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Thesis

Approved:

Advisor
Dr. Stefan Forcey

Faculty Reader
Dr. W. Stuart Clary

Faculty Reader
Dr. Jeffrey Riedl

Department Chair
Dr. Timothy Norfolk

Accepted:

Dean of the College
Name of Coll Dean

Dean of the Graduate School
Name of Grad Schl Dean

Date

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ABSTRACT

In this paper we continue a study of polytopes built from nested objects. We will use posets to build our polytopes, and give a SAGE program to compute aspects of these poset associahedra.

# CHAPTER I

# INTRODUCTION

Humankind's infatuation with polytopes (classically line segments, polygons, and polyhedra) is almost as old, if not older, than mathematics itself. There is evidence that Pythagorean triples were known to Babylonians, while this simply stated result about special polytopes is taught to schoolchildren the world around. One might even make the case that the square taught ancient Greeks about irrational numbers, and that these mysterious entities had to wait many hundreds of years for Dedekind to coherently place them within our view of mathematics. Euclid's Elements is almost entirely devoted to manipulating small dimensional polytopes, culminating in an investigation of the five regular solids. One need only inquire how symmetric the Platonic Solids are to see a beautiful application of group theory.

While much as been written about polytopes over the millennia, one modern branch of the theory is devoted to studying polytopes whose faces have combinatorial significance. Although relatively new, many novel and useful results have been derived under this framework. Our purpose is not to give an overview of recent activity, but expand the domain of a particular class of combinatorially significant polytopes, the graph associahedra of M. Carr and S. Devadoss. However, it would be a shame to completely omit the wonderful motivational examples that spurred our development.

This paper will be an introduction to the realm of polytopes whose faces have combinatorial meaning currently known as poset associahedra. In Chapter II, we give a brief overview of polytopes whose faces were long known to have combinatorial significance, including simplices and Stasheff's associahedron. This will culminate in a brief introduction to the polytopes we will be generalizing, the graph associahedra of Carr and Devadoss.

Our ultimate goal will be to show that our poset associahedra are structures known as abstract polytopes. In Chapter III, we focus on general abstract polytopes first giving the standard definition by Schulte. We then discuss this definition, remarking that convex polytopes are examples of abstract polytopes.

Finally, in Chapter IV, we present the main result, that poset associahedra are abstract polytopes. First, we give the definition of the poset associahedron $\mathcal{K}P$ and necessary poset terminology, given a poset $P$.

# CHAPTER II

# MOTIVATION

The most basic polytope with combinatorial significance one can imagine is the $n$-dimensional simplex, $\Delta^n$. As will be a common theme, we give a convex hull representation of $\Delta^n$ in Euclidean $(n+1)$-space. First, we label the standard basis $\mathcal{E}^n = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$.

**Definition 1.** The *$n$-simplex* $\Delta^n$ is the convex hull of the standard basis vectors of $\mathbb{R}^{n+1}$.

The main reason $\Delta^n$ is combinatorial is the bijection between subsets of $\mathcal{E}^{n+1}$ and faces of $\Delta^n$. Namely, for any subset $S$ of $\mathcal{E}^{n+1}$, the corresponding face of $\Delta^n$ is $\Delta^n_S = \operatorname{conv}\{e_i \mid e_i \in S\}$. In fact, the face lattice of $\Delta^n$ is isomorphic to the standard Boolean lattice of subsets. We note that $\Delta^n$ is contained in the hyperplane $\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n + \mathbf{x}_{n+1} = 1$, and is thus $n$-dimensional.

There is another polytope whose faces have been shown to be in bijection with combinatorial objects, but first we need some background. Let $Y_n$ be the set of planar, binary trees with $n$ internal nodes. We distinguish between internal *nodes* and exterior *leaves* of $t$. We label the leaves of the tree $0, 1, \ldots, n$ and the internal nodes $1, 2, \ldots, n$, so that the internal node $i$ is between leaves $i-1$ and $i$. Finally, for nodes $t_i$ of $t$, we define the function $a(t_i)$ to be the number of leaves to the left of node $i$, and $b(t_i)$ to be the number of leaves to the right of node $i$.

**Definition 2.** For any tree $t \in Y_{n+1}$, we define the function $M : Y_{n+1} \to \mathbb{R}^{n+1}$ by $M(t) = \langle a(t_i)b(t_i)\rangle_{i=1}^{n+1}$. The *n-associahedron* is then defined to be $\mathcal{K}^n = \text{conv}\{M(t) \mid t \in Y_{n+1}\}$.

Although the associahedron may traditionally be thought of in terms of parenthesizations of a word, it is the bijection between $Y_n$ and parenthesizations of words of length $n+1$ that makes this realization of the associahedron easy to describe.

For example, there are exactly five parenthesizations of the expression $x_1x_2x_3x_4$, in bijection with the five binary trees on three nodes. These expressions are $(x_1(x_2x_3))x_4$, $x_1((x_2x_3)x_4)$, $x_1(x_2(x_3x_4))$, $(x_1x_2)(x_3x_4)$, and $((x_1x_2)x_3)x_4$, which get coordinates (2,1,3), (3,1,2), (3,2,1), (1,4,1), and (1,2,3) respectively.

It is relatively simple to see that these vertices of $\mathcal{K}^2$, while embedded in $\mathbb{R}^3$, lie on the hyperplane $x_1 + x_2 + x_3 = 6$. It is a general fact that this realization of $\mathcal{K}^{n-1}$ is contained in the hyperplane $x_1 + x_2 + \ldots + x_n = \frac{n(n+1)}{2}$, implying that the associahedron $\mathcal{K}^n$ is actually $n$-dimensional.

Another combinatorial polytope which is older than the associahedron is the permutohedron. In order to give its convex hull realization, we will use $\text{Sym}(n)$, the symmetric group on $n$ letters and its action on the set $[n] = \{1, 2, \ldots, n\}$. For each $\sigma \in \text{Sym}(n)$, let $v_\sigma = \langle \sigma(1), \sigma(2), \ldots, \sigma(n)\rangle \in \mathbb{R}^n$. Thus the vector $\langle 1, 2, \ldots, n\rangle$ is associated with the identity element, $e$, of $\text{Sym}(n)$.

**Definition 3.** The **permutohedron** $\mathcal{P}^n = \text{conv}\{v_\sigma \mid \sigma \in \text{Sym}(n+1)\}$.

The permutohedron is defined using the group of permutations, just as the associahedron is defined using bracketings, which relate to associativity. However, it is easier to see that for any permutation $\sigma \in \text{Sym}(n)$, the point $v_\sigma$ lies in the hyperplane $x_1 + x_2 + \ldots + x_n = \frac{n(n+1)}{2}$, the same $(n-1)$-dimensional subspace of $\mathbb{R}^n$ as the $(n-1)$-associahedron, thus $\mathcal{P}^n$ is $n$-dimensional.

While these intriguing families of polytopes each deal with quite disparate phenomena, one simple unification was demonstrated by Carr, Devadoss in 2004, dubbed *graph associahedra*. Since this thesis is concerned with an attempt to generalize graph associahedra, it will be beneficial to understand the base process in order to justify the techniques used to build poset associahedra. We first present the necessary definitions (From ???).

**Definition 4.** For a graph $G$,

(1) A *tube* $t$ of $G$ is a connected subgraph of $G$ such that the edge $\{u, v\} \in t$ whenever $u, v \in G_t$.

(2) Two tubes $t, u$ are *compatible* if one contains the other or if they are disjoint and do not contain both endpoints of any edge not in $t \cup u$.

(3) A *tubing* $G_T$ of $G$ is a collection of tubes which are pairwise compatible. A $k$*-tubing* is simply a tubing $G_T$ containing $k$ tubes, i.e., $|G_T| = k$.

We remark here that the prior generalization of graph associahedra to *pseudograph associahedra* (CITE) allows $G$ to be a pseudograph, that is, a graph which may have multiple edges and loops. In this case, we only change component (1) of the above definition by forcing tubes to include *an* edge $\{u, v\}$ whenever the tube contains both endpoints. With these related definitions, we are now ready to present the fundamental polytopes we will later generalize.

**Definition 5.** For a (pseudo)graph $G$, the *(pseudo)graph associahedron* $\mathcal{K}G$ is a polytope whose face poset is isomorphic to the poset of tubings of $G$ ordered by reverse inclusion. In particular, codimension $k$ faces of $\mathcal{K}G$ are in bijection with tubings of cardinality $k$.

The authors gave a method to construct $\mathcal{K}G$ by truncating specific faces of a particular product of simplices, and we refer the interested reader to (CITE). However, our project will not be as ambitious, for we will be content to show that poset associahedra are abstract polytopes, a term to be defined soon.

It is by incredible design that each of the families of polytopes described above – the $n$-dimensional simplices, associahedra, and permutohedra – were shown to be graph associahedra of the edgeless graph, path graph, and complete graph respectively, each with $n + 1$ nodes.

# CHAPTER III

# ABSTRACT POLYTOPES

In this chapter, we will set the stage. Our ultimate goal is to show that, from any finite poset $P$, we are able to construct a particular abstract polytope, $\mathcal{K}P$, whose faces are in bijection with special subposets of $P$. However, we would first like to clarify the often unfamiliar subject of abstract polytopes before revealing the techniques to construct $\mathcal{K}P$.

Convex polytopes have been studied since antiquity, and as a result, are rather well-understood. However, with the advent of objects such as the Kepler-Poinsot star polyhedra, it became clear that not all polytopes could be understood using the traditional techniques. Finally, Schulte in 1980 gave a definition of *abstract polytope* that came to be accepted as the proper framework to study unconventional constructions. We begin by listing and discussing four axioms that define an abstract polytope.

**Definition 6.** An *abstract d-polytope* $\mathcal{P}$ is a partially-ordered set whose elements are called *faces* satisfying the four properties below.

(**AP I**) The poset $\mathcal{P}$ has a least a least and greatest face.

(**AP II**) Each flag of $\mathcal{P}$ has $d+2$ elements including the least and greatest faces.

(**AP III**) $\mathcal{P}$ is strongly flag-connected.

(**AP IV**) Every 1-section of $\mathcal{P}$ is isomorphic to the face lattice of a line segment.

Our first axiom, (AP I), is extremely simple in nature: it merely assures us that we have least and greatest elements. The former is conventionally thought of as the null-polytope which has dimension $-1$, and the polytope $\mathcal{P}$ itself the latter. Recalling the $n$-simplex $\Delta^n$ whose faces are in bijection with subsets of an $n+1$ element set, the least and greatest faces of $\Delta^n$ correspond to the empty subset and entire set. We mention that any poset satisfying (AP I) is called *bounded*. Furthermore, faces of $\mathcal{P}$ which lie strictly between the least and greatest faces are called *proper*, while the least and greatest faces are *improper*.

