EQUIVALENCE OF ASSOCIATIVE STRUCTURES OVER A BRAIDING

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ABSTRACT. It is well known that the existence of a braiding in a monoidal category $\mathcal{V}$ allows many structures to be built upon that foundation. These include a monoidal 2-category $\mathcal{V}$-Cat of enriched categories and functors over $\mathcal{V}$, a monoidal bicategory $\mathcal{V}$-Mod of enriched categories and modules, a category of operads in $\mathcal{V}$ and a 2-fold monoidal category structure on $\mathcal{V}$. We ask, given a braiding on $\mathcal{V}$, what non-equal structures of a given kind from this list exist which are based upon the braiding. For example, what non-equal monoidal structures are available on $\mathcal{V}$-Cat, or what non-equal operad structures are available which base their associative structure on the braiding in $\mathcal{V}$. The basic question is the same as asking what non-equal 2-fold monoidal structures exist on a given braided category. The main results are that the possible 2-fold monoidal structures are classified by a particular set of four strand braids which we completely characterize, and that these 2-fold monoidal categories are divided into two equivalence classes by the relation of 2-fold monoidal equivalence.

CONTENTS

1. INTRODUCTION

The key requirement of a 2-fold monoidal structure on a category is that a second multiplication must be a functor which preserves the structure of the first multiplication. When the two multiplications are identical, this translates into the existence of a coherent interchange transformation from one four-operand product to another

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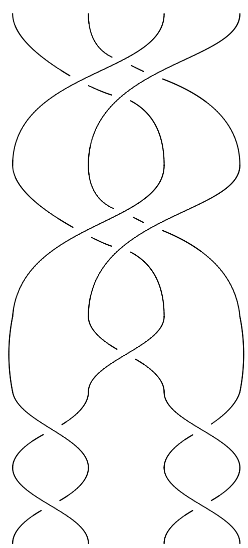
Thanks to XY-pic for the diagrams.

which differs from the first only in that the middle two operands are exchanged. The central goal of this paper is to study and classify the braids on four strands which can play the role of an interchange transformation in a braided category. The value of this classification is principally to provide a solid framework for proofs about structures based upon a braiding. It turns out that only certain braids can represent interchange transformations. Thus our results can be used to provide cases for proofs, either by treating all the cases up to braid equivalence or more often just by treating representative cases of categorical equivalence classes. As an example we discuss the general existence of a braiding on the category of enriched categories.

When unit conditions are also obeyed the classification of braids which can underlie an interchange becomes quite simple. These we designate as unital associative braids. The first main result is that these braids are precisely those given in terms of standard generators by

$$b_{n^\pm} = (\sigma_2\sigma_1\sigma_3\sigma_2)^{\pm n}\sigma_2^{\pm 1}(\sigma_1\sigma_3)^{\mp n}$$

for n a non-negative integer. For example b_{2^+} appears as:



A key set of geometrical facts can be observed about these braids. We refer to the strands of a braid by their initial positions. A sub-braid will refer to the braid resulting from the deletion of a subset of the strands of a braid.

1.1. **Lemma.** *If n is odd then deleting the outer two strands in the braid $b_{n^\pm}$ leaves the two strand sub-braid $\sigma_1^{\pm n}$, while deleting the inner two strands gives the sub-braid $\sigma_1^{\pm(n+1)}$. If n is even then deleting the outer two strands in the braid $b_{n^\pm}$ leaves the two strand sub-braid $\sigma_1^{\pm(n+1)}$, while deleting the inner two strands gives the sub-braid $\sigma_1^{\pm n}$.*

Proof. Consider the upper portion of the braid $b_{n^\pm}$ given by $(\sigma_2\sigma_1\sigma_3\sigma_2)^{\pm n}$. The outer two strands and the inner pair of strands both are crossed $\pm n$ times. If n is even then the upper portion is pure and so the next generator $\sigma_2^{\pm 1}$ is applied to the inner two strands. If n is odd then the upper portion has the associated permutation which sends $\{1\ 2\ 3\ 4\} \to \{3\ 4\ 1\ 2\}$ and so the next generator $\sigma_2^{\pm 1}$ is applied to the outer two strands. Note that the lower portion of the braid given by $(\sigma_1\sigma_3)^{\mp n}$ contributes no further crossings to either the outer or inner sub-braids. $\square$

1.2**. Corollary.** *The braids $b_{n^\pm}$ and $b_{m^\pm}$ are equivalent if and only if $m = n$ and the superscript signs are the same.*

Proof. For two braids to be equivalent it is necessary that all their corresponding sub-braids be equivalent. If n, m are both odd (or both even) and the signs are the same then the implication is clear by Lemma 1.1. Let n, m be both odd (or both even) with the signs not the same. Then if we assume that $b_{n^\pm}$ and $b_{m^\pm}$ are equivalent then use of Lemma 1.1 leads to the absurd implication $1 = -1$. Let n be odd and m be even with the superscript signs the same. Then if the sub-braids formed by the outer strands are equal we have that $m = n + 1$. Then $m + 1 = n + 2 \neq n$ so the inner sub-braids are not equal. Finally let n be odd and m be even with the superscript signs not the same. If the braids are equivalent then $m = -(n + 1)$ but both m and n are required to be non-negative, so this is a contradiction. $\square$

We begin with a review of the category of enriched categories over a braided category, since the construction of the opposite of an enriched category in this setting will provide the evidence of sufficiency of our conditions which a braid must meet to be associative. At the end of the second section we generalize the results mentioned by Joyal and Street who point out that the category of enriched categories is in general not braided and that taking the opposite is not an involution. In the third section we review the axioms of a 2-fold monoidal category and demonstrate the necessity of the conditions for our main result. Next we turn to ask which of the 2-fold monoidal categories we have described as arising from a certain braided category are equivalent as 2-fold monoidal categories. Our result is that the relation of equivalence of 2-fold monoidal categories splits our associative braids into into two equivalence classes, represented by σ_2 and σ_2^{-1}. The third section ends with a list of obstructions which prevent a braid from having the associative property–i.e. which prevent it from being equivalent to one of the braids described by our main result. Finally we relate these results to operad structures in a braided category.

2. Braiding and Enrichment

First we briefly review the definition of a category enriched over a monoidal category $\mathcal{V}$. Enriched functors and enriched natural transformations make the collection of enriched categories into a 2-category $\mathcal{V}$-Cat. The definitions and proofs can be found in more or less detail in [Kelly, 1982] and [Eilenberg and Kelly, 1965] and of course in [Mac Lane, 1998]. Some are included here for easy reference.

2.1. **Definition.** A *monoidal category* is a category $\mathcal{V}$ together with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an object I such that

(1) $\otimes$ is associative up to the coherent natural isomorphisms α. The coherence axiom is given by the usual commuting pentagonal diagram as in [Mac Lane, 1998].
(2) In this paper, I is a strict 2-sided unit for $\otimes$.

2.2. **Definition.** A (small) $\mathcal{V}$ *-Category* $\mathcal{A}$ is a set $|\mathcal{A}|$ of *objects*, a *hom-object* $\mathcal{A}(A,B) \in |\mathcal{V}|$ for each pair of objects of $\mathcal{A}$, a family of *composition morphisms* $M_{ABC} : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$ for each triple of objects, and an *identity element* $j_A : I \to \mathcal{A}(A,A)$ for each object. The composition morphisms are subject to the associativity axiom which states that the following pentagon commutes

$$\begin{tikzcd}
 & (\mathcal{A}(C,D)\otimes\mathcal{A}(B,C))\otimes\mathcal{A}(A,B) \arrow[dl, "M\otimes 1"] \arrow[rr, "\alpha"] & & \mathcal{A}(C,D)\otimes(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)) \arrow[dr, "1\otimes M"] & \\
\mathcal{A}(B,D)\otimes\mathcal{A}(A,B) \arrow[drr, "M"] & & & & \mathcal{A}(C,D)\otimes\mathcal{A}(A,C) \arrow[dll, "M"] \\
 & & \mathcal{A}(A,D)) & &
\end{tikzcd}$$

and to the unit axioms which state that both the triangles in the following diagram commute

$$\begin{tikzcd}
I\otimes\mathcal{A}(A,B) \arrow[dr, "="] \arrow[dd, "j_B\otimes 1"'] & & \mathcal{A}(A,B)\otimes I \arrow[dl, "="'] \arrow[dd, "1\otimes j_A"] \\
 & \mathcal{A}(A,B) & \\
\mathcal{A}(B,B)\otimes\mathcal{A}(A,B) \arrow[ur, "M_{ABB}"] & & \mathcal{A}(A,B)\otimes\mathcal{A}(A,A) \arrow[ul, "M_{AAB}"']
\end{tikzcd}$$

In general a $\mathcal{V}$-category is directly analogous to an (ordinary) category enriched over **Set**. If $\mathcal{V} = \textbf{Set}$ then these diagrams are the usual category axioms. Basically, composition of morphisms is replaced by tensoring and the resulting diagrams are required to commute. The next two definitions exhibit this principle and are important since they give us the setting in which to construct a category of $\mathcal{V}$-categories.

2.3. **Definition.** For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-*functor* $T : \mathcal{A} \to \mathcal{B}$ is a function $T : |\mathcal{A}| \to |\mathcal{B}|$ and a family of morphisms $T_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(TA,TB)$ in $\mathcal{V}$ indexed by pairs $A, B \in |\mathcal{A}|$. The usual rules for a functor that state $T(f \circ g) = Tf \circ Tg$ and $T1_A = 1_{TA}$ become in the enriched setting, respectively, the commuting diagrams

$$\begin{tikzcd}
\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \arrow[r, "M"] \arrow[d, "T\otimes T"] & \mathcal{A}(A,C) \arrow[d, "T"] \\
\mathcal{B}(TB,TC)\otimes\mathcal{B}(TA,TB) \arrow[r, "M"] & \mathcal{B}(TA,TC)
\end{tikzcd}$$

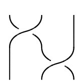
and

$$\begin{tikzcd}
 & \mathcal{A}(A,A) \arrow[dd, "T_{AA}"] \\
I \arrow[ur, "j_A"] \arrow[dr, "j_{TA}"'] & \\
 & \mathcal{B}(TA,TA)
\end{tikzcd}$$

$\mathcal{V}$-functors can be composed to form a category called $\mathcal{V}$-Cat. This category is actually enriched over **Cat**, the category of (small) categories with Cartesian product.

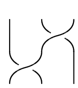
2.4**. Definition.** A *braiding* for a monoidal category $\mathcal{V}$ is a family of natural isomorphisms $c_{XY} : X \otimes Y \to Y \otimes X$ such that the following diagrams commute. They are drawn next to their underlying braids.

(1)



$$\begin{tikzcd}
 & (X\otimes Y)\otimes Z \arrow[r, "\alpha_{XYZ}"] \arrow[dl, "c_{XY}\otimes 1"] & X\otimes(Y\otimes Z) \arrow[dr, "c_{X(Y\otimes Z)}"] & \\
(Y\otimes X)\otimes Z \arrow[dr, "\alpha_{YXZ}"] & & & (Y\otimes Z)\otimes X \arrow[dl, "\alpha_{YZX}"] \\
 & Y\otimes(X\otimes Z) \arrow[r, "1\otimes c_{XZ}"] & Y\otimes(Z\otimes X) &
\end{tikzcd}$$

(2)



$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "\alpha^{-1}_{XYZ}"] \arrow[dl, "1\otimes c_{YZ}"] & (X\otimes Y)\otimes Z \arrow[dr, "c_{(X\otimes Y)Z}"] & \\
X\otimes(Z\otimes Y) \arrow[dr, "\alpha^{-1}_{XZY}"] & & & Z\otimes(X\otimes Y) \arrow[dl, "\alpha^{-1}_{ZXY}"] \\
 & (X\otimes Z)\otimes Y \arrow[r, "c_{XZ}\otimes 1"] & (Z\otimes X)\otimes Y &
\end{tikzcd}$$

A braided category is a monoidal category with a chosen braiding. We will assume a strict unit in the monoidal categories considered here which implies a strict respect for units by the braiding. That is, $c_{IA} = c_{AI} = 1_A$.

