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POLYTOPES AND ALGEBRAS OF GRAFTED TREES II. FAN GRAPHS AND PTERAHEDRA.

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Abstract. We answer an open question by showing that certain sets of combinatorial trees are naturally the minimal elements of face posets of convex polytopes. The polytopes that constitute our main results are well known in other contexts. They are graph-associahedra of the fan graphs (we call them pterahedra.) Tree species considered here include ordered and unordered binary trees. Compositions of these trees were recently introduced as bases for one-sided Hopf algebras. Thus our results show a path to extending those algebras to differential graded Hopf algebras based on the polytope faces.

1. Introduction

Painted trees which combine elements of the associahedra and the permutohedra are studied in [10]. In that source we combine the trees that represent vertices of these two polytope sequences to describe a composition operation on coalgebras. For instance: given the Loday-Ronco Hopf algbra $\mathcal{Y}Sym$ from [15], and the Malvenuto-Reutenauer Hopf algebra of permutations $\mathfrak{S}Sym$ from [16], one composition of these is denoted $\mathfrak{S}Sym \circ \mathcal{Y}Sym$. It inherits the structure of a graded cofree coalgebra automatically, and the structure of a (one-sided) graded Hopf algebra by virtue of the interaction between the operads and the coproducts involved.

We noticed in [9] that in 3 dimensions our compositions of $\mathcal{Y}Sym$ and $\mathfrak{S}Sym$ appeared as the vertices of polyhedra (although the corresponding trees are mislabeled in that source, which we correct here). Here we start by defining a larger set of grafted, painted trees which contains the trees that form the graded basis for $\mathfrak{S}Sym \circ \mathcal{Y}Sym$. This larger set comes with a canonical lattice structure. We prove that this lattice is in fact isomorphic to the face lattices of a sequence of convex polytopes.

The composition operation is not commutative. We also look at the composition $\mathcal{Y}Sym \circ \mathfrak{S}Sym$, and conjecture that it also has an underlying sequence of polytopes

1.1. **Main Results.** We show that a sequence of sets of painted trees, with their relations, is isomorphic as posets to face lattices of convex polytopes. In Theorem 3.6 we show that forests of plane rooted trees grafted to weakly ordered trees are isomorphic to the fan-graph-associahedra, or pterahedra.

In Section 4 we show how our bijections transfer the algebraic structure from [10] to the vertices of the pterahedra. In Proposition 5.5 we show that a new, alternative product on these vertices is associative.

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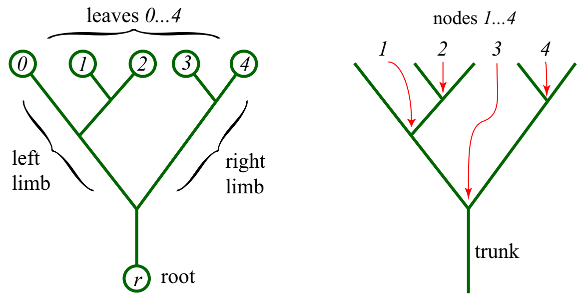
2. Definitions

This section of definitions is taken, almost verbatim, from [11]. It is included here for completeness. *Trees* are connected, acyclic graphs. We begin our discussion, following [8], with rooted, plane, binary trees.

- Rooted: There is a vertex of degree one which is designated the root. The other vertices of degree one are called leaves.
- Plane: Each tree is drawn in the plane with no edges crossing. Two plane trees are only equal if they can be made into identical pictures by scaling portions of the plane, without any reflection or rotation in any dimension.
- Binary: The vertices that are not degree one are all of degree three. These vertices are called nodes.

Here is a *plane rooted binary tree*, often called a binary tree when the context is clear:



The leaves are ordered left to right as shown by the circled labels. The node ordering corresponds to the order of "gaps between leaves:" the n^{th} node is the one where a raindrop would be caught which fell just to the left of the n^{th} leaf. The branches are the edges with a leaf. Non-leafed edges are referred to as internal edges. The nodes are also partially ordered vertically; we say two nodes are comparable in this partial order if they both lie on a path to the root. The root is maximal in this partial order. In the preceding picture, we have for instance that node 3 is greater than node 1 which is greater than node 2. The set of plane rooted binary trees with n nodes and $n+1$ leaves is denoted $\mathcal{Y}_n$. The cardinality of these sets are the Catalan numbers:

$$|\mathcal{Y}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

We will also need to consider rooted plane trees with nodes of larger degree than three. An $(n+1)$-leaved rooted tree with only one node (it will have degree $n+2 \geq 3$), or, for $n = 0$, a single leaf tree with zero nodes, is called a *corolla*, denoted $\mathfrak{C}_n$. This notation for the (set of one) corolla with $n+1$ leaves is the same as used for the set of one *left comb* in [10]. In the current paper we have decided that the corollas are more easily recognized than the combs.

2.1. **Ordered and Painted binary trees.** Many variations of the idea of the binary tree have proven useful in applications to algebra and topology. First, an *ordered tree* (sometimes called *leveled*) is a binary tree that has a vertical linear ordering of the n nodes as well as horizontal. This vertical linear ordering extends the partial vertical ordering given by distances to the root. See Figure 1 for ways to draw ordered trees.

This ordering allows a well-known bijection between the ordered trees with n nodes, denoted $\mathfrak{S}_n$, and the permutations on $[n]$.

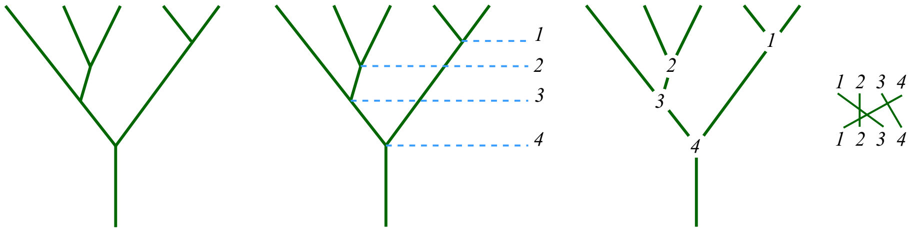


FIGURE 1. Drawing an ordered tree, in three different styles. The corresponding permutation σ is (3,2,4,1), in the notation $(\sigma(1), \sigma(2), \sigma(3), \sigma(4).)$

We will also consider *forests* of trees. In this paper, all forests will be a linearly ordered list of trees, drawn left to right. This linear ordering can also be seen as an ordering of all the nodes of the forest, left to right. On top of that, we can also order all the nodes of the forest vertically, giving a *vertically ordered forest*, which we often shorten to *ordered forest*. This initially gives us four sorts of forests to consider, shown in Figure 2.

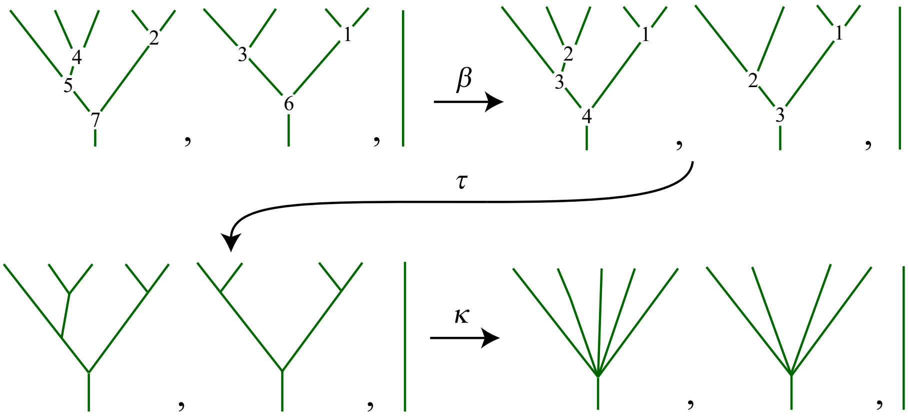


FIGURE 2. Following the arrows: a (vertically) ordered forest, a forest of ordered trees, a forest of binary trees, and a forest of corollas.

Also shown in Figure 2 are three canonical, forgetful maps between the types of forests.

2.1. **Definition.** *We define β to be the function that takes an ordered forest F and gives a forest of ordered trees. The output $\beta(F)$ will have the same list of trees as F, and for a tree t in $\beta(F)$ the vertical order of the nodes of t will respect the vertical order of the nodes in F. That is, for two nodes a, b of t we have $a \leq b$ in t iff $a \leq b$ in F.*

We define the τ to be the function that takes an ordered tree and outputs the tree itself, forgetting all of the vertical ordering of nodes (except for the partial ordering based on distance from the root.) We define κ to be the function that takes a tree and gives the corolla with the same number of leaves.

Note that τ and κ are immediately both functions on forests, simply by applying them to each tree in turn. Also note that τ and κ are described in [10], but that there κ yields a left comb rather than a corolla.

Now we define larger sets of trees that generalize the binary ones. First we drop the word binary; we will consider plane rooted trees with nodes that have any degree larger than two. Then, from the non-binary vertically ordered trees we further generalize by allowing more than one node to reside at a given level. Instead of corresponding to a permutation, or total ordering, these trees will correspond to an ordered partition, or weak ordering, of their nodes.

2.2. **Definition.** *A* weakly ordered tree *is a plane rooted tree with a weak ordering of its nodes that respects the partial order of proximity to the root.*

Recall that this means all sets of nodes are comparable–but some are considered as tied when compared, forming a block in an ordered partition of the nodes. The linear ordering of the blocks of the partition respects the partial order of nodes given by paths to the root.

For a weakly ordered tree with $n+1$ leaves the ordered partition of the nodes determines an ordered partition of $S = \{1, \ldots, n\}$, where S is the set of "gaps between leaves," as described in [18]. (Recall that a gap between two adjacent leaves corresponds to the node where a raindrop would eventually come to rest; S is partitioned into the subsets of gaps that all correspond to nodes at a given level.) Weakly ordered trees are drawn using nodes with degree greater than two, and using numbers and dotted lines to show levels as in Figure 3. Note that an ordered tree is a (special) weakly ordered tree.

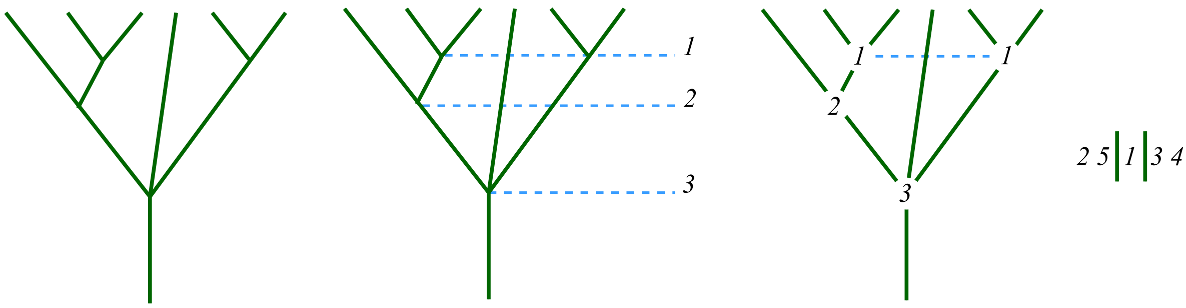


FIGURE 3. Drawing a weakly ordered tree, in three different styles. The corresponding ordered partition is $(\{2,5\},\{1\},\{3,4\})$.

As well as forests of weakly ordered trees we also consider weakly ordered forests. This gives us three more sorts of forests to consider, shown in Figure 4. As indicated in that figure, the maps β, τ and κ are easily extended to forests of the non-binary and/or weakly ordered trees: β forgets the weak ordering of the forest to create a forest of weakly ordered trees, τ forgets the weak ordering, and κ forgets the partial order to create corollas.

