# PRODUCT OF YOUNG DIAGRAMS IN 2-FOLD MONOIDAL CATEGORY

A Thesis

Submitted to the Graduate School

of

Tennessee State University

in

Partial Fulfillment of the Requirements

for the Degree of

Master of Science

Graduate Research Series No._________

GOVINA MAKAKA EYUM

May 2007

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To the Graduate School

We are submitting a thesis by Govina M. Eyum entitled “PRODUCTS OF YOUNG DIAGRAMS IN A 2-FOLD MONOIDAL CATEGORY”. We recommend that it be accepted in partial fulfillment of requirements for the degree, Master of Science in Mathematical Sciences.

Chairperson

Committee Member

Committee Member

Committee Member

Accepted for the Graduate School

Dean of the Graduate School

## ACKNOWLEDGEMENTS

I would like to begin by thanking the ALMIGHTY LORD for giving me this opportunity, strength and knowledge to work on this Thesis. A special thanks to Dr. Stefan Forcey for devoting his time to make this research a reality. To my wife,Judith Eyum, a many "Thank You" for being there for me through all times. To my children Bryant and Priscilia, many thanks for making me a "Daddy".

# ABSTRACT

Govina M.Eyum "Products of Young Diagrams in 2-fold Monoidal Category" (under the direction of DR. STEFAN FORCEY).

An example of Iterated monoidal category is the Young diagrams (boxes) as objects in the category. Young diagrams are presented as a decreasing sequence of non-negative integers in two ways: The sequence that gives the height of the columns and that that gives length of the row. We let the tensor product $\otimes_1$ be the product that adds the length of rows of two objects (horizontal stacking) and $\otimes_2$ be the product that adds the height of columns of two objects (vertical stacking). These stacking are functorial with respect to morphisms(defined as the $\leqq$ relations). This thesis presents the higher product of Young's diagram in 2-fold monoidal category i.e functors are $\boxtimes_1$ and $\boxtimes_2$. The products in this case are **higher products of Young diagram** with emphasis on multiplying the objects, a technique that still makes use of the tensor products $\otimes_1$ and $\otimes_2$ normally used during stacking. An example will be, given $A \leqq B$ and $C \leqq D$, then $A \boxtimes_i C \leqq B \boxtimes_i D$ for $i = 1, 2$. Furthermore, this category will be shown to be a strict 2-fold monoidal category with both $\boxtimes_1$ and $\boxtimes_2$ to be commutative, associative and having an identity.

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# CHAPTER I: INTRODUCTION ( BACKGROUND )

This thesis is based on "Iterated Monoidal Category in 2-fold. The typical example here is the Young diagram in 2-fold monoidal category. Before explaining what Young digram is, it is imperative to define Category, functors, morphisms, natural transformation and monoidal category to better understand the various operations using Young diagram. **Category** C simply consists of a class of objects ob(C) and a class of morphisms hom(C). Letting F be the morphism and a and b to be objects, then $F : a \to b$. "a" is the unique source object while "b" is the target object. Now given three objects a,b,and c, $\ni$ a binary operation $hom(a, b) \times hom(b, c) \to hom(a, c)$ called composition of morphisms. The composition $f : a \to b$ and $g : b \to c$ can be written as $g \circ f$ or $gf$ such that the following axioms hold;

(i) Associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$, then $h \circ (g \circ f) = (h \circ g) \circ f$ and

(ii) Identity: For every object x, $\exists$ a morphism $1_x : x \to x$ called the identity morphism for x. Example: For every morphism $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$. Examples of category include group, posets, top, Ab category, sets etc.

A **morphism**$f : A \to B$ in a category is called an isomorphism if there is an inverse $f^{-1} : A \to B$ so $f^{-1} \circ f = 1_A$ and $f \circ f = 1_B$.

From the definition of category, we can now define a **functor** as a special type of mapping between categories or in other words as morphism of categories. Let C and D be categories, a functor F from C to D is a mapping that (a) associates to each object X in C an object F(X) in D, and (b) associates to each morphism $f : X \to Y$ in C a morphism $F(f) : F(X) \to F(Y)$ in D such that the following properties hold:

(i) $F(1_x) = 1_F(X)$ for every object $X \varepsilon C$.

(ii)$F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g :\to Z$. It is important to note that functors preserve the identity morphisms and composition of morphisms.

**Natural Transformation** is a way of transforming one functor into another while preserving the internal structure. It is the composition of morphisms or rather morphism of functors. Now let F and G be two functors between C and D categories, then a natural transformation $\eta$ from F to G associates to every object X in C a morphism $\eta_X : F(X) \to G(X)$ in D such that for every morphism $f : X \to Y$ in C we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$. This equation can be expressed by a commutative diagram.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

**Products of Category**: The product of two categories B and C $(B \times C)$is a constructed from B and C by letting $b \varepsilon B$ and $c \varepsilon C$. The pair $< b, c >$ of objects and arrow $< b, c > \to < b', c' >$ of $B \times C$ is a pair $< f, g >$ of arrows $f : b \to b'$ and $g : c \to c'$ and composite of such two arrows $< b, c > \xrightarrow{<f,g>} < b', c' > \xrightarrow{<f',g'>} < b", c" >$ is defined in terms of the composites in B and C by $< f', g' > \circ < f, g > = < f' \circ f, g' \circ g >$. The functors $B \xleftarrow{P} B \times C \xrightarrow{Q} C$ , called the projections of the product, are defined on (objects and) arrows by $P < f, g >= f$, and $Q < f, g >= g$. In this paper, we will assume that all monoidal category are strict.

Now a **A Strict Monoidal Category** $< B, \otimes, e >$ is a category B with a bi-functor $\otimes : B \times B \to B$ which is associative, $\otimes(\otimes \times 1) = \otimes(\otimes \times 1) : B \otimes B \times B$, and with an object e which is a left and right unit for $\otimes$,$\otimes(e \times 1) = id_B = \otimes(1 \times e)$.

**Twice Monoidal (2-fold) category** is a monoidal category with $\boxtimes_1$ and a second operation $\boxtimes_2$ such that we have the following

(a) Natural transformation $\eta$

(b) Unit conditions (id)

(c) Associative conditions and

(d) and their diagrams commute.

The natural transformation:

$$\eta_{ABCD} : (A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$$

satisfies the external associativity and external unit conditions. The internal unit condition is given by $\eta_{ABII} = \eta_{IIAB} = id_{A \otimes_2 B}$, and the external unit conditions given by $\eta_{AIBI} = \eta_{IAIB} = id_{A \otimes_1 B}$. The internal associativity conditions gives the commutative diagram:

$$\begin{tikzcd}
((U \otimes_2 V) \otimes_1 (W \otimes_2 X)) \otimes_1 (Y \otimes_2 Z) \arrow[r, "\eta_{UVWX} \otimes_1 1_{Y \otimes_2 Z}"] \arrow[d, "1_{U \otimes_2 V} \otimes_1 \eta_{WXYZ}"] & ((U \otimes_1 W) \otimes_2 (V \otimes_1 X)) \otimes_1 (Y \otimes_2 Z) \arrow[d, "\eta_{(U \otimes_1 W)(V \otimes_1 X)YZ}"] \\
(U \otimes_2 V) \otimes_1 ((W \otimes_2 Y)) \otimes_2 (X \otimes_2 Z) \arrow[r, "\eta_{UV(W \otimes_1 Y)(X \otimes_1 Z)}"] & (U \otimes_1 (W \otimes_1 Y) \otimes_2 (V \otimes_1 X \otimes_1 Z))
\end{tikzcd}$$

The external associativity condition gives the commutative diagram:

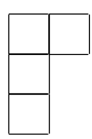
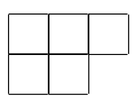
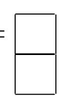
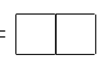
$$\begin{tikzcd}
((U \otimes_2 V) \otimes_2 W) \otimes_1 ((X \otimes_2 Y) \otimes_2 Z) \arrow[r, "\eta_{(U \otimes_2 V)W(X \otimes_2 Y)Z}"] \arrow[d, "\eta_{UV(\otimes_2 W)X(Y \otimes_2 Z)}"] & ((U \otimes_2 V) \otimes_1 (X \otimes_1 Y)) \otimes_2 (W \otimes_2 Z) \arrow[d, "\eta_{UVXY} \otimes_2 1_{W \otimes_1 Z}"] \\
(U \otimes_1 X) \otimes_2 ((V \otimes_2 W)) \otimes_1 (Y \otimes_2 Z) \arrow[r, "1_{U \otimes_1 X} \otimes_1 \eta_{VWYZ}"] & (U \otimes_1 X) \otimes_2 ((V \otimes_1 Y) \otimes_2 (W \otimes_1 Z))
\end{tikzcd}$$