The next axiom, (AP II), states that *flags* of abstract polytopes all have the same cardinality. Here, a flag of $\mathcal{P}$ is any maximal *chain*, that is, totally ordered subset of faces, ordered by inclusion. We simply stipulate that every maximal chain of faces contains the same number of faces. In the statement of the axiom, the integer $d$ is the dimension of $\mathcal{P}$, which we will formalize shortly.

As an example, any maximal chain of subsets of $[n+1]$ corresponds to some flag of the $n$ dimensional simplex $\Delta^n$. We note that posets satisfying AP II are called *graded* posets. On any graded poset, we may define a *rank function* that agrees with the dimension of traditional polytopes.

**Definition 7.** Given a poset $P$ satisfying (API) and (APII), we define the *rank* of $P$ to be two less than the cardinality of a maximal chain.

Since we are working with face lattices $P$ satisfying AP I, they have a greatest element, the polytope itself, and a least element, the null polytope. Thus, if we look at the face lattice of $\Delta^2$, a triangle, a maximal chain would consist of the null polytope, a vertex, a line segment, and the triangle itself. This chain has cardinality 4, so we see the face poset has rank 2, agreeing with the typical notion of dimension. Essentially, by subtracting two from the cardinality of a maximal chain, we are compensating for the least face and 0-dimensional elements.

Before we can explain the third axiom, that we want abstract polytopes to be strongly flag-connected, we must first define flag-connectedness for our specific posets.

**Definition 8.** A poset $P$ of rank $n$ satisfying axioms (AP I) and (AP II) is *flag-connected* if $n \leq 1$, or if $n \geq 2$ and for every pair of proper elements $x, y \in P$, there exists a sequence of proper elements $p_1, p_2, \ldots, p_k$ of $P$ with $x \perp p_1 \perp p_2 \perp \ldots \perp p_k \perp y$ where $x \perp p_1$ means $x$ and $p_1$ are *comparable*, that is, $x \leq p_1$ or $p_1 \leq x$.

To be *strongly* flag-connected, we stipulate that special portions of the poset in question are flag-connected.

**Definition 9.** Given two elements $a \leq b$ of a poset $P$, we define the *section* $b/a$ to be all elements lying between $a$ and $b$; that is, $b/a = \{x \in P \mid a \leq x \leq b\}$. Noting that sections of posets satisfying (AP I) and (AP II) themselves satisfy these axioms and are thus graded, we call any section of rank $k$ a *$k$-section*.

We are now able to precisely define strong flag-connectedness in posets:

**Definition 10.** A poset $P$ is *strongly flag-connected* if every section of $P$ is flag-connected.

While it may seem like an odd restriction, or perhaps not much stronger than regular (flag-)connectedness, it is this property that does not allow certain constructions to be considered abstract polytopes. For example, if we join two triangular regions together at a vertex point, the poset of this structure is not strongly flag-connected, and thus not an abstract polytope, despite being connected in a weaker sense.

Our final axiom, that 1-sections of the poset must be isomorphic to line segments, may be easier to parse if phrased slightly differently: that each 1-section $F/H$ must contain 4 elements, including the maximum and minimum faces $F$ and $H$. This leaves just two proper faces $G_1$ and $G_2$ in $F/H$, with $G_1$ and $G_2$ not comparable.

Let's take a moment to apply this axiom to the face lattice of any 3-polytope, in particular a convex polyhedron $\mathcal{P}$. If our maximum face is $\mathcal{P}$ itself, then our minimum must be a 1-face, that is, edge, of $\mathcal{P}$. Then it is quite believable that we will find precisely two 2-faces of $\mathcal{P}$ which contain this edge. Instead of choosing our maximum face to be $\mathcal{P}$, let us instead choose our minimum face to be the null polytope of dimension $-1$ contained in all polytopes. Then our maximum will be any 1-face of $\mathcal{P}$, whereby we find exactly two 0-faces (vertices) of $\mathcal{P}$ – the endpoints of our edge – containing the null polytope and contained in this 1-face.

We note that with these definitions in hand, we may more succinctly state the requirements of an abstract polytope as a bounded, graded, strongly flag-connected poset whose 1-sections contain four elements.

# CHAPTER IV

# CONSTRUCTION

Our goal is to produce a unique lattice $\mathcal{K}P$ given any finite poset $P$. Thus for the rest of this paper, we mean finite poset when we say poset. The lattice $\mathcal{K}P$ will consist of what we call 'tubings' of the poset $P$, which will be defined shortly. Eventually, we will see that this lattice $\mathcal{K}P$ is the face lattice of an abstract polytope. As such, we will often make no distinction between the tubings of $P$ and the faces of $\mathcal{K}P$, and use the two terms interchangeably. Before we are able to construct such a face lattice, we need some poset-related terminology and definitions.

**Definition 11.** Given a poset $P$, an *induced subposet* of $P$ is a subset $Q$ of $P$ with partial ordering of $Q$ induced from $P$ so that $x \leq y$ in $Q$ if and only if $x \leq y$ in $P$.

We note that there is a weaker concept of subposet, but here, we will use the term *subposet* to mean *induced subposet*. Although the lattice $\mathcal{K}P$ is a lattice of subposets of $P$, it is not the lattice of *all* subposets of $P$. We first distinguish a special class of subposets called *lower sets* of $P$.

**Definition 12.** A subset $S$ of elements of a poset $P$ is a *lower set* if, whenever $x \leq y$ for some $y \in S$, then $x \in S$. The *lower set induced by* $S$ is simply the subposet $S_{\downarrow} = \{x \in P \mid x \leq s \text{ for some } s \in S\}$. An *upper set* of $P$ is defined analogously, with "$\leq$" replaced by "$\geq$" in the preceding definition. We write $S^{\uparrow}$ for the *upper set induced by* $S$, where $S$ is any subset of $P$.

Visually, the lower set induced by $S$ is simply the elements of $P$ which are both below something in $S$ in the Hasse diagram of $P$ and are comparable to something in $S$. While we will see upper sets used later, lower sets play a key role in construction of the polytope $\mathcal{K}P$, and we will make use of a certain class of lower sets.

**Definition 13.** Given an element $a$ of a poset $P$, the *boundary of* $a$ is the set $\partial a = \{x \in P \mid x < a\}$. Equivalently, $\partial a = a_{\downarrow} \setminus \{a\}$.

Ultimately, we are extending the concept of a *tube* of graph to that of a poset. Tubes of a graph were subgraphs induced by sets of vertices, so that if a graph tube contained an edge, it contained the endpoints of the edge. If we keep in mind the incidence structure of a graph, that is, the poset of containment relationships between edges and vertices, we see that graph tubes are lower sets of this incidence poset.

Thinking still of the incidence poset of a graph, a graph tube had to contain at least one edge between a pair of nodes if such an edge existed and the tube contained that pair of nodes. We would like tubes of a poset to obey an analogous rule, except we may deal with poset elements of a larger rank, or even posets without a well-defined rank function. This motivates our next definition.

**Definition 14.** Given an element $a$ of poset $P$, we define the *$P$-bundle of $a$* to be the set of elements $\mathfrak{b}_a = \{x \in P \mid \partial x = \partial a\}$. It will usually be clear from context the poset of which we're speaking, and simply write *bundle*. We call a bundle $\mathfrak{b}_a$ *trivial* if $\mathfrak{b}_a = \{a\}$, and *nontrivial* otherwise. We may at times refer to an element $x$ of $P$ as trivial or nontrivial, depending on whether the bundle $\mathfrak{b}_x$ is trivial or nontrivial.

The bundle of $a$ is the set of all elements that share their boundary with $a$. Colloquially, elements of $\mathfrak{b}_a$ are indistinguishable as far as elements below $a$ are concerned. This is an extension of the concept of multiple edges between a pair of nodes in a graph setting. Note that the relation "$x$ is in the same bundle as $y$" is reflexive, symmetric, and transitive, so that any poset is naturally partitioned into bundles of elements.

**Definition 15.** Given a poset $P$, we let $\mathfrak{B}$ denote the set of nontrivial bundles of $P$.

We will see that much of our investigation into poset associahedra can be carried out by focusing on elements of a poset $P$ in nontrivial $P$-bundles. Though we may make use of certain subposets, these are always induced subposets of an overall parent poset, and as such, little confusion arises if we only carry around the set $\mathfrak{B}$ of nontrivial bundles of $P$.

**Definition 16.** A subset $S \subseteq P$ of poset elements is said to be *filled* if $\mathfrak{b}_a \cap S \neq \emptyset$ whenever $\partial a \subseteq S$.

This is a natural generalization of forcing tubes to contain at least one edge between nodes whenever the tube contains both endpoints. In the incidence poset of a graph, a set which is not filled would contain the endpoints of a set of edges, but not include any of these edges. Thus, this set is not a graph tube.

**Definition 17.** A subset $S$ of a poset $P$ is *connected* if, for every pair of elements $x, y \in S$, there exists a sequence of elements $p_1, p_2, \ldots, p_k$ of $S$ with $x \perp p_1 \perp p_2 \perp \ldots \perp p_k \perp y$ where $x \perp p_1$ means $x$ and $p_1$ are *comparable*, that is, $x \leq p_1$ or $p_1 \leq x$.

We say $\Delta \subseteq S$ is a *connected component* of $S \subseteq P$ if no subset of $S$ properly containing $\Delta$ is connected. We denote the set of connected components of $S$ by $CC(S)$.

With this handful of concepts, we are able to define the building blocks of the poset associahedron.

**Definition 18.** A *tube* of a poset $P$ is a subposet of $P$ whose underlying set is a filled, connected, lower set of $P$ not containing all of $P$. A *(proper) tubing of $P$* is a set of tubes which are pairwise nested or disjoint and for which the union of any subset of constituent tubes is a filled, proper subset of $P$. A *$k$-tubing* of $P$ is a tubing containing $k$ tubes. Tubes containing the maximum number of tubes are called *maximal* tubings of $P$.

**Definition 19.** Given a poset $P$, the *poset associahedron* $\mathcal{K}P$ is the lattice of tubings of $P$, ordered by reverse inclusion so that $T_1 < T_2$ if $T_2 \subset T_1$.

**Definition 20.** For any poset $P$, we include in $\mathcal{K}P$ the *empty tubing* $\hat{1}$ and the *universal tubing* $\hat{0}$ for which $\hat{0} < T < \hat{1}$ for all proper tubings $T$ of $P$.

The empty and universal tubings of $P$ fill the role of the maximum and minimum, respectively, of $\mathcal{K}P$ so that $\mathcal{K}P$ will always satisfy the first axiom to be an abstract polytope. By the ordering on $\mathcal{K}P$ of reverse subset inclusion, we see that 1-tubings of $P$, in natural bijection with the tubes of $P$, fill the role of *facets* of $\mathcal{K}P$. That is, 1-tubings are codimension 1 faces of $\mathcal{K}P$. In general, a $k$-tubing of $P$ corresponds to a codimension $k$ face of $\mathcal{K}P$. Thus, 1-tubings are facets of $\mathcal{K}P$, while maximal tubings are vertices of $\mathcal{K}P$.

