MINIMIZING POLLUTION THROUGH SEMI-ANTAGONISTIC EQUILIBRIUM POINTS

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ABSTRACT

A game theoretic model between a firm and regulator is studied in an effort to minimize pollution. This paper focuses on locating semi-antagonistic equilibrium points to find where player payoffs and pollution are each minimized amongst all equilibrium points. The impact of the variables governing the model on the existence of these sorts of points is also studied. Finally, games will be adjusted in such ways that the semi-antagonistic equilibrium point disappears and becomes an equilibrium where player payoffs are maximized, but pollution remains minimized.

ACKNOWLEDGEMENTS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In economics, an externality is a cost or benefit produced by a transaction between two parties and is imposed upon a third party unrelated to this transaction. The paper "When Does Compromise Prevent More Pollution?" [1] analyzes a model which contains the negative externality of pollution. By changing how much pollution each of the firm and regulator are meant to clean up, Clemons et al. show where the players of this model, the firm creating the pollution and the regulating agency that wants to control pollution, will find equilibrium. In other words, they look for wherever the firm and regulator each fulfill their interest of greatest utility in different situations.

Pollution, represented in their model as a frequency or likelihood of being produced, is a component of each player's payoff. This can affect each player's strategy depending on how often pollution is believed to occur. However, this model at face value only takes into consideration the wish of each player to maximize their own utility. Only because the model is built with the idea that the firm and regulator will clean it up does pollution become a factor. The effects of pollution are actually felt by the environment.

So, instead of looking for conditions in the model under which the firm and regulator would benefit the most, the intent of this paper is to find locations where the environment benefits the most. The firm and regulator factor into their decisions the cost of cleaning up all pollution, so the environment eventually is meant to recover fully. The benefit to the environment then can be defined as the amount of pollution that never reaches the environment. Maximizing the environment's utility would mean searching for the conditions under which the most pollution is prevented.

The model makes no mention of the time it would take for all pollution to be cleaned up. It simply attaches a variable of monetary cost to the cleanup of pollution. The actual process of cleanup could be over any span of time. During this time, the pollution may have a negative impact on the environment. Some of these negative factors could be reversed over time and reflected in the cost of pollution, letting the cost encompass both the cleanup of pollution and the restoration of damaged areas. In this time, there still may be effects that can not be recovered, such as the extinction of a local species. A cost can not be associated with these kinds of events, and so it would be important to avoid those costs. This is the motivation behind minimizing the onset of pollution. By having less pollution to begin with, cleanup is likely to take less time and the negative effects of pollution may be less severe.

Surely enough, pollution would be minimized the most when both the regulator and firm invest in a filter. However, we can not expect the firm and regulator to always make this investment. If we assume the firm and regulator still wish to maximize their own utility, then the real problem is to find when the firm and regulator reach an equilibrium that also results in the environment benefiting the most. One corollary by Clemons et al. [1] describes conditions under which the mixed or pure strategies end up providing a worse payoff for the environment, or prevent less pollution as they had put it. An alternative hypothesis offered by Dr. Jerzy Filar [?] is that these equilibrium points which minimize pollution have the same properties as those described in his paper "Semi-Antagonistic Eqilibrium Points and Action Costs" [2]. These points would potentially be equilibrium that both provide greatest utility to the environment and provide least utility to both firm and regulator amongst all equilibria in a game. This paper tests the hypothesis of Filar by searching for these points and using them as a lens through which other observations about the game can be made.

CHAPTER II

THE MODEL

	Firm controls (y)	Firm does not control $(1-y)$
Regulator controls (x)	$f_1 = (1-\tau)(P_F - C_F) - (1-\delta)^2 p\Delta(1-\alpha_1)$ $r_1 = \tau(P_F - C_F) - C_R - (1-\delta)^2 p\Delta\,\alpha_1$	$f_3 = (1-\tau)P_F - (1-\delta)p\Delta(1-\alpha_3)$ $r_3 = \tau P_F - C_R - (1-\delta)p\Delta\,\alpha_3$
Reg. does not control $(1-x)$	$f_2 = (1-\tau)(P_F - C_F) - (1-\delta)p\Delta(1-\alpha_1)$ $r_2 = \tau(P_F - C_F) - (1-\delta)p\Delta\,\alpha_1$	$f_4 = (1-\tau)P_F - p\Delta(1-\alpha_4)$ $r_4 = \tau P_F - p\Delta\,\alpha_4$

Figure 2.1: Model of Pollution Control Between Regulator and Firm [1] [1]

The payoff matrix by Clemons et al. [1] is reintroduced above. We retain many of the same assumptions made in its design. As such, the game is still between two players, the regulator and the firm, the two players choose independently whether to control pollution or not, and production by the firm remains constant and at a

[1]The variables in the model are as follows: τ: the corporate tax rate on profit minus any cost of filters; P_F: the profit achieved by the firm, before tax and before paying for any voluntary preventative filters and mandatory cleanup costs; C_F: the cost to the firm of optional filter; C_R: the cost to the regulator of optional filter; δ: the effectiveness of the optional filters; p: probability or fraction of pollution released as a result of production; Δ: expected cost of cleanup; α_1: the percentage of cleanup costs paid by the regulator if the firm opts for extra controls; α_3: the percentage of cleanup costs paid by the regulator when the firm does not apply extra controls but the regulator does; α_4: the percentage of cleanup costs paid by the regulator if neither firm nor regulator opt for extra controls.[1]

practical optimum level for the entire game.

Clemons et al. studied this payoff matrix in four different cases to see which kinds of equilibria, pure and mixed, would show up under certain conditions. These cases were differentiated by different values of α_n, the portion of pollution whose cleanup would be the responsibility of the regulator, according to different real-world scenarios. This variable was framed as a penalty against the firm with smaller values of α_n representing a larger penalty. There is no α_2 in the model as it it is assumed that by opting to control, the firm is guaranteed some level of cleanup by the regulator. In that case, $\alpha_2 = \alpha_1$. The four cases studied were a uniform α_n ($\alpha_1 = \alpha_3 = \alpha_4$, known as Case 1), a smaller penalty for no control ($\alpha_1 = \alpha_3 > \alpha_4$, Case 2), a larger penalty for no control ($\alpha_1 = \alpha_3 < \alpha_4$, Case 3), and a larger penalty for no control by the firm ($\alpha_1 > \alpha_3 = \alpha_4$, Case 4). In each of these cases, Clemons et al. had shown what types of equilibria would show up with given ratios of control cost to cleanup for the regulator and firm.

Our primary approach to study this model will be an experimental one. We will develop a program that takes the data from these different cases and finds the conditions under which *semi-antagonistic equilibrium points* exist, a point whose qualities will be examined in the next chapter. Due to this technological approach, it would be most useful to work with variables that are bounded so that we can describe cases on a closed space and along their entire domains. As $p, \delta, \alpha_i \in [0, 1]$, these variables are some of the best suited to be changed. We will choose to work with these variables on the range $(0, 1)$ instead in order to avoid awkward conditions such as guaranteed or absolute pollution or a perfect filter. We also assume that $\tau \in (0, 1)$ as it would be unlikely for a tax to be for all and more of a firm's revenue or to be negative. Further details about the construction of the experimental program will be covered in a later chapter. For the sake of comparison with the examples given by Clemons et al. in each of the above cases, α_n will first remain fixed to fit the previously outlined cases while p, δ, and τ are adjusted at uniform steps between 0 and 1. The program will also be run with a fixed τ and adjustable α_n in the uniform case. Case 3, the case of higher penalties for control by any player, is not studied in depth here, having been mentioned by Clemons et al. as being counterintuitive. [1]

CHAPTER III

SEMI-ANTAGONISTIC EQUILIBRIUM POINTS

The points we want to find will be of a special type of equilibrium point. Filar names these *semi-antagonistic equilibrium points*, henceforth referred to as SAEPs [2]. He hypothesizes that these points are supposedly where the firm and regulator are at equilibrium and pollution is minimized [**?**]. The definition of these points is restated as follows: an SAEP is an equilibrium point (x, y) of a bimatrix game (A, B) that solves the following set of inequalities:

(a) $e_i Ay \leq xAy; \qquad i = 1, 2, .., m,$

(b) $xBe_j \leq xBy; \qquad j = 1, 2, .., n,$

(c) $x(-B)e_j \geq v(-B); \; j = 1, 2, .., n,$

(d) $e_i Ay \leq v(A); \qquad i = 1, 2, .., m,$

where $v(A)$ (respectively $v(-B)$) is the minimax value of the matrix game A $(-B)$ and e_i (e_j) is the i^{th} (j^{th}) vector of the m (n)-dimensional unit basis. The values $v(A)$ and $v(-B)$ are each calculated by considering A and $-B$ as zero-sum games.

