POLYTOPES AND ALGEBRAS OF GRAFTED TREES: STELLOHEDRA

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Abstract. We show that certain sets of combinatorial trees are naturally the minimal elements of face posets of convex polytopes. The polytopes that constitute our main results are well known in other contexts. First we see the classical permutahedra, and then certain generalized permutahedra: specifically the graph-associahedra of the star graphs which are known collectively as the *stellohedra*. As an aside we show that the stellohedra also appear as certain lifted generalized permutahedra: graph composihedra for complete graphs. Tree species considered here include ordered and unordered binary trees and ordered lists (labeled corollas). Compositions of these trees were recently introduced as bases for one-sided Hopf algebras. Thus our results show how to represent those algebras using the graph tubings of star graphs, the vertices of the stellohedra. We also show an alternative associative algebra structure on the vertices of the stellohedra.

1. Introduction

The mathematical operation of grafting trees, root to leaf, is the key feature for the structure of more than a few important operads and Hopf algebras. In 1998 Loday and Ronco found a Hopf algebra of plane binary trees, initiating the study of these type of structures [18]. The Loday-Ronco algebra is the image of the Malvenuto-Reutenauer Hopf algebra of permutations [21] and projects to the algebra of quasisymmetric polynomials. Alternatively, since the Loday-Ronco algebra is self-dual, it can project to the divided power Hopf algebra. The authors of [18] showed Hopf algebra maps which factor the descent map from permutations to Boolean subsets. The first factor is the Tonks projection from the vertices of the permutohedron to the vertices of the associahedron. Chapoton put this latter fact into context when he found that the Hopf algebras of vertices are subalgebras of larger ones based on the faces of the respective polytopes [7]. Chapoton's algebras are the associated graded structures to algebras already described by Loday and Ronco in [20].

In [19] the authors describe the product of planar binary trees in terms of the Tamari order. In 2005 and 2006 Aguiar and Sottile characterized operations in these algebras by using Möbius functions (of the Tamari order and of the weak Bruhat order) to obtain new bases, in respectively [2],[1]. Their work gave a nice way to construct a basis of primitive elements, using the irreducible trees. In ForSpr:2010 we characterize the same operations in terms of inclusions (into the larger polytopes) of products of polytope faces.

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In [15] we used the following notation: $\mathfrak{S}Sym$ for the Malvenuto-Reutenauer Hopf algebra, $\mathcal{Y}Sym$ for the Loday-Ronco Hopf algebra, and $\mathfrak{C}Sym$ for the divided power Hopf algebra. We defined the idea of grafting with two colors, preserving the colors after the graft in order to have two-tone, or painted, trees with various structures possible in each colored region. Here we review the definitions, adding some generality and defining poset structures on each set of painted trees. We are able to conclude that most of the coalgebras defined in [15] have underlying geometries of polytope sequences.

The *stellohedra*, or star-graph-associahedra, were first defined using the latter terminology by Carr and Devadoss in [6]. The former terminology was introduced in [24], where these polytopes were studied as special cases of nestohedra. In [13] the 3d version of the stellohedron appears graphically, as the domain and range quotient of the multiplihedra for the complete graphs. These quotients are the *composihedra* and *cubeahedra* respectively, but this source does not identify them as stellohedra. Also in [13] it is claimed without proof that grafted trees represent these quotients in all dimensions, although the corresponding trees in that source are associated in error to the wrong polytope. (We correct the mistake here; compare our Figure 9 and 7 to Figures 3 and four of that source.)

In [22] the authors do actually prove that the stellohedra for all dimensions are in fact the cubeahedra of complete graphs (which we will review). Also in [22] the stellohedra are recognized as the secondary polytopes of pairs of nested concentric n-dimensional simplices. The stellohedra have also been seen as special cases of signed-tree associahedra in [23].

1.1. **Main Results.** We show that three sequences of our sets of painted trees, with their relations, are isomorphic as posets to face lattices of convex polytopes. In Theorem 3.2 we show that weakly ordered forests grafted to weakly ordered trees are isomorphic to the permutohedra. In Theorem 3.10 we show that forests of corollas grafted to weakly ordered trees are isomorphic to the star-graph-associahedra, or stellohedra. In Theorem 3.12 and Theorem 3.13 we show that weakly ordered forests grafted to a corolla are also isomorphic to stellohedra. In Theorem 3.11 we show that the stellahedra again appear as graph-composihedra for the complete graphs.

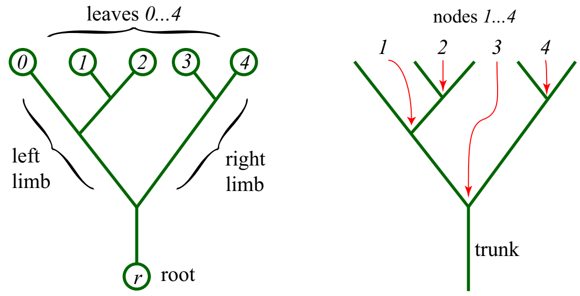
In Section 4 we show how our bijections transfer the algebraic structure from [15] to the vertices of the stellohedra. In Proposition 5.3 we show that a new, less forgetful, product on these vertices is associative.

2. Definitions

Trees are connected, acyclic graphs. We begin our discussion, following [12], with rooted, plane, binary trees.

- Rooted: There is a vertex of degree one which is designated the root. The other vertices of degree one are called leaves.
- Plane: Each tree is drawn in the plane with no edges crossing. Two plane trees are only equal if they can be made into identical pictures by scaling portions of the plane, without any reflection or rotation in any dimension.
- Binary: The vertices that are not degree one are all of degree three. These vertices are called nodes.

Here is a *plane rooted binary tree*, often called a binary tree when the context is clear:



The leaves are ordered left to right as shown by the circled labels. The node ordering corresponds to the order of "gaps between leaves:" the n^{th} node is the one where a raindrop would be caught which fell just to the left of the n^{th} leaf. The branches are the edges with a leaf. Non-leafed edges are referred to as internal edges. The nodes are also partially ordered vertically; we say two nodes are comparable in this partial order if they both lie on a path to the root. The root is maximal in this partial order. In the preceding picture, we have for instance that node 3 is greater than node 1 which is greater than node 2. The set of plane rooted binary trees with n nodes and $n+1$ leaves is denoted $\mathcal{Y}_n$. The cardinality of these sets are the Catalan numbers:

$$|\mathcal{Y}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

We will also need to consider rooted plane trees with nodes of larger degree than three. An $(n+1)$-leaved rooted tree with only one node (it will have degree $n+2 \geq 3$), or, for $n = 0$, a single leaf tree with zero nodes, is called a *corolla*, denoted $\mathfrak{C}_n$. This notation for the (set of one) corolla with $n+1$ leaves is the same as used for the set of one *left comb* in [15]. In the current paper we have decided that the corollas are more easily recognized than the combs.

2.1. **Ordered and Painted binary trees.** Many variations of the idea of the binary tree have proven useful in applications to algebra and topology. First, an *ordered tree* (sometimes called *leveled*) is a binary tree that has a vertical linear ordering of the n nodes as well as horizontal. This vertical linear ordering extends the partial vertical ordering given by distances to the root. See Figure 1 for ways to draw ordered trees. This ordering allows a well-known bijection between the ordered trees with n nodes, denoted $\mathfrak{S}_n$, and the permutations on $[n]$.

We will also consider *forests* of trees. In this paper, all forests will be a linearly ordered list of trees, drawn left to right. This linear ordering can also be seen as an ordering of all the nodes of the forest, left to right. On top of that, we can also order all the nodes of the forest vertically, giving a *vertically ordered forest*, which we often shorten to *ordered forest*. This initially gives us four sorts of forests to consider, shown in Figure 2.

Also shown in Figure 2 are three canonical, forgetful maps between the types of forests.

2.1. **Definition.** *We define β to be the function that takes an ordered forest F and gives a forest of ordered trees. The output $\beta(F)$ will have the same list of trees as F, and for*

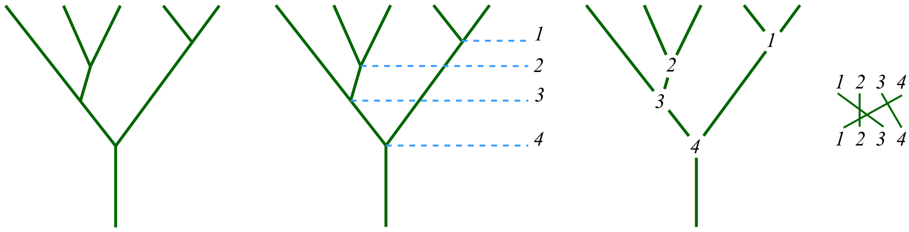


FIGURE 1. Drawing an ordered tree, in three different styles. The corresponding permutation σ is (3,2,4,1), in the notation $(\sigma(1), \sigma(2), \sigma(3), \sigma(4).)$

a tree t in $\beta(F)$ the vertical order of the nodes of t will respect the vertical order of the nodes in F. That is, for two nodes a, b of t we have $a \leq b$ in t iff $a \leq b$ in F.

We define τ to be the function that takes an ordered tree and outputs the tree itself, forgetting all of the vertical ordering of nodes (except for the partial ordering based on distance from the root.) We define κ to be the function that takes a tree and gives the corolla with the same number of leaves.

Note that τ and κ are immediately both functions on forests, simply by applying them to each tree in turn. Also note that τ and κ are described in [15], but that there κ yields a left comb rather than a corolla.

Now we define larger sets of trees that generalize the binary ones. First we drop the word binary; we will consider plane rooted trees with nodes that have any degree larger than two. Then, from the non-binary vertically ordered trees we further generalize by allowing more than one node to reside at a given level. Instead of corresponding to a permutation, or total ordering, these trees will correspond to an ordered partition, or weak ordering, of their nodes.

2.2. **Definition.** *A* weakly ordered tree *is a plane rooted tree with a weak ordering of its nodes that respects the partial order of proximity to the root.*

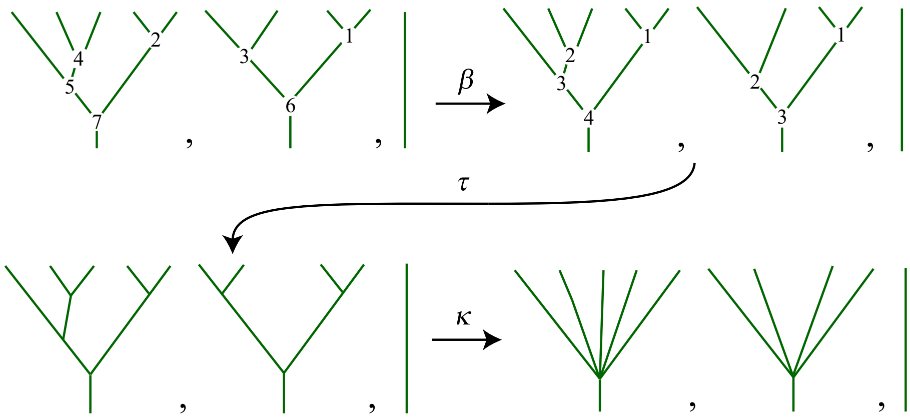


FIGURE 2. Following the arrows: a (vertically) ordered forest, a forest of ordered trees, a forest of binary trees, and a forest of corollas.

Recall that this means all sets of nodes are comparable–but some are considered as tied when compared, forming a block in an ordered partition of the nodes. The linear ordering of the blocks of the partition respects the partial order of nodes given by paths to the root.

For a weakly ordered tree with $n+1$ leaves the ordered partition of the nodes determines an ordered partition of $S = \{1, \ldots, n\}$, where S is the set of "gaps between leaves," as described in [27]. (Recall that a gap between two adjacent leaves corresponds to the node where a raindrop would eventually come to rest; S is partitioned into the subsets of gaps that all correspond to nodes at a given level.) Weakly ordered trees are drawn using nodes with degree greater than two, and using numbers and dotted lines to show levels as in Figure 3. Note that an ordered tree is a (special) weakly ordered tree.

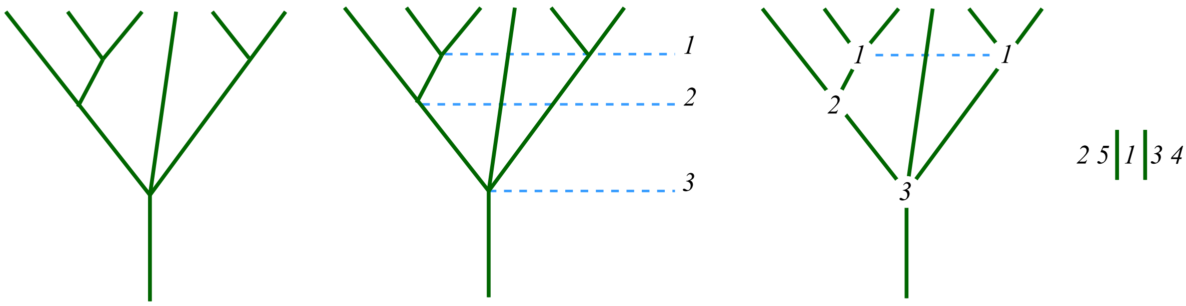


FIGURE 3. Drawing a weakly ordered tree, in three different styles. The corresponding ordered partition is $(\{2,5\}, \{1\}, \{3,4\})$.

As well as forests of weakly ordered trees we also consider weakly ordered forests. This gives us three more sorts of forests to consider, shown in Figure 4. As indicated in that figure, the maps β, τ and κ are easily extended to forests of the non-binary and/or weakly ordered trees: β forgets the weak ordering of the forest to create a forest of weakly ordered trees, τ forgets the weak ordering, and κ forgets the partial order to create corollas.

The trees we focus on in this paper are constructed by grafting together combinations of ordered trees, binary trees, and corollas. Visually, this is accomplished by attaching the roots of one of the above forests to the leaves of one of the above types of trees. We use two colors, which we refer to as "painted" and "unpainted." The forest is described as unpainted, and the base tree (which the forest is grafted to) is painted. At a graft the leaf is identified with the root, and in the diagram that point is drawn as a change in color (and thickness, for easy recognition) of the resulting edge. (Note that in earlier papers such as [11] our mid-edge change in color is described instead as a new node of degree two.)

We refer to the result as a (partly) *painted tree*, regardless of the types of upper (unpainted) and lower (painted) portions. Notice that in a painted tree the original trees (before the graft) are still easily observed since the coloring creates a boundary, called the *paint-line* halfway up the edges where the graft was performed. Thus the paint line separates the painted tree into a single tree of one color and a forest of trees of another color. In Figure 5 we show all 12 ways to graft one of our types of partially ordered forest with one of our types of tree.