In this sense, the maximum of $\mathcal{K}P$ should be a 0-tubing, which we may naturally identify with the empty set. This is why we call $\hat{1}$ the empty tubing. It is certainly true that every tubing contains the empty tubing, and thus this identification is indeed natural. On the other hand, the universal tubing has no "nice" identification. It should be a tubing containing all other tubings as a subset, while having cardinality one greater than any maximal tubing. This inability to consider $\hat{1}$ as anything but the maximum of $\mathcal{K}P$, among other factors, will often require great care when working with sections of $\mathcal{K}P$ of the form $F/\hat{0}$ for any tubing $F > \hat{0}$ of $P$.

In this paper our primary focus is to show, for any poset $P$, that the poset associahedron $\mathcal{K}P$ is an abstract polytope. In what follows, we consider only finite posets $P$, and note that by definition 20 above of the empty and universal tubings, $\mathcal{K}P$ satisfies axiom (API).

Ultimately, this paper is concerned merely with the structure of $\mathcal{K}P$. We will develop a wealth of results in order to deal with the axioms that $\mathcal{K}P$ must satisfy to be an abstract polytope. Surprisingly, to accomplish this goal, we will be concerned almost entirely with the set $\mathfrak{B}$ of nontrivial bundles of elements of $P$. As such, we currently develop several auxiliary results on the structure of $\mathcal{K}P$ by focusing solely on the influence of nontrivial bundles on tubings of $\mathcal{K}P$. Our first attempt in this vein is the simple observation that, should a poset contain nontrivial elements, then any proper tubing cannot contain *all* nontrivial elements in $P$.

We mention first a piece of notation. Many of the objects of interest will be a set of sets, for example, tubings are sets of tubes, with tubes sets of elements of a poset, or the set $\mathfrak{B}$ of nontrivial bundles of $P$, with an element of $\mathfrak{B}$ some bundle, that is, subset of $P$. As such, it will be convenient to write $\cup\mathfrak{B}$ when we really mean $\cup_{B\in\mathfrak{B}}B$, or given a tubing $T$, we write $\cup T$ for $\cup_{t\in T}t$.

**Lemma 4.0.1.** *Let $T$ be a nonempty, proper tubing of a poset $P$. If $P$ contains an element in a nontrivial bundle, then there exists at least one element of $P$ which is in a nontrivial bundle and not contained in any tube in $T$.*

*Proof.* This is a nice introductory result which will hopefully become quite clear. To see the reason why it must be true, assume a poset $P$ contains a mixture of trivial and nontrivial elements. Fix an element $y \in P$ with $\mathfrak{b}_y = \{y\}$, that is, no other elements of $P$ have the same boundary as $y$. Then the fate of $y$ is completely determined by $\partial y$ in the sense that, if $\partial y$ is in a tube, or in the union of tubes in a tubing, $y$ must also be in this tube or tubing. This need not be the case for an element in a nontrivial bundle, for one may always choose some other element of the bundle to include in such a tube or tubing.

Now, to specifically address the lemma, let $P$ be a poset which contains nontrivial elements, and let $T$ be a proper tubing of $P$, that is, $\cup T$ is a proper subset of $P$. If $P$ contains no trivial elements, then it is clear that we've contradicted our assumption that $\cup T$ is proper in $P$, and we many now assume that $P$ contains a mixture of trivial and nontrivial elements. Note that any element $x$ in a trivial bundle must be comparable to some element in $P$, otherwise $\{x\}$ is a connected component of $P$ (which must have at least three elements at this point), forcing $x$ to be in a nontrivial bundle, namely the bundle of minimal elements of $P$. We may also assume that some element $x$ in a trivial bundle is in the upper set of some element $m$ in a nontrivial bundle. Otherwise, all trivial elements would be in the lower set induced by the set of all nontrivial elements, again contradicting the assumption that $T$ is a proper tubing.

Now, since $T$ is a proper tubing, at least one element must not be in $\cup T$, and by our assumption, these elements are in trivial bundles. In particular, there exists some $y \in P \setminus \cup T$ with $\partial y$ comprised of elements in $\cup T$, simply because $P$ is a finite poset. This immediately violates the assumption that the union of a particular subset of tubes of $T$, those tubes which collectively contain $\partial y$, is filled in $P$, contradicting our assumption that $T$ is a tubing. $\square$

This notion that, given a tubing $T$ of a poset, we may always find an element in a nontrivial bundle which is not in $\cup T$ is a very useful observation. In fact, when we restrict our attention to maximal tubings, we may draw an even stronger conclusion, uniqueness of the nontrivial element not included. This will be a key tool, but it will be helpful to develop a preliminary result which will also be useful in other situations.

For this preliminary result, we fix a poset $Q$, and a generic element $m$ in a nontrivial bundle of $Q$. We've just shown that any tubing $T$ of $Q$ must exclude at least one nontrivial element, here we assume that $m \notin \cup T$, so that $m$ is not in any tube in $T$. Since tubes of a poset are lower sets, this necessarily implies that the upper set $m^{\uparrow}$ of $m$ cannot intersect any tube in $T$, since otherwise, we would be forced to include $m$ in such a tube. In our attempt to glean information about maximal tubings excluding $m$, we will show that the connected components of $Q \setminus m^{\uparrow}$ form a tubing $Q$, and even more informative, any maximal tubing of $Q$ which excludes $m$ must include these connected components as a subtubing.

**Lemma 4.0.2.** *Given a poset $Q$ and an element $m$ in a nontrivial bundle of $Q$, the set of connected components of $Q \setminus m^{\uparrow}$ form a tubing of $Q$. Furthermore, given any tubing $T'$ of $Q$ with $\cup T' \subseteq Q \setminus m^{\uparrow}$, then the union of $T'$ with the set of connected components of $Q \setminus m^{\uparrow}$ is also a tubing of $Q$.*

*Proof.* Fix an element $m$ of $Q$ in a nontrivial $Q$-bundle and let $T_C = CC(Q \setminus m^{\uparrow})$ be the set of connected components of $Q \setminus m^{\uparrow}$. We fix an arbitrary connected component $t \in T_C$, and show that $t$ is a connected, filled, lower set of $Q$, that is, a tube of $Q$.

It is clear that $t$ is connected by definition. To see that $t$ is a lower set of $Q$, fix some $x \in Q$ with $x < y$ for some $y \in t$. Then clearly $x \in t$, since, to be less than $y$, $x$ and $y$ must be in the same connected component of $Q \setminus m^{\uparrow}$, that is, $t$.

Finally, we show that $t$ is filled in $Q$. Suppose $\partial y \subseteq t$ for some $y \in Q$. Thus we cannot have $\mathfrak{b}_y \subseteq m^{\uparrow}$ since this would imply $m \in \partial y \subseteq t \subseteq Q \setminus m^{\uparrow}$. Since we've defined connectedness of a poset in terms of comparability, it is clear that any bundle $\mathfrak{b}_y$ together with its boundary $\partial\mathfrak{b}_y$ is a connected subset of $Q$. Now we see $\mathfrak{b}_y$ intersects $Q \setminus m^{\uparrow}$, with $\partial y$ in the connected component $t$ of $Q \setminus m^{\uparrow}$ so that $\mathfrak{b}_y$ must intersect $t$. Thus $t$ is filled and $T_C$ is comprised of tubes. In order to show that $T_C$ is a tubing, we must now show that tubes in $T_C$ are pairwise nested or disjoint, and the union of any subset of tubes in $T_C$ is filled.

It is clear that since tubes in $T_C$ are connected components of $Q \setminus m^{\uparrow}$ they partition $Q \setminus m^{\uparrow}$ and are pairwise disjoint. Given any subset $\Delta \subseteq T_C$, we now show that $\cup\Delta$ is filled in $Q$ and suppose that $\partial y \subseteq \cup\Delta$ for some $y \in Q$. We mentioned above that a bundle together with its boundary is a connected subset of $Q$, so that in fact $\partial y$ is contained in a single tube: we cannot spread the boundary across several connected components, since $\partial y \subseteq \cup\Delta \subseteq Q \setminus m^{\uparrow}$. Now having $\partial y$ contained in a single tube, we know that $\mathfrak{b}_y$ intersects this tube, and $\cup\Delta$ is filled. Thus the set of connected components of $Q \setminus m^{\uparrow}$ form a tubing of $Q$.

We now fix any tubing $T'$ of $Q$ with $\cup T' \subseteq Q \setminus m^{\uparrow}$, wishing to show that $T_C \cup T'$ is a tubing of $Q$. We first show that tubes in $T_C \cup T'$ are pairwise nested or disjoint. However, we need only focus on pairs of tubes with one in $T_C$ and the other in $T'$, given that both $T_C$ and $T'$ are tubings. Since tubes in $T_C$ are connected components of $Q \setminus m^{\uparrow}$, it is clear that any tube $t' \in T'$ is contained in exactly one connected component of $P$, and disjoint from all others.

We now argue that the union of any subset of $T_C \cup T'$ is filled in $Q$. It is not hard to believe this is true, since it seems impossible that the addition of larger, filled sets to $T'$ could make any new subset union unfilled, and this is exactly the case.

Specifically, any subset of $T_C \cup T'$ is the union of $\Delta \subseteq T_C$ and $\Delta' \subseteq T'$. Note that if $\partial y$ is a subset of either $\Delta$ or $\Delta'$ alone there is nothing left to show, since $T_C$ and $T'$ are tubings. Then supposing $y \in Q$ with $\partial y \subseteq \cup\{t \mid t \in \Delta \cup \Delta'\}$, we may assume $\partial y$ intersects $\cup\Delta$. In this case $\partial y \subseteq \cup\Delta$ since $\partial y$ is connected and $\Delta$ is a set of connected components. Thus we see that $T_C \cup T'$ is a tubing of $Q$ as desired.

□

We are now in a position to prove a very useful result about arbitrary maximal tubings of a poset. Namely, Lemma 4.0.1 tells us that no tubing of a poset $Q$ can contain every nontrivial element. When we consider any maximal tubing $T$ of $Q$, we will see that maximality forces $T$ to include all but one nontrivial element.

**Lemma 4.0.3.** *Given a proper maximal tubing $T$ of a poset $Q$, there exists a unique element $m$ of $Q$ such that $m$ is in a nontrivial bundle and $m$ is not in any tube in $T$. Furthermore, we have that $\cup T = Q \setminus m^{\uparrow}$.*

*Proof.* Given the maximal tubing $T$ of poset $Q$, Lemma 4.0.1 guarantees the existence of an element in a nontrivial bundle and not in $\cup T$. Among such elements, we choose $m$ so that the upper set $m^{\uparrow}$ induced by $m$ contains no elements besides $m$ in a nontrivial bundle. This is not a stringent requirement, for if we chose $m$ otherwise, we could simply take instead an $\hat{m} \in m^{\uparrow}$ fitting our needs, by finiteness of $Q$.