Joyal and Street proved the coherence theorem for braided categories in [Joyal and Street, 1993], the immediate corollary of which is that in a free braided category generated by a set of objects, a diagram commutes if and only if all legs having the same source and target have the same underlying braid.

2.5. **Definition.** A *symmetry* is a braiding such that the following diagram commutes

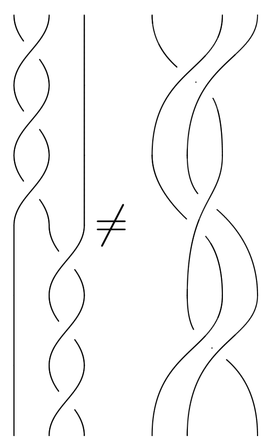
$$\begin{tikzcd} X \otimes Y \arrow[rr, "1"] \arrow[dr, "c_{XY}"'] & & X \otimes Y \\ & Y \otimes X \arrow[ur, "c_{YX}"'] & \end{tikzcd}$$

In other words $c_{XY}^{-1} = c_{YX}$. A symmetric category is a monoidal category with a chosen symmetry.

As pointed out by Joyal and Street, it is true that c^{-1} is a (non-equivalent) braiding whenever c is. It should be noted that there is immediately an obstruction to other potential braidings based on the original. For sake of efficiency we use notation $c^n_{AB} = c_{AB} \circ c_{BA} \circ c_{AB} \circ \cdots \circ c_{AB}$ where there are n instances of c. It appears at first that if c_{AB} is a braiding then $c' = c^{\pm(2n+1)}$ is potentially a braiding for any n, but actually we find that:

2.6. **Lemma.** *for $n \geq 1$, $c' = c^{\pm(2n+1)}$ is a braiding if and only if c is a symmetry. In that case of course $c^{\pm(2n+1)}$ is also a symmetry.*

Proof. The obstruction arises from the the braided coherence theorem applied to the hexagonal diagrams with $c^{\pm(2n+1)}$ in place of the original instances of c. Observe that for $n = 1$ when we test the potential braiding the first hexagonal diagram has legs with the following underlying braid inequality:



Indeed we have that the required equality of braids for the first hexagonal axiom can never hold for $n \geq 1$. We check the positive powers of c and note that the negative powers are shown similary. For $c' = c^{2n+1}$ the braid inequality underlying the hexagon in terms of the standard braid generators is $\sigma_1^{2n+1}\sigma_2^{2n+1} \neq \sigma_1\sigma_2(\sigma_2\sigma_1\sigma_1\sigma_2)^n$. It is easy to see this inequality since the semigroup of positive braids embeds into the braid group of the same number of strands, as shown in [Garside, 1969]. Thus any two positive braids are equivalent in the braid group if and only if they are equivalent in the positive semigroup, i.e. related by a chain of braid relations. For three strand braids the only possible braid relation is the standard $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Note that in the braid words representing the three strand braids in question there are no instances of either side of this relation, and so both are in a unique positive form, and so clearly not equal. $\square$

If $\mathcal{V}$ is braided then we can define additional structure on $\mathcal{V}$-Cat. First there is a left opposite of a $\mathcal{V}$-category which has $|\mathcal{A}^{op}| = |\mathcal{A}|$ and $\mathcal{A}^{op}(A, A') = \mathcal{A}(A', A)$. The composition morphisms are given by

$$\begin{tikzcd}
\mathcal{A}^{op}(A', A'') \otimes \mathcal{A}^{op}(A, A') \arrow[d, equal] \\
\mathcal{A}(A'', A') \otimes \mathcal{A}(A', A) \arrow[d, "c_{\mathcal{A}(A'',A')\otimes\mathcal{A}(A',A)}"'] \\
\mathcal{A}(A', A) \otimes \mathcal{A}(A'', A') \arrow[d, "M_{AA'A''}"'] \\
\mathcal{A}(A'', A) \arrow[d, equal] \\
\mathcal{A}^{op}(A, A'')
\end{tikzcd}$$

It is clear from this that $(\mathcal{A}^{op})^{op} \neq \mathcal{A}$. The pentagon diagram for the composition morphisms commutes since the braids underlying its legs are the two sides of the braid relation, also known as the Yang-Baxter equation. The unit morphism is the same as the original $j_A : I \to \mathcal{A}(A, A) = \mathcal{A}^{op}(A, A)$. The unit axioms are obeyed due to the fact that $c_{IA} = c_{AI} = 1_A$. The right opposite denoted $\mathcal{A}^{po}$ is given by the same definition of composition and unit morphisms, but using c^{-1}. It is clear that $(\mathcal{A}^{po})^{op} = (\mathcal{A}^{op})^{po} = \mathcal{A}$. The second structure is a product for $\mathcal{V}$-Cat, that is, a 2-functor

$$\otimes^{(1)} : \mathcal{V}\text{-Cat} \times \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat}.$$

We will always denote the product(s) in $\mathcal{V}$-Cat with a superscript in parentheses that corresponds to the level of enrichment of the components of their domain. The product(s) in $\mathcal{V}$ should logically then have a superscript (0) but we have suppressed this for brevity and to agree with our sources. The product of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has $\left|\mathcal{A} \otimes^{(1)} \mathcal{B}\right| = |\mathcal{A}| \times |\mathcal{B}|$ and $(\mathcal{A} \otimes^{(1)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$.

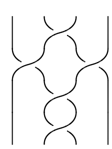
The unit morphisms for the product $\mathcal{V}$-categories are the composites

$$I \cong I \otimes I \xrightarrow[j_A \otimes j_B]{} \mathcal{A}(A, A) \otimes \mathcal{B}(B, B)$$

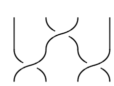
The composition morphisms
$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A}\otimes^{(1)}\mathcal{B})((A',B'),(A'',B''))\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A',B')) \to (\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A'',B''))$
may be given canonically by

$$\begin{array}{c}
(\mathcal{A}\otimes^{(1)}\mathcal{B})((A',B'),(A'',B''))\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A',B')) \\
\Big\| \\
(\mathcal{A}(A',A'')\otimes\mathcal{B}(B',B''))\otimes(\mathcal{A}(A,A')\otimes\mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle (1\otimes\alpha^{-1})\circ\alpha} \\
\mathcal{A}(A',A'')\otimes((\mathcal{B}(B',B'')\otimes\mathcal{A}(A,A'))\otimes\mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle 1\otimes(c_{\mathcal{B}(B',B'')\mathcal{A}(A,A')}\otimes 1)} \\
\mathcal{A}(A',A'')\otimes((\mathcal{A}(A,A'))\otimes\mathcal{B}(B',B''))\otimes\mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle \alpha^{-1}\circ(1\otimes\alpha)} \\
(\mathcal{A}(A',A'')\otimes\mathcal{A}(A,A'))\otimes(\mathcal{B}(B',B''))\otimes\mathcal{B}(B,B')) \\
\Big\downarrow{\scriptstyle M_{AA'A''}\otimes M_{BB'B''}} \\
(\mathcal{A}(A,A'')\otimes\mathcal{B}(B,B'')) \\
\Big\| \\
(\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A'',B''))
\end{array}$$

That $(\mathcal{A}\otimes^{(1)}\mathcal{B})^{op} \neq \mathcal{A}^{op}\otimes^{(1)}\mathcal{B}^{op}$ follows from the following braid inequality:



$\neq$



Now consider more carefully the morphisms of $\mathcal{V}$ that make up the composition morphism for a product enriched category, especially those that accomplish the "middle exchange" [Kelly, 1982] of the interior hom-objects, that is, all but the terminal pair of instances of the original composition M_{ABC}. In the symmetric case, any other combination of instances of α and c with the same domain and range would be equal, due to symmetric coherence. In the merely braided case, there at first seems to be a much larger range of available choices. There is a canonical epimorphism $\sigma : B_n \to S_n$ of the braid group on n strands onto the permutation group. The permutation given by σ is that given by the strands of the braid on the n original positions. For instance on a standard generator of B_n, σ_i, we have $\sigma(\sigma_i) = (i \;\; i+1)$. Candidates for multiplications would seem to be those defined using any braid $b \in B_4$ such that $\sigma(b) = (2\ 3)$. It is clear that the composition morphism would be defined as above, with a series of instances of α and c such that the underlying braid is b, followed in turn by $M_{AA'A''} \otimes M_{BB'B''}$ in order to complete the composition. That $M_{AA'A''} \otimes M_{BB'B''}$ will have the correct domain on which to operate is guaranteed by the permutation condition on b.

For the unit $\mathcal{V}$-category $\mathcal{I}$ to be indeed a unit for one of the multiplications in question requires that in the underlying braid of the composition dropping either the first and third strand or the second and fourth strand leaves the identity on two strands. The unit axioms of the product categories are satisfied as long as dropping either the first two or the last two strands leaves again the identity on two strands. This is also due to the naturality of compositions of α and c and the unit axioms obeyed by $\mathcal{A}$ and $\mathcal{B}$. The remaining things to be checked are associativity of composition and functoriality of the associator.

For the associativity axiom to hold the following diagram must commute, where the initial bullet represents

$$[(\mathcal{A}\otimes^{(1)}\mathcal{B})((A'',B''),(A''',B'''))\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A',B'),(A'',B''))]\otimes(\mathcal{A}\otimes^{(1)}\mathcal{B})((A,B),(A',B'))$$

and the last bullet represents $[\mathcal{A} \otimes^{(1)} \mathcal{B}]((A,B),(A''',B'''))$.

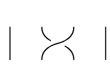
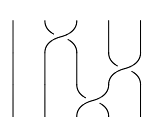
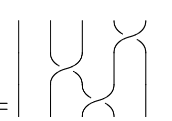
$$\begin{tikzcd}
 & \bullet \arrow[rr, "\alpha"] \arrow[dl, "M\otimes 1"] & & \bullet \arrow[dr, "1\otimes M"] & \\
\bullet \arrow[drr, "M"] & & & & \bullet \arrow[dll, "M"] \\
 & & \bullet & &
\end{tikzcd}$$

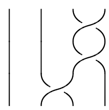
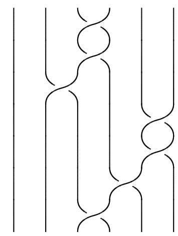
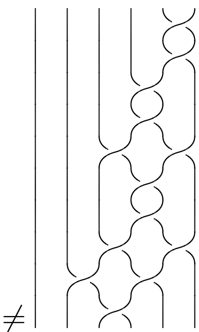
In $\mathcal{V}$ let $X = \mathcal{A}(A,A')$, $X' = \mathcal{A}(A',A'')$, $X'' = \mathcal{A}(A'',A''')$, $Y = \mathcal{B}(B,B')$, $Y' = \mathcal{B}(B',B'')$ and $Y'' = \mathcal{B}(B'',B''')$. The exterior of the following expanded diagram (where we leave out some parentheses for clarity and denote various composites of α and c by unlabeled arrows) is required to commute.