The first two of these inequalities verify that the point in question is a Nash equilibrium point. The last two inequalities ensure that each player is earning no more than they would in the zero-sum component of the bimatrix model corresponding to that player's payoff matrix. If all four inequalities are satisfied, then neither player can increase their own payoff nor decrease their opponent's payoff by changing strategy, assuming the other player's strategy is fixed. As mentioned by Filar, this description of SAEPs mimics the description of points found in Aumann's *almost strictly competitive games*, having the properties of both equilibrium points and *twisted equilibrium points*, where the objective of players is to minimize their opponents utility rather than maximize their own [3]. *Almost strictly competitive games* are games which contain only *twisted equilibrium points*, which are equilibrium points where a unilateral change in strategy results in no smaller of a payoff for an opponent.

$$\begin{pmatrix} 2,\ 1 & 0,\ 0 \\ 0,\ 0 & 1,\ 2 \end{pmatrix} \begin{pmatrix} 2,\ 1 & 5,\ 0 & 0,\ \text{-}1 \\ 4,\ 1 & 0,\ 0 & 0,\ \text{-}1 \end{pmatrix} \begin{pmatrix} 4,\ 4 & 0,\ 5 \\ 5,\ 0 & 1,\ 1 \end{pmatrix} \begin{pmatrix} 6,\ 9 & 3,\ 8 \\ 5,\ 6 & 4,\ 7 \end{pmatrix}$$

Figure 3.1: Four different matrix games used to demonstrate how to find semi-antagonistic equilibrium points.

The four games above shall be used to demonstrate how SAEPs are determined to exist or not. The first bimatrix game from the left represents a form of the "battle of the sexes" game. There are three equilibria in this game - two pure and one mixed. The pure equilibria are located at $(x, y) = (1, 1)$ and $(x, y) = (0, 0)$. The mixed equilibrium can be found to be $(x, y) = (\frac{2}{3}, \frac{1}{3})$. The mixed equilibrium here is an SAEP. To demonstrate this, we check each inequality to see whether it is satisfied. The first inequality is satisfied when a unilateral change in the row player's strategy results in no greater payoff for the row player. This is true since, given the column player's strategy, the row player always earns the same utility. The second inequality is satisfied when a unilateral change in the column player's strategy results in no greater payoff for the column player and is true in this case in a similar way as previous.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad -B = \begin{pmatrix} \text{-1} & 0 \\ 0 & \text{-2} \end{pmatrix}$$

Figure 3.2: The A and $-B$ matrices for the "battle of the sexes" game.

The bimatrix game must then be split into its separate payoff matrices for each player to find $v(A)$ and $v(-B)$. For game A, $(x, y) = (\frac{1}{3}, \frac{1}{3})$, and for game -B, $(x, y) = (\frac{2}{3}, \frac{2}{3})$. Therefore, $v(A) = \frac{2}{3}$ and $v(-B) = \frac{-2}{3}$. Returning to the inequalities, the third condition is satisfied when the given strategy of the row player and any pure strategy of the column player in the $-B$ game provides no worse utility than the value of $v(-B)$. Similarly, the fourth condition is satisfied when the given strategy of the column player and any pure strategy of the row player in the A game provides no better utility than the value of $v(A)$. In A, the given strategy of the column player is such that any pure strategy by the row player gives the same value, which is $v(A)$. The third condition is satisfied similarly with all values being $v(-B)$. Having satisfied all inequalities, the mixed equilibrium is found to be an SAEP. The pure strategies both fail at this stage, showing that each player could choose a strategy that results in a lower payoff for their opponent.

The second game has a single equilibrium point at $x_0 = (0, 1), y_0 = (1, 0, 0)$ due to the dominant strategy of the column player. However, the column player can reduce the row player's payoff by adjusting their strategy to $y_0 = (0, 0, 1)$ which demonstrates that the fourth inequality is always unsatisfied. $v(A)$ in this game is 0, but the payoffs available to the row player with the column player's given strategy are 2 and 4, clearly better than 0.

The third game, a version of "prisoner's dilemma", also has a single equilibrium of $(x, y) = (0, 0)$. This point is certainly Nash and is also twisted, so it must be an SAEP. To see this, notice that if the row player's goal is to maximize their own utility, they would always play down, and the same is true if their goal is to minimize their opponent's utility. Due to the symmetry of the game, the column player would make similar decisions. In this case, all of the game's equilibrium points are SAEPs, which would make it fit the conditions for being Aumann's almost strictly competitive game.

The last game contains pure equilibria at $(x, y) = (1, 1)$ and $(x, y) = (0, 0)$ and mixed equilibrium at $(x, y) = (\frac{1}{2}, \frac{1}{2})$. The pure equilibrium at $(x, y) = (0, 0)$ can be seen to be an SAEP. It is easy to see that this point is a twisted equilibrium,

Knowing that SAEPs only occur when both players' utilites are maximized against their opponent's strategy and when both players' utilities can not be reduced by a change in their opponents strategy, we can represent the existence of an SAEP in the pollution game at each pure equilibria as satisfying four inequalities.

I: Nash conditions: $f_1 \geq f_3, r_1 \geq r_2$, Antagonistic conditions: $r_1 \leq r_3, f_1 \leq f_2$

II: . . . . . $f_2 \geq f_4, r_2 \geq r_1$, ... $r_2 \leq r_4, f_2 \leq f_1$

III: . . . . . $f_3 \geq f_1, r_3 \geq r_4$, ... $r_3 \leq r_1, f_3 \leq f_4$

IV: . . . . . $f_4 \geq f_2, r_4 \geq r_3$, ... $r_4 \leq r_2, f_4 \leq f_3$

The Nash conditions are two conditions that must at least be satisfied for the point to be a Nash equilibrium. The antagonistic conditions would be the conditions fulfilled by the players if their intent was to minimize their opponent's utility. As they have been written here, the conditions parallel each other. For example, in pure equilibrium type I, the firm plays to the Nash condition of maximizing their utility between f_1 and f_3, opting for the larger of the two. At the same time, they play to minimize the utility of the regulator between r_1 and r_3. We can substitute in the equations from the payoff matrix and have a better idea of what conditions need to be fulfilled.

3.1 Type I

Type I SAEPs are equilibria where the regulator and firm both choose to always control pollution through purchase of a filter.

Our Nash condition for the firm yields

$$
\begin{aligned}
f_1 \geq f_3: \quad 0 \leq f_1 - f_3 &= (1-\tau)(P_F - C_F) \\
&- (1-\delta)^2 p\Delta(1-\alpha_1) - [(1-\tau)P_F - (1-\delta)p\Delta(1-\alpha_3)] \\
&= -C_F(1-\tau) - (1-\delta)p\Delta([1-\delta][1-\alpha_1] - [1-\alpha_3]) \\
&= -C_F(1-\tau) - (1-\delta)p\Delta(\alpha_3 - \delta - \alpha_1 + \alpha_1\delta).
\end{aligned} \tag{3.1}
$$

Our Nash condition for the regulator yields

$$
\begin{aligned}
r_1 \geq r_2: \quad 0 \leq r_1 - r_2 &= \tau(P_F - C_F) - C_R - (1-\delta)^2 p\Delta\alpha_1 \\
&- \quad [\tau(P_F - C_F) - (1-\delta)p\Delta\alpha_1] \\
&= \quad -C_R + (1-\delta)p\Delta\alpha_1(1 - [1-\delta]) \\
&= \quad -C_R + (1-\delta)p\Delta\alpha_1\delta.
\end{aligned}
\tag{3.2}
$$

Our antagonistic condition for the firm yields

$$
\begin{aligned}
r_1 \leq r_3: \quad 0 \geq r_1 - r_3\tau(P_F - C_F) - C_R& \\
- \quad (1-\delta)^2 p\Delta\alpha_1 - [\tau P_F - C_R - (1-\delta)p\Delta\alpha_3]& \\
= \quad -\tau C_F - (1-\delta)p\Delta([1-\delta]\alpha_1 - \alpha_3)& \\
= \quad -\tau C_F - (1-\delta)p\Delta(\alpha_1 - \alpha_3 - \alpha_1\delta).&
\end{aligned}
\tag{3.3}
$$

Our antagonistic condition for the regulator yields

$$
\begin{aligned}
f_1 \leq f_2: \quad 0 \geq f_1 - f_2 = (1-\tau)(P_F - C_F)& \\
- \quad (1-\delta)^2 p\Delta(1-\alpha_1) - [(1-\tau)(P_F - C_F) - (1-\delta)p\Delta(1-\alpha_1)]& \\
= \quad -(1-\delta)p\Delta(1-\alpha_1)([1-\delta] - 1) = (1-\delta)p\Delta(1-\alpha_1)\delta.&
\end{aligned}
\tag{3.4}
$$

All of these conditions must be satisfied in order for a type I SAEP to exist. In 3.4, there is a product that must be nonpositive in order to be satisfied. However, each factor in this product is positive. As such, it must be that either $\delta = 0$, $\delta = 1$, $\alpha_1 = 1$, $p = 0$, or $\Delta = 0$. Since extreme values are not used in our study, we will never observe this equilibrium.