The trees we focus on in this paper are constructed by grafting together combinations of ordered trees, binary trees, and corollas. Visually, this is accomplished by attaching the roots of one of the above forests to the leaves of one of the above types of trees. We use two colors, which we refer to as "painted" and "unpainted." The forest is described as unpainted, and the base tree (which the forest is grafted to) is painted. At a graft the leaf is identified with the root, and in the diagram that point is drawn as a change in color (and thickness, for easy recognition) of the resulting edge. (Note that in earlier papers such as [7] our mid-edge change in color is described instead as a new node of degree two.)

We refer to the result as a (partly) *painted tree*, regardless of the types of upper (unpainted) and lower (painted) portions. Notice that in a painted tree the original trees (before the graft) are still easily observed since the coloring creates a boundary, called the *paint-line* halfway up the edges where the graft was performed. Thus the paint line separates the painted tree into a single tree of one color and a forest of trees of another color. In Figure 5 we show all 12 ways to graft one of our types of partially ordered forest with one of our types of tree.

2.3. **Definition.** *The maps β, τ and κ are now extended to the painted trees, just by applying them to the unpainted forest and/or to the painted tree beneath. We indicate this by writing a fraction: $\frac{f}{g}$ for two of our three maps, or the identity map, as seen in Figure 5. That is, $\frac{f}{g}$ indicates applying f to the forest and g to the painted base tree, for $f, g \in \{\beta, \tau, \kappa, 1\}$.*

2.2. **General painted trees.** Now our definition of painted trees is expanded to include any of our types of forest grafted to any of our types of tree. On top of that we will also permit a further broadening of the allowed structure of our painted trees. The paint-line, where the graft occurs, is allowed to coincide with nodes, where branching occurs. We call it a *half-painted node*. In terms of the grafting of a forest onto a tree our description depends on the type of forest. If the forest is weakly ordered, or is a forest of weakly ordered trees, then we see each half-painted node as grafting on a single tree at its least node, after removing its trunk and root. If the forest is only partially

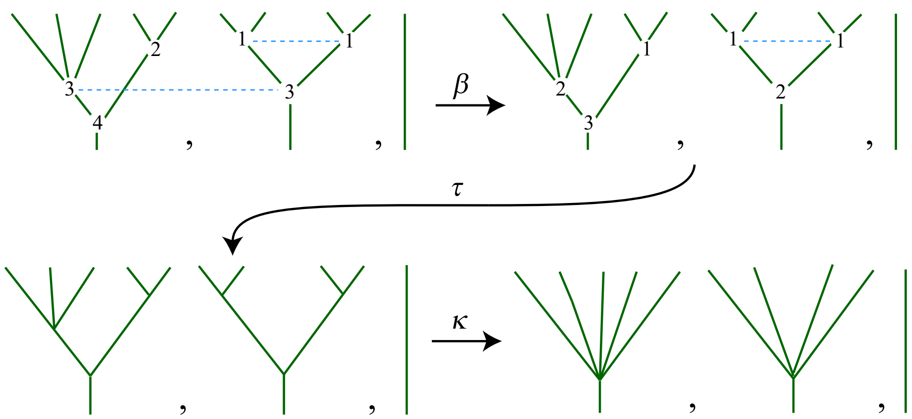


FIGURE 4. Following the arrows: a (vertically) weakly ordered forest, a forest of weakly ordered trees, a forest of plane rooted trees, and a forest of corollas. Note that the forests in Figure 2 are special cases of these.

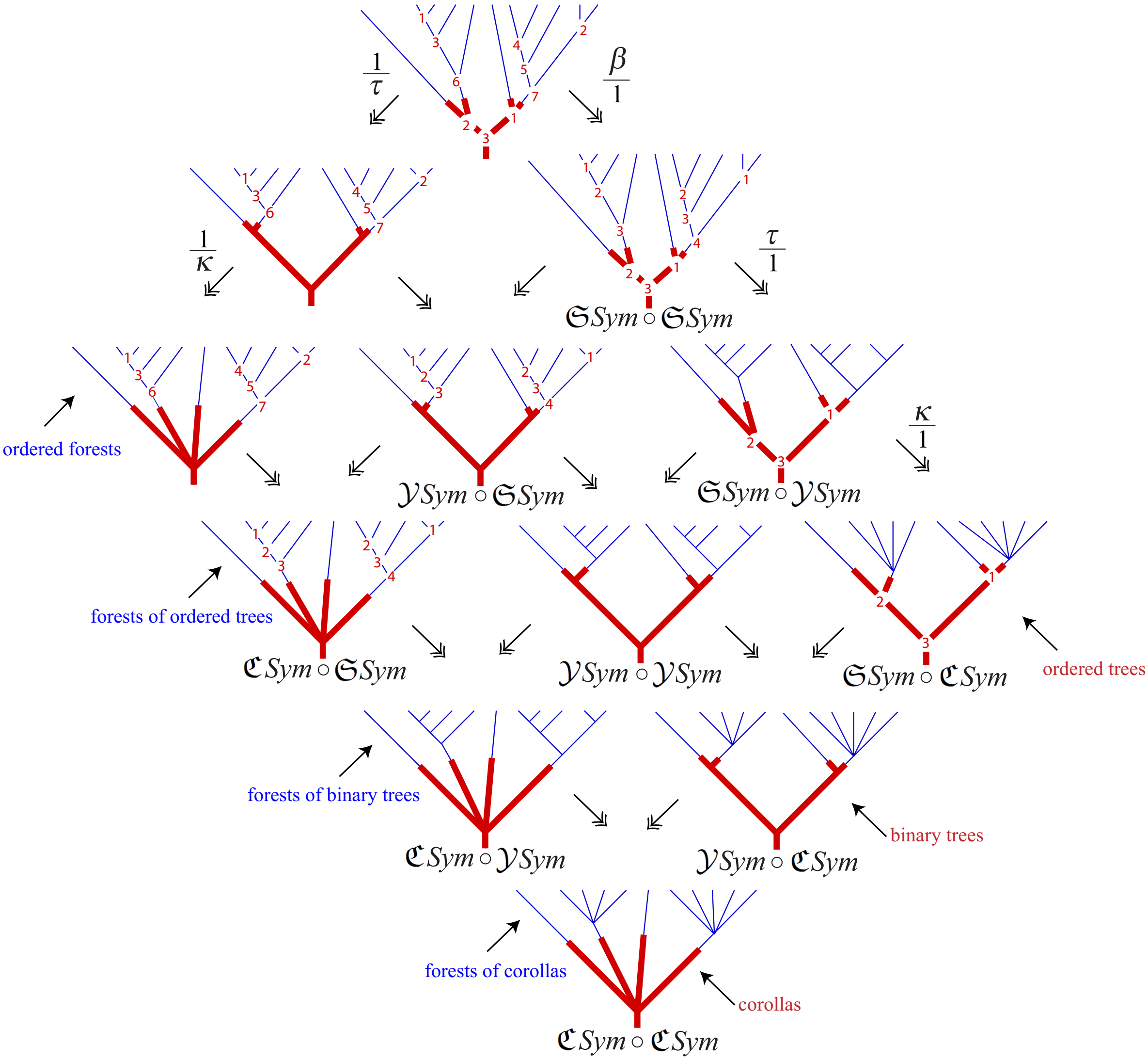


FIGURE 5. Varieties of grafted, painted trees. Each diagonal shares a type of tree on the bottom (painted) or a type of forest grafted on, as indicated by the labels. Since these trees are all binary, they correspond to vertex labels of the polytope sequences whose 3d versions are shown in Figure 9. The forgetful maps are shown with example input and output. Parallel arrows all denote the same map, except of course that the identity is context dependent.

ordered (i.e. of binary trees or corollas) then we see the half-painted nodes as (possibly) several roots of several trees simultaneously grafted to a given leaf. See the examples in Figures 7 and 6.

For these general painted trees we can again extend the "fractional" maps using β, τ and κ. We reiterate from above how the half-painted nodes are interpreted, since that determines the input for the "numerator" map. Specifically $\frac{\beta}{g}$ operates by taking as input for β the weakly ordered forest of trees, one tree for each half-painted node. That is, $\frac{\beta}{1}$ treats the half-painted nodes as being the location of a single tree that is grafted on without a trunk. This description is the same for $\frac{\tau}{g}$. In contrast however, the map $\frac{\kappa}{g}$ takes as input the forest found by listing all the unpainted trees while assuming each has a visible trunk, some of which are simultaneously grafted at the same half-painted node. Examples of these maps are shown in Figure 7, where we show 12 general painted trees that consist of one of the four general types of forest and one of the three general types of trees. Figure 6 is a detail from Figure 7 showing how the actions of the projections differ.

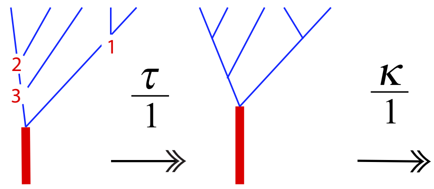


FIGURE 6. Action of the projections, detail from Figure 7. At first there is a single (weakly) ordered tree attached at the half-painted node; at last there are two (corolla) trees attached at the same node. In the center, where the unpainted portion is one or two binary trees, we can see it as either without any contradiction.

2.3. **The face poset relations.**

2.3.1. *Partial ordering of nodes and gaps.* Each of our 12 types of painted tree comes with a canonical vertical partial ordering of its nodes (branch points), produced by concatenating the orders that exist before the graft. Each new partial ordering is a refinement of the partial ordering given by distances to the root of the newly minted painted tree, and also preserves the relations that existed before the graft. We add the rules that 1) all half-painted nodes must be forced to remain at the same level, that is, incomparable to each other (or tied in a weak order); and 2) that nodes below the paint line will never surpass half painted nodes, and neither of the former will surpass unpainted nodes in the partial order. Furthermore, this ordering of nodes implies an ordering of the gaps between leaves of the tree. Some gaps share a node. Two gaps that share a node are considered to be incomparable in the partial order (or tied in a weak order).

Now we can define 12 separate posets whose elements are trees: one poset on each of our 12 types of painted trees shown in Figure 7. Note that the simplest painted tree with n leaves has one half-painted node: n single leafed unpainted trees all grafted to a painted trunk, the node coinciding with the paint line. This *half-painted corolla* can be interpreted as one of any of the 12 painted tree varieties, and it will be the unique maximal element in all 12 posets.

