Advanced Combinatorics. Last HW!

Recall the species $\mathcal{L}$ of lists, or linear orders. There is exactly one "empty list," corresponding to the one way we can fill in a row of chairs with 0 people. When we are required to have at least one person–e.g., when we want to count the ways to create an ordered line of people–then we say there are 0 ways to do it with 0 people. For a set U we define:

$$\mathcal{L}_+(U) = \begin{cases} \text{linear orders on } U, & U \neq \emptyset \\ \emptyset, & U = \emptyset \end{cases}$$

Then:

$$\mathcal{L}_+(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \text{ since } |\mathcal{L}_+([n])| = n!, \quad n > 0.$$

In contrast to:

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The species $\mathcal{C}$ of non-empty cycles (cyclical orders), Y of binary trees with labeled leaves, X of singletons and E of sets are defined as in the book and the notes.

For exponential generating functions (e.g.f.) we have:

$$\mathcal{C}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \text{ which is implied by } |\mathcal{C}([n])| = (n-1)!, \quad n > 0.$$

$$Y(x) = \frac{1-\sqrt{1-4x}}{2}$$

$$X(x) = x \text{ and } E(x) = e^x.$$

One more: if a cycle can be empty (think filling seats at a round table) then the species could be called $\mathcal{C}^-$. We have

$$\mathcal{C}^-(x) = 1 + \mathcal{C}(x) = 1 - \ln(1-x).$$

I. For each of the following problems:

a) Count the possibilities for $n = 0, 1, 2, 3$. Draw to illustrate (you may draw structure types and count by multiplying.)

b) Find the species F that describes the situation using a combination of the above basic species.

c) Find the e.g.f. $F(x)$ by combining the e.g.f's using the recipe you found in (b.)

d) Put the e.g.f. into the computer (series: $F(x)$ in wolfram alpha) and list the coefficients for $n = 0, 1, 2, 3, 4, 5$.

e) Look up the sequence at OEIS.org. Is there a known formula?

(in class) Consider the ways to take n people and arrange them around a round table of any size (possibly empty), and an ordered line of those waiting for a seat (possibly empty). Network type: several nodes linked in a ring, plus a queue of nodes disconnected from the ring. (Or, the queue is doubly connected with the queue leader linked to each ring node.)

(in class) Consider the ways to take n people and arrange them into any number of non-empty ordered lines, and then arrange those lines into a cycle. The first person in each line carries a red flag. Network type: a directed ring of nodes, each the lead node of a secondary chain: the secondary chains can consist of just the lead node.

(in class) Consider the ways to take n people and arrange them around a round table of any size (non-empty), and an ordered line of those waiting for a seat (non-empty).

1. Consider the ways to take n people and arrange them around a round table of any size (not empty), and an ordered line of those waiting for a seat (possibly empty).

2. Consider the ways to take n people and arrange them around a round table of any size (possibly empty), and an ordered line of those waiting for a seat (not empty).

3. Consider the ways to take n people and arrange them around a round table of any size (possibly empty), and an unordered set of those waiting for a seat (possibly empty). Network type: several nodes in a ring all online (linked to a server), plus some more nodes that are unlinked and currently offline.

4. Consider the ways to take n people and arrange them around 5 round tables (tables may be empty).

5. Consider the ways to take n books and arrange them in ordered rows on any number of shelves of a bookshelf (shelves must not be empty, and there is at least one shelf). Network type: a double-linked linearly ordered chain of any number of nodes ≥ 1, each the lead node of a secondary chain: the secondary chains can consist of just the lead node.