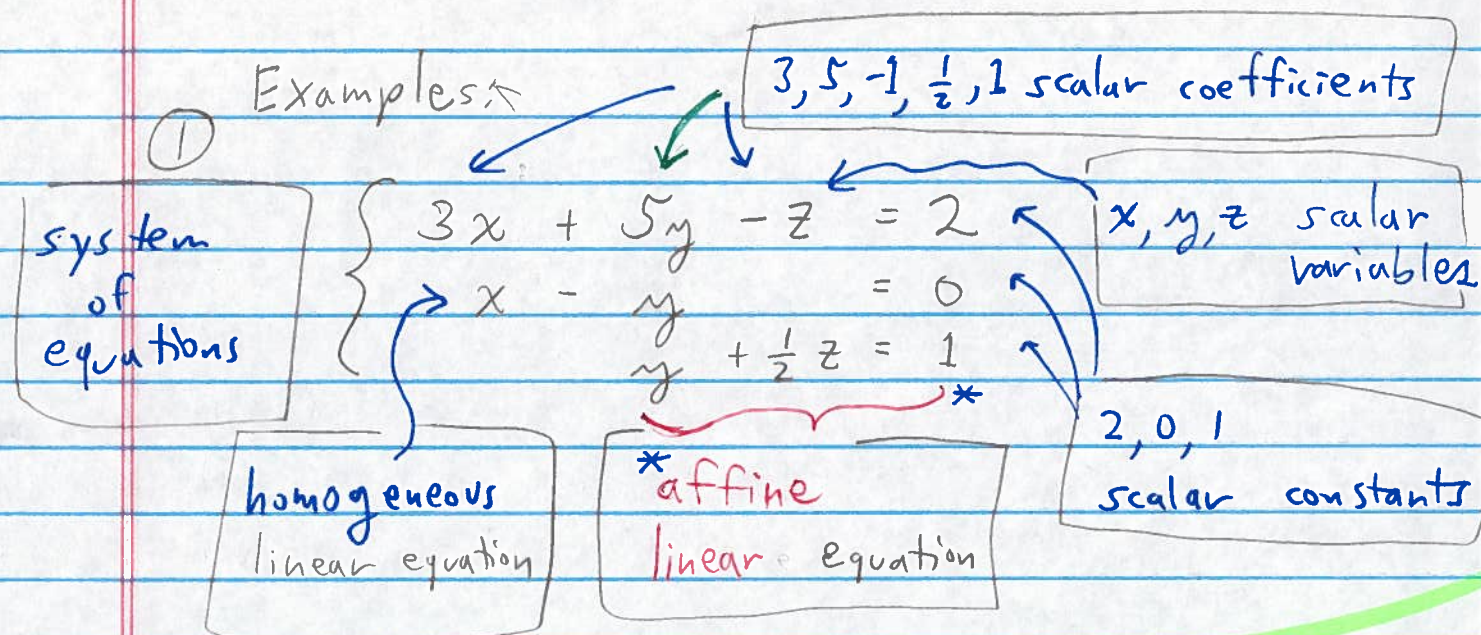


Linear Algebra

Chp. 1



②

homogeneous system of linear equations

$$\begin{cases} 3x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_3 - x_2 = 0 \end{cases}$$

(Alternate scalar variables x_1, x_2, x_3, \dots)

Solve: simultaneous solution (x_1, x_2, x_3)

(1) Subtract equations: $x_1 + x_3 = 0$

$$- (x_1 + x_3 - x_2 = 0)$$

$$\Rightarrow x_2 = 0$$

(2) Substitute back: $3x_1 - 0 = 0$ $0 + x_3 = 0$

$$\Rightarrow x_1 = 0$$
 $x_3 = 0$

$(x_1, x_2, x_3) = (0, 0, 0)$ makes all 3 true.

Solving with a matrix of coefficients

→ same as combining (subtracting) equations to eliminate variables

→ Three allowed moves, to break it down:

① Switch 2 rows (just like reordering equations)
 $R_3 \leftrightarrow R_5$

Row
Reduction
Moves

② Replace a row with a multiple of itself $R_5 \leftarrow -\frac{2}{3} R_5$

③ Replace a row with a combination of itself with another row.

$$R_7 \leftarrow R_7 + 2 R_5$$

Ex: $3x_1 - x_2 = 0$
 $x_1 + x_3 = 2$
 $x_1 + x_3 - x_2 = 1$

Matrix A

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array} \right]$$

Rows:
Equations

$\leftarrow R_1$
 $\leftarrow R_2$
 $\leftarrow R_3$

$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & -1 & 0 & 0 \end{array} \right]$$

