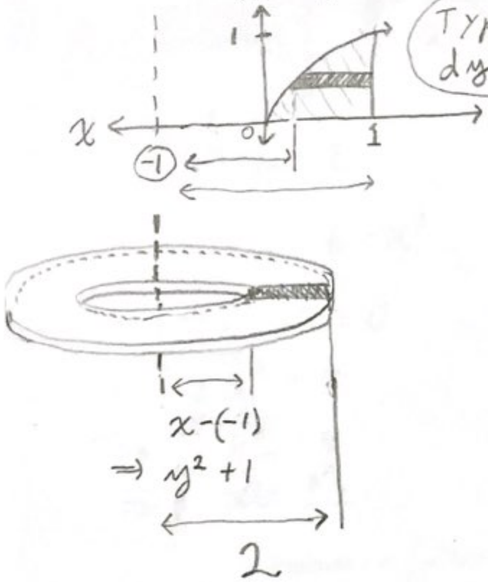
Test Total

Name ______________________________

Final Exam **Calculus II** **Dr. Kreider**
Fall 2016 **For full credit, show your work and use correct notation**

1. Let R be the region enclosed by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$. Let S be the solid obtained by rotating R about the line $x = -1$.

(a) Set up but do not evaluate the integral for the volume of S using the washer method.



Type II
dy

$y = \sqrt{x}$

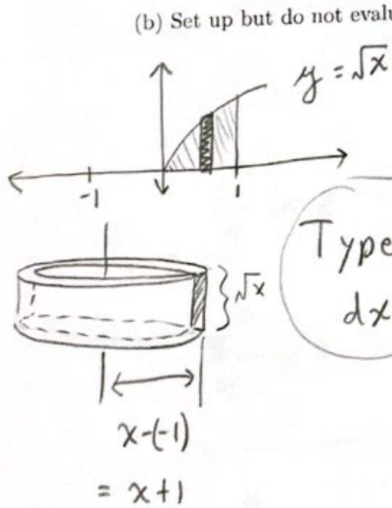
$\boxed{x = y^2}$

$$V = \int_{y_1}^{y_2} \pi \left(r_2^2 - r_1^2\right) dy$$

$$= \int_0^1 \pi\left(2^2 - (y^2+1)^2\right) dy$$

$$= \boxed{\int_0^1 \pi\left(4 - (y^2+1)^2\right) dy} = \frac{32\pi}{15}$$

(b) Set up but do not evaluate the integral for the volume of S using the shell method.



Type I
dx

$$V = \int_{x_1}^{x_2} 2\pi r h \, dx$$

$$= \boxed{\int_0^1 2\pi (x+1)\sqrt{x} \, dx}$$

Pg 1: 10 pts

$$= \frac{32\pi}{15}$$

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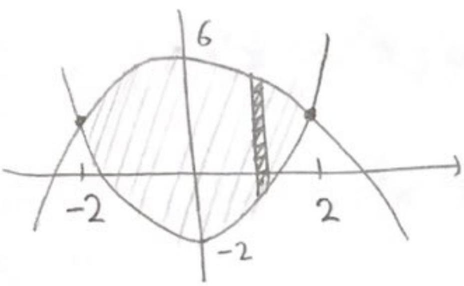
2. Find the area of the region between the curves $y = x^2 - 2$ and $y = 6 - x^2$.