2.3. **Definition.** *The maps β, τ and κ are now extended to the painted trees, just by applying them to the unpainted forest and/or to the painted tree beneath. We indicate this by writing a fraction: $\frac{f}{g}$ for two of our three maps, or the identity map, as seen in Figure 5. That is, $\frac{f}{g}$ indicates applying f to the forest and g to the painted base tree, for $f, g \in \{\beta, \tau, \kappa, 1\}$.*

2.2. **General painted trees.** Now our definition of painted trees is expanded to include any of our types of forest grafted to any of our types of tree. On top of that we will also permit a further broadening of the allowed structure of our painted trees. The paint-line, where the graft occurs, is allowed to coincide with nodes, where branching occurs. We call it a *half-painted node*. In terms of the grafting of a forest onto a tree our description depends on the type of forest. If the forest is weakly ordered, or is a forest of weakly ordered trees, then we see each half-painted node as grafting on a single tree at its least node, after removing its trunk and root. If the forest is only partially ordered (i.e. of binary trees or corollas) then we see the half-painted nodes as (possibly) several roots of several trees simultaneously grafted to a given leaf. See the examples in Figures 7 and 6.

For these general painted trees we can again extend the “fractional” maps using β, τ and κ. We reiterate from above how the half-painted nodes are interpreted, since that determines the input for the “numerator” map. Specifically $\frac{\beta}{g}$ operates by taking as input for β the weakly ordered forest of trees, one tree for each half-painted node. That is, $\frac{\beta}{1}$ treats the half-painted nodes as being the location of a single tree that is grafted on without a trunk. This description is the same for $\frac{\tau}{g}$. In contrast however, the map $\frac{\kappa}{g}$ takes as input the forest found by listing all the unpainted trees while assuming each has a visible trunk, some of which are simultaneously grafted at the same half-painted node. Examples of these maps are shown in Figure 7, where we show 12 general painted trees

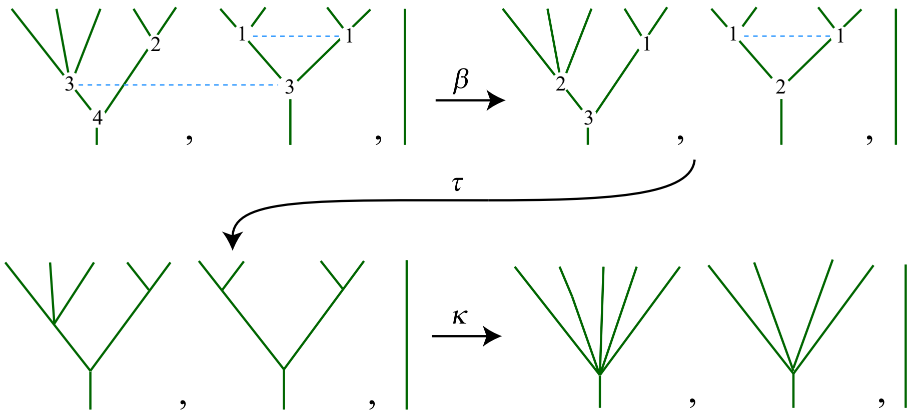


FIGURE 4. Following the arrows: a (vertically) weakly ordered forest, a forest of weakly ordered trees, a forest of plane rooted trees, and a forest of corollas. Note that the forests in Figure 2 are special cases of these.

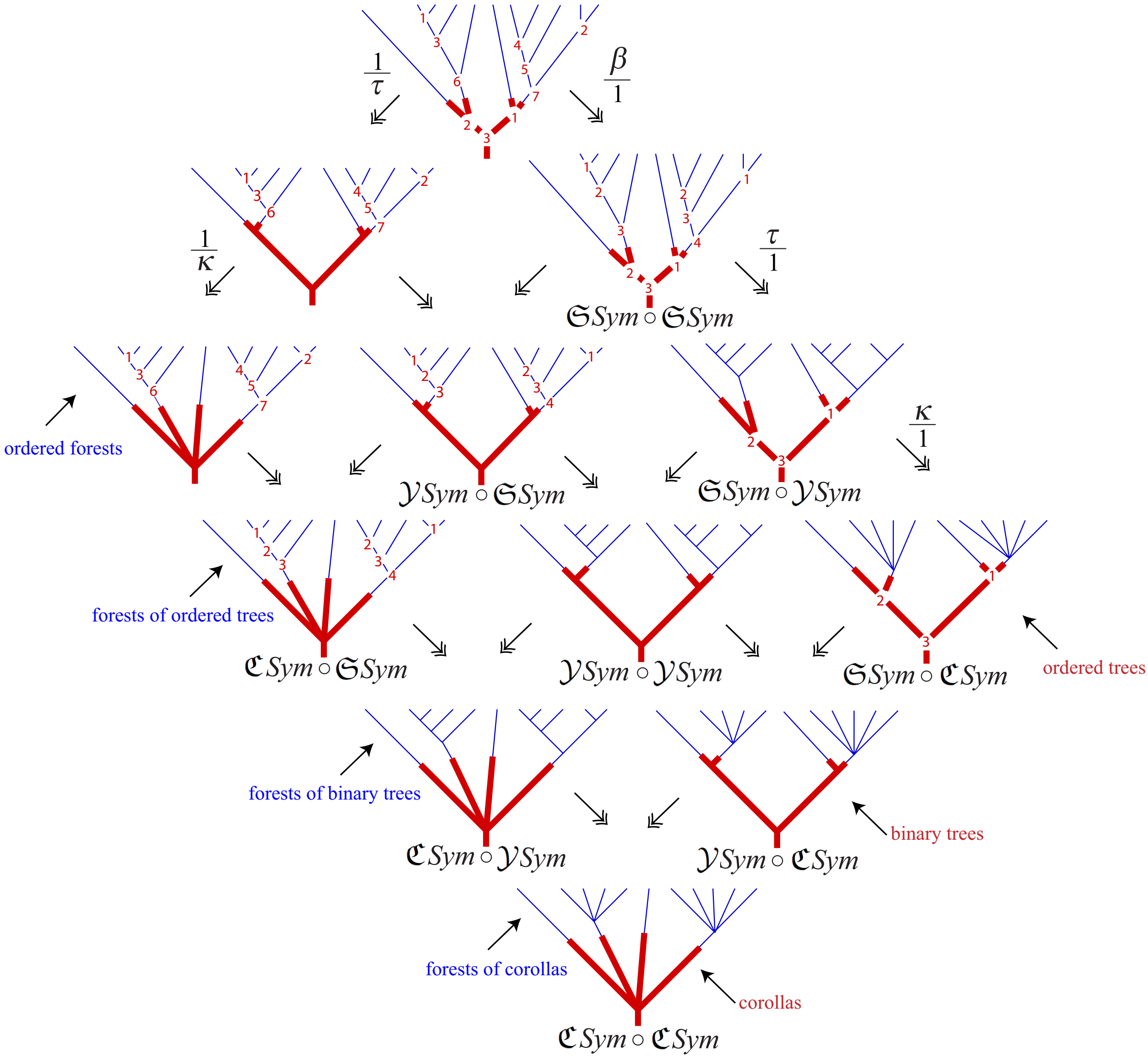


FIGURE 5. Varieties of grafted, painted trees. Each diagonal shares a type of tree on the bottom (painted) or a type of forest grafted on, as indicated by the labels. Since these trees are all binary, they correspond to vertex labels of the polytope sequences whose 3d versions are shown in Figure 9. The forgetful maps are shown with example input and output. Parallel arrows all denote the same map, except of course that the identity is context dependent.

that consist of one of the four general types of forest and one of the three general types of trees. Figure 6 is a detail from Figure 7 showing how the actions of the projections differ.

2.3. **The face poset relations.**

2.3.1. *Partial ordering of nodes and gaps.* Each of our 12 types of painted tree comes with a canonical vertical partial ordering of its nodes (branch points), produced by

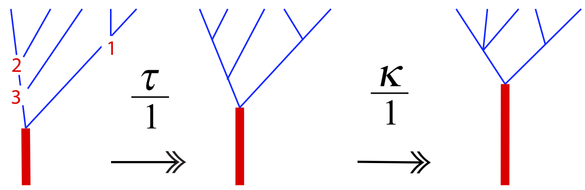


FIGURE 6. Action of the projections, detail from Figure 7. At first there is a single (weakly) ordered tree attached at the half-painted node; at last there are two (corolla) trees attached at the same node. In the center, where the unpainted portion is one or two binary trees, we can see it as either without any contradiction.

concatenating the orders that exist before the graft. Each new partial ordering is a refinement of the partial ordering given by distances to the root of the newly minted painted tree, and also preserves the relations that existed before the graft. We add the rules that 1) all half-painted nodes must be forced to remain at the same level, that is, incomparable to each other (or tied in a weak order); and 2) that nodes below the paint line will never surpass half painted nodes, and neither of the former will surpass unpainted nodes in the partial order. Furthermore, this ordering of nodes implies an ordering of the gaps between leaves of the tree. Some gaps share a node. Two gaps that share a node are considered to be incomparable in the partial order (or tied in a weak order).

Now we can define 12 separate posets whose elements are trees: one poset on each of our 12 types of painted trees shown in Figure 7. Note that the simplest painted tree with n leaves has one half-painted node: n single leafed unpainted trees all grafted to a painted trunk, the node coinciding with the paint line. This *half-painted corolla* can be interpreted as one of any of the 12 painted tree varieties, and it will be the unique maximal element in all 12 posets.

2.4. **Definition.** Given two painted trees s and t that are of the same painted type (i.e. they share the same types of tree and forest, below and above the paint line) we define the *painted growth preorder*, where :

$$s \prec t$$

if $s = t$ or if s is formed from t using a series of pairs $(a, b)_i$ of the following two moves, each pair performed in the following order:

a) "growing" internal edges of t : introducing new internal edges or increasing the length of some internal edges (either painted or unpainted). This is precisely described as a possible refinement of the vertical partial order of gaps between leaves, by adding relations to the partial order between previously incomparable elements. Relations may not be deleted by the growth (nor ties formed in a weak order), but if the growing of painted edges occurs at a collection of half-painted nodes in t then the partial order may be preserved rather than strictly refined. Note that internal edges can grow where there was no internal edge before, such as at a half-painted node or any node that had degree larger than three. Note also that the rules for painted trees must be obeyed by the growing process–for instance an unpainted edge cannot grow from a completely painted node

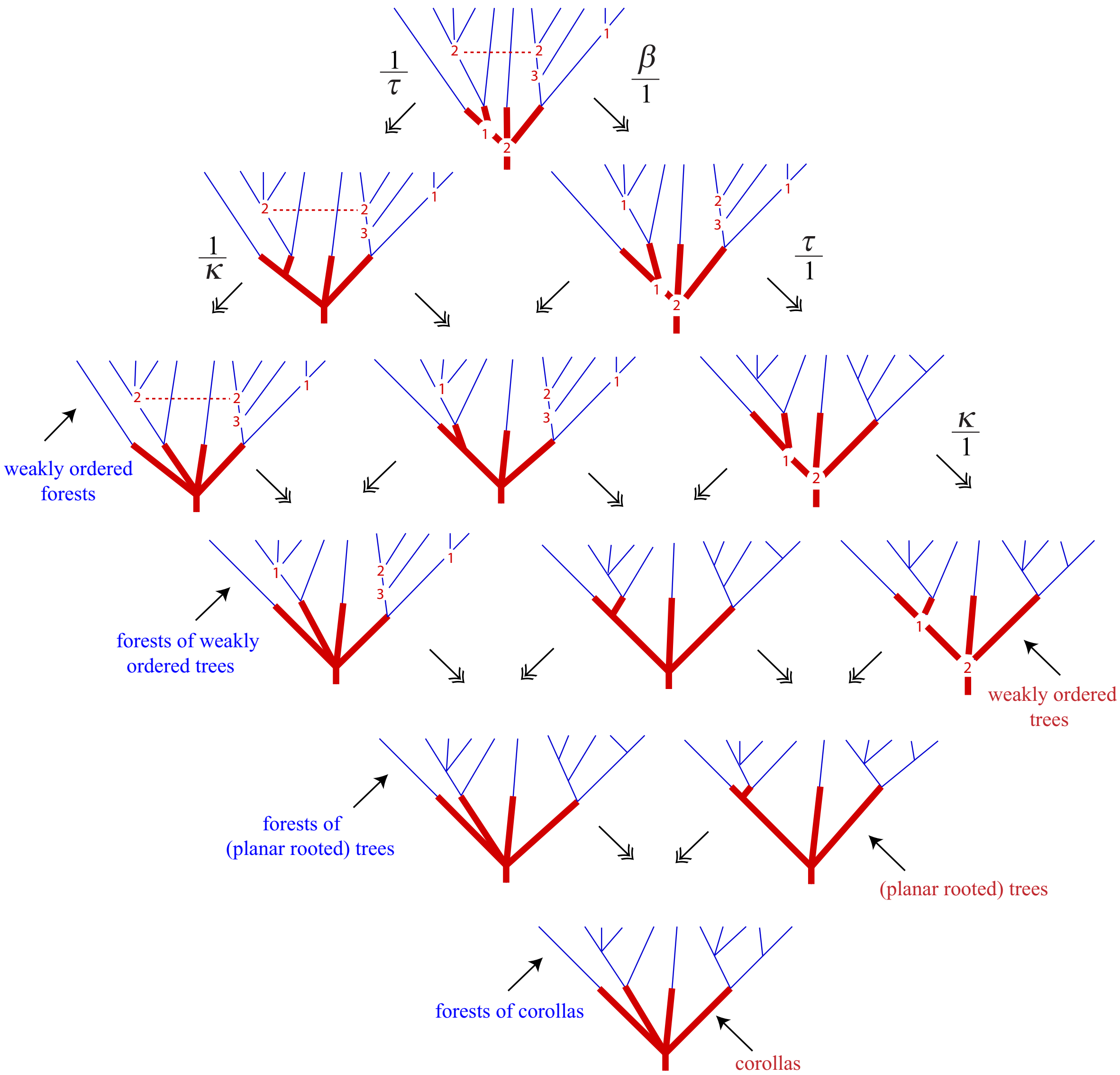


FIGURE 7. More varieties of grafted, painted trees. These correspond to face labels of the polytope sequences whose 3d versions are shown in Figure 9. Parallel arrows all denote the same map. Note that the trees in Figure 5 are special cases–vertex trees, or minimal in the face lattice–of the types illustrated in this figure.

(and vice versa), and if some painted edges are grown from a half-painted node then all the edges possible must be formed, to allow the paint line to be drawn horizontally.

b) ...followed by “throwing away,” or forgetting, any superfluous structure introduced by the edge growing. This is described precisely by taking the tree that results from growing edges, and applying to it the forgetful map (from the set of $\beta, \tau, \kappa, 1$ and their fractions and compositions) that is needed to ensure that the result is in the original type of the painted tree t. E.g. if the original type had weakly ordered forests grafted to weakly ordered trees, we only apply the identity. However if t originally was a forest of weakly ordered trees grafted to a weakly ordered tree we should apply $\frac{\beta}{1}$.