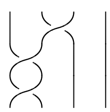
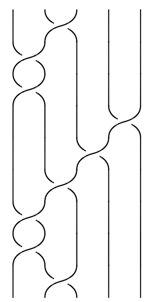
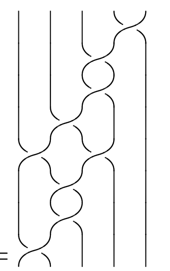
$$\begin{tikzcd}
 & {[X''\otimes Y''\otimes X'\otimes Y']\otimes(X\otimes Y)} \arrow[dl] \arrow[dr] & \\
{[X''\otimes X'\otimes Y''\otimes Y']\otimes(X\otimes Y)} \arrow[d] & & {(X''\otimes Y'')\otimes[X'\otimes Y'\otimes X\otimes Y]} \arrow[d] \\
{(X''\otimes X')\otimes(Y''\otimes Y')\otimes X\otimes Y} \arrow[d] & & {(X''\otimes Y'')\otimes[X'\otimes X\otimes Y'\otimes Y]} \arrow[d] \\
{[(X''\otimes X')\otimes X]\otimes[(Y''\otimes Y')\otimes Y]} \arrow[d, "(M\otimes 1)\otimes(M\otimes 1)"] \arrow[drr, "\alpha\otimes\alpha"] & & {X''\otimes Y''\otimes(X'\otimes X)\otimes(Y'\otimes Y)} \arrow[d] \\
{[\mathcal{A}(A',A''')\otimes X]\otimes[\mathcal{B}(B',B''')\otimes Y]} \arrow[d, "M\otimes M"] & & {[X''\otimes(X'\otimes X)]\otimes[Y''\otimes(Y'\otimes Y)]} \arrow[d, "(1\otimes M)\otimes(1\otimes M)"] \\
{\mathcal{A}(A,A''')\otimes\mathcal{B}(B,B''')} & & {[X''\otimes\mathcal{A}(A,A'')]\otimes[Y''\otimes\mathcal{B}(B,B'')]} \arrow[ll, "M\otimes M"']
\end{tikzcd}$$

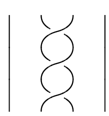
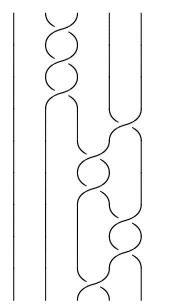
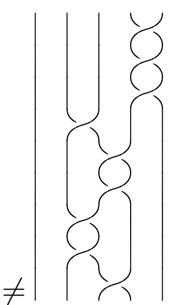
The bottom region commutes by the associativity axioms for $\mathcal{A}$ and $\mathcal{B}$. We are left needing to show that the underlying braids are equal for the two legs of the upper region. Again these basic nodes must be present regardless of the choice of braid by which the composition morphism is defined. Notice that the right and left legs have the following underlying braids in B_6 for some examples of various choices of b in B_4. We call the two derived six strand braids Lb and Rb respectively. Lb is algorithmically described as a copy of b on the first 4 strands followed by a copy of b on the 4 "strands" that result from pairing as the edges of two ribbons strands 1 and 2, and strands 3 and 4, along with the remaining two strands 5 and 6. Rb is similarly described, but the initial copy of b is on the last 4 strands, and the ribbon edge pairing is on the pairs (4,5) and (5,6). The first example for b is the one used in the original definition of $\otimes^{(1)}$ given above.

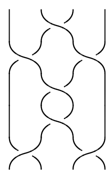
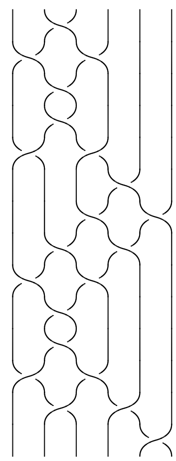
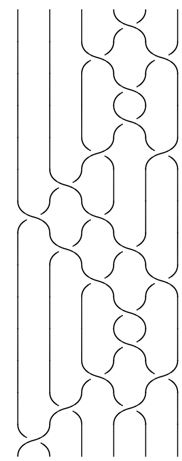
2.7. **Example.**

$b_{(1)} =$  Associativity follows from:  $=$ 

$b_{(2)} =$  Associativity does not follow since:  $\neq$ 

$b_{(3)} =$  Associativity follows from:  $=$ 

$b_{(4)} =$  Associativity does not follow since:  $\neq$ 

$b_{(5)} =$  Associativity does follow since:  $=$ 

Before turning to check on functoriality of the associator, we note that $b_{(3)}$ is the braid underlying the composition morphism of the product category $(\mathcal{A}^{op})^{op} \otimes^{(1)} \mathcal{B}$ where the product is defined using $b_{(1)}$. This provides the hint that the two derived braids $Lb_{(3)}, Rb_{(3)} \in B_6$ are equal because of the fact that the opposite of a $\mathcal{V}$-category is a valid $\mathcal{V}$-category. In fact we can describe sufficient conditions for Lb to be equivalent to Rb by describing the braids b that underlie the composition morphism of a product category given generally by $(((\mathcal{A}^{op})^{\cdots op} \otimes^{(1)} (\mathcal{B}^{op})^{\cdots op})^{op})^{\cdots op}$ where the number of *op* exponents is arbitrary in each position. Those braids are alternately described as lying in $H\sigma_2 K \subset B_4$ where H is the cyclic subgroup generated by the braid $\sigma_2\sigma_1\sigma_3\sigma_2$ and K is the subgroup generated by the two generators $\{\sigma_1, \sigma_3\}$. The latter subgroup K is isomorphic to $Z \times Z$. The first coordinate corresponds to the number of *op* exponents on $\mathcal{A}$ and the second component to the number of *op* exponents on $\mathcal{B}$. Negative integers correspond to the right opposites, *po*. The power of the element of H corresponds to the number of *op* exponents on the product of the two enriched categories, that is, the number of *op* exponents outside the parentheses. That $b \in H\sigma_2 K$ implies $Lb = Rb$ follows from the fact that the composition morphisms belonging to the opposite of a $\mathcal{V}$-category obey the pentagon axiom. An exercise of some value is to check consistency of the definitions by constructing an inductive proof of the implication based on braid group generators. This is not a necessary condition for $Lb = Rb$, since for example the equation holds for $b = (\sigma_2\sigma_1\sigma_3\sigma_2)^n$, but it may be when the additional requirement that $\sigma(b) = (2\ 3)$ is added. More work needs to be done to determine the necessary conditions and to study the structure and properties of the braids that meet these conditions. Of course we will see shortly that when the unit conditions are obeyed then there is a necessary and sufficient condition. Note that the unit conditions are not met by $b_{(2)}$ and $b_{(3)}$ above.

Functoriality of the associator is necessary because here we need a 2-natural transformation $\alpha^{(1)}$. This means we have a family of $\mathcal{V}$-functors indexed by triples of $\mathcal{V}$-categories. On objects

$\alpha^{(1)}_{\mathcal{ABC}}((A,B),C) = (A,(B,C))$. In order to guarantee that $\alpha^{(1)}$ obey the coherence pentagon for hom-object morphisms, we define it to be *based upon* α in $\mathcal{V}$. This means precisely that:

$$\alpha^{(1)}_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}} : [(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A,B),C)((A',B'),C')) \to [\mathcal{A}\otimes^{(1)}(\mathcal{B}\otimes^{(1)}\mathcal{C})]((A,(B,C))(A',(B',C')))$$

is equal to

$$\alpha_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')} : (\mathcal{A}(A,A')\otimes\mathcal{B}(B,B'))\otimes\mathcal{C}(C,C') \to \mathcal{A}(A,A')\otimes(\mathcal{B}(B,B')\otimes\mathcal{C}(C,C')).$$

This definition guarantees that the $\alpha^{(1)}$ pentagons for objects and for hom-objects commute: the first trivially and the second by the fact that the α pentagon commutes in $\mathcal{V}$. We must also check for $\mathcal{V}$-functoriality. The unit axioms are trivial – we consider the more interesting axiom. The following diagram must commute, where the first bullet represents

$$[(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A',B'),C'),((A'',B''),C''))\otimes[(\mathcal{A}\otimes^{(1)}\mathcal{B})\otimes^{(1)}\mathcal{C}](((A,B),C),((A',B'),C'))$$

and the last bullet represents

$$[\mathcal{A}\otimes^{(1)}(\mathcal{B}\otimes^{(1)}\mathcal{C})]((A,(B,C)),(A'',(B'',C''))).$$

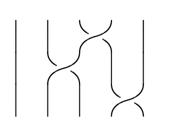
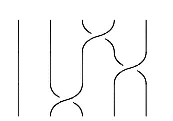
$$\begin{tikzcd}
\bullet \arrow[r, "M"] \arrow[d, "\alpha^{(1)}\otimes\alpha^{(1)}"] & \bullet \arrow[d, "\alpha^{(1)}"] \\
\bullet \arrow[r, "M"'] & \bullet
\end{tikzcd}$$

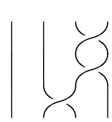
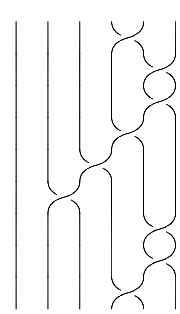
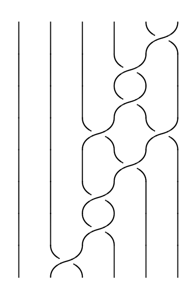
In $\mathcal{V}$ let $X = \mathcal{A}(A',A'')$, $Y = \mathcal{B}(B',B'')$, $Z = \mathcal{C}(C',C'')$, $X' = \mathcal{A}(A,A')$, $Y' = \mathcal{B}(B,B')$ and $Z' = \mathcal{C}(C,C')$ Then expanding the above diagram (where we leave out some parentheses for clarity and denote various composites of α and c by unlabeled arrows) we have

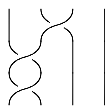
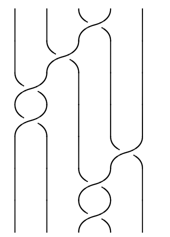
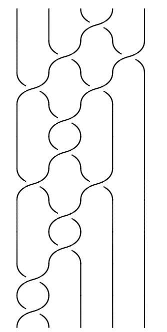
$$
\begin{tikzcd}[column sep=small]
 & (X \otimes Y) \otimes Z \otimes (X' \otimes Y') \otimes Z' \arrow[dl] \arrow[dr] & \\
X \otimes (Y \otimes Z) \otimes X' \otimes (Y' \otimes Z') \arrow[d] & & (X \otimes Y) \otimes (X' \otimes Y') \otimes Z \otimes Z' \arrow[d] \\
X \otimes X' \otimes (Y \otimes Z) \otimes (Y' \otimes Z') \arrow[d] & & [X \otimes Y \otimes X' \otimes Y'] \otimes (Z \otimes Z') \arrow[d] \\
(X \otimes X') \otimes [Y \otimes Z \otimes Y' \otimes Z'] \arrow[d] & & [(X \otimes X') \otimes (Y \otimes Y')] \otimes (Z \otimes Z') \arrow[d, "(M \otimes M) \otimes M"] \arrow[dll, "\alpha"] \\
(X \otimes X') \otimes [(Y \otimes Y') \otimes (Z \otimes Z')] \arrow[dr, "M \otimes (M \otimes M)"] & & [\mathcal{A}(A, A'') \otimes \mathcal{B}(B, B'')] \otimes \mathcal{C}(C, C'') \arrow[dl, "\alpha"] \\
 & \mathcal{A}(A, A'') \otimes [\mathcal{B}(B, B'') \otimes \mathcal{C}(C, C'')] &
\end{tikzcd}
$$

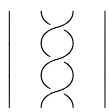
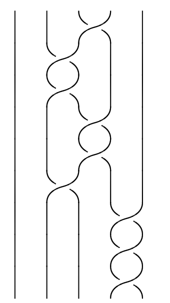
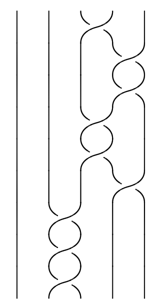
The bottom quadrilateral commutes by naturality of α. The top region must then commute for the diagram to commute. These basic nodes must be present regardless of the choice of braid by which the composition morphism is defined. Notice that the right and left legs have the following underlying braids for some examples of various choices of b. The two derived braids in B_6 we will refer to as $L'b$ and $R'b$. $L'b$ is formed from b by first pairing strands 2 and 3 , as well as strands 5 and 6 and performing b on the resulting four (groups of) strands. Then b is performed exactly on the last 4 strands. $R'b$ is derived in an analogous way as seen in the following examples. The first is the one used in the original definition of $\otimes^{(1)}$ given above.