CHAPTER II: CASE STUDY OF THE YOUNG DIAGRAMS

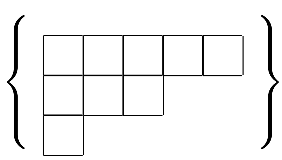
Young diagrams are the objects of a 2-fold monoidal category. The Young diagram can be presented by a decreasing sequence of nonnegative integers by the following two ways;

a) The sequence that gives the heights of the columns often denoted by $\otimes_2$ or called the **vertical stacking**.

b) The sequence that gives the lengths of the rows often denoted by $\otimes_1$ or called the **horizontal stacking.** These two operations $\otimes_1$ and $\otimes_2$ are functorial up to morphism. The morphisms are lexicographic ordering. i.e if $A < B$ means that the first time there is a difference between column heights, hence B is taller. An example here will be to consider the following diagrams.

$A =$  $B =$  $C =$  $D =$ 

(A) $A \otimes_1 B \rightarrow A \otimes_2 B$. Then

$A \otimes_1 B =$ 

and

$$A \otimes_2 B = \left\{ \ydiagram{3,2,2,1,1} \right\}$$

This gives

$$\left\{ \ydiagram{5,3,1} \right\} \leq \left\{ \ydiagram{3,2,2,1,1} \right\}$$

(B) Given $(A \otimes_2 B) \otimes_1 (C \otimes_2 D) \to (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$ Then following operations

$$(A \otimes_2 B) \otimes_1 (C \otimes_2 D) = \left\{ \ydiagram{5,5,3,3,1,1,1,1} \right\}$$

and

$$(A \otimes_1 C) \otimes_2 (B \otimes_1 D) = \left\{ \ydiagram{5,4,4,3,1,1,1,1} \right\}$$

From the above diagrams, since the height of the fourth column of $(A \otimes_1 C) \otimes_2 (B \otimes_1 D)$

is more than that of $(A \otimes_2 B) \otimes_1 (C \otimes_2 D)$ hence,

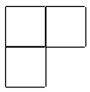
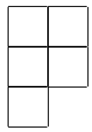
$$\left\{ \ydiagram{5,5,3,3,1,1,1,1} \right\} \leq \left\{ \ydiagram{5,4,4,3,1,1,1,1} \right\}$$

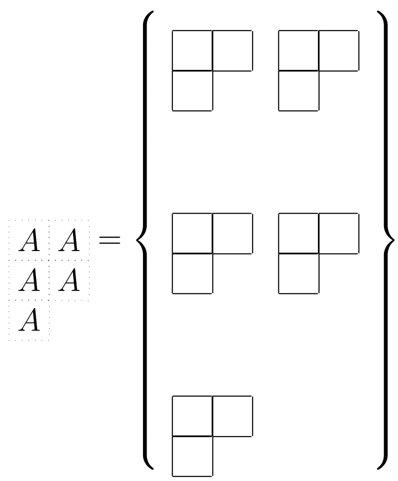
# CHAPTER III

## INTRODUCTION TO HIGHER PRODUCTS OF YOUNG DIAGRAM

In chapter one, we saw how the various operations of the Young diagrams are carried out. If we were to consider $A \boxtimes_i B, i = 1, 2, 3, ..,$ then the operation is different as oppose to $A \otimes_i B, i = 1, 2, 3, ...$ whereby $\otimes_1$ adds the length of the rows and $\otimes_2$ adds the height of the columns. So, in this chapter, we will consider our new category Young diagrams, $\boxtimes_1$ and $\boxtimes_2$ with Young diagrams being the objects and the morphism as lexicographic ordering. The new products are $\boxtimes_1$ , $\boxtimes_2$ with identity $I = \square$ (single box). Now lets define our new products $\boxtimes_1$ and $\boxtimes_2$ to be horizontal and vertical multiplication of the Young diagram respectively. Since Young diagram operations are done in pairs, it is very possible to multiply two diagrams to get the result we would expect. How are these operations done? Horizontal and vertical multiplications are done by; **(i)** packing each box of say diagram A with a copy of say diagram B or vice versa. This results in a "replicate" of whatever is copied. (ii)These replicates are either horizontally stacked followed by vertical stacking or vice versa depending on the method of multiplication. An example will be to consider two diagrams, A and B. Take diagram $A = 3$ boxes and diagram $B = 5$ boxes. Copying diagram A onto

B, each box of B will contain 3 boxes (copies of A) and adding them up will result in a total of 15 boxes(i.e $3 \times 5$). i.e

$A =$  and $B =$  then the process of packing will be as follows:



Since we are dealing with a monoidal category, we want to show that the various axioms mentioned in the background hold to form a total ordered structure of the Young diagram. We expect the morphisms between these objects to be well-ordered given by lexicographic ordering. If this is the case, then we will be able to show that $\boxtimes_1$, $\boxtimes_2$, and the identity (single box) are functorial. To define horizontal and vertical multiplication, we will consider **(ii)** only since both begin with **(i)**. (A) **Horizontal Multiplication** $(\boxtimes_1)$ : Defined by vertical stacking $\otimes_2$ followed by horizontal stacking

$\otimes_1$. To show these operations, consider

$$A = \ydiagram{2,1} \quad B = \ydiagram{2,2,1}$$

Then

$$A \boxtimes_1 B = \begin{ytableau} A & A \\ A & A \\ A \end{ytableau} = \left\{ \begin{array}{c} \ydiagram{2,1} \\ \otimes_2 \\ \ydiagram{2,1} \\ \otimes_2 \\ \ydiagram{2,1} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{2,1} \\ \otimes_2 \\ \ydiagram{2,1} \end{array} \right\}$$

(B) **Vertical Multiplication** ($\boxtimes_2$): Defined by horizontal stacking $\otimes_1$ followed by vertical stacking $\otimes_2$. Using diagrams A and B as above, then

$$A \boxtimes_2 B = \begin{ytableau} A & A \\ A & A \\ A \end{ytableau} = \begin{array}{c} \left\{ \ydiagram{2,1} \otimes_1 \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \otimes_1 \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \end{array}$$

CHAPTER IV

# 2-FOLD MONOIDAL IN HIGHER PRODUCTS OF YOUNG DIAGRAMS

For any 2-fold monoidal category with objects a, b, c, and d, we want to show that there is a morphism between two composite objects with the same operands. To do this, lets look at some theorems adopted from the paper "Operads in Iterated Monoidal Category", Forcey et al.

a) **Theorem 4.13** says that inequality of stacking holds for any four (4) Young diagrams.

b) **Theorem 3.6** (Coherence theorem for n-fold Monoidal Categories) it states: Let X and Y be objects of $M_n(K)$. Then

(a) There is at most one morphism $X \to Y$

(b) A necessary and sufficient condition for the existence of a morphism $X \to Y$ is that for any two elements x, y in $A, B, C...$, if $x\Box_i y$ in X, then in Y either $x\Box_j y$ for some $j \geq i$ or $y\Box_j x$ for some $j > i$. To apply this theorem, lets consider this operation; $X = (B\Box_2 C)\Box_1 A \to B\Box_2 A\Box_2 C = Y$ in $M_2(3)$(2-fold monoidal category with three objects). Is there a morphism? The answer is obviously YES, since

$B\Box_1 A$ in X and $B\Box_2 A$ in Y,

$C\Box_1 A$ in X and $A\Box_2 C$ in Y,

$B\Box_2 C$ in X and $B\Box_2 C$ in Y

but there is no morphism from X to $Z = A\Box_2 C\Box_2 B$ since $B\Box_2 C$ in A, while $C\Box_2 B$ in Z.

Note that the first theorem asserts that any diagram built out of the natural transformations $\eta^{ij}$ must commute. The necessity of the conditions in the second part of the Coherence Theorem is forced by existence of the restriction functors $R_{a,b} : M_n(K) \to M_n(a,b)$, i.e if there is a morphism $A \to B$ in $M_n(K)$, then there must be a morphism $R_{a,b}(A) \to R_{a,b}(B)$ in $M_n(a,b)$.

