1. Associative products on stellohedra and pterohedra

Let G be a simple finite graph. The *suspension* of G is the graph $\mathfrak{S}G$ whose set of nodes is obtained by adding a node 0 to the set $\mathrm{Nod}(G)$, of nodes of G, and whose edges are defined as follows:

$$\mathrm{Edg}(\mathfrak{S}G)(i,j) := \begin{cases} \mathrm{Edg}(G)(i,j), & \text{for } i,j \in \mathrm{Edg}(G), \\ \#, & \text{for } i = 0 \text{ and } j \in \mathrm{Nod}(G). \end{cases}$$

Note that the star graph S_n is the suspension of the graph C_n with has n nodes and no edge, while the fan graph $F_{1,n}$ is the suspension of the line graph L_n.

The reconnected complement of $\{0\}$ in $\mathfrak{S}G$ is the complete graph $\mathcal{K}_n$ for any graph G with n nodes. In the present section, we use the shuffle product, which defines an associative structure on the vector space spanned by all the vertices of permutohedra, in order to introduce associative products (of degree -1) on the vector spaces spanned by the vertices of stellohedra and pterohedra.

1.1. **Preliminaries.**

Let $\mathbb{K}$ denote a field. For any set X, we denote by $\mathbb{K}[X]$ the $\mathbb{K}$-vector space spanned by X.

For $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$. For any set of natural numbers $U = \{u_1, \dots, u_k\}$ and any integer $r \in \mathbb{Z}$, we denote by $U + r$ the set $\{u_1 + r, \dots, u_k + r\}$.

Let G be a simple finite graph with set of nodes $\mathrm{Nod}(G) = \{j_1, \dots, j_r\} \subseteq \mathbb{N}$, we denote by $G + n$ the graph G, with the set of nodes coloured by $\mathrm{Nod}(G) + n$, obtained by replacing the node j_i of G by the node $j_i + n$, for $1 \leq i \leq r$.

Let G be a simple finite graph, whose set of nodes is $[n]$, we identify a tube $t = \{v_1, \dots, v_k\}$ of G with the tube $t(h) = \{v_1 + h, \dots, v_k + h\}$ of $G + h$. For any tubing $T = \{t_i\}$ of G, we denote by $T(h)$ the tubing $\{t_i(h)\}$ of $G + h$.

For $n \geq 1$, we denote by Σ_n the group of permutations of n elements. For any set $U = \{u_1, \dots, u_n\}$ with n elements, an element $\sigma \in \Sigma_n$ acts naturally on U and induces a total order $u_{\sigma(1)} < \cdots < u_{\sigma(n)}$ on U.

For nonnegative integers n and m, let $\mathrm{Sh}(n,m)$ denote the set of (n,m)-shuffles, that is the set of permutations σ in the symmetric group Σ_{n+m} satisfying that:

$$\sigma(1) < \cdots < \sigma(n) \qquad \text{and} \qquad \sigma(n+1) < \cdots < \sigma(n+m).$$

For $n = 0$, we define $\mathrm{Sh}(0,m) := \{1_m\} =: \mathrm{Sh}(m,0)$, where 1_m is the identity of the group Σ_m. More in general, for any composition $(n_1, \dots, n_r)$ of n, we denote by $\mathrm{Sh}(n_1, \dots, n_r)$ the subset of all permutations σ in Σ_n such that $\sigma(n_1 + \cdots + n_i + 1) < \cdots < \sigma(n_1 + \cdots + n_{i+1})$, for $0 \leq i \leq r - 1$.

The concatenation of permutations $\times : \Sigma_n \times \Sigma_m \hookrightarrow \Sigma_{n+m}$ is the associative product given by:

$$\sigma \times \tau(i) := \begin{cases} \sigma(i), & \text{for } 1 \leq i \leq n, \\ \tau(i-n)+n, & \text{for } n+1 \leq i \leq n+m, \end{cases}$$

for any pair of permutations $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$.

The well-known associativity of the shuffle states that:

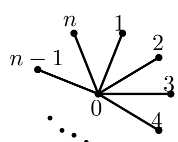
$$\mathrm{Sh}(n+m,r)\cdot(\mathrm{Sh}(n,m)\times 1_r) = \mathrm{Sh}(n,m,r) = \mathrm{Sh}(n,m+r)\cdot(1_n \times \mathrm{Sh}(m,r)),$$

where $\cdot$ denotes the product in the group Σ_{p+q+r}.