For examples of (non-covering) relations in the 12 posets see the trees in respective locations of figure 5 and Figure 7: the latter are all greater than the former in the same positions. Several more covering relations for some of our 12 classes of general painted trees are shown in Figure 8.

3. Bijections

The painted growth relation is reflexive and transitive by construction, for all 12 types. We conjecture also that in all 12 of cases the the painted growth preorder is in fact a poset, and moreover we conjecture that all the posets thus defined are realized as the face posets of sequences of convex polytopes. Four of the cases have been proven in previous work. These four appear as the highlighted diamond in Figure 9. The polytope sequences are the cubes, associahedra, composihedra and multiplihedra. The latter three are shown (with pictures of painted trees) in [16]; the fact that the cubes result from forgetting all the branching sructure is equuivalent to the fact that cubes arise when both of two product spaces are associative, as pointed out in [5], also (with design tubings) in [9].

In this section three more sequences of our sets of painted trees, with their relations, will be shown to be isomorphic as posets to face lattices of convex polytopes. Two of these are the species whose structure types are: a forest of corollas grafted to a weakly ordered tree (stellohedra) or a weakly ordered forest grafted to a corolla (stellohedra again). The third is the species whose structure type is the weakly ordered forest grafted to a weakly ordered tree (permutohedra). In the companion to this paper, [4], we show that another structure is also a polytope: a forest of plane rooted trees grafted to a weakly ordered tree (pterahedra). There remain four cases in Figure 9 that we leave as a conjecture. (These latter four are the ones which do not have a label naming them under their picture).

Some of our proofs and corollaries will use the concept of tubings, which we review next.

3.1. Tubes, tubings and marked tubings.

The definitions and examples in this section are largely taken from [17]. They are based on the original definitions in [6], with only the slight change of allowing a universal tube, as in [8].

3.1. Definition. Let G be a finite connected simple graph. A *tube* is a set of nodes of G whose induced graph is a connected subgraph of G. We will often refer to the induced graph itself as the tube. Two tubes u and v may interact on the graph as follows:

(1) Tubes are *nested* if $u \subset v$.
(2) Tubes are *far apart* if $u \cup v$ is not a tube in G, that is, the induced subgraph of the union is not connected, (equivalently none of the nodes of u are adjacent to a node of v).

Tubes are *compatible* if they are either nested or far apart. We call G itself the *universal tube*. A *tubing* U of G is a set of tubes of G such that every pair of tubes in U is compatible; moreover, we force every tubing of G to contain (by default) its universal tube. By the term *k-tubing* we refer to a tubing made up of k tubes, for $k \in \{1, \dots, n\}$.

When G is a disconnected graph with connected components $G_1, \dots, G_k$, an additional condition is needed: If u_i is the tube of G whose induced graph is G_i, then any tubing of G cannot contain all of the tubes $\{u_1, \dots, u_k\}$. However, the universal tube is still included despite being disconnected. Parts (a)-(c) of Figure 12 from [8] show examples of allowable tubings, whereas (d)-(f) depict the forbidden ones.

3.2**. Theorem.** [6, Section 3] *For a graph G with n nodes, the* graph associahedron *$\mathcal{K}G$ is a simple, convex polytope of dimension $n-1$ whose face poset is isomorphic to the set of tubings of G, ordered by the relationship $U \prec U'$ if U is obtained from U' by adding tubes.*

The vertices of the graph associahedron are the n-tubings of G. Faces of dimension k are indexed by $(n-k)$-tubings of G. In fact, the barycentric subdivision of $\mathcal{K}G$ is precisely the geometric realization of the described poset of tubings.

To describe the face structure of the graph associahedra we need a definition from [6, Section 2].

3.3**. Definition.** *For graph G and a collection of nodes t, construct a new graph $G^*(t)$ called the* reconnected complement*: If V is the set of nodes of G, then $V - t$ is the set of nodes of $G^*(t)$. There is an edge between nodes a and b in $G^*(t)$ if $\{a, b\} \cup t'$ is connected in G for some $t' \subseteq t$.*

3.4**. Example.** *Figure 13 illustrates some examples of graphs along with their reconnected complements.*

For a given tube t and a graph G, let $G(t)$ denote the induced subgraph on the graph G.

3.5**. Theorem.** [6, Theorem 2.9] *Let V be a facet of $\mathcal{K}G$, that is, a face of dimension $n-2$ of $\mathcal{K}G$, where G has n nodes. Let t be the single, non-universal, tube of V . The face poset of V is isomorphic to $\mathcal{K}G(t) \times \mathcal{K}G^*(t)$.*

We will consider a related operation on graphs. The *suspension* of G is the graph $\mathfrak{S}G$ whose set of nodes is obtained by adding a node 0 to the set $\mathrm{Nod}(G)$, of nodes of G, and whose edges are defined as all the edges of G together with the edges $0, v$ for v $\in Nod(G)$.

The reconnected complement of $\{0\}$ in $\mathfrak{S}G$ is the complete graph $\mathcal{K}_n$ for any graph G with n nodes. Note that the star graph St_n is the suspension of the graph C_n with has n nodes and no edge, while the fan graph $F_{1,n}$ is the suspension of the line graph L_n.

It turns out that this construction of the graph multiplihedra is a special case of a more general construction on certain polytopes called the *generalized permutahedra* as defined by Postnikov in [25]. The *lifting* of a generalized permutahedron, and a nestohedron in particular, is a way to get a new generalized permutahedron of one greater dimension from a given example, using a factor of $q \in [0, 1]$ to produce new vertices from some of the old ones [3]. This procedure was first seen in the proof that Stasheff's multiplihedra complexes are actually realized as convex polytopes [11].

Soon afterwards the lifting procedure was applied to the graph associahedra–well-known examples of nestohedra first described by Carr and Devadoss. We completed an initial study of the resulting polytopes, dubbed *graph multiplihedra*, published as [8].

This application raised the question of a general definition of lifting using q. At the time it was also unknown whether the results of lifting, then just the multiplihedra and the graph-multiplihedra, were themselves generalized permutahedra. These questions were both answered in the recent paper of Ardila and Doker [3]. They defined nesto-multiplihedra and showed that they were generalized permutohedra of one dimension higher in each case.

We refer the reader to Ardila and Doker [3] for the general definitions. Here we need only the following definitions, from [8]. Combinatorially, lifting of a graph associahedron occurs when the notion of a tube is extended to include markings.

3.6. **Definition.** A *marked tube* of P is a tube with one of three possible markings:

(1) a *thin* tube given by a solid line,
(2) a *thick* tube given by a double line, and
(3) a *broken* tube given by fragmented pieces.

Marked tubes u and v are *compatible* if

(1) they form a tubing and
(2) if $u \subset v$ where v is not thick, then u must be thin.

A *marked tubing* of P is a tubing of pairwise compatible marked tubes of P.

A partial order is now given on marked tubings of a graph G. This poset structure is then used to construct the *graph multiplihedron* below.

3.7. **Definition.** The collection of marked tubings on a graph G can be given the structure of a poset. A marked tubing $U \prec U'$ if U is obtained from U' by a combination of the following four moves. Figure 14 provides the appropriate illustrations, with the top row depicting U' and the bottom row U.

(1) *Resolving markings*: A broken tube becomes either a thin tube (14a) or a thick tube (14b).
(2) *Adding thin tubes*: A thin tube is added inside either a thin tube (14c) or broken tube (14d).
(3) *Adding thick tubes*: A thick tube is added inside a thick tube (14e).
(4) *Adding broken tubes*: A collection of compatible broken tubes $\{u_1, \ldots, u_n\}$ is added simultaneously inside a broken tube v only when $u_i \Subset v$ *and* v becomes a thick tube; two examples are given in (14f) and (14g).

Here is the key theorem from [8]

3.8. **Theorem.** *For a graph G with n nodes, the* graph multiplihedron $\mathcal{J}G$ *is a convex polytope of dimension n whose face poset is isomorphic to the set of marked tubings of G with the poset structure given above.*

There are two important quotient polytopes mentioned in [8]: $\mathcal{J}G_d$ and $\mathcal{J}G_r$ for a given graph G. The former is called the *graph composihedron.* Its faces correspond to marked tubings, but for which no thin tubes are allowed to be inside another thin tube.

In terms of equivalence of tubings, the face poset of $\mathcal{J}G_d$ is isomorphic to the poset $\mathcal{J}G$ modulo the equivalence relation on marked tubings generated by identifying any two tubings $U \sim V$ such that $U \prec V$ in $\mathcal{J}G$ precisely by the addition of a thin tube inside another thin tube, as in Figure 14(c). It is defined via geometric realization in [8].

$\mathcal{J}G_r$ has faces which correspond to marked tubings, but for which no thick tubes are allowed to be inside another thick tube. In terms of equivalence of tubings, the face poset of $\mathcal{J}G_r$ is isomorphic to the poset $\mathcal{J}G$ modulo the equivalence relation on marked tubings generated by identifying any two tubings $U \sim V$ such that $U \prec V$ in $\mathcal{J}G$ precisely by the addition of a thick tube, as in Figure 14(e). It is defined via geometric realization in [8]. For connected graphs G, the $\mathcal{J}G_r$ is combinatorially equivalent to the graph cubeahedron $\mathbb{C}G$, as defined in [9].

The graph cubeahedron $\mathbb{C}K_n$ is described in [9] as comprising the *design-tubings* on the complete graph. In Figure 23 we show the correspondence between labels of vertices: range-equivalence classes of marked tubings and design tubings. The isomorphism claimed in [9] is easily described: design tubes (square tubes) correspond to the nodes not inside any thin or broken tube; while round tubes in the design tubing correspond to thin tubes. Broken tubes contain any nodes not in any tube of the design tubing.

For this reason we refer to the entire class of polytopes $\mathcal{J}G_r$ as the *(general) graph cubeahedra.* In fact the description of $\mathbb{C}G$ using design tubings which is given in [9] is not difficult to extend to graphs with multiple components: we only need to introduce the universal (round) tube. For example, the graph cubeahedron for the edgeless graph is the hypercube with a single truncated vertex.

The four well-known examples of polytopes from Figure 9 can be seen as tubing posets, as pointed out in [8]. The multiplihedra $\mathcal{J} = \mathcal{J}P$ have face posets equivalent to the marked tubes on path graphs P. The composihedra are the domain quotients of these: $\mathcal{J}P_d$; and the associahedra are the range quotients of these: $\mathcal{J}P_r$. The cubes show up as the result of taking both quotients simultaneously.

3.2. **Permutohedra.** First we prove that the poset of painted trees made by grafting a weakly ordered forest to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with n leaves this polytope is the permutohedron $\mathcal{P}_n$. It is well known (see [18]) that the permutohedron has faces indexed by the weak orders, which in turn may be represented by weakly ordered trees. The face poset is the partial ordering of these trees by refinements.

3.9. **Theorem.** *There is an isomorphism φ from the poset of $(n+1)$-leaved weakly ordered trees to the painted growth preorder of n-leaved weakly ordered forests grafted to weakly ordered trees.*

Proof. The isomorphism and its inverse are described as switching between the paint line and an extra branch. Given a weakly ordered tree t, we find $\varphi(t)$ by adding a paint line at the level of left-most node of t, and then deleting the left-most branch of t. This process clearly describes a set isomorphism, since the inverse is easy to perform. Given a weakly ordered forest grafted to a weakly ordered tree, painted to make the graft visible, we attach a new branch to the left of the tree, precisely at the level of the paint line. Then the paint is forgotten and the unpainted result is a weakly ordered tree.

Next we argue that the isomorphism just described respects the poset structures. If $a \prec b$ for two weakly ordered trees, we have that the weak ordering of the nodes of a is a refinement of the weak ordering for b. We can visualize this refinement as the growing of some internal edges of a to break ties between nodes that were at the same level. If the refinement involves breaking a tie that does not include the left-most node, then the same growing produces the same relation between the painted tree images $\varphi(a)$ and $\varphi(b)$. If the growing does break a tie involving the left-most node, then the image of b may differ from that of a only in that the set of nodes of $\varphi(a)$ which coincide with the paint line will be a subset of those in $\varphi(b)$. This can be seen as a growing of some painted edges at some half-painted nodes. □

This theorem immediately implies that the poset of n-leaved weakly ordered forests grafted to weakly ordered trees is isomorphic to the face poset of the n-dimensional permutohedron. That is because the poset of $(n+1)$-leaved weakly ordered trees is well known to represent the face poset of the permutohedron (via seeing each tree as a weak order of $[n]$, that is, an ordered partition.)

A corollary, from [8], is that the poset of n-leaved weakly ordered forests grafted to weakly ordered trees is isomorphic to the face poset of the n-dimensional graph multiplihedron of the complete graph.

3.3. **Stellohedra.** Now we prove that the poset of painted trees made by grafting a forest of corollas to a weakly ordered base tree is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is the graph-associahedron $\mathcal{K}G$ where G is the star graph St_n.

Recall that the star graph St_n is defined as follows: we use the set $\{0, 1, 2, \ldots, n\}$ as the set of nodes. Edges are $\{0, i\}$ for $i = 1, \ldots, n$.

3.10**. Theorem.** *The poset of tubings on the star graph St_n is isomorphic to the poset of n-leaved forests of corollas grafted to weakly ordered trees.*

Proof. We use the facts shown in [6]: that the permutohedron is combinatorially equivalent to the graph-associahedron of the complete graph, and that the simplex is combinatorially equivalent to the graph-associahedron of the of the edgeless graph, which in turn is equivalent to the Boolean lattice of subsets of its nodes. Recall that the permutahedron is also indexed by the weakly ordered trees, leading to an isomorphism between tubings and trees as seen in Figures 15 and 16.

We first note that any tubing T of the star graph includes a unique smallest tube t_0 which contains node 0. All other tubes of T are either contained in t_0 or contain t_0, since the node 0 is adjacent to all other nodes. The tubes contained in t_0 form a tubing of an edgeless graph. The tubes containing t_0 form a tubing on the reconnected complement of t_0, which is the complete graph on the nodes not in t_0.

Thus the bijection from the tubing on the star graph to the tree is formed of the bijection from tubings on a complete graph to weakly ordered trees, together with the bijection from tubings on an edgeless graph to subsets of $[n]$. The tube t_0 plays the same role as the paint line in the corresponding tree. The nodes $1, \ldots, n$ of the star graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. The tubing outside of t_0 maps to the painted weakly ordered tree, the tubing inside t_0 maps to a subset of $[n]$ which is the subset of the gaps-between-leaves that end in nodes of unpainted corollas, and nodes that are inside t_0 but not inside any smaller tube determine the gaps-between-leaves that coincide with the paint line. An example is seen in Figure 17.