Given a 2-fold monoidal category $A \boxtimes_i B$, $i = 1, 2$, we can check for commutativity and then we will prove that

$A \boxtimes_1 B = B \boxtimes_1 A$ and $A \boxtimes_2 B = B \boxtimes_2 A$. An example will be to consider the following diagrams;

$$A = \ydiagram{2,1} \quad B = \ydiagram{2,1,1}$$

Then

$$A \boxtimes_1 B = \begin{array}{cc} A & A \\ A & \\ A & \end{array} = \left\{ \begin{array}{c} \ydiagram{2,1} \\ \otimes_2 \\ \ydiagram{2,1} \\ \otimes_2 \\ \ydiagram{2,1} \end{array} \right\} \otimes_1 \left\{ \ydiagram{2,1} \right\} = \left\{ \ydiagram{2,2,2,1,1,1} \right\} \otimes_1 \left\{ \ydiagram{2,1} \right\}$$

$$\Longrightarrow A \boxtimes_1 B = \left\{ \ydiagram{4,3,2,1,1,1} \right\}$$

The order here is $\{6, 3, 2, 1\}$

Now

$$B \boxtimes_1 A = \begin{array}{cc} B & B \\ B & \end{array} = \left\{ \begin{array}{c} \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \end{array} \right\} \otimes_1 \left\{ \ydiagram{2,1,1} \right\} = \left\{ \ydiagram{2,2,1,1,1,1} \right\} \otimes_1 \left\{ \ydiagram{2,1,1} \right\}$$

$$\implies B \boxtimes_1 A = \left\{ \ydiagram{4,3,2,1,1,1} \right\}$$

The order here is $\{6, 3, 2, 1\}$. Hence, $A \boxtimes_1 B = B \boxtimes_1 A$.

Lets also check that $A \boxtimes_2 B = B \boxtimes_2 A$. Lets consider

$$A \boxtimes_2 B = \begin{array}{ll} A & A \\ A & \\ A & \end{array} = \begin{array}{c} \left\{ \ydiagram{2,1} \otimes_1 \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \end{array} = \begin{array}{c} \left\{ \ydiagram{4,2} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \end{array} = \left\{ \ydiagram{4,2,2,2,1,1} \right\}$$

The order here is $\{6, 4, 1, 1\}$. And

$$B \boxtimes_2 A = \begin{array}{ll} B & B \\ B & \end{array} = \begin{array}{c} \left\{ \ydiagram{2,1,1} \otimes_1 \ydiagram{2,1,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1,1} \right\} \end{array} = \begin{array}{c} \left\{ \ydiagram{4,2,2} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1,1} \right\} \end{array} = \left\{ \ydiagram{4,2,2,2,1,1} \right\}$$

The sequence is $\{6, 4, 1, 1\}$ This shows that $A \boxtimes_2 B = B \boxtimes_2 A$

The proof is obvious in both cases i.e $\boxtimes_1$ and $\boxtimes_2$ since we had defined the two categories given two objects A and B to be copies of A onto B and vice versa, this means that if we had copied A onto B in $B \boxtimes_1 A$ and in $B \boxtimes_2 A$ then this will be exactly the same as we had done with $A \boxtimes_1 B$ and $A \boxtimes_2 B$. The lexicographic order will be the same in either cases. Our main concern now will be to show why $A \boxtimes_1 B \leq A \boxtimes_2 B$.i.e $\{6, 3, 2, 1\} < \{6, 4, 1, 1\}$. From the sequence we can see that the height of the second column of $A \boxtimes_2 B$ is one more than that of $A \boxtimes_1 B$. Why is this?

Proof: Using theorem 3.6

Lets consider $X = A \boxtimes_1 B$ and $Y = A \boxtimes_2 B$. Now X is stacked in the following order

$X = (A \otimes_2 A \otimes_2 A) \otimes_1 (A)$

Y is stacked in the order $Y = (A \otimes_1 A) \otimes_2 (A) \otimes_2 (A)$. If we were to number the $A_s$ accordingly, then $X = (A^1 \otimes_2 A^2 \otimes_2 A^3) \otimes_1 (A^4)$ and $Y = (A^1 \otimes_1 A^4) \otimes_2 (A^2) \otimes_2 (A^3)$. Also considering the following pattern;

a) If two boxes are in the same row $\otimes_1$ or column $\otimes_2$ of B, then keep the same order in X and Y and $i = j$ where i and j are the subscripts of their product in X and Y respectively.

b) If box x is above and left of box y and in the same order, then $i < j$

c) Else if box x is below and left of box y and order is switched, then $i < j$. These two patterns will help us conclude this proof. Lets investigate this. $A^1 \otimes_1 A^4$ in X and $A^1 \otimes_1 A^4$ in Y then $i = j$

$A^1 \otimes_1 A^2$ in X and $A^1 \otimes_1 A^2$ in Y then $i = j$

$A^1 \otimes_1 A^3$ in X and $A^1 \otimes_1 A^3$ in Y then $i = j$

$A^3 \otimes_1 A^4$ in X and $A^4 \otimes_2 A^3$ in Y then $i < j$

Since the order of $A^3$ and $A^4$ are switched and the functor between them is $\otimes_2$, this implies the column of Y compared to X will be greater according to theorem 3.6. This ends the proof. We can use this same process to check for any number of objects.

Our products $\boxtimes_1$ and $\boxtimes_2$ respect ordering. How can we show that this true? Let $A \leq B$ and $C \leq D$, then $A \boxtimes_i C < B \boxtimes_i D$ for $i = 1, 2$ by well-ordering . Now consider an example

$$A = \ydiagram{1,1} \quad B = \ydiagram{2,1,1} \quad C = \ydiagram{2,1} \quad D = \ydiagram{2,2,1}$$

then

$$A \boxtimes_1 C = \begin{array}{cc} A & A \\ A & \end{array} = \left\{ \begin{array}{c} \ydiagram{1,1} \\ \otimes_2 \\ \ydiagram{1,1} \end{array} \right\} \otimes_1 \left\{ \ydiagram{1,1} \right\} = \left\{ \ydiagram{2,2,1,1} \right\}$$

The order here $\{4, 2\}$

and

$$B \boxtimes_1 D = \begin{array}{cc} B & B \\ B & B \\ B & \end{array} = \left\{ \begin{array}{c} \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \end{array} \right\} = \left\{ \ydiagram{2,2,2,1,1,1,1,1,1} \right\} \otimes_1 \left\{ \ydiagram{2,2,1,1,1,1} \right\}$$

