PAINTED TREES AND PTERAHEDRA

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Lisa Tredway Berry

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PAINTED TREES AND PTERAHEDRA

Lisa Tredway Berry

Thesis

Approved:

Advisor
Dr. Stefan Forcey

Faculty Reader
Dr. W. Stuart Clary

Faculty Reader
Dr. Hung Nguyen

Department Chair
Dr. Timothy Norfolk

Accepted:

Dean of the College
Dr. Chand Midha

Dean of the Graduate School
Dr. George Newkome

Date

ABSTRACT

Associahedra can be realized by taking the convex hull of coordinates derived from binary trees. Similarly, permutahedra can be found using leveled trees. In this paper we will introduce a new type of painted tree, $(T \circ Y)_n$ where n is the number of interior nodes. We create these painted trees by composing binary trees on leveled trees. We define a coordinate system on these trees and take the convex hull of these points. We explore the resulting polytope and prove, using a bijection to tubings, that for $n \leq 4$ the poset of the painted face trees with $n+1$ leaves is isomorphic to the face poset of an n-dimensional polytope, specifically $\mathcal{K}F_{1,n}$, the graph-associahedron for a fan graph, $F_{1,n}$.

ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION

Mathematicians study objects and puzzles, looking for descriptive patterns and equations. Often these patterns and equations can then be used to further science and engineering applications. Binary trees are just one of these objects mathematicians study. Binary trees have been used in the study of waves [1], quantum physics [2] and phylogenetics (evolutionary development) [3].

Loday [4] showed that associahedra can be formed by taking the convex hull of coordinates derived from rooted, planar binary trees. Similarly, permutahedra can be found using leveled trees. Forcey [5] studied the multiplihedra realized by painted trees made of binary trees. In this paper we look at a new type of painted tree made by composing binary trees on leveled trees.

We define the *pterahedron* to be the poset of painted face trees, which we show to be the face poset of a polytope. We will prove, using a bijection to tubings, that the poset of the painted face trees with $n+1$ leaves is isomorphic to the face poset of an n-dimensional polytope, specifically $\mathcal{K}F_{1,n}$, the graph-associahedron for a fan graph, $F_{1,n}$. We will realize the pterahedron in a new way for $n \leq 4$, taking the convex hull of coordinates derived from painted trees.

CHAPTER II

TREES

2.1 Binary Trees

Trees are connected, oriented graphs of nodes and edges. We begin our discussion with rooted, planar, binary trees.

- Rooted: Each node has a single root edge.
- Planar: Each tree can be drawn in one plane with no edges crossing. Two planar trees are only equal if they can be made into identical pictures by scaling portions of the plane, without any reflection or rotation in any dimension.
- Binary: Each node has exactly two leaf edges.

We will refer to these trees as simply "binary" trees. Y_n is the set of all binary trees with n nodes and $n+1$ leaves. As Loday and Ronco [6] discussed, the number of trees in the set Y_n is the Catalan number C_n.

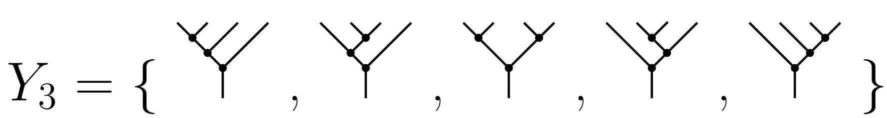
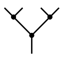
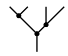
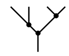


Figure 2.1: The set Y_3.

2.2 Leveled Trees

A *leveled* tree (sometimes called an *ordered* tree) is similar to a binary tree but with the additional condition that each node must be on a distinct horizontal level. Thus  is a binary tree, but not a leveled tree. Imposing the condition of the leveled nodes leads to the distinct leveled trees  and . Clearly, a leveled tree with n nodes must have n levels. T_n is the set of all leveled trees with n nodes and $n+1$ leaves. As Loday and Ronco [6] discussed, there are $n!$ trees in the set T_n.

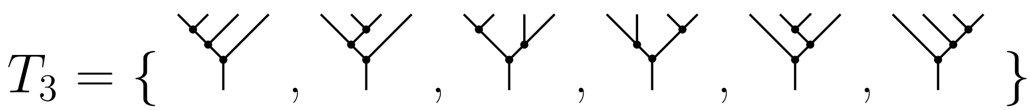


Figure 2.2: The set T_3.

2.3 Painted Trees

The trees we discuss in this paper are constructed by composing leveled trees with binary trees. (See figure 2.4 for an illustration of this composition.) We distinguish between the two layers of the tree by painting the lower (leveled) portion of the tree. Since these painted trees are leveled trees, T_k, composed with binary trees, Y_j, we will refer to the painted trees as $(T \circ Y)_n$.