The fact that this bijection preserves the ordering again follows easily from the definitions. □

The isomorphism in 3d is shown pictorially in Figure 18.

Next we show that the stellohedra can also be seen as the domain and range quotients $\mathcal{J}G_d$ and $\mathcal{J}G_r$ of the multiplihedron $\mathcal{J}G$ where G is the complete graph.

3.11**. Theorem.** *The graph-composihedron for a complete graph $\mathcal{K}_n$ is combinatorially equivalent to the stellohedron for the star-graph St_{n+1}.*

Proof. We can most easily see the isomorphism by using the stellohedra just found in Theorem 3.10 that is by showing an isomorphism to painted trees.

We show a bijection from the graph-composihedron to the forests of corollas grafted to weakly ordered trees. The nodes $1, \ldots, n$ of the complete graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree.

A broken tube t_0 plays the same role as the half-painted nodes in the corresponding tree. The nodes $1, \ldots, n$ of the complete graph correspond to the gaps-between-leaves $1, \ldots, n$ of the tree. The nodes inside the single thin tube, if it exists, map to the unordered gaps-between-leaves that end in nodes of the unpainted corollas. Nodes that are inside a broken tube but not inside any thin tube determine the gaps-between-leaves that coincide with the paint line. Any nodes outside of all the thin or broken tubes map to the weakly ordered base tree, and this mapping is via the previously mentioned bijection between weak orders and tubings on the complete graph. Note that the reconnected complement of the largest thin or broken tube is a complete graph of the right size. An example of the bijection is seen in Figure 19.

The fact that this bijection preserves the ordering follows easily from the definitions. Note that the relations are simpler than in general for marked tubes since the tubings must all be completely nested. Adding or resolving a tube (and forgetting tubing structure that is ignored by the quotient from the marked tubing to the cubahedron or composihedron) corresponds to growing edges and forgetting the corresponding tree structure. 3d examples are seen in Figure 21. See Figure 24 for some isomorphic chains. □

Moreover, we will show that the poset of painted trees made by grafting a weakly ordered forest to a base corolla is the face poset of a polytope. It turns out that for painted trees with $n+1$ leaves this polytope is again the graph-associahedron $\mathcal{K}G$ where G is the star graph St_n.

First, however, we show a bijection from the range-quotients of the complete graph multiplihedron (the complete graph-cubeahedron) to the weakly ordered forests grafted to corollas.

3.12. **Theorem.** *The poset of $n+1$-leaved weakly ordered forests grafted to corollas is combinatorially equivalent to the graph-cubeahedron for a complete graph $\mathcal{K}_n$.*

Proof. This proof follows the pattern of the previous one. Now however any nodes outside of the broken t_0 tube are mapped to the painted corolla base tree. The tubing inside a largest thin tube (which contains a clique) maps to the gaps-between-leaves that end in nodes of the unpainted weakly ordered forest, and nodes that are inside a broken tube but not inside any thin tube determine the gaps-between-leaves that coincide with the paint line. An example of the bijection is seen in Figure 20.

The fact that this bijection preserves the ordering is seen just as in the proof of Theorem 3.11. 3d examples are seen in Figure 22. See Figure 24 for some isomorphic chains. □

We now can finish with the following:

3.13. **Theorem.** *The weakly ordered forests grafted to corollas are isomorphic to the stellohedra.*

Proof. By Theorem 62 of [22], the graph-cubeahedron for a complete graph $\mathcal{K}_n$ is combinatorially equivalent to the stellohedron for the star-graph St_{n+1}. If the star graph St_{n+1} has node 0 as its center, and the nodes of the complete graph are $1, \dots, n$, then a square tube on the complete graph is mapped to itself, as a round tube; and round tubes on the complete graph are mapped to their complement plus the node 0 on the star graph. We demonstrate this isomorphism in Figure 24. Thus the theorem is shown, by composition with the isomorphism in our Theorem 3.12. □

3.4. **Enumeration.** The number of vertices of the stellahedron is worked out in several places, including [14], where the formula is given:

$$v(n) \;=\; \sum_{k=0}^{n} k!\tbinom{n}{k} \;=\; \sum_{k=0}^{n} n!/k!\,,$$

which is sequence A000522 in the OEIS [26]. This is the binomial transform of the factorials.

Now for the facets. We will use the following, possibly well-known

3.14. **Theorem.** *The bipartite graph associahedron $\mathcal{K}K_{m,n}$ has $2^{m+n}+(m+n)-(2^m+2^n)$ facets.*

To see this, we will count subsets of nodes which give valid tubes. We will over-count and then correct. Let $K_{m,n} = (V_1 \cup V_2, E)$ where $|V_1| = m$ and $|V_2| = n$. Note that the only subsets S of nodes which do not give valid tubes are S such that $S \subseteq V_1$ (or V_2) with $|S| > 1$. These are simply empty graphs with $|S| > 1$ nodes, and do not constitute valid tubes. For the moment, let $S \subseteq V_1$ where $|V_1| = m$.

Let $M = \#\{S \subseteq V_1 : |S| > 1\}$. Now

$$M = \sum_{2}^{k} \binom{m}{k} = 2^m - (m+1)$$

and by the above, there are $2^n-(n+1)$ "bad subsets" that can be chosen from V_2 for a total of $2^m+2^n-(m+n-2)$ bad subsets of $V=V_1\cup V_2$.

It follows that we may choose any of $2^{m+n}-2$ proper, nonempty subsets of V, and subtract off the bad choices for the total number of tubes. Thus, $K_{m,n}$ has

$$\begin{aligned}2^{m+n}-2-(2^m+2^n-(m+n-2)) &= 2^{m+n}-2-(2^m+2^n)+m+n+2\\ &= 2^{m+n}+(m+n)-(2^m+2^n)\end{aligned}$$

tubes.

Note that, for $K_{1,n-1}$, we see that star graphs on n nodes have

$$\begin{aligned}2^n+n-(2^{n-1}+2) &= 2^n-2^{n-1}+n-2\\ &= 2^{n-1}(2-1)+n-2\\ &= 2^{n-1}+n-2\end{aligned}$$

tubes.

4. Associative products on the Stellohedra

4.1. **Preliminaries.** Let $\mathbb{K}$ denote a field. For any set X, we denote by $\mathbb{K}[X]$ the $\mathbb{K}$-vector space spanned by X. In what follows the set X will be the collection of painted trees that has a forest of corollas grafted to a (weakly) ordered tree, or alternately the collection of tubings on the star graphs.

For $n\geq 1$, we denote by Σ_n the group of permutations of n elements. For any set $U=\{u_1,\dots,u_n\}$ with n elements, an element $\sigma\in\Sigma_n$ acts naturally on the left on U and induces a total order $u_{\sigma^{-1}(1)}<\cdots<u_{\sigma^{-1}(n)}$ on U.

For nonnegative integers n and m, let $\mathrm{Sh}(n,m)$ denote the set of (n,m)-shuffles, that is the set of permutations σ in the symmetric group Σ_{n+m} satisfying that:

$$\sigma(1)<\cdots<\sigma(n) \qquad \text{and} \qquad \sigma(n+1)<\cdots<\sigma(n+m).$$

For $n=0$, we define $\mathrm{Sh}(0,m):=\{1_m\}=:\mathrm{Sh}(m,0)$, where 1_m is the identity of the group Σ_m. More in general, for any composition $(n_1,\dots,n_r)$ of n, we denote by $\mathrm{Sh}(n_1,\dots,n_r)$ the subset of all permutations σ in Σ_n such that $\sigma(n_1+\cdots+n_i+1)<\cdots<\sigma(n_1+\cdots+n_{i+1})$, for $0\leq i\leq r-1$.

The concatenation of permutations $\times:\Sigma_n\times\Sigma_m\hookrightarrow\Sigma_{n+m}$ is the associative product given by:

$$\sigma\times\tau(i):=\begin{cases}\sigma(i), & \text{for } 1\leq i\leq n,\\ \tau(i-n)+n, & \text{for } n+1\leq i\leq n+m,\end{cases}$$

for any pair of permutations $\sigma\in\Sigma_n$ and $\tau\in\Sigma_m$.

The well-known associativity of the shuffle states that:

$$\mathrm{Sh}(n+m,r)\cdot(\mathrm{Sh}(n,m)\times 1_r)=\mathrm{Sh}(n,m,r)=\mathrm{Sh}(n,m+r)\cdot(1_n\times\mathrm{Sh}(m,r)),$$

where $\cdot$ denotes the product in the group Σ_{p+q+r}.

For $n\geq 1$, recall that the star graph St_n is a simple connected graph with set of nodes $\mathrm{Nod}(St_n)=\{0,1,\dots,n\}$ and whose edges are given by $\{0,i\}$ for $i=1,\dots,n$.

That is St_n is the suspension of the graph with n nodes and no edges.

4.2. **Notation.** *For any maximal tubing T of St_n such that $T \neq \{\{1\}, \dots, \{n\}\}$, there exists a unique integer $0 \leq r \leq n$, a unique family of integers $1 \leq u_1 < \cdots < u_r \leq n$ and an order $\{u_{r+1}, \dots, u_n\}$ on the set $\{1, \dots, n\} \setminus \{u_1, \dots, u_r\}$ such that*

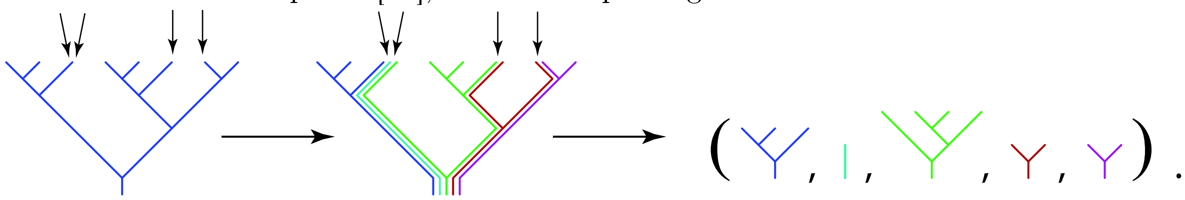
$$T = \{\{u_1\}, \dots, \{u_r\}, t_0, t_0 \cup \{u_{r+1}\}, \dots, t_0 \cup \{u_{r+1}, \dots, u_{n-1}\}\},$$

where $t_0 := \{0, u_1, \dots, u_r\}$.

We denote such tubing T by $Tub_r(u_1, \dots, u_n)$, where u_n is the unique vertex which does not belong to any tube of T. We denote the tubing $\{\{1\}, \dots, \{n\}\} = Tub_n(1, \dots, n)$ by Tub_n.

In [15] there are defined coproducts on nine of the families of painted trees shown in Figure 5 (the ones with labels denoting their membership in a composition of coalgebras). Eight of these, all but the composition of coalgebras $\mathfrak{S}Sym \circ \mathfrak{S}Sym$, are shown to possess various Hopf algebraic structures in [15]. For the example of $\mathfrak{S}Sym \circ \mathfrak{C}Sym$ this product can be easily transferred to the vertices of the stellohedra via the bijection we have developed here. It is also feasible to extend the definition of this product to all the faces of the stellohedra, which we point out in Section 6.

Recall from [15] the concept of splitting a tree, given a multiset of its leaves. Here, modified from an example in [13], is a 4-fold splitting into an ordered list of 5 trees:



Note that a k-fold splitting, which is given by a size k multiset of the $n+1$ leaves, corresponds to a (k, n) shuffle. The corresponding shuffle is described as follows: $\sigma(i)$ for $i \in 1, \dots, k$ is equal to the sum of the numbers of leaves in the resulting list of trees 1 through i. For instance in the above example the shuffle is $(3, 4, 8, 10, 1, 2, 5, 6, 7, 9, 11)$.

In [15] the product of two painted trees in the vector space $\mathbb{K}[X]$ is described as a sum over splits of the first tree, where after each split the resulting list of trees is grafted to the leaves of the second tree. Thus this product can be seen as a sum over shuffles. In fact if we illustrate the products using the graph tubings, then shuffles are actually more easily made visible than splittings.

Here we mainly want to show some examples of the products, referring the reader to [15] for the full definitions (in terms of the trees). For that purpose we show single terms in the product, each term relative to a shuffle. Figure 25 shows a sample product, relative to the given shuffle, and illustrating the splitting as well. In Figure 26 we show the same sample product, pictured using the tubings on the star graphs: both vertices in the stellohedra.

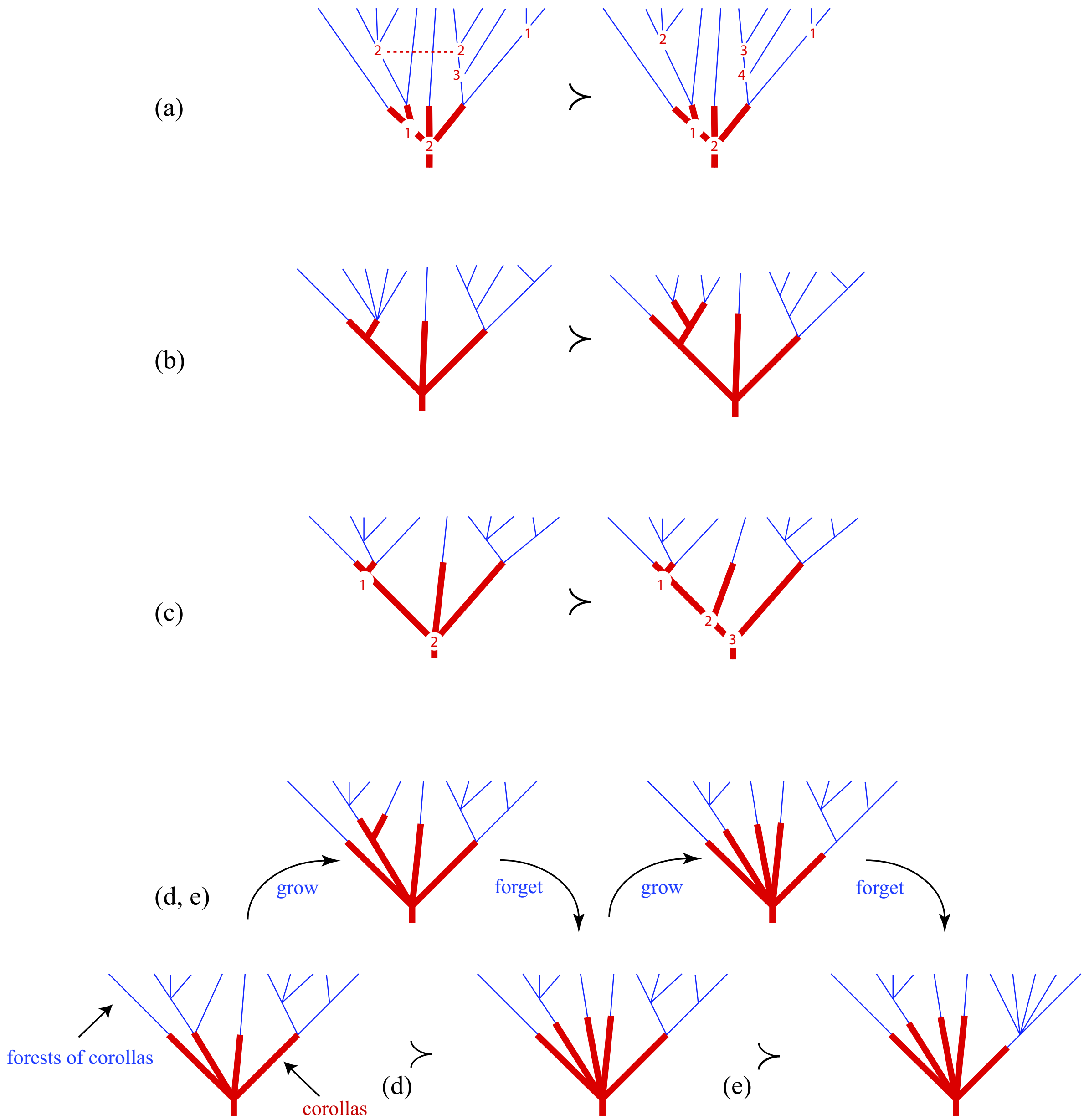


FIGURE 8. Some covering relations. In the first (a) we are looking at weakly ordered forests grafted to a weakly ordered trees, so growing an edge is a covering relation. In the next relation (b) we are looking at rooted trees above and below the graft–again no forgetting is needed. Relation (c) is in the stellohedron. At the bottom for both covering relations the forgetful map is κ. Relation (e) is in the stellohedron, although it is also true in the cube.