2.4. **Definition.** Given two painted trees s and t that are of the same painted type (i.e. they share the same types of tree and forest, below and above the paint line) we define the *painted growth preorder*, where :

$$s \prec t$$

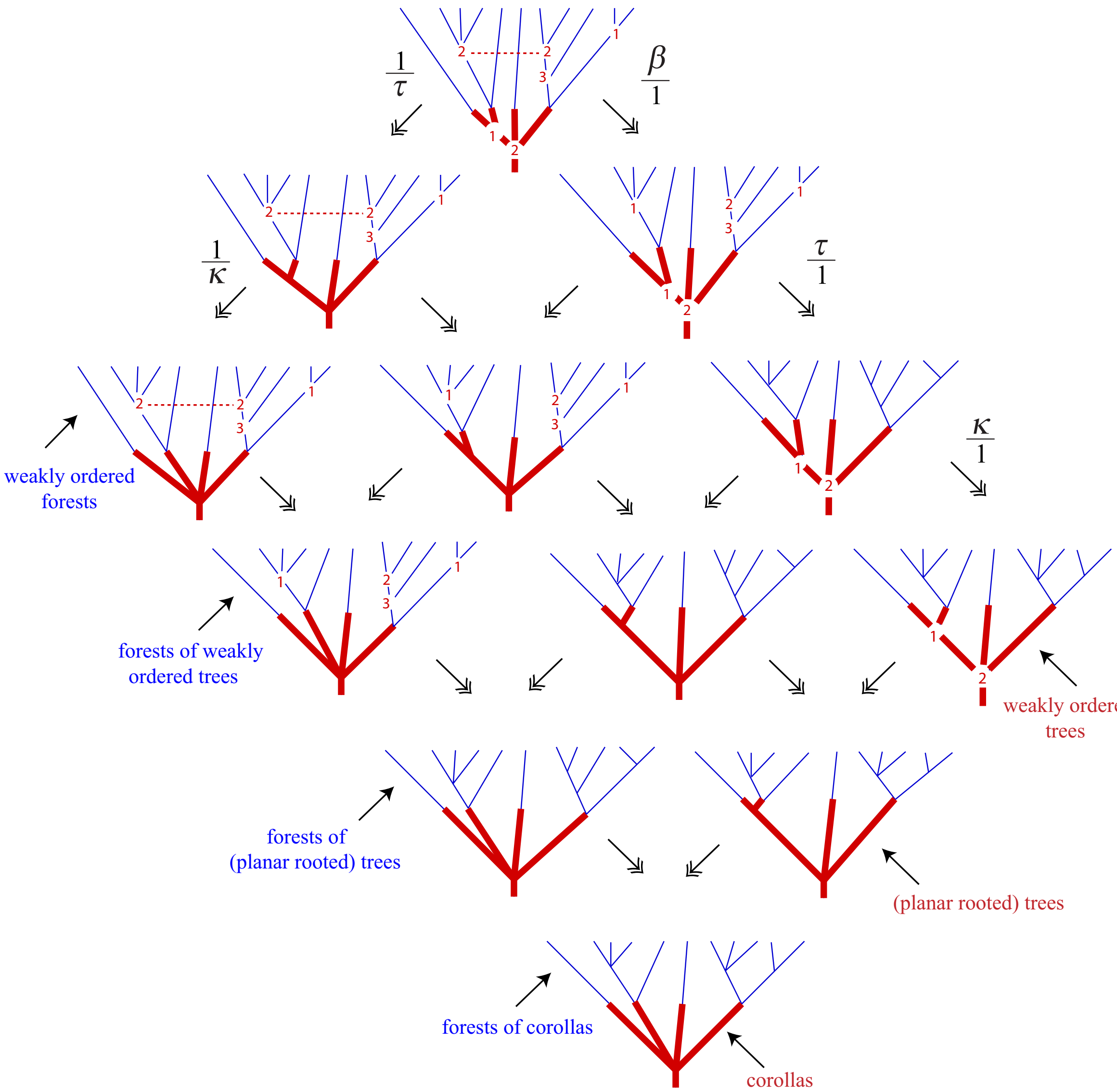


FIGURE 7. More varieties of grafted, painted trees. These correspond to face labels of the polytope sequences whose 3d versions are shown in Figure 9. Parallel arrows all denote the same map. Note that the trees in Figure 5 are special cases–vertex trees, or minimal in the face lattice–of the types illustrated in this figure.

if $s = t$ or if s is formed from t using a series of pairs $(a, b)_i$ of the following two moves, each pair performed in the following order:

a) "growing" internal edges of t : introducing new internal edges or increasing the length of some internal edges (either painted or unpainted). This is precisely described as a possible refinement of the vertical partial order of gaps between leaves, by adding relations to the partial order between previously incomparable elements. Relations may not be deleted by the growth (nor ties formed in a weak order), but if the growing of painted edges occurs at a collection of half-painted nodes in t then the partial order may be preserved rather than strictly refined. Note that internal edges can grow where there was no internal edge before, such as at a half-painted node or any node that had degree larger than three. Note also that the rules for painted trees must be obeyed by the growing process–for instance an unpainted edge cannot grow from a completely painted node (and vice versa), and if some painted edges are grown from a half-painted node then all the edges possible must be formed, to allow the paint line to be drawn horizontally.

b) ...followed by "throwing away," or forgetting, any superfluous structure introduced by the edge growing. This is described precisely by taking the tree that results from growing edges, and applying to it the forgetful map (from the set of $\beta, \tau, \kappa, 1$ and their fractions and compositions) that is needed to ensure that the result is in the original type of the painted tree t. E.g. if the original type had weakly ordered forests grafted to weakly ordered trees, we only apply the identity. However if t originally was a forest of weakly ordered trees grafted to a weakly ordered tree we should apply $\frac{\beta}{1}$.

For examples of (non-covering) relations in the 12 posets see the trees in respective locations of figure 5 and Figure 7: the latter are all greater than the former in the same positions. Several more covering relations for some of our 12 classes of general painted trees are shown in Figure 8.

3. Bijections

The painted growth relation is reflexive and transitive by construction, for all 12 types. We conjectured in [11] that in all 12 of cases the the painted growth preorder is in fact a poset, and moreover we conjecture that all the posets thus defined are realized as the face posets of sequences of convex polytopes. Five of the cases have been proven in previous work. These five appear in Figure 9. The polytope sequences are the cubes, associahedra, composihedra, multiplihedra and stellohedra. The associahedra, composihedra, and multiplihedra are shown (with pictures of painted trees) in [12]. The fact that the cubes result from forgetting all the branching structure is equuivalent to the fact that cubes arise when both of two product spaces are associative, as pointed out in [2], also (with design tubings) in [5].

In this section one of our sequences of posets will be shown to be isomorphic to face lattices of convex polytopes. This is the species whose structure types are a forest of plane rooted trees grafted to a weakly ordered tree. The polytope, called the pterahedra, turns out to be a certain family of graph associahedra. Our proofs and corollaries will use the concept of tubings, which we review next.

3.1. **Tubes, tubings.** The definitions and examples in this section are largely taken from [13]. They are based on the original definitions in [3], with only the slight change of allowing a universal tube, as in [4].

3.1. **Definition.** Let G be a finite connected simple graph. A *tube* is a set of nodes of G whose induced graph is a connected subgraph of G. We will often refer to the induced graph itself as the tube. Two tubes u and v may interact on the graph as follows:

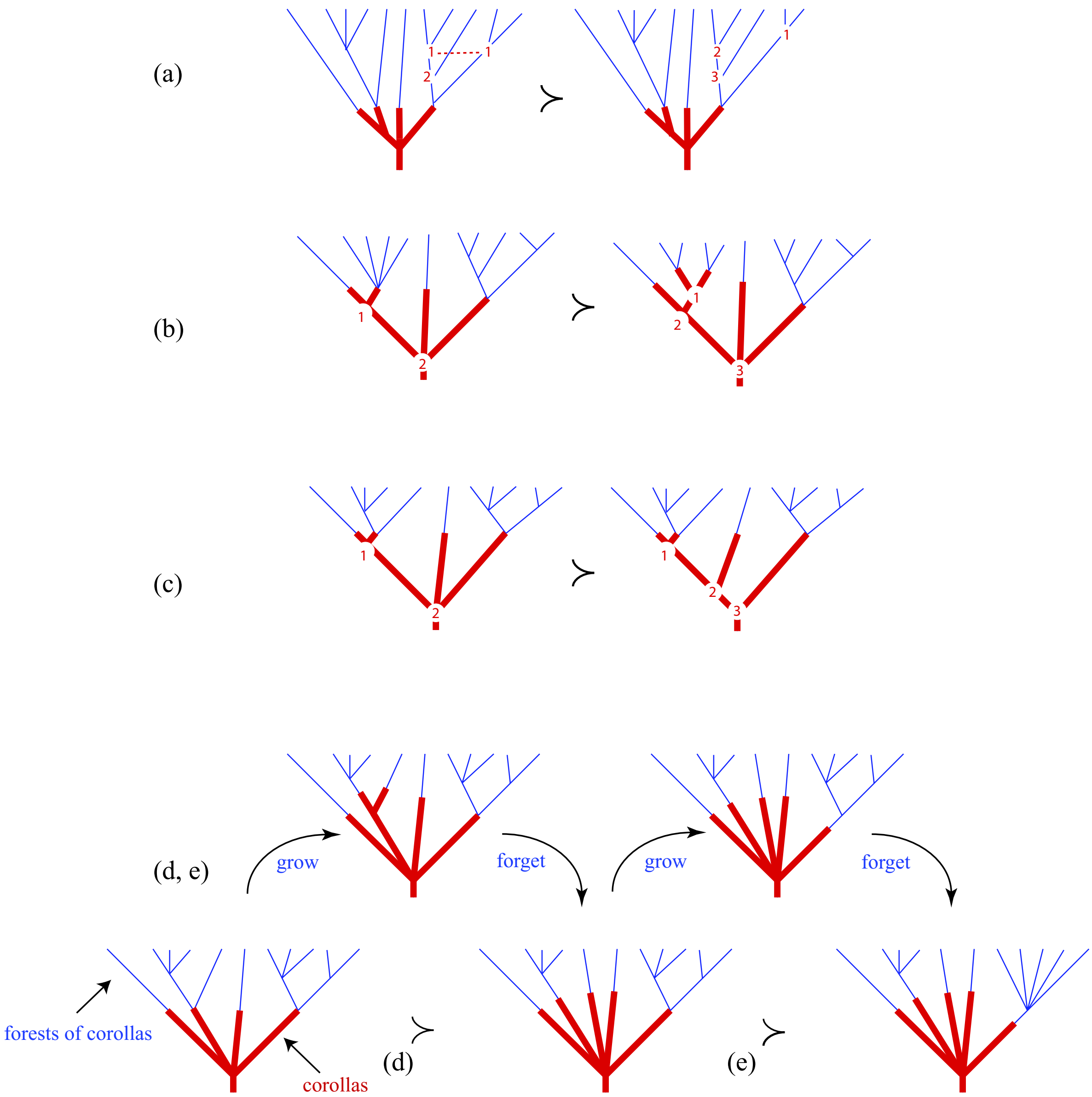


Figure 8. Some covering relations. In the first (a) we are looking at weakly ordered trees grafted to a (non-binary) tree, so growing an edge is a covering relation. Relations (b) and (c) are in the pterahedron. At the bottom for both covering relations the forgetful map is κ. Relation (e) is in the stellohedron, although it is also true in the cube.

