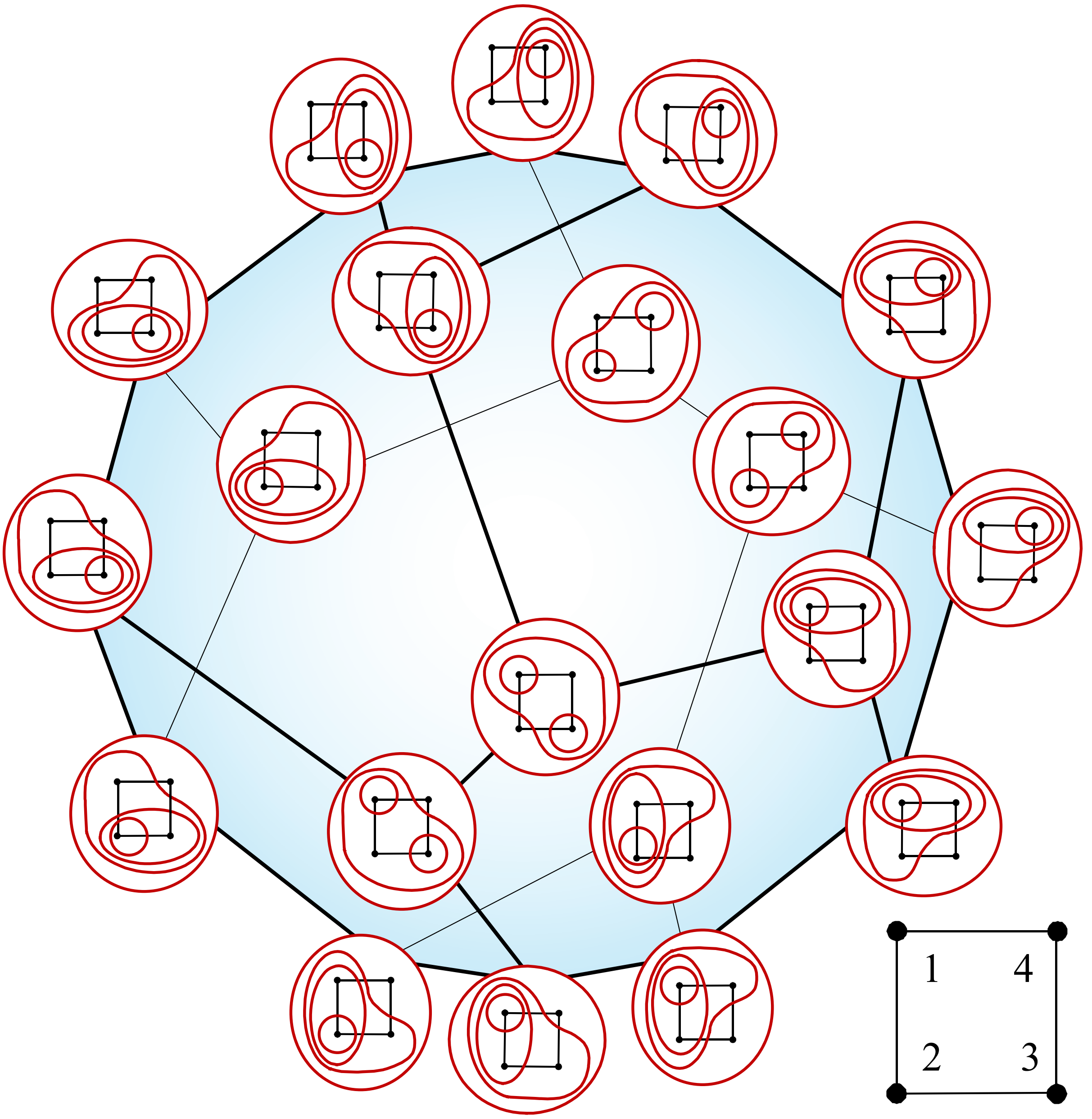
Pre-submission notes: Extending the Tamari lattice to graph associahedra

Hello fellow authors! I would like to post here some preliminary definitions and results, especially since I see by the submitted abstracts the potential for quite a bit of constructive comparison and contrast. The best way to quickly check for overlapping ideas is to compare Hasse diagrams of lattices, so I'll try to include some examples up front. -Stefan Forcey