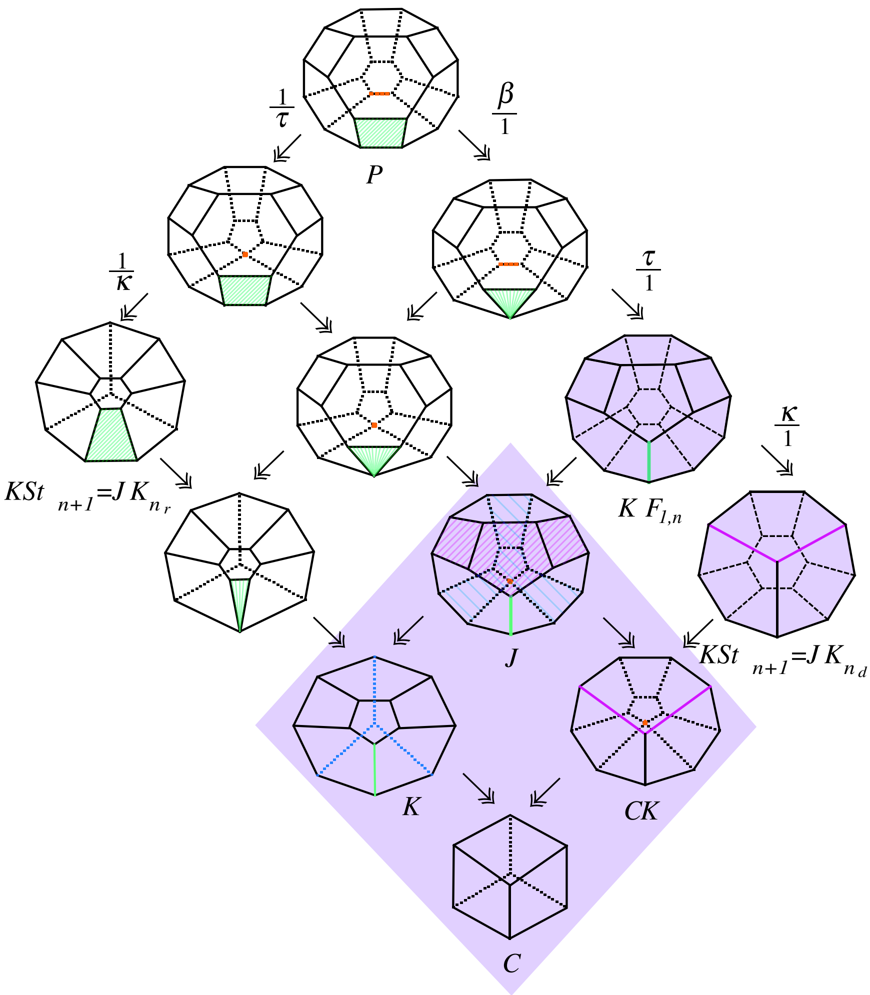


FIGURE 9. The 3d terms of some new and old polytope sequences. The four in the shaded diamond are the cube, associahedron $\mathcal{K}$, multiplihedron $\mathcal{J}$ and composihedron $\mathcal{CK}$. The other two shaded are the pterahedron $\mathcal{P}_t = \mathcal{K}F_{1,n}$ (fan graph associahedron) and the stellahedron $\mathcal{K}St$. The topmost is the permutohedron. The furthest to the left is again the stellohedron. The other four, unlabeled, are conjectured to be polytopes (clearly they are in three dimensions–the conjecture is about all dimensions.) Each of these corresponds to the type of tree shown in Figure 5, in the corresponding position.

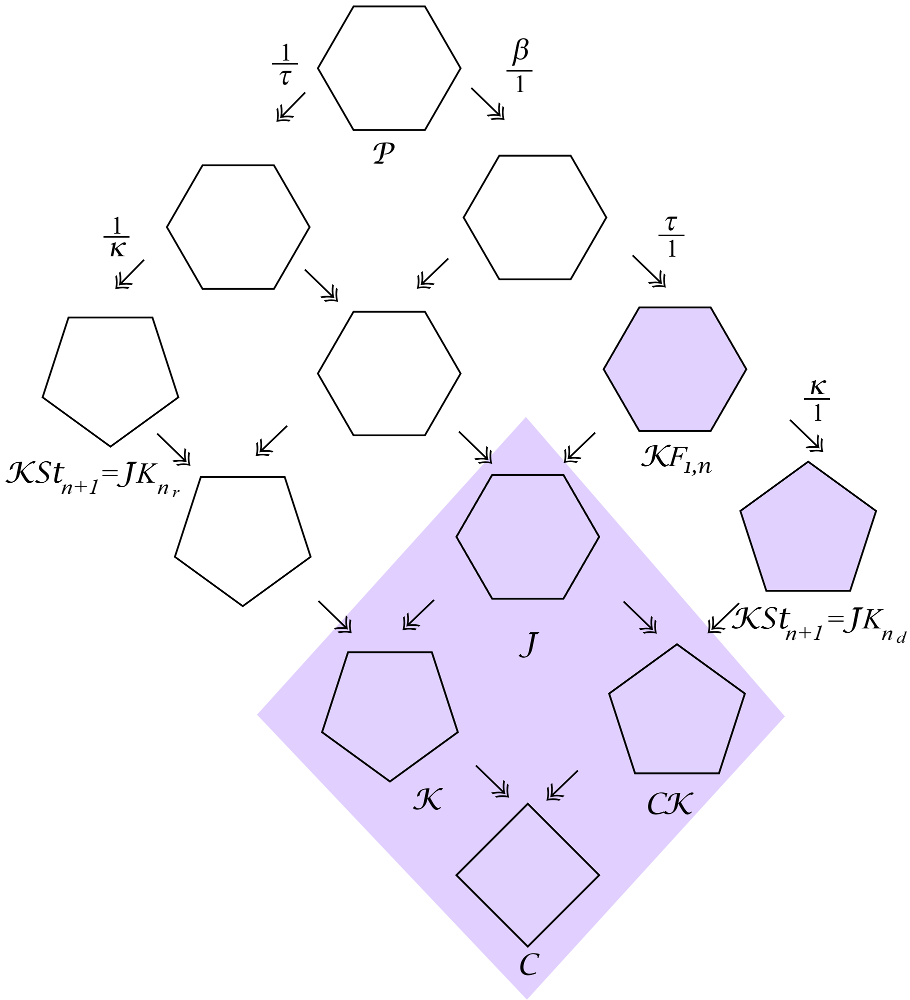


FIGURE 10. These are the 2d terms in the same sequences as in Figure 9.

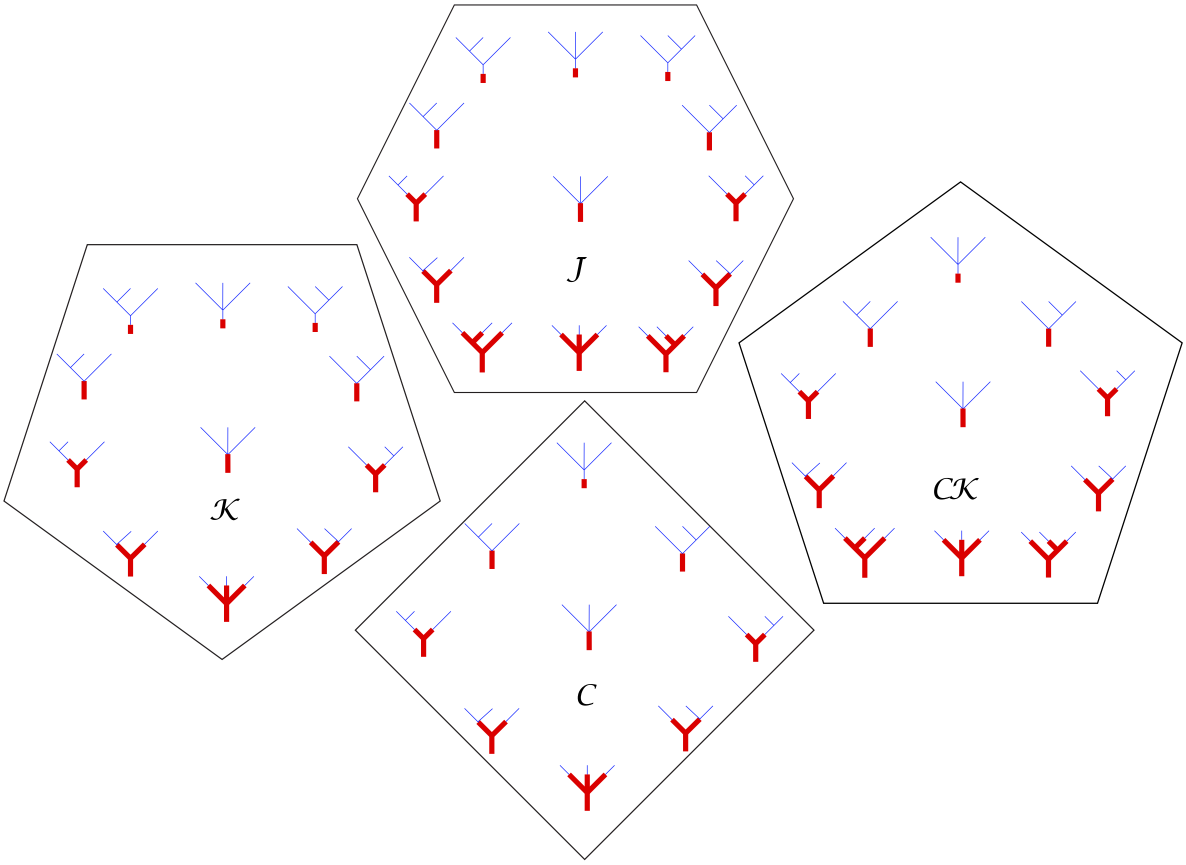


FIGURE 11. These are the 2d terms with their faces labeled. The same labels are used in 2d no matter where the shape occurs in figure 10.

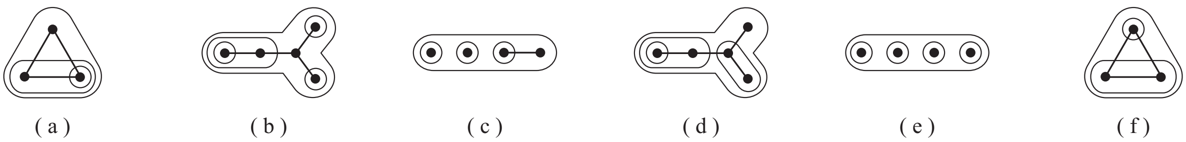


FIGURE 12. (a)-(c) Allowable tubings and (d)-(f) forbidden tubings, figure from [8].

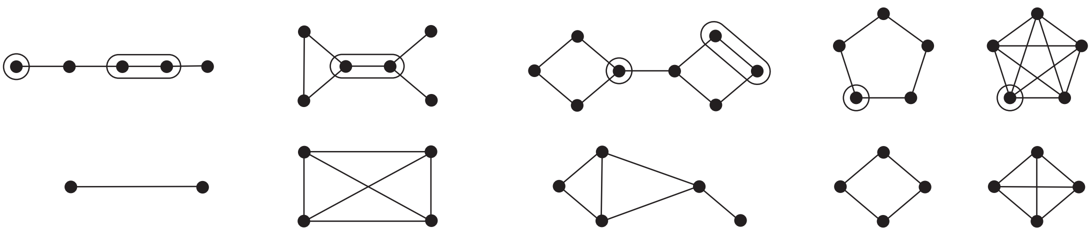


FIGURE 13. Examples of tubes and their reconnected complements. Figure from [17].

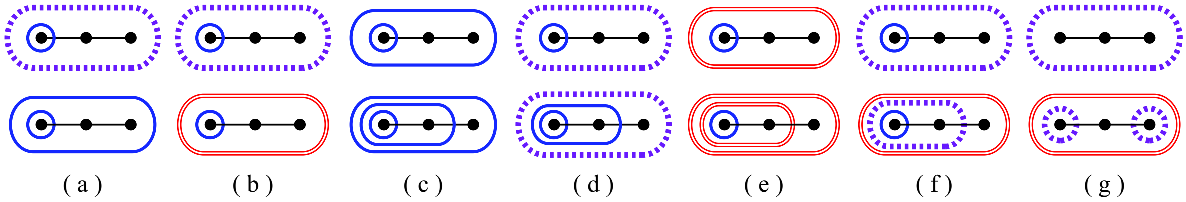


FIGURE 14. The top row are the tubings and bottom row their refinements. Figure based on original in [8].