2: 10 pts



$$A = \int_{-2}^{2} \left(6 - x^2 - (x^2 - 2)\right) dx$$

$$= \int_{-2}^{2} (8 - 2x^2)\, dx$$

$$= \left[8x - \frac{2x^3}{3} \right]_{-2}^{2}$$

$$= 16 - \frac{16}{3} - \left(-16 + \frac{16}{3}\right) = \boxed{\frac{64}{3}}$$

Type I.

$$x^2 - 2 = 6 - x^2$$

$$\Rightarrow 2x^2 - 8 = 0$$

$$\Rightarrow 2x^2 = 8$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

3. Evaluate $I = \int z^3 \ln z \, dz$

3: 10 pts

$u = \ln z \quad dv = z^3 dz$

$du = \frac{1}{z} dz \quad v = \frac{z^4}{4}$

$$= uv - \int v\, du$$

$$= \frac{z^4}{4} \ln z - \int \frac{z^4}{4}\left(\frac{1}{z}\right) dz$$

$$= \frac{z^4}{4} \ln z - \int \frac{1}{4} z^3\, dz$$

$$= \boxed{\frac{z^4}{4} \ln z - \frac{z^4}{16} + C}$$

Pg 2: 20 pts

4. Evaluate $I = \int \sin^3 x \cos^6 x \, dx$

4: 10 pts

$$= \int \sin^2 x \cos^6 x \sin x \, dx$$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$$= \int (1-\cos^2 x) \cos^6 x \sin x \, dx$$

$$= \int (1-u^2) u^6 (-1) \, du$$

$$= \int (-u^6 + u^8) \, du$$

$$= -\frac{u^7}{7} + \frac{u^9}{9} + c = \boxed{-\frac{\cos^7 x}{7} + \frac{\cos^9 x}{9} + c}$$

5. Determine whether the improper integral $I = \int_1^6 \frac{7}{(x-1)^4} \, dx$ converges or diverges. If it converges, find its value.

5: 10 pts

$$= \lim_{t \to 1^+} \int_t^6 \frac{7}{(x-1)^4} \, dx$$

$u = x - 1$
$du = dx$

$$= \lim_{t \to 1^+} \int_t^6 \frac{7}{u^4} \, du$$

x-values

$$= \lim_{t \to 1^+} \int_t^6 7u^{-4} \, du$$

$$= \lim_{t \to 1^+} \left[\frac{7u^{-3}}{-3} \right]_{x=t}^{x=6}$$

$$= \lim_{t \to 1^+} \left[\frac{7}{-3} (x-1)^{-3} \right]_t^6$$

$$= \frac{-7}{3} \lim_{t \to 1^+} \left(\frac{1}{5^3} - \frac{1}{(t-1)^3} \right)$$

$$= -\frac{7}{3} \left(\frac{1}{5^3} - \infty \right)$$

$$= \boxed{\infty}$$

$\boxed{\text{diverges}}$

Pg 3: 20 pts

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6. Evaluate $I = \int \frac{t^5}{\sqrt{t^2+16}}\, dt$

6: 10 pts

Method 1.

$$\boxed{\begin{aligned} u &= t^2+16, \quad t^2 = u-16 \\ du &= 2t\,dt \quad t^4 = (u-16)^2 \\ \tfrac{1}{2} du &= t\,dt \end{aligned}}$$

$$= \int \frac{(u-16)^2}{\sqrt{u}} \left(\tfrac{1}{2}\right) du$$

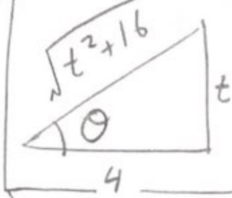
$$= \frac{1}{2}\int \frac{u^2 - 32u + 256}{u^{1/2}}\, du$$

$$= \frac{1}{2}\int u^{3/2} - 32u^{1/2} + 256u^{-1/2}\, du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 32\left(\frac{2}{3}\right)u^{3/2} + 256\left(\frac{2}{1}\right)u^{1/2}\right)$$

$$= \boxed{\frac{1}{5}(t^2+16)^{5/2} - \frac{32}{3}(t^2+16)^{3/2} + 256(t^2+16)^{1/2} + C}$$

Method 2.