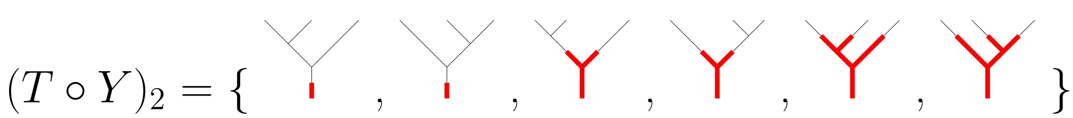


Figure 2.3: The set $(T \circ Y)_2$.

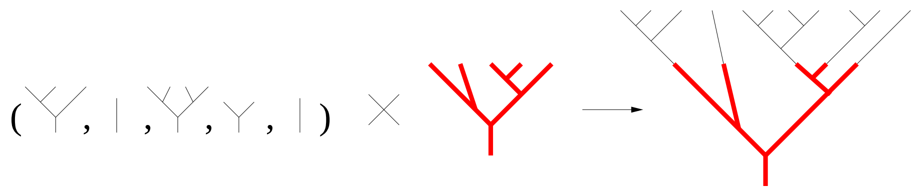


Figure 2.4: Construction of a painted tree in $(T \circ Y)_{10}$.

2.4 The Cardinality of $(T \circ Y)_n$

As in the sets Y_n and T_n, the number of nodes in each tree in the set $(T \circ Y)_n$ is constant. Thus if there are k nodes in the leveled/painted portion of a tree we must consider that there are

- $k!$ ways to make the leveled/painted portion of this tree with k nodes,
- $k+1$ leaves on the top of the leveled/painted portion of this tree, and
- $n-k$ remaining nodes to be distributed among the $k+1$ binary trees that will go on the leveled/painted leaves.

Thus the number of trees in $(T \circ Y)_n$ is

$$\sum_{k=0}^{n} \Big[\ k! \sum_{\substack{\gamma_0+\ldots+\gamma_k \\ =n-k}} \Big(\prod_{i=0}^{k} C_{\gamma_i}\Big)\ \Big].$$

As an example, the number of trees in the set $(T \circ Y)_4$ is

$$
\begin{aligned}
|(T \circ Y)_4| = {} & 0!\ [C_4] \\
& + 1!\ [C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3] \\
& + 2!\ [C_2C_0C_0 + C_1C_1C_0 + C_1C_0C_1 + C_0C_2C_0 + C_0C_1C_1 + C_0C_0C_2] \\
& + 3!\ [C_1C_0C_0C_0 + C_0C_1C_0C_0 + C_0C_0C_1C_0 + C_0C_0C_0C_1] \\
& + 4!\ [C_0C_0C_0C_0C_0] \\
& = 1(14) + 1(14) + 2(9) + 6(4) + 24(1) \\
& = 14 + 14 + 18 + 24 + 24 \\
& = 94
\end{aligned}
$$

We have computed the cardinalities of $(T \circ Y)_n$ for $n = 0$ to 9 and they are shown in table 2.1. See appendix A for the computations of these numbers.

Table 2.1: The number of trees in the set $(T \circ Y)_n$, $n = 0$ to 9

n	$\lvert(T \circ Y)_n\rvert$	n	$\lvert(T \circ Y)_n\rvert$
0	1	5	464
1	2	6	2652
2	6	7	17,562
3	22	8	133,934
4	94	9	1,162,504

Examination of the computations of $|(T \circ Y)_n|$ leads to an interesting discovery. If we strip off the factorial factors in $|(T \circ Y)_n|$ and build a triangle of just the sums of C_{γ_i} products, it appears we are building the Catalan triangle.

1									
1	1								
2	2	1							
5	5	3	1						
14	14	9	4	1					
42	42	28	14	5	1				
132	132	90	48	20	6	1			
429	429	297	165	75	27	7	1		
1430	1430	1001	572	275	110	35	8	1	
4862	4862	3432	2002	1001	429	154	44	9	1

For example, as seen previously and in appendix A,

$$(T \circ Y)_4 = 94 = 0!(\mathbf{14}) + 1!(\mathbf{14}) + 2!(\mathbf{9}) + 3!(\mathbf{4}) + 4!(\mathbf{1})$$

leads to the values of the $n = 4$ row in the triangle above. In fact, Zoque [7] states that the entries of the Catalan triangle, often called *ballot numbers*, count “the number of ordered forests with m binary trees and with total number of ℓ internal vertices” where m and ℓ are indices into the triangle. These forests describe exactly the sets of binary trees we are grafting onto the leaf edges of individual leveled trees, which are counted by the sums of C_{γ_i} products. Thus we know that the ballot numbers are equivalent to the sums of C_{γ_i} products and can be used in the calculation of $|(T \circ Y)_n|$ for all values of n.