Knowing that $m$ is not in $\cup T$, it follows that $m^{\uparrow}$ is not a subset of $\cup T$, which we may rephrase as the statement that $\cup T \subseteq Q \setminus m^{\uparrow}$. Now our previous Lemma 4.0.2 states that the connected components of $Q \setminus m^{\uparrow}$ form a tubing of $Q$, which we denote by $T_C$ so that $T_C = CC(Q \setminus m^{\uparrow})$. Furthermore, since $\cup T \subseteq Q \setminus m^{\uparrow}$, we have $T \cup T_C$ is a tubing of $Q$. However, maximality of $T$ as a tubing of $Q$ means no proper tubings of $Q$ properly contain $T$, from which we conclude that $T_C \subseteq T$. Observing that $\cup T_C = Q \setminus m^{\uparrow}$, we see $\cup T = Q \setminus m^{\uparrow}$ and conclude that every nontrivial element of $Q$ except $m$ is in $\cup T$ since $m$ is the only nontrivial element in $m^{\uparrow}$.

□

When looking at an arbitrary poset, there is generally very little one can say. Our results above have shown that, despite the many forms a poset may take, once we are dealing with a maximal tubing $T$ of $Q$, we are actually given a very specific piece of information, that a unique element in some nontrivial bundle is not in $\cup T$. Thus, there is an interplay between elements, maximal among those in nontrivial bundles, and the possibilities for $\cup T$. It may come as a surprise then, that maximal tubings pose "just the right amount" of restriction.

Specifically, we would like 1-sections of $\mathcal{K}P$ to contain precisely four elements at all times. We will see this is quite easily verified for 1-sections which do not contain $\hat{0}$, the unique minimal element of $\mathcal{K}P$. What will require more work, and the insight provided by our discussion of elements in nontrivial bundles, is verifying that 1-sections of the form $F/\hat{0}$ contain precisely two maximal tubings of $P$. We will work to show precisely this result, broken into steps. Any 1-section of the form $F/\hat{0}$ is bounded below by $\hat{0}$ of dimension $-1$ in $\mathcal{K}P$, and bounded above by an "almost maximal" tubing $F$, of cardinality one less than the cardinality of a maximal tubing of $P$. Thus, in order to consist of four total elements, the 1-section $F/\hat{0}$ must contain exactly two maximal tubings, each containing the tubing $F$.

Thus, fixing an "almost maximal" tubing $F$ of $P$, we see that since $F$ isn't maximal, there exists some tube $t$ of $P$ so that $T = F \cup \{t\}$ is a maximal tubing of $P$. Having in mind the requirement that $F/\hat{0}$ must contain exactly two maximal tubings, we must show that there exists exactly one other tube $t' \neq t$ of $P$ so that $T' = F \cup \{t'\}$ is also a maximal tubing of $P$. We will prove this using the idea of a unique "replacement" tube. That is, given any maximal tubing $T$ of $P$ and any tube $t$ in $T$, we will show that there exists a unique tube $t' \neq t$ so that $T' = T \setminus \{t\} \cup \{t'\}$ is a maximal tubing of $P$.

Our next result will lead almost effortlessly to this statement, although it may appear weaker. We first set the stage, as we will be required to pay close attention to the relationships between various tubes comprising the maximal tubing $T$ of $Q$. By the definition of tubing, we have a powerful tool: tubes in a tubing must be pairwise nested or disjoint. This leads us to single out a special class of tubes:

**Definition 21.** Given a tubing $T$ of a poset $P$ and a tube $t$ in $T$, we say $t$ is *maximal in* $T$ if, for any tube $t'$ in $T$, either $t$ and $t'$ are disjoint or $t$ contains $t'$. Likewise, $t$ is *minimal in* $T$ if, for any tube $t'$ in $T$, either $t$ and $t'$ are disjoint, or $t'$ contains $t$.

We have seen maximal tubes earlier, when discussing maximal tubings. For example, given a maximal tubing $T$ of $P$, we saw that $\cup T = P \setminus m^{\uparrow}$ for some unique element $m$ in a nontrivial bundle of $P$. We showed that the set of connected components of $P \setminus m^{\uparrow}$ form a subtubing of $T$, and it is not hard to see that each connected component is maximal in $T$, since every tube in $T$, being a connected subset of $P \setminus m^{\uparrow}$, must either be disjoint from or contained in any connected component.

We also remark, for tubes $t$ which are not maximal in a tubing $T$, there exists a least upper bound in $T$: some $\hat{t} \in T$ for which there exist no tube $t' \in T$ such that $t \subset t' \subset \hat{t}$. This is a straightforward observation, since tubes containing $t$ must intersect nontrivially, and are hence nested. It does not require a formal proof, but it will be used later, so we emphasize this remark now.

**Lemma 4.0.4.** *Given any tube $t$ which is not maximal in a tubing $T$, there exists a unique tube $\hat{t}$ in $T$ such that $t \subset t' \subseteq \hat{t}$ implies $t' = \hat{t}$.*

For tubes $t$ which are not minimal in $T$, we may say just as much about tubes $t''$ where $t'' \subset t$, but our conclusion is slightly more complicated, and may only be precisely stated for tubes in a maximal tubing. We will need to think more deeply about this situation.

Consider a tube $t$ of a maximal tubing $T$ which is not minimal in $T$. By the definition of a tube, $t$ is necessarily an induced subposet of $P$, and we call attention to this fact by thinking of $t$ as the poset $P_t$. It should not be surprising that the set of tubes properly contained in $P_t$ form a tubing of $P_t$, and furthermore, this subtubing $T_t = \{t'' \in T \mid t'' \subset t\}$ is actually a maximal tubing of $P_t$. Otherwise, maximality of the tubing $T$ of $P$ would be violated. Thus we may apply Lemma 4.0.3 to obtain the unique element $m_t \in P_t$ such that $m_t \notin \cup T_t$ with $m_t$ in a nontrivial bundle of $P_t$. We know that $m_t^{\uparrow}$ contains no elements of $P_t$ besides $m_t$ which belong to nontrivial bundles, and it is here that we may lose uniqueness of "greatest" tubes properly contained in $t$. Lemma 4.0.2 shows that $CC(P_t \setminus m_t^{\uparrow})$ is a subtubing of $T$. Further, $CC(P_t \setminus m^{\uparrow})$ is comprised of tubes which are maximal with respect to "being properly contained in $t$," since all tubes contained in $P_t$ must not intersect $m_t^{\uparrow}$, hence are contained in one of these connected components.

Of the remarks above, one in particular will be used repeatedly in the future, and we give a proof of this collection of facts now.

**Lemma 4.0.5.** *Given a tube $t$ in a tubing $T$ of a poset $P$, then the collection $T_t$ of tubes properly contained in $t$ is a tubing of $t$ when considering $t$ as a poset. Furthermore, if $T'$ is any tubing of $t$ of which $T_t$ is a subtubing, then $T \cup T'$ is a tubing of $P$.*

*Proof.* In a sense, this is a generalization of Lemma 4.0.2, where we saw that the set of connected components of a particular lower set form a tubing, and that this tubing gets along nicely with tubings of a specific form.

We begin by fixing a tube $t$ in a tubing $T$ of a poset $P$, and to emphasize the fact that we're considering $t$ as a poset, we denote this poset by $P_t$. We let $T_t = \{t' \in T \mid t' \subset t\}$ be the set of tubes in $T$ properly contained in $P_t$. It is trivial that $T_t$ is a tubing of $P_t \subset P$, and the reader may verify that all required properties are inherited instantly from $P$, given only that $T_t$ is a tubing of $P$ and that $P_t$ is a subset of $P$. The real purpose of this lemma is to verify the less than obvious claim that for any tubing $T'$ of $P_t$ which contains $T_t$ as a subtubing, we have $T' \cup T_t$ is a tubing of $P$.

Fix any tubing $T'$ of $P_t$ such that $T_t \subseteq T'$. We first verify that tubes in $T'$ are tubes of $P$, and fix any tube $t' \in T'$ so that $t'$ is a proper subset of $P_t$. It is clear that $t'$ is connected in $P$, and it is not much harder to show that a lower set of $P_t$, where $P_t$ is itself a lower set of $P$, is a lower set of $P$ also.

Finally, to verify that tubes in $T'$ are tubes of $P$, we now show that the tube $t'$ is filled in $P$: for any $y \in P$ with $\partial y \subseteq t'$, we must have that $\mathfrak{b}_y \cap t' \neq \emptyset$. Now let $y$ be any element of $P$ with $\partial y \subseteq t' \subset P_t$. Since $P_t$ is filled in $P$, we then have some $y' \in \mathfrak{b}_y$ with $y' \in P_t$. Thus, we now have that $y' \in P_t$ with $\partial y' = \partial y \subseteq t'$, so that $\mathfrak{b}_y \cap t' = \mathfrak{b}_{y'} \cap t' \neq \emptyset$, since $t'$ is filled in $P_t$.

It remains to show that $T \cup T'$ is a tubing of $P$. We simply need that tubes in $T \cup T'$ are pairwise nested or disjoint, and that the union of any subset of $T \cup T'$ is filled in $P$.

First, to see that tubes in $T \cup T'$ are pairwise nested or disjoint, we recall the tubings involved. We have $T$, the tubing of $P$, with $T_t$ the set of tubes in $T$ properly contained in $t \in T$. Finally, we have that $T_t \subseteq T'$, with $T'$ some tubing of $t$. Thus $T \cap T' = T_t$, where $T, T'$ and $T_t$ are all tubings, hence contain pairwise nested or disjoint tubes. Finally, it is clear that any tube in $T \setminus T_t$ is either disjoint from or contains $t$, where $t$ contains all tubes in $T'$. Thus tubes in $T \cup T'$ are pairwise nested or disjoint.

Finally, we verify that the union of any subset of $T \cup T'$ is filled. We think of $T \cup T'$ as the disjoint union $(T \setminus T_t) \sqcup T'$, and write a generic subset of $T \cup T'$ as $\Delta \cup \Delta'$, with $\Delta \subseteq T \setminus T_t$ and $\Delta' \subseteq T'$. The discussion above, in which we showed that tubes in $T \cup T'$ are pairwise nested or disjoint, leads us to conclude that $\cup\Delta$ either contains or is disjoint from $\cup\Delta'$. The case in which $\cup\Delta' \subseteq \cup\Delta$ is trivial since $\Delta$ is a subset of the tubing $T$, and we assume that $\cup\Delta$ and $\cup\Delta'$ are disjoint. We will show that $\partial y$ is contained entirely in one of $\Delta$ or $\Delta'$, from which the conclusion follows since $\Delta$ and $\Delta'$ are subsets of tubings.