1.2. **Stellohedra.** For $n \geq 1$, recall that the star graph S_n is a simple connected graph with set of nodes $\mathrm{Nod}(S_n) = \{0,1,\dots,n\}$ and whose edges are given by

$$\mathrm{Edg}(S_n)(i,j) := \begin{cases} \{\#\}, & \text{for } 0 \in \{i,j\} \text{ and } i \neq j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is S_n is the suspension of the graph with n nodes and no edge:



Notation 1.3. For any maximal tubing T of S_n such that $T \neq \{\{1\},\dots,\{n\}\}$, there exists a unique integer $0 \leq r \leq n$, a unique family of integers $1 \leq u_1 < \dots < u_r \leq n$ and an order $\{u_{r+1},\dots,u_n\}$ on the set $\{1,\dots,n\} \setminus \{u_1,\dots,u_r\}$ such that

$$T = \{\{u_1\},\dots,\{u_r\}, t_0, t_0 \cup \{u_r+1\},\dots, t_0 \cup \{u_{r+1},\dots,u_{n-1}\}\},$$

where $t_0 := \{0,u_1,\dots,u_r\}$.

We denote such tubing T by $\mathrm{Tub}_r(u_1,\dots,u_n)$, where u_n is the unique vertex which does not belong to any tube of T. We denote the tubing $\{\{1\},\dots,\{n\}\}$ by Tub_n.

Definition 1.4. Let T be a maximal tubing of S_n and W be a maximal tubing of S_m such that $T = \mathrm{Tub}_r(u_1,\dots,u_n)$ and $V = \mathrm{Tub}_s(v_1,\dots,v_m)$. For any $(n-r,m-s)$- shuffle $\sigma \in S_{n+m-r-s}$ define the maximal tubing $T *_\sigma W$ of S_{n+m} as follows:

$$T *_\sigma W := \mathrm{Tub}_{r+s}(u_1,\dots,u_r,v_1,\dots,v_s,w_{\sigma(1)},\dots,w_{\sigma(n+m-(r+s))}),$$

where:

$$w_i := \begin{cases} u_{r+i}, & \text{for } 1 \leq i \leq n-r, \\ v_{s+i+r-n}, & \text{for } n-r < i \leq n+m-r-s. \end{cases}$$

If $T = \mathrm{Tub}_n$, then $\sigma = 1_m$ and

$$\mathrm{Tub}_n *_{1_m} W = \mathrm{Tub}_{n+s}(1, \dots, n, v_1 + n, \dots, v_m + n).$$

In a similar way, we have that

$$T *_{1_n} \mathrm{Tub}_m = \mathrm{Tub}_{r+m}(u_1, \dots, u_r, n+1, \dots, n+m, u_{r+1}, \dots, u_n).$$

Using Definition 1.4, we define a shuffle product on the vector space $\mathbb{K}[\mathcal{MT}(\mathrm{St})] := \bigoplus_{n\geq 1} \mathbb{K}[\mathcal{MT}(\mathrm{St}_n)]$, where $\mathcal{MT}(\mathrm{St}_n)$ denotes the set of maximal tubings on S_n, which correspond to the vertices of stellohedra, as follows.

Definition 1.5. Let $T \in \mathcal{MT}(\mathrm{St}_n)$ and $W \in \mathcal{MT}(\mathrm{St}_m)$ be two maximal tubings. The product $T * W \in \mathcal{MT}(\mathrm{St}_{n+m})$ is defined as follows:

(1) If $T = \mathrm{Tub}_r(u_1, \dots, u_n) \neq \mathrm{Tub}_n$ and $W = \mathrm{Tub}_s(w_1, \dots, w_m) \neq \mathrm{Tub}_m$, then:

$$T * W := \sum_{\sigma \in \mathrm{Sh}(n-r, m-s)} T *_\sigma W,$$

where $\mathrm{Sh}(n-r, m-s)$ denotes the set of $(n-r, m-s)$-shuffles in $S_{n+m-(r+s)}$.

(2) If $T = \mathrm{Tub}_n$ and $W = \mathrm{Tub}_m$, then:

$$T * W = \mathrm{Tub}_{n+m}.$$

Note that

(1) If $T = \mathrm{Tub}_n$ and $W = \mathrm{Tub}_s(w_1, \dots, w_m) \neq \mathrm{Tub}_m$, then:

$$T * W := \mathrm{Tub}_{n+s}(\{1\}, \dots, \{n\}, w_1, \dots, w_m).$$

(2) If $T = \mathrm{Tub}_r(u_1, \dots, u_n) \neq \mathrm{Tub}_n$ and $W = \mathrm{Tub}_m$, then:

$$T * W := \mathrm{Tub}_{r+m}(u_1, \dots, u_r, \{n+1\}, \dots, \{n+m\}, u_{r+1}, \dots, u_n).$$

