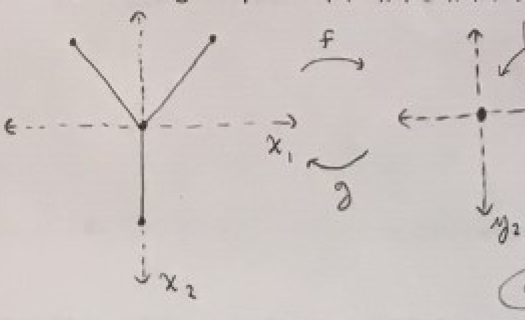
Topology Quiz 7. Name: Key

1) Show that "Y" is homotopic to ".". That is, find a homotopy equivalence from the "Y" given by connecting the points (0,-1), (0,0), (1,1) and (-1,1) with three line segments; to the single point (0,0).



Only one point $\vec{y} = \vec{0}$

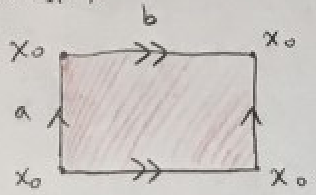
$$\boxed{\begin{aligned} f(\vec{x}) &= \vec{0} \\ g(\vec{y}) &= \vec{y} = \vec{0} \\ F(\vec{x}, t) &= t\vec{x} \\ G(\vec{y}, t) &= \vec{y} \end{aligned}}$$

all continuous

Check:
$gf(\vec{x}) = \vec{0}$
$fg(\vec{y}) = \vec{0} = \vec{y}$
$F(\vec{x}, 0) = \vec{0}$
$G(\vec{y}, 0) = \vec{y} = \vec{0}$
$F(\vec{x}, 1) = \mathbb{1} = G(\vec{y}, 1)$

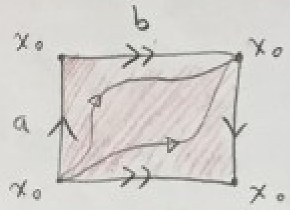
Find the fundamental groups of the torus, Klein bottle, two-holed torus, punctured torus and thrice punctured sphere; each as a group presentation.

2) Find $\pi_1(T^2)$.



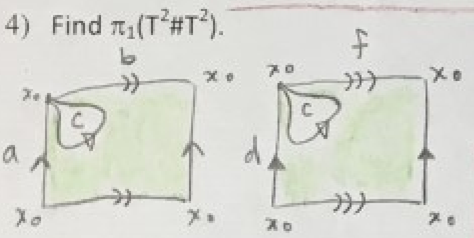
$$\pi_1 = \langle a, b \mid ab = ba \rangle$$

3) Find $\pi_1(K^2)$.



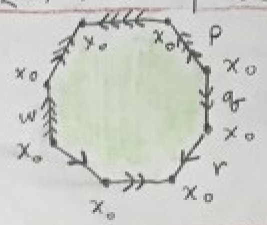
$$\pi_1 = \langle a, b \mid ab = ba^{-1} \rangle$$

4) Find $\pi_1(T^2 \# T^2)$.



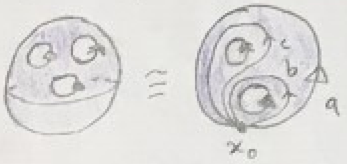
$$\pi_1 = \langle a, b, c, d, f \mid acb = ba,\ dcf = fd \rangle$$
$$\cong \langle a, b, d, f \mid ba^{-1}b^{-1}a = fd^{-1}f^{-1}d \rangle$$

OR



$$\pi_1 = \langle p, q, r, w \mid qrq^{-1}r^{-1} = pwp^{-1}w^{-1} \rangle$$

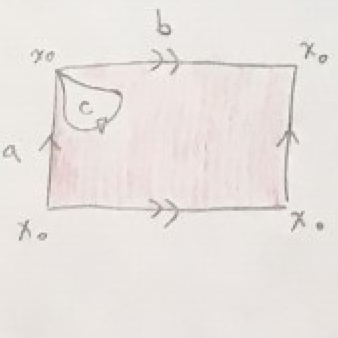
5) Find $\pi_1(S^2 - 3D^2)$.



$$\pi_1 = \langle a, b, c \mid a = bc \rangle$$
$$\cong \langle b, c \mid \emptyset \rangle$$

The group isomorphism here is ϕ, defined by:
$\phi(a) = bc$
$\phi(b) = b$
$\phi(c) = c$
$\phi(e) = e$ (identity, empty word)
$\phi(x^{-1}) = (\phi(x))^{-1}$

6) Find $\pi_1(T^2 - D^2)$.



$$\pi_1 = \langle a, c, b \mid acb = ba \rangle$$
$$\cong \langle a, b \mid \emptyset \rangle$$

The group isomorphism here is ϕ, defined by:
$\phi(a) = a$
$\phi(b) = b$
$\phi(c) = a^{-1}bab^{-1}$
$\phi(e) = e$ (identity, empty word)
$\phi(x^{-1}) = (\phi(x))^{-1}$

These last two are in fact homotopy equivalent!

(Both retract onto a pair of circles)