Linear. Test 2, Review.

Also study the quizzes, and homework problems!

Consider the following subsets of $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 0 \\ x-y \\ 3y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \; T = \left\{ \begin{bmatrix} x \\ 7y \\ y+3 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \; U = \left\{ \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

Which is a subspace? Recall: subspaces are subsets that can be written as spans, and subspaces are planes or lines containing the origin $\mathbf{0}$ (or just the origin, or the whole space.)

Consider the following functions from $\mathbb{R}^2 \to \mathbb{R}^3$:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x-y \\ 3y \end{bmatrix}, \; T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 7y \\ y+3 \end{bmatrix}, \; U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix}$$

Which is a linear transformation? Recall: lin. transformations can always be described by matrix multiplication, which uses the input vector to make a linear combination of the columns. They take $\mathbf{0}$ to $\mathbf{0}$ and obey $T(\mathbf{u}+2\mathbf{v}) = T(\mathbf{u}) + 2T(\mathbf{v})$. They take a space to a subspace called the range, the span of the column vectors. Find a matrix representation of S using the standard bases, and find $N(S)$ and $R(S)$, as spans of bases. Bonus: Describe a function from $\mathbb{R}^1 \to \mathbb{R}^2$ whose range is a subspace, but which is not linear.

Consider the following sets of polynomials in $\mathcal{P}_3$.

$$\mathcal{A} = \{x-1, x, x^2+1\}, \; \mathcal{B} = \{5x^2, x, x^3+2, 3\}, \; \mathcal{C} = \{3x^2, x^2-1, x+2, 3\},$$

Which one is a basis for $\mathcal{P}_3$? Which one is lin. dep.? Recall the 2 out of 3 theorem: any two implies the third (and thus a basis for V,) out of {lin. indep., spans V, has the same number of items as $\dim(V)$.} Note also, these questions are equivalent to the same questions about their coordinate vectors in $\mathbb{R}^4$ with respect to the standard basis.

 Consider the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 6 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 10 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the rank, nullity, null space, and range for each, as spans of bases. Which of them are 1-1? Which are onto? Which two of them cannot represent the derivative from $\mathcal{P}_3$ to $\mathcal{P}_4$? Which two of them cannot represent the derivative from $\mathcal{P}_4$ to $\mathcal{P}_3$? Recall that rank + nullity = dim(domain) = number of columns. Recall that if a matrix represents a transformation then it will have the right number of rows and columns, and that it will have the same rank and nullity as that transformation.

Consider the three bases for $\mathcal{P}_3$:

$$\mathcal{E} = \{1, x, x^2, x^3\},\ \mathcal{B} = \{5x^2, x, x^3, 3\},\ \mathcal{C} = \{x^3 + 3x^2 + 1, x^2 - 2, x - 7, 2\}$$

Find the representatives of the derivative: $T : \mathcal{P}_3 \to \mathcal{P}_3$, where $T(f) = f'$.

$[T]_{\mathcal{B}}^{\mathcal{E}}$

$[T]_{\mathcal{C}}^{\mathcal{B}}$

Find the representative of the lin. trans.: $T : \mathcal{P}_3 \to \mathcal{P}_3$, where $T(f) = f'' + xf''$. Also find the null space $N(T)$ and the range $R(T)$ as spans of bases of polynomials. Is T onto? 1-1? Recall the 2 out of 3 theorem for matrices: any two implies the third out of {1-1,onto,square}.

$[T]_{\mathcal{C}}^{\mathcal{E}}$