Linear. Test 2, Review.

Also study the quizzes, and homework problems!

Consider the following subsets of $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 0 \\ x-y \\ 3y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \; T = \left\{ \begin{bmatrix} x \\ 7y \\ y+3 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \; U = \left\{ \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

Which is a subspace? Recall: subspaces are subsets that can be written as spans, and subspaces are planes or lines containing the origin $\mathbf{0}$ (or just the origin, or the whole space.)

$$\boxed{S = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}}$$

Consider the following functions from $\mathbb{R}^2 \to \mathbb{R}^3$:

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x-y \\ 3y \end{bmatrix}, \; T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 7y \\ y+3 \end{bmatrix}, \; U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix}$$

Which is a linear transformation? Recall: lin. transformations can always be described by matrix multiplication, which uses the input vector to make a linear combination of the columns. They take $\mathbf{0}$ to $\mathbf{0}$ and obey $T(\mathbf{u}+2\mathbf{v}) = T(\mathbf{u}) + 2T(\mathbf{v})$. They take a space to a subspace called the range, the span of the column vectors. Find a matrix representation of S using the standard bases, and find $N(S)$ and $R(S)$, as spans of bases. Bonus: Describe a function from $\mathbb{R}^1 \to \mathbb{R}^2$ whose range is a subspace, but which is not linear.

$\boxed{S.}$

$$[S]_{\mathcal{E}} = \left[\left[S\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{\mathcal{E}}, \left[S\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]_{\mathcal{E}} \right]$$

$$\boxed{= \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}}$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{N(S) = \vec{0}}, \quad \boxed{R(S) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}}$$

Bonus: $F([x]) = \begin{bmatrix} x^3 \\ 5x^3 \end{bmatrix}$

2 Consider the following sets of polynomials in $\mathcal{P}_3$.

$$\mathcal{A} = \{x-1, x, x^2+1\},\ \mathcal{B} = \{5x^2, x, x^3+2, 3\},\ \mathcal{C} = \{3x^2, x^2-1, x+2, 3\},$$

Which one is a basis for $\mathcal{P}_3$? Which one is lin. dep.? Recall the 2 out of 3 theorem: any two implies the third (and thus a basis for V,) out of {lin. indep., spans V, has the same number of items as $\dim(V)$.} Note also, these questions are equivalent to the same questions about their coordinate vectors in $\mathbb{R}^4$ with respect to the standard basis.

$\mathcal{B}$ is a basis for $\mathcal{P}_3$

$\mathcal{C}$ is lin. dep.

(A is a basis for $\mathcal{P}_2$)

Note that $\mathcal{B}$ is lin. indep

either by def:

$$c_1 5x^2 + c_2 x + c_3(x^3+2) + c_4 3 = 0$$

$$\Rightarrow \quad c_1 5 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

$$2c_3 + 3c_4 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$$

or matrix of coordinate vectors

$$\begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

has determinant $= 15$

$\neq 0$

Therefore since $\mathcal{B}$ has 4 vectors, it also spans.

$\Rightarrow$ basis.

Consider the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 6 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 10 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the rank, nullity, null space, and range for each, as spans of bases. Which of them are 1-1? Which are onto? Which two of them cannot represent the derivative from $\mathcal{P}_3$ to $\mathcal{P}_4$? Which two of them cannot represent the derivative from $\mathcal{P}_4$ to $\mathcal{P}_3$? Recall that rank + nullity = dim(domain) = number of columns. Recall that if a matrix represents a transformation then it will have the right number of rows and columns, and that it will have the same rank and nullity as that transformation.

$$A|\vec{0} \longrightarrow \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$\mathbb{R}^4 \to \mathbb{R}^5$

rank = 3 $\Rightarrow$ not onto
nullity = 1 $\quad$ not 1-1

$$N(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

cannot rep derivative $\mathcal{P}_4 \to \mathcal{P}_3$.

$$R(A) = \text{Span}\left\{\begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$$B|\vec{0} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 7 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -35 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$\mathbb{R}^5 \to \mathbb{R}^4$

rank = 4 $\Rightarrow$ onto
nullity = 1 $\quad$ not 1-1

cannot rep derivative $\mathcal{P}_3 \to \mathcal{P}_4$.

$$\hookrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -39 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$N(B) = \text{span}\left\{\begin{bmatrix} 39 \\ -5 \\ 2 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$$R(B) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \end{bmatrix}\right.$$

$$C|\vec{0} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbb{R}^5 \to \mathbb{R}^4$

rank = 3 $\Rightarrow$ not onto
nullity = 2 $\quad$ not 1-1

$$\hookrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

cannot be either derivative since nullity = 2.

$$N(C) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$R(C) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right.$$

$$\Rightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \\ x_3 &= -\tfrac{1}{2} x_5 \\ x_4 &= 0 \\ x_5 &= x_5 \end{aligned}$$

4 Consider the three bases for $\mathcal{P}_3$:

$$\mathcal{E} = \{1, x, x^2, x^3\},\ \mathcal{B} = \{5x^2, x, x^3, 3\},\ \mathcal{C} = \{x^3 + 3x^2 + 1, x^2 - 2, x - 7, 2\}$$

Find the representatives of the derivative: $T : \mathcal{P}_3 \to \mathcal{P}_3$, where $T(f) = f'$.

$[T]^{\mathcal{E}}_{\mathcal{B}}$

f	$5x^2$	x	x^3	3
f'	$10x$	1	$3x^2$	0

$$\Rightarrow \boxed{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$[T]^{\mathcal{B}}_{\mathcal{C}}$

f	x^3+3x^2+1	x^2-2	$x-7$	2
f'	$3x^2+6x$	$2x$	1	0

$$\Rightarrow \boxed{\begin{bmatrix} 3/5 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}}$$

Find the representative of the lin. trans.: $T : \mathcal{P}_3 \to \mathcal{P}_3$, where $T(f) = f'' + xf''$. Also find the null space $N(T)$ and the range $R(T)$ as spans of bases of polynomials. Is T onto? 1-1? Recall the 2 out of 3 theorem for matrices: any two implies the third out of {1-1,onto,square}.

$[T]^{\mathcal{E}}_{\mathcal{C}}$

f	x^3+3x^2+1	x^2-2	$x-7$	2
$T(f)$	$6x+6+x(6x+6)$ $= 6x^2+12x+6$	$2+2x$	0	0

$$\Rightarrow \begin{bmatrix} 6 & 2 & 0 & 0 \\ 12 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$N(T)$:
$$\left[\begin{array}{cccc|c} 6 & 2 & 0 & 0 & 0 \\ 12 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\leadsto \left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \\ 12 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\leadsto \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\leadsto \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
\Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$$

$$N(T) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \boxed{\operatorname{span}\{x-7, 2\}}$$

$R(T)$

$$= \operatorname{span}\left\{ \begin{bmatrix} 6 \\ 12 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\boxed{= \operatorname{span}\{6 + 12x + 6x^2, 2 + 2x\}}$$

T is not 1-1 and T is not onto.