$$\begin{aligned} t &= 4\tan\theta \\ dt &= 4\sec^2\theta\, d\theta \\ \sqrt{t^2+16} &= \frac{4}{\cos\theta} \end{aligned}$$

$$\cos\theta = 4(t^2+16)^{-\frac{1}{2}}$$

$$= \int \frac{4^5 \tan^5\theta \cos\theta}{4} 4\sec^2\theta\, d\theta$$

$$= \int \frac{4^5 \sin^5\theta}{\cos^6\theta}\, d\theta$$

$$\boxed{\begin{aligned} u &= \cos\theta \\ -du &= \sin\theta\, d\theta \\ \sin^4\theta &= (1-u^2)^2 \end{aligned}}$$

$$= \int \frac{-4^5(1-u^2)^2}{u^6}\, du$$

$$= \int -4^5(1-2u^2+u^4)/u^6\, du$$

$$= \int -4^5(u^{-6} - 2u^{-4} + u^{-2})\, du$$

$$= -4^5\left(\frac{u^{-5}}{-5} - \frac{2u^{-3}}{-3} + \frac{u^{-1}}{-1}\right) + C$$

= (sub back in, same answer!)

7. Find the arc length of the curve $y = 1 + 2x^{3/2}$ for $1 \le x \le 3$.

7: 10 pts

$$\text{Let } \begin{cases} x = t \\ y = 1 + 2t^{3/2} \end{cases} \Rightarrow \begin{aligned} x' &= 1 \\ y' &= 3t^{1/2} \end{aligned}$$

$$L = \int_1^3 \sqrt{1 + (3t^{1/2})^2}\, dt$$

$$= \int_1^3 \sqrt{1+9t}\, dt$$

$$\left[\begin{aligned} u &= 1+9t \\ du &= 9\,dt \\ \tfrac{1}{9}du &= dt \end{aligned}\right.$$

Pg 4: 20 pts

$$= \int_{x=1}^{x=3} u^{1/2} \frac{1}{9}\, du$$

$$= \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_{x=1}^{x=3} = \frac{1}{9}\left[\frac{2}{3}(1+9t)^{3/2}\right]_1^3 = \boxed{\frac{2}{27}\left(28^{3/2} - 10^{3/2}\right)}$$

$$\approx 8.6325$$

a)

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8. Evaluate $I = \int \frac{3x-2}{x^2+12x+37}\,dx$

7: 10 pts

$$= \int \frac{3x-2}{(x+6)^2+1}\,dx \quad \left[\begin{array}{l} u = x+6 \rightarrow x = u-6 \\ du = dx \end{array}\right]$$

$$= \int \frac{3(u-6)-2}{u^2+1}\,du$$

$$= \int \frac{3u-20}{u^2+1}\,du$$

$$\left[\begin{array}{l} w = u^2+1 \\ dw = 2u\,du \\ \frac{1}{2}dw = u\,du \end{array}\right] = \int \frac{3u}{u^2+1}\,du - 20\int \frac{1}{u^2+1}\,du$$

$$= \int \frac{3}{2}\left(\frac{1}{w}\right)dw \quad -20\tan^{-1}u + c$$

$$= \frac{3}{2}\ln|w|$$

$$= \boxed{\frac{3}{2}\ln\left|(x+6)^2+1\right| - 20\tan^{-1}(x+6) + c}$$

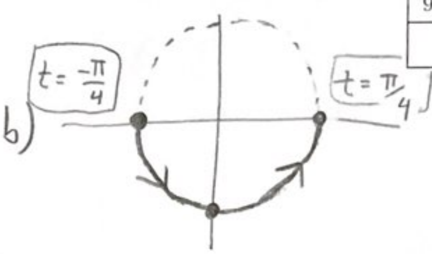
9. Consider the parametric curve $x = \sin 2t$, $y = -\cos 2t$ for $-\pi/4 \le t \le \pi/4$.
(a) Find the Cartesian form of the curve.
(b) Sketch the curve. Label the starting point and ending point, and draw an arrow on the curve to indicate the direction of travel.
(c) Find an equation for the curve's tangent line at the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