Columns

coeffs of x_1, x_2, x_3

augmented constants

$$R_3 \leftarrow -\frac{1}{3} R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array} \right]$$

$$R_2 \leftarrow R_2 - R_1$$

$$R_3 \leftarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -3 & -3 \end{array} \right]$$

$$R_1 \leftarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array} \right]$$

$$R_3 \leftarrow R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{array} \right]$$

$$R_1 \leftarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5/3 \end{array} \right]$$

... so
$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = 1 \\ x_3 = \frac{5}{3} \end{cases}$$

Check: these make all three original equations true

$$3\left(\frac{1}{3}\right) - 1 = 0 \quad \checkmark$$

$$\frac{1}{3} + \frac{5}{3} = 2 \quad \checkmark$$

$$\frac{1}{3} + \frac{5}{3} - 1 = 1 \quad \checkmark$$

→ Geometric meaning and 3 types of solution.

For any number of linear equations,
with any number of variables,
there can only be one of 3 possibilities

→ Zero solutions

→ One solution

→ ∞ solutions

Why?

1) dimension of a space

is a counting number $0, 1, 2, 3, 4, 5, 6, \dots$
which describes a collection of
points. It tells:

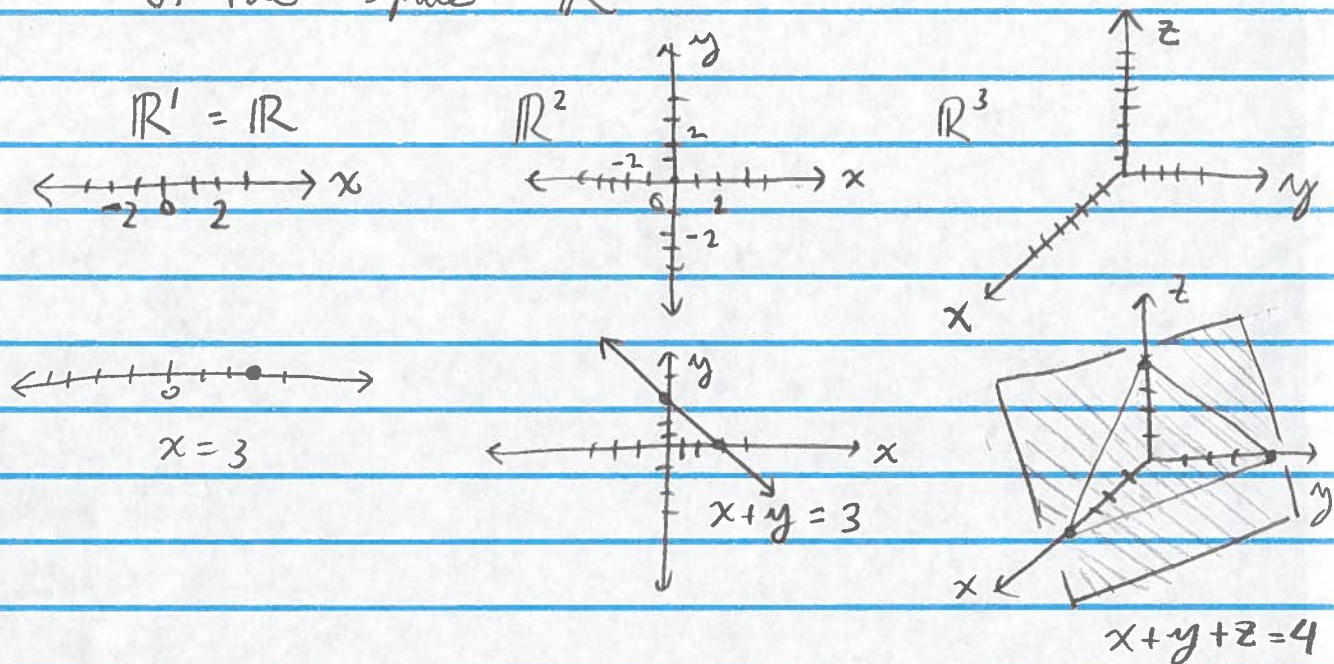
→ the number of independent, free
decisions for perpendicular motions