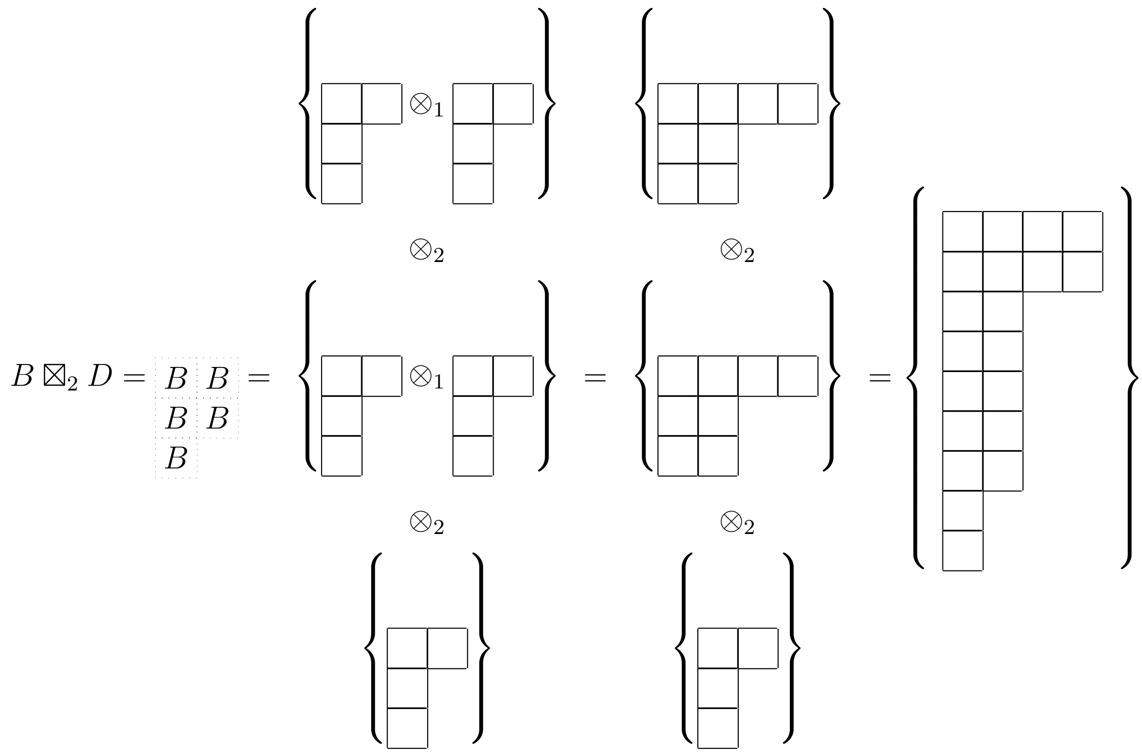
Hence,

$$B \boxtimes_1 D = \left\{ \ydiagram{4,4,3,2,2,2,1,1,1} \right\}$$

The order here is $\{9, 6, 3, 2\}$. Therefore, $A \boxtimes_1 C \leq B \boxtimes_1 D$. Now, we want to show that $A \boxtimes_2 C \leq B \boxtimes_2 D$.

$$A \boxtimes_2 C = \ytableaushort{AA,A} = \begin{array}{c} \left\{ \ydiagram{1,1} \otimes_1 \ydiagram{1,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{1,1} \right\} \end{array} = \left\{ \ydiagram{2,2,1,1} \right\}$$

The order is the same as in $A \boxtimes_1 C$ i.e $\{4, 2\}$.




The order is $\{9, 7, 2, 2\}$. Therefore, $A \boxtimes_2 C \leq B \boxtimes_2 D$. To proof this, we examine the example above by considering the fact that, if we let A to be approximately close to (i.e the proceeding height of A is a few box(s) less than) B but not equal to B and C to be approximately close to D but not equal to D, then we can see that in both cases we have vertical stacking followed by horizontal stacking. The height of $B \boxtimes_1 D$ will be taller than that of $A \boxtimes_1 C$. Hence, we can conclude that $A \boxtimes_i C < B \boxtimes_i D$. If $A = B$ and $C = D$, then we expect the morphism between these objects to be equality. i.e

$$A \boxtimes_1 C = \left\{ \ydiagram{4,4,3,2,2,2,1,1,1} \right\} = B \boxtimes_1 D$$

which definitely gives an equality. By theorem 3.6, the two boxes $A = B$ and $C = D$ are in the same row $\otimes_1$ or column $\otimes_2$ , hence have the same order and $i = j$.

This chapter is concerned with the composition of morphisms. By theorem 4.13, given $\eta_{ABCD}$, with $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \to (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$, therefore the morphism we will expect will be an inequality. We are also going to check for internal and external associativity as mentioned in our background study. Now lets consider an example

$$A = \ydiagram{2,1,1} \quad B = \ydiagram{3,2} \quad C = \ydiagram{2,1} \quad D = \ydiagram{2}$$

then $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \leq (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$

by Theorem 3.6. Using a diagram as an example, we have;

$$A \boxtimes_2 B = \begin{ytableau} A & A & A \\ A & A \end{ytableau} = \begin{matrix} \left\{ \ydiagram{2,1,1} \otimes_1 \ydiagram{2,1,1} \otimes_1 \ydiagram{2,1,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1,1} \otimes_1 \ydiagram{2,1,1} \right\} \end{matrix} = \left\{ \begin{matrix} \ydiagram{6,3,3} \\ \otimes_2 \\ \ydiagram{4,2,2} \end{matrix} \right\}$$

Now this finally gives

$$A \boxtimes_2 B = \ydiagram{6,4,3,3,2,2}$$

Now

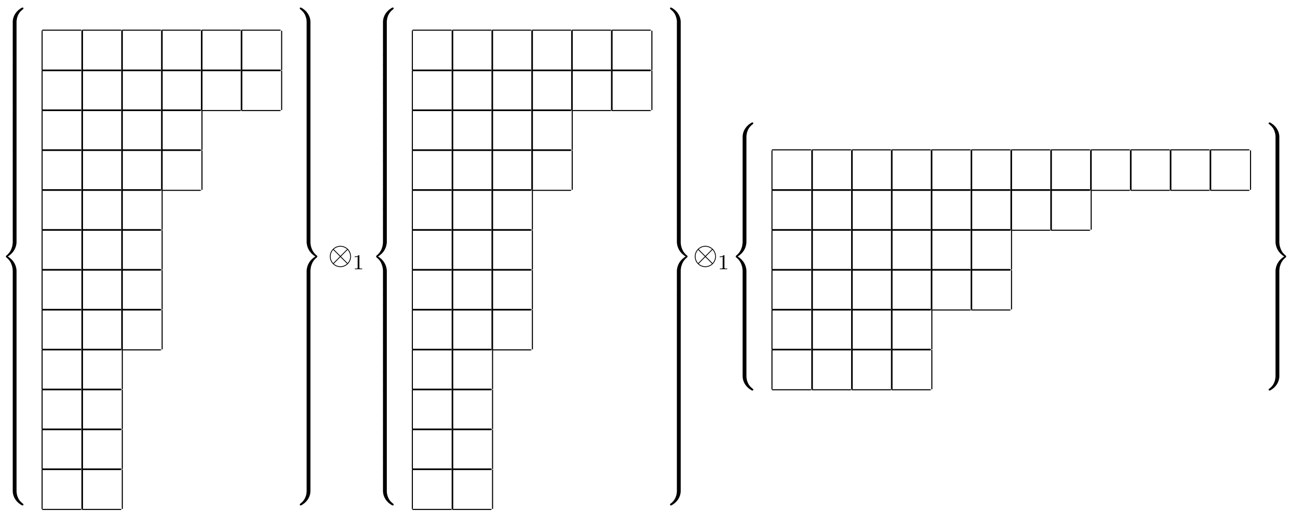
$$C \boxtimes_2 D = \begin{array}{|c|c|}\hline C & C \\ \hline\end{array} = \left\{ \ydiagram{2,1} \quad \otimes_1 \quad \ydiagram{2,1} \right\} = \ydiagram{4,2}$$

Hence

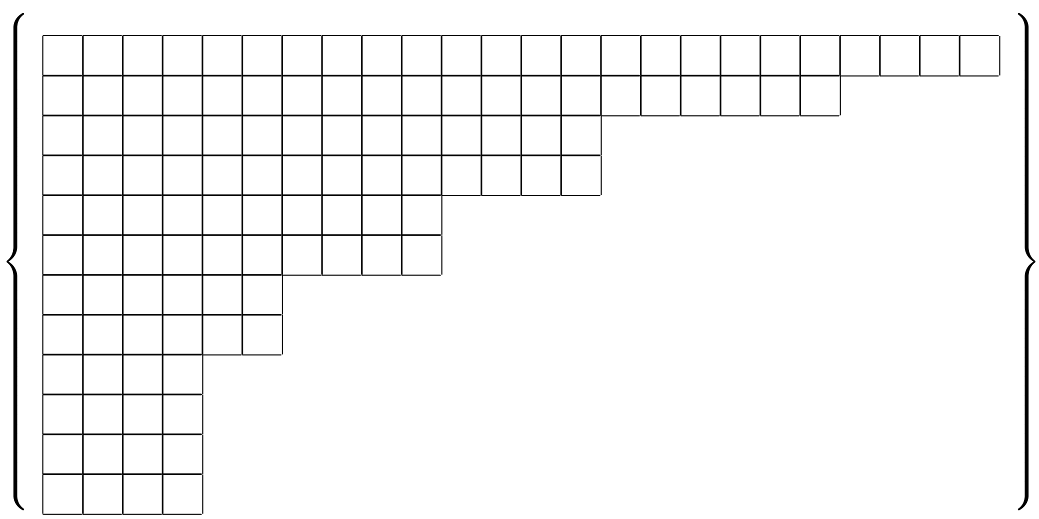
$$(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) = \begin{array}{|c|c|c|c|}\hline A \boxtimes_2 B & A \boxtimes_2 B & A \boxtimes_2 B & A \boxtimes_2 B \\ \hline A \boxtimes_2 B & A \boxtimes_2 B & & \\ \cline{1-2}\end{array} =$$

$$\left\{ \begin{array}{c} \ydiagram{6,4,3,3,2,2} \\ \otimes_2 \\ \ydiagram{6,4,3,3,2,2} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{6,4,3,3,2,2} \\ \otimes_2 \\ \ydiagram{6,4,3,3,2,2} \end{array} \right\} \otimes_1 \left\{ \ydiagram{6,4,3,3,2,2} \right\} \otimes_1 \left\{ \ydiagram{6,4,3,3,2,2} \right\}$$