Proposition 1.6. *The graded vector space $\mathbb{K}[\mathcal{MT}(St)]$, equipped with the product $*$ is an associative algebra.*

Proof. Suppose that $T = \mathrm{Tub}_r(u_1, \dots, u_n) \in \mathcal{MT}(\mathrm{St}_n)$, $V = \mathrm{Tub}_s(v_1, \dots, v_m) \in \mathcal{MT}(\mathrm{St}_m)$ and $W = \mathrm{Tub}_z(w_1, \dots, w_p) \in \mathcal{MT}(\mathrm{St}_p)$ are three maximal tubings, where eventually $T = \mathrm{Tub}_n$, $V = \mathrm{Tub}_m$ or $W = \mathrm{Tub}_p$.

Applying the associativity of the shuffle, we get that:

$$(T * V) * W =$$

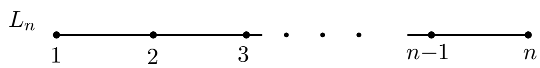
$$\sum_{\sigma \in \mathrm{Sh}(n-r, m-s, p-z)} \mathrm{Tub}_{r+s+z}(u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_z, x_{\sigma(1)}, \dots, x_{\sigma(n+m+p-(r+s+z))}) =$$

$$T * (V * W),$$

which ends the proof. $\square$

1.7. **Pterohedra.** Recall that the line graph L_n is the graph whose set of nodes is $\mathrm{Nod}(L_n) = \{1, \dots, n\}$, with edges

$$\mathrm{Edg}(L_n)(i,j) = \begin{cases} \{\#\}, & \text{for } |i-j| = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$



There exists a canonical bijection between the set of maximal tubings of L_n and the set $\mathcal{Y}_n$ of planar binary rooted trees with $n+1$ leaves, as showed by S. Forcey and D. Springfield in [1]. The authors showed that the associative product defined on the space spanned by all planar binary trees by J.-L. Loday and M. Ronco in [2], may be translated easily to the vector space spanned by the maximal tubings on the linear graphs. We describe briefly this product, for the proofs of the results we refer to [1].

Let T be a maximal tubing of L_n and W be a maximal tubing of W_m, the product $T \circ W$ is defined by the formula:

$$T \circ W = \sum_{\substack{S|_{[n]} = T \\ S|_{[m]+n} = W}} S,$$

where the sum is taken over all maximal tubings S of L_{n+m} such that S restricted to the set of vertices $[n]$ is T or $T \cup [n]$ and S restricted to the set of vertices $[m] + n$ is W or $W \cup [m]$, modulo the identification of the vertex $n + i$ of W with the vertex i of S for $1 \leq i \leq m$.

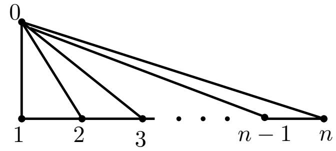
For example, the product of $T = \{\{1\}, \{3\}\}$ of L_3 with $W = \{\{1\}\}$ of L_2 is given by:

$$\begin{aligned} T \circ W = \{\{1\},\{3\},\{1,2,3\},\{1,2,3,4\}\} \ + \ \{\{1\},\{3\},\{3,4\},\{1,2,3,4\}\} \ + \\ \{\{1\},\{4\},\{3,4\},\{1,2,3,4\}\} \ + \{\{1\},\{4\},\{4,5\},\{3,4,5\}\} \ + \\ \{\{1\},\{3\},\{3,4\},\{3,4,5\}\} \ + \ \{\{1\},\{4\},\{3,4\},\{3,4,5\}\}. \end{aligned}$$

For $n \geq 1$, let us consider the simple connected graph $F_{1,n} = \mathfrak{S}(L_n)$ with nodes $\mathrm{Nod}(F_{1,n}) = \{0, 1, \dots, n\}$ and the edges given by

$$\mathrm{Edg}(F_{1,n})(i,j) := \begin{cases} \{\#\}, & \text{for } 0 \in \{i,j\} \text{ or } |i-j| = 1, \text{ and } i \neq j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

That is, $F_{1,n}$ is the graph:



Remark 1.8. For any maximal tubing T of $F_{1,n}$, there exists:

(1) a unique integer $0 \leq r$ and unique families of nonnegative integers $j_1, \dots, j_r$ and $n_1, \dots, n_r$ such that

$$1 \leq j_1 < j_1 + n_1 + 1 < j_2 < j_2 + n_2 + 1 < \cdots < j_r \leq j_r + n_r \leq n,$$