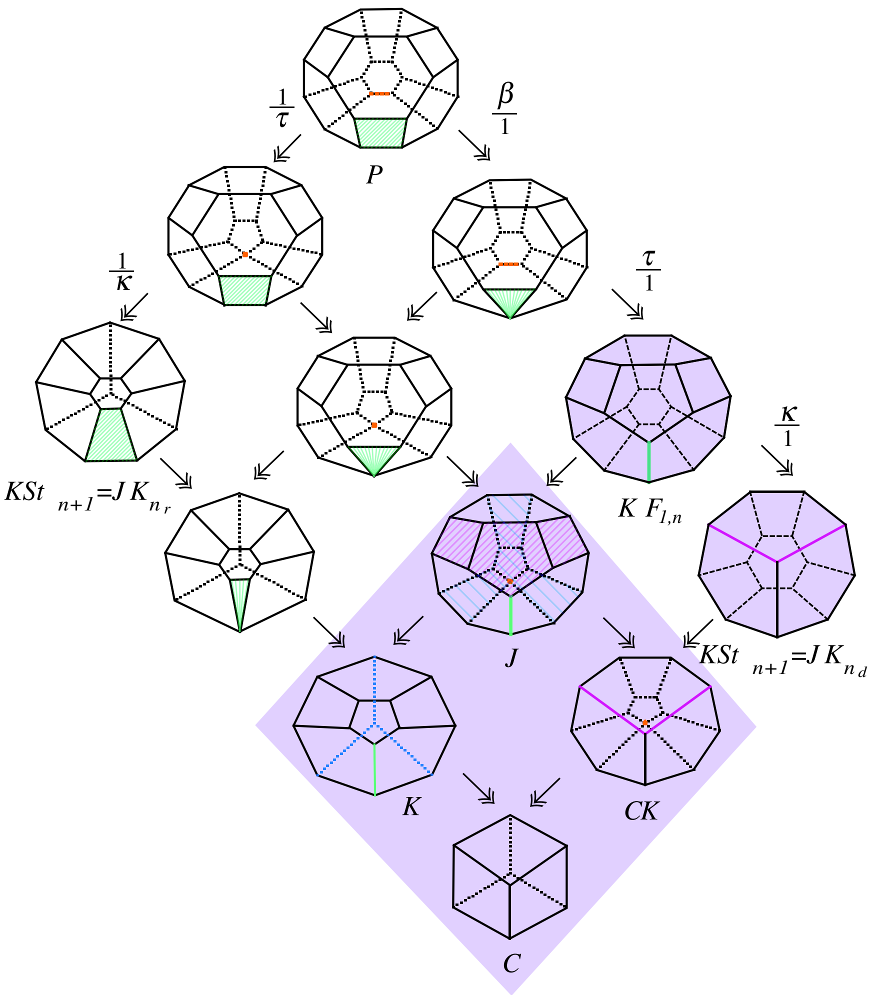


FIGURE 9. The 3d terms of some new and old polytope sequences. The four in the shaded diamond are the cube, associahedron $\mathcal{K}$, multiplihedron $\mathcal{J}$ and composihedron $\mathcal{CK}$. The other two shaded are the pterahedron $\mathcal{P}_t = \mathcal{K}F_{1,n}$ (fan graph associahedron) and the stellahedron $\mathcal{K}S$. The topmost is the permutohedron and the others are conjectured to be polytopes (clearly they are in three dimensions–the conjecture is about all dimensions.) Each of these corresponds the type of tree shown in Figure 5, in the corresponding position.

(1) Tubes are *nested* if $u \subset v$.
(2) Tubes are *far apart* if $u \cup v$ is not a tube in G, that is, the induced subgraph of the union is not connected, (equivalently none of the nodes of u are adjacent to a node of v).

Tubes are *compatible* if they are either nested or far apart. We call G itself the *universal tube.* A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible; moreover, we force every tubing of G to contain (by default) its universal tube. By the term *k-tubing* we refer to a tubing made up of k tubes, for $k \in \{1, \ldots, n\}$.

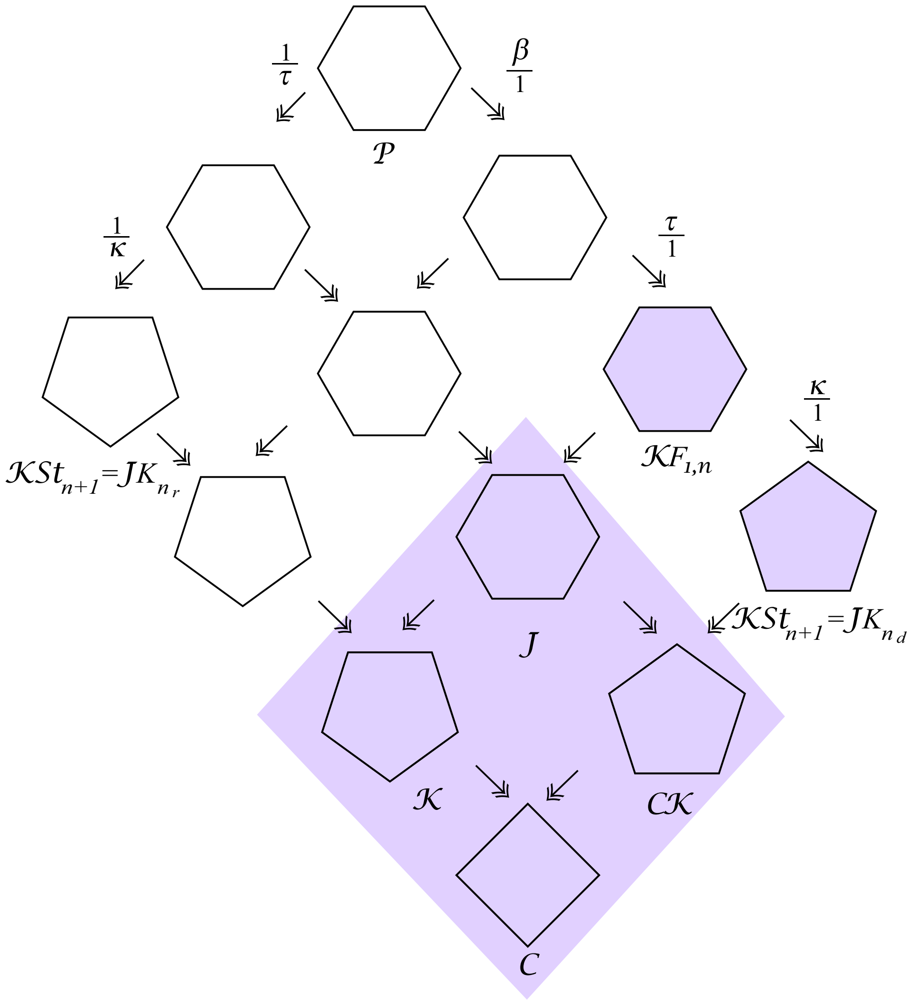


FIGURE 10. These are the 2d terms in the same sequences as in Figure 9.

When G is a disconnected graph with connected components $G_1, \ldots, G_k$, an additional condition is needed: If u_i is the tube of G whose induced graph is G_i, then any tubing of G cannot contain all of the tubes $\{u_1, \ldots, u_k\}$. However, the universal tube is still included despite being disconnected. Parts (a)-(c) of Figure 12 from [4] show examples of allowable tubings, whereas (d)-(f) depict the forbidden ones.

3.2. **Theorem.** [3, Section 3] *For a graph G with n nodes, the* graph associahedron *$\mathcal{K}G$ is a simple, convex polytope of dimension $n-1$ whose face poset is isomorphic to the set of tubings of G, ordered by the relationship $U \prec U'$ if U is obtained from U' by adding tubes.*

The vertices of the graph associahedron are the n-tubings of G. Faces of dimension k are indexed by $(n-k)$-tubings of G. In fact, the barycentric subdivision of $\mathcal{K}G$ is precisely the geometric realization of the described poset of tubings. Many of the face vectors of graph associahedra for path-like graphs have been found, as shown in [17]. This source also contains the face vectors for the cyclohedra.

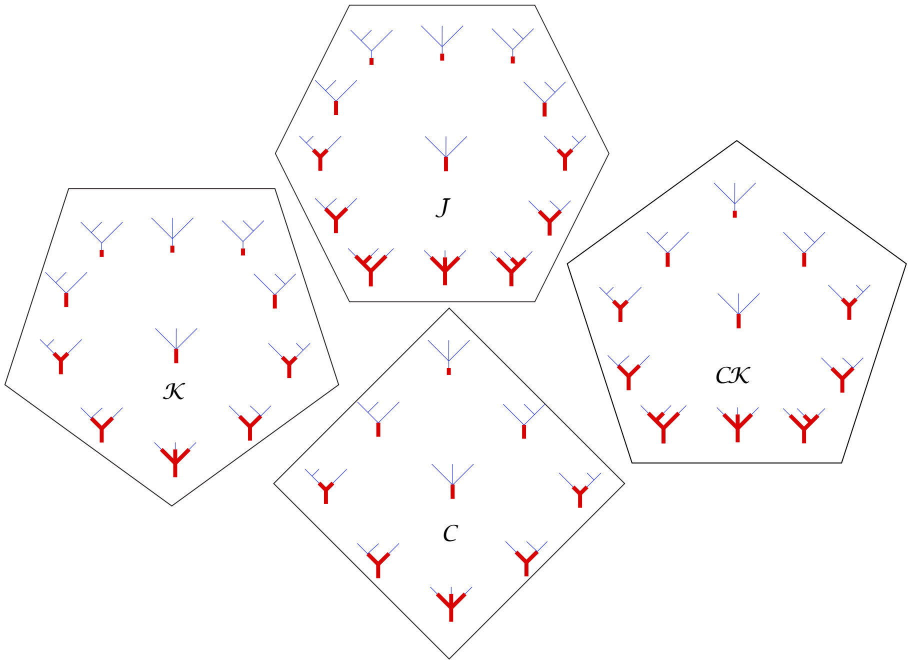


FIGURE 11. These are the 2d terms with their faces labeled. The same labels are used in 2d no matter where the shape occurs in figure 10.

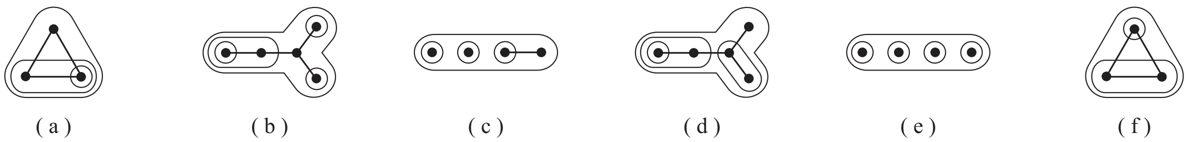


FIGURE 12. (a)-(c) Allowable tubings and (d)-(f) forbidden tubings, figure from [4].

To describe the face structure of the graph associahedra we need a definition from [3, Section 2].

3.3. **Definition.** *For graph G and a collection of nodes t, construct a new graph $G^*(t)$ called the* reconnected complement*: If V is the set of nodes of G, then $V - t$ is the set of nodes of $G^*(t)$. There is an edge between nodes a and b in $G^*(t)$ if $\{a, b\} \cup t'$ is connected in G for some $t' \subseteq t$.*

3.4. **Example.** *Figure 13 illustrates some examples of graphs along with their reconnected complements.*

For a given tube t and a graph G, let $G(t)$ denote the induced subgraph on the graph G.

3.5. **Theorem.** [3, Theorem 2.9] *Let V be a facet of $\mathcal{K}G$, that is, a face of dimension $n - 2$ of $\mathcal{K}G$, where G has n nodes. Let t be the single, non-universal, tube of V . The face poset of V is isomorphic to $\mathcal{K}G(t) \times \mathcal{K}G^*(t)$.*

We will consider a related operation on graphs. The *suspension* of G is the graph $\mathfrak{S}G$ whose set of nodes is obtained by adding a node 0 to the set $\mathrm{Nod}(G)$, of nodes of

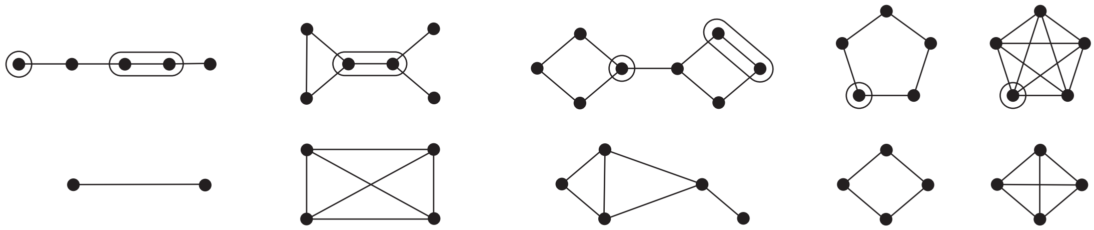


FIGURE 13. Examples of tubes and their reconnected complements. Figure from [13].