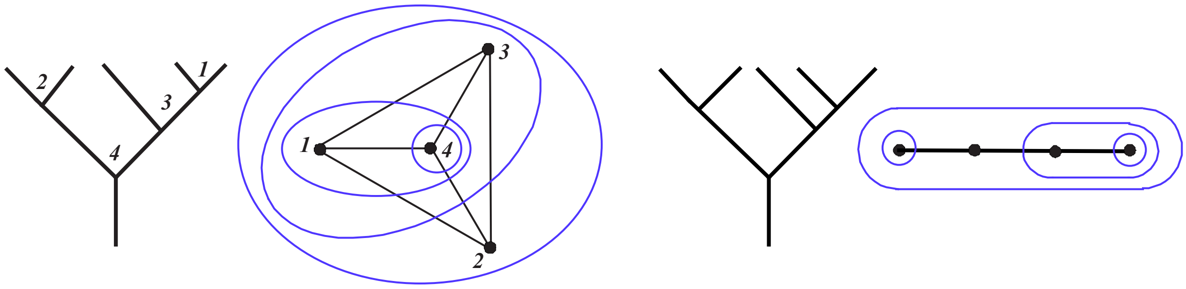


FIGURE 15. The permutation $\sigma = (2431) \in S_4$ pictured as an ordered tree and as a tubing of the complete graph; An unordered binary tree, and its corresponding tubing. Figure from [17].

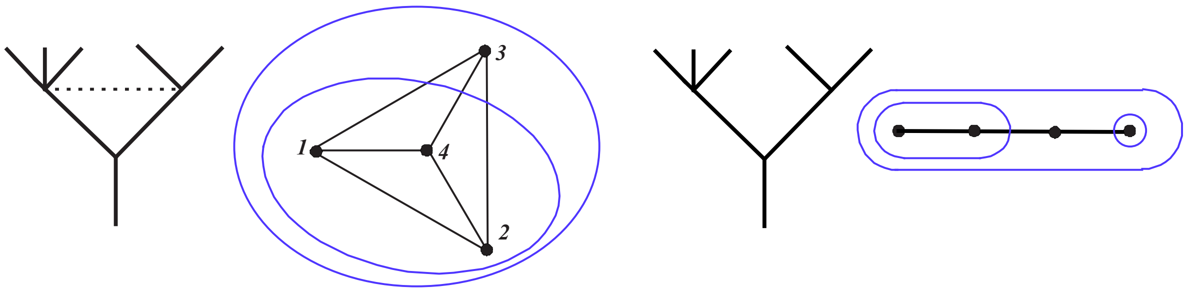


FIGURE 16. The ordered partition $(\{1, 2, 4\}, \{3\})$ pictured as a leveled tree and as a tubing of the complete graph; the underlying tree, and its corresponding tubing. Figure from [17].

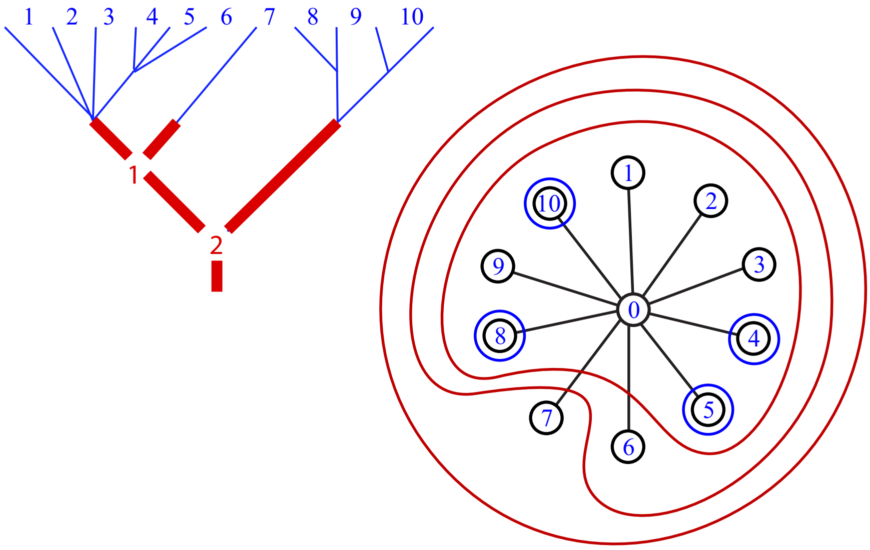


FIGURE 17. A tubing on the star graph and its bijective image in the corollas over weakly ordered trees.

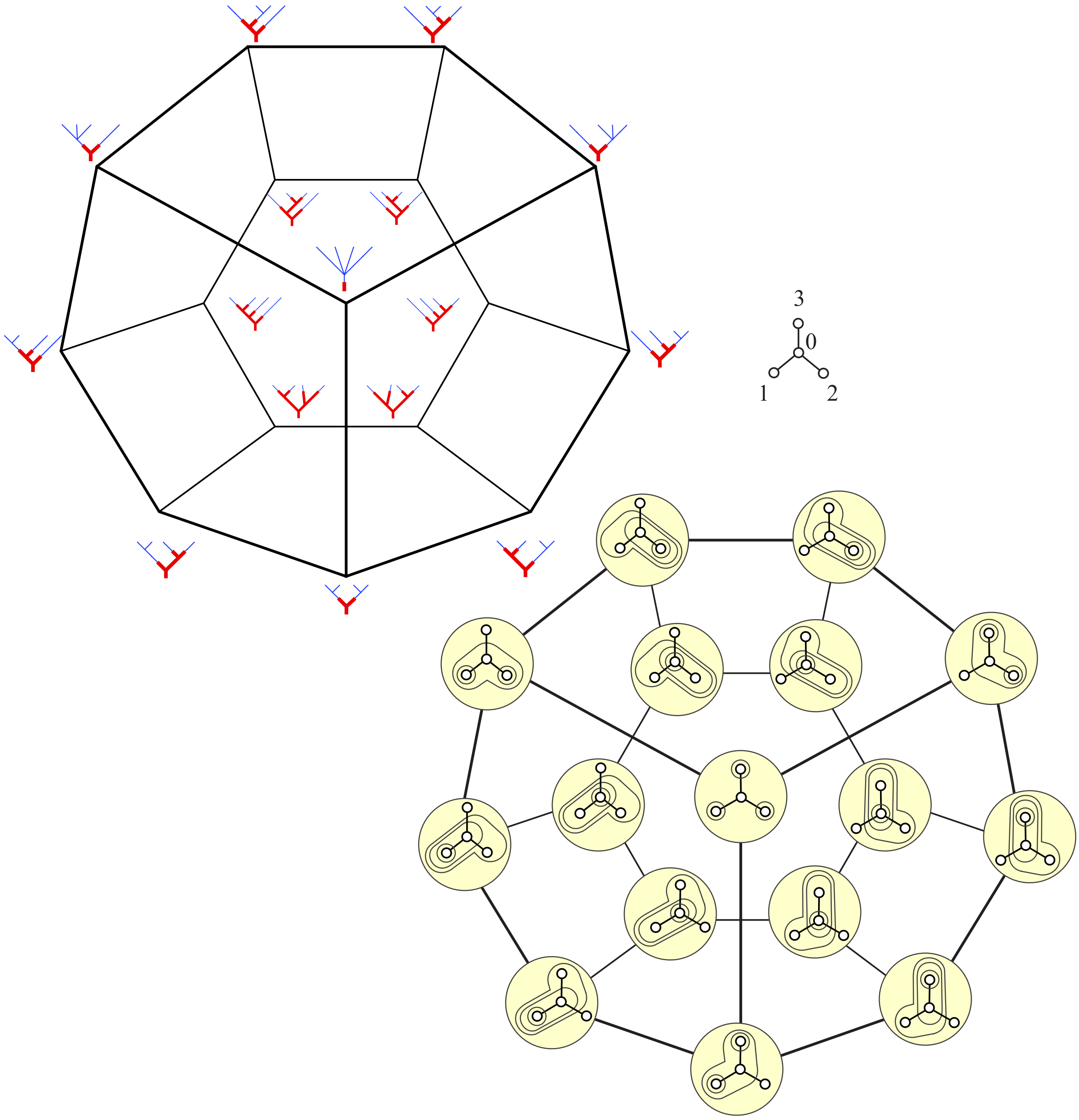


FIGURE 18. Two pictures of the stellohedron.

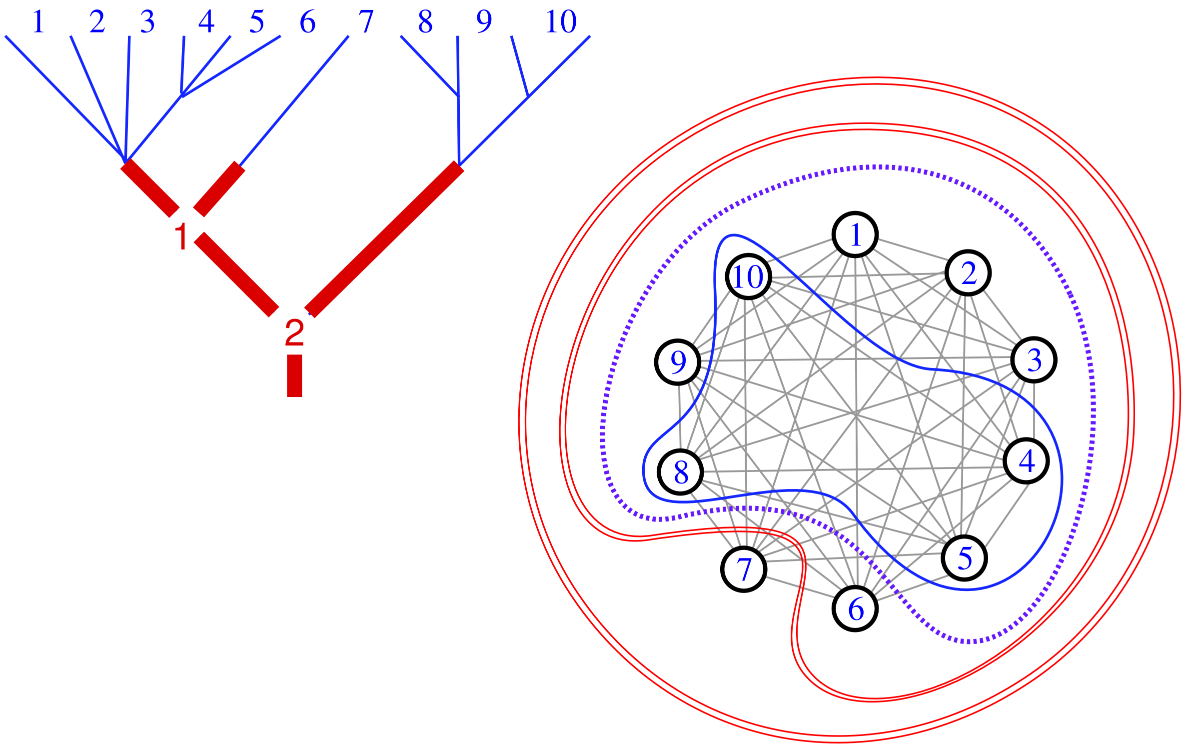


Figure 19. A marked tubing on the complete graph, representing an element of the complete-graph-composihedron (no structure is shown inside the thin tube) and its bijective image in the forest of corollas over a weakly ordered tree.

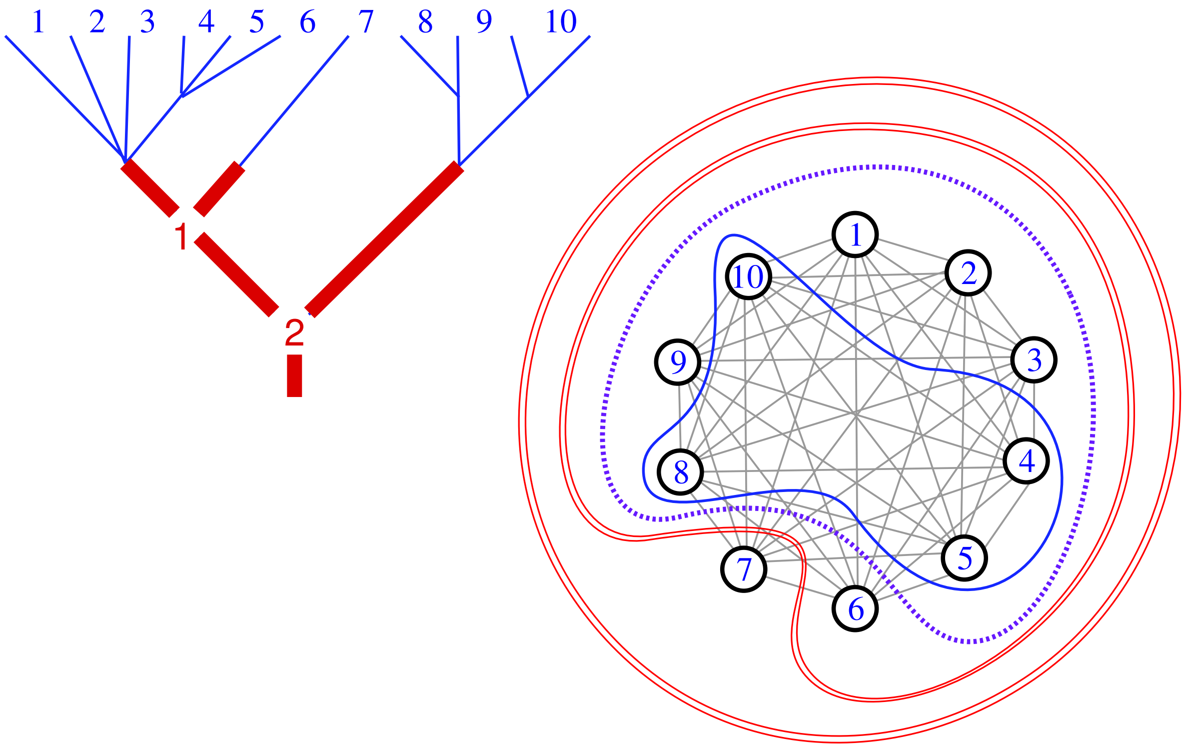


FIGURE 20. A marked tubing on the complete graph, representing an element of the complete-graph-cubahedron (no structure is shown outside the broken tube) and its bijective image in the weakly ordered forests over corollas.