(2) for any $1 \leq i \leq r$, a unique maximal tubing $\tilde{T}_i$ of the line graph $L_{n_i+1} + j_i$,

(3) a total order $u_1, \dots, u_N$ on the set

$$\{1, \dots, n\} \setminus \{j_1, \dots, j_1 + n_1, j_2, \dots, j_2 + n_2, \dots, j_r, \dots, j_r + n_r\},$$

where $N = n - (n_1 + \cdots + n_r + r)$,

such that t is a tube of T if, and only if, t satisfies one of the following conditions:

(1) t is a tube of $\tilde{T}_i$ or $t = \{j_i, j_i + 1, \dots, j_i + n_i\}$, for some $1 \leq i \leq r$,
(2) $t = t_0 := \{0, j_1, \dots, j_1 + n_1, j_2, \dots, j_r + n_r\}$,
(3) $t = t_0 \cup \{u_1, \dots, u_k\}$, for some $1 \leq k < N$.

Example 1.9. For $n = 8$, consider the tubing

$$T = \{\{2\}, \{5\}, \{7\}, \{2,3\}, \\ \{5,6,7\}, \{0,2,3,5,6,7\}, \{0,2,3,5,6,7,4\}, \{0,2,3,5,6,7,4,8\}\}.$$

We have that $r = 2$, $j_1 = 2$, $j_2 = 5$, $n_1 = 1$, $n_2 = 2$, $N = 3$ and $(u_1, u_2, u_3) = (4, 8, 1)$.

The tubing $\tilde{T}_1 = \{\{2\}\}$, while the tubing $\tilde{T}_2 = \{\{5\}, \{7\}\}$.

Notation 1.10. Let $U = \{u_1 < \cdots < u_p\} \subseteq \{1, \dots, n\}$.

(1) For any permutation $\omega \in \Sigma_p$, we denote by U_ω the set U ordered by ω, that is $U = (u_{\omega(1)}, \dots, u_{\omega(p)})$. When $U = \emptyset$, we assume that $\Sigma_\emptyset = \{(1)\}$.
(2) The complement U^c of U in $\{1, \dots, n\}$ may be written in a unique way as:

$$U^c = \{j_1, j_1 + 1, \dots, j_1 + n_1, j_2, \dots, j_2 + n_2, \dots, j_r, \dots, j_r + n_r\},$$

for a unique $r \geq 0$ and unique families of integers $(j_1, \dots, j_r)$ and $(n_1, \dots, n_r)$ satisfying that

$$1 \leq j_1 < j_1 + n_1 + 1 < j_2 < \cdots < j_r < j_r + n_r \leq n.$$

We denote by $r(U)$ the integer r, by $\underline{j(U)}$ the collection $(j_1, \dots, j_r)$ and by $\overline{n(U)}$ the family $(n_1, \dots, n_r)$.
(3) By Remark 1.8, we have that any tubing T of $F_{1,n}$ is determined uniquely by a subset U of $\{1, \dots, n\}$, a permutation $\omega \in \Sigma_{|U|}$, and a family $(\tilde{T}_1, \dots, \tilde{T}_r)$, where $\tilde{T}_i$ is a maximal tubing of the linear graph L_{n_i+1}. We denote such a tubing by $T(U, \omega, \{\tilde{T}_j\}_{1 \leq j \leq r})$.

Note that for $U = \emptyset$, we have that $r(\emptyset) = 1$, $\underline{j(\emptyset)} = (1)$ and $\overline{n(\emptyset)} = n - 1$. So, for any maximal tubing T of $F_{1,n}$, we have:

$$T(\emptyset, (1), T) = T \cup [n].$$

On the other hand, for $U = [n]$ and $\omega \in \Sigma_n$, we get that

$$T([n], \omega, \emptyset) = \{\{0\}, \{0, \omega(1)\}, \{0, \omega(1), \omega(2)\}, \dots, \{0, \omega(1), \dots, \omega(n-1)\}.$$

Definition 1.11. Let $T = T(U, \omega, \{\tilde{T}_j\}_{1\leq j\leq r})$ be a maximal tubing of Pt_n, and $W = T(V, \tau, \{\tilde{W}_j\}_{1\leq j\leq s})$ be a maximal tubing of Pt_m, we define $T * W$ as follows:

(1) If either $n \in U$ or $1 \in V$, then

$$T * W := \sum_{\sigma\in \mathrm{Sh}(|U|,|V|)} T(U \cup (V + n), \sigma \cdot (\omega \times \tau), \{\tilde{T}_j\}_{1\leq j\leq r} \cup \{\tilde{W}_k(n)\}_{1\leq k\leq s}),$$

where $\{\tilde{T}_j\}_{1\leq j\leq r} \cup \{\tilde{W}_k(n)\}_{1\leq k\leq s} = \{\tilde{T}_1, \dots, \tilde{T}_r, \tilde{W}_1(n), \dots, \tilde{W}_s(n)\}$.