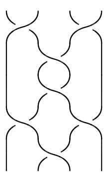
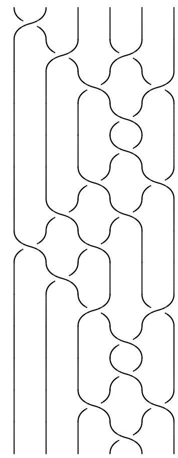
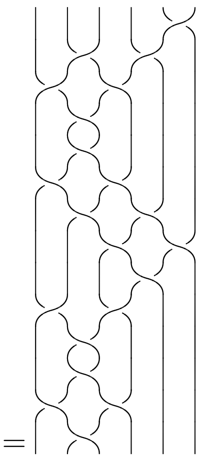
2.8**. Example.**

$b_{(1)} =$  Functoriality follows from:  $=$ 

$b_{(2)} =$  Functoriality follows from:  $=$ 

$b_{(3)} =$  Functoriality does not follow since:  $\neq$ 

$b_{(4)} =$  Functoriality does not follow since:  $\neq$ 

$b_{(5)} =$  Functoriality does follow since:  $=$ 

A comparison with the previous examples is of interest. Braids $b_{(2)}$ and $b_{(3)}$ are 180 degree rotations of each other. Notice that the second braid in the set of functoriality examples leads to an equality that is actually the same as for the third braid in the set of associativity examples. To see this the page must be rotated by 180 degrees. Similarly, the inequality preventing braid $b_{(2)}$ from being associative is the 180 degree rotation of the inequality preventing braid $b_{(3)}$ from being functorial. Braid $b_{(1)}$ and braid $b_{(5)}$ are each their own 180 degree rotation (we took advantage of the latter fact in drawing $L'b_{(5)}$ and $R'b_{(5)}$ above), and the two braids proving each to be the underlying braid of an associative composition morphism are the same two that show each to underlie a functorial associator. Braid $b_{(4)}$ is its own 180 degree rotation, and the two braids preventing it from being associative are the same two that obstruct it from being functorial. Thus there is a certain kind of duality between the requirements of associativity of the enriched composition and the functoriality of the associator. The full meaning of this duality becomes more clear in the study of enrichment over iterated monoidal categories, where we see that in a braided category two potentially different multiplications have collapsed into one.

If we were considering a strictly associative monoidal category $\mathcal{V}$ then the condition of a functorial associator would become a condition of a well defined composition morphism. Including the coherent associator is somewhat more enlightening.

The question now is whether there are braids underlying the composition of a product of enriched categories besides the braids $b_{(1)}$ and $b_{(5)}$ above (and their inverses) which fulfill both obligations. The answer is yes. We have defined functions $L, R, L', R' : B_4 \to B_6$.

2.9**. Definition.** An *associative unital* braid on four strands is one for which the permutation associated to the braid is (2 3), for which both $Lb = Rb$ and $L'b = R'b$ in B_6, and for which the unit conditions are satisfied: deleting any one of the pairs of strands (1, 2); (3, 4); (1, 3) , or (2, 4) results in the 2 strand identity braid. An *interchange candidate* braid is an element of B_4 which has the correct permutation and obeys the unit conditions.

To find associative braids we need only use the duality structure that exists on $\mathcal{V}$-Cat. By $\mathcal{A}^{op^n}$ is denoted the n^{th} (left) opposite of $\mathcal{A}$. By $\otimes^{(1)}$ and $\otimes'^{(1)}$ we denote the standard multiplications defined respectively with braid $b_{(1)}$ and its inverse.

2.10. **Theorem.** *The multiplication of enriched categories given by*

$$\mathcal{A} \otimes^{(1)}_{1^-} \mathcal{B} = (\mathcal{A}^{op} \otimes'^{(1)} \mathcal{B}^{op})^{po}$$

is a valid monoidal product on $\mathcal{V}$-Cat. Furthermore, so are the multiplications

$$\mathcal{A} \otimes^{(1)}_{n^-} \mathcal{B} = (\mathcal{A}^{op^n} \otimes'^{(1)} \mathcal{B}^{op^n})^{po^n}$$

as well as those with underlying braids that are the inverses of these, denoted

$$\mathcal{A} \otimes^{(1)}_{n^+} \mathcal{B} = (\mathcal{A}^{po^n} \otimes^{(1)} \mathcal{B}^{po^n})^{op^n}.$$

Proof. The first multiplication is mentioned alone since the middle exchange in its composition morphism has the underlying braid shown above as braid $b_{(5)}$. Thus we have already demonstrated its fitness as a monoidal product. However this can be more efficiently shown just by noting that the category given by the product is certainly a valid enriched category, and that for three operands we have an associator from the isomorphism given by the following:

$$\begin{aligned}
&(\mathcal{A} \otimes^{(1)}_{1^-} \mathcal{B}) \otimes^{(1)}_{1^-} \mathcal{C} \\
&= (((\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op})^{po})^{op} \otimes^{(1)} \mathcal{C}^{op})^{po} \\
&= ((\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op}) \otimes^{(1)} \mathcal{C}^{op})^{po} \\
&\cong (\mathcal{A}^{op} \otimes^{(1)} (\mathcal{B}^{op} \otimes^{(1)} \mathcal{C}^{op}))^{po} \\
&= (\mathcal{A}^{op} \otimes^{(1)} ((\mathcal{B}^{op} \otimes^{(1)} \mathcal{C}^{op})^{po})^{op})^{po} \\
&= \mathcal{A} \otimes^{(1)}_{1^-} (\mathcal{B} \otimes^{(1)}_{1^-} \mathcal{C})
\end{aligned}$$

The associator implicit in the isomorphism here is the same as the one defined above as based upon α in $\mathcal{V}$, since the object sets of the domain and range are identical and since:

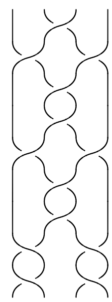
$$[(\mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op})^{po}]((A,B)(A',B')) = [\mathcal{A} \otimes^{(1)} \mathcal{B}]((A,B)(A',B'))$$

It is clear that this can be repeated with all the left opposites and right opposites raised to the n^{th} degree. Recall that the unit $\mathcal{V}$-category $\mathcal{I}$ has only one object 0 and $\mathcal{I}(0,0) = I$, the unit in $\mathcal{V}$. That $\mathcal{I}$ is indeed a unit for the multiplications in question follows from the facts that $\mathcal{I}^{op} = \mathcal{I} = \mathcal{I}^{po}$ which are in turn evident from facts $c_{IA} = c_{AI} = 1_A$. Thus we have that, using any of the above multiplications including the standard ones $\otimes^{(1)} = \otimes^{(1)}_{0^+}$ and $\otimes'^{(1)} = \otimes^{(1)}_{0^-}$ defined respectively with braid $b_{(1)}$ and its inverse, $\mathcal{V}$-Cat is a monoidal 2-category. $\square$

The braids underlying these new multiplications are not hard to describe directly. Suppressing the associators, the instances of the braiding forming the middle exchange for $\otimes^{(1)}_{1^+}$ are as follows, using X', Y', X, Y to stand for hom objects as above:

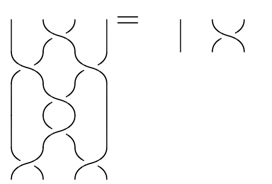
$$(c_{XX'} \otimes c_{YY'}) \circ (1_X \otimes c^{-1}_{YX'} \otimes 1_{Y'}) \circ (c^{-1}_{(X' \otimes Y')(X \otimes Y)}),$$

with underlying braid $b_{(5)}$. Note that $b_{(5)}$ has also been denoted b_{1^-}. Another is the following braid which underlies $\otimes^{(1)}_{2^+}$:



Note that this is precisely the braid b_{2^+} shown in the introduction. In fact the construction of the new products leads to the observation that the braid underlying the middle exchange in the composition for the product $\mathcal{A} \otimes^{(1)}_{n^\pm} \mathcal{B}$ is the previously defined braid $b_{n^\pm}$.

2.11. *Remark.* Note that if in the definition of $\otimes^{(1)}_{n^\pm}$ we replace $\otimes'^{(1)}$ with $\otimes^{(1)}$ or vice versa, then we have another valid multiplication, but with a braid underlying the middle exchange in its composition morphisms equivalent to that found in $\otimes^{(1)}_{(n-1)^\pm}$. For example:



Simply having a valid multiplication in a *specific* $\mathcal{V}$-Cat, with a middle exchange built out of instances of the the associators and the braiding, does not imply that the underlying braid of the middle exchange is associative. In order to argue that the braids $b_{n^\pm}$ are associative, we need to show that having a multiplication of $\mathcal{V}$-categories which is valid for an arbitrary braided base does imply an underlying associative braid. We will also show that this condition of being equivalent to some $b_{n^\pm}$ is necessary, offer some quick checks to determine when this condition holds, and investigate when the resulting monoidal categories are equivalent. All these steps are best taken in the context of iterated monoidal categories. Before we switch to that arena, we digress briefly from the main task to investigate whether the new underlying braids have any effect on the related structure of the category of enriched categories.

2.1. **Obstructions to braiding in $\mathcal{V}$-Cat .** Notice that in the case of symmetric $\mathcal{V}$ the axioms of enriched categories for $\mathcal{A} \otimes^{(1)} \mathcal{B}$ and the existence of a coherent 2-natural associator follow from the coherence of symmetric categories and the enriched axioms for $\mathcal{A}$ and $\mathcal{B}$. It remains to consider just why it is that $\mathcal{V}$-Cat is braided if and only if $\mathcal{V}$ is symmetric, and that if so then $\mathcal{V}$-Cat is symmetric as well. This fact is stated in [Joyal and Street, 1993]. We choose to give a proof here which covers all possible associative braids explicitly, and all potential braidings on $\mathcal{V}$-Cat based on any odd power of the braiding on $\mathcal{V}$, by appealing to information from the theory of knots and links. This is opposed to arguments based on the fact that a braiding transports over a tensor equivalence, and on Theorem 3.7. Our choice allows us to demonstrate how low dimensional topology can inform category theory as well as vice versa. A braiding $c^{(1)}$ on $\mathcal{V}$-Cat is a 2-natural transformation so $c^{(1)}_{\mathcal{AB}}$ is a $\mathcal{V}$-functor $\mathcal{A} \otimes^{(1)} \mathcal{B} \to \mathcal{B} \otimes^{(1)} \mathcal{A}$. On objects $c^{(1)}_{\mathcal{AB}}((A,B)) = (B,A)$. Now to be precise we define $c^{(1)}$ to be based upon c to mean that

$$c^{(1)}_{\mathcal{AB}_{(A,B)(A',B')}} : (\mathcal{A} \otimes^{(1)} \mathcal{B})((A,B),(A',B')) \to (\mathcal{B} \otimes^{(1)} \mathcal{A})((B,A),(B',A'))$$

is exactly equal to

$$c_{\mathcal{A}(A,A')\mathcal{B}(B,B')} : \mathcal{A}(A,A') \otimes \mathcal{B}(B,B') \to \mathcal{B}(B,B') \otimes \mathcal{A}(A,A')$$