→ the number of real numbers needed
to describe a single point location
in that space

degrees
of freedom

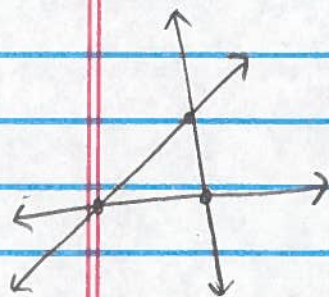
coordinates,
axes \mathbb{R}

2) Each single (affine) linear equation describes the points in a hyperplane of the space \mathbb{R}^d

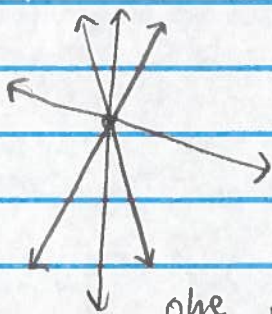


3) Several hyperplanes in \mathbb{R}^d

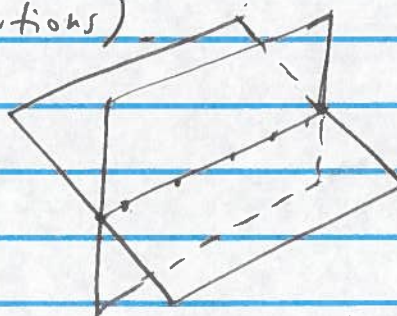
- can either:
- not share a common point of intersection (0 solutions)
 - share exactly one common point (needs at least d hyperplanes, but no guarantee)
 - or, all intersect in a lower-dimensional plane, line, or hyperplane (∞ solutions)



no point in common - no solution



one point solution

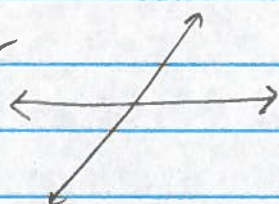


line of solution points $\rightarrow \infty$ solutions

Goal # 1 : be able to look at equations & know what the picture is, and vice versa: look at the picture and know things about the equations.

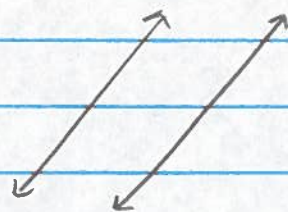
Two lines in \mathbb{R}^2 :

either



crossing = different slopes

OR



parallel = same slope

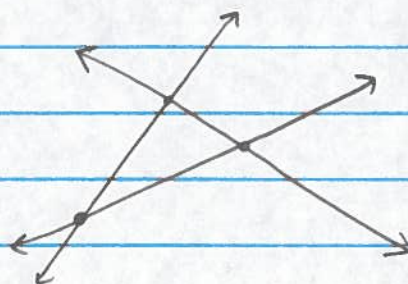
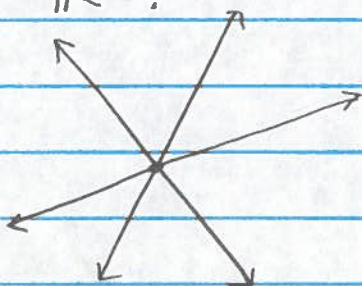
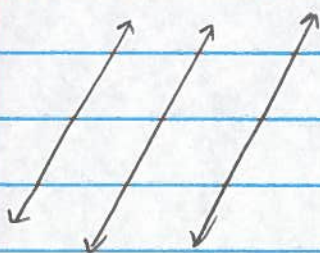
ex:

$$y = \frac{2}{3}x + 1, \quad y = \frac{2}{3}x - 5$$

$$3y - 2x = 1, \quad 3y - 2x = -5$$

Slope = coefficients

Three lines in \mathbb{R}^2 :



Four lines: 9 different pictures

Five lines: 47

Six lines: 791

Seven lines: 37,830

Eight lines: 4,134,940

Nine lines: Unknown

In general n lines?

→ open research question.

Back to solution method: matrix $A_{m \times n}$ has $\begin{matrix} \bullet m \text{ rows} \\ \bullet n \text{ columns} \end{matrix}$

→ Recall, from a system of (affine) linear equations we write a matrix (augmented) of scalar coefficients and solve using row reduction moves.

→ Two matrices are row equivalent, $A \sim B$, when you get from A to B by row reduction moves.

→ a pivot in a matrix B is a "1" in a row of B with

- all "0"s before it, in its row
- and
- all "0"s above and below, in its column

→ The row reduced echelon form of A (r.r.e.f.) is a matrix $B \sim A$ where each row of B is either all 0's or has a pivot 1 and the pivots in earlier (higher) rows are in earlier (further left) columns, and "0" rows are at bottom.

ex: $B = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 5 & -2 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ } lots of numbers after (but not above or below) pivots

↑ ↑ ↑
3 pivots, and one row all "0"

→ If $A \sim B$ in r.r.e.f., a pivot column of A is a column of A where that column in B has a pivot

→ a system is solved when its matrix A of coefficients is put in r.r.e.f. B (the moves are also done on the augmented column of constants, but that column doesn't have to be in r.r.e.f.)