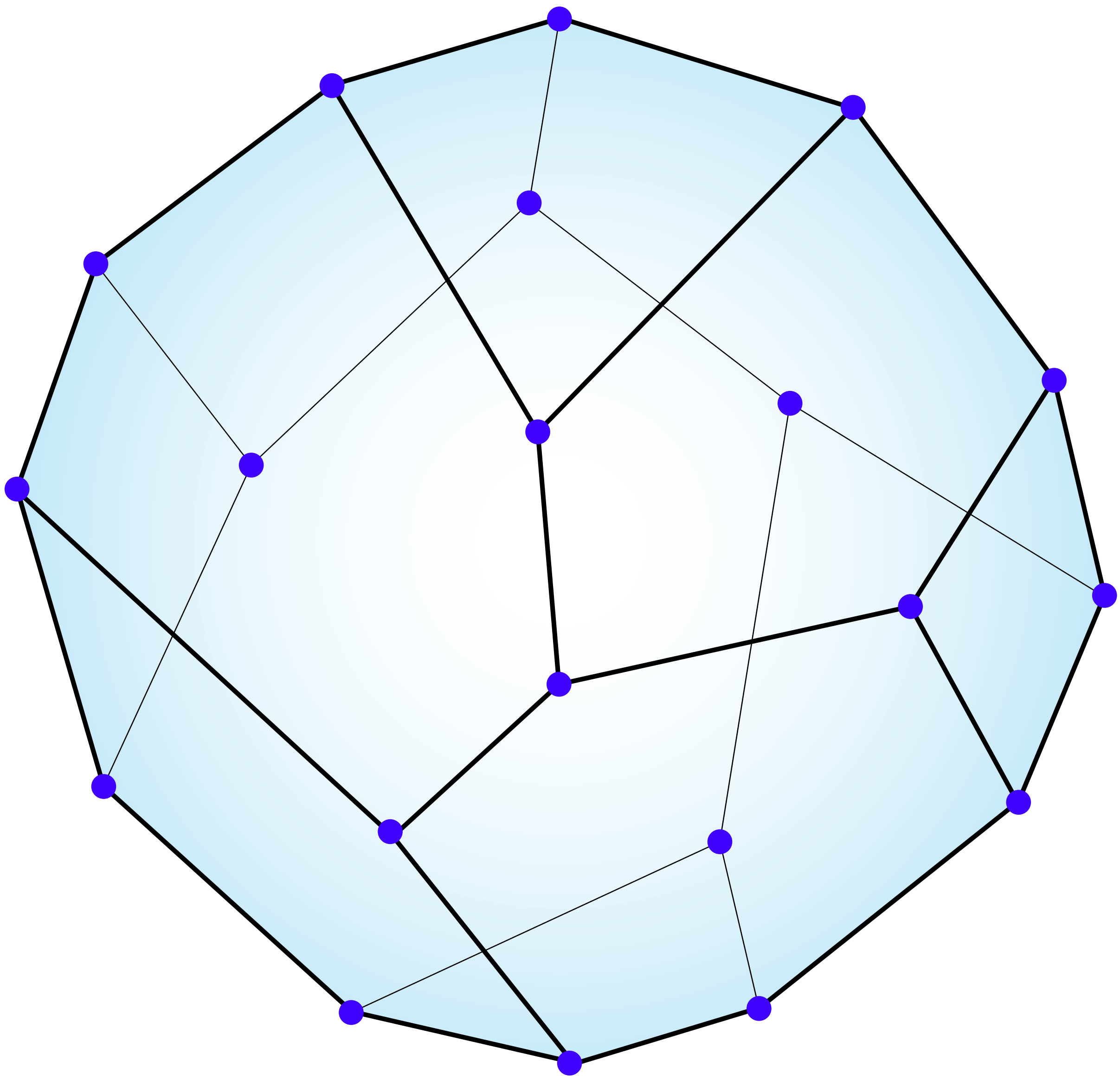
Abstract: We have uncovered several new poset structures on the skeletons of the graph associahedra. Given a numbering of the nodes of a connected graph we define the ordered graph lattice. This ordering generalizes both the weak order on permutations and the Tamari ordering of binary trees. The 1-skeletons of certain graph associahedra are Hasse diagrams of the quotient lattice of the weak order on the symmetric group. The 1-skeleton of the cyclohedron, seen as a lattice of graph tubings, gives a new poset structure different than Reading's Cambrian lattice on the same set. We investigate chains and antichains in our new lattices, and conjecture about connections to classical graph invariants. We also review some of the applications of Mbius inversion on the Tamari lattice to the Loday-Ronco Hopf algebra and describe analogous operations on an algebra based on the cyclohedra.