3.2 Type II

Type II SAEPs are equilibria where the regulator chooses to not control pollution, but the firm does choose to control pollution through purchase of a filter.

Our Nash condition for the firm yields

$$\begin{aligned} f_2 \geq f_4: \quad 0 \leq f_2 - f_4 &= (1-\tau)(P_F - C_F) \\ &- (1-\delta)p\Delta(1-\alpha_1) - [(1-\tau)P_F - p\Delta(1-\alpha_4)] \\ &= -C_F(1-\tau) - p\Delta([1-\delta][1-\alpha_1] - [1-\alpha_4]) \\ &= -C_F(1-\tau) - p\Delta(\alpha_4 - \delta - \alpha_1 + \alpha_1\delta). \end{aligned} \tag{3.5}$$

Our Nash condition for the regulator yields

$$r_2 \geq r_1: \quad 0 \leq r_2 - r_1 = -(r_1 - r_2) = C_R - (1-\delta)p\Delta\alpha_1\delta. \tag{3.6}$$

Our antagonistic condition for the firm yields

$$\begin{aligned} r_2 \leq r_4: \quad & 0 \geq r_2 - r_4 \\ &= \tau(P_F - C_F) - (1-\delta)p\Delta\alpha_1 - [\tau P_F - p\Delta\alpha_4] \\ &= -\tau C_F - p\Delta([1-\delta]\alpha_1 - \alpha_4) \\ &= -\tau C_F - p\Delta(\alpha_1 - \alpha_4 - \delta\alpha_1). \end{aligned} \tag{3.7}$$

Our antagonistic condition for the regulator yields

$$f_2 \leq f_1: \quad 0 \geq f_2 - f_1 = -(f_1 - f_2) = -(1-\delta)p\Delta(1-\alpha_1)\delta. \tag{3.8}$$

The only thing worth immediate mention here is that since 3.4 will always be unsatisfied, 3.8 will be always satisfied for the same reasons.

3.3 Type III

Type III SAEPs are equilibria where the firm chooses to not control pollution, but the regulator does choose to control pollution through purchase of a filter.

Our Nash condition for the firm yields

$$\begin{aligned} f_3 \geq f_1: \quad & 0 \leq f_3 - f_1 = -(f_1 - f_3) \geq 0 \\ \rightarrow \quad & C_F(1-\tau) + (1-\delta)p\Delta(\alpha_3 - \delta - \alpha_1 + \alpha_1\delta). \end{aligned} \tag{3.9}$$

Our Nash condition for the regulator yields

$$\begin{aligned} r_3 \geq r_4: \quad & 0 \leq r_3 - r_4 \\ = \quad & \tau P_F - C_R - (1-\delta)p\Delta\alpha_3 - [\tau P_F - p\Delta\alpha_4] \\ = \quad & -C_R - p\Delta([1-\delta]\alpha_3 - [1-\alpha_4]) \\ = \quad & -C_R - p\Delta(\alpha_3 + \alpha_4 - 1 - \alpha_3\delta). \end{aligned} \tag{3.10}$$

Our antagonistic condition for the firm yields

$$r_3 \leq r_1: \quad 0 \geq r_3 - r_1 = -(r_1 - r_3) = \tau C_F + (1-\delta)p\Delta(\alpha_1 - \alpha_3 - \alpha_1\delta). \tag{3.11}$$

Our antagonistic condition for the regulator yields

$$\begin{aligned} f_3 \leq f_4: \quad 0 \geq f_3 - f_4 &= (1-\tau)P_F - (1-\delta)p\Delta(1-\alpha_3) \\ &- [(1-\tau)P_F - p\Delta(1-\alpha_4)] \\ &= -p\Delta([1-\delta][1-\alpha_3] - [1-\alpha_4]) \\ &= -p\Delta(\alpha_4 - \alpha_3 - \delta + \alpha_3\delta). \end{aligned} \tag{3.12}$$

Similar to case I, there is a reason why this type of equilibrium will not show up in our program. In 3.12, we have a product that must be negative. Since p and Δ are strictly positive, we can divide them out of the inequality and rearrange the remaining terms to produce $\delta(1-\alpha_3) + \alpha_3 \leq \alpha_4$. Since all of our cases have $\alpha_3 \geq \alpha_4$ and $\delta \in (0,1)$, this will not be satisfied.

It may be worthwhile to mention again that Case 3 ($\alpha_1 = \alpha_3 < \alpha_4$) is left unstudied. Under those conditions, it may be possible for a type III equilibrium to exist. In that case, δ would also have to be very small to reduce 3.12 towards $\alpha_3 \leq \alpha_4$, which would then be true.

3.4 Type IV

Type IV SAEPs are equilibria where neither the regulator nor firm choose to ever control pollution through purchase of a filter.

Our Nash condition for the firm yields

$$\begin{aligned} f_4 \geq f_2: \quad 0 \leq f_4 - f_2 &= -(f_2 - f_4) \\ &= C_F(1-\tau) + p\Delta(\alpha_4 - \delta - \alpha_1 + \alpha_1\delta). \end{aligned} \tag{3.13}$$

Our Nash condition for the regulator yields

$$r_4 \geq r_3: \quad 0 \leq r_4 - r_3 = -(r_3 - r_4) = C_R + p\Delta(\alpha_3 + \alpha_4 - 1 - \alpha_3\delta). \tag{3.14}$$

Our antagonistic condition for the firm yields

$$r_4 \leq r_2: \quad 0 \geq r_4 - r_2 = -(r_2 - r_4) = \tau C_F + p\Delta(\alpha_1 - \alpha_4 - \delta\alpha_1). \tag{3.15}$$

Our antagonistic condition for the regulator yields

$$f_4 \leq f_3: \quad 0 \geq f_4 - f_3 = -(f_3 - f_4) = p\Delta(\alpha_4 - \alpha_3 - \delta + \alpha_3\delta). \tag{3.16}$$

These conditions that are satisfied when type II or type IV SAEPs exist will be discussed more in the results section. For now, it is enough to say that none of the conditions to either of those types are immediately threatened to always be unsatisfied in any of our cases. What we can claim now at least is that a type I or III SAEP will never be encountered in our tests of each case, though we might have found some type III SAEP if had looked into Case 3.

CHAPTER IV

PROGRAM AND RESULTS

To find the conditions under which SAEPs exist in the pollution model, a program was built whose input would be the conditions of the model and whose output would be a scatter plot of where such points exist with axes of p, δ, and τ between 0 and 1. The program iterates through uniform step sizes of these variables to produce an $n \times n \times n$ array whose nonzero values correspond to the type of equilibrium point associated with the SAEP, be it pure or mixed. By default, the program searches for any SAEP in games with as few as one equilibrium point. Revisions to the program will demonstrate changes in the plot under different search criteria. The program then graphs this data and associates different colors with different types of equilibria, giving different colors for different strategies.

4.1 The Program

The program, coded in MATLAB, begins with a small set of code that sets the variables P_F, C_F, C_R, Δ, and α_n. There are separate sets of code with different values corresponding to each case studied. A step size is defined as $\frac{1}{n}$. These variables are then all sent into an m-file whose purpose is to produce the scatter plot.

The m-file's first task is to define several matrices whose purpose later will be to hold values of p, δ, τ, and the type of equilibrium associated with the SAEP for the scatter plot to be created. Then a new *step* variable is defined to be one less the inverse of the step size. An array of zeros is defined with dimensions $step \times step \times step$ with element (i, j, k) corresponding to the i-th step of p, j-th step of δ, and k-th step of τ. This array will record a value that represents each type of pure and mixed equilibria for each set of p, δ, and τ that produces an SAEP. The only purpose of this array is to be called after the whole program has been run if there is a desire to see what type of SAEP might exist for a specific set of p, δ, and τ. A counter *good* starts at 1 and will expand the size of the matrices holding values of p, δ, τ, and the equilibrium type for each "good" set of conditions producing an SAEP. Several nested "for" loops of similar structure are then made for each of p, δ, and τ whose counters are i, j, and k respectively. Each "for" loop runs from 1 to *step* and sets its variable equal to the product of its counter and the step size. Inside the deepest loop, the variables $P_F, C_F, C_R, \Delta, \alpha_n, p, \delta$, and τ are sent into a second m-file.