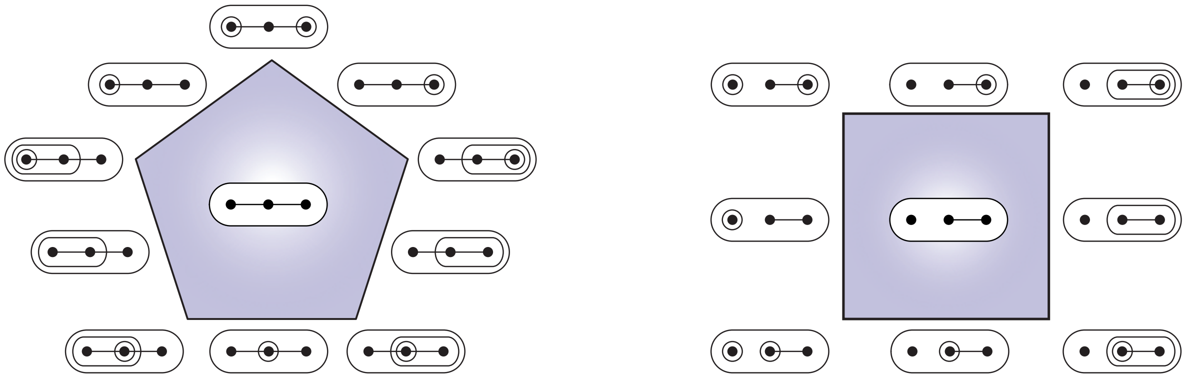


FIGURE 14. Graph associahedra of a path and a disconnected graph. The 3-cube is found as the graph-associahedron of two disjoint edges on four nodes, but no simple graph yields the 4-cube. Figure from [13].

G, and whose edges are defined as all the edges of G together with the edges $0, v$ for v $\in Nod(G)$.

The reconnected complement of $\{0\}$ in $\mathfrak{S}G$ is the complete graph $\mathcal{K}_n$ for any graph G with n nodes. Note that the star graph S_n is the suspension of the graph C_n with has n nodes and no edge, while the fan graph $F_{1,n}$ is the suspension of the line graph L_n.

3.2. **Pterahedra.** Now we prove that the poset of painted trees made by grafting a forest of plane rooted trees to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is the graph-associahedron $\mathcal{K}G$ where G is the fan graph $F_{1,n}$.

Recall that the fan graph $F_{1,n}$ is defined as follows: we use the set $\{0, 1, 2, \ldots, n\}$ as the set of nodes. Edges are $\{i, i+1\}$ for $i = 1, \ldots, n-1$, together with $\{0, i\}$ for $i = 1, \ldots, n$.

3.6. **Theorem.** *The poset of tubings on the fan graph $F_{1,n}$ is isomorphic to the poset of n-leaved forests of plane trees grafted to weakly ordered trees.*

Proof. We first note that any tubing T of the fan graph includes a unique smallest tube t_0 which contains node 0. All other tubes of T are either contained in t_0 or contain t_0, since the node 0 is adjacent to all other nodes. The tubes contained in t_0 form a tubing of a graph which is a (possibly) disconnected set of path graphs. The tubes containing

t_0 form a tubing on the reconnected complement of t_0, which is the complete graph on the nodes not in t_0.

There is an isomorphism between the poset of weakly ordered trees and the tubings on the complete graph: this is pictured in Figures 15 and 16 from [14]. Also in those pictures is shown the isomorphism between plane trees and tubings on the path graph.

Thus the bijection from the tubing on the fan graph to the tree is formed of the bijection from tubings on a complete graph to weakly ordered trees, together with the bijection from tubings on a path graph to plane trees. The tube t_0 plays the same role as the paint line in the corresponding tree. The nodes $1, \ldots, n$ of the fan graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. The tubing outside of t_0 maps to the painted weakly ordered tree, the tubings inside t_0 map to the unpainted trees, and nodes that are inside t_0 but not inside any smaller tube determine the gaps-between leaves that coincide with the paint line. Examples are seen in Figure 17.

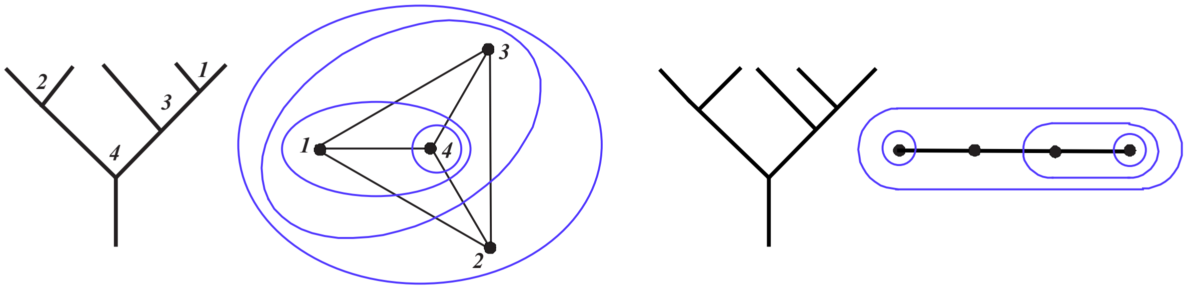


FIGURE 15. The permutation $\sigma = (2431) \in S_4$ pictured as an ordered tree and as a tubing of the complete graph; An unordered binary tree, and its corresponding tubing. Figure from [13].

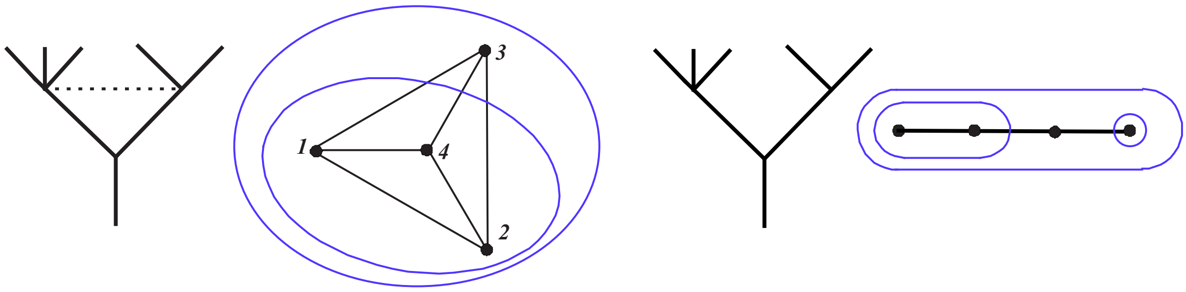


FIGURE 16. The ordered partition $(\{1,2,4\},\{3\})$ pictured as a leveled tree and as a tubing of the complete graph; the underlying tree, and its corresponding tubing. Figure from [13].

The fact that this bijection preserves the ordering follows easily from the definitions. Just note that adding a tube to a tubing of the fan graph corresponds to growing an internal edge in the tree. Adding a tube far outside of t_0 corresponds to growing an edge in the painted base. Adding a tube containing node 0 just inside t_0 (so that it becomes the new t_0) corresponds to growing painted edge(s) from a half-painted node. Adding a tube just inside t_0 that does not contain node 0 corresponds to growing unpainted

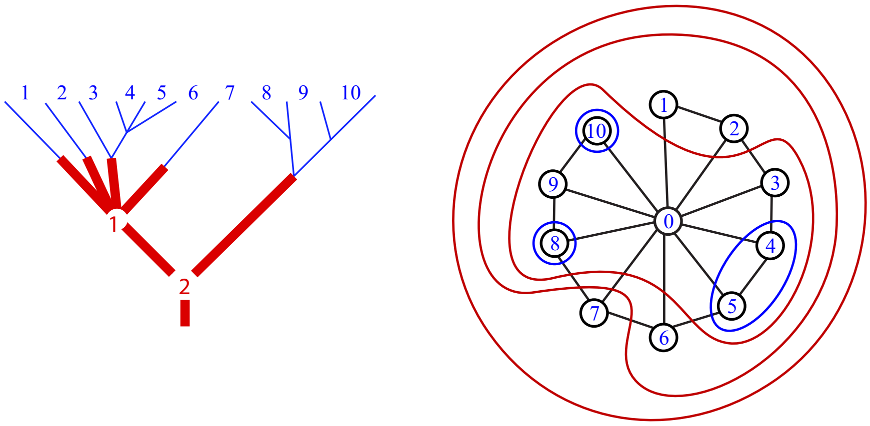


FIGURE 17. Example of the bijection in Theorem 3.6.

edge(s) from a half-painted node. Adding a tube further inside of t_0 (that does not contain node 0) corresponds to growing an edge in the unpainted forest. □

The isomorphism in 3d is shown pictorially in Figure 3.2.

3.3. **Enumeration.** As well as uncovering the equivalence between the pterahedra and the fan-graph associahedra, we found some new counting formulas for the vertices and facets of the pterahedra.

First the vertices of the pterahedra, which are forests of binary trees grafted to an ordered tree. If there are k nodes in the ordered, painted portion of a tree, then there are:

- $k!$ ways to make the ordered portion of this tree with k nodes,
- $k+1$ leaves of the ordered portion of this tree, and
- $n-k$ remaining nodes to be distributed among the $k+1$ binary trees that will go on the leveled/painted leaves.

Thus the number of vertices of the pterahedron, labeled by trees with n nodes, is:

$$v(n) = \sum_{k=0}^{n} [\ k! \sum_{\substack{\gamma_0+\ldots+\gamma_k \\ =n-k}} (\prod_{i=0}^{k} C_{\gamma_i})\ ].$$