Now let $y$ be any element of $P$ with $\partial y \subseteq \bigcup(\Delta \cup \Delta')$. Given $\partial y \subseteq \bigcup(\Delta \cup \Delta')$, then certainly $\partial y \subseteq \bigcup(\Delta \cup \{t\})$, since $\cup\Delta' \subseteq t$. We then conclude that $\mathfrak{b}_y$ intersects $\bigcup(\Delta \cup \{t\})$, since $\Delta \cup \{t\}$ is a subset of $T$ hence its union must be filled. If $\mathfrak{b}_y$ intersects $\cup\Delta$, we're done since $\Delta$ is a subset of the tubing $T$. Otherwise, we have that $\mathfrak{b}_y$ intersects $t$ with $\cup\Delta' \subseteq t$. Then certainly $\partial y \subseteq \cup\Delta'$ since $\cup\Delta$ and $\cup\Delta'$ are disjoint. □

During this proof, we attained a result which will be widely useful. We highlight this achievement now.

**Corollary 4.0.6.** *Given a tube $t$ in a maximal tubing $T$ of $P$, denote by $T_t$ the set of tubes properly contained in $t$. Then $T_t$ is a maximal tubing of $t$, and there exists a unique element $m_t$ in a nontrivial $t$-bundle for which $t \setminus \cup T_t = m_t^{\uparrow}$. Furthermore, $m_t$ is the unique element in the upper set $m_t^{\uparrow}$ which is in a nontrivial $t$-bundle.*

*Proof.* We've just seen that any tubing $T'$ of $t$ which contains $T_t$ is compatible with any tubing $T$ of $P$ containing $t$ in the sense that $T \cup T'$ is also a tubing of $P$.

However, if $T_t$ isn't a maximal tubing of $t$ and such a $T'$ properly containing $T_t$ exists, then certainly $T \cup T'$ would be a tubing of $P$ properly containing $T$, contradicting maximality of $T$ as a tubing of $P$. Then we must have that $T_t$ is a maximal tubing of $t$, and we may apply Lemma 4.0.3 from which the rest of the statement follows. $\square$

Our first result toward showing that $\mathcal{K}P$ has the structure of an abstract polytope will be verifying that flags of $\mathcal{K}P$ contain $d+2$ faces. However, from the axiom's point of view, the specific value of $d$ is largely irrelevant. All that is important is that flags are of the same length.

To this end, we will show that maximal tubings of $P$ have the same cardinality. This will imply that any flag, that is, maximal chain of tubings, contains the same number of tubings. However, in order to attain this result, we will show that maximal tubings of $P$ contains a specific number of tubes. This will be our dimension formula, and it is obtained knowing only the number of nontrivial bundles of elements of $P$, and how many elements are collectively contained in this set of nontrivial bundles.

**Theorem 4.0.7.** *Let $\mathfrak{B}$ be the set of nontrivial bundles of a poset $P$. Then maximal tubings of $P$ contain $|\cup \mathfrak{B}| - |\mathfrak{B}|$ tubes.*

*Proof.* Fix a poset $P$, and let $\mathfrak{B}$ be the set of nontrivial bundles of elements of $P$. Let $T$ be any maximal tubing of $P$. For any tube $t$ in $T$, let $T_t = \{t' \in T \mid t' \subset t\} \subseteq T$ be the subtubing of $T$ consisting of all tubes properly contained in $t$. For the entirety of this proof, we will mean elements in a nontrivial bundle when we speak of elements of $P$, that is, elements in $\cup\mathfrak{B}$.

Now for each tube $t$ in $T$, we consider the set of elements of $P$ which are "added" to $\cup T$ by $t$. That is, elements $x \in \cup\mathfrak{B}$ for which $t$ is the smallest tube in $T$ which contains $x$. We denote this set of new elements "added" to $\cup T$ via $t$ by $\nu(t) = \{x \in \cup\mathfrak{B} \mid x \in t \setminus \cup T_t\}$, where $T_t$ is the set of tubes in $T$ properly contained in $t$. For this discussion, we think of $\nu(t)$ as an induced subposet of $P$, induced by the elements listed above. Note that in general, since $\nu(t)$ is concerned only with elements in nontrivial bundles, it is quite possibly a proper subset of $t \setminus \cup T_t$.

Now, Lemma 4.0.3 tells us that a unique nontrivial element is not in $\cup T$, and it should be obvious that $\nu(t)$ and $\nu(t')$ are disjoint for all distinct tubes $t, t'$ in $T$. Thus, we conclude

$$\sum_{t\in T} |\nu(t)| = |\cup\mathfrak{B}| - 1. \tag{4.1}$$

It should also be obvious that $\nu(t)$ is nonempty for every tube $t$ in $T$, so that we now consider the sum

$$\begin{aligned}\sum_{t\in T} \big(|\nu(t)| - 1\big) &= \sum_{t\in T} |\nu(t)| - |T| &(4.2)\\ &= |\cup\mathfrak{B}| - 1 - |T| &(4.3)\end{aligned}$$

which was obtained using Equation 4.1. Now we can determine the size of $T$, isolating

$$|T| = |\cup\mathfrak{B}| - 1 - \sum_{t\in T} \big(|\nu(t)| - 1\big). \tag{4.4}$$

We would like to show that $|T| = |\cup\mathfrak{B}| - |\mathfrak{B}|$, which will follow if we can verify

$$\sum_{t\in T} \big(|\nu(t)| - 1\big) = |\mathfrak{B}| - 1 \tag{4.5}$$

for the generic maximal tubing $T$ of $P$.

Since we have noted that $|\nu(t)| \geq 1$ for all tubes $t$ in $T$, our goal will be most easily achieved by focusing on tubes $t^*$ for which $|\nu(t^*)| > 1$. We denote the set of tubes in $T$ for which this is true by $T^*$, so that $T^* = \{t \in T : |\nu(t)| > 1\} \subseteq T$. In fact, it is easily verified that tubes in $T \setminus T^*$ have a trivial effect on Equation 4.5. Thus, we now focus on $T^*$ and the conclusion follows once we show

$$\sum_{t^* \in T^*} \big(|\nu(t^*)| - 1\big) = |\mathfrak{B}| - 1. \tag{4.6}$$

To start, by Lemma 4.0.5 we know that $T_t$ is a maximal tubing of the tube $t$. Thus, Lemma 4.0.3 tells us that there is a unique element in a nontrivial $t$-bundle in $t \setminus \cup T_t$ which we denote by $m_t$. Furthermore, we have that $t \setminus \cup T_t = t \cap m_t^{\uparrow}$, where we know that $m_t$ is in a nontrivial $P$-bundle since it is in a nontrivial $t$-bundle. Thus, we know that $m_t \in \nu(t)$ and that $m_t$ is the unique minimum of $\nu(t)$.

We now have a combinatorial interpretation of $|\nu(t)| - 1$ as the size of the set $\nu(t) \setminus \{m_t\}$. We will use this interpretation to verify Equation 4.6, by showing that each nonminimal nontrivial bundle in $\mathfrak{B}$ intersects exactly one $\nu(t)$ in exactly one element.

As an introduction, consider Figure 4.1, in which we have a tubing $T$ of poset $P$. On the right, we consider the set $\mathfrak{B}$ of nontrivial bundles, collapsing each nontrivial bundle to a point. We use the ordering $\leq$ on $P$ to order $\mathfrak{B}$ by declaring $\mathfrak{b}_x \leq \mathfrak{b}_y$ if there exists $x, y \in P$ with $x \in \mathfrak{b}_x,\ y \in \mathfrak{b}_y$ and $x \leq y$. It is easy to verify

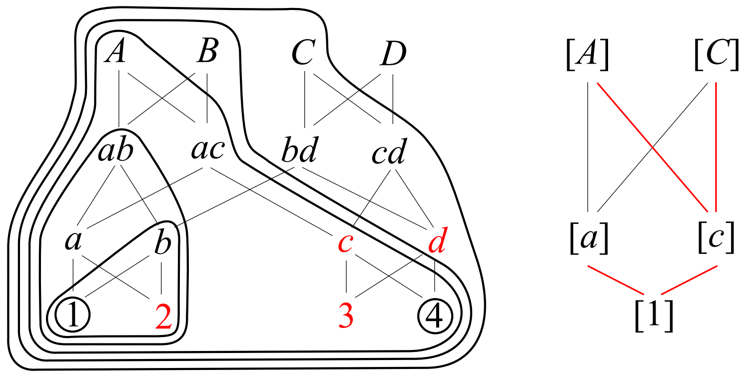



Figure 4.1: A tubing and associated tree on the nodes of quotient poset.

that $(\mathfrak{B}, \leq)$ is a poset, given that $(P, \leq)$ is a poset (In fact, verifying that $\leq$ is still antisymmetric on $\mathfrak{B}$ is a good exercise).

Writing the set of tubes $T = \{1, 4, b, ab, A, AB, ABC\}$ where, for example, by the tube $ABC$ we really mean the lower set $\{ABC\}_{\downarrow} \subseteq P$ induced by $\{ABC\}$. Now we see that $T^* = \{b, A, ABC\}$ because, for example, $\nu(b) = \{2, b\}$ of cardinality greater than 1. Thus, we see that the element $b$ in the nontrivial bundle $\mathfrak{b}_a$ is nonminimal in $\nu(b)$. Likewise, the elements $c$ and $A$ in bundles $\mathfrak{b}_c$ and $\mathfrak{b}_A$ are nonminimal in $\nu(A)$. Finally, the element $C$ is nonminimal in $\nu(ABC)$. Thus, all $|\mathfrak{B}| - 1 = 4$ nonminimal bundles in $\mathfrak{B}$ are represented uniquely by the above elements. Note that here the number of elements in nontrivial bundles is $|\cup \mathfrak{B}| = 12$ while there are $|\mathfrak{B}| = 5$ nontrivial bundles, while the maximal tubing has 7 tubes, as desired.

We must first show that the poset $(B, \leq)$ has a unique minimum bundle whenever $\mathfrak{B} \neq \emptyset$. The minimal elements $\{x \in P \mid \partial x = \emptyset\}$ form a nontrivial bundle which is certainly minimal among nontrivial bundles, provided $P$ doesn't have a unique minimal element. If $P$ does have a unique minimal element, then there is some least element which has more than one upper cover, since $\mathfrak{B} \neq \emptyset$. This set of upper covers are the minimal nontrivial bundle in $\mathfrak{B}$, since having a unique minimum means $P$ must be a connected poset.

We haven't considered the case when $\mathfrak{B} = \emptyset$. In this situation, $\mathcal{K}P$ should have dimension $|\cup \mathfrak{B}| - |\mathfrak{B}| = 0$ by the dimension formula. This is also implied by Lemma 4.0.1, in which we saw that any proper tubing must not contain every nontrivial element. However, if $\mathfrak{B} = \emptyset$, then we see that $P$ has no proper tubings, and must have dimension 0.