The purpose of the second m-file is to determine whether an SAEP exists by plugging the variables into the formulas of the pollution game. A matrix *eqlb* is defined to hold payoffs to the regulator and firm, the type of equilibrium, and the amount of pollution for every strategy set that is determined to be a Nash equilibrium. Then, following Filar's definition of SAEP, the values of each equilibrium for the zero-sum games for regulator and firm are found. Each Nash equilibrium is tested to see if its strategy set satisfies the third and fourth inequalities of the definition of an SAEP. Once one such equilibrium is found, its type is reported back along with a counter to indicate that such an equilibrium was found.

If an SAEP was found, the first m-file records the values of p, δ, τ, and the type of equilibrium. Then the *good* counter in incremented so that the matrices storing those values can be expanded. After increasing the counter, or if no SAEP is found, the next step in the nesting "for" loops begin. When all of the "for" loops have finished, a scatter plot is generated with each coordinate (p, δ, τ) being plotted and colored according to type.

Within the first m-file is comment-blocked code which can be turned on to produce a matrix similar to the one which holds the type, but instead stores a single value, resulting in a plot with a single color. Within the second m-file are several lines of optional code to further restrict what points are recorded. There is an "if" statement which can be turned on to only accept an equilibrium when it also produces the least pollution of any valid SAEP. There is another "if" statement that requires the existence of more than one valid equilibrium in order for the point to be recorded. A third "if" loop will only allow a specific type to be accepted, which can be used if there is a desire to graph only one type of equilibrium. The program itself can be modified such that instead of a fixed α_n and variable τ, there is a fixed τ and variable α_n, as we show in Case 1.

4.2 Case 1

The first case studied is the uniform case ($\alpha_1 = \alpha_3 = \alpha_4$). The same conditions as the original first case of Clemons et al. are used.

Case 1:

$P_f = 1000$

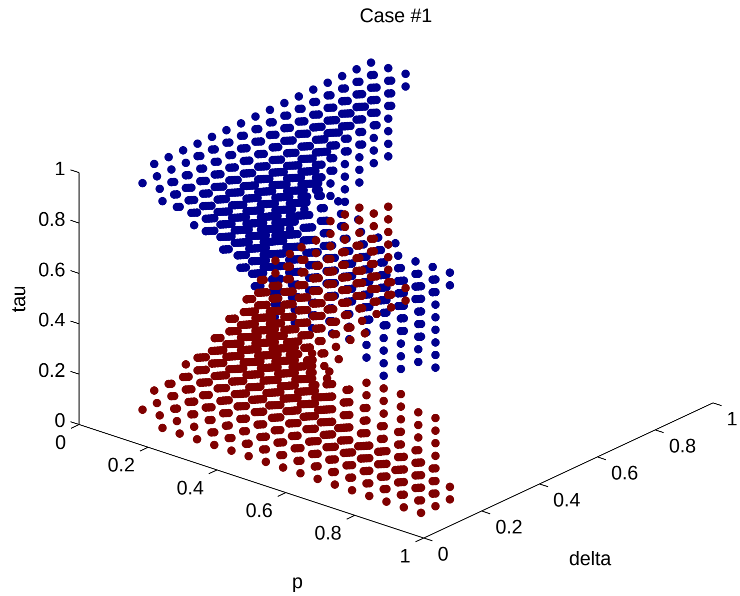
$\Delta = 500$

$C_F = 40$

$C_R = 40$

$\alpha_1 = \alpha_3 = \alpha_4 = 0.5$

$\tau, \delta, p \in (0, 1)$



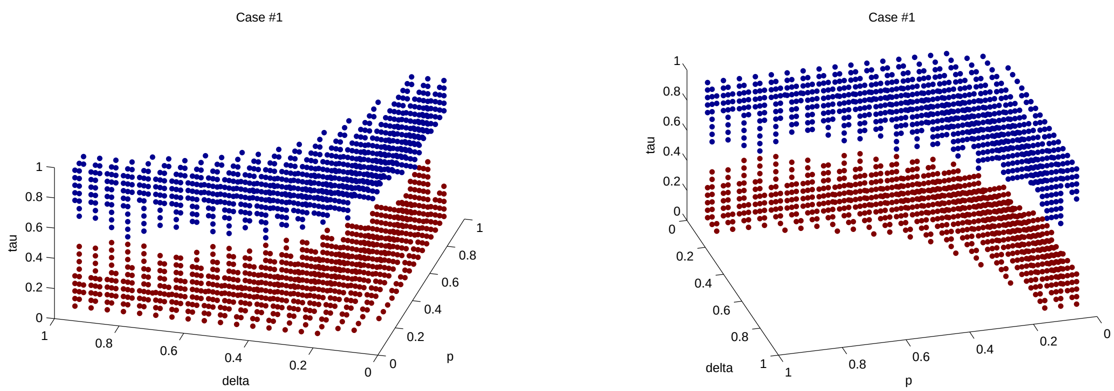


Figure 4.1: Conditions allowing SAEPs in Case 1.

The first observation to make is the apparent symmetry of the graph. There is an observed symmetry along the planes $p = \delta$ and $\tau = 0.5$. The $p = \delta$ symmetry demonstrates that SAEPs will exist for low values each of p and δ or for high values of one and small values of the other. In other words, SAEPs exist when filters are poor and pollution is very likely and when filters are excellent and pollution is rare. No SAEPs exist when filters are strong against a high chance of pollution.

The second aspect of note is that there are only two types of equilibria shown in this graph. For $\tau > 0.5$, the type II equilibrium is the only type observed. For $\tau < 0.5$, only type IV is observed. In both cases, the regulator opts to not control pollution. It should be noted that for $\alpha_n = 0.5$, there appears to be a sort of gap in the data at $\tau = 0.5$ and that the type of equilibrium changes across this value of τ. This will be demonstrated further later on.

To demonstrate why this makes sense, recall the Nash and antagonistic conditions of each type of point. In case 1, 3.7 becomes $-\tau C_F + p\Delta\delta\alpha_1 \leq 0$. This inequality is likely to be satisfied for high values of τ. In the same way, 3.15 is likely to be satisfied for low values of τ. These same conditions demonstrate how high values of δ and p can cause the inequality to become unsatisfied, hence the large amount of space with no SAEP for greater values of these variables.

When the same conditions are used, but the program now is asked to find only points which have strictly worse pollution than some other equilibria, we would hope, according to our hypothesis, that a graph produced under such constraints would have no points.

Figure 4.2 demonstrates that just by playing to an SAEP does not guarantee that pollution is minimal amongst all available equilibria. Because the program has been set to record this point only if its pollution is strictly worse than some other equilibrium strategy in the game, we know that these are nontrivial points where the only strategy is the SAEP. It would also be interesting to impose this same restriction of more than one equilibrium point available upon the original data set.

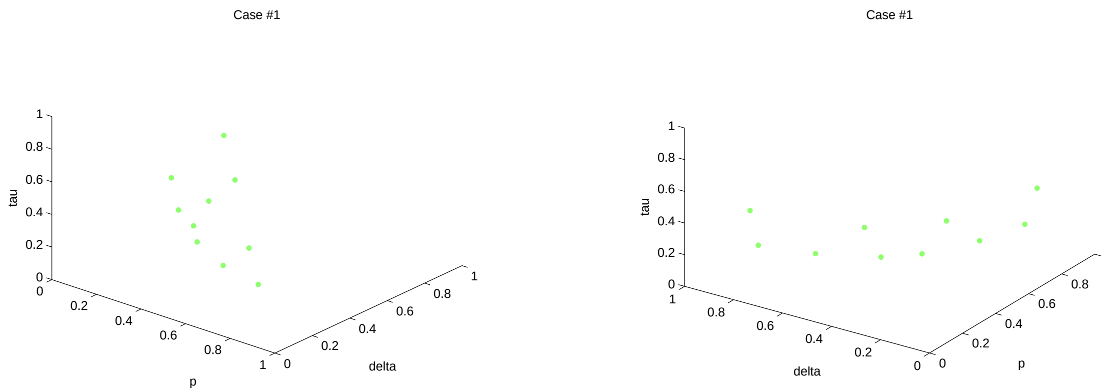


Figure 4.2: Conditions allowing SAEPs in Case 1 restricted to SAEPs without minimal pollution.