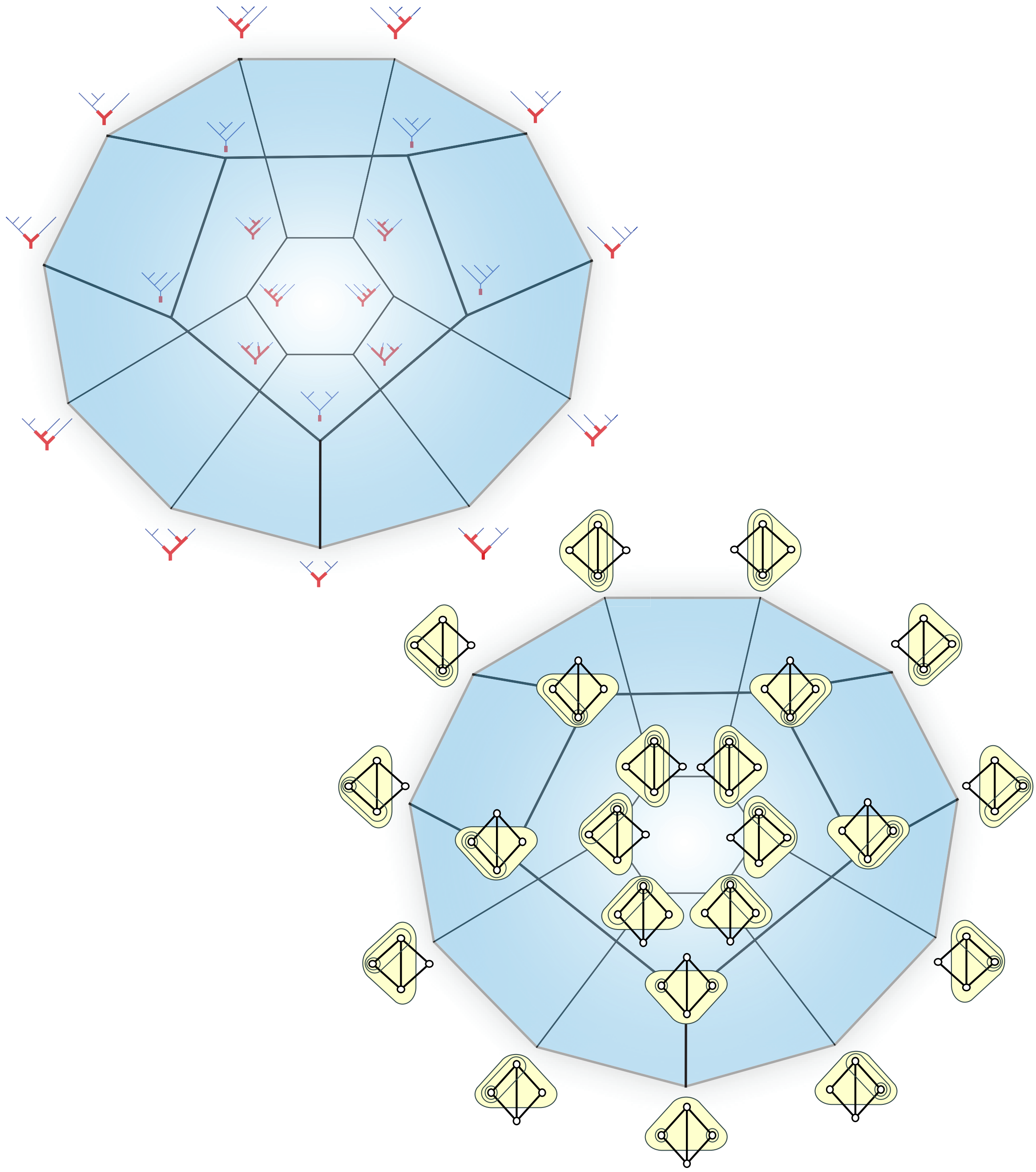


FIGURE 18. Two pictures of the pterahedron.

where C_j is the jth Catalan number. As an example, the number of trees with $n = 4$ is:

$$\begin{aligned}
&0!\ [C_4] \\
&+1!\ [C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3] \\
&+2!\ [C_2C_0C_0 + C_1C_1C_0 + C_1C_0C_1 + C_0C_2C_0 + C_0C_1C_1 + C_0C_0C_2] \\
&+3!\ [C_1C_0C_0C_0 + C_0C_1C_0C_0 + C_0C_0C_1C_0 + C_0C_0C_0C_1] \\
&+4!\ [C_0C_0C_0C_0C_0] \\
&= 1(14) + 1(14) + 2(9) + 6(4) + 24(1) \\
&= 14 + 14 + 18 + 24 + 24 \\
&= 94
\end{aligned}$$

We have computed the cardinalities for $n = 0$ to 9 and they are shown in table 1.

TABLE 1. The number $v(n)$ vertices of the pterahedra, $n = 0$ to 9

n	$v(n)$	n	$v(n)$
0	1	5	464
1	2	6	2652
2	6	7	17,562
3	22	8	133,934
4	94	9	1,162,504

There does not seem to be an entry for this sequence in the OEIS.

Examination of the computations of $v(n)$ leads to an interesting discovery. If we strip off the factorial factors in $v(n)$ and build a triangle of just the sums of C_{γ_i} products, it appears we are building the Catalan triangle.

$$\begin{array}{cccccccccc}
1 \\
1 & 1 \\
2 & 2 & 1 \\
5 & 5 & 3 & 1 \\
14 & 14 & 9 & 4 & 1 \\
42 & 42 & 28 & 14 & 5 & 1 \\
132 & 132 & 90 & 48 & 20 & 6 & 1 \\
429 & 429 & 297 & 165 & 75 & 27 & 7 & 1 \\
1430 & 1430 & 1001 & 572 & 275 & 110 & 35 & 8 & 1 \\
4862 & 4862 & 3432 & 2002 & 1001 & 429 & 154 & 44 & 9 & 1
\end{array}$$

For example,

$$v(4) = 94 = 0!(\mathbf{14}) + 1!(\mathbf{14}) + 2!(\mathbf{9}) + 3!(\mathbf{4}) + 4!(\mathbf{1})$$

leads to the values of the $n = 4$ row in the triangle above. In fact, Zoque [19] states that the entries of the Catalan triangle, often called *ballot numbers*, count "the number of ordered forests with m binary trees and with total number of ℓ internal vertices" where m and ℓ are indices into the triangle. These forests describe exactly the sets of binary trees we are grafting onto the leaf edges of individual leveled trees, which are counted by the sums of C_{γ_i} products. Thus we know that the ballot numbers are equivalent to the sums of C_{γ_i} products and can be used in the calculation of $v(n)$ for all values of n.

The formula for the entries of the Catalan triangle [**?**] leads to a simpler formula for $v(n)$, namely

$$v(n) = \sum_{k=0}^{n} k! \tfrac{(2n-k)!(k+1)}{(n-k)!(n+1)!},$$

Lastly, by considering the Catalan triangle as a matrix as in [1], we can say that the sequence of cardinalities $v(n)$ for all n is the Catalan transform of the factorials $(n-1)!$.

This means that the ordinary generating function for $v(n)$ is:

$$\sum_{k=1}^{\infty}(k-1)!\left(\frac{1-\sqrt{1-4x}}{2}\right)^{k}.$$

Also, it will be helpful to have a formula for the number of facets for the fan graphs, $F_{m,n}$. Recall $F_{m,n}$ is defined to be the graph join of $\bar{K}_m$ the empty graph on m nodes, and P_n the path graph on n nodes. Thus, $F_{m,n}$ has $m+n$ vertices, n of which comprise a subgraph isomorphic to the path graph on m nodes, the other n vertices connected to each of these m. Thus, $F_{m,n}$ has $m-1+mn = m(n+1)-1$ edges.

Now, counting tubes in this case is again a matter of counting subsets of vertices whose induced subgraph is connected. The structure of $F_{m,n}$ makes it useful to let V_m denote those vertices coming from the empty graph of m nodes, and likewise V_n those from the path graph on n nodes.

It is clear that some tubes are simply tubes of the path graph P_n, hence there are at least $\frac{n(n+1)}{2}-1$ tubes. We must not forget that V_n is itself now a tube since it is a proper subset of nodes of $F_{m,n}$.These tubes include every subset of V_n that is a valid tube of $F_{m,n}$.

It is simple to see that the only subsets of V_m that are valid tubes are precisely the singletons, since no pair of vertices in V_m are connected by an edge. Thus $F_{m,n}$ has at least $\frac{n(n+1)}{2}+m$ tubes.

The remaining possibility for tubes must include at least one node from V_m as well as at least one node from V_n. This produces all (possibly improper) tubes, since any subset of $V = V_m \cup V_n$ satisfying this criterion is connected. It is straightforward to see that there are exactly $(2^m-1)(2^n-1)$ tubes arising in this fashion. Now, however, we must subtract 1 from the above since we have allowed ourselves to count V as a tube, although it is not proper.

Hence we count

$$\frac{n(n+1)}{2}+(2^m-1)(2^n-1)+m-1$$

as the number of tubes of $F_{m,n}$ and the number of facets of the corresponding graph associahedron. For the pterahedra, where $m=1$, the formula becomes:

$$\frac{n(n+1)}{2}+2^n-1.$$

Interestingly, this is the same number of facets as possessed by the multiplihedron $\mathcal{J}(n)$, as seen in [7], where we enumerate the facets by describing their associated trees.

4. Associative products on the Pterohedra

4.1. **Preliminaries.** Let $\mathbb{K}$ denote a field. For any set X, we denote by $\mathbb{K}[X]$ the $\mathbb{K}$-vector space spanned by X.

For $n \geq 1$, we denote by Σ_n the group of permutations of n elements. For any set $U = \{u_1, \dots, u_n\}$ with n elements, an element $\sigma \in \Sigma_n$ acts naturally on the left on U and induces a total order $u_{\sigma^{-1}(1)} < \cdots < u_{\sigma^{-1}(n)}$ on U.

For nonnegative integers n and m, let $\mathrm{Sh}(n, m)$ denote the set of (n, m)-shuffles, that is the set of permutations σ in the symmetric group Σ_{n+m} satisfying that:

$$\sigma(1) < \cdots < \sigma(n) \qquad \text{and} \qquad \sigma(n+1) < \cdots < \sigma(n+m).$$

For $n = 0$, we define $\mathrm{Sh}(0, m) := \{1_m\} =: \mathrm{Sh}(m, 0)$, where 1_m is the identity of the group Σ_m. More in general, for any composition $(n_1, \dots, n_r)$ of n, we denote by $\mathrm{Sh}(n_1, \dots, n_r)$ the subset of all permutations σ in Σ_n such that $\sigma(n_1 + \cdots + n_i + 1) < \cdots < \sigma(n_1 + \cdots + n_{i+1})$, for $0 \le i \le r - 1$.

The concatenation of permutations $\times : \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$ is the associative product given by:

$$\sigma \times \tau(i) := \begin{cases} \sigma(i), & \text{for } 1 \le i \le n, \\ \tau(i-n) + n, & \text{for } n+1 \le i \le n+m, \end{cases}$$

for any pair of permutations $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$.

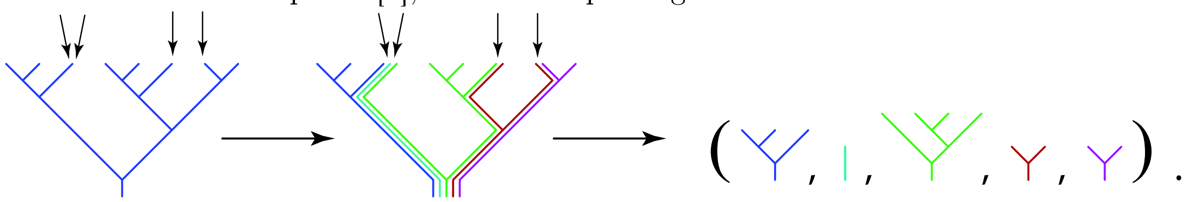
The well-known associativity of the shuffle states that:

$$\mathrm{Sh}(n+m, r) \cdot (\mathrm{Sh}(n, m) \times 1_r) = \mathrm{Sh}(n, m, r) = \mathrm{Sh}(n, m+r) \cdot (1_n \times \mathrm{Sh}(m, r)),$$

where $\cdot$ denotes the product in the group Σ_{p+q+r}.

In [10] there are defined products and coproducts on some of the painted trees shown in Figure 5. These products can be easily transferred to the vertices of the pterohedra via the bijection we have developed here.

Recall from [10] the concept of splitting a tree, given a multiset of its leaves. Here, modified from an example in [9], is a 4-fold splitting into an ordered list of 5 trees:



Note that a k-fold splitting, which is given by a size k multiset of the $n+1$ leaves, corresponds to a (k, n) shuffle. The corresponding shuffle is described as follows: $\sigma(i)$ for $i \in 1, \dots, k$ is equal to the sum of the numbers of leaves in the resulting list of trees 1 through i. For instance in the above example the shuffle is $\sigma = (3, 4, 8, 10, 1, 2, 5, 6, 7, 9, 11)$.

In [10] the product of two trees is described as a sum over splits of the first tree, where after each split the resulting list of trees is grafted to the leaves of the second tree. Thus this product can be seen as a sum over shuffles. In fact if we illustrate the products using the graph tubings, then shuffles are actually more easily made visible than splittings.

Here we mainly want to show some examples of the products, referring the reader to [10] for the full definitions (in terms of the trees). For that purpose we show single terms in the product, each term relative to a shuffle. Figure 19 shows a sample product, relative to the given shuffle, and illustrating the splitting as well. In Figure 20 we show the same sample product, pictured using the tubings on the star graphs: both vertices in the pterahedra.