(2) If $n \notin U$ and $1 \notin V$, then

$$T * W = \sum_{\sigma\in \mathrm{Sh}(|U|,|V|)} T(U \cup (V + n), \sigma \cdot (\omega \times \tau), \{\tilde{T}_j\}_{1\leq j\leq r} \circ \{\tilde{W}_k(n)\}_{1\leq k\leq s}),$$

where

$$\{\tilde{T}_j\}_{1\leq j\leq r} \circ \{\tilde{W}_k(n)\}_{1\leq k\leq s} = \{\tilde{T}_1, \dots, \tilde{T}_r \circ \tilde{W}_1(n_r + 1), \dots, \tilde{W}_s(n)\},$$

and the tubing $\tilde{T}_r \circ \tilde{W}_1(n_r + 1)$ is the product ot two maximal tubings in $L_{n_r+m_1+2}$.

Proposition 1.12. *The binary operation $*$ introduced in Definition 1.11 is an associative product.*

Proof. The proof is similar to the one of Proposition 1.6, using the associativity of the shuffle product and of the product $\circ$ defined on the space spanned by binary planar rooted trees.

Let $T = T(U, \sigma, \{\tilde{T}_j\}_{1\leq j\leq r})$, $W = T(V, \tau, \{\tilde{W}_k\}_{1\leq k\leq s})$ and $Z = T(X, \delta, \{\tilde{Z}_j\}_{1\leq j\leq p})$ be three maximal tubings of $F_{1,n}$.

(1) If either $n \in U$ or $1 \in V$, and either $m \in V$ or $1 \in X$, then:

$$(T * W) * Z = \sum_{\omega} T(U \cup (V + n) \cup (X + n + m), \qquad \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1\leq j\leq r} \cup \{\tilde{W}_k(n)\}_{1\leq k\leq s} \cup \{\tilde{Z}_h(n + m)\}_{1\leq h\leq p}) =$$
$$T * (W * Z),$$

(2) If $n \notin U$ and $1 \notin V$, and either $m \notin V$ or $1 \notin X$, then

$$(T * W) * Z = \sum_{\omega} T(U \cup (V+n) \cup (X+n+m), \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1 \leq j \leq r} \circ \{\tilde{W}_k(n)\}_{1 \leq k \leq s} \cup \{\tilde{Z}_h(n+m)\}_{1 \leq h \leq p}) =$$
$$T * (W * Z).$$

A similar result holds for $n \notin U$ or $1 \notin V$, and $m \in V$ and $1 \in X$, where the sums are taken over all the permutations $\omega \in \mathrm{Sh}(|U|, |V|, |X|)$.

(3) Suppose that $n \in U$, $\{1, m\} \subseteq V$ and $1 \in X$.

(a) If $s \geq 1$, then

$$(T * W) * Z = \sum_{\omega} T(U \cup (V+n) \cup (X+n+m), \omega \cdot (\sigma \times \tau \times \delta),$$
$$\{\tilde{T}_j\}_{1 \leq j \leq r} \circ \{\tilde{W}_k(n)\}_{1 \leq k \leq s} \circ \{\tilde{Z}_h(n+m)\}_{1 \leq h \leq p}) =$$
$$T * (W * Z),$$

where the sum is taken over all the permutations $\omega \in \mathrm{Sh}(|U|, |V|, |X|)$.

(b) If $W = T(\emptyset, (1), \tilde{W})$ for some maximal tubing $\tilde{W}$ of L_m, then

$$(T * W) * Z = \sum_{\omega} T(U \cap (X+n+m), \omega(\sigma \times \delta),$$
$$\tilde{T}_1, \ldots, \tilde{T}_r \circ \tilde{W} \circ \tilde{Z}_1, \ldots, \tilde{Z}_p) =$$
$$T * (W * Z),$$

where the sum is taken over all permutations $\omega \in \mathrm{Sh}(|U|, |X|)$.

$\square$

References

[1] S. Forcey, D. Springfield, *Geometric combinatorial algebras: cyclohedron and simplex*, J. Algebraic Comb. 32 (2010) no. 4, 597–627.

[2] J.-L. Loday , M. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. 139 (2) (1998) 293–309.