* Chp. 5 Eigen-stuff

Def: When $T: V \to V$ is a lin. trans. and we find a specific vector $\vec{x} \in V$ such that $\vec{x} \neq \vec{0}$

and $T(\vec{x}) = c\vec{x}$ for some constant c

then we call $\vec{x}$ an eigenvector with eigenvalue c (often use $c = \lambda$)
(λ can be 0, but $\vec{x} \neq \vec{0}$)

(if T is just multiplying every vector by a constant, then every vector in V is an eigenvector, with that constant λ its eigenvalue.)

However, most lin. trans. $T: V \to V$ have only certain eigenvectors and eigenvalues. Find them!

Steps:

1) We work with $A = [T]_{\mathcal{B}}^{\mathcal{B}}$

2) Let $A\vec{x} = \lambda\vec{x}$ $(\vec{x} \neq \vec{0})$

then $\Rightarrow A\vec{x} = (\lambda I)\vec{x}$ $\left(\lambda I = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \right)$

$\Rightarrow A\vec{x} - (\lambda I)\vec{x} = \vec{0}$

$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

So $\vec{x} \neq \vec{0}$ and $\vec{x} \in N(A - \lambda I)$

$\Rightarrow \det(A - \lambda I) = 0$

3) this gives us an algebraic equation to solve for λ. Then plug back in to find $\vec{x}$.

ex) Let $T: \mathcal{P}^2 \to \mathcal{P}^2$ be given by $T(f(x)) = 2x f'(x) + 3x f''(x)$

Find the eigenvalues and their corresponding eigenvectors for T.

$\vec{e}_i \in \mathcal{E}$	$f'(x)$	$f''(x)$	$T(\vec{e}_i)$
1	0	0	0
x	1	0	$2x$
x^2	$2x$	2	$4x^2+6x$

1) $A = [T]_{\mathcal{E}}^{\mathcal{E}} = \left[[0]_{\mathcal{E}} \; [2x]_{\mathcal{E}} \; [4x^2+6x]_{\mathcal{E}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

2) $\det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = 0$

$= \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$

$= -\lambda(2-\lambda)(4-\lambda) = 0$ ← This is called the characteristic (polynomial) equation.

$\Rightarrow \boxed{\lambda = 0, 2, 4}$

3) Solve $(A - \lambda I)\vec{x} = \vec{0}$

$\boxed{\lambda = 0}$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x_1 = x_1 \text{ free} \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

$\vec{x} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \text{Span}\{1\}$

$\boxed{\lambda = 2}$

$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x_1 = 0 \\ x_2 = x_2 \\ x_3 = 0 \end{cases}$

$\vec{x} \in \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \text{Span}\{x\}$

$\boxed{\lambda = 4}$

$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = 0 \\ x_2 = 3x_3 \\ x_3 = x_3 \end{cases}$

$x \in \text{Span}\left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\} = \text{Span}\{3x + x^2\}$