9: 10 pts

a)
$$\begin{array}{l} x^2 = \sin^2 2t \\ y^2 = \cos^2 2t \\ \hline x^2+y^2 = 1 \end{array}$$

t	x	y
$-\pi/4$	-1	0
0	0	-1
$\pi/4$	1	0

b)



c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin 2t}{2\cos 2t} = \tan 2t$

at $x = \frac{\sqrt{2}}{2}$, $\sin 2t = \frac{\sqrt{2}}{2}$

$\Rightarrow 2t = \frac{\pi}{4}$

$\Rightarrow t = \frac{\pi}{8}$

$m = \tan\left(2\frac{\pi}{8}\right) = 1$

$y - \left(-\frac{\sqrt{2}}{2}\right) = 1\left(x - \frac{\sqrt{2}}{2}\right)$

$\boxed{y = x - \sqrt{2}}$

Pg 5: 20 pts

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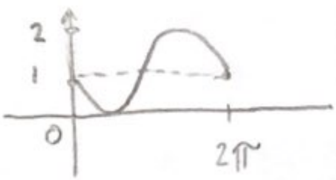
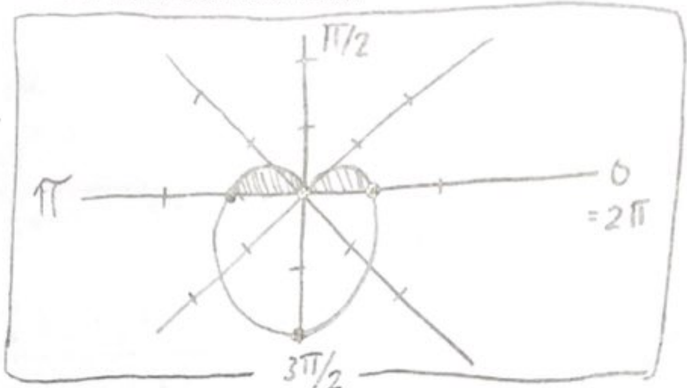
10. Consider the polar curve $r = 1 - \sin\theta$. (a) Sketch the curve. (b) Set up but do not evaluate the integral for the area that lies inside the curve and is above the x-axis.

10: 10 pts





$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$\boxed{= \int_0^{\pi} \frac{1}{2}(1-\sin\theta)^2 \, d\theta}$$

11. Determine if the series $S = \sum_{n=1}^{\infty} \frac{2^n+3}{5^n+2}$ converges or diverges. Indicate the test/s you used.

11: 10 pts

limit comparison to $\sum \left(\frac{2}{5}\right)^n$, convergent geometric $\left(\frac{2}{5} < 1\right)$

$$\lim_{n\to\infty} \frac{2^n+3}{5^n+2} \cdot \left(\frac{5^n}{2^n}\right)$$

$$= \lim_{n\to\infty} \frac{10^n + 3(5^n)}{10^n + 2(2^n)}$$

$$= \lim_{n\to\infty} \frac{1 + 3\left(\frac{5}{10}\right)^n}{1 + 2\left(\frac{2}{10}\right)^n}$$

Pg 6: 20 pts

$$= 1$$

and $0 < 1 < \infty \Rightarrow \boxed{\text{converges}}$

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12. Determine if the series $S = \sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$ converges absolutely, converges conditionally or diverges. Indicate the test/s you used.

12: 10 pts

Ratio test

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{3(2n+1)!}{(2n+2+1)!}$$

$$= \lim_{n\to\infty} \frac{3(2n+1)!}{(2n+3)(2n+2)(2n+1)!}$$

$$= \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0 < 1 \Rightarrow \boxed{\text{converges absolutely}}$$

13. Find the interval and radius of convergence for the power series $f(x) = \sum_{n=0}^{\infty} \frac{2^n(x-1)^n}{2n^2+1}$.

