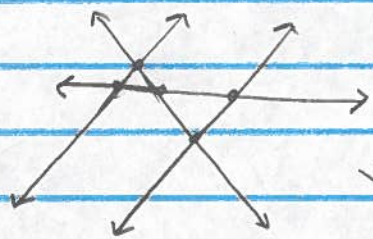


How to: Reverse engineer, to write a quiz!
 Create a system that gives the picture-type:



augmented B: two pivots,
 random augment

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{array} \right]$$

Notice: no
 set of 3 lines
 here has a
 solution!

do some random
 row reduction: \swarrow

$$\begin{array}{l} R_1 \leftarrow R_1 + 2 \cdot R_2 \\ R_3 \leftarrow R_3 - 3 \cdot R_2 \end{array} \quad \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & 5 \end{array} \right]$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_4 \leftarrow R_4 - R_1$$

$$R_3 \leftarrow R_3 + R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & -1 & 5 \\ -1 & -2 & -2 \end{array} \right] \Rightarrow$$

$$\begin{cases} x + 2y = 7 \\ x + 3y = 9 \\ x - y = 5 \\ -x - 2y = -2 \end{cases}$$

System!

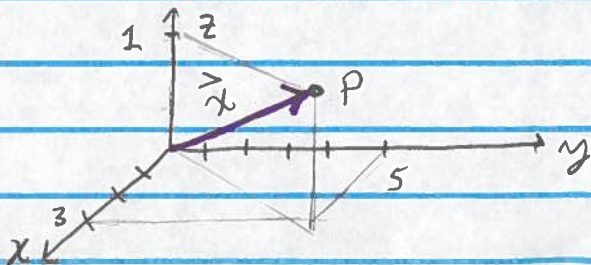
Now, solve that system the usual
 way, for practice.

Points & Vectors in \mathbb{R}^n

Any point in \mathbb{R}^n can also be written as a
 (thought of as) a vector in \mathbb{R}^n .

$$\mathbb{R}^3 \text{ point } P = (3, 5, 1)$$

$$\text{vector } \vec{x} = \langle 3, 5, 1 \rangle = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$



Points are better for describing location,
so the numbers are called coordinates.

Vectors also describe location, but can also
describe moving in that direction, or
a force pulling in that direction, so the
numbers are called components.

We can add components to add vectors,
and scale vectors by multiplying components.

$$2 \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$$

Recall dot product $\langle 0, -3, 6 \rangle \cdot \langle 3, 5, 1 \rangle = 0 - 15 + 6 = -9$

With variables: $\langle 3, 5, 1 \rangle \cdot \langle x, y, z \rangle = 3x + 5y + 1z$

Rows of coefficients $\begin{bmatrix} 3 & 5 & 1 \\ 2 & 0 & -4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 2 \end{pmatrix}$

is a way to write
the system

$$3x + 5y + z = -9$$

$$2x - 4z = 2$$

so solve $\left[\begin{array}{ccc|c} 3 & 5 & 1 & -9 \\ 2 & 0 & -4 & 2 \end{array} \right] R_1 \leftarrow R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 2 & 0 & -4 & 2 \end{array} \right]$

$R_2 \leftarrow R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & -10 & -14 & 24 \end{array} \right] R_2 \leftarrow R_2 / -10 \left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & 1 & 1.4 & -2.4 \end{array} \right] R_1 \leftarrow R_1 - 5R_2 \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 1.4 & -2.4 \end{array} \right]$

$$\begin{aligned} \Rightarrow x_1 - 2x_3 &= -1 & \Rightarrow x_1 &= -1 + 2x_3 \\ x_2 + 1.4x_3 &= -2.4 & x_2 &= -2.4 - 1.4x_3 \\ x_3 &= x_3 \text{ (free!)} & x_3 &= x_3 \end{aligned}$$

OR

$$\begin{aligned} x &= -1 + 2z \\ y &= -2.4 - 1.4z \\ z &= z \text{ (free!)} \end{aligned}$$

specific \rightarrow

general \leftarrow

$$\begin{aligned} x &= 1 \\ y &= -2.4 \\ z &= 0 \end{aligned}$$

pick any value

OR

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ -1.4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2.4 \\ 0 \end{pmatrix}$$

\hookrightarrow this version of the general answer for a system is called a linear combination of constant vectors, with one variable coefficient. In general there is one vector for each free variable plus one constant vector (no variable).

Matrix operations : useful for shortcuts.

- 1) Matrix times vector $A_{m \times n}$ times $\vec{x} \in \mathbb{R}^n$; $A\vec{x} \in \mathbb{R}^m$
 \rightarrow multiply components and sum (dot product)
 for each row of A (length n) and all of \vec{x} .

ex.

$$\begin{aligned} A_{2 \times 4}, \quad \vec{x} \in \mathbb{R}^4, \quad & \begin{matrix} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 5 & 4 & -1 & 0 \end{bmatrix} & \begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix} & = & \begin{pmatrix} 3+6+0-2 \\ 15+8-1+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 22 \end{pmatrix} \\ A & \quad \vec{x} & & A\vec{x} \in \mathbb{R}^2 \end{matrix} \end{aligned}$$

2) Matrix times matrix

$A_{m \times n}$ times $B_{n \times q}$ gives AB , $m \times q$.
↑ = = ↑ ↑

→ find the entries of AB (in say row i and column j) by multiplying and summing (dot product) row i of A times column j of B .

