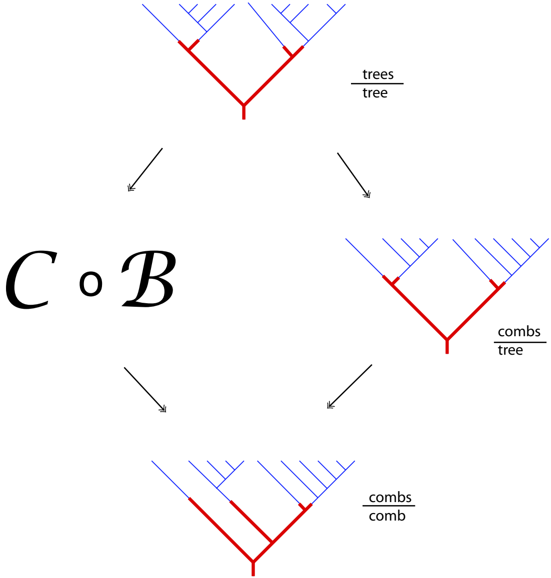

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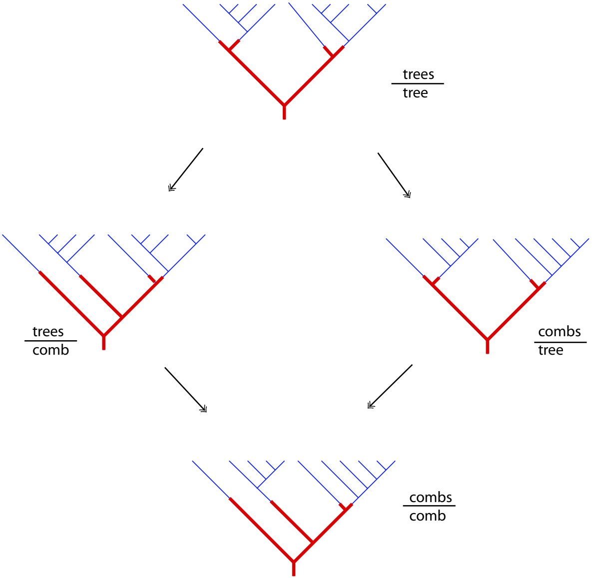

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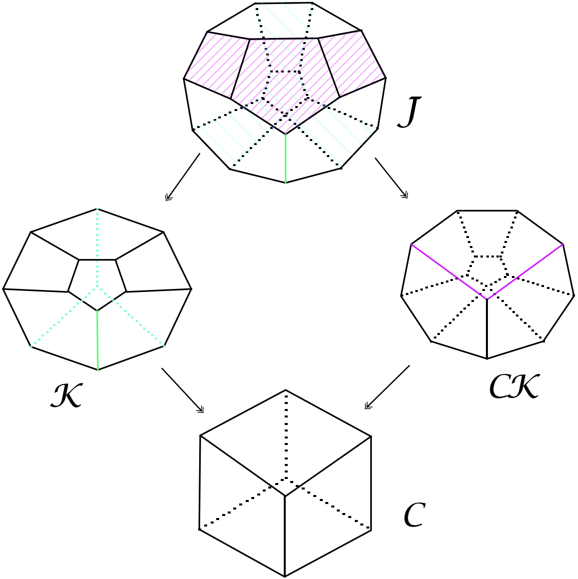
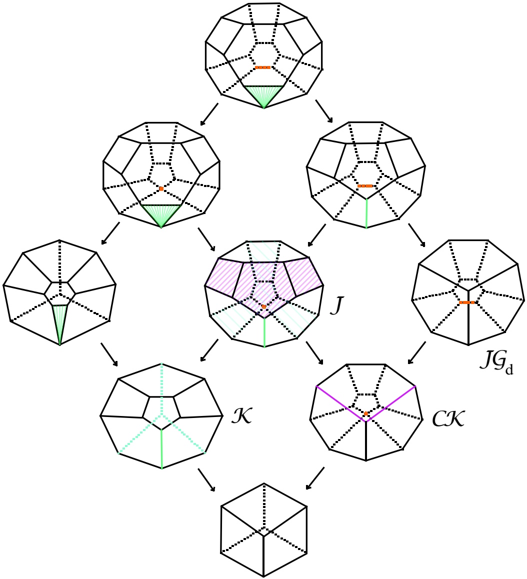


More polytopes.