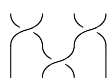
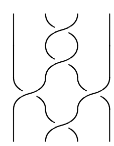
This potential braiding must be checked for $\mathcal{V}$-functoriality. Again the unit axioms are trivial and we consider the more interesting associativity of hom-object morphisms property. The following diagram must commute

$$\begin{tikzcd}
(\mathcal{A} \otimes^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes (\mathcal{A} \otimes^{(1)} \mathcal{B})((A,B),(A',B')) \arrow[r, "M"] \arrow[d, "c^{(1)} \otimes c^{(1)}"] & (\mathcal{A} \otimes^{(1)} \mathcal{B})((A,B),(A'',B'')) \arrow[d, "c^{(1)}"] \\
(\mathcal{B} \otimes^{(1)} \mathcal{A})((B',A'),(B'',A'')) \otimes (\mathcal{B} \otimes^{(1)} \mathcal{A})((B,A),(B',A')) \arrow[r, "M"] & (\mathcal{B} \otimes^{(1)} \mathcal{A})((B,A),(B'',A''))
\end{tikzcd}$$

Let $X = \mathcal{A}(A',A'')$, $Y = \mathcal{B}(B',B'')$, $Z = \mathcal{A}(A,A')$ and $W = \mathcal{B}(B,B')$ Then expanding the above diagram using the composition defined as above (denoting various composites of α by unlabeled arrows) we have

$$\begin{tikzcd}
 & (X\otimes Y)\otimes(Z\otimes W) \arrow[dl, "c_{XY}\otimes c_{ZW}"'] \arrow[dr] & \\
(Y\otimes X)\otimes(W\otimes Z) \arrow[d] & & X\otimes((Y\otimes Z)\otimes W) \arrow[d, "1\otimes(c_{YZ}\otimes 1)"] \\
Y\otimes((X\otimes W)\otimes Z) \arrow[d, "1\otimes(c_{XW}\otimes 1)"] & & X\otimes((Z\otimes Y)\otimes W) \arrow[d] \\
Y\otimes((W\otimes X)\otimes Z) \arrow[d] & & (X\otimes Z)\otimes(Y\otimes W) \arrow[d, "M_{AA'A''}\otimes M_{BB'B''}"] \arrow[dll, "c_{(X\otimes Z)(Y\otimes W)}"'] \\
(Y\otimes W)\otimes(X\otimes Z) \arrow[dr, "M_{BB'B''}\otimes M_{AA'A''}"'] & & \mathcal{A}(A,A'')\otimes\mathcal{B}(B,B'') \arrow[dl, "c"] \\
 & \mathcal{B}(B,B'')\otimes\mathcal{A}(A,A'') &
\end{tikzcd}$$

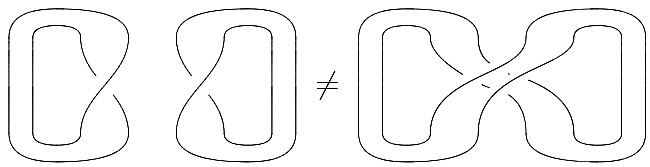
The bottom quadrilateral commutes by naturality of c. The top region must then commute for the diagram to commute, but the left and right legs have the following underlying braids

 $\neq$ 

Thus as noted in [Joyal and Street, 1993] neither braid $b_{(1)}$ nor its inverse can in general give a monoidal structure with a braiding based on the original braiding. In fact, it is easy to show more.

2.12. **Theorem.** *Composition morphisms for product enriched categories with any underlying braid b will fail to produce a braiding in $\mathcal{V}$-Cat based upon the braiding c in $\mathcal{V}$. Moreover, this failure will also be the case for attempts to produce a braiding in $\mathcal{V}$-Cat based upon any (odd) power c^{2n+1}.*

Proof. Notice that in the above braid inequality each side of the inequality consists of the braid which underlies the definition of the composition morphism, in this case $b_{(1)}$, and an additional braid which underlies the segment of the preceding diagram that corresponds to a composite of $c^{(1)}$. In terms of braid generators the left side of the braid inequality begins with $\sigma_1\sigma_3$ corresponding to $c_{XY}\otimes c_{ZW}$ and the right side of the braid inequality ends with $\sigma_2\sigma_1\sigma_3\sigma_2$ corresponding to $c_{(X\otimes Z)(Y\otimes W)}$. Since the same braid b must end the left side as begins the right side, then for the diagram to commute we require $b\sigma_1\sigma_3 = \sigma_2\sigma_1\sigma_3\sigma_2 b$. This implies $\sigma_1\sigma_3 = b^{-1}\sigma_2\sigma_1\sigma_3\sigma_2 b$, or that the braids $\sigma_1\sigma_3$ and $\sigma_2\sigma_1\sigma_3\sigma_2$ are conjugate in B_4. Conjugate braids have precisely the same link as their closures, but the closure of $\sigma_1\sigma_3$ is an unlinked pair of circles whereas the closure of $\sigma_2\sigma_1\sigma_3\sigma_2$ is the Hopf link.



If we instead let

$$c^{(1)}_{\mathcal{AB}_{(A,B)(A',B')}} = c^{2n+1}_{\mathcal{A}(A,A')\mathcal{B}(B,B')}$$

then the requirement becomes that the braids $(\sigma_1\sigma_3)^{2n+1}$ and $(\sigma_2\sigma_1\sigma_3\sigma_2)^{2n+1}$ are conjugate in B_4. Both braids have as closure a link of two components–two copies of the $(2n+1, 2)$-torus knot. However the first closure is two unlinked copies of the knot while in the second closure the two (cabled) copies are linked with linking number $2n+1$. Thus the braids cannot be conjugate, and so the braids underlying the legs of the functoriality diagram will not be equal for any choice of middle exchange. □

2.13**. Corollary.** *It is also interesting to note that the braid inequality above is the 180 degree rotation of the one which implies that* $(\mathcal{A} \otimes^{(1)} \mathcal{B})^{op} \neq \mathcal{A}^{op} \otimes^{(1)} \mathcal{B}^{op}$. *Thus the proof also implies that the latter inequality holds for product enriched categories with any braid b underlying their composition morphisms, as well as any power of op as the exponent.*

2.14. *Remark.* It is quickly seen that if c is a symmetry then in the second half of the braid inequality the upper portion of the braid consists of c_{YZ} and $c_{ZY} = c^{-1}_{YZ}$ so in fact equality holds. In that case then the derived braiding $c^{(1)}$ is a symmetry simply due to the definition.

3. 2-fold Monoidal Categories

In this section we closely follow the authors of [Balteanu et.al, 2003] in defining a notion of iterated monoidal category. For those readers familiar with that source, note that we vary from their definition only by including associativity up to natural coherent isomorphisms. This includes changing the basic picture from monoids to something that is a monoid only up to a monoidal natural transformation. The main reason for this more general approach is to allow for (strong) associativity in related braided categories, following [Forcey, 2004] in which it is pointed out that the axioms for a multiplication on $\mathcal{V}$-Cat mirror those which are about to follow.

3.1**. Definition.** A *monoidal functor* $(F, \eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(I) = I$ together with a natural transformation

$$\eta_{AB} : F(A) \otimes F(B) \to F(A \otimes B),$$

which satisfies the following conditions

(1) Internal Associativity: The following diagram commutes

$$\begin{tikzcd}
(F(A)\otimes F(B))\otimes F(C) \arrow[r, "\eta_{AB}\otimes 1_{F(C)}"] \arrow[d, "\alpha"] & F(A\otimes B)\otimes F(C) \arrow[d, "\eta_{(A\otimes B)C}"] \\
F(A)\otimes (F(B)\otimes F(C)) \arrow[d, "1_{F(A)}\otimes\eta_{BC}"] & F((A\otimes B)\otimes C) \arrow[d, "F\alpha"] \\
F(A)\otimes F(B\otimes C) \arrow[r, "\eta_{A(B\otimes C)}"] & F(A\otimes (B\otimes C))
\end{tikzcd}$$

(2) Internal Unit Conditions: $\eta_{AI} = \eta_{IA} = 1_{F(A)}$.

Given two monoidal functors $(F, \eta) : \mathcal{C} \to \mathcal{D}$ and $(G, \zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF, \xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\otimes GF(B) \xrightarrow{\zeta_{F(A)F(B)}} G\big(F(A)\otimes F(B)\big) \xrightarrow{G(\eta_{AB})} GF(A\otimes B).$$

It is easy to verify that ξ satisfies the internal associativity condition above by subdividing the necessary commuting diagram into two regions that commute by the axioms for η and ζ respectively and two that commute due to their naturality. **MonCat** is the monoidal category of monoidal categories and monoidal functors, with the usual Cartesian product as in **Cat**.

A *monoidal natural transformation* $\theta : (F, \eta) \to (G, \zeta) : \mathcal{D} \to \mathcal{E}$ is a natural transformation $\theta : F \to G$ between the underlying ordinary functors of F and G such that the following diagram commutes

$$\begin{tikzcd}
F(A)\otimes F(B) \arrow[r, "\eta"] \arrow[d, "\theta_A\otimes\theta_B"] & F(A\otimes B) \arrow[d, "\theta_{A\otimes B}"] \\
G(A)\otimes G(B) \arrow[r, "\zeta"] & G(A\otimes B)
\end{tikzcd}$$

3.2**. Definition.** For our purposes a 2-*fold monoidal category* is a tensor object in **MonCat**. This means that we are given a monoidal category $(\mathcal{V}, \otimes_1, \alpha^1, I)$ and a monoidal functor $(\otimes_2, \eta) : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ which satisfies

(1) External Associativity: the following diagram describes a monoidal natural isomorphism α^2 in **MonCat**.

$$\begin{tikzcd}
\mathcal{V}\times\mathcal{V}\times\mathcal{V} \arrow[r, "(\otimes_2,\eta)\times 1_{\mathcal{V}}"] \arrow[d, "1_{\mathcal{V}}\times(\otimes_2,\eta)"'] & \mathcal{V}\times\mathcal{V} \arrow[d, "(\otimes_2,\eta)"] \arrow[dl, Rightarrow, "\alpha^2"] \\
\mathcal{V}\times\mathcal{V} \arrow[r, "(\otimes_2,\eta)"'] & \mathcal{V}
\end{tikzcd}$$

(2) External Unit Conditions: the following diagram commutes in **MonCat**

$$\begin{tikzcd}
\mathcal{V} \times I \arrow[r, "\subseteq"] \arrow[dr, "\cong"'] & \mathcal{V} \times \mathcal{V} \arrow[d, "(\otimes_2, \eta)"] & I \times \mathcal{V} \arrow[l, "\supseteq"'] \arrow[dl, "\cong"] \\
& \mathcal{V} &
\end{tikzcd}$$

(3) Coherence: The underlying natural transformation α^2 satisfies the usual coherence pentagon.