1. Examples of new lattices.

Example 1.1. *A lattice whose Hasse diagram is made up of the 1-skeletons of the cyclohedra–here shown in dimension 3:*

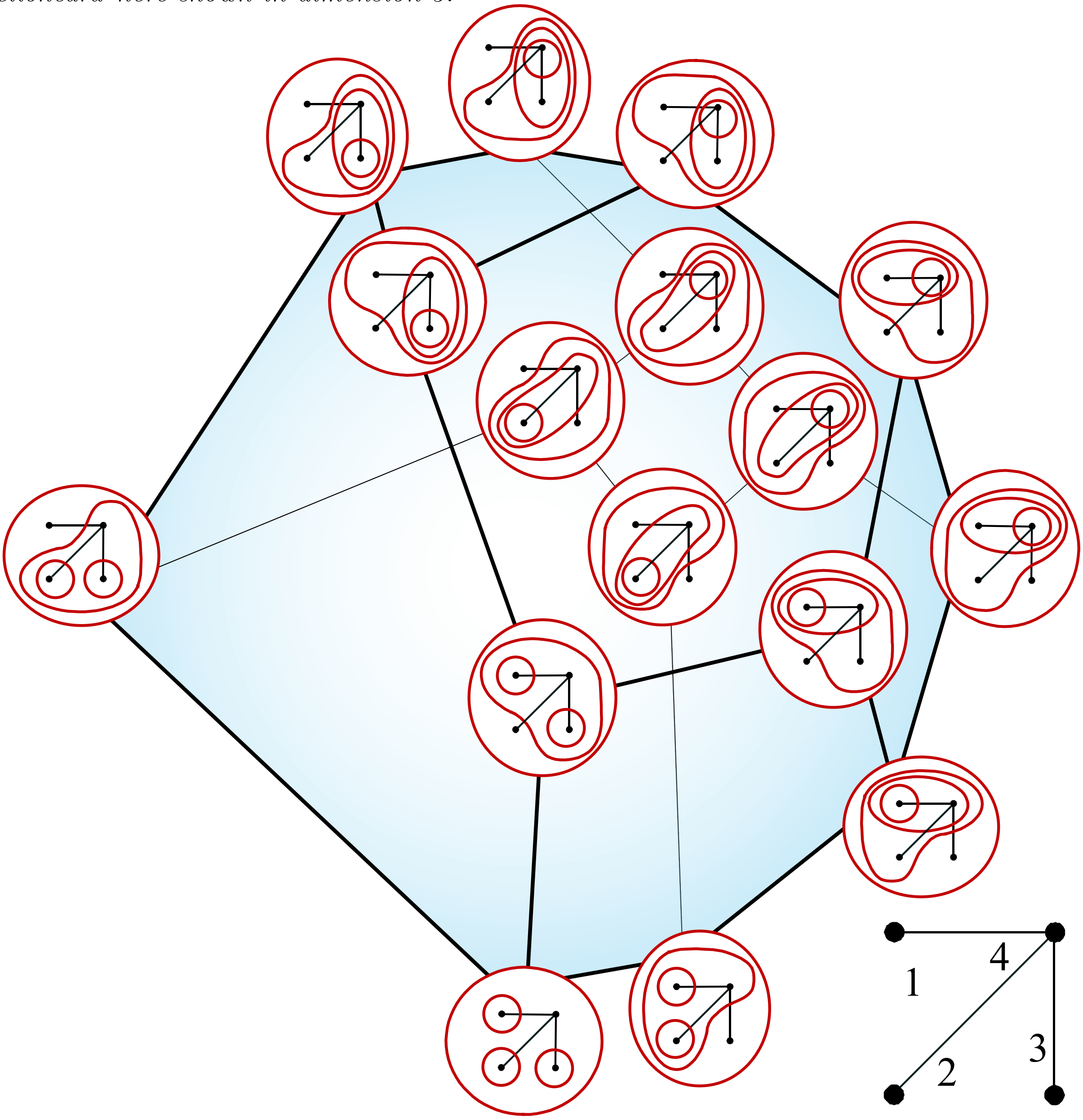


This Hasse diagram is labeled by tubings of the cycle graph, with nodes labeled 1–4.