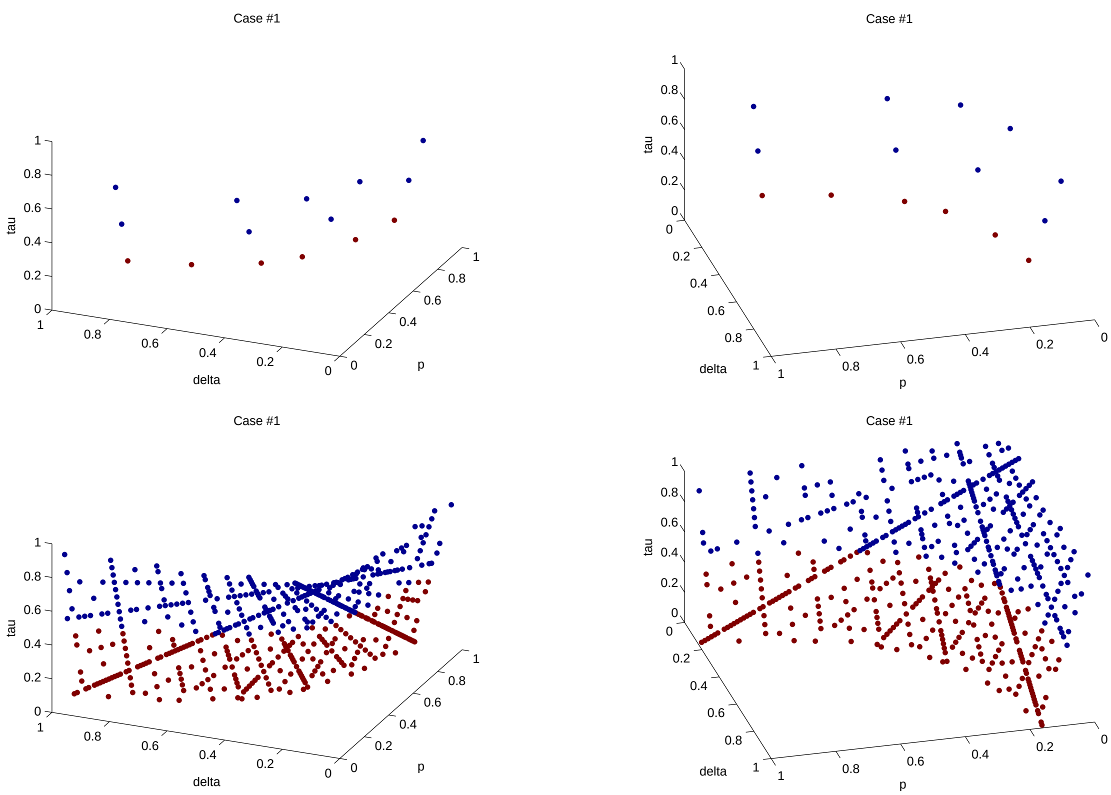


Figure 4.3: Conditions allowing SAEPs in Case 1 restricted to points with multiple equilibria. Top: step sizes of 1/20. Bottom: step sizes of 1/100.

The same step size produces a graph that looks similar to the one with non-minimal pollution. In fact, many of the same points can be seen. By reducing the step size, a more visible pattern appears. This graph takes the shape of a sort of curved plane. It appears that for $\tau > 0.5$, the points plotted are from the face of the original plot closer to the origin, while for $\tau < 0.5$, the points plotted are from the face of the original plot farther from the origin. There also are lines of points in this shape that reveal a sort of skeleton for this plot.

Because of the α_n being uniform (i.e. $\alpha_1 = \alpha_3 = \alpha_4$), there is an easy opportunity to demonstrate what happens if instead of fixing α_n and changing τ, we fix τ and change α_n. We choose to look at this because both α_n and τ are parameters that can be controlled by the regulator.

Just as with an iteration of τ, we can see that the same sort of symmetry exists across $p = \delta$. We also see the return of the gap between equilibria at $\tau = \alpha_n$. For $\alpha_n > 0.5$, type IV SAEPs are observed, and for $\alpha_n < 0.5$, type II are observed, which matches the behavior apparent in the original graph that showed the firm choosing to control pollution only when taxes are greater than the α_n penalty. Unique to this graph is the existence of mixed equilibria, something not yet seen in previous plots. These mixed equilibria all occupy the previously empty space with higher pollution chances and filter strengths. More of these mixed equilibria may have appeared in previous plots with a smaller step size.

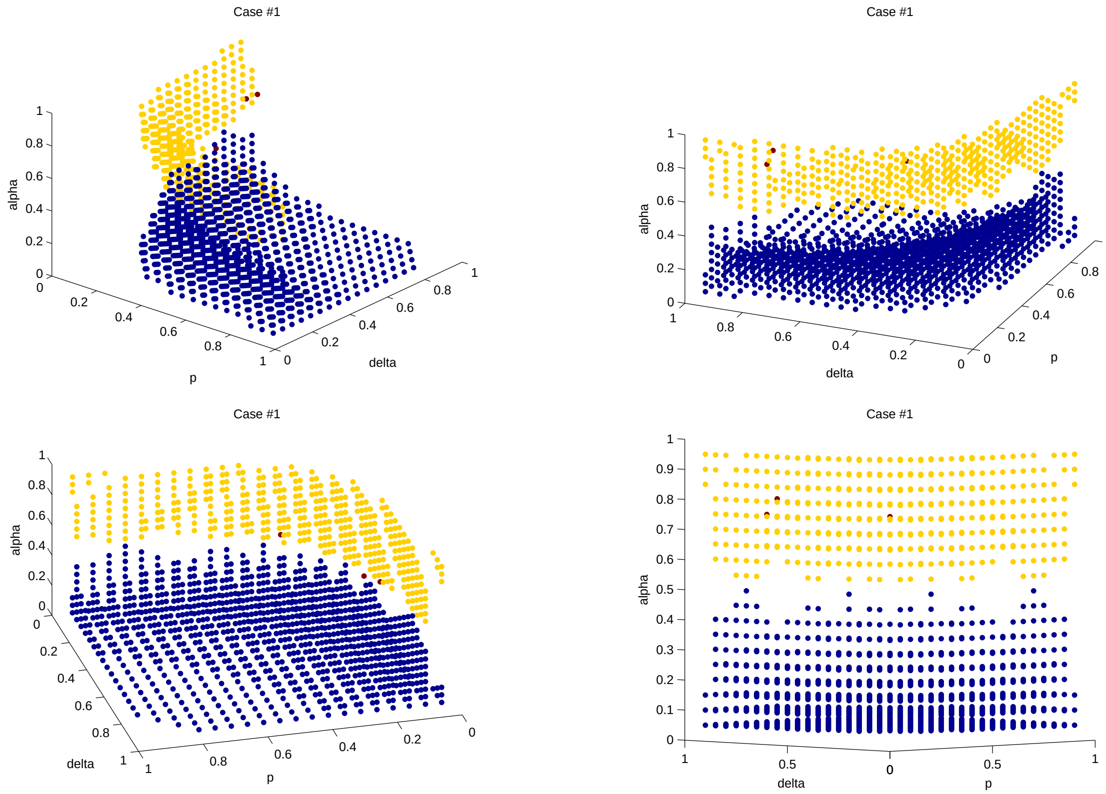


Figure 4.4: Conditions allowing SAEPs in Case 1 with fixed τ and varied α_n.

4.3 Case 2

The second case studied is when penalty against the firm is reduced when nobody opts to control pollution ($\alpha_1 = \alpha_3 > \alpha_4$).