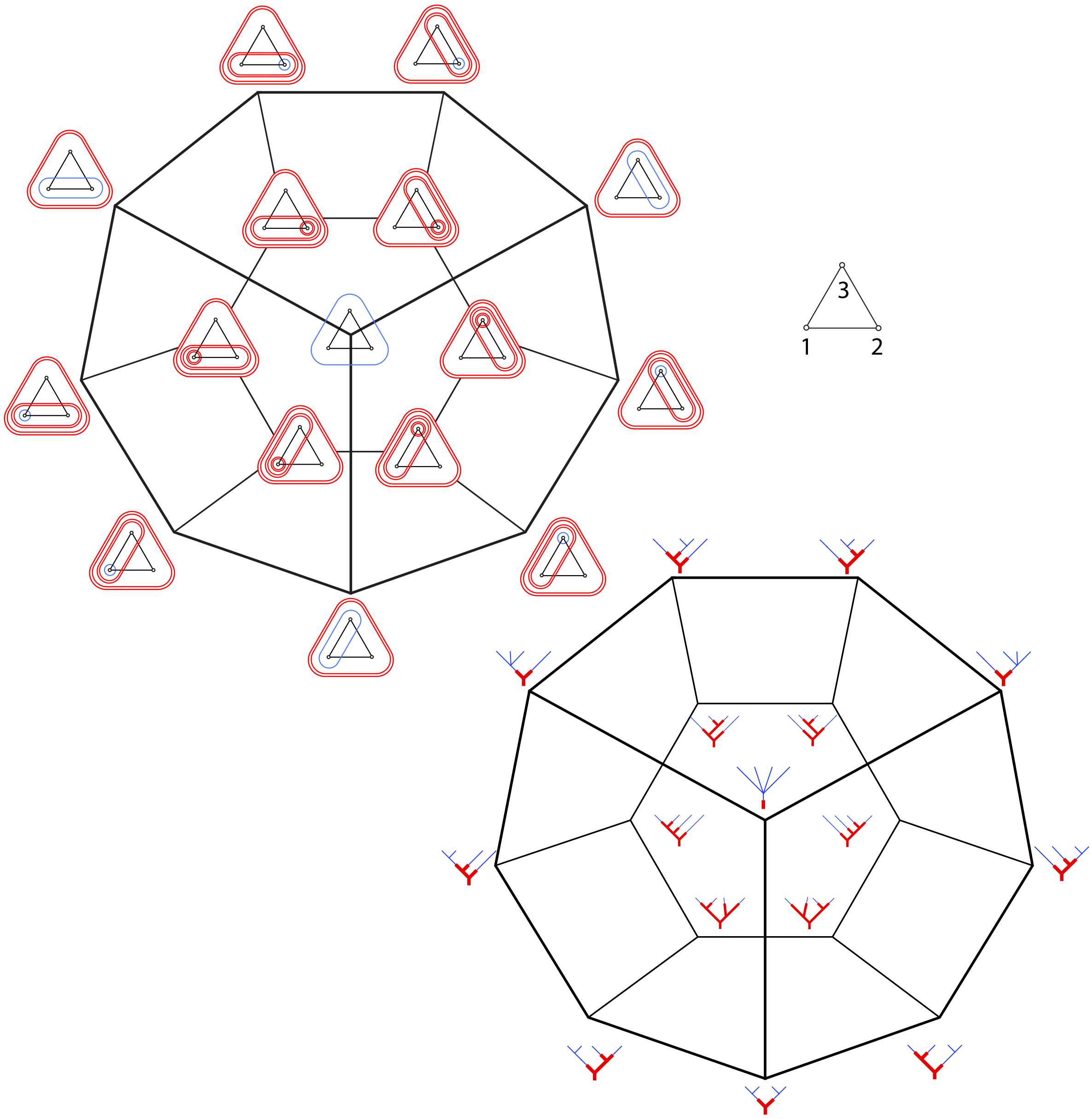


FIGURE 21. Another stellahedra bijection: when K_n is the complete graph then $\mathcal{J}K_{n_d}$ (the complete graph composihedron) is the stellohedron.

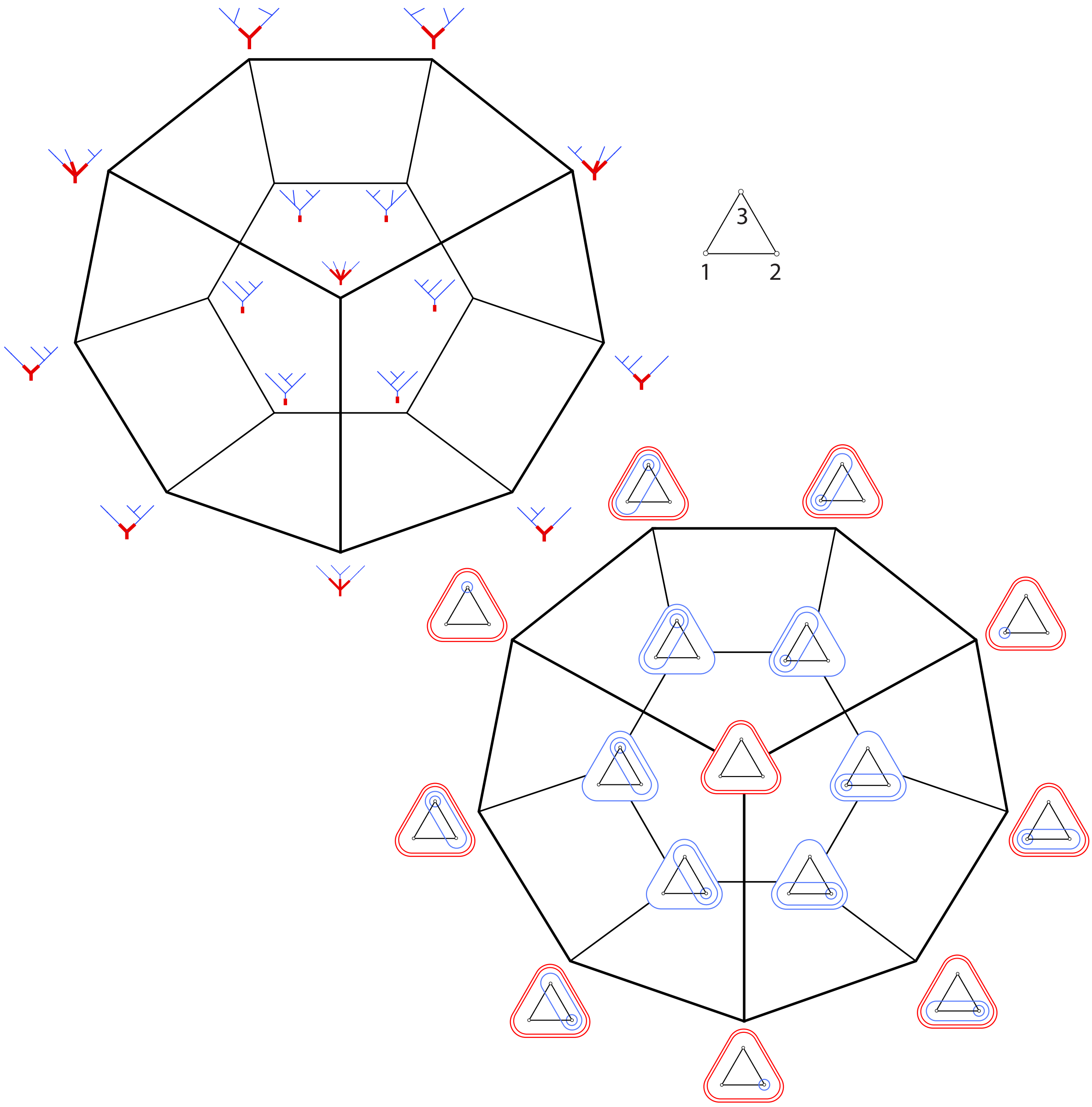


FIGURE 22. Another stellahedra bijection: the composition of ordered forests with corollas, seen in bijection with the complete graph cubeahedron.

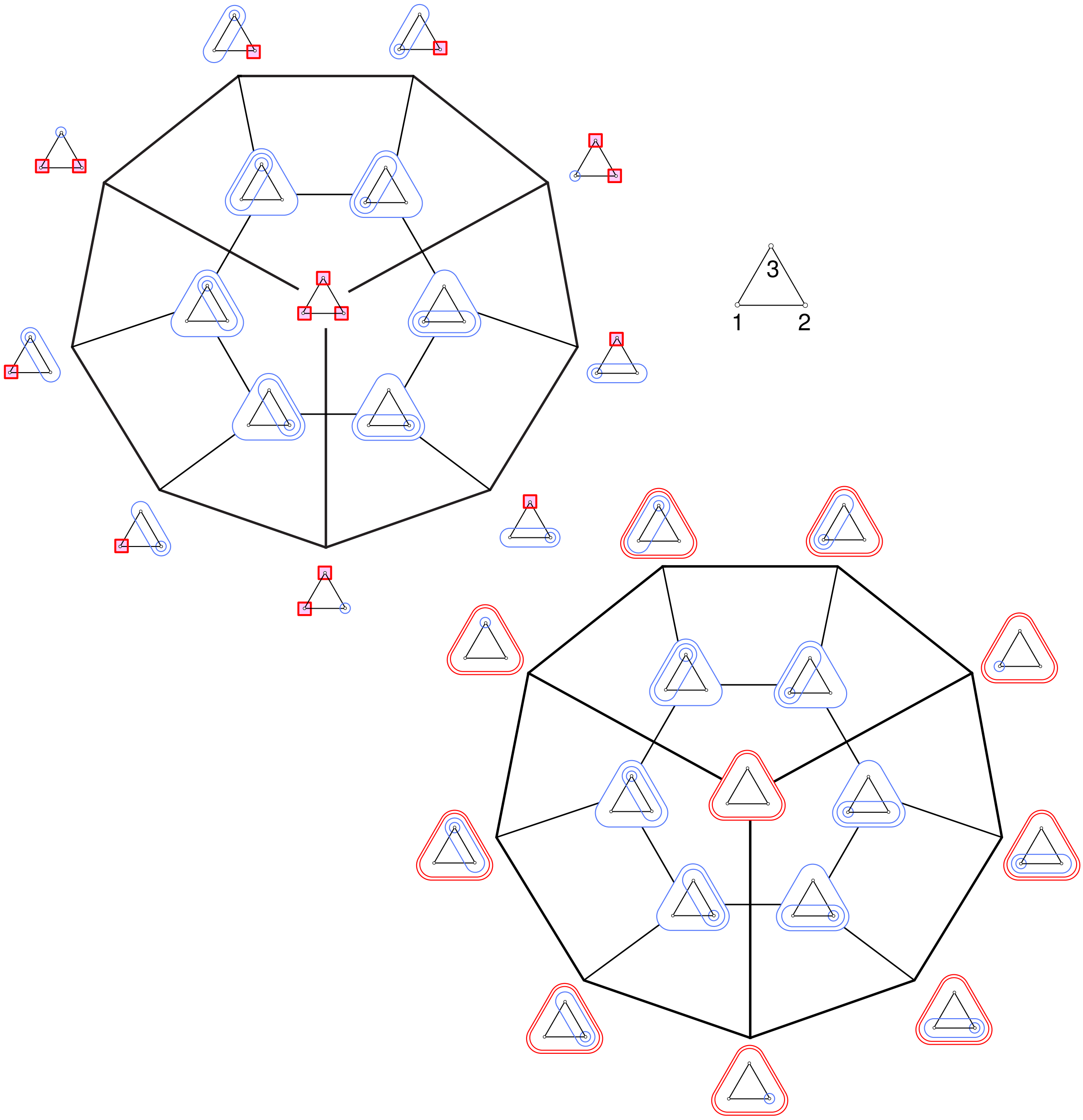


FIGURE 23. Another stellahedra bijection: the complete graph cubeahedron indexed by design tubings.

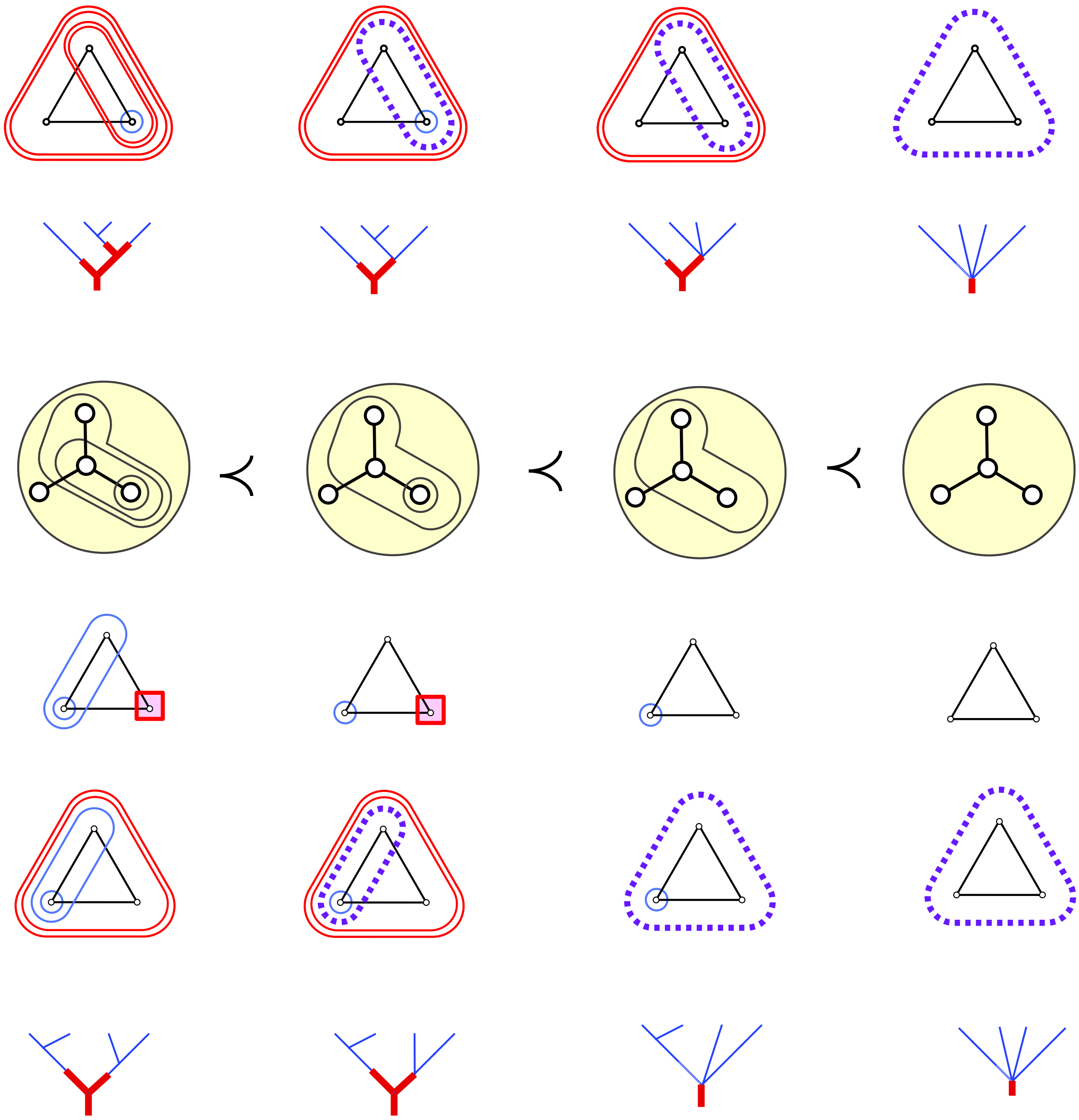


FIGURE 24. Here we bring together six isomorphic flags from the 3d stellohedra shown above.