Our first observation to verify Equation 4.0.7 is that $\nu(t)$ intersects any $P$-bundle in at most one element for each $t \in T$. From Lemma 4.0.5, we may consider $T_t$ as a maximal tubing of $t$. Thus, we may apply Lemma 4.0.3 to receive $m_t$, the unique element in a nontrivial $t$-bundle which is in $t \setminus \cup T_t$. Thus, if $\nu(t)$ contained distinct elements from the same $P$-bundle, they would necessarily belong to the same $t$-bundle, contradicting Lemma 4.0.3 since $\nu(t)$ is a subset of $t \setminus \cup T_t$.

To verify that

$$\sum_{t^* \in T^*} \big(|\nu(t^*)| - 1\big) = |\mathfrak{B}| - 1,$$

we establish a bijection between

$$\bigcup_{t^* \in T^*} \nu(t^*) \setminus \{m_{t^*}\} \quad \text{and} \quad \mathfrak{B} \setminus \mathfrak{b}_0,$$

where $\mathfrak{b}_0$ is the unique minimal element in $\mathfrak{B}$. An injection from the left set into the right set is straightforward, since we simply send $x \in \nu(t) \setminus \{m_t\}$ to the bundle $\mathfrak{b}_x$, where $\mathfrak{b}_x$ cannot be minimal in $\mathfrak{B}$ since $x$ is not minimal in $\nu(t)$.

Now, injecting $\mathfrak{B} \setminus \{\mathfrak{b}_0\}$ into $\bigcup_{t^* \in T^*} \nu(t^*) \setminus \{m_{t^*}\}$ is not as simple. Given a nonminimal bundle $\mathfrak{b}_x$ in $\mathfrak{B}$, we need some unique $y \in \mathfrak{b}_x$ with $y \in \nu(t) \setminus \{m_t\}$ for some $t \in T^*$. We focus first on existence of such an element, then uniqueness.

Consider the set of tubes in $T$ which intersect the bundle $\mathfrak{b}_x$. Since these tubes contain $\partial x$, they must be linearly ordered by subset inclusion. We claim that the smallest tube $t$ which intersects $\mathfrak{b}_x$ is such that $\mathfrak{b}_x$ intersects $\nu(t)$ in an element $y$ which is not minimal in $\nu(t)$. If $y$ were minimal in $\nu(t)$, then we recall $\cup T_t = t \setminus y^{\uparrow}$ so that $\partial y$ is a subset of $\cup T_t$ and yet $\mathfrak{b}_y$ doesn't intersect $\cup T_t$. This would violate the assumption that $T$ is a tubing since the union $\cup T_t$ isn't filled.

To see that this element must be unique, we assume that we have $y, z \in \mathfrak{b}_x$ with $y$ nonminimal in $\nu(t)$ and $z$ in $\nu(t')$ and show that $z$ must be minimal in $\nu(t')$. First, since $y$ is nonminimal in $\nu(t)$, we see that $\partial y$ intersects $\nu(t)$. Thus, since $\nu(t)$ and $\nu(t')$ cannot intersect, we must have $t$ contained in $t'$ since each must contain the boundary $\partial y$. Now it is straightforward to see that $z$ is minimal in $\nu(t')$ since $\nu(t')$ doesn't intersect $\partial y$.

$\square$

Having that the poset assosciahedron $\mathcal{K}P$ satisfies axiom (AP II), we work towards the remaining to axioms which must be satisfied. The last two axioms will not require much more work than we have developed up to this point, although the results themselves will be fairly detailed. We also postpone working on the third axiom in favor of the fourth. Specifically, we will derive helpful results on our way to see that $\mathcal{K}P$ satisfies the fourth axiom, that 1-sections of $\mathcal{K}P$ contain four elements. These results will be of great use to verify that $\mathcal{K}P$ is strongly flag-connected, and this is the main reason why we pursue the fourth axiom currently.

It will turn out that what distinguishes tubings of $P$ is how nontrivial elements are incorporated into the tubing. That is not to say that trivial elements have no affect on $\mathcal{K}P$, for they will contribute to connectivity, generally allowing tubes to encompass larger portions of $P$ than when trivial elements are removed from $P$. Be that as it may, we will be able to say much more about $\mathcal{K}P$ if we understand more fully the role of nontrivial elements in the formation of $\mathcal{K}P$.

We have just seen that all maximal tubings of a poset have the same cardinality. Recalling several relationships between tubes in a given tubing, we are now ready to prove the current lemma, that for any tube $t$ which is maximal in a maximal tubing $T$ of $Q$, there exists a unique tube $t'$ of $Q$ such that $T' = T \setminus \{t\} \cup \{t'\}$ is a maximal tubing of $Q$. This will ultimately lead to the fourth axiom $\mathcal{K}P$ must satisfy to be an abstract polytope, that 1-sections of $\mathcal{K}P$ contain exactly four elements.

**Lemma 4.0.8.** *Given a maximal proper tubing $T$ of a poset $Q$ and a tube $t$ maximal in $T$, then there exists a unique tube $t'$ of $Q$ such that $T' = T \setminus \{t\} \cup \{t'\}$ is a maximal tubing of $Q$. Furthermore, if $T_t$ is the set of tubes in $T$ properly contained in $t$ so that $\cup T_t = t \setminus m_t^{\uparrow}$, then the replacement $t'$ does not contain $m_t$.*

*Proof.* We need to construct a tubing $T' = T \setminus \{t\} \cup \{t'\}$. In particular, we must have $T \cap T' = T \setminus \{t\}$. Now, recalling Lemma 4.0.3, we may write $\cup T = P \setminus m^{\uparrow}$ and $\cup T' = P \setminus \hat{m}^{\uparrow}$ where $m$ and $\hat{m}$ are in nontrivial $P$-bundles, since $T$ and $T'$ are maximal tubings of $P$. This, paired with Lemma 4.0.2, immediately tells us that $\hat{m} \neq m$, otherwise $t \in T'$ since $t$ is a connected component of $P \setminus m^{\uparrow}$.

Thus, since $m$ is the unique element in $m^{\uparrow}$ in a nontrivial $P$-bundle where $\hat{m} \neq m$, we conclude that $\hat{m} \in \cup T$. However, we also note that $\cup T' = P \setminus \hat{m}^{\uparrow}$, in particular, $\hat{m} \notin \cup T'$. Thus, any tube in $T$ containing $\hat{m}$ is not in $T'$, while $T \setminus T' = \{t\}$. We then conclude that $\hat{m}$ is in $t$, the tube to be replaced, and not in any tube properly contained in $t$. That is, $\hat{m} \in t \setminus \cup T_t$.

We now note, by Lemma 4.0.5, that $T_t$ is a maximal tubing of the poset $t$, so that $\cup T_t = t \setminus m_t^{\uparrow}$, or equivalently, $t \setminus \cup T_t = m_t^{\uparrow}$. Recall also that $m_t$ is the unique element in $m_t^{\uparrow}$ in a nontrivial $t$-bundle. Now, applying the result of the preceding paragraph, we have that $\hat{m} \in m_t^{\uparrow}$, and consequently, that $\hat{m}$ is in a trivial $t$-bundle if $\hat{m} \neq m_t$. However, since $\cup T' = P \setminus \hat{m}^{\uparrow}$, we must have that $\hat{m}$ is in a nontrivial $P$-bundle, in fact, the only element in $\hat{m}^{\uparrow}$ in a nontrivial $P$-bundle. We now claim that there is a unique element in $m_t^{\uparrow}$ satisfying these criteria: an element who is the only element in its own upper set in a nontrivial $P$-bundle which is contained in $m_t^{\uparrow}$. Note that up to this point, we have simply investigated properties of $\hat{m}$ needed to find the maximal tubing $T' = T \setminus \{t\} \cup \{t'\}$, where $\cup T' = P \setminus \hat{m}^{\uparrow}$. Thus, showing that such an element $\hat{m}$ is unique will ensure that the replacement tube $t'$ for $t$ is unique.

Recall the elements to whom we have called attention. We have $m$, the unique element in a nontrivial $P$-bundle which is not in $\cup T$. We also have $m_t$, the unique element in a nontrivial $t$-bundle which is not in $\cup T_t$, so that $m_t^\uparrow \setminus \{m_t\}$ contains only elements in trivial $t$-bundles. Note that, looking in $m_t^\uparrow$, an element in a nontrivial $P$-bundle exists, namely $m_t$, since $t$-bundles must be subsets of $P$-bundles, so that in fact $m_t$ is in a nontrivial $P$-bundle. We now show that there exists a unique element $\hat{m}$ in $m_t^\uparrow$ with $\hat{m}$ the only element in $\hat{m}^\uparrow$ in a nontrivial $P$-bundle. Note that if $m_t$ is the unique element in $m_t^\uparrow$ in a nontrivial $P$-bundle, there is nothing left to show that $\hat{m} = m_t$ is unique.

We thus suppose there exist $\hat{m}, \tilde{m}$ satisfying the criteria (minus uniqueness) of $\hat{m}$ above. Further, we assume that neither is $m_t$. Since $\hat{m}, \tilde{m} \in m_t^\uparrow \setminus \{m_t\}$, they must belong to distinct trivial $t$-bundles, and we've assumed they belong to nontrivial $P$-bundles. Since they belong to nontrivial $P$-bundles but trivial $t$-bundles, there must exist some $\hat{m}'$ and $\tilde{m}'$ belonging to the same nontrivial $P$-bundle as $\hat{m}$ and $\tilde{m}$ respectively, with $\hat{m}'$ and $\tilde{m}'$ not in $t$. But then we have violated uniqueness of $m$ as the sole element of $P$ not in $\cup T$ belonging to a nontrivial $P$-bundle. Thus, we have singled out $\hat{m}$ to fill the role of the unique element in a nontrivial $P$-bundle with $\hat{m} \notin \cup T'$, on our quest to construct the tubing $T' = T \setminus \{t\} \cup \{t'\}$. The interested reader will note that whenever $\hat{m} \neq m_t$, then $\hat{m}$ must be in the same $P$-bundle as $m$. We have yet to construct $t'$, but the preceding paragraphs have ensured that, should we find a compatible replacement, it is indeed unique.

In fact, at this point, constructing $t'$ is almost trivial. We have showed that, when replacing $t \in T$ with $t'$ so that $T' = T \setminus \{t\} \cup \{t'\}$, there is a unique $\hat{m}$ with

$\cup T' = P \setminus \hat{m}^{\uparrow}$ so that $T \cap T' = T \setminus \{t\}$. Then, since $T'$ is a maximal tubing of $P$, we have $CC(P \setminus \hat{m}^{\uparrow}) \subseteq T'$, while $T' \setminus T = \{t'\}$. Finally, since $CC(P \setminus m^{\uparrow}) \neq CC(P \setminus \hat{m}^{\uparrow})$, we see that $t'$ is the unique tube in $CC(P \setminus \hat{m}^{\uparrow})$ and not in $CC(P \setminus m^{\uparrow})$.