This plot has two significant regions. The larger blue region, representing type II equilibria, has a parabolic curve shape when looked at top-down. As long as τ and p are close to 1, SAEPs occur for extreme values of δ. As p decreases, values of δ closer to 0.5 create conditions allowing SAEPs to exist. The region ends at lower values of τ. The behavior of the curve can be explained by looking at 3.6, whose second term on the left becomes small at extreme values of *delta*, allowing the

Case 2:

$P_f = 1000$

$\Delta = 250$

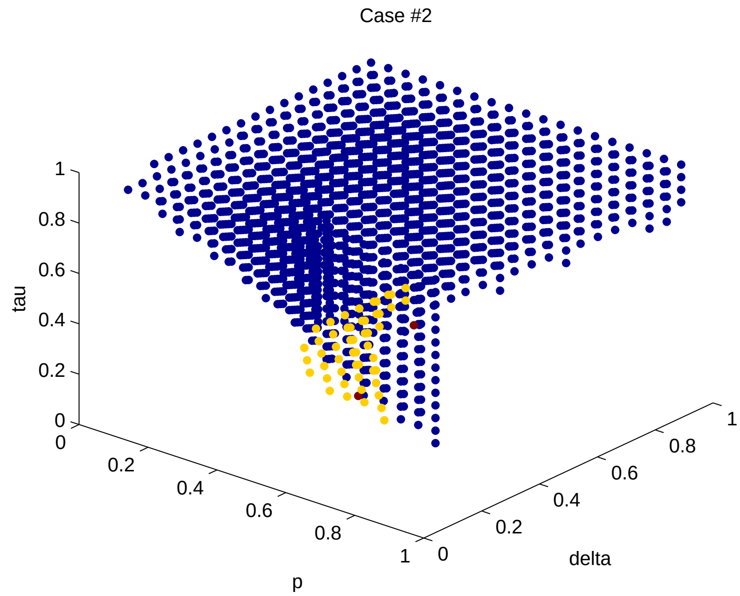
$C_F = 55$

$C_R = 7$

$\alpha_1 = \alpha_3 = 0.3$

$\alpha_4 = 0.2$

$\tau, \delta, p \in (0, 1)$



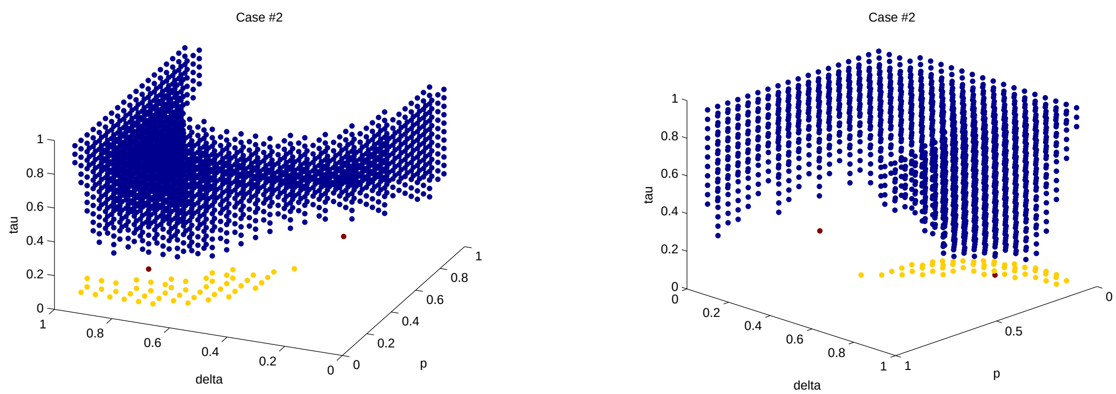


Figure 4.5: Conditions allowing SAEPs in Case 2.

inequality to be satisfied. Additionally, when τ becomes small, it becomes harder for 3.5 and 3.7 to be satisfied.

The island of yellow at the bottom represents SAEPs for which neither player controls pollution. This island resides in an area of low p and τ and high δ. We can see that 3.15 becomes hard to satisfy without low τ or high δ. Provided δ is large, 3.14 tends to require low p to be satisfied. Once again, we have two large regions in which only two types of equilibria are present, both being ones where the regulator opts to not control. Some mixed equilibria are also present and do not appear to have any sort of pattern to where they exist.

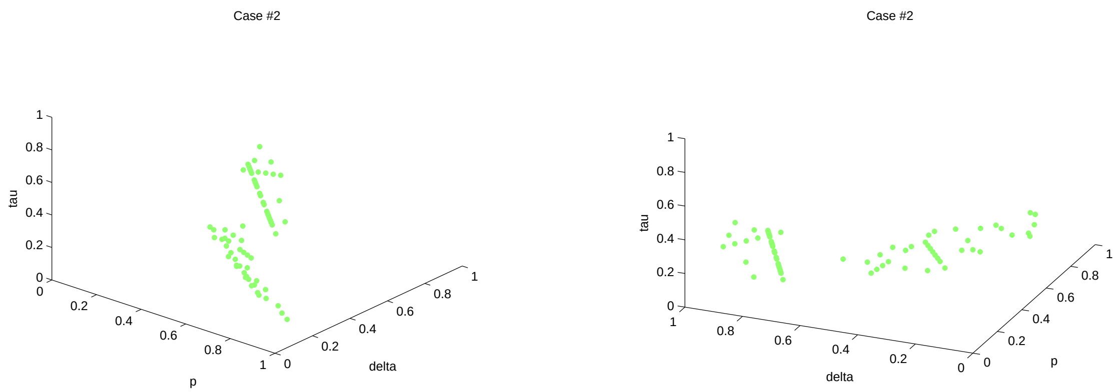


Figure 4.6: Conditions allowing SAEPs in Case 2 restricted to mixed equilibrium SAEPs at a step size of 1/250.

If we choose to limit the program again to SAEPs where pollution is minimized, it will produce mostly the same graph. There is a noticeable set of points missing from the upper region at higher values of δ.

This makes for an even more visible counterexample to the hypothesis that SAEPs would lead to points with lower pollution. If we now change the constraints of the program to again ignore comparative levels of pollution, but to instead look for points with multiple equilibria, we produce a somewhat similar graph.

It makes sense that the graphs of the previous figure are similar to these graphs as the previous figure contained only points which were strictly better than some other point, requiring there to be more than one. Thus, even by restricting to

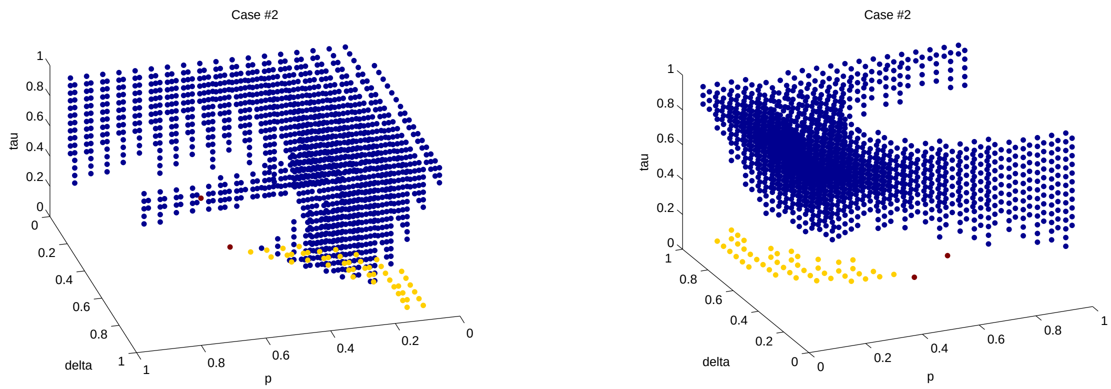


Figure 4.7: Conditions allowing SAEPs in Case 2 where pollution is minimized.

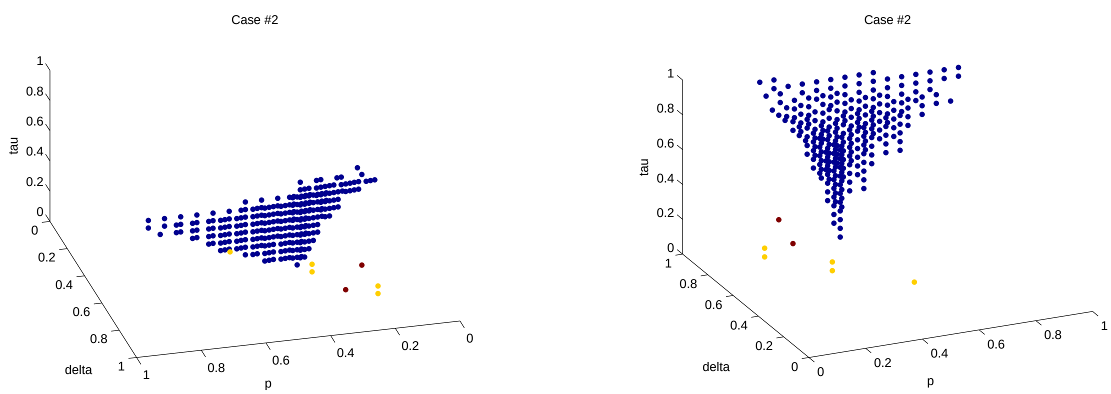


Figure 4.8: Conditions allowing SAEPs in Case 2 where pollution is strictly maximized.

cases with multiple equilibria, SAEPs are not guaranteed to minimize pollution. For all mixed equilibria and pure equilibria with no players controlling pollution shown in the graph, these points would not be included if we restricted our search to points with minimal pollution.