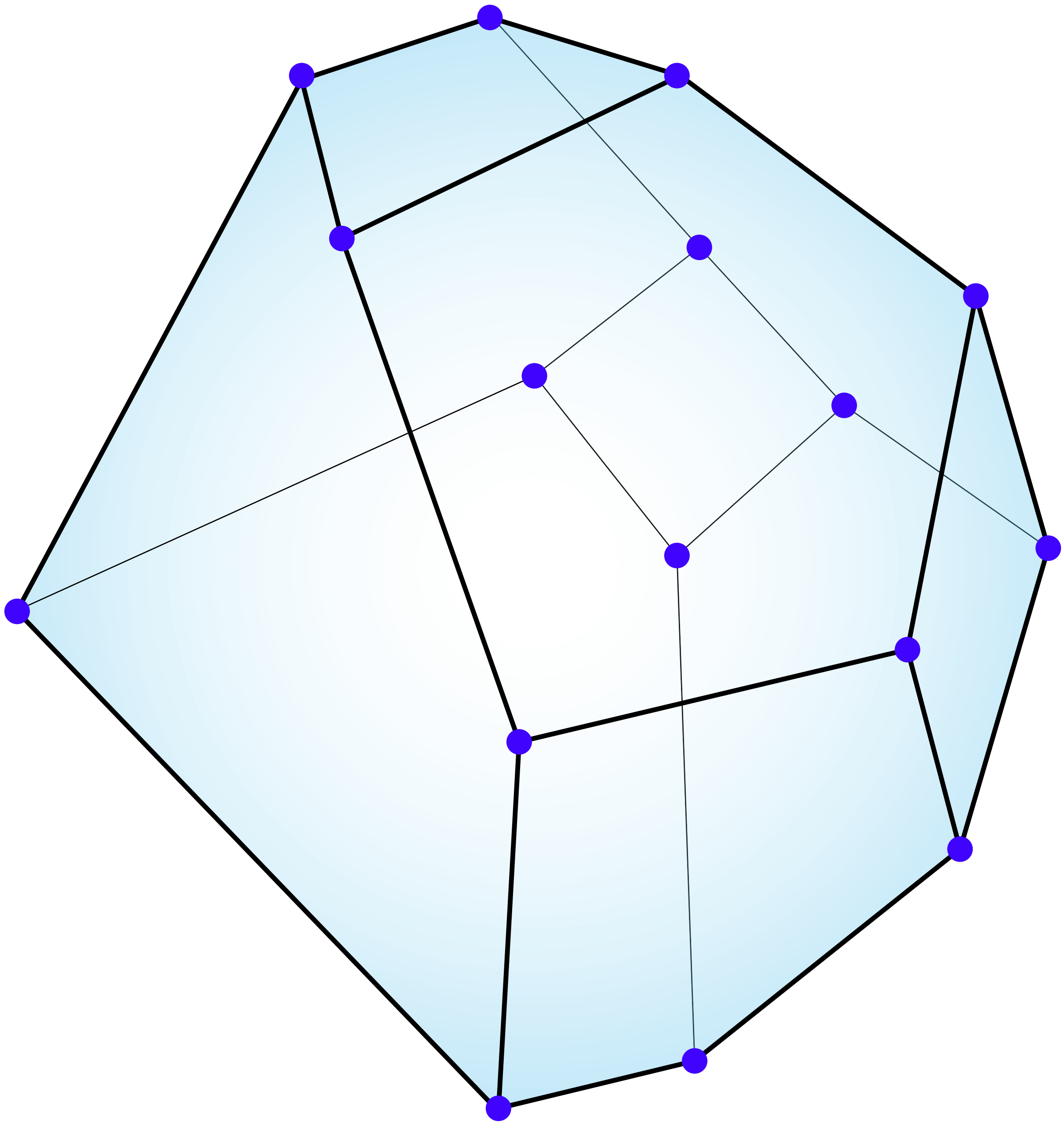


This is the same Hasse diagram as above with unlabeled nodes.

Example 1.2. *A lattice whose Hasse diagram is made up of the 1-skeletons of the stellohedra–here shown in dimension 3:*