Stacking vertically and horizontally equals



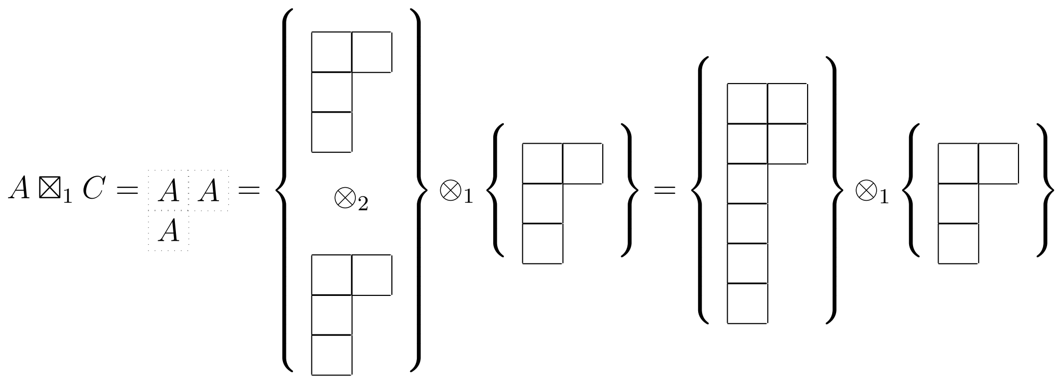

$\Rightarrow$



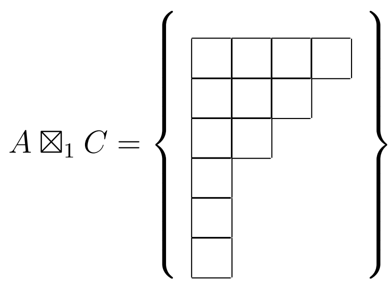
The "diagram stacking" if restricted to the sequence decreasing across the columns, then the sequence of the above diagram will be read as;

$\{12, 12, 12, 12, 8, 8, 6, 6, 6, 6, 4, 4, 4, 4, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1\}$

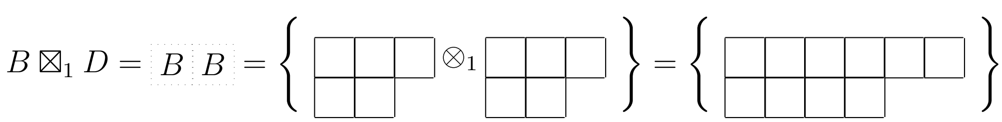
On the other hand, $(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D) \Rightarrow$