Explicitly this means that we are given a second associative binary operation $\otimes_2 : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, for which I is also a two-sided unit. We are also given a natural transformation called the *interchanger*

$$\eta_{ABCD} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D).$$

The internal unit conditions give $\eta_{ABII} = \eta_{IIAB} = 1_{A \otimes_2 B}$, while the external unit conditions give $\eta_{AIBI} = \eta_{IAIB} = 1_{A \otimes_1 B}$. The internal associativity condition gives the commutative diagram

$$\begin{tikzcd}
((U \otimes_2 V) \otimes_1 (W \otimes_2 X)) \otimes_1 (Y \otimes_2 Z) \arrow[r, "\eta_{UVWX} \otimes_1 1_{Y \otimes_2 Z}"] \arrow[d, "\alpha^1"] & \big((U \otimes_1 W) \otimes_2 (V \otimes_1 X)\big) \otimes_1 (Y \otimes_2 Z) \arrow[d, "\eta_{(U \otimes_1 W)(V \otimes_1 X) YZ}"] \\
(U \otimes_2 V) \otimes_1 ((W \otimes_2 X) \otimes_1 (Y \otimes_2 Z)) \arrow[d, "1_{U \otimes_2 V} \otimes_1 \eta_{WXYZ}"] & ((U \otimes_1 W) \otimes_1 Y) \otimes_2 ((V \otimes_1 X) \otimes_1 Z) \arrow[d, "\alpha^1 \otimes_2 \alpha^1"] \\
(U \otimes_2 V) \otimes_1 \big((W \otimes_1 Y) \otimes_2 (X \otimes_1 Z)\big) \arrow[r, "\eta_{UV(W \otimes_1 Y)(X \otimes_1 Z)}"] & (U \otimes_1 (W \otimes_1 Y)) \otimes_2 (V \otimes_1 (X \otimes_1 Z))
\end{tikzcd}$$

The external associativity condition gives the commutative diagram

$$\begin{tikzcd}
((U \otimes_2 V) \otimes_2 W) \otimes_1 ((X \otimes_2 Y) \otimes_2 Z) \arrow[r, "\eta_{(U \otimes_2 V) W (X \otimes_2 Y) Z}"] \arrow[d, "\alpha^2 \otimes_1 \alpha^2"] & \big((U \otimes_2 V) \otimes_1 (X \otimes_2 Y)\big) \otimes_2 (W \otimes_1 Z) \arrow[d, "\eta_{UVXY} \otimes_2 1_{W \otimes_1 Z}"] \\
(U \otimes_2 (V \otimes_2 W)) \otimes_1 (X \otimes_2 (Y \otimes_2 Z)) \arrow[d, "\eta_{U(V \otimes_2 W) X (Y \otimes_2 Z)}"] & ((U \otimes_1 X) \otimes_2 (V \otimes_1 Y)) \otimes_2 (W \otimes_1 Z) \arrow[d, "\alpha^2"] \\
(U \otimes_1 X) \otimes_2 \big((V \otimes_2 W) \otimes_1 (Y \otimes_2 Z)\big) \arrow[r, "1_{U \otimes_1 X} \otimes_2 \eta_{VWYZ}"] & (U \otimes_1 X) \otimes_2 ((V \otimes_1 Y) \otimes_2 (W \otimes_1 Z))
\end{tikzcd}$$

Just as in [Balteanu et.al, 2003] we now define a 2-fold monoidal functor $(F, \lambda^1, \lambda^2)$ between 2-fold monoidal categories. It is a functor $F : \mathcal{V} \to \mathcal{W}$ together with two natural transformations:

$$\lambda^1_{AB} : F(A) \otimes_1 F(B) \to F(A \otimes_1 B)$$

$$\lambda^2_{AB} : F(A) \otimes_2 F(B) \to F(A \otimes_2 B)$$

satisfying the same associativity and unit conditions as in the case of monoidal functors. In addition we require that the following hexagonal interchange diagram commutes:

$$\begin{tikzcd}
(F(A)\otimes_2 F(B))\otimes_1(F(C)\otimes_2 F(D)) \arrow[rr, "\eta_{F(A)F(B)F(C)F(D)}"] \arrow[d, "\lambda^2_{AB}\otimes_1\lambda^2_{CD}"] & & (F(A)\otimes_1 F(C))\otimes_2(F(B)\otimes_1 F(D)) \arrow[d, "\lambda^1_{AC}\otimes_2\lambda^1_{BD}"] \\
F(A\otimes_2 B)\otimes_1 F(C\otimes_2 D) \arrow[d, "\lambda^1_{(A\otimes_2 B)(C\otimes_2 D)}"] & & F(A\otimes_1 C)\otimes_2 F(B\otimes_1 D) \arrow[d, "\lambda^2_{(A\otimes_1 C)(B\otimes_1 D)}"] \\
F((A\otimes_2 B)\otimes_1(C\otimes_2 D)) \arrow[rr, "F(\eta_{ABCD})"] & & F((A\otimes_1 C)\otimes_2(B\otimes_1 D))
\end{tikzcd}$$

We can now refer to the category **2−MonCat** of 2-fold monoidal categories and 2-fold monoidal functors.

The authors of [Balteanu et.al, 2003] remark that we have natural transformations

$$\eta_{AIIB} : A\otimes_1 B \to A\otimes_2 B \qquad \text{and} \qquad \eta_{IABI} : A\otimes_1 B \to B\otimes_2 A.$$

If they had insisted a 2-fold monoidal category be a tensor object in the category of monoidal categories and *strictly monoidal* functors, this would be equivalent to requiring that $\eta = 1$. In view of the above, they note that this would imply $A\otimes_1 B = A\otimes_2 B = B\otimes_1 A$ and similarly for morphisms. This is shown by what is usualy referred to as the Eckmann-Hilton argument.

Joyal and Street [Joyal and Street, 1993] considered a similar concept to Balteanu, Fiedorowicz, Schwänzl and Vogt's idea of 2-fold monoidal category. The former pair required the natural transformation η_{ABCD} to be an isomorphism and showed that the resulting category is naturally equivalent to a braided monoidal category. As explained in [Balteanu et.al, 2003], given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\otimes_2$, and defining the braiding for the remaining operation $\otimes_1$ to be the composite

$$A\otimes_1 B \xrightarrow{\eta_{IABI}} B\otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B\otimes_1 A.$$

In [Balteanu et.al, 2003] it is shown that a 2-fold monoidal category with $\otimes_1 = \otimes_2$, η an isomorphism and

$$\eta_{AIBC} = \eta_{ABIC} = 1_{A\otimes B\otimes C}$$

is a braided monoidal category with the braiding $c_{BC} = \eta_{IBCI}$.

Also note that for $\mathcal{V}$ braided the interchange given by $\eta_{ABCD} = 1_A\otimes c_{BC}\otimes 1_D$ gives a 2-fold monoidal category where $\otimes_1 = \otimes_2$. This interchanger has the underlying braid $\sigma_2 \in B_4$. In this setting we ask whether, given a braiding, there are alternate 2-fold monoidal structures on $\mathcal{V}$, with $\otimes_1 = \otimes_2$. This is the same question as asking whether there are other associative unital braids besides $b_{0+} = b_{(1)} = \sigma_2$ and its inverse.

3.3. **Lemma.** *An interchange which is constructed from associators and instances of a braiding obeys the axioms of a 2-fold monoidal category (where $\otimes_1 = \otimes_2$) if and only if it has as its underlying braid an associative unital braid b.*

Proof. This is by the coherence theorem for braided categories. $Lb = Rb$ is precisely the internal associativity and $L'b = R'b$ is precisely the external associativity. The unit conditions for the interchanger are precisely the conditions on the braid described by the fact that deleting certain pairs of strands yields the identity braid. □

In [Forcey, 2004] it is shown that the 2-category of categories enriched over a 2-fold monoidal category is itself monoidal. The middle exchange for the product is given by an instance of η^{12}. In fact the theorem has a true converse only when the composition morphisms are invertible.

3.4. **Lemma.** *Let $\mathcal{V}$ be a 2-fold monoidal groupoid, $A, B, C, D \in |\mathcal{V}|$. A natural transformation $\eta_{ABCD} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$ is an interchanger if and only if it is a middle exchange for a product on $\mathcal{V}$-Cat.*

Proof. In [Forcey, 2004] it is demonstrated that each axiom of a 2-fold monoidal category (along with naturality) implies the corresponding axiom of a monoidal 2-category of enriched categories. Each axiom of a 2-fold monoidal category can be expressed as a commuting diagram which appears as a sub-diagram of the corresponding axiom of the monoidal 2-category of enriched categories. To see the converse we need merely redraw each full planar diagram to have the 2-fold monoidal category axiom appear on the exterior. Since the compositions M_{ABC} are invertible the direction of certain interior arrows may be reversed in order that the sub-diagrams have the correct source and target. □

Now we are ready to state and prove the main result.

3.5. **Theorem.** *A braid $b \in B_4$ is associative and unital if and only if it is equivalent to one of the braids $b_{n^\pm}$.*

Proof. By Lemma 3.4 we have that given an arbitrary braided groupoid $\mathcal{V}$ and given a middle exchange formed from the braiding (and which is thus a natural isomorphism), then that middle exchange is also an interchanger for a 2-fold monoidal structure on $\mathcal{V}$ with $\otimes_1 = \otimes_2$. Since we have shown $\otimes^{(1)}_{n^\pm}$ to be valid for an arbitrary braided base category $\mathcal{V}$ then it is true for the cases in which $\mathcal{V}$ is a groupoid. Thus by Theorem 2.10 the middle exchanges given by, suppressing the associators,

$$\eta_{ABCD} = (c^{\mp n}_{CA} \otimes c^{\mp n}_{DB}) \circ (1_C \otimes c^{\pm 1}_{DA} \otimes 1_B) \circ (c^{\pm n}_{(A \otimes B)(C \otimes D)}),$$

with underlying braid $b_{n^\pm}$, are indeed each an interchanger. Therefore by Lemma 3.3 the braids $b_{n^\pm}$ are associative and unital.

For the converse we assume that b is associative and unital and therefore by Lemma 3.3 it underlies an interchanger $\eta_{(b)}$ formed of instances of the braiding c in a braided category $\mathcal{V}$. We focus on the two strand sub-braids of b underlying ${\eta_{(b)}}_{AIIB}$

and $\eta_{(b)_{IABI}}$. It is seen by group operations that a selection of these two underlying braids uniquely determines the braid b. First assume that we have chosen a braid to underlie $\eta_{(b)_{IABI}}$, i.e. the inner sub-braid of b. Consider the internal associativity axiom but with $U = Z = W = I$. Now the top horizontal arrow of the diagram has the underlying identity braid on three strands, the left vertical side of the diagram has the underlying braid of $\eta_{(b)_{IXYI}}$ placed to the right of a single strand, and the right vertical has the underlying braid of $\eta_{(b)_{I,V\otimes X,Y,I}}$. This latter is just the choice we made for the inner two strand sub-braid of b, with the first strand doubled. The bottom horizontal arrow has the underlying braid of $\eta_{(b)_{IVYX}}$. This last three strand sub-braid of b is thus determined by the assumption that the diagram commutes, the braided coherence theorem, and the operation of taking the inverse in the braid group B_3. Thus we have found the three strand sub-braid of b formed by deleting the first strand. Next we assume that we have chosen a braid to underlie $\eta_{(b)_{AIIB}}$. Then we again use the internal associativity diagram this time with $V = X = Y = I$ to determine the underlying braid of $\eta_{(b)_{UIWZ}}$, i.e. the three strand sub-braid of b formed by deleting the second strand. Finally we set $W = V = I$ in the internal associativity diagram and use the choice of the outer two strand sub-braid of b (underlies the top arrow, with a pair of extra unbraided strands) and our predetermined three strand sub-braids (underlie the left and bottom with an extra strand and a doubled strand respectively). Thus by group operations in B_4 we can determine the braid underlying the right vertical side, which is the braid b underlying $\eta_{(b)_{UXYZ}}$.