The formula for the entries of the Catalan triangle [8] leads to a simpler formula for the cardinality of $(T \circ Y)_n$, namely

$$|(T \circ Y)_n| = \sum_{k=0}^{n} [\ k! \tfrac{(2n-k)!(k+1)}{(n-k)!(n+1)!}],$$

making it possible to calculate $|(T \circ Y)_n|$ for any value of n.

Lastly, with the discovery of the Catalan triangle in the $|(T \circ Y)_n|$ computations, we can say that the sequence of cardinalities $|(T \circ Y)_n|$ for all n is, according to Barry [9], the Catalan transform of the factorials.

CHAPTER III

POLYTOPES

A *polytope* is the bounded intersection of a finite number of half-spaces. More simply, it is a geometric object with straight edges that exists in any number of dimensions. A polygon is a polytope in two dimensions, a polyhedron is a polytope in three dimensions, and an n-polytope is a polytope in n dimensions. The elements of a polytope (e.g., vertices and edges) are referred to as faces. An m-face is an m-dimensional element. For example, a vertex is a 0-face, an edge is a 1-face and a facet is an $(n-1)$-face.

The *convex hull* of a set of points is the minimal convex set containing all of the points. The convex hull of a finite set of points forms a polytope. For example, the convex hull of the points $\{(0,0),(0,2),(2,0),(2,2)\}$ is a 2-dimensional square. It is frequently of interest whether all of the points in a set are vertices of the convex hull. For example, if we include $(1,1)$ in the set above, then the convex hull is unchanged and the new point is in the interior of the square. Thus the point $(1,1)$ is contained in the convex hull but is **not** a vertex of the resulting polytope.

3.1 Associahedron

Loday and Ronco [4] described a method for realizing an associahedron from the binary trees of the set Y_n by deriving coordinates from each tree and taking the convex hull of the resulting set of points. They performed the following steps for each tree in the set.

1. Number the nodes of a given tree from left to right by $1, 2, ..., n$. (Number the spaces between the leaves. Nodes are ordered as the spaces directly above them.)

2. For each node in the tree, let a_i be the number of leaves to the left of the ith node and b_i be the number of leaves to the right of the ith node.

3. Let the point associated with the tree be $(a_1 b_1, a_2 b_2, ..., a_n b_n)$.

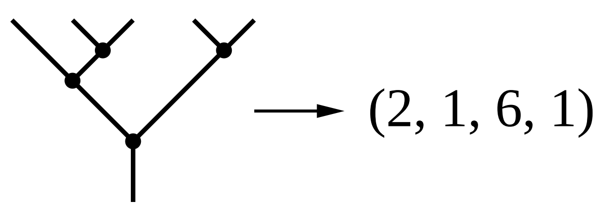


Figure 3.1: Example of coordinates derived from a binary tree.

Observe in the example of figure 3.1 that the third node has three leaves to the left and two leaves to the right. Hence the third coordinate entry is 6.

The convex hull of these points in $\mathbf{R}^n$ is an associahedron. Further, each point associated with a tree in Y_n is a vertex of the resulting polytope. Interestingly, the resulting polytope is $(n-1)$-dimensional and contained in a hyper-plane of $\mathbf{R}^n$. Observe in the plot of figure 3.2 that when $n = 3$ the associahedron is a two-dimensional pentagon suspended in $\mathbf{R}^3$.

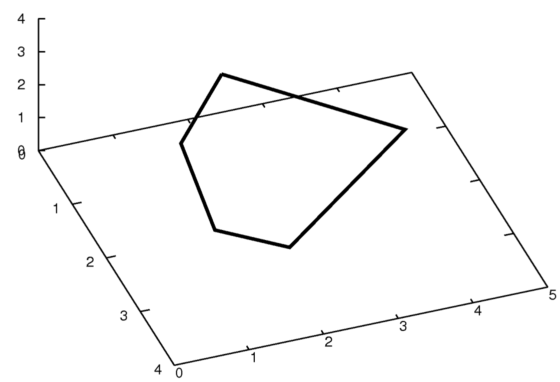


Figure 3.2: Plot of coordinates associated with Y_3.

3.2 Permutahedron

Forcey [10] described a well-known bijection between leveled trees with n nodes and the permutations of n. We use this bijection to derive coordinates from leveled trees in a straightforward manner, given that the trees have a node on each level. Again, consider a tree's nodes, 1 to n, from left to right. Additionally, number the node levels, 1 to n, from top to bottom. Let each a_i in the point $(a_1, a_2, ..., a_n)$ be the level of the ith node.