Hence

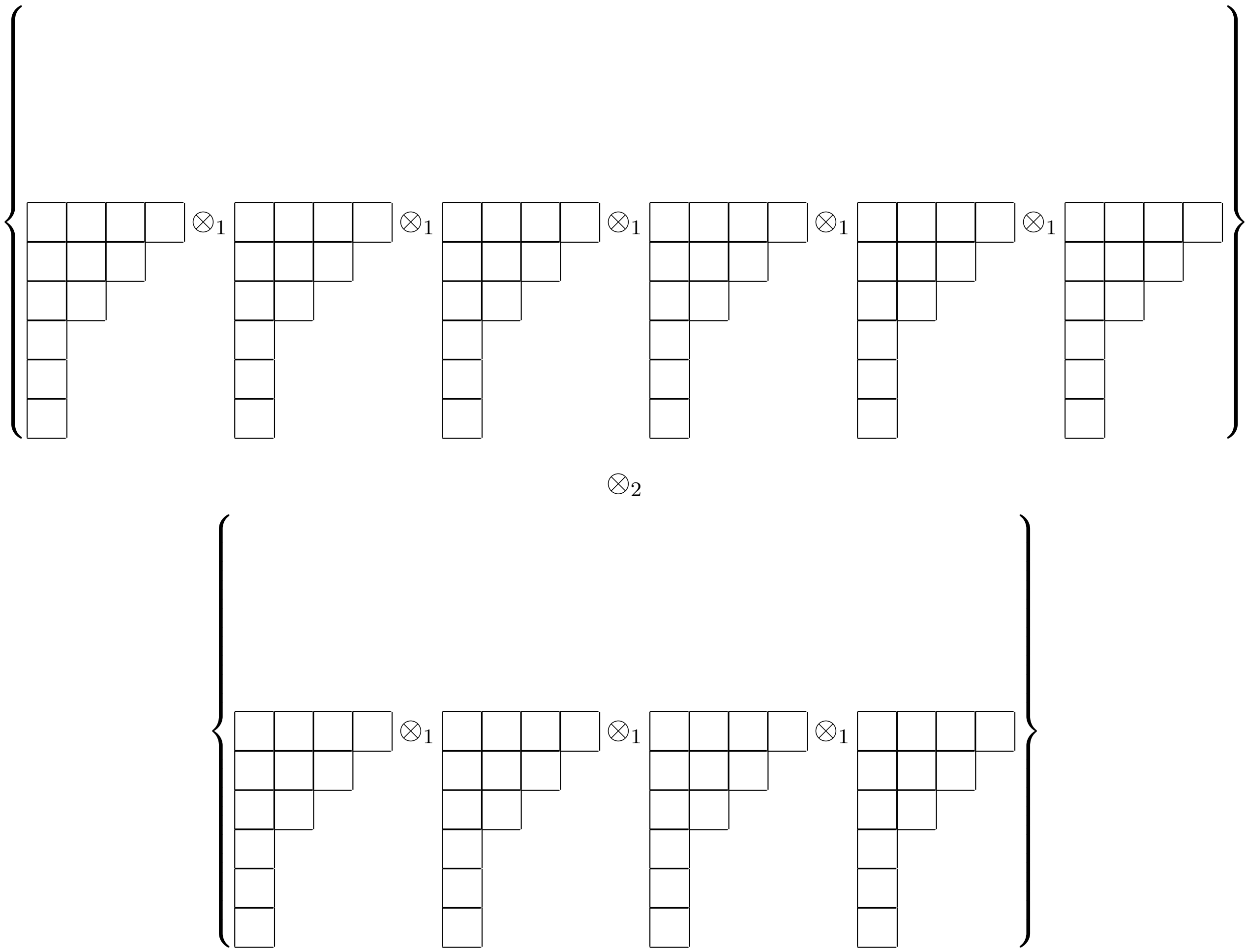



and

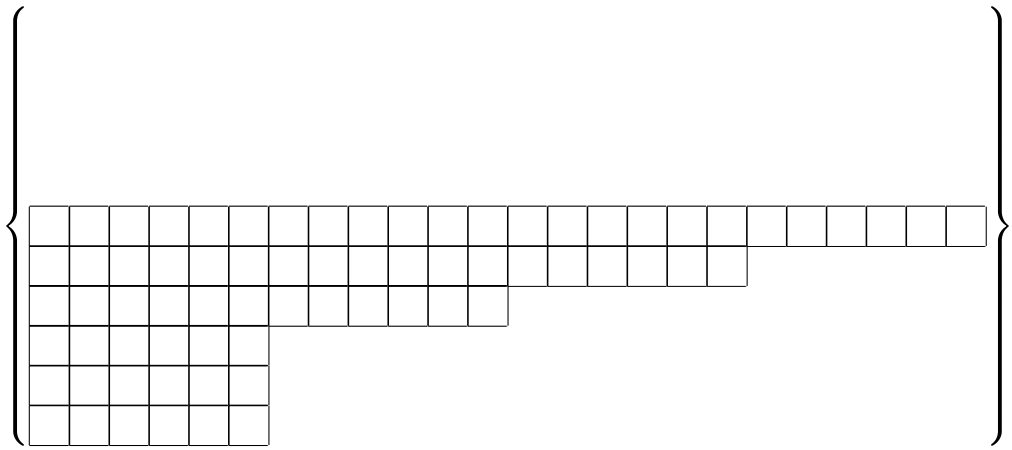



Hence

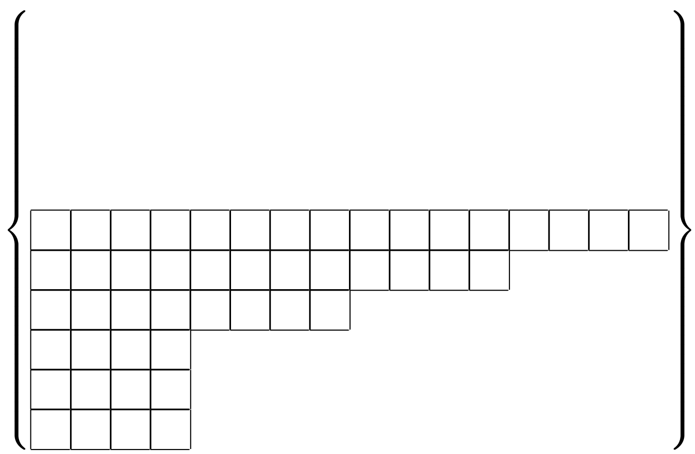
$$(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D) = \begin{array}{cccccc} A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C \\ A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C & A \boxtimes_1 C & & \end{array} =$$



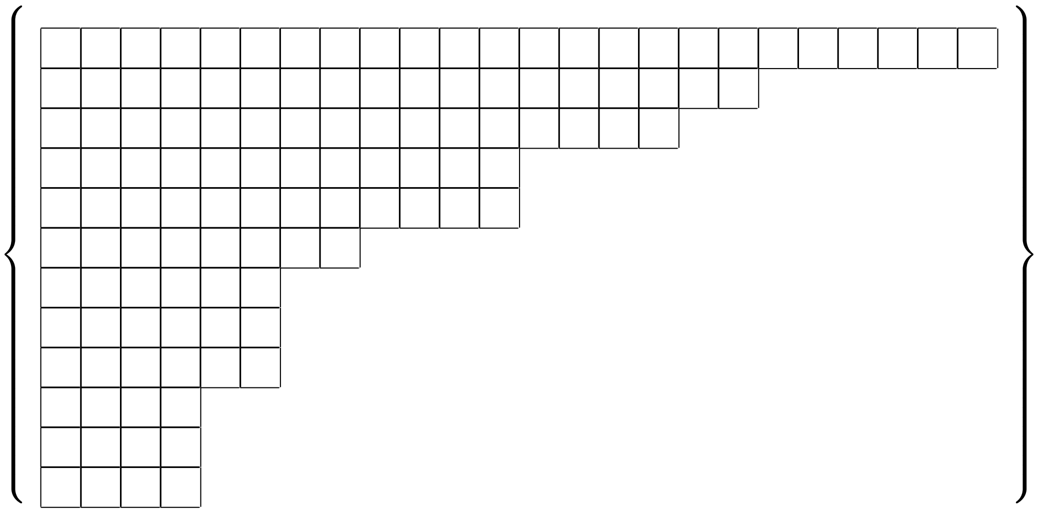
$\Rrightarrow$



$$\bigotimes_2$$



Therefore, $(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$



The sequence decreases across the column from left to right as follows {12,12,12,12,9,9,6,6,5,5,5,5,3, 3,3,3,2,2,1,1,1,1,1,1}

Now comparing the operation $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \rightarrow (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$, we expect $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \leq (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$ as the result shows i.e The height at the 5th column of $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D)$ is less than that of $(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$, hence $h((A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D)) < h((A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D))$ by lexicographic order.

Note that the morphism of $(A \boxtimes_1 B) \boxtimes_2 C \boxtimes_1 D)) \leftarrow ((A \boxtimes_2 C) \boxtimes_1 (B \boxtimes_2 D)$ will be $(\leq)$.

Lets use our example to prove that $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \leq (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$.

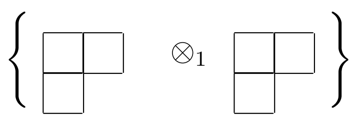
Proof:

Let $X = (A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D)$ and $Y = (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$. By Theorem 3.6, multiplication of any two objects involves $\otimes_1$ and $\otimes_2$. Since our 2-fold monoidal category involves $\boxtimes_1$ and $\boxtimes_2$, we want to find a way of using the theorem to prove the morphism between these categories. Lets once again look at the **A** copied onto **B** in the following way. We then have;

$$A \boxtimes_2 B = \begin{array}{ccc} A & A & A \\ A & A & \end{array} = \begin{array}{c} \{\boxed{A^1} \otimes_1 \boxed{A^2} \otimes_1 \boxed{A^3}\} \\ \otimes_2 \\ \{\boxed{A^4} \otimes_1 \boxed{A^5}\} \end{array}$$

and **C** copied onto **D**, we have

$$C \boxtimes_2 D = \begin{array}{cc} C & C \end{array} =$$



Now taking the partial stacking of **A** and stack it in each box of **C** copied that is a copy of **D** gives the following;

$X = \{\{(A^1 \otimes_1 A^2 \otimes_1 A^3) \otimes_2 (A^4 \otimes_1 A^5)\} \otimes_1 \{(A^6 \otimes_1 A^7 \otimes_1 A^8) \otimes_2 (A^9 \otimes_1 A^{10})\} \otimes_2 \{(A^{11} \otimes_1 A^{12} \otimes_1 A^{13}) \otimes_2 (A^{14} \otimes_1 A^{15})\}\} \otimes_1 \{\{(A^{16} \otimes_1 A^{17} \otimes_1 A^{18}) \otimes_2 (A^{19} \otimes_1 A^{20})\} \otimes_1 \{(A^{21} \otimes_1 A^{22} \otimes_1 A^{23}) \otimes_2 (A^{24} \otimes_1 A^{25})\} \otimes_2 \{(A^{26} \otimes_1 A^{27} \otimes_1 A^{28}) \otimes_2 (A^{29} \otimes_1 A^{30})\}\}$

Note: Copying **A** onto C's depend on the type of stacking chosen. In this case, horizontal is considered, and so same has to be applied in Y and not vice versa.

Doing the same for Y, we have the following pattern; $Y = \{\{(A^1 \otimes_2 A^2) \otimes_1 A^3\} \otimes_1 \{(A^4 \otimes_2 A^5) \otimes_1 A^6\} \otimes_1 \{(A^7 \otimes_2 A^8) \otimes_1 A^9\} \otimes_2 \{(A^{10} \otimes_2 A^{11}) \otimes_1 A^{12}\} \otimes_1 \{(A^{13} \otimes_2 A^{14}) \otimes_1 A^{15}\}\} \otimes_1 \{\{(A^{16} \otimes_2 A^{17}) \otimes_1 A^{18}\} \otimes_1 \{(A^{19} \otimes_2 A^{20}) \otimes_1 A^{21}\} \otimes_1 \{(A^{22} \otimes_2 A^{23}) \otimes_1 A^{24}\} \otimes_2 \{(A^{25} \otimes_2 A^{26}) \otimes_1 A^{27}\} \otimes_1 \{(A^{28} \otimes_2 A^{29}) \otimes_1 A^{30}\}\}$ By theorem 3.6, and following the patterns of X and Y, by matching the orders, we can see that the tensor products vary considerably. Using the theorem 3.6 to show the morphism between these objects will make this proof prodigious. The tensor products can be checked in terms of X and Y.

In X, $\otimes_1 = 20$ and in Y, $\otimes_1 = 17$. This means we will have more combinations of $\otimes_1$ in X than we would with Y.

$\otimes_2 = 9$ in X while $\otimes_2 = 12$ in Y. This means we will have more combinations of $\otimes_2$ in Y than we would with X.

Comparing both, we have for

$\otimes_1 \Rightarrow X = Y + 3$ (1) and for

$\otimes_2 \Rightarrow X = Y - 3.$ (2)

Adding both sides of the equations (i.e 1 and 2) gives $2X = 2Y \Rightarrow X = Y$.

$X = Y$ means both have the same number of morphism between the objects. How do we know if the morphism between these objects are equality or inequality?