$$\text{Formula: } (AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

ex: $\begin{bmatrix} 3 & 0 & 1 & 2 \\ 4 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$
 $A_{2 \times 4} \nearrow \quad \quad \quad B_{4 \times 3} \nearrow$

$$AB = \begin{bmatrix} 3+0+1+2 & 0+0-1+0 & 6+0+0-2 \\ 4+0+0+2 & 0-1+0+0 & 8-1+0-1 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 4 \\ 6 & -1 & 6 \end{bmatrix}$$

(2×3)

3) Matrix + Matrix, scalar times matrix

A, B both $m \times n$

$$\begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 3 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 3 & 2 & -2 \end{bmatrix}$$

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$5 \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 10 & 0 \\ 20 & 5 & -10 \end{bmatrix}$$

$$(cA)_{ij} = c(A_{ij})$$

4) Matrix transpose: $A_{m \times n} \rightarrow A^t_{n \times m}$

$$(A^t)_{ij} = A_{ji} \quad \text{"rows become columns"}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix} \rightarrow A^t = \begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 0 \end{bmatrix}$$

5) Matrix determinant: $A_{n \times n} \rightarrow \det(A) \in \mathbb{R}$
(square A) \rightarrow scalar,

$$\det A = \sum_{j=1}^n (-1)^{j+1} A_{1j} \det(M_{1j})$$

where M_{1j} is the matrix made from A by deleting row 1 and column j ,
 \rightarrow Recursive! we also need: $\det(c) = c$
for any scalar c , which is a 1×1 matrix.

$$\det \begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & -3 & -1 \end{bmatrix} \quad \begin{cases} n=3 \\ k=1 \text{ to } 3 \end{cases}$$

$$= (-1)^2 3 \det \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix} + (-1)^3 1 \det \begin{bmatrix} 4 & 6 \\ 0 & -1 \end{bmatrix} + (-1)^4 2 \det \begin{bmatrix} 4 & 5 \\ 0 & -3 \end{bmatrix}$$

$$= 3((-1)^2 5 \det(-1) + (-1)^3 6 \det(-3)) \\ + -1((-1)^2 4 \det(-1) + (-1)^3 6 \det(0)) \\ + 2((-1)^2 4 \det(-3) + (-1)^3 5 \det(0))$$

$$= 3(5(-1) - 6(-3)) - (4(-1) - 6(0)) + 2(4(-3) - 5(0))$$

$$= 3(13) + 4 + 2(-12)$$

$$= \boxed{19}$$

- Using any row i

$$\det A = \sum_{j=1}^n (-1)^{i+j} A_{ij} \det(M_{ij})$$

↑
going along row i

$(-1)^{i+j}$ has checkerboard pattern

$$\begin{bmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \\ + & - & + & - & + & \\ \vdots & & & & & \ddots \end{bmatrix} \quad (\text{odd} + \text{odd} = \text{even})$$

- or you can use a column!
- So, if A has a row of zeros, or a column of zeros, then $\det(A) = 0$.
- If A is triangular (either all zeros above or below the main diagonal (upper left to lower right)) then $\det(A) =$ multiplying all the main diagonal entries A_{ii} .

$$\det \begin{bmatrix} 3 & 0 & 4 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = -24$$

More determinant facts and shortcuts

- $2 \times 2 \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

- row equivalence moves change the det.

1) Switching 2 rows \rightarrow multiply by -1 .

$$A \sim B \quad \text{by} \quad R_i \leftrightarrow R_k$$

$$\Rightarrow \det(A) = (-1) \det(B)$$

2) Scaling 1 row \rightarrow multiply by: $\frac{1}{\text{scalar}}$

$$A \sim B \quad \text{by} \quad R_i \leftarrow c R_i$$

$$\Rightarrow \det(A) = \frac{1}{c} \det(B)$$

3) Adding a multiple of one row to another \rightarrow no change

$$A \sim B \quad \text{by} \quad R_i \leftarrow R_i + c R_k$$

$$\Rightarrow \det(B) = \det(A)$$

- $\det(A^t) = \det(A)$

- $\det(AB) = \det(A) \det(B)$

Note:
 $AB \neq BA$
 but
 $\det(AB) = \det(BA)$

ex) $\det \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} (-1) \det \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow \frac{1}{2} R_3}$

$$(-1) 2 \det \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} = -2 \det \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2.5 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -2(-6) = \boxed{12}$$

Identity matrix and inverse matrix

- The identity matrix $I_{n \times n}$, sometimes written I_n or just I , has entries
- 1 on the main diagonal
 - 0 off the main diagonal

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- For a square matrix $A_{n \times n}$

$$AI = IA = A \quad \text{row times column}$$

- Only some square matrices $A_{n \times n}$ are invertible; which means that there exists another square matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

- A is invertible if and only if $\det(A) \neq 0$.

- to find A^{-1} , augment A with I (all at the same time) and row reduce

$$[A | I] \sim [I | A^{-1}]$$

ex) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $\det(A) = -1$

find A^{-1} : $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array} \right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array} \right]$

$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$ check $AA^{-1} = I = A^{-1}A$

- $\det(A^{-1}) = 1/\det(A)$
- When A^{-1} does not exist, we say A is singular (or non-invertible)
- For invertible A , augmenting A with a column of constants (vector) \vec{b} is the same as solving a system with variables $(x_1, x_2, \dots, x_n) = \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and n equations.
System: $A\vec{x} = \vec{b}$
Then the solution (one unique solution) is $\vec{x} = A^{-1}\vec{b}$.

Handout: solving $A\vec{x} = \vec{b}$.