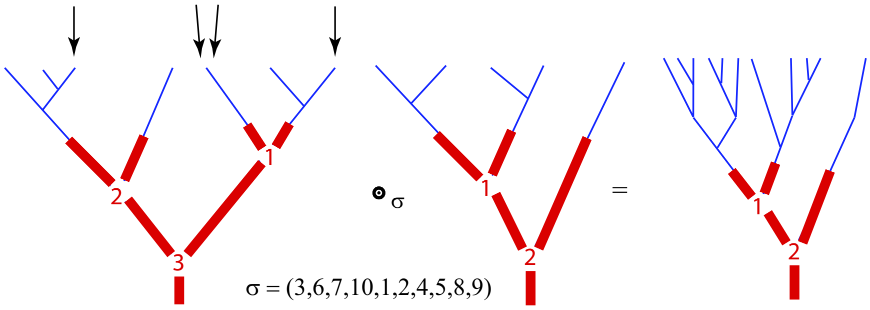


FIGURE 19. Example of one term in the multiplication of two painted trees.

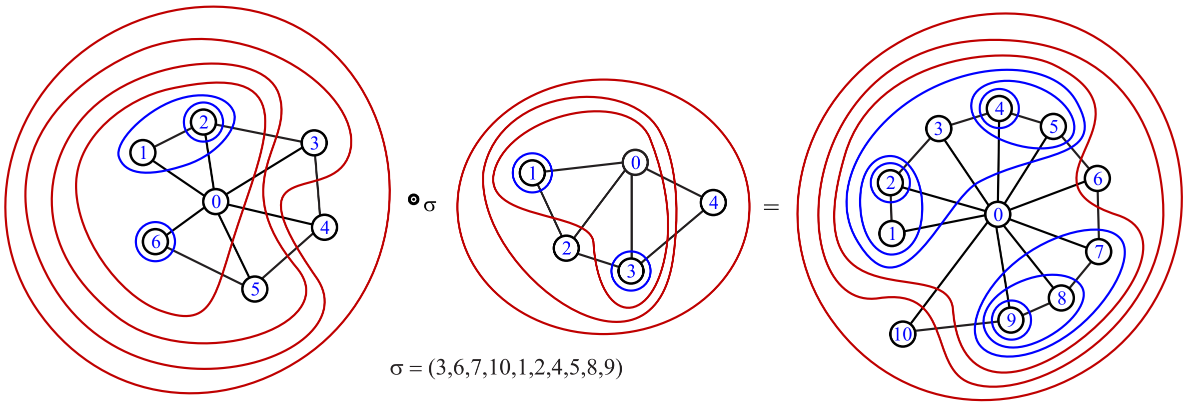


FIGURE 20. Example of one term in the multiplication of two painted trees; the same term as in Figure 19 but shown here with tubings.

5. Alternative product

For $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$. For any set of natural numbers $U = \{u_1, \dots, u_k\}$ and any integer $r \in \mathbb{Z}$, we denote by $U + r$ the set $\{u_1 + r, \dots, u_k + r\}$.

Let G be a simple finite graph with set of nodes $\mathrm{Nod}(G) = \{j_1, \dots, j_r\} \subseteq \mathbb{N}$, we denote by $G + n$ the graph G, with the set of nodes colored by $\mathrm{Nod}(G) + n$, obtained by replacing the node j_i of G by the node $j_i + n$, for $1 \leq i \leq r$.

Let G be a simple finite graph, whose set of nodes is $[n]$, we identify a tube $t = \{v_1, \dots, v_k\}$ of G with the tube $t(h) = \{v_1 + h, \dots, v_k + h\}$ of $G + h$. For any tubing $T = \{t_i\}$ of G, we denote by $T(h)$ the tubing $\{t_i(h)\}$ of $G + h$. In the present section,

we use the shuffle product, which defines an associative structure on the vector space spanned by all the vertices of permutohedra, in order to introduce associative products (of degree -1) on the vector spaces spanned by the vertices of pterahedra.

5.1. **The product.** Recall that the line (or path) graph L_n is the graph whose set of nodes is $\mathrm{Nod}(L_n) = \{1, \dots, n\}$, with edges $\{i, j\}$ for $|i-j| = 1$.

There exists a canonical bijection between the set of maximal tubings of L_n and the set $\mathcal{Y}_n$ of planar binary rooted trees with $n+1$ leaves, as showed by S. Forcey and D. Springfield in [13]. The authors showed that the associative product defined on the space spanned by all planar binary trees by J.-L. Loday and M. Ronco in [15], may be translated easily to the vector space spanned by the maximal tubings on the linear graphs. We describe briefly this product, for the proofs of the results we refer to [13].

Let T be a maximal tubing of L_n and W be a maximal tubing of L_m, the product $T \circ W$ is defined by the formula:

$$T \circ W = \sum_{\substack{S|_{[n]}=T \\ S|_{[m]+n}=W}} S,$$

where the sum is taken over all maximal tubings S of L_{n+m} such that S restricted to the set of vertices $[n]$ is T or $T \cup [n]$ and S restricted to the set of vertices $[m]+n$ is W or $W \cup [m]$, modulo the identification of the vertex $n+i$ of W with the vertex i of S for $1 \leq i \leq m$.

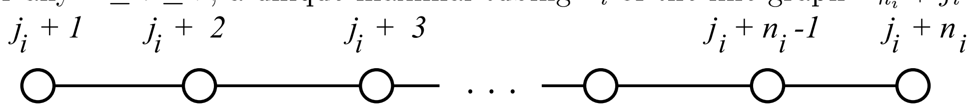
For example, the product of $T = \{\{1\}, \{3\}\}$ of L_3 with $W = \{\{1\}\}$ of L_2 is given by:

$$\begin{aligned} T \circ W = \{\{1\}, \{3\}, \{1,2,3\}, \{1,2,3,4\}\} \ + \ \{\{1\}, \{3\}, \{3,4\}, \{1,2,3,4\}\} \ + \\ \{\{1\}, \{4\}, \{3,4\}, \{1,2,3,4\}\} \ + \{\{1\}, \{4\}, \{4,5\}, \{3,4,5\}\} \ + \\ \{\{1\}, \{3\}, \{3,4\}, \{3,4,5\}\} \ + \ \{\{1\}, \{4\}, \{3,4\}, \{3,4,5\}\}. \end{aligned}$$

For $n \geq 1$, let us consider the simple connected graph $F_{1,n} = \mathfrak{S}(L_n)$.
That is, $F_{1,n}$ is the fan graph.

5.1**. Remark.** For any maximal tubing T of $F_{1,n}$, there exists:

(1) a unique integer $0 \leq r$ and unique families of nonnegative integers $j_1, \dots, j_r$ and $n_1, \dots, n_r$ such that
$$0 \leq j_1 < j_1 + n_1 < j_2 < j_2 + n_2 < \cdots < j_r < j_r + n_r \leq n,$$
(2) for any $1 \leq i \leq r$, a unique maximal tubing $\tilde{T}_i$ of the line graph $L_{n_i} + j_i =$



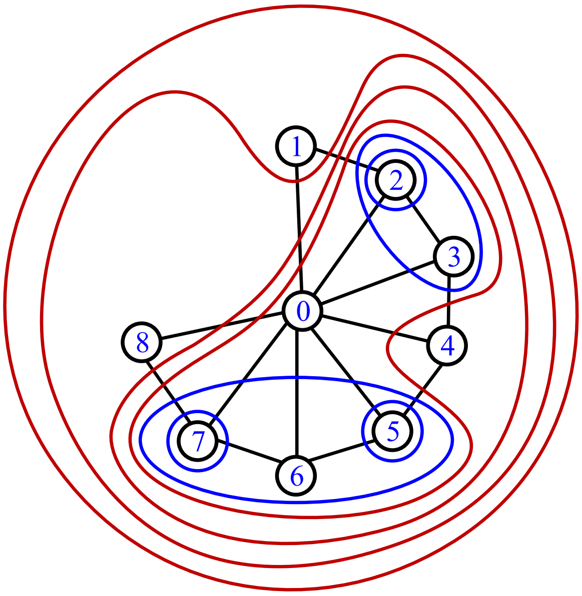
(3) a total order $u_1, \dots, u_N$ on the set
$$\{1, \dots, j_1, j_1 + n_1 + 1, \dots, j_2, j_2 + n_2 + 1, \dots, j_3, \dots, j_r, j_r + n_r + 1, \dots, n\}$$
where $N = n - (n_1 + \cdots + n_r)$,

such that t is a tube of T if, and only if, t satisfies one of the following conditions:

(1) t is a tube of $\tilde{T}_i$ or $t = \{j_i + 1, \dots, j_i + n_i\}$, for some $1 \leq i \leq r$,
(2) $t = t_0 := \{0, j_1 + 1, \dots, j_1 + n_1, j_2 + 1, \dots, j_r + n_r\}$,
(3) $t = t_0 \cup \{u_1, \dots, u_k\}$, for some $1 \leq k < N$.

5.2. **Example.** For $n = 8$, consider the tubing

$$T = \{\{2\},\{5\},\{7\},\{2,3\},\\ \{5,6,7\},\{0,2,3,5,6,7\},\{0,2,3,5,6,7,4\},\{0,2,3,5,6,7,4,8\}\}.$$



We have that $r = 2$, $j_1 = 1$, $j_2 = 4$, $n_1 = 2$, $n_2 = 3$, $N = 3$ and $(u_1, u_2, u_3) = (4, 8, 1)$. The tubing $\tilde{T}_1 = \{\{2\}\}$, while the tubing $\tilde{T}_2 = \{\{5\},\{7\}\}$.

5.2. **Notation.** *Let* $U = \{u_1 < \cdots < u_p\} \subseteq \{1, \ldots, n\}$.

(1) *For any permutation* $\omega \in \Sigma_p$, *we denote by* U_ω *the set* U *ordered by* ω, *that is* $U = (u_{\omega^{-1}(1)}, \ldots, u_{\omega^{-1}(p)})$. *When* $U = \emptyset$, *we assume that* $\Sigma_\emptyset = \{(1)\}$.

(2) *The complement* U^c *of* U *in* $\{1, \ldots, n\}$ *may be written in a unique way as:*

$$U^c = \{j_1 + 1, \ldots, j_1 + n_1, j_2 + 1, \ldots, j_2 + n_2, \ldots, j_r + 1, \ldots, j_r + n_r\},$$

for a unique $r \geq 0$ *and unique families of positive integers* $(j_1, \ldots, j_r)$ *and* $(n_1, \ldots, n_r)$ *satisfying that*

$$0 \leq j_1 < j_1 + n_1 < j_2 < \cdots < j_r < j_r + n_r \leq n.$$

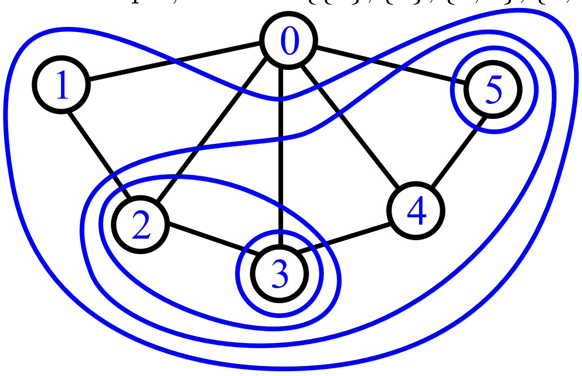
We denote by $r(U)$ *the integer* r *the number of blocks of consecutive integers in* U^c, *by* $\underline{j(U)}$ *the collection* $(j_1, \ldots, j_r)$ *and by* $\overline{n(U)}$ *the family* $(n_1, \ldots, n_r)$.