Since the cross products $\otimes_i$ for $i = 1, 2$ differ considerably, we assume the stacking process will also differ despite both having the same number of morphism i.e $i > j = j > i$. The difference in the stacking process gives the difference in X and Y by lexicographic order. Hence $X \leq Y$, end of proof.

The natural transformation:$\eta_(ABCD) : (A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \to (A \boxtimes_1 C) \boxtimes_2$

$(B \boxtimes_1 D)$ satisfies the internal unit condition given $\eta_{ABII} = \eta_{IIAB} = id_(A \otimes_1 B)$. The unit is a single box as mentioned earlier. This means $C = \square$ and $D = \square$ are a unit. Therefore we get an identity. The morphism will be an equality. The external unit condition is given by $\eta_{AIBI} = \eta_{IAIB} = id_(A \otimes_1 B)$. A representation will be $A \boxtimes_2 I = A = A \boxtimes_1 I$ i.e

$$A = \ydiagram{2,1,1} \qquad I = \square$$

then

$$A \boxtimes_2 I = \boxed{A} = \left\{ \ydiagram{2,1,1} \right\} = A = I \boxtimes_1 A$$

# CHAPTER VI: CANCELLATION THEOREM

**Cancellation Theorem:** For any four objects (boxes) A, B, C, and D in a category with $A \leq B$ and $C \leq D$, then the morphism in a strict 2-fold monoidal category i.e $(A \boxtimes_i B) \boxtimes_j (C \boxtimes_i D) \to (A \boxtimes_j C) \boxtimes_i (B \boxtimes_j D)$ (if $i = 1$ then $j = 2$ and vice versa) is an equality if any two internal or external objects are in a rectangular or square shape. The rectangular or square shape cancels the effects of the mapping (functor) between these objects. In other words, the rectangular or square shape subdivides the Young diagram that is copied i.e pick two objects A and B. If A is rectangular and we are given $A \boxtimes_i B$ for $i = 1, 2$, then A copied onto B simple subdivides B to form B' where B and B' both are similar objects.

If we consider A,C and D to be the same as in chapter V and take $B = \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \end{array}$

then $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \to (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$ will be as follows;

$$A \boxtimes_2 B = \begin{array}{cc} A & A \end{array} = \left\{ \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} \otimes_1 \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array} \right\} = \left\{ \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \phantom{x} & \phantom{x} \\ \hline \end{array} \right\}$$

and

$$C \boxtimes_2 D = \begin{array}{cc} C & C \\ C & C \\ C & \end{array} = \begin{array}{c} \left\{ \ydiagram{2,1} \otimes_1 \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \otimes_1 \ydiagram{2,1} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \end{array} = \begin{array}{c} \left\{ \ydiagram{4,2} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{4,2} \right\} \\ \otimes_2 \\ \left\{ \ydiagram{2,1} \right\} \end{array} = \left\{ \ydiagram{4,4,2,2,2,1} \right\}$$

Therefore,

$$(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) = \begin{array}{cc} C \boxtimes_2 D & C \boxtimes_2 D \\ C \boxtimes_2 D & C \boxtimes_2 D \end{array} = \left\{ \begin{array}{c} \ydiagram{4,4,2,2,2,2,1} \\ \otimes_2 \\ \ydiagram{4,4,2,2,2,2,1} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{4,4,2,2,2,2,1} \\ \otimes_2 \\ \ydiagram{4,4,2,2,2,2,1} \end{array} \right\}$$

$\Rightarrow$

$$\left\{ \ydiagram{8,8,8,8,4,4,4,4,4,4,2,2} \right\}$$

The sequence is $\{12, 12, 10, 10, 4, 4, 4, 4\}$. Now $(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$ be as follows;

$$A \boxtimes_1 C = \begin{array}{cc} A & A \\ A & \end{array} = \left\{ \begin{array}{c} \ydiagram{1,1} \\ \otimes_2 \\ \ydiagram{1,1} \end{array} \right\} \otimes_1 \left\{ \ydiagram{1,1} \right\} = \left\{ \ydiagram{2,2,1,1} \right\}$$

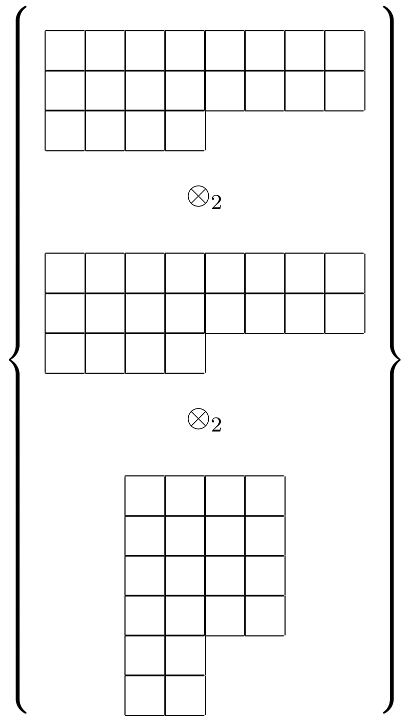
and

$$B \boxtimes_1 D = \begin{array}{cc} B & B \\ B & B \\ B & \end{array} = \left\{ \begin{array}{c} \ydiagram{2} \\ \otimes_2 \\ \ydiagram{2} \\ \otimes_2 \\ \ydiagram{2} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{2} \\ \otimes_2 \\ \ydiagram{2} \end{array} \right\} = \left\{ \ydiagram{4,4,2} \right\}$$

then

$$
(A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D) =
\begin{array}{cc}
B \boxtimes_1 D & B \boxtimes_1 D \\
B \boxtimes_1 D & B \boxtimes_1 D \\
B \boxtimes_1 D & \\
B \boxtimes_1 D &
\end{array}
=
\begin{array}{c}
\left\{ \ydiagram{4,4,2} \otimes_1 \ydiagram{4,4,2} \right\} \\
\otimes_2 \\
\left\{ \ydiagram{4,4,2} \otimes_1 \ydiagram{4,4,2} \right\} \\
\otimes_2 \\
\left\{ \ydiagram{4,4,2} \right\} \\
\otimes_2 \\
\left\{ \ydiagram{4,4,2} \right\}
\end{array}
$$

$\Rightarrow$

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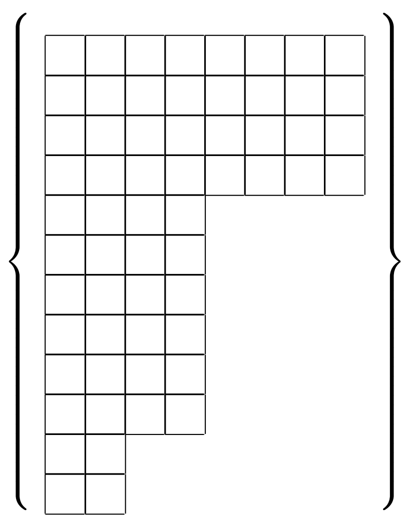


The sequence is $\{12, 12, 10, 10, 4, 4, 4, 4\}$. Hence $(A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) = (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D)$.

Proof:

Let $X = (A \boxtimes_2 B) \boxtimes_1 (C \boxtimes_2 D) \rightarrow (A \boxtimes_1 C) \boxtimes_2 (B \boxtimes_1 D) = Y$.

Using the same way as the proof above to get the patterns for X and Y gives the following;

$X = \{(A^1 \otimes_1 A^2) \otimes_2 (A^3 \otimes_1 A^4) \otimes_2 (A^5 \otimes_1 A^6) \otimes_2 (A^7 \otimes_1 A^8) \otimes_2 (A^9 \otimes_1 A^{10}) \otimes_2 (A^{11} \otimes_1 A^{12})\} \otimes_1 \{(A^{13} \otimes_1 A^{14}) \otimes_2 (A^{15} \otimes_1 A^{16}) \otimes_2 (A^{17} \otimes_1 A^{18}) \otimes_2 (A^{19} \otimes_1 A^{20}) \otimes_2 (A^{21} \otimes_1 A^{22})\} \otimes_1 \{(A^{23} \otimes_1 A^{24}) \otimes_2 (A^{25} \otimes_1 A^{26}) \otimes_2 (A^{27} \otimes_1 A^{28}) \otimes_2 (A^{29} \otimes_1 A^{30})\}$

and

$Y = \{\{(A^1 \otimes_2 A^2) \otimes_1 A^3\} \otimes_1 \{(A^4 \otimes_2 A^5) \otimes_1 A^6\} \otimes_1 \{(A^7 \otimes_2 A^8) \otimes_1 A^9\} \otimes_1 \{(A^{10} \otimes_2 A^{11}) \otimes_1 A^{12}\}\} \otimes_2 \{\{(A^{13} \otimes_2 A^{14}) \otimes_1 A^{15}\} \otimes_1 \{(A^{16} \otimes_2 A^{17}) \otimes_1 A^{18}\} \otimes_1 \{(A^{19} \otimes_2 A^{20}) \otimes_1 A^{21}\} \otimes_1 \{(A^{22} \otimes_2 A^{23}) \otimes_1 A^{24}\}\} \otimes_2 \{\{(A^{25} \otimes_2 A^{26}) \otimes_1 A^{27}\} \otimes_1 \{(A^{28} \otimes_2 A^{29}) \otimes_1 A^{30}\}\}$.