Then the r.r.e.f. B is returned to equations as follows:

- each column corresponds to an original variable x, y, z or $x_1, x_2, x_3, x_4, \dots$ (except the augment column, which is constants).
- each pivot in B is a determined variable of the solution: it will be on the left of an equation.
- each non-pivot column of B is a free variable, it can be any real number.

ex: B (augment)

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & -2 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7$

↑
↑
pivots

Next we solve the non-free equations, one for each pivot.

$$x_1 = x_1 \quad (\text{free!})$$

$$\rightarrow x_2 - 2x_5 + 5x_7 = 3$$

$$x_3 = x_3 \quad (\text{free!})$$

$$\rightarrow x_4 + x_5 + 3x_7 = 1/4$$

$$x_5 = x_5 \quad (\text{free!})$$

$$x_6 = x_6 \quad (\text{free!})$$

$$x_7 = x_7 \quad (\text{free!})$$

↑ Five free variables = 5 dimensional solution

→

$$x_1 = x_1$$

$$x_2 = 3 + 2x_5 - 5x_7$$

$$x_3 = x_3$$

$$x_4 = \frac{1}{4} - x_5 - 3x_7$$

$$x_5 = x_5$$

$$x_6 = x_6$$

$$x_7 = x_7$$

This is the final general solution. There are ∞ solution points since choosing any values for the free variables gives a specific solution.

Specific solution example:

$$x_1 = 0 \leftarrow \text{pick any!}$$

$$x_2 = ? \leftarrow \text{find: } 3 + 2(-2) - 5(0) = -1$$

$$x_3 = 1 \leftarrow \text{pick any!}$$

$$x_4 = ? \leftarrow \text{find: } \frac{1}{4} - (-2) - 3(0) = \frac{9}{4}$$

$$x_5 = -2 \leftarrow \text{pick any!}$$

$$x_6 = 3 \leftarrow \text{pick any!}$$

$$x_7 = 0 \leftarrow \text{pick any!}$$

$$x_1 = 0$$

$$x_2 = -1$$

$$x_3 = 1$$

$$x_4 = \frac{9}{4}$$

$$x_5 = -2$$

$$x_6 = 3$$

$$x_7 = 0$$

→ Other possibilities:

- only one unique solution: when every column is a pivot column, and any row in B of "0"s ends in an augment of 0 in that row.
- Zero solutions: when there is a row of "0"s in B but the augment column is not 0 in that row.

ex:

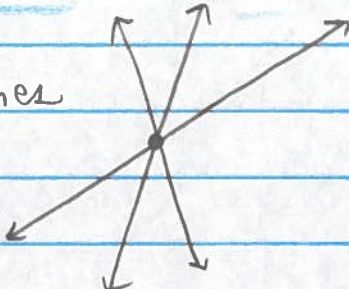
$$\begin{array}{c|cccccc|c} & \text{B} & & & & & & \\ \hline 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}$$

$$0 = 5$$

no solution

So now we know some facts to conclude:

This set of lines
in \mathbb{R}^2



has only one
solution (x, y)
So...

...it has a matrix $A_{3 \times 2}$, (3 equations, 2 variables)
3 ↑ rows, 2 ↑ columns

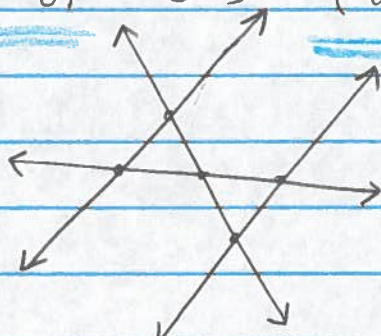
(with an extra augmented column)

and together they row reduce to r.r.e.f. B ,
(with augment),

that has 2 pivots (both columns)

and a row of "0"s (with 0 in augment).

This set of lines
in \mathbb{R}^2



has no
solutions!

... So it has a matrix $A_{4 \times 2}$ (4 equations, 2 vars)
(with an extra augment column)

which row reduces to r.r.e.f. B ,

(with augment)

that has at least one row of
"0"s, with a nonzero entry in the
augment of that row.

... And, it does have 2 pivots. Why? Just pick
two crossing lines to be two rows. One solution!