Next, in order to limit the choices we can make for the underlying braids of $\eta_{(b)_{IABI}}$ and $\eta_{(b)_{AIIB}}$, we utilize Joyal and Street's result that for any interchanger a braiding is given by:

$$A \otimes_1 B \xrightarrow{\eta_{IABI}} B \otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B \otimes_1 A.$$

Thus by Lemma 2.6 and braided coherence we have the equation

$$\eta^{-1}_{(b)_{BIIA}} \circ \eta_{(b)_{IABI}} = c^{\pm 1}_{AB} \quad \text{or} \quad \eta_{(b)_{IABI}} = \eta_{(b)_{BIIA}} \circ c^{\pm 1}_{AB}.$$

Now in order for the permutation associated to b to be $(2\ 3)$, $\eta_{(b)_{IABI}}$ must be an odd power of c or c^{-1}. Therefore our choice for the underlying braids of $\eta_{(b)_{IABI}}$ and $\eta_{(b)_{AIIB}}$ is reduced respectively to a choice of an odd integer and a choice of one of its neighboring integers. The latter choice of ± 1 is the choice of the exponent of c in the above equation.

Now by Lemma 1.1 these possible choices for the underlying braids of $\eta_{(b)_{AIIB}}$ and $\eta_{(b)_{IABI}}$ are all actually represented by one of the $b_{n^{\pm}}$.

Therefore if any braid b is associative and unital then it is equivalent to one of the braids $b_{n^{\pm}}$.

□

The next item on the agenda is to investigate the equivalence of the various 2-fold monoidal structures which can be constructed from a braiding, with differing underlying associative braids.

3.1. **Equivalence of 2-fold Monoidal Categories.** By finding interchangers which are formed from a braiding we have actually defined a collection of functors from the category of braided categories to the category of 2-fold monoidal categories. The complete classification of associative unital braids is a well defined parameterization of this family.

3.6**. Definition.** For b an associative unital braid, the functor F_b takes each braided category $\mathcal{V}$ to itself, seen as a 2-fold monoidal category with interchanger $\eta_{(b)}$. A braided tensor functor f with $\phi : f(A) \otimes f(B) \to f(A \otimes B)$ is taken by F_b to a 2-fold monoidal functor $F_b(f)$ which has the same definition on objects and morphisms and for which $\lambda^1 = \lambda^2 = \phi$. Braided natural tranformations are also taken to themselves.

3.7**. Theorem.** *Given an associative unital braid b the functor F_b is equivalent to either $F_{b_{0+}}$ or to $F_{b_{0-}}$ but not to both.*

Proof. It is directly implied in [Joyal and Street, 1993] that given a 2-fold monoidal category $\mathcal{V}$ with $\otimes_1 = \otimes_2$ and with strong interchanger η then that category is equivalent to the 2-fold monoidal category $\mathcal{V}'$ with the same objects and morphisms but with interchanger given by

$$\eta'_{ABCD} = 1_A \otimes (\eta^{-1}_{CIIB} \circ \eta_{IBCI}) \otimes 1_D$$

For $\mathcal{V}$ braided and in terms of an original interchanger $\eta_{(b)}$ based on a braiding c with b associative and unital, we have seen that $\eta'_{ABCD} = 1_A \otimes c^{\pm 1}_{BC} \otimes 1_D$. Thus $\mathcal{V}' = F_{b_{0\pm}}(\mathcal{V})$. The 2-fold monoidal functorial equivalence $U_{\mathcal{V}} : F_{b_{0\pm}}(\mathcal{V}) \to F_b(\mathcal{V})$ is the identity on objects and morphisms. Explicitly $U_{\mathcal{V}}$ has $\lambda^2_{AB} = 1_{A\otimes B}$ and $\lambda^1_{AB} = \eta_{(b)_{AIIB}}$. This allows us to define in the target category:

$$\eta_{U_{\mathcal{V}}(A)U_{\mathcal{V}}(B)U_{\mathcal{V}}(C)U_{\mathcal{V}}(D)} = \eta_{(b)_{ABCD}}.$$

The required hexagonal interchange diagram commutes due to braided coherence, using the braid equalities mentioned in Remark 2.11.

For b such that $\eta^{-1}_{(b)_{CIIB}} \circ \eta_{(b)_{IBCI}} = c$, i.e. $b \in \{b_{n^+} \mid n \text{ is even }\} \bigcup \{b_{n^-} \mid n \text{ is odd }\}$, the family of functors $U_{\mathcal{V}}$ make up a natural isomorphism $U : F_{b_{0+}} \to F_b$.

For b such that $\eta^{-1}_{(b)_{CIIB}} \circ \eta_{(b)_{IBCI}} = c^{-1}$, i.e. $b \in \{b_{n^+} \mid n \text{ is odd }\} \bigcup \{b_{n^-} \mid n \text{ is even }\}$, the family of functors $U_{\mathcal{V}}$ make up a natural isomorphism $U : F_{b_{0-}} \to F_b$.

There is not in general a natural isomorphism from $F_{b_{0-}}$ to $F_{b_{0+}}$. If there were then the hexagonal interchange diagram for 2-fold monoidal functors with $A = D = I$ would become the diagram of braided equivalence between $\mathcal{V}$ with braiding c and $\mathcal{V}$ with braiding c^{-1}. There is not in general a braided equivalence between $\mathcal{V}$ with braiding c and $\mathcal{V}$ with braiding c^{-1} since any λ^2 (in general based upon c) would have to satisfy $\lambda^2 \circ c^{-1} = c \circ \lambda^2$ which is precluded by the braided coherence theorem and the fact that B_2 is abelian.

□

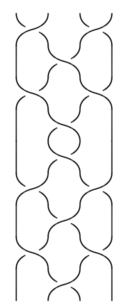
Thus the associative braids can be divided into two equivalence classes by the relation given by $b \equiv b'$ if F_b is equivalent to F'_b. The two classes are canonically represented by the braids b_{0^+} and b_{0^-}. It would be an interesting future study to consider the braid groups B_n for $n \geq 4$ modulo that equivalence relation on the first four strands. With that in mind we turn to examine some shortcuts to determining whether a given braid is associative and unital.

3.2. **Obstructions to being associative.** The general scheme is to find extra conditions on the interchanger $\eta_{(b)}$ which together with the unit conditions and the associativity conditions will force the underlying braid b to have easily checked characteristics. Then we can find families of unital braids in B_4 which cannot underlie an interchanger, i.e. which are not equivalent to any braid $b_{n^\pm}$.

3.8**. Theorem.** *Given an interchange candidate braid b with the property that deleting either the 2nd or 3rd strand gives the identity braid on three strands, then b is associative if and only if $b = \sigma_2$, the second generator of B_4, or its inverse.*

Proof. This follows the logic of [Balteanu et.al, 2003]. Letting $\eta = \eta_{(b)}$ be the interchanger based on the braiding of $\mathcal{V}$ with underlying braid b, note that deleting a strand in b corresponds to replacing the respective object in the product $A \otimes B \otimes C \otimes D$ with the identity I. Now let $V = W = I$ in the internal associativity diagram to see that due to the hypotheses on b we have that $\eta_{UXYZ} = 1_U \otimes \eta_{IXYZ}$. Then let $X = Y = I$ in the internal associativity diagram to see that $\eta_{UVWZ} = \eta_{UVWI} \otimes 1_Z$. Together these two facts imply that $\eta_{ABCD} = 1_A \otimes \eta_{IBCI} \otimes 1_D$. Then if we take $U = Z = W = 0$ in the internal associativity law we get the first axiom of a braided category for $c'_{BC} = \eta_{IBCI}$, and letting $U = Z = X = 0$ in the internal associativity diagram gives the other one. This then implies that either $c' = c$ or $c' = c^{-1}$, since no other combinations of c give a braiding. Therefore $\eta_{ABCD} = 1_A \otimes c^{\pm 1}_{BC} \otimes 1_D$ which has the underlying braid $\sigma_2^{\pm 1}$. The converse is also clear from this discussion, since all the implications can be reversed. Of course, we already have the converse since the braids $b_{0^\pm}$ are associative. □

This sort of obstruction can rule out candidate braids such as the braid $b_{(4)}$ in the last section. It also rules out all but one element each of the left and right $\sigma_2^{\pm(2n-1)}$-cosets of the Brunnian braids in B_4, where the Brunnian braids are those pure braids where any strand deletion gives the identity braid. Even more broadly this obstruction rules out braids such as:

$b =$ 

3.9**. Theorem.** *Let b be an interchange candidate braid with the property that deleting both the inner two strands leaves the identity sub-braid on the remaining two strands. Then if b is associative it follows that deleting either the second or the third strand will result in the three strand identity sub-braid on either of the remaining subsets of strands.*

Proof. Let $\eta_{(b)}$ be the interchanger based on the braiding, with underlying associative braid b. We are given that ${\eta_{(b)}}_{AIIB} = 1_{A\otimes B}$ and must demonstrate that ${\eta_{(b)}}_{AIBC} = {\eta_{(b)}}_{ABIC} = 1_{A\otimes B\otimes C}$. The conclusion about the deletion of the second strand is shown by considering the internal associativity diagram with $V = X = Y = I$. The conclusion about the deletion of the third strand is shown by considering the internal associativity diagram with $V = W = Y = I$. An alternative proof just uses the main results to check all the associative braids which fit the hypothesis. $\square$

This obstruction rules out all but one element each of the left and right $\sigma_2^{\pm(2n-1)}$-cosets of the 2-trivial or 2-decomposable braids in B_4. These latter braids are a generalization of the Brunnian braids in which deletion of any 2 strands results in a trivial braid.

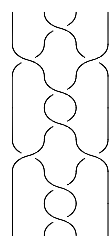
Notice that the longer associative braids $b_{n^\pm}$ for $n > 0$ give examples of interchangers that do not fit the conditions of the obstruction theorems so far. They also serve as examples of interchangers η such that η_{IBCI} is not a braiding. Recall however that they do give a braiding via $c'_{AB} = \eta^{-1}_{BIIA} \circ \eta_{IABI}$ as predicted by Joyal and Street. The latter condition also serves as a source of obstructions on its own. According to their theorem, any associative braid will have the property that dropping the outer two strands will give a two strand braid with one more or one less crossing of the same handedness than the two strand braid achieved by dropping the inner two strands. Indeed this condition rules out some of the same braids just mentioned, namely the Brunnian cosets of higher powers of σ_2 in B_4.

The next sort of obstruction is found by slightly weakening the extra conditions. This will allow us to rule out a larger, different class of candidates, but they will be a little bit harder to recognize.