Using Theorem 3.6, by trying to get pairs as in the first proof, will make this proof prodigious. We can check the tensor products $\otimes_1$ and $\otimes_2$ in X and Y. We see that both have the same number of tensor products. i.e

In $\otimes_1$, $X = 17$ and $Y = 17$.

In $\otimes_2$, $X = 12$ and $Y = 12$. Since both $\otimes_i$ for $i = 1, 2$ are the same, it means the rectangular Young diagram simple subdivides the Young diagram that it is copied. It doesn't matter what tensor product is used i.e $A \otimes_1 C = A \otimes_2 C$. So without loss of generality, $\otimes_1 = \otimes_2$, hence $X = Y$, which ends the proof.

It is important to note that the rectangular or square shape acts as if it were a unit.

# CHAPTER VII: ASSOCIATIVITY

We have seen the various axioms in a 2-fold monoidal category and some of the proofs using Young diagrams as examples. To conclude this thesis, we will like to check the associative property as a composition of morphisms. Lets consider an example;

$$A = \ydiagram{2,1,1} \quad B = \ydiagram{3,2} \quad C = \ydiagram{1,1}$$

then $(A \boxtimes_1 B) \boxtimes_1 C = A \boxtimes_1 (B \boxtimes_1 C)$ will be as follows;

$$A \boxtimes_1 B = \begin{ytableau} A & A & A \\ A & A \end{ytableau} = \left\{ \begin{array}{c} \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \ydiagram{2,1,1} \\ \otimes_2 \\ \ydiagram{2,1,1} \end{array} \right\} \otimes_1 \left\{ \ydiagram{2,1,1} \right\} = \left\{ \ydiagram{6,5,3,2,2,2} \right\}$$

and so

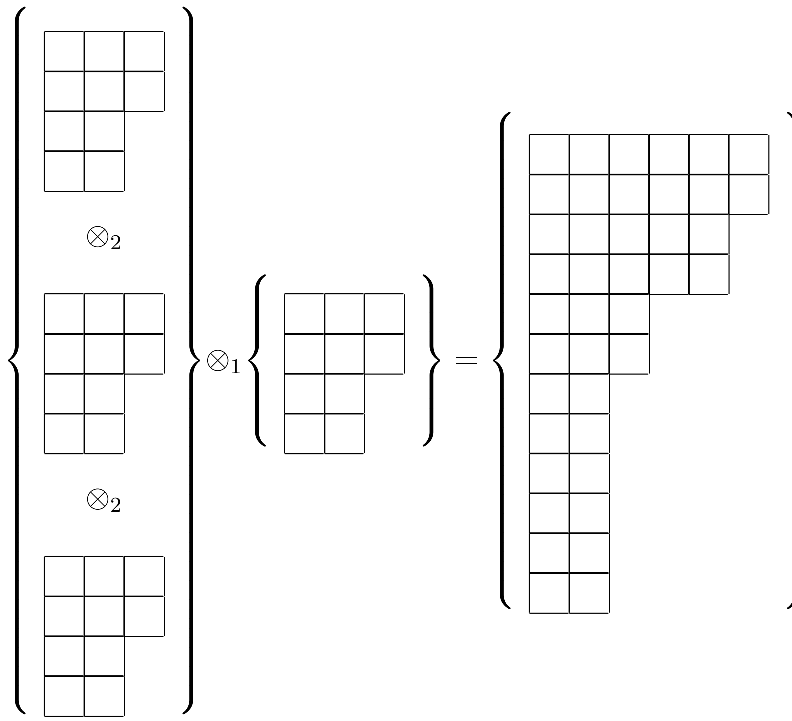
$$(A \boxtimes_1 B) \boxtimes_1 C = \begin{array}{c} A \boxtimes_1 B \\ A \boxtimes_1 B \end{array} = \left\{ \begin{array}{c} \ydiagram{6,5,3,2,2,2} \\ \otimes_2 \\ \ydiagram{6,5,3,2,2,2} \end{array} \right\} = \left\{ \ydiagram{6,6,5,5,3,3,2,2,2,2,2,2} \right\}$$

The sequence here is $\{12, 12, 6, 4, 4, 2\}$. On the other hand, we have

$$A \boxtimes_1 (B \boxtimes_1 C) \Rightarrow \qquad B \boxtimes_1 C = \begin{array}{c} B \\ B \end{array} = \left\{ \begin{array}{c} \ydiagram{3,2} \\ \otimes_2 \\ \ydiagram{3,2} \end{array} \right\} = \left\{ \ydiagram{3,3,2,2} \right\}$$

then

$$A \boxtimes_1 (B \boxtimes_1 C) = \begin{array}{ll} B \boxtimes_1 C & B \boxtimes_1 C \\ B \boxtimes_1 C & \\ B \boxtimes_1 C & \end{array} =$$



The sequence is also $\{12, 12, 6, 4, 4, 2\}$. Hence $(A \boxtimes_1 B) \boxtimes_1 C = A \boxtimes_1 (B \boxtimes_1 C)$.

Proof:

Using theorem 3.6, we can show that $(A \boxtimes_1 B) \boxtimes_1 C = A \boxtimes_1 (B \boxtimes_1 C)$.

Let $X = (A \boxtimes_1 B) \boxtimes_1 C$ and $Y = A \boxtimes_1 (B \boxtimes_1 C)$. Making copies of **A** onto **B** in X, we have the following;

$$A \boxtimes_1 B = \begin{array}{lll} A & A & A \\ A & A & \end{array} = \left\{ \begin{array}{c} \boxed{A^1} \\ \otimes_2 \\ \boxed{A^2} \end{array} \right\} \otimes_1 \left\{ \begin{array}{c} \boxed{A^3} \\ \otimes_2 \\ \boxed{A^4} \end{array} \right\} \otimes_1 \left\{ \boxed{A^5} \right\}$$

and stacking the partial copies of **A** onto **C** gives the following;

$X = \{(A^1 \otimes_2 A^2) \otimes_1 (A^3 \otimes_2 A^4) \otimes_1 A^5\} \otimes_2 \{(A^6 \otimes_2 A^7) \otimes_1 (A^8 \otimes_2 A^9) \otimes_1 A^10\}$

On the other hand, B copied onto C is;

$$B \boxtimes_1 C = \begin{array}{c} B \\ B \end{array} = \left\{ \begin{array}{c} \ydiagram{3,2} \\ \\ \otimes_2 \\ \\ \ydiagram{3,2} \end{array} \right\}$$

and stacking **A** in each box of **B** that is copied onto **C** gives;

$Y = \{(A^1 \otimes_2 A^2) \otimes_1 (A^3 \otimes_2 A^4) \otimes_1 A^5\} \otimes_2 \{(A^6 \otimes_2 A^7) \otimes_1 (A^8 \otimes_2 A^9) \otimes_1 A^{1}0\}$. We can see here that cross products $\otimes_i$ for $i = 1, 2$ are the same in X and Y and the objects are ordered the same, which enables the morphism between these objects to be the same. Hence $X = Y$, end of proof.

Note: We can get pairs to show how i differs from j, but it will result to too many combinations. Now we can also show that $(A \boxtimes_2 B) \boxtimes_2 C = A \boxtimes_2 (B \boxtimes_2 C)$, using the various Young diagrams A, B, and C as above. The proof will be exactly the same as above. The difference will only be the size of the Young diagrams.

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