ex) Find $N(T)$ and $R(T)$ as spans of polynomials, where $T: P^3 \to P^2$ is given by $T(f(x)) = 3f'(x) + f''(x)$

$$[T]_\mathcal{E}^\mathcal{E} = \left[[T(1)]_\mathcal{E} \quad [T(x)]_\mathcal{E} \quad [T(x^2)]_\mathcal{E} \quad [T(x^3)]_\mathcal{E} \right]$$

$$= \left[[0]_\mathcal{E} \quad [3]_\mathcal{E} \quad [6x+2]_\mathcal{E} \quad [9x^2 + 6x]_\mathcal{E} \right]$$

$$= \begin{bmatrix} 0 & 3 & 2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

r.r.

$$\left[\begin{array}{cccc|c} 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & -2/3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

↑ free   ︶ pivots

$\left. \begin{array}{l} C_1 = C_1 \\ C_2 = 0 \\ C_3 = 0 \\ C_4 = 0 \end{array} \right\rangle$ $N(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \boxed{\text{span} \{ 1 \}}$

$$R(T) = \text{span} \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 9 \end{pmatrix} \right\}$$

$$= \boxed{\text{span} \{ 3, \ 2+6x, \ 6x + 9x^2 \}}$$

nullity $(T) = \boxed{1}$,   rank $(T) = \boxed{3}$