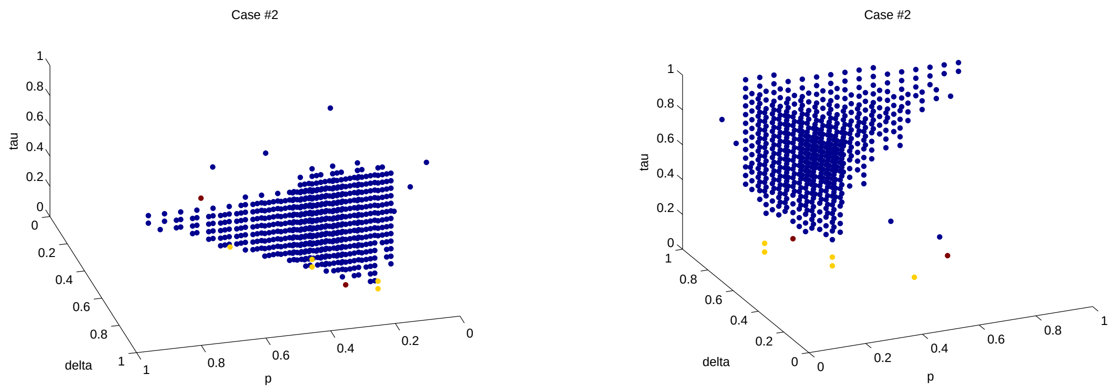


Figure 4.9: Conditions allowing SAEPs in Case 2 restricted to points with multiple equilibria.

4.4 Case 4

The final case studied is when the firm is penalized for not controlling pollution regardless of the regulator's choice to control ($\alpha_1 > \alpha_3 = \alpha_4$).

This plot has a similar theme to Case 2 in that there are two regions, the larger one for higher τ values representing the firm choosing to control pollution, the smaller island for lower values of τ representing the firm choosing to not control, and neither region containing points where the regulator chooses to control. So, across all cases studied, there have been no pure equilibria found where the regulator chooses to control pollution. There also appear to be no mixed equilibria, but by shrinking the step size, we may be able to find them. In fact, what we will see is that the mixed equilibria SAEPs are found along the region where neither player controls pollution, δ is very high, and $0.3 > p > 0.2$.

Case 2:

$P_f = 1000$

$\Delta = 250$

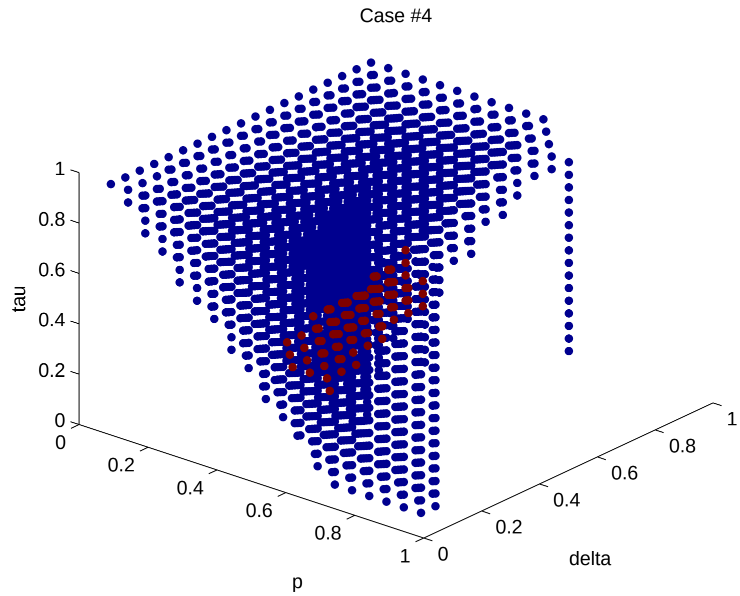
$C_F = 80$

$C_R = 30$

$\alpha_1 = 0.5$

$\alpha_3 = \alpha_4 = 0.3$

$\tau, \delta, p \in (0, 1)$



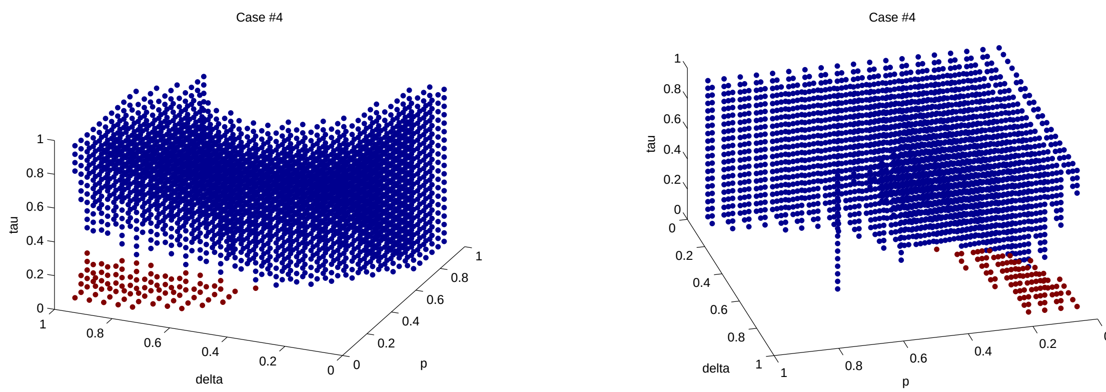


Figure 4.10: Conditions allowing SAEPs in Case 4.

In Case 2, we saw that a wedge from the larger type II region had been lost when we restricted the graph to points with minimized pollution. The same loss occurs with the same restriction in place on Case 4. The odd columns of values at certain values of δ and p are also lost. All of these points lost show up in the graph when pollution is instead sought to be maximized. In this way, 4.10 can be thought of as the union of 4.12 and 4.13.

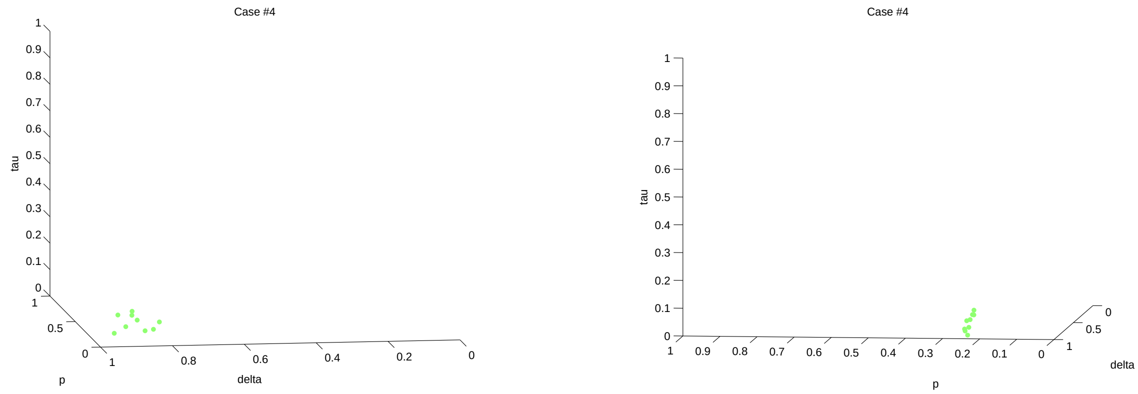


Figure 4.11: Conditions allowing SAEPs in Case 4 restricted to mixed equilibrium SAEPs at a step size of 1/250.

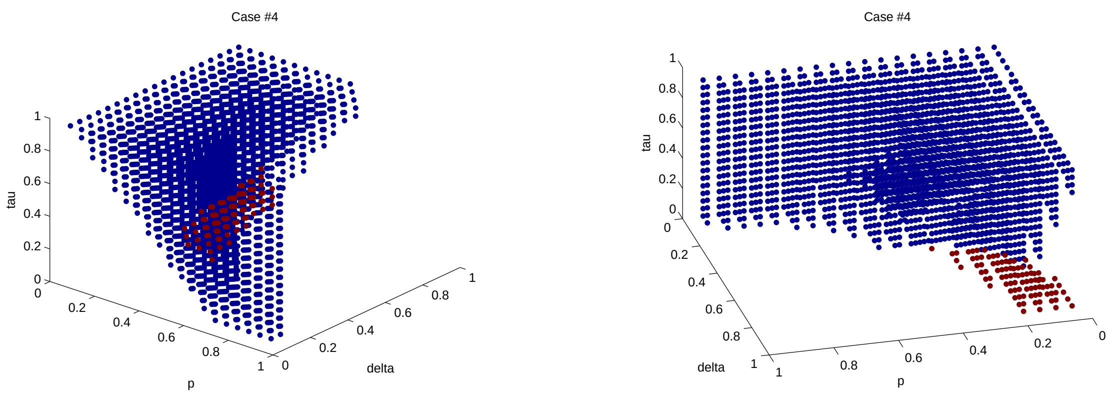


Figure 4.12: Conditions allowing SAEPs in Case 4 where pollution is minimized.