□

The above argument became rather detailed and technical, but it contains a fact which we will call upon when we see that $\mathcal{K}P$ is flag-connected. It concerns the element $\hat{m}$, for which $\cup T' = P \setminus \hat{m}^{\uparrow}$, and the reader is advised to keep its properties in mind: If $\cup T_t = t \setminus m_t^{\uparrow}$, then $\hat{m}$ is the unique element in $m_t^{\uparrow}$ satisfying the requirement that $\hat{m}^{\uparrow} \setminus \{\hat{m}\}$ consists of elements in trivial $P$-bundles.

We are now able to state a crucial result, that we may find a unique replacement for *any* tube in a maximal tubing, not just those which are maximal in the tubing. The latter case was just taken care of, and it required all of our preliminary results. Even then, the proof itself required some fairly heavy lifting. We will see very shortly that this work leads almost effortlessly to verifying that all tubes in a maximal tubing have a unique replacement tube.

As an example, Figure 4.2 shows various tubings of a fixed poset. In the center is the tubing whose tubes we would like to individually replace. The replacement tubes are outlined in blue, and we see four new tubings, corresponding to replacing each of the four original tubes individually. These new tubings contain all of the tubes in the original tubing except for the tube which has been replaced. The interested reader will see several examples of the process of replacing a tube which is maximal.

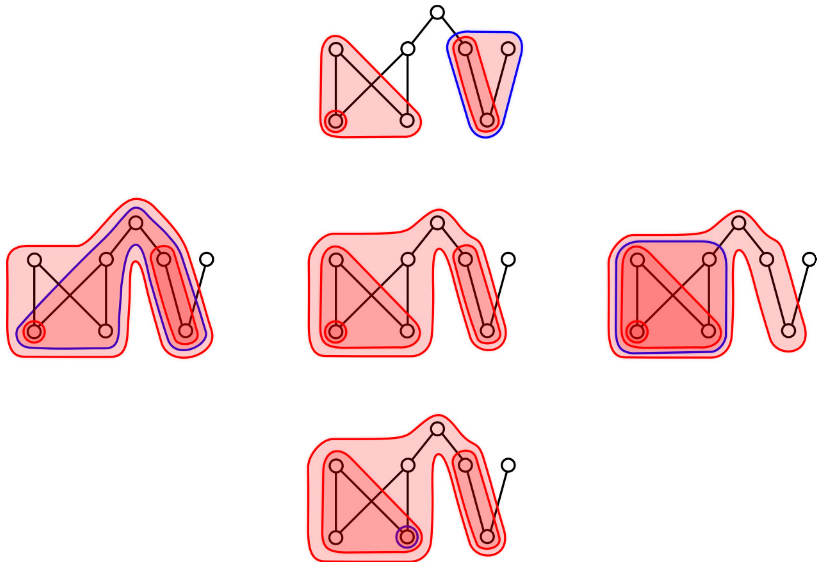


Figure 4.2: Replacement tubes and the new tubings they create.

**Lemma 4.0.9.** *Given a maximal tubing $T$ of a poset $Q$, then for any tube $t \in T$, there exists a unique tube $t'$ of $Q$ such that $T' = T \setminus \{t\} \cup \{t'\}$ is a maximal tubing of $Q$.*

*Proof.* Fix any maximal tubing $T$ of $Q$, and any tube $t$ in $T$. The case when the tube $t$ is maximal in $T$ was covered in Lemma 4.0.8, and we now assume that $t$ is not maximal in $T$.

We noted earlier in Lemma 4.0.4 that $t$, an arbitrary nonmaximal tube in $T$, can be associated with a unique tube $\hat{t}$ which is the smallest tube in $T$ properly containing $t$. We will use this observation and consider $\hat{t}$ as a poset, so that the subtubing $T_{\hat{t}} = \{t'' \in T \mid t'' \subset \hat{t}\} \subseteq T$ is a maximal tubing of $\hat{t}$, by Lemma 4.0.5. Viewed in this light, we now see that $t$ is maximal in $T_{\hat{t}}$, so that Lemma

4.0.8 now applies, and we are granted a unique replacement tube $t'$ for $t$, so that $T'_{\hat{t}} = T_{\hat{t}} \setminus \{t\} \cup \{t'\}$ is a maximal tubing of $\hat{t}$. Thus, the tube $t'$ is a replacement for $t$ in the tubing $T_{\hat{t}}$, and consequently in $T$ as well, with uniqueness guaranteed by Lemma 4.0.5. □

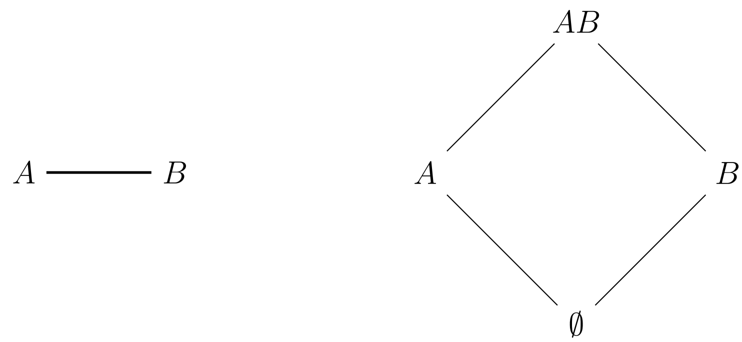
We have, up to this point, shown that the poset associahedron $\mathcal{K}P$ satisfied axioms I, II, and III that a poset must satisfy in order to be an abstract polytope. We have also done much work needed to verify that the fourth and final axiom is also satisfied, that every 1-section of $\mathcal{K}P$ contains exactly four elements. We give this proof now.

**Theorem 4.0.10.** *In the poset associahedron $\mathcal{K}P$, any 1-section is isomorphic to the face lattice of a line segment.*

We begin with a brief description of the line segment and its associated face lattice. This lattice has exactly four elements: the line segment itself as a maximal element, the null polytope as a minimal element, and two proper faces corresponding to the end points of the segment. These four faces are arranged as a rank 1 lattice, with the endpoints lying under the whole segment and over the null polytope.

*Proof.* Fix any two faces $F < H$ of $\mathcal{K}P$ with rank $H =$ rank $F + 2$. Considering the 1-section $H/F$ of $\mathcal{K}P$, we want exactly two proper faces, $G_1$ and $G_2$, in $H/F$, each with rank one more than that of $F$.

Consider the case when $F \neq \hat{0}$, where our assertion is extremely straight forward. By the construction of the lattice $\mathcal{K}P$, we have proper tubings $H \subseteq F$




(a) The line segment $AB$ (b) The face lattice of $AB$

associated with the faces $F$ and $H$. Furthermore, since $H/F$ is a 1-section, we have that $|F| = |H| + 2$. This means that $F = H \cup \{t_1, t_2\}$ for some tubes $t_1, t_2$ of $P$. Thus, $H \cup \{t_1\}$ and $H \cup \{t_2\}$ are precisely the two proper faces in $H/F$, lying strictly between $F$ and $H$.

We have essentially covered the case when $F = \hat{0}$ in Lemma 4.0.9, where we showed that it is always possible to find a unique replacement for any tube in a maximal tubing. Viewed from this light, our final class of 1-sections are those $F/\hat{0}$, where the face $F$ corresponds to an 'almost maximal' tubing of $P$, that is, tubings $F$ for which any proper tubing containing $F$ is maximal. The assertion of Lemma 4.0.9 ensures us precisely two distinct, proper faces in the 1-section $F/\hat{0}$. □

We now focus on the third and final axiom posets must satisfy to be an abstract polytope, namely that $\mathcal{K}P$ is strongly flag-connected. For this, we must show that every section of $\mathcal{K}P$ is flag-connected.

Recall that given a face $F$ of an abstract polytope $\mathcal{P}$ and another face $G$

containing $F$, the section $G/F = \{H \in \mathcal{P} \mid F \leq H \leq G\}$ is simply the set of faces of $\mathcal{P}$ contained in $G$ and containing $F$. Note that $G/F$ inherits the structure of an abstract polytope from $\mathcal{P}$, that is, each axiom of abstract polytopes is satisfied by the section $G/F$ since $\mathcal{P}$ satisfies that axiom. Given a section $G/F$ of rank $n$, $G/F$ is trivially flag-connected if $n \leq 1$. If $n \geq 2$, then $G/F$ is flag-connected if, for any two proper faces $H$ and $H'$ of $G/F$, we can "connect" $H$ and $H'$ by a sequence of faces, proper in $G/F$, such that each face contains or is contained in the preceding.

**Theorem 4.0.11.** *The poset associahedron $\mathcal{K}P$ is strongly flag-connected for all posets $P$.*

*Proof.* Let $n = \text{rank } \mathcal{K}P$, and fix two faces, $G$ and $F$, of $\mathcal{K}P$ with $F < G$. By the definition above, if rank $G/F \leq 1$, then $G/F$ is trivially flag-connected. Thus we assume rank $G/F \geq 2$, and wish to show that $G/F$ is flag-connected. As was the case showing that 1-sections contain four elements, we have two drastically different cases to consider, concerning the bottom face of our section. When $F$ corresponds to a proper tubing of $P$, we will have a much easier time seeing that $G/F$ is flag-connected. However, we must also consider when $F = \hat{0}$, at which time we will make use of the structure imposed on maximal tubings of $P$.

We first consider the case when $F \neq \hat{0}$. Since $\mathcal{K}P$ is the poset of tubings of $P$ and $F \neq \hat{0}$, the faces $F$ and $G$ correspond to tubings of $P$. Now, since $F < G$, we have that $G \subseteq F$, that is, $G$ is a subtubing of $F$, by the reverse inclusion ordering on $\mathcal{K}P$. Our assumption that rank $G/F \geq 2$ means that $|F| \geq |G| + 3$. In order to handle this case, we make several remarks concerning the structure of $G/F$.

Now, since $F$ is a proper tubing of $P$, the section $G/F$ consists of tubings which are contained in $F$. Likewise, these tubings in $G/F$ contain $G$ as a subtubing. In fact, we may think of the section $G/F$ as comprised of tubings of the form $G \cup \Delta$, where $\Delta$ is any subset of $F \setminus G$. This is simply because it is trivial to verify that any subtubing of a tubing of $P$ is itself a tubing of $P$, and we may think of $G/F$ as the set of all subtubings of $F$ which contain $G$ as a subtubing. Thus we see that $G/F$ is flag-connected if and only if the $n$-dimensional simplex is flag-connected, which it certainly is, being a convex polytope.

Now we are left with the more difficult case, when $F = \hat{0}$. In this case, we remark that $G/\hat{0}$ contains several maximal tubings of $P$. We have seen that maximal tubings, above all others, are endowed with many nice properties. We will show that one may always move between any pair $H_1, H_k$ of maximal tubings of $P$ by repeatedly subtracting then adding single tubes. This creates a sequence of maximal and almost maximal tubings, with each tubing containing or contained in the preceding.