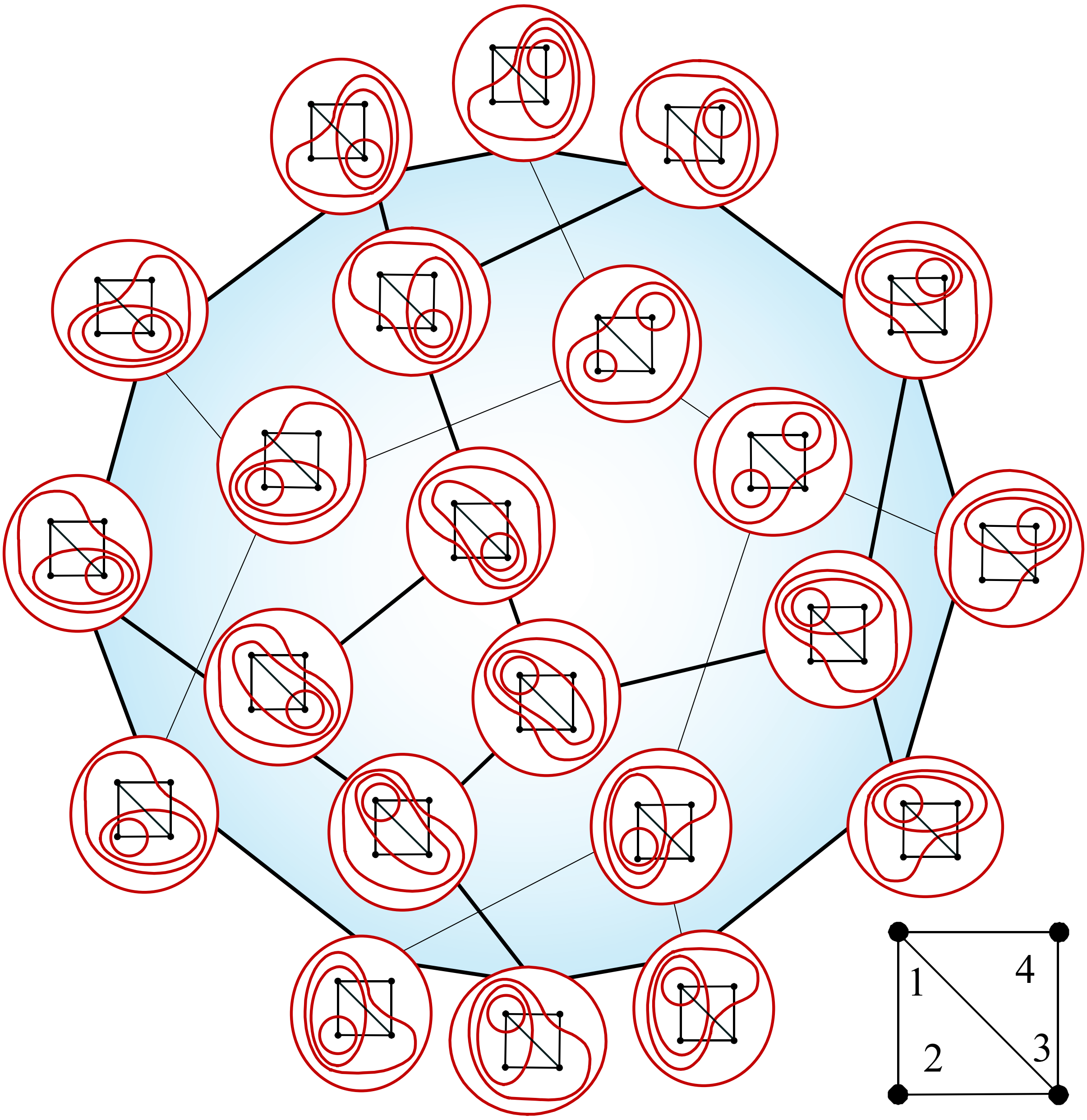


This Hasse diagram is labeled by tubings of the star graph, with nodes labeled 1–4.



This is the same Hasse diagram as above with unlabeled nodes.

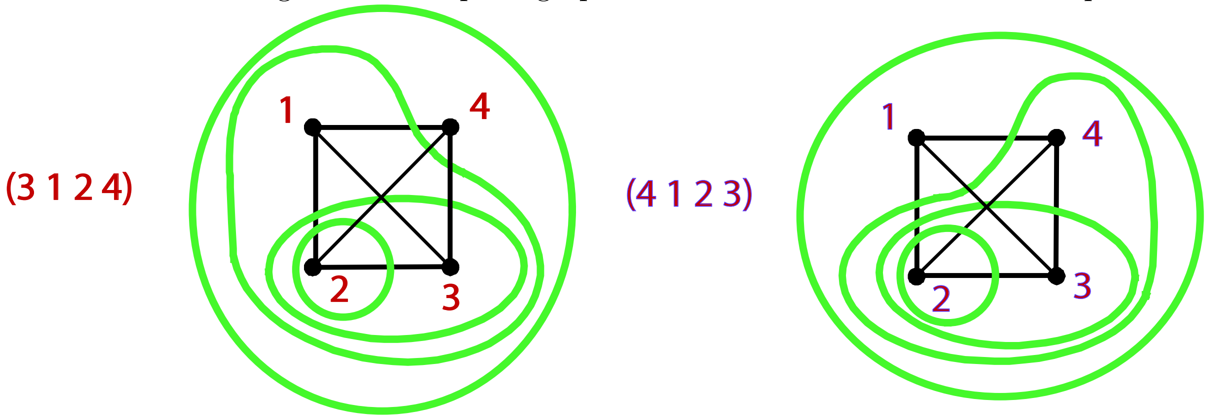
Example 1.3. *A lattice whose Hasse diagram is made up of the 1-skeletons of a barred cycle–here shown in dimension 3:*



This Hasse diagram is labeled by tubings of the barred cycle graph, with nodes labeled 1–4.