When the program is restricted to points where only multiple equilibria exist, it appears as if only type II equilibria persist, as in 4.14. Like Case 2, this plot is very similar to that of the maximized pollution plot. 4.15 and 4.16 show that by shrinking the step size, more points from the other pure equilibria and even some mixed equilibria can be found.

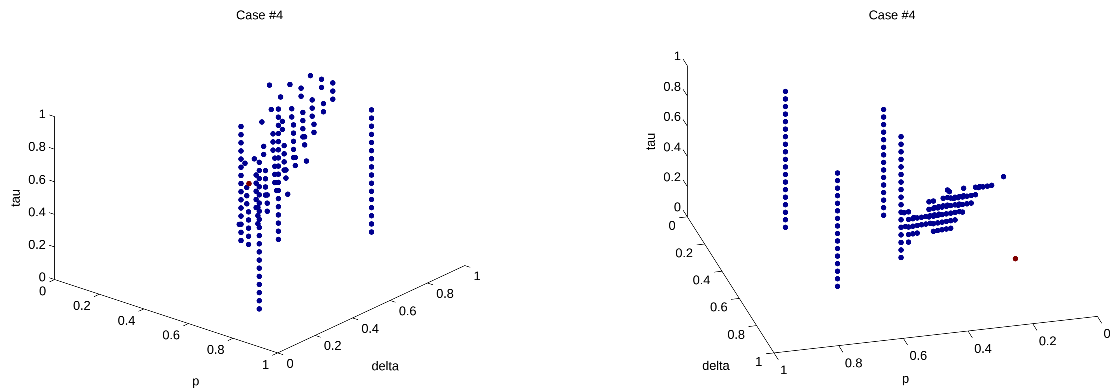


Figure 4.13: Conditions allowing SAEPs in Case 4 where pollution is strictly maximized.

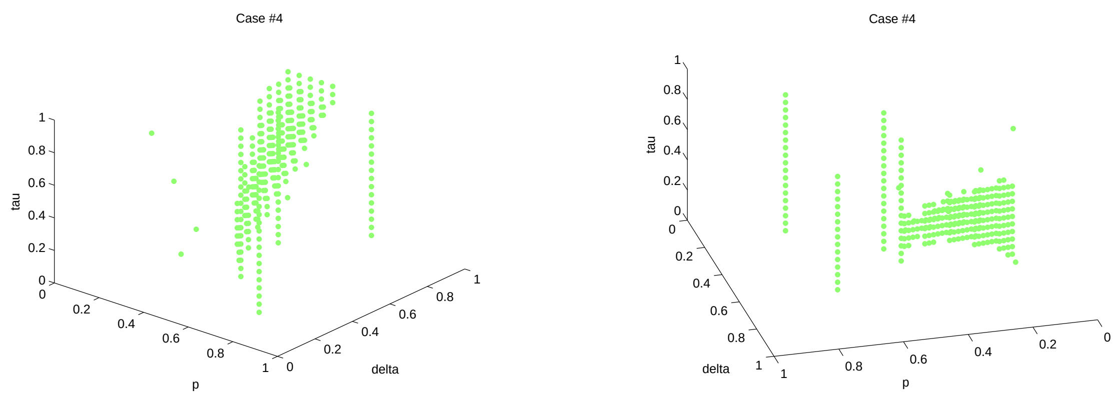


Figure 4.14: Conditions allowing SAEPs in Case 4 restricted to points with multiple equilibria. All visible points are type II.

4.5 Conclusions

As far as can be seen with the various trials of the program, we can confirm our claim from the previous chapter that no type I or type III SAEP will be encountered in these cases. This means that amongst conditions that allow for the existence of an SAEP, the only time a regulator in this model opts to control pollution is in some mixed equilibrium scenario. By and large, it tends to be that SAEPs occur in pure

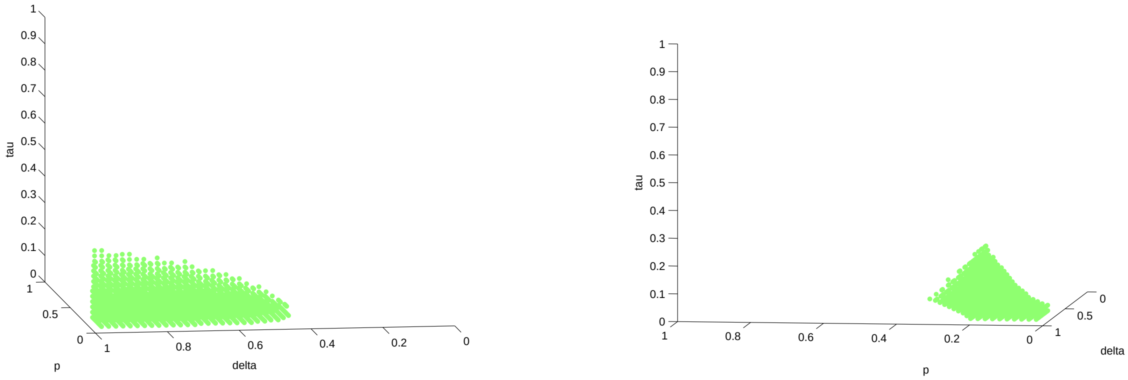


Figure 4.15: Conditions allowing SAEPs in Case 4 restricted to points with multiple equilibria with a type IV SAEP.

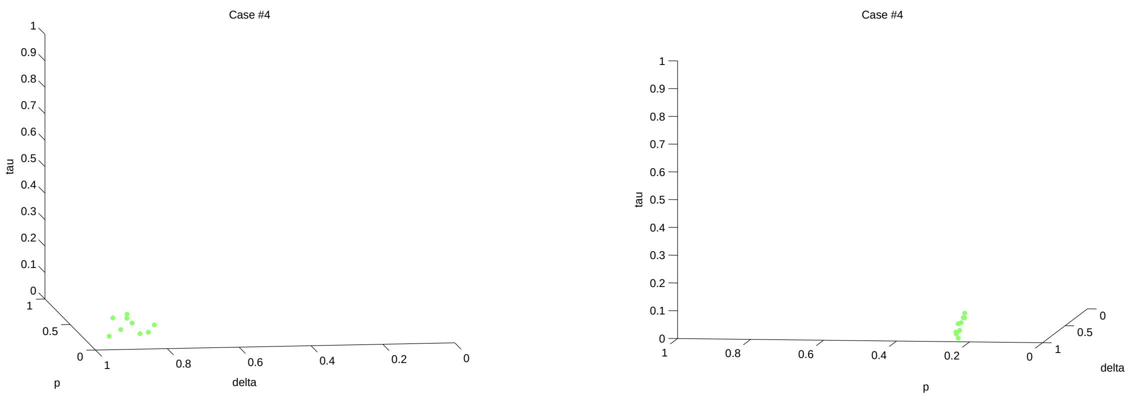


Figure 4.16: Conditions allowing SAEPs in Case 4 restricted to points with multiple equilibria where the SAEP is at a mixed equilibrium.

strategy cases where the regulator does not control and the firm chooses either to do so or to not. It was also shown in each case how there were points where an SAEP existed along with at least one other equilibrium point and the SAEP did not have the lowest pollution. Therefore, playing to an SAEP will not guarantee minimizing pollution in a game, counter to what we had hypothesized.

If not to minimize pollution, what good then would it be to play into an SAEP in this model? Traditionally, the goal of a player in a game is to maximize their own utility against unilateral changes in an opponent's strategy, but as it was mentioned previously, another goal could be to minimize the opponent's utility. By playing to the SAEP, a player accomplishes both of those goals. One way that this could be framed is that by playing into an SAEP, both players have the security of knowing that a decision on behalf of their opponent can only benefit them. Since the regulator in this model chooses before the game the value of τ and the α_n, and because δ and p chosen by nature could be determined prior to making this choice, the regulator could force the game into a point where SAEP exist, allowing them a strategy that protects themselves from firms who choose to deviate from this equilibrium. They could additionally make this point one in which the optimal strategy for the firm at the SAEP is to invest in pollution control, thereby at least encouraging some pollution abatement to take place, even though it isn't by them.

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