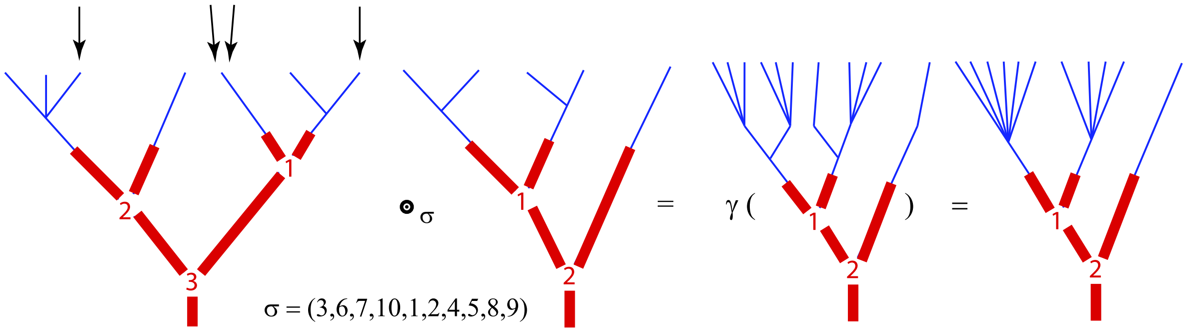


FIGURE 25. One term in the product defined in [15], relative to the given shuffle.

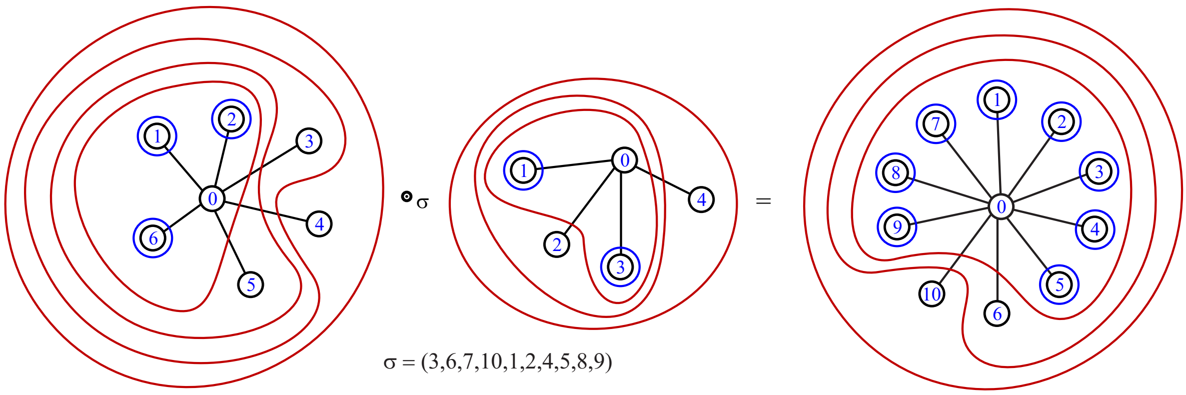


FIGURE 26. The same product as pictured in Figure 25, shown here using tubings on the star graphs.

5. Alternative product on the Stellohedra vertices

For $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$. For any set of natural numbers $U = \{u_1, \dots, u_k\}$ and any integer $r \in \mathbb{Z}$, we denote by $U + r$ the set $\{u_1 + r, \dots, u_k + r\}$.

Let G be a simple finite graph with set of nodes $\mathrm{Nod}(G) = \{j_1, \dots, j_r\} \subseteq \mathbb{N}$, we denote by $G + n$ the graph G, with the set of nodes colored by $\mathrm{Nod}(G) + n$, obtained by replacing the node j_i of G by the node $j_i + n$, for $1 \leq i \leq r$.

Let G be a simple finite graph, whose set of nodes is $[n]$, we identify a tube $t = \{v_1, \dots, v_k\}$ of G with the tube $t(h) = \{v_1 + h, \dots, v_k + h\}$ of $G + h$. For any tubing $T = \{t_i\}$ of G, we denote by $T(h)$ the tubing $\{t_i(h)\}$ of $G + h$. In the present section,

we use the shuffle product, which defines an associative structure on the vector space spanned by all the vertices of permutohedra, in order to introduce associative products (of degree -1) on the vector spaces spanned by the vertices of stellohedra.

5.1. **Definition.** Let T be a maximal tubing of St_n and V be a maximal tubing of St_m such that $T = \mathrm{Tub}_r(u_1, \ldots, u_n)$ and $V = \mathrm{Tub}_s(v_1, \ldots, v_m)$. For any $(n-r, m-s)$-shuffle $\sigma \in S_{n+m-r-s}$ define the maximal tubing $T *_\sigma V$ of S_{n+m} as follows:

$$T *_\sigma V := \mathrm{Tub}_{r+s}(u_1, \ldots, u_r, v_1{+}n, \ldots, v_s{+}n, w_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(n+m-(r+s))}),$$

where:

$$w_i := \begin{cases} u_{r+i}, & \text{for } 1 \leq i \leq n-r, \\ v_{s+i+r-n}{+}n, & \text{for } n-r < i \leq n+m-r-s. \end{cases}$$

If $T = \mathrm{Tub}_n$, then $\sigma = 1_m$ and

$$\mathrm{Tub}_n *_{1_m} V = \mathrm{Tub}_{n+s}(1, \ldots, n, v_1 + n, \ldots, v_m + n).$$

In a similar way, we have that

$$T *_{1_n} \mathrm{Tub}_m = \mathrm{Tub}_{r+m}(u_1, \ldots, u_r, n+1, \ldots, n+m, u_{r+1}, \ldots, u_n).$$

In Figure 27 we illustrate the following example.

$$\mathrm{Tub}_3(1,2,6,5,3,4) *_\sigma \mathrm{Tub}_2(1,3,2,4) = \mathrm{Tub}_5(1,2,6,7,9,5,8,3,10,4)$$

...where $\sigma = (1,3,5,2,4)$. For comparison see a product with the same operands in Figure 26.

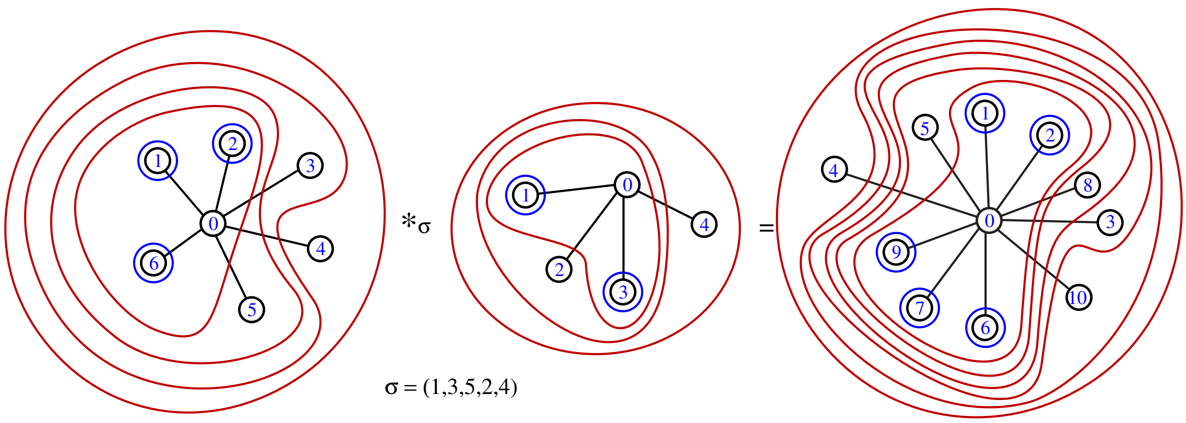


Figure 27. Example of Definition 5.1.

Using Definition 5.1, we define a shuffle product on the vector space $\mathbb{K}[\mathcal{MT}(\mathrm{St})] := \bigoplus_{n\geq 1} \mathbb{K}[\mathcal{MT}(\mathrm{St}_n)]$, where $\mathcal{MT}(\mathrm{St}_n)$ denotes the set of maximal tubings on St_n, which correspond to the vertices of stellohedra, as follows.

5.2. **Definition.** Let $T \in \mathcal{MT}(\mathrm{St}_n)$ and $V \in \mathcal{MT}(\mathrm{St}_m)$ be two maximal tubings. The product $T * V \in \mathcal{MT}(\mathrm{St}_{n+m})$ is defined as follows:

(1) If $T=\mathrm{Tub}_r(u_1,\dots,u_n)\neq \mathrm{Tub}_n$ and $V=\mathrm{Tub}_s(v_1,\dots,v_m)\neq \mathrm{Tub}_m$, then:

$$T*V:=\sum_{\sigma\in \mathrm{Sh}(n-r,m-s)} T*_\sigma V,$$

where $\mathrm{Sh}(n-r,m-s)$ denotes the set of $(n{-}r,m{-}s)$-shuffles in $S_{n+m-(r+s)}$.

(2) If $T=\mathrm{Tub}_n$ and $V=\mathrm{Tub}_m$, then:

$$T*V=\mathrm{Tub}_{n+m}.$$

Note that

(1) If $T=\mathrm{Tub}_n$ and $V=\mathrm{Tub}_s(v_1,\dots,v_m)\neq \mathrm{Tub}_m$, then:

$$T*V:=\mathrm{Tub}_{n+s}(1,\dots,n,v_1+n,\dots,v_m+n).$$

(2) If $T=\mathrm{Tub}_r(u_1,\dots,u_n)\neq \mathrm{Tub}_n$ and $V=\mathrm{Tub}_m$, then:

$$T*V:=\mathrm{Tub}_{r+m}(u_1,\dots,u_r,n+1,\dots,n+m,u_{r+1},\dots,u_n).$$

5.3. **Proposition.** *The graded vector space $\mathbb{K}[\mathcal{MT}(St)]$, equipped with the product $*$ is an associative algebra.*

Proof. Suppose that the elements $T=\mathrm{Tub}_r(u_1,\dots,u_n)$ in $\mathcal{MT}(\mathrm{St}_n)$, $V=\mathrm{Tub}_s(v_1,\dots,v_m)$ belongs to $\mathcal{MT}(\mathrm{St}_m)$ and $W=\mathrm{Tub}_z(w_1,\dots,w_p)$ belongs to $\mathcal{MT}(\mathrm{St}_p)$ are three maximal tubings, where eventually $T=\mathrm{Tub}_n$, $V=\mathrm{Tub}_m$ or $W=\mathrm{Tub}_p$.

Applying the associativity of the shuffle, we get that:

$$(T*V)*W=$$
$$\sum \mathrm{Tub}_{r+s+z}(u_1,\dots v_r,v_1{+}n,\dots,w_1{+}n+m,\dots,x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n+m+p-(r+s+z))})=$$
$$T*(V*W),$$

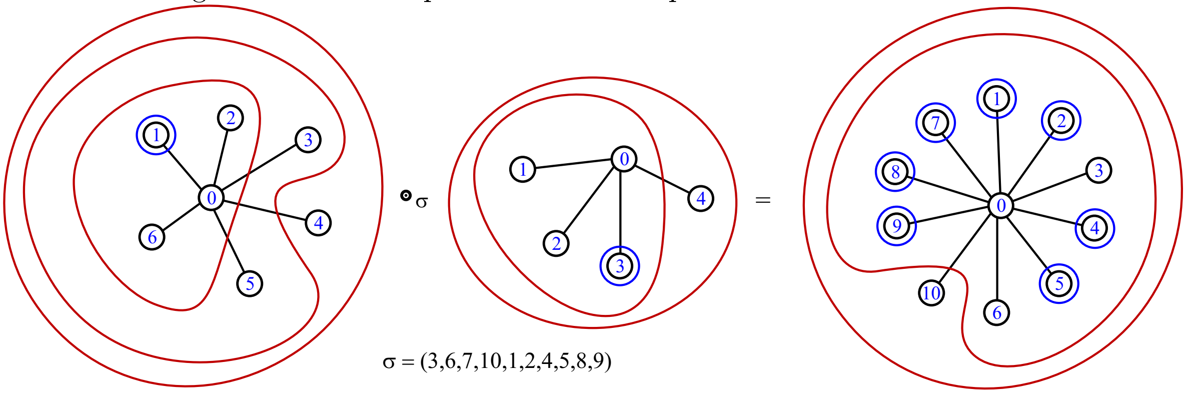
where we sum over all permutations $\sigma\in \mathrm{Sh}(n{-}r,m-s,p{-}s)$ and

$$x_i:=\begin{cases} u_{r+i}, & \text{for } 1\le i\le n{-}r,\\ v_{s+i+r-n}{+}n, & \text{for } n{-}r<i\le n+m{-}r{-}s,\\ w_{z+i+r+s-n-m}{+}n+m, & \text{for } n+m{-}r{-}s<i\le n+m+p{-}r{-}s{-}z,\end{cases}$$

which ends the proof. □

6. Questions

One advantage of seeing the product in $\mathfrak{S}Sym \circ \mathfrak{C}Sym$ using the (maximal) tubings of the star graph is that it is immediately clear how to extend the product to all tubings of the star graphs: that is, to all the faces of the stellohedron. For any shuffle, we simply enforce that the nodes in the product term corresponding to the nodes in the first operand (picked out by that shuffle) are all singleton tubes. The remaining nodes reflect the tubing of the second operand. For example:



We conjecture that this product, defined as a sum over shuffles, defines an algebra of faces of the stellohedra. There are already extensions of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ to Hopf algebras based on all of the faces of the permutohedron and associahedron. These were first described by Chapoton, in [7], along with a Hopf algebra of the faces of the hypercubes. We realize the first two Hopf algebras using the graph tubings in [17]. They are denoted $\mathfrak{S}\tilde{S}ym$ and $\mathcal{Y}\tilde{S}ym$ respectively, and so we refer to Chapoton's algebra of the faces of the cube as $\mathfrak{C}\tilde{S}ym$.

Immediately the question is raised: how can we relate the coalgebra $\mathfrak{S}\tilde{S}ym \circ \mathfrak{C}\tilde{S}ym$ to our conjectured algebra of stellohedra faces? Can either or both be given the full structure of a Hopf algebra?

Further questions arise as we look at the other polytopes in our set of 12 sequences. (Of course, recall that 4 of them are only conjecturally convex polytope sequences.) For instance, via our bijection there is a Hopf algebra based on the weakly ordered forests grafted to corollas–which we would like to characterize in terms of known examples. Also, by following the projections in Figure 9, we could easily define products on the composihedron and hypercube faces. We conjecture that these too are associative products. More questions are raised: for example, how can we relate the product thus discovered on the faces of the hypercube to Chapoton's Hopf algebra $\mathfrak{C}\tilde{S}ym$?

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but encourages reformers of the NSA who are working to ensure that protections of civil liberties keep pace with intelligence capabilities.

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