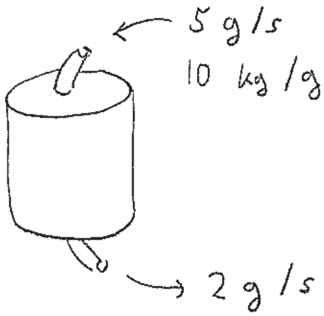
Differential Equations. Review for Test 1, **Fall '19.** Also study all the homework and quizzes, as well as examples in class notes.

Note: Some questions on the actual test may state "Set up the differential equation only." Since you don't know which kind, for practice do both the set-up and the solution.

Note: Don't forget that the answer will have an unknown constant or constants, unless it is an IVP.

1. A 100 gallon tank initially contains 60 gallons of water with 20 kg of sugar in solution. An input pipe adds 10 kg of sugar per gallon, at the rate of 5 gallons per second. An output pipe drains 2 gallons of well-stirred mixture per second.
   - Set up the diff. eq. for finding $A(t)$, the amount of sugar in the tank after t seconds.



(IVP) $\boxed{\begin{aligned} A' &= 50 - 2\left(\frac{A}{3t+60}\right) \\ A(0) &= 20 \end{aligned}}$ ← denominator is volume

   - Solve to get the formula for $A(t)$.

$$A' + \frac{2}{3t+60}A = 50$$

$$\left[\mu(t) = e^{\int \frac{2}{3t+60}dt} = e^{\frac{2}{3}\ln(t+20)} = (t+20)^{2/3}\right]$$

$$\Rightarrow \quad A'(t+20)^{2/3} + A\frac{2}{3t+60}(t+20)^{2/3} = 50(t+20)^{2/3}$$

$$\Rightarrow \quad \frac{d}{dt}\left(A(t+20)^{2/3}\right) = 50(t+20)^{2/3}$$

$$\Rightarrow \quad A(t+20)^{2/3} = 50\left(\frac{3}{5}\right)(t+20)^{5/3} + C$$

$$\Rightarrow \quad A = 30(t+20) + C(t+20)^{-2/3}$$

$$\Rightarrow \quad A = 600 + 30t + C(t+20)^{-2/3}$$

$$\Rightarrow \quad \boxed{A = 600 + 30t - 4273.5(t+20)^{-2/3}}$$

$$\left[\begin{aligned} A(0) &= 20 \\ 20 &= 600 + \frac{C}{20^{2/3}} \\ C &= -4273.5 \end{aligned}\right]$$

   - When will the tank be full, and how much sugar will it contain then?

Volume $= 100$

$\Rightarrow \quad 3t + 60 = 100$

$\boxed{t = \frac{40}{3}}$

$\boxed{A\left(\frac{40}{3}\right) = 587.4 \text{ kg}}$

2. Solve the diff. eq. $y' - y = e^x y^2$. Is it linear, Bernoulli or separable?

$\boxed{\text{Bernoulli}}$

$$y' y^{-2} - y^{-1} = e^x$$
$$\Rightarrow -u' - u = e^x$$
$$\Rightarrow u' + u = -e^x$$
$$\Rightarrow u' e^x + u e^x = -e^{2x}$$
$$\Rightarrow \frac{d}{dx}(u e^x) = -e^{2x}$$
$$\Rightarrow u e^x = -\frac{1}{2} e^{2x} + c$$

$$\left[\begin{array}{l} u = y^{-1} \\ du = -y^{-2} dy \\ \frac{du}{dx} = -y^{-2}\frac{dy}{dx} \\ -u' = y^{-2} y' \end{array}\right] \Rightarrow y = \frac{1}{u}$$

$$\left[\mu(x) = e^{\int 1 dx} = e^x\right]$$

$$\Rightarrow u = -\frac{1}{2} e^x + c e^{-x} \Rightarrow \boxed{y = \frac{1}{-\frac{1}{2} e^x + c e^{-x}}}$$

3. Solve the diff. eq. $y' = y(xy^3 - 1)$. Is it linear, Bernoulli or separable?

$\boxed{\text{Bernoulli}}$

$$y' + y = x y^4$$
$$\Rightarrow y' y^{-4} + y^{-3} = x$$
$$\Rightarrow -\frac{1}{3} u' + u = x$$
$$\Rightarrow u' - 3u = -3x$$

$$\left[\begin{array}{l} u = y^{-3} \\ du = -3 y^{-4} dy \\ -\frac{1}{3} u' = y^{-4} y' \end{array}\right] \Rightarrow y = u^{-1/3}$$

$$\left[\mu(x) = e^{\int -3 dx} = e^{-3x}\right]$$

$$\Rightarrow u e^{-3x} = \int -3x e^{-3x} dx$$
$$\Rightarrow u = x + \frac{1}{3} + c e^{3x}$$

$$\boxed{y = \left(x + \frac{1}{3} + c e^{3x}\right)^{-1/3}}$$

$$\left[\begin{array}{l} \int -3x e^{-3x} dx \\ = x e^{-3x} + \frac{1}{3} e^{-3x} + c \end{array}\right]$$

4. Consider the differential equation $x^2 y'' - 7xy' + 7y = 0$.
By using $y = x^m$ find two solutions of the above equation. Write a (family of) solutions that uses the constants c_1, c_2.