(3) *By Remark 5.1, we have that any tubing* T *of* $F_{1,n}$ *is determined uniquely by a subset* U *of* $\{1, \ldots, n\}$, *a permutation* $\omega \in \Sigma_{|U|}$, *and a family* $(\tilde{T}_1, \ldots, \tilde{T}_r)$, *where* $\tilde{T}_i$ *is a maximal tubing of the linear graph* L_{n_i}. *We denote such a tubing by* $T(U, \omega, \{\tilde{T}_j\}_{1 \leq j \leq r})$.

Note that for $U = \emptyset$, we have that $r(\emptyset) = 1$, $\underline{j(\emptyset)} = (0)$ and $\overline{n(\emptyset)} = n$. So, for any maximal tubing T of L_n, we have:

$$T(\emptyset, (1), T) = T \cup [n] = T \cup \{1, \ldots, n\}.$$

For example, for $T = \{\{3\},\{5\},\{2,3\},\{2,3,4,5\}\}$ in L_5, we get that $T(\emptyset,(1),T) =$



On the other hand, for $U = [n]$ and $\omega \in \Sigma_n$, we get that

$$T([n],\omega,\emptyset) = \{\{0\},\{0,\omega^{-1}(1)\},\{0,\omega^{-1}(1),\omega^{-1}(2)\},\dots,\{0,\omega^{-1}(1),\dots,\omega^{-1}(n-1)\}.$$

For instance

$$T = T([6],(4,2,5,6,1,3),\emptyset) = \{\{0\},\{0,5\},\{0,5,2\},\{0,5,2,6\},\{0,5,2,6,1\},\{0,5,2,6,1,3\}\},$$

in L_6.

5.3. **Definition.** Let $T = T(U,\omega,\{\tilde{T}_j\}_{1\leq j\leq r})$ be a maximal tubing of Pt_n, and $W = T(V,\tau,\{\tilde{W}_j\}_{1\leq j\leq s})$ be a maximal tubing of Pt_m, we define $T * W$ as follows:

(1) If either $n \in U$ or $1 \in V$, then

$$T * W := \sum_{\sigma\in \mathrm{Sh}(|U|,|V|)} T(U\cup(V+n),\sigma\cdot(\omega\times\tau),\{\tilde{T}_j\}_{1\leq j\leq r}\cup\{\tilde{W}_k+n\}_{1\leq k\leq s}),$$

where $\{\tilde{T}_j\}_{1\leq j\leq r}\cup\{\tilde{W}_k+n\}_{1\leq k\leq s} = \{\tilde{T}_1,\dots,\tilde{T}_r,\tilde{W}_1+n,\dots,\tilde{W}_s+n\}$.

(2) If $n \notin U$ and $1 \notin V$, then

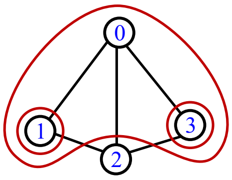
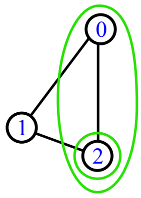
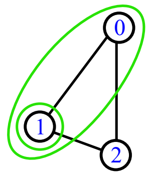
$$T * W = \sum_{\sigma\in \mathrm{Sh}(|U|,|V|)} T(U\cup(V+n),\sigma\cdot(\omega\times\tau),\{\tilde{T}_j\}_{1\leq j\leq r}\circ\{\tilde{W}_k+n\}_{1\leq k\leq s}),$$

where

$$\{\tilde{T}_j\}_{1\leq j\leq r}\circ\{\tilde{W}_k+n\}_{1\leq k\leq s} = \{\tilde{T}_1,\dots,\tilde{T}_r\tilde{*}\tilde{W}_1+n,\tilde{W}_2+n,\dots,\tilde{W}_s+n\},$$

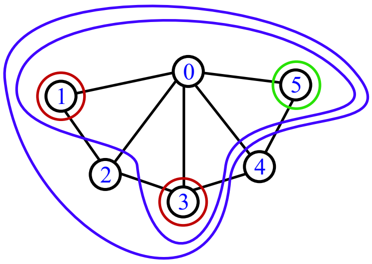
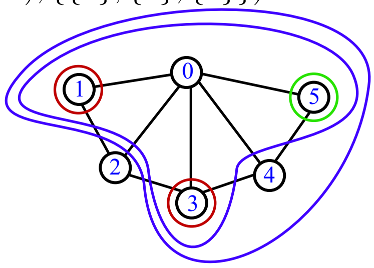
and the tubing $\tilde{T}_r\tilde{*}\tilde{W}_1+n_r$ denotes the product of the maximal tubings $\tilde{T}_r \in L_{n_r}+j_r$ and $\tilde{W}_1+n \in L_{m_1}+n$ maximal tubings in $L_{n_r+m_1}+j_r$.

5.4. **Example.** Consider the tubings $T = \{\{1\},\{3\},\{0,1,3\}\} = T(\{2\},(1),\{\tilde{T}_1 = \{1\},\tilde{T}_2 = \{3\}\})$ in Pt_3, $W = \{\{2\},\{0,2\}\} = T(\{1\},(1),\tilde{W}_1 = \{2\})$ in Pt_2 and $Z = \{\{1\},\{0,1\}\}$ in Pt_2, that is $Z = T(\{2\},(1),\emptyset)$,

$T =$  $W =$  $Z =$ 

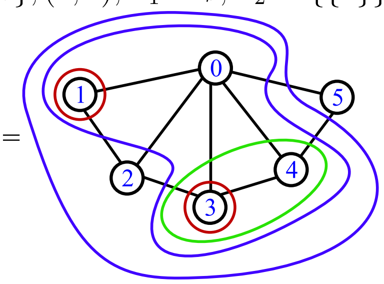
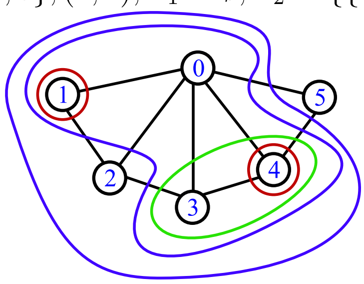
We have that:

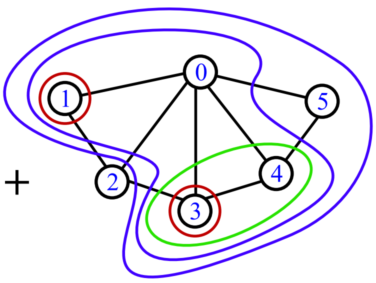
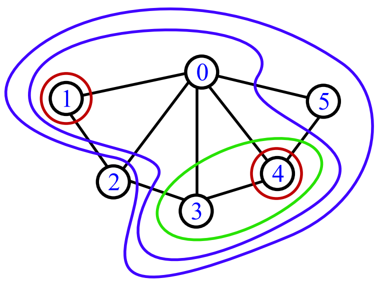
(1) $T * W = \{\{1\},\{3\},\{5\},\{0,1,3,5\},\{0,1,3,5,2\}\} +$
$\{\{1\},\{3\},\{5\},\{0,1,3,5\},\{0,1,3,5,4\}\} =$
$T(\{2,4\},(1,2),\{\{1\},\{3\},\{5\}\}) + T(\{2,4\},(2,1),\{\{1\},\{3\},\{5\}\})$.

$T \ * \ W =$  $+$ 

and

(2) $T*Z = \{\{1\},\{3\},\{3,4\},\{0,1,3,4\},\{0,1,2,3,4\}\}+\{\{1\},\{3\},\{3,4\},\{0,1,3,4\},\{0,1,3,4,5\}\}$
$+\{\{1\},\{4\},\{3,4\},\{0,1,3,4\},\{0,1,2,3,4\}\}+\{\{1\},\{4\},\{3,4\},\{0,1,3,4\},\{0,1,3,4,5\}\}$
$= T(\{2,5\},(1,2),\tilde{Z}_1=\emptyset,\tilde{Z}_2=\{\{3\}\}) + T(\{2,5\},(2,1),\tilde{Z}_1=\emptyset,\tilde{Z}_2=\{\{3\}\})$
$+ T(\{2,5\},(1,2),\tilde{Z}_1=\emptyset,\tilde{Z}_2=\{\{4\}\}) + T(\{2,5\},(2,1),\tilde{Z}_1=\emptyset,\tilde{Z}_2=\{\{4\}\})$.

$T * Z =$  $+$ 

$+$  $+$ 

5.5. **Proposition.** *The binary operation $*$ introduced in Definition 5.3 is an associative product.*

Proof. The proof uses the associativity of the shuffle product and of the product $\circ$ defined on the space spanned by binary planar rooted trees.

Let $T = T(U, \sigma, \{\tilde{T}_j\}_{1\leq j\leq r})$, $W = T(V, \tau, \{\tilde{W}_k\}_{1\leq k\leq s})$ and $Z = T(X, \delta, \{\tilde{Z}_j\}_{1\leq j\leq p})$ be three maximal tubings of $F_{1,n}$.

(1) If either $n \in U$ or $1 \in V$, and either $m \in V$ or $1 \in X$, then:

$$(T * W) * Z = \sum_{\omega} T(U \cup (V + n) \cup (X + n + m), \qquad \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1\leq j\leq r} \cup \{\tilde{W}_k + n\}_{1\leq k\leq s} \cup \{\tilde{Z}_h + n + m\}_{1\leq h\leq p}) =$$
$$T * (W * Z),$$

(2) If $n \notin U$ and $1 \notin V$, and either $m \notin V$ or $1 \notin X$, then

$$(T * W) * Z = \sum_{\omega} T(U \cup (V + n) \cup (X + n + m), \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1\leq j\leq r} \circ \{\tilde{W}_k + n\}_{1\leq k\leq s} \cup \{\tilde{Z}_h + n + m\}_{1\leq h\leq p}) =$$
$$T * (W * Z).$$

A similar result holds for $n \notin U$ or $1 \notin V$, and $m \in V$ and $1 \in X$, where the sums are taken over all the permutations $\omega \in \mathrm{Sh}(|U|, |V|, |X|)$.

(3) Suppose that $n \in U$, $\{1, m\} \subseteq V$ and $1 \in X$.

(a) If $s \geq 1$, then

$$(T * W) * Z = \sum_{\omega} T(U \cup (V + n) \cup (X + n + m), \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1\leq j\leq r} \circ \{\tilde{W}_k + n\}_{1\leq k\leq s} \circ \{\tilde{Z}_h + n + m\}_{1\leq h\leq p}) =$$
$$T * (W * Z),$$

where the sum is taken over all the permutations $\omega \in \mathrm{Sh}(|U|, |V|, |X|)$.

(b) If $W = T(\emptyset, (1), \tilde{W})$ for some maximal tubing $\tilde{W}$ of L_m, then

$$(T * W) * Z = \sum_{\omega} T(U \cap (X + n + m), \omega(\sigma \times \delta),$$
$$\tilde{T}_1, \ldots, \tilde{T}_r \circ \tilde{W} \circ \tilde{Z}_1, \ldots, \tilde{Z}_p) =$$
$$T * (W * Z),$$

where the sum is taken over all permutations $\omega \in \mathrm{Sh}(|U|, |X|)$.

$\square$

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H98230-14-0121.[1] The second author's specific position on the NSA is published in [6]. Suffice it to say here that the second author appreciates NSA funding for open research and education, but encourages reformers of the NSA who are working to ensure that protections of civil liberties keep pace with intelligence capabilities.

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