2. Definitions

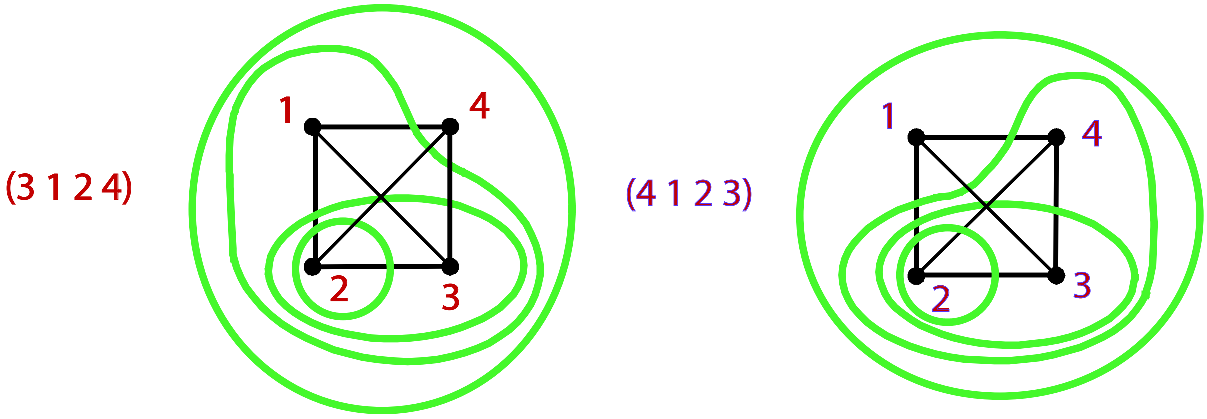
2.1. **Ordering permutations.** First we review the pictures of permutations arising from Devadoss's tubings of the complete graph on n numbered nodes. Two examples:



The nodes are the inputs for the permutation, and the output is the relative circle size. In the first example the image of 2 is 1, and so we put the smallest circle around 2.

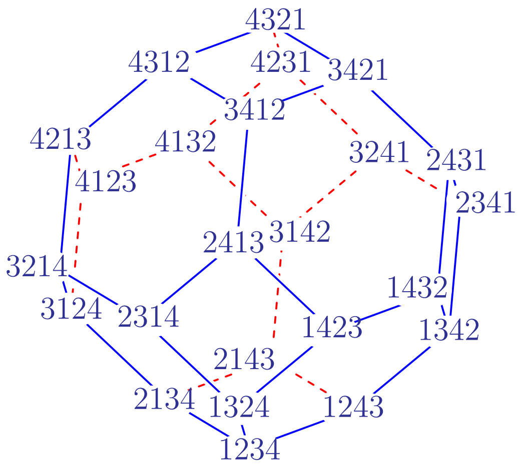
Next we give a new way to describe the weak order on permutations.

In order to describe the ordering we give the covering relations. Write down the sets of nodes in the circles: the tubes. Only one pair of tubes will differ. Compare the two numbered nodes of these which are in no smaller tubes. Here $1 < 4$.



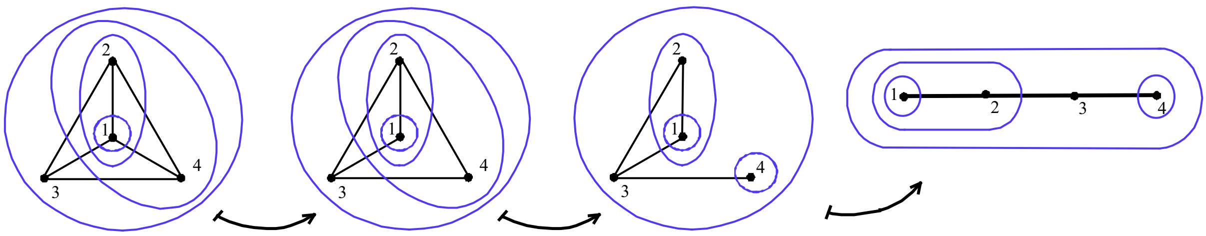
{2}		{2}
{3 2}		{3 2}
{3 2 1}	<	{4 3 2}

The 1-skeleton of $\mathcal{P}_n$ as a lattice.



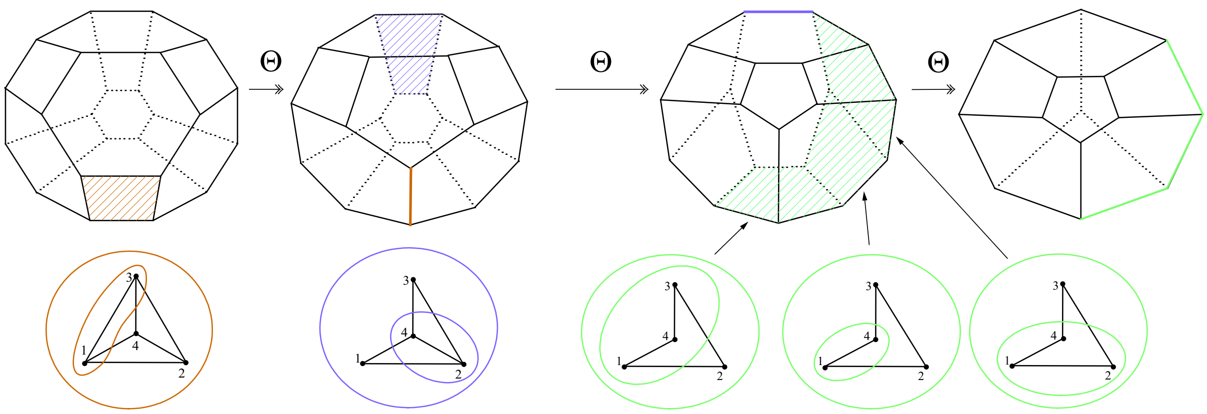
3. New lattices.