3.10. **Theorem.** *Let b be an interchange candidate braid with the property that deleting either the first or the fourth strand results in a 3-strand braid that is just a power of the braid generator on what were the middle two strands: $\sigma_i^{\pm n}$; $i = 2$ or $i = 1$ respective of whether the first or fourth strand was deleted. Then b is associative implies that $n = 1$.*

Proof. The strand deletion conditions on the underlying braid b of η are equivalent to assuming that $\eta_{IBCD} = \eta_{IBCI} \otimes 1_D$ and that $\eta_{ABCI} = 1_A \otimes \eta_{IBCI}$. Of course the power of the generator σ_i being ± 1 is equivalent to saying that η_{IBCI} is the braiding c or its inverse. Hence we need only show that the assumptions imply that η_{IBCI} is a braiding. This is seen immediately upon letting $U = Z = W = 0$ in the internal associativity axiom to get the first axiom of a braiding and letting $U = Z = X = 0$ to get the other one. $\square$

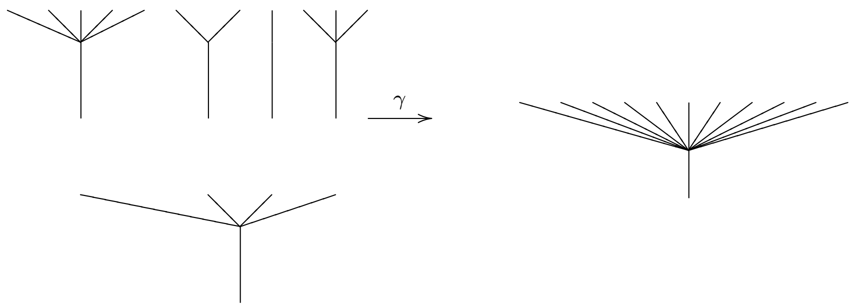
This theorem can directly rule out candidates which satisfy the Joyal and Street condition that $c_{AB} = \eta^{-1}_{BIIA} \circ \eta_{IABI}$ and the first or last strand deletion condition given here, but which fail to give a single crossing braid upon that removal. The simplest example is this braid:



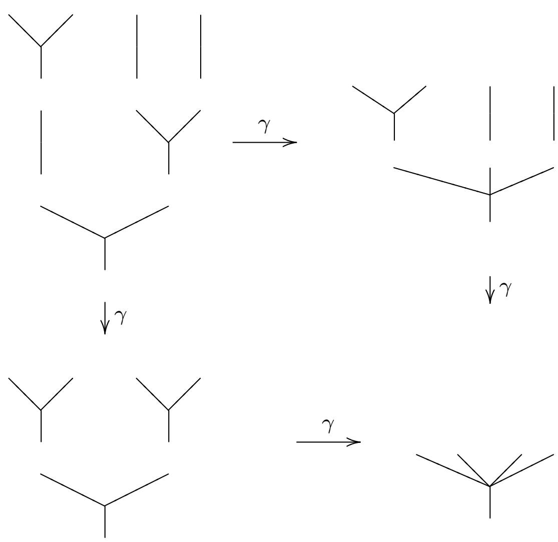
It is also true that a candidate braid which yields a single crossing after deletion of the first and fourth strands, if associative, must then obey the condition that deleting the first or last strand frees the other of those two from any crossings. This can be most easily seen by use of the main result; we simply check all four examples of associative unital braids which have inner two strand sub-braids a single crossing. They are $b_{0^\pm}$ and $b_{1^\pm}$.

4. Implications for operads.

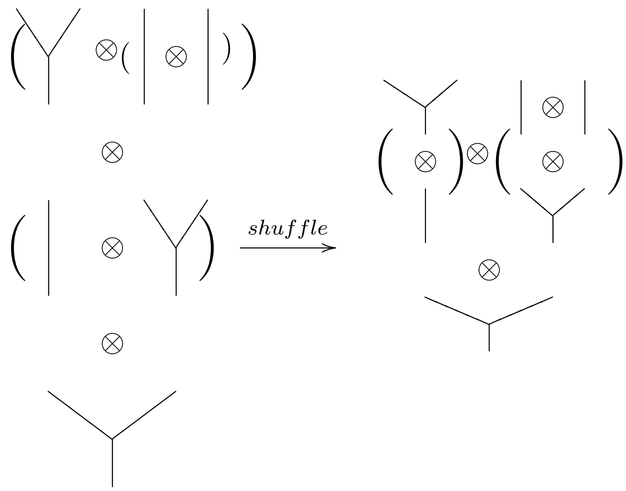
So far herein we have completely characterized families of interchangers based on a braiding which can define either a 2-fold monoidal structure on a category or a monoidal structure on a 2-category. Another common use of a braiding is to define a monoidal structure on a category of collections, as in the theory of operads. Operads in a 2-fold monoidal category are defined as monoids in a certain category of collections in [Forcey, 2007]. Here we repeat the basic ideas and the expanded definition in terms of commuting diagrams. The two principle components of an operad are a collection, historically a sequence, of objects in a monoidal category and a family of composition maps. Operads are often described as parameterizations of n-ary operations. Peter May's original definition of operad in a symmetric (or braided) monoidal category [May, 1972] has a composition γ that takes the tensor product of the n^{th} object (n-ary operation) and n others (of various arity) to a resultant that sums the arities of those others. The n^{th} object or n-ary operation is often pictured as a tree with n leaves, and the composition appears like this:



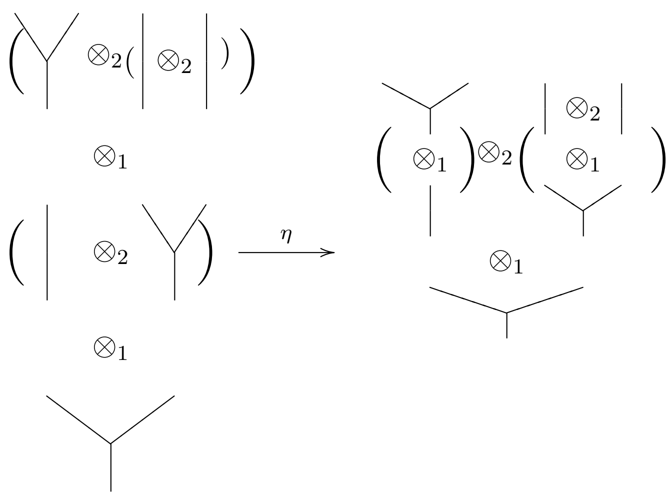
By requiring this composition to be associative we mean that it obeys this sort of pictured commuting diagram:



In the above pictures the tensor products are shown just by juxtaposition, but now we would like to think about the products more explicitly. If the monoidal category is not strict, then there is actually required another leg of the associativity diagram, where the tensoring is reconfigured so that the composition can operate in an alternate order. Here is how that rearranging looks in a symmetric (braided) category, where the shuffling is accomplished by use of the symmetry (braiding):



We now foreshadow our definition of operads in an iterated monoidal category with the same picture as above but using two tensor products, $\otimes_1$ and $\otimes_2$. It becomes clear that the true nature of the shuffle is in fact that of an interchange transformation.



To see this just focus on the actual domain and range of η which are the upper two levels of trees in the pictures, with the tensor product $(|\otimes_2|)$ considered as a single object.

Now we are ready to give the technical definitions. We begin with the definition of 2-fold operad in an n-fold monoidal category, as in the above picture, and then mention how it generalizes the case of operad in a braided category.

4.1**. Definition.** Let $\mathcal{V}$ be a strict 2-fold monoidal category. A 2-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J}: I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$

$$\gamma^{12}: \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms:

(1) Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s = \sum_{u=1}^{s} j_u$ and let $h_s = \sum_{u=1+g_{s-1}}^{g_s} i_u$. The η labeling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the interchange.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \mathcal{C}(j_s) \right) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2 \mathcal{C}(i_t) \right) \arrow[r, "\gamma^{12} \otimes_1 \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_1 \eta"'] & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2 \mathcal{C}(i_t) \right) \arrow[d, "\gamma^{12}"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \left(\mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{u=1}^{j_s} {}_2 \mathcal{C}(i_{u+g_{s-1}}) \right) \right) \right) \arrow[r, "\mathrm{id} \otimes_1 (\otimes_2^k \gamma^{12})"'] & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2 \mathcal{C}(h_s) \right) \arrow[u, "\gamma^{12}"']
\end{tikzcd}$$

(2) Respect of units is required just as in the symmetric case. The following unit diagrams commute.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 (\otimes_2^k I) \arrow[r, equal] \arrow[d, "1 \otimes_1 (\otimes_2^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_1 (\otimes_2^k \mathcal{C}(1)) \arrow[ur, "\gamma^{12}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_1 \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_1 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_1 \mathcal{C}(k) \arrow[ur, "\gamma^{12}"] &
\end{tikzcd}$$

Now the problem of describing the various sorts of operads in a braided monoidal category becomes more clear, as a special case. Here again we let $\otimes = \otimes_1 = \otimes_2$. The family of 2-fold structures based on associative braids gives rise to a family of monoidal structures on the category of collections, and thus to a family of operad structures.

In the operad picture the underlying braid of an operad structure only becomes important when we inspect the various ways of composing a product with 4 levels of trees in the heuristic diagram, such as $\mathcal{C}(2)\otimes(\mathcal{C}(1)\otimes\mathcal{C}(1))\otimes(\mathcal{C}(1)\otimes\mathcal{C}(1))\otimes(\mathcal{C}(1)\otimes\mathcal{C}(1))$. For this composition to be well defined we require the internal associativity of the interchange that is used to rearrange the terms. When we consider composing a product with 3 levels of trees in the heuristic diagram, but with a base term $\mathcal{C}(n)$ with $n \geq 3$, such as: $\mathcal{C}(3) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1) \otimes \mathcal{C}(1)) \otimes (\mathcal{C}(1) \otimes \mathcal{C}(1) \otimes \mathcal{C}(1))$, then we see that the external associativity of η is also required.

Thus the same theorems proven above for associative and nonassociative families of braids apply here as well, in deciding whether a certain braid based shuffling of the terms in an operad product is allowable. The point is that not all shuffles using a braiding make sense, and the viewpoint of the 2-fold monoidal structure is precisely what is needed to see which shuffles do make sense. By seeing various shuffles as being interchanges on a fourfold product rather than braidings on a simple binary product, we are able to describe an infinite family of distinct compositions of the braiding each leading to well defined operad structure. The underlying braids are precisely those we denoted $b_{n^{\pm}}$. In summary, structures based on a braiding are at worst ill-defined, at best defined up to equivalence, unless a 2-fold monoidal structure is chosen. Often in the literature the default is understood to be the simplest such structure where $\eta_{ABCD} = 1_A \otimes c_{BC} \otimes 1_D$, but to be careful this choice should be made explicit. We have directly addressed operads and multiplication of enriched categories. The results herein should also be applied to $\mathcal{V}$-Act, the category of categories with an action of a monoidal category, as well as to $\mathcal{V}$-Mod, the bicategory of enriched categories and modules.

References

[Baez and Dolan, 1998] J. C. Baez and J. Dolan, Categorification, in “Higher Category Theory”, eds. E. Getzler and M. Kapranov, Contemp. Math. 230 , American Mathematical Society, 1-36, (1998).

[Balteanu et.al, 2003] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, Iterated Monoidal Categories, Adv. Math. 176 (2003), 277-349.

[Borceux, 1994] F. Borceux, Handbook of Categorical Algebra 1: Basic Category Theory, Cambridge University Press, 1994.

[Day, 1970] B.J. Day, On closed categories of functors, Lecture Notes in Math 137 (Springer, 1970) 1-38

[Eilenberg and Kelly, 1965] S. Eilenberg and G. M. Kelly, Closed Categories, Proc. Conf. on Categorical Algebra, Springer-Verlag (1965), 421-562.

[Forcey, 2004] S. Forcey, Enrichment Over Iterated Monoidal Categories, Algebraic and Geometric Topology 4 (2004), 95-119.

[Forcey, 2007] S. Forcey ,J.Siehler, and E.Seth Sowers, Operads in iterated monoidal categories, *to appear in* Journal of Homotopy and Related Structures 2 (2007),

[Garside, 1969] F. Garside, The Braid Group and Other Groups, Quart. J. Math Oxford 20 (1969), 235-254.

[Joyal and Street, 1993] A. Joyal and R. Street, Braided tensor categories, Advances in Math. 102(1993), 20-78.

[Kelly, 1982] G. M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Society Lecture Note Series 64, 1982.

[Mac Lane, 1998] S. Mac Lane, Categories for the Working Mathematician 2nd. edition, Grad. Texts in Math. 5, 1998.

[Mac Lane, 1965] S. MacLane, Categorical algebra, Bull. A. M. S. 71(1965), 40-106.

[May, 1972] J. P. May, The geometry of iterated loop spaces, Lecture Notes in Mathematics, Vol. 271, Springer, 1972

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