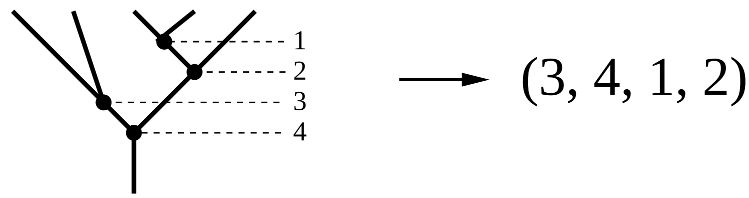


Figure 3.3: Example of coordinates derived from a leveled tree.

Observe in the example of figure 3.3 that the leftmost node is at the third level from the top. Hence the first coordinate entry is 3. Clearly the coordinates $(a_1, a_2, ..., a_n)$ are a permutation of the numbers 1 to n.

The coordinates associated with all the trees of T_n make up the set of all permutations of the numbers 1 to n. The resulting convex hull of these points is the permutahedron. Observe in the plot of figure 3.4 that, as with the associahedron, when $n = 3$ the permutahedron is a two-dimensional hexagon suspended in $\mathbf{R}^3$.

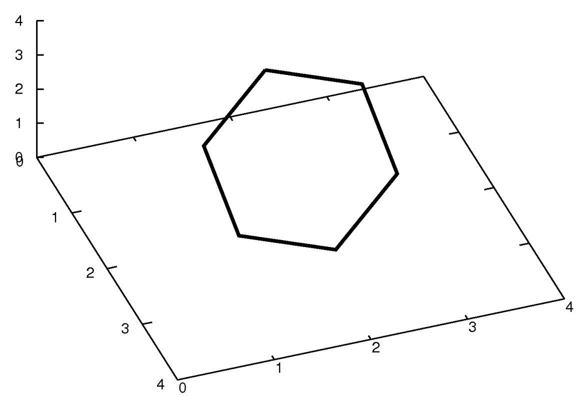


Figure 3.4: Plot of coordinates associated with T_3.

CHAPTER IV

TREES AND TUBINGS

We can combinatorially describe a polytope by looking at its *face poset*, or the set of all of its faces partially ordered by inclusion. In order to show that the poset of painted trees is actually the face poset of a polytope, we describe a bijection between painted trees with $n+1$ leaves and tubings on a fan graph with $n+1$ vertices. First we define tubings.

Let G be a finite, connected graph. As Devadoss [11] described, a *tube* is a proper nonempty set of vertices of G whose induced graph is a proper, connected subgraph of G. Two tubes u_1 and u_2 are *nested* if $u_1 \subset u_2$, *intersecting* if $u_1 \cap u_2 \neq \emptyset$ but neither tube is nested in the other, and *adjacent* if $u_1 \cup u_2$ induces a connected subgraph of G. Two tubes are *compatible* if they do not intersect and they are not adjacent. A set of tubes is called a *tubing* if each pair of tubes is compatible.

(a) (b)

Figure 4.1: (a) Valid tubings and (b) invalid tubings

4.1 Vertex Tree Bijection

The trees we have discussed thus far are *vertex trees*, that is, they are the trees from which we derive the polytope's vertex coordinates. Recall that vertex tree nodes are trivalent only, with one edge on the root-side and two edges on the leaf side. Paint begins mid-edge (i.e., between nodes) only. We will call where paint begins the *paint line*.

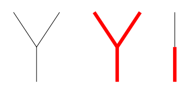


Figure 4.2: Vertex-tree nodes and a mid-edge paint line

We start with a bijection between vertex trees with n nodes and tubings with the maximal $n+1$ tubes (including the all-inclusive tube). We will address the proof of this bijection at the end of section 4.2. Note that, as is standard, we will draw tubings without the implied all-inclusive tube. Also note that we use the word "vertex" in relation to graphs and "node" with trees.

Tubing $\rightarrow$ Tree

1. Number the graph vertices, 0 to n, in a counter-clockwise fashion beginning with the top vertex. Identify the smallest tube containing the 0-vertex. In this paper we shall call the smallest tube containing the 0-vertex t.

2. Draw $n + 1$ dots for the tree leaves. Number spaces between the leaves 1 to n.

3. Note that the node under space i between tree leaves corresponds to vertex i of the graph.

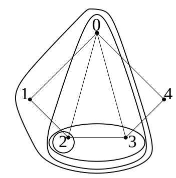
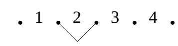
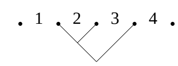
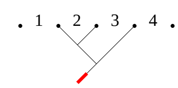