(1) Generalize permutations by deleting graph edges from the complete graph.
(2) If a circle no longer surrounds a connected subgraph, split it into two.
(3) Note: sometimes several permutations will be mapped to the same graph tubing.

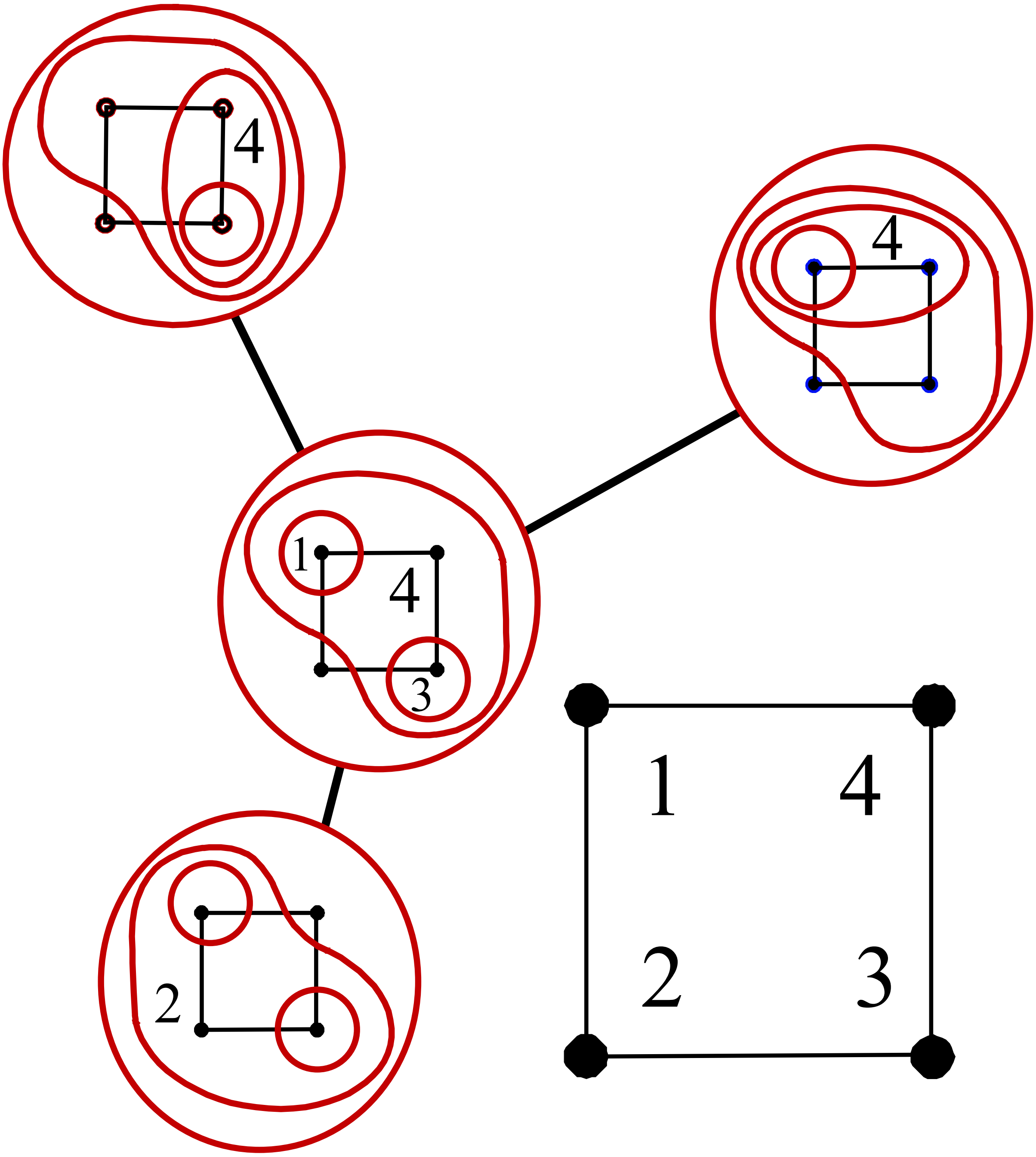


The result of deleting the same edges in all the pictures of $\mathfrak{S}_n$ is still a polytope.