Composition of coalgebras

Given two graded coalgebras we combine them in a way reminiscent of species composition.
Let $\mathcal{C}$ and $\mathcal{D}$ be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

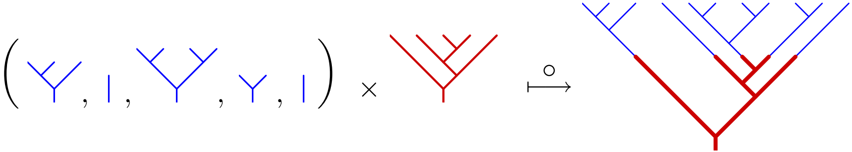
$$\mathcal{D} \circ \mathcal{C} \; := \; \bigoplus_{n \geq 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes (n+1)} \,. \tag{1}$$

Examples

The motivating example is when $\mathcal{C}$ and $\mathcal{D}$ are spaces of rooted trees. Then $\circ$ may be interpreted as some rule for grafting $n+1$ trees from $\mathcal{C}$ onto the leaves of a tree in $\mathcal{D}_n$.

Example

Suppose $\mathcal{C} = \mathcal{D} = \mathcal{Y}Sym$ and consider some $(c_0, \ldots, c_n) \times d \in \left(\mathcal{Y}_{\bullet}^{\,n+1}\right) \times \mathcal{Y}_n$. Then defining $\circ$ by grafting with color coding:

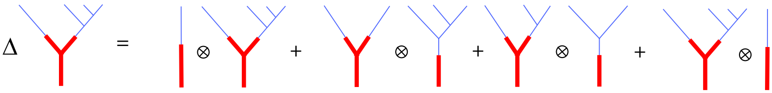


One sided Hopf algebras from compositions.

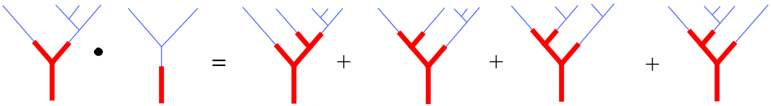
If our composition contains one of the Hopf operads, we get lots of free extra structure: a one-sided Hopf algebra, a Hopf module and a comodule algebra.

Examples from trees.

Here is an example of the coproduct in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:



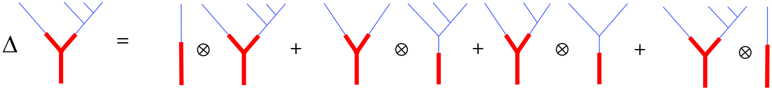
Here is an example of the product in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:



Combs of combs

The coproduct is the usual splitting of trees:

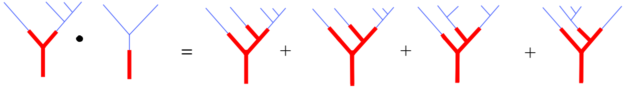
$$\Delta\ F_{13} \quad =$$



$$= \quad F_1 \otimes F_{13} \quad + F_{11} \otimes F_3 \quad + F_{12} \otimes F_2 \quad + F_{13} \otimes F_1$$

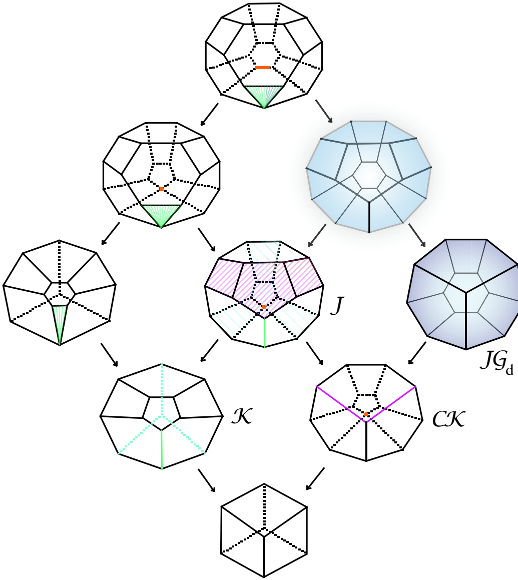
Here is the product:

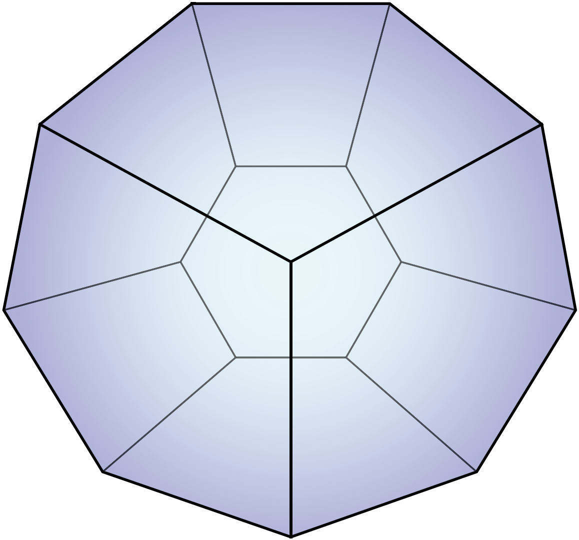
$$F_{13} \bullet F_2 \quad =$$



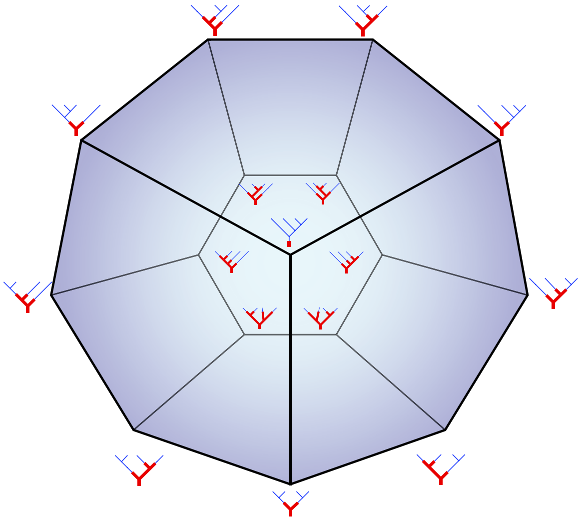
$$F_{113} \quad + F_{113} \quad + F_{122} \quad + F_{131}$$

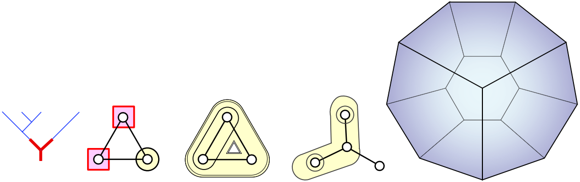
More polytopes.





$\mathcal{S} \circ \mathcal{C}$

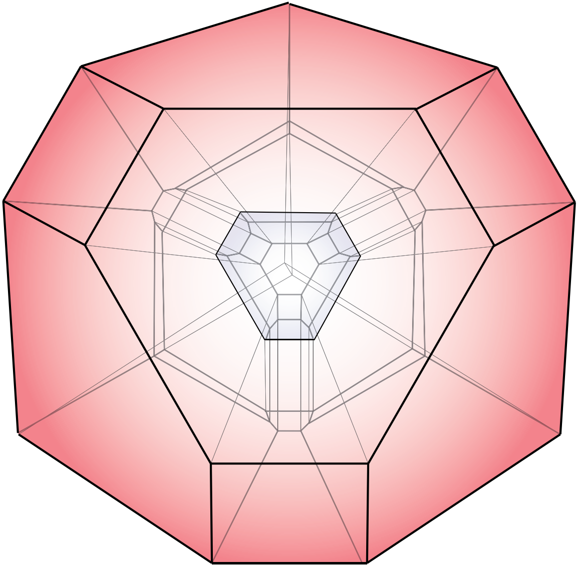




This polytope has been seen before!
Stellohedron = Complete-graph-cubeahedron
Number of vertices =

$$\sum_{k=0}^{n} \frac{n!}{k!}$$

Thanks!



Thanks!

Questions and comments?