Definition: (Hilbert + Dedekind)

An abstract plane geometry is:

I. (incidence)

$\boxed{D1}$ a set of points $\mathcal{P} = \{A, B, C, \dots\}$

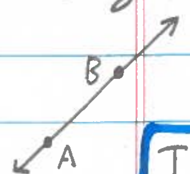
$\boxed{D2}$ a set of lines $\mathcal{L} = \{l, m, q, \dots\}$

*(all named points and lines are distinct, except in C2, C3, C5, C6)

with: $\boxed{S1}$ a relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$, called incidence

obeying $\boxed{I1}$ $\forall A, B \in \mathcal{P}, \exists!\, l \in \mathcal{L}$ s.t. $(A, l), (B, l) \in \mathcal{I}$.

(any two points lie on exactly one line; and so we equate $l = \overleftrightarrow{AB} = \{C \in \mathcal{P} \mid (C, l) \in \mathcal{I}\}$)



$\boxed{I2}$ $\forall l \in \mathcal{L}, \exists A, B \in \mathcal{P}$ s.t. $(A, l), (B, l) \in \mathcal{I}$.

$\boxed{I3}$ $\exists l, m \in \mathcal{L}$ s.t. $l \neq m$. (gives "2D" plane.)

II. (order)

with: $\boxed{S2}$ a relation $\mathcal{B} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$, called betweenness.

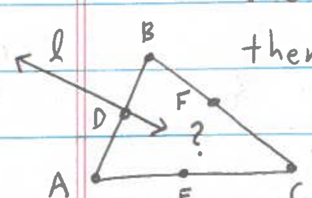
→ $(A, B, C) \in \mathcal{B}$ is stated: B is between A and C.

obeying $\boxed{O1}$ $\forall A, B, C \in \mathcal{P}, (A, B, C) \in \mathcal{B} \Rightarrow (C, B, A) \in \mathcal{B}$, and all three points are on the same line $l = \overleftrightarrow{AC}$.

$\boxed{O2}$ $\forall A, C \in \mathcal{P}, \exists B \in \mathcal{P}$ s.t. $(A, B, C) \in \mathcal{B}$. (gives ∞ points)

$\boxed{O3}$ $\forall A, B, C$ on line l, exactly one point is between the others.

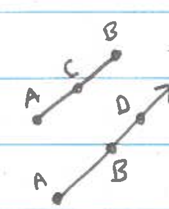
$\boxed{O4}$ $\forall A, B, C \in \mathcal{P}, l \in \mathcal{L}$, if A, B, C are not all on one line, and l does not contain any of them, then: { if l contains D with $(A, D, B) \in \mathcal{B}$ then l must contain E with $(A, E, C) \in \mathcal{B}$ or contain F with $(B, F, C) \in \mathcal{B}$.



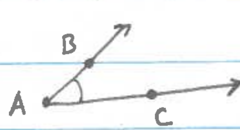
[definition break]

→ segment $\overline{AB} = \{A, B\} \cup \{C \mid (A, C, B) \in \mathcal{B}\}$

→ ray $\overrightarrow{AB} = \overline{AB} \cup \{D \mid (A, B, D) \in \mathcal{B}\}$

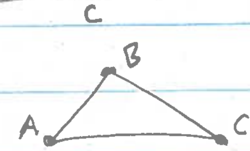


→ angle $\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$



→ triangle $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}$

(often we include the interior)



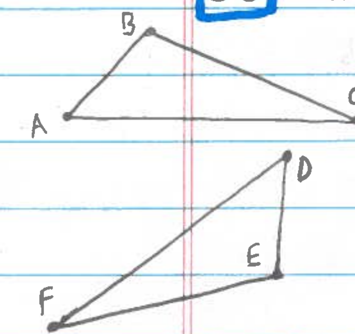
→ $\mathcal{S} \subseteq \mathcal{P}$ is convex means $A, B \in \mathcal{S} \Rightarrow \overline{AB} \subseteq \mathcal{S}$.

III. (congruence)

with: $\boxed{S3}$ an equivalence relation on segments denoted $\overline{AB} \cong \overline{CD}$

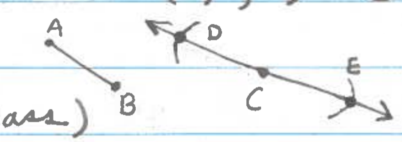
$\boxed{S4}$ an equivalence relation on angles denoted $\angle ABC \cong \angle DEF$

$\boxed{S5}$ an equivalence relation on triangles denoted $\triangle ABC \cong \triangle DEF$ where order matters: 2 triangles are congruent when there exists matching between the two ordered lists of sets of 3 points, and $\triangle ABC \cong \triangle DEF$ in that order means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$ and $\angle BAC \cong \angle EDF$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$.



obeying $\boxed{C1}$ $\forall \overline{AB}$ and $(C, l) \in \mathcal{I}, \exists!\, D, E$ on l s.t. $(D, C, E) \in \mathcal{B}$ and $\overline{AB} \cong \overline{CD} \cong \overline{CE}$.

(copy segment with compass)

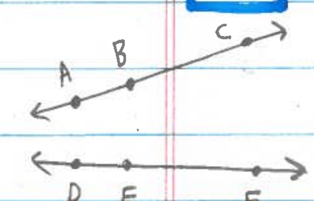


$\boxed{C2}$ Segment congruence is transitive:

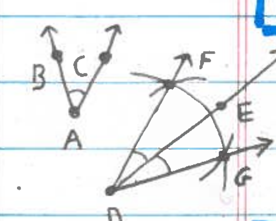
$\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF} \Rightarrow \overline{AB} \cong \overline{EF}$.

$\boxed{C3}$ Segment congruence is additive:

$\forall A, B, C, D, E, F \in \mathcal{P}$, If $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$ and $(A, B, C) \in \mathcal{B}$ on line l and $(D, E, F) \in \mathcal{B}$ on line m, then $\overline{AC} \cong \overline{DF}$.



$\boxed{C4}$ $\forall \angle BAC$ and $\overrightarrow{DE}, \exists!\, \overrightarrow{DF}$ and $\overrightarrow{DG}$ s.t. $\overleftrightarrow{FG}$ contains one point of $\overleftrightarrow{DE}$ between F and G, (we say F and G are on opposite sides of $\overleftrightarrow{DE}$) and s.t. $\angle EDF \cong \angle EDG \cong \angle BAC$. (copy angle with compass)

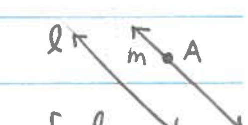


$\boxed{C5}$ Angle congruence is transitive.

$\boxed{C6}$ $\forall A, B, C, D, E, F \in \mathcal{P}$, If $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$ then $\triangle ABC \cong \triangle DEF$. (SAS axiom).

IV. (parallels)

$\boxed{P1}$ $\forall$ line l and point A not on l, $\exists!\, m \in \mathcal{L}$ s.t. m contains A and m contains no points of l.



V. (Dedekind continuity)

$\boxed{DC1}$ $\forall$ line l partitioned into $l = S_1 \cup S_2$ convex sets, $\exists!\, A$ on l s.t. $S_i = \overrightarrow{AB}$ and $S_j = \overrightarrow{AC} - \{A\}$ with $(B, A, C) \in \mathcal{B}$, where $\{i, j\} = \{1, 2\}$.