These are the graph associahedra, discovered by M. Carr and S. Devadoss. The edge deletions correspond to cellular projections.



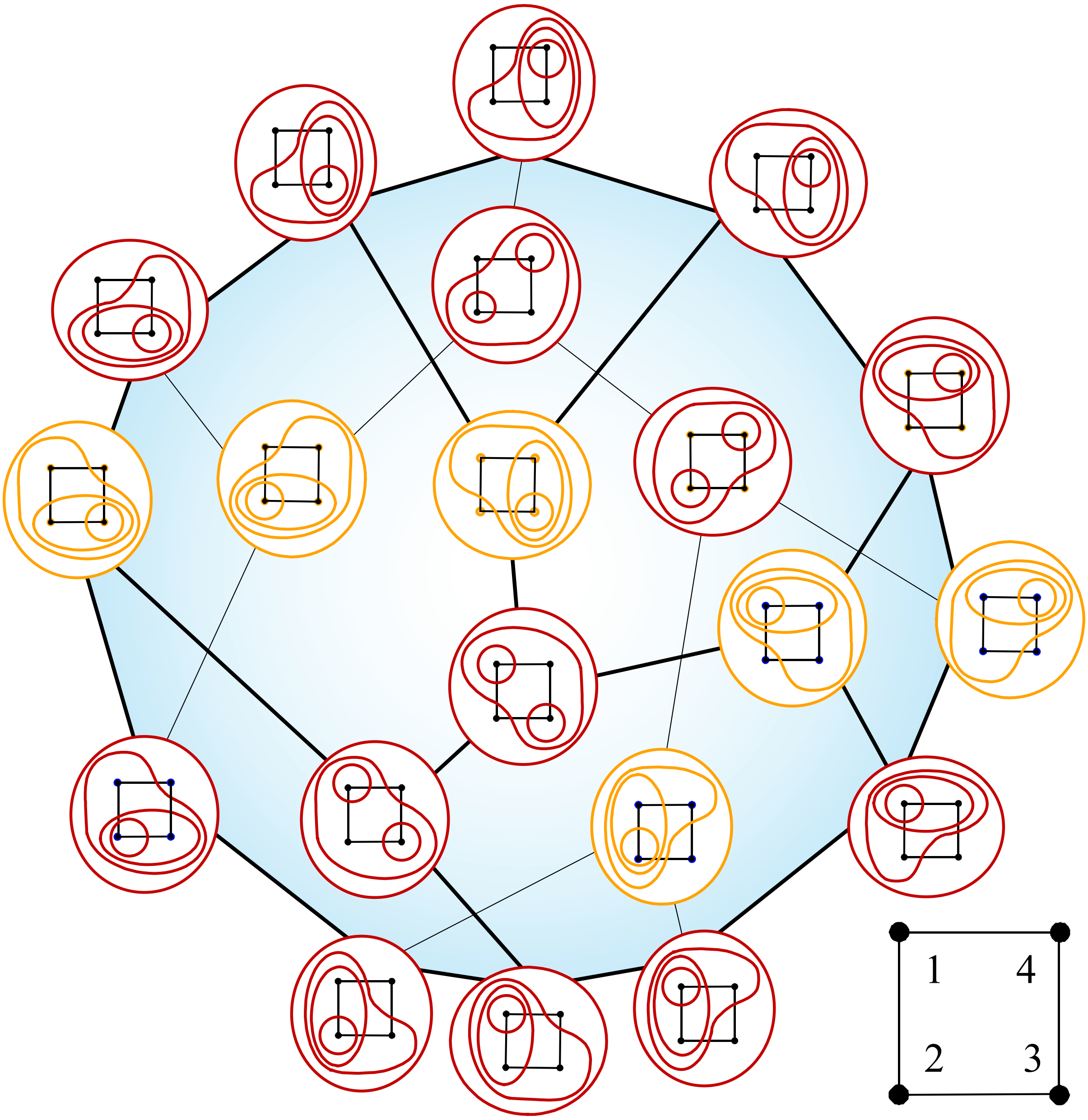
If the graph remains connected, then the 1-skeleton of each of these is still a lattice. The covering relations are analogous to those for permutations. Write down the sets of nodes in the circles: the tubes. Only one pair of tubes will differ. Compare the two numbered nodes of these which are in no smaller tubes.



3.1. **Main Result.** The projection function from $\mathcal{P}_n$ to our new lattice forms a lattice congruence.

Definition: A lattice congruence is a projection of lattices that preserves least upper bounds and greatest lower bounds.

Here is a maximal antichain in the cyclohedron lattice:



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