13: 10 pts

Ratio test

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{2(n+1)^2+1} \cdot \frac{2n^2+1}{2^n(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2(x-1)(2n^2+1)}{2(n+1)^2+1} \right|$$

$$= |x-1| \lim_{n\to\infty} \frac{2(2n^2+1)}{2(n^2+2n+1)+1}$$

$$= |x-1| \lim_{n\to\infty} \frac{4n^2+2}{2n^2+4n+2+1}$$

$$= |x-1| \frac{4}{2} < 1$$

$$\Rightarrow |x-1| < \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$$

Ends:

$x = \frac{1}{2}$: $\sum \frac{2^n(-\frac{1}{2})^n}{2n^2+1} = \sum \frac{(-1)^n}{2n^2+1}$

converges by alt series

$x = \frac{3}{2}$: $\sum \frac{2^n(\frac{1}{2})^n}{2n^2+1} = \sum \frac{1}{2n^2+1}$

converges by comparison to the p-series $\sum \frac{1}{n^2}$ since $2n^2+1 > n^2$ $\Rightarrow \frac{1}{2n^2+1} < \frac{1}{n^2}$.

Pg 7: 20 pts

$$\boxed{R = \frac{1}{2} \quad \text{I.O.C.} = \left[\frac{1}{2}, \frac{3}{2}\right]}$$

14. Use any method to find the first 4 nonzero terms of the Maclaurin series for $f(x) = \frac{1}{(1-2x)^2}$.

14: 10 pts

Method 1

Use $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$

$\Rightarrow \frac{1}{(1-2x)^2} = \sum_{n=0}^{\infty} n(2x)^{n-1}$

Then

$n=0 \longrightarrow 0(2x)^{-1} = 0$

$n=1 \longrightarrow 1(2x)^0 = \boxed{1}$

$n=2 \longrightarrow 2(2x)^1 = \boxed{4x}$

$n=3 \longrightarrow 3(2x)^2 = \boxed{12x^2}$

$n=4 \longrightarrow 4(2x)^3 = \boxed{32x^3}$

Method 2.

Use $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = f(x)$

$f^{(0)}(x) = (1-2x)^{-2}$

$f^{(1)}(x) = -2(1-2x)^{-3}(-2)$

$f^{(2)}(x) = 6(1-2x)^{-4}(-2)(-2)$

$f^{(3)}(x) = -24(1-2x)^{-5}(-2)(-2)(-2)$

$f^{(0)}(0) = 1$ | $1x^0/0! = \boxed{1}$

$f^{(1)}(0) = 4$ | $4x^1/1! = \boxed{4x}$

$f^{(2)}(0) = 24$ | $24x^2/2! = \boxed{12x^2}$

$f^{(3)}(0) = 24(8)$ | $24(8)x^3/3! = \boxed{32x^3}$

15. Find the first 4 nonzero terms of the Maclaurin series for $f(x) = \frac{e^{x^2} - (1+x^2)}{x}$.

15: 10 pts

Method 1

Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + x^2 + \sum_{n=2}^{\infty} \frac{(x^2)^n}{n!}$

$\Rightarrow e^{x^2} - (1+x^2) = \sum_{n=2}^{\infty} \frac{(x^2)^n}{n!}$

$\Rightarrow \frac{e^{x^2}-(1+x^2)}{x} = \sum_{n=2}^{\infty} \frac{x^{2n}}{x\,n!} = \sum_{n=2}^{\infty} \frac{x^{2n-1}}{n!}$

$n=2 \longrightarrow x^{4-1}/2! = \boxed{\frac{x^3}{2}}$

$n=3 \longrightarrow x^{6-1}/3! = \boxed{\frac{x^5}{6}}$

$n=4 \longrightarrow x^{8-1}/4! = \boxed{\frac{x^7}{24}}$

$n=5 \longrightarrow x^{10-1}/5! = \boxed{\frac{x^9}{120}}$

Method 2

Use $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = f(x)$

(first few terms are 0, start at $n=2$)

Pg 8: 20 pts