The section $G/\hat{0}$ of $\mathcal{K}P$ is simply the set of tubings containing $G$ as a subtubing, ordered by reverse inclusion. Note that since we're still assuming that $G/\hat{0}$ is of rank at least 2, we may assume that $G \subseteq H_1 \cap H_k$ has cardinality at most $|H_1| - 2$, so that the process of moving between maximal and almost maximal tubings will use only faces which are proper in $G/\hat{0}$.

Fix any distinct proper tubings $H_1$ and $H_k$ of $G/\hat{0}$. Without loss of generality, we may assume that $H_1$ and $H_k$ are maximal tubings of $P$, since any tubing is contained in at least one maximal tubing in $G/\hat{0}$.

Building our sequence, we will focus on the intersection of consecutive tubings with $H_k$, our destination. To ensure that we stay among proper faces in $G/\hat{0}$, we must ensure that intermediate maximal tubings properly contain $G$ as a subtubing. Loosely speaking, we do so by continually growing the intersection of intermediate tubings with $H_k$, or keeping this intersection constant.

We start by choosing a tube which is maximal in $H_1 \setminus H_k$. There may be several maximal tubes, but any will work. We would like to remove a tube maximal in $H_1 \setminus H_k$, and add a tube from $H_k \setminus H_1$, although this alone will generally require intermediate steps. It is an intricate argument, and we start by calling attention to $\hat{t}$, the unique smallest tube in $H_1$ containing the chosen tube as a proper subset. We let $\hat{t} = P$ if the chosen tube is maximal in all of $H_1$. Thus, when $\hat{t} \neq P$, we have $\hat{t}$ in $H_1 \cap H_k$ since the chosen tube is maximal in $H_1 \setminus H_k$.

This enables us to choose a distinguished element $m_k$, which is the unique element in a nontrivial $\hat{t}$-bundle contained in $\hat{t}$ but not tubes in $H_k$ properly contained in $\hat{t}$. We then choose $t \in H_1 \setminus H_k$ to be the largest tube containing $m_k$, and this is the tube we seek to replace. We also have an element $m_1$, defined similarly using tubes in $H_1$. We now see that tubes maximal in $H_1 \setminus H_k$ must be a connected component of $\hat{t} \setminus m_1^{\uparrow}$, while tubes maximal in $H_k \setminus H_1$ must be a connected component of $\hat{t} \setminus m_k^{\uparrow}$, by Lemma 4.0.3. This leads us to conclude the important fact that $m_1 \neq m_k$.

Figure 4.3 is an example of choosing such an element $m_k$, where we are moving

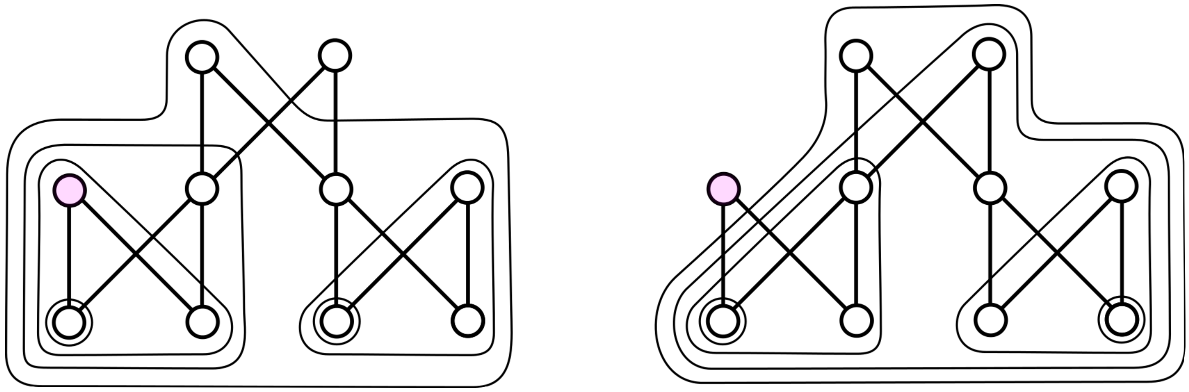


Figure 4.3: Two maximal tubings.

from the left to the right tubing. In the figure specifically, our first goal is to replace the largest tube in $H_1$ with the largest tube in $H_k$, using the highlighted element $m_2$. Generally, replacing $t$ will require several intermediate steps during which we will prepare to replace $t$ with an acceptable tube in $H_k \setminus H_1$. Ultimately, this preparation entails replacing tubes in $H_1 \setminus H_k$ containing $m_k$ with tubes that do not contain $m_k$. This process is shown in Figure 4.4, and we note that consecutive tubings each contain the intersection $H_1 \cap H_k$, and that these tubings are proper in $G/\hat{0}$.

Tubes in $H_1 \setminus H_k$ containing $m_k$ form a chain linearly ordered by subset inclusion, and we denote these tubes by $C_k$, with $t$ the largest tube in $C_k$. Let $t_0$ be the smallest tube in $C_k$ so that $m_k \in t_0 \setminus \cup T_{t_0} = m_0^{\uparrow}$, with $m_0$ in a nontrivial $t_0$-bundle. We can show that $m_k^{\uparrow} \setminus \{m_k\}$ consists of elements in trivial $\hat{t}_0$-bundles, where $\hat{t}_0$ is the smallest tube in $H_1$ properly containing $t_0$. In doing so, we are able to apply the remark after Lemma 4.0.8 and we then know that the replacement for $t_0$ doesn't contain $m_k$.

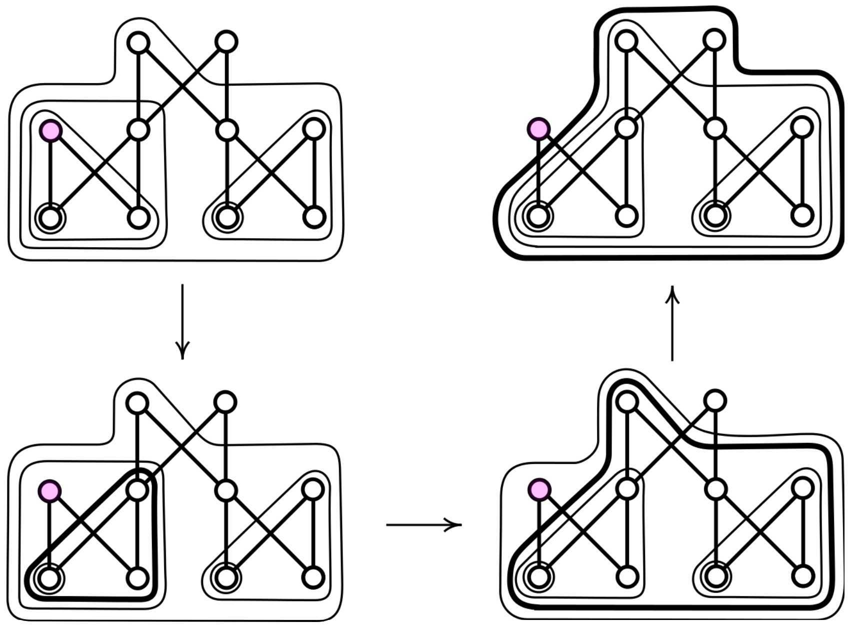


Figure 4.4: Exposing the desired element. Bold tubes are newly replaced.

Now, to see that $m_k^{\uparrow} \setminus \{m_k\}$ consists of elements in trivial $\hat{t}_0$-bundles, we note that $\hat{t}_0$ is contained in $\hat{t}$, hence $\hat{t}_0$-bundles are simply subsets of $\hat{t}$ bundles. But recall that $\cup_{t'' \in H_k} t'' = \hat{t} \setminus m_k^{\uparrow}$, so that elements in $(\hat{t} \cap m_k^{\uparrow}) \setminus \{m_k\}$ are in trivial $\hat{t}$-bundles, by Lemma 4.0.3, from which the claim follows. Thus the replacement for the smallest tube in $C_k$ doesn't contain $m_k$, again by the remark following 4.0.8. Of course, having replaced the smallest element of $C_k$, we can then replace the next smallest tube in $C_k$ with a tube not containing $m_k$ by the same analysis since all tubes in $C_k$ are a subset of $\hat{t}$. Repeating this process, we eventually replace $t$, the last step of Figure 4.4.

Now, having replaced tubes properly contained in $t$ which contain $m_k$ before replacing $t$, we claim that the replacement for $t$ is a tube in $H_k \setminus H_1$. This is straightforward, since the set of largest tubes in $H_k$ contained in $\hat{t}$ are the set of connected components of $\hat{t} \setminus m_k^{\uparrow}$, by Lemma 4.0.3. Upon replacing $t$, we arrive at a tubing $H_2$ in which tubes contained in $\hat{t}$ are subsets of $\hat{t} \setminus m_k^{\uparrow}$. Now maximality of $H_2$ implies that the set of connected components of $\hat{t} \setminus m^{\uparrow}$ is a subtubing of $H_2$ by Lemma 4.0.3, so that $CC(\hat{t} \setminus m^{\uparrow}) \subseteq H_2 \cap H_k$.

We have thus moved closer to $H_k$, since $H_1 \cap H_k$ is a proper subset of $H_2 \cap H_k$. This inclusion truly is proper, since the largest tubes in $H_1 \setminus H_k$ must have been the connected components of $\hat{t} \setminus m_1^{\uparrow}$, thus $CC(\hat{t} \setminus m_k^{\uparrow}) \not\subseteq H_1$.

We now have all the tools to construct a desired sequence of successively nested tubings, starting with $H_1$ and ending at $H_k$. The algorithm given leads to an intermediate maximal tubing $H_2$, with $H_1 \cap H_k \subset H_2 \cap H_k$. Thus, given the arbitrary maximal tubings $H_1$ and $H_k$, we may always construct a tubing $H_2$ with the intersection $H_1 \cap H_k$ proper in $H_2 \cap H_k$. If $H_2 \neq H_k$, we may of course apply this algorithm to $H_2$, yielding an $H_3$ with the intersection $H_2 \cap H_k$ proper in $H_3 \cap H_k$, building a sequence of successively nested tubings from $H_1$ which must terminate at $H_k$, since $\mathcal{K}P$ only has finite dimension.

□

We have finally completed our goal, verifying that the poset associahedron $\mathcal{K}P$ satisfies the four axioms and is thus an abstract polytope. It is remarkable that noteworthy properties concerning the large scale structure of $\mathcal{K}P$ – that it is a bounded, ranked, strongly flag-connected poset whose 1-sections contain four elements – has been ascertained by considering the relationship between tubings of $P$

and nontrivial bundles of elements of $P$.

This result merely scratches the surface of poset associahedra, and much more investigation is possible. Chief among conjectures is that $\mathcal{K}P$ is not only an abstract polytope, but isomorphic to the face lattice of a convex polytope.