$$y' = m x^{m-1}, \quad y'' = m(m-1) x^{m-2}$$
$$\Rightarrow m(m-1) x^{m-2} x^2 - 7x m x^{m-1} + 7x^m = 0$$
$$\Rightarrow m(m-1) x^m - 7m x^m + 7x^m = 0$$
$$\Rightarrow m^2 - m - 7m + 7 = 0$$
$$\Rightarrow m^2 - 8m + 7 = 0$$
$$\Rightarrow (m-7)(m-1) = 0 \quad \text{So } m = 7, \; m = 1$$
$$\Rightarrow \boxed{y = c_1 x + c_2 x^7}$$

5. Solve the differential equation (IVP): $x^3y' = y - xy$; $y(1) = 7$. Is it Bernoulli or separable?

$$x^3 y' = y(1-x)$$

$$\Rightarrow \quad y' = y\frac{(1-x)}{x^3} \qquad \boxed{\text{separable}}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{1-x}{x^3}\,dx$$

$$\Rightarrow \ln|y| = \int (x^{-3} - x^{-2})\,dx$$

$$\Rightarrow \ln|y| = x^{-2}/{-2} - x^{-1}/{-1} + c$$

$$\Rightarrow |y| = e^{(-1/2x^2 + 1/x + c)}$$

$$|y| = e^{(-1/2x^2 + 1/x)}e^c$$

$$\Rightarrow y = \pm e^c e^{(1/x - 1/2x^2)}$$

$$\Rightarrow y = c_1 e^{(1/x - 1/2x^2)}$$

$$\Rightarrow 7 = c_1 e^{1/2} \Rightarrow c_1 = \frac{7}{\sqrt{e}}$$

$$\Rightarrow \boxed{y = \frac{7}{\sqrt{e}} e^{(1/x - 1/2x^2)}}$$

6. Solve the differential equation generally: $y' = 2^x(1 + y^2)$. Is it linear, Bernoulli or separable?

Your answer should be solved for y, and will have an unknown constant.

$\boxed{\text{Separable}}$

$$\frac{dy}{dx} = 2^x(1+y^2)$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int 2^x\,dx$$

$$\Rightarrow \tan^{-1}(y) = \frac{2^x}{\ln 2} + c$$

$$\Rightarrow \boxed{y = \tan\left(\frac{2^x}{\ln 2} + c\right)}$$

7. Solve the differential equation generally: $y' + \frac{1}{x}y = \sqrt{x^2+1}$. Is it linear, Bernoulli or separable?

Your answer should be solved for y, and will have an unknown constant.

$\boxed{\text{linear}}$

$$\left[\ \mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x\ \right]$$

$$\Rightarrow \frac{d}{dx}(xy) = x\sqrt{x^2+1}$$

$$\Rightarrow xy = \int x\sqrt{x^2+1}\,dx$$

$$\Rightarrow \boxed{y = \frac{1}{x}\left(\frac{1}{3}(x^2+1)^{3/2} + c\right)}$$

$$\left[\ \int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + c = \frac{1}{3}(x^2+1)^{3/2} + c \quad \middle| \quad u = x^2+1,\ du = 2x\,dx,\ \frac{1}{2}du = x\,dx\ \right]$$

8. Consider the differential equation: $(e^x \cos y + y^2)dx + (2yx - e^x \sin y)dy = 0$; $y(2) = 0$.
   - Show whether this diff. eq. is exact or inexact.
   - Solve it (IVP).

$$M_y = -e^x \sin y + 2y = N_x = 2y - e^x \sin y \Rightarrow \boxed{\text{exact}}$$

$$\Rightarrow \int \left(\begin{array}{c|c} f_x = e^x \cos y + y^2 & f_y = 2yx - e^x \sin y \\ \hline f = e^x \cos y + y^2 x + g(y) & f = y^2 x + e^x \cos y + h(x) \end{array} \right.$$

$$\Rightarrow \quad f(x,y) = e^x \cos y + y^2 x$$

$\Rightarrow$ Setting potential = constant :

$$e^x \cos y + y^2 x = C$$

$$\left.\begin{aligned} &\Rightarrow e^2 \cos(0) + 0^2(2) = C \\ &\Rightarrow C = e^2 \end{aligned}\right\} \text{IVP}$$

$$\Rightarrow \boxed{e^x \cos y + y^2 x = e^2}$$

9. Solve the differential equation generally: $y' = \dfrac{2e^y + x^3 + 1}{-xe^y}$.
   - Show whether this diff. eq. is exact or inexact.

   Your answer should be an implicit equation with an unknown constant.

Answer next page!

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2e^y + x^3 + 1}{-xe^y}$$

$$\Rightarrow \quad -xe^y\,dy = (2e^y + x^3 + 1)\,dx$$

$$\Rightarrow \quad \underbrace{(2e^y + x^3 + 1)}_{M}\,dx + \underbrace{xe^y}_{N}\,dy = 0$$

$M_y = 2e^y \neq N_x = e^y$, so $\boxed{\text{not exact}}$

$$\left[\ \mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx} = e^{\int \frac{e^y}{xe^y}\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x\ \right]$$

$$\Rightarrow \quad x(2e^y + x^3 + 1)\,dx + x(xe^y)\,dy = 0$$

$$\Rightarrow \quad (2xe^y + x^4 + x)\,dx + x^2e^y\,dy = 0$$

$$\begin{array}{l|l} \Rightarrow \quad f_x = 2xe^y + x^4 + x & f_y = x^2e^y \\ \hline \int\!\big(\quad f = x^2e^y + \frac{x^5}{5} + \frac{x^2}{2} + g(y) & f = x^2e^y + h(x) \end{array}$$

So we form our potential and set equal to constant (since it is a level curve!)

$$\boxed{x^2e^y + \frac{x^5}{5} + \frac{x^2}{2} = C}$$