Research Statement
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Summary

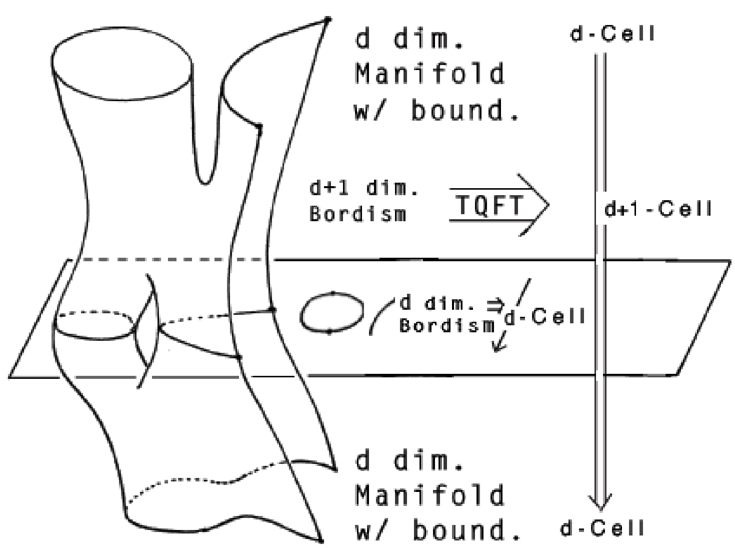
My studies involve the development of extended topological field theories and the relationship of categorical enrichment to iterated loop spaces, as well as applications of these concepts to other areas of mathematics and physics. Enrichment is a form of categorification, the process of replacing sets with categories. My results have shed light upon a set of questions about categorification and homotopy theory that begs for further inquiry. I defined strict and weak enrichment over iterated monoidal n-categories and proved theorems about the structure of the resulting collections of objects. I have related these results to homotopy theory and to the theory of weak n–categories. I plan to further develop these relationships and to extend them to alternative forms of enrichment. I plan to use my results to construct new TQFT's, which in turn I plan to use in the study of knots and in physics applications.

The major focuses of my research thus far are interrelated as in the following diagram, where the directions of arrows indicate flow of information and the numerals correspond to sections in the following text. In the text I put explicit statements of my *results in italics* and precise descriptions of my **future plans in bold.**

$$\begin{tikzcd}
\begin{array}{c}\text{knot}\\ \text{concordance}\end{array} \arrow[dd] & & & & \begin{array}{c}\text{enriched}\\ \text{categories}\end{array} \arrow[dll] \arrow[dd, leftrightarrow] \arrow[drr, leftrightarrow] & & \\
& \text{(IV)} & \begin{array}{c}\text{(I)}\\ \text{extended}\\ \text{TQFT}\end{array} \arrow[ull] \arrow[dll] & \text{(II)} & & \text{(III)} & \begin{array}{c}\text{homotopy theory:}\\ \text{iterated loop spaces}\\ \text{(and their operads)}\end{array} \\
\begin{array}{c}\text{loop quantum gravity}\\ \text{and string theory}\end{array} & & & & \begin{array}{c}\text{(weak)}\\ n\text{–categories}\end{array} \arrow[ull] \arrow[urr, leftrightarrow] & &
\end{tikzcd}$$

I. Extended Topological Quantum Field Theories

Naively a TQFT is a topological invariant of spaces that is functorial with respect to cobordisms. This idea is motivated by physics where we are used to picturing a pair of spaces forming the boundary of a space–time cobordism. Let $d+1$ be the dimension of the space–time. Technically a $d+1$ dimensional TQFT is a multiplicative functor between bordism categories. An n–category has arrows (1–cells) between objects (0–cells), 2–cells between 1–cells, and so on. A multidimensional TQFT that has an n–category as its range is known as an extended topological field theory. The picture that we hope to realize in some fashion is as follows:



We studied a new variation on the definition of field theory that retains the gluing properties of a TQFT but defines the invariants as limits of functors rather than as functors themselves. I investigated using sheaf cohomology, the limit of a presheaf, on various spaces of decompositions of the bordisms. The first step is to establish a source of algebraic input that will be successful in uncovering subtle topological data. Typical algebraic input has evolved over the years, from groups to their representations, from categories of representations to more generally defined monoidal categories, and from categories with structure to categories enriched over categories with structure. This last stage is what I hope to help develop.

II. Iterated Enrichment and Higher Dimensional Algebra

The definition of enriched category generalizes the usual definition of category by replacing the hom–sets of morphisms between each two objects by hom–objects in some monoidal category $\mathcal{V}$. The collection of enriched categories over a given monoidal category, with enriched functors and natural transformations, is the 2–category known as $\mathcal{V}$–Cat. $\mathcal{V}$–Cat is known to inherit structure from $\mathcal{V}$: if $\mathcal{V}$ is symmetric then so is $\mathcal{V}$–Cat, if $\mathcal{V}$ is merely braided then $\mathcal{V}$–Cat is merely monoidal. These facts are important to the construction of extended TQFT's since accurately modeling various dimensions requires the range category to have various levels of structure, especially if it is desired that the field theory reflect the embedding of the cobordism as well as the topology.

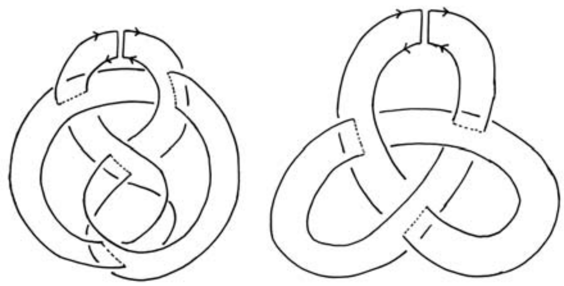
My contributions to enrichment theory are related to this inheritance of properties. *I defined enrichment over a very general type of monoidal category with extra structure; a k–fold monoidal, or iterated monoidal category. I then proved that for $\mathcal{V}$ k-fold monoidal $\mathcal{V}$–Cat is a $(k-1)$–fold monoidal 2–category.* **By detailing additional constraints on the structure of $\mathcal{V}$ I plan to describe a new source of braided monoidal 2–categories and bicategories.** *Next I recursively defined higher dimensional enrichment. $\mathcal{V}$–n–Cat is the collection of categories enriched over $\mathcal{V}$–$(n-1)$–Cat. I defined the enriched higher morphisms for these objects and showed that $\mathcal{V}$–2–Cat is a $(k-2)$–fold monoidal strict 3–category.* **I am preparing an inductive proof that for $\mathcal{V}$ k–fold monoidal we have that $\mathcal{V}$–n–Cat is a $(k-n)$–fold monoidal strict $(n+1)$–category.** The algebraic extension of these results is to weaken the concept of higher dimensional enrichment by replacing equalities with isomorphic higher morphisms. Since a strict n–category is just a category enriched over $(n-1)$–Cat, this weakening has a direct application to the definition of weak n–categories, something that is still not satisfactorily understood. **I am currently preparing a definition of weak enrichment based on Stasheff's associahedra. I will also contribute to this field of research next June as I have been invited to the workshop on "n–categories: Foundations and Applications" organized by John Baez and Peter May at the IMA.** The topological extension of my results is to understand their implications for homotopy theory.

III. Enrichment and Delooping

The effort to link category theory and geometry, especially categorical and homotopical coherence, has recently been aided by the work of Balteanu, Fiedorowicz, Schwänzl, and Vogt who show a direct correspondence between k–fold monoidal categories and k–fold loop spaces through the categorical nerve in [2]. As I pursued my plan to relate the enrichment functor to topology, *I noticed that the concept of higher dimensional enrichment is important in its relationship to double, triple and further iterations of delooping.* This is evident immediately from the proof that under enrichment categorical dimension increases while monoidalness decreases. In ΩX points (objects) are paths (1–cells) in X, and loop spaces are always provided with a multiplication of points by concatenation of loops. **In the future I plan to characterize the nerves of $\mathcal{V}$–n–Cat. Then I plan to extend this characterization to the weak version of enrichment.** The nerves of bicategories have been defined by Duskin in [6] and may provide the link to loop spaces in this context. Other categories based on a monoidal category $\mathcal{V}$ are $\mathcal{V}$–Act, the bicategory $\mathcal{V}$–Mod and internal categories in $\mathcal{V}$. **I am currently developing the potential of $\mathcal{V}$–Act and $\mathcal{V}$–Mod to be generalized in my scheme and thus to reveal their import to homotopy theory.**

IV. Applications

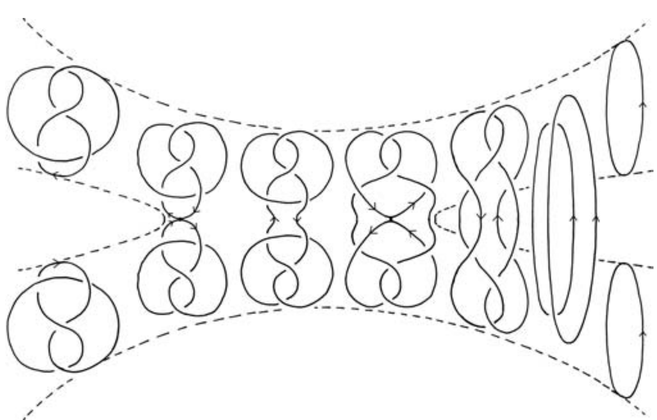
Next I return to the question of field theories that reflect embedding. Baez and Langford have described the use of braided bicategories to model braided surfaces [1]. **I plan to apply their construction to a braided $\mathcal{V}$–Mod and look for useful invariants of embedded surfaces, especially knot cobordisms.** The knot concordance group has elements equivalence classes of knots related in that two equivalent knots cobound a cylinder that is smoothly embedded in Euclidean 4–space. The group operation is connected sum. The knots that are concordant to the identity unknot are the slice knots–they occur as cross sections of a sphere smoothly embedded in 4–space. *Here are examples for the figure eight and the trefoil of a drawing* *algorithm that I constructed to show by picture how the connected sum of a knot and its orientation reversed mirror image forms a special slice knot called a ribbon knot.* The inverse knot is drawn behind the original by placing the reversed crossings each beside the corresponding original crossing and then connecting with arcs in the background.



It is unknown whether all slice knots are in fact ribbon knots. Amphicheiral knots represent elements of order two in the group, but it is unknown whether there are any elements of order two without an amphicheiral representative. The trefoil represents an element of infinite order, but it is unknown whether there are elements of order $n \geq 3$. It seems worth the effort to attack some of these questions with the new invariants available through quantum field theories. The question of order, for instance, might be solved if it was possible to construct a field theory for which the image of a potential cobordism between the unknot and a triple connected sum could be shown not to exist. Another way to extend the ideas of TQFT is to define field theories that are invariants of gropes and other CW complexes. The work of Conant and Teichner in [4] desribes a filtration of the knot concordance group by grope cobordism, and this might be the correct place to try to solve problems using such field theories as tools.

The connection between TQFT and quantum gravity is that field theories often occur as topological state sums, which are invariants calculated from but independent of triangulations on the cobordism manifold. These state sums are the starting point for loop quantum gravity in the work of several theorists, including Crane, Barrett [3], and Smolin [8]. There is also an intriguing possibility of applying the sheaf theoretic version of field theory to string theory. It has been suggested that the transport of strings and higher dimensional branes be described in terms of n–gerbes. An 0–gerbe is a sheaf, and a 1–gerbe assigns groupoids to the open sets. I would like to define gerbe theoretic TQFT and investigate its relationship to the K–theory of gerbes.

Kauffman notes a connection between the knot concordance group and string theory in [7]. The connected sum of a knot and its inverse is cobordant to the unknot in a way that suggests a particle-antiparticle collision. The cobordism is seen as a movie of cross-sectional stills. We include the connected sum as an initial saddle and neglect to cap off one of the unknots that results.



I would like to define extended topological field theories on this sort of picture, as well as understand any physical significance its higher dimensional analogues may reflect. There may be some illumination on the subject in the recent work of Conant and Teichner on grope cobordism and Feynman diagrams [5].

References

[1] J. Baez, L. Langford, Higher-Dimensional Algebra IV:2–Tangles, to appear in Adv. Math.

[2] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, Iterated Monoidal Categories, Adv. Math. 176 (2003), 277-349.

[3] J.W. Barrett, L. Crane, Relativistic Spin Networks and Quantum Gravity, /gr-qc/9709028

[4] J. Conant, P. Teichner, Grope Cobordism of Classical Knots, math.GT/0012118

[5] J. Conant, P. Teichner, Grope Cobordism and Feynman Diagrams, math.GT/0209075

[6] J.W. Duskin, Simplicial Matrices and the Nerves of Weak n–Categories I: Nerves of Bicategories, Theory and Applications of Categories, 9, No. 10 (2002), 198-308.

[7] L. Kauffman, Knots and Physics 3rd ed., World Scientific Press, 2001

[8] L. Smolin, Linking Topological Quantum Field Theory and Nonperturbative Quantum Gravity, J. Math. Phys., 36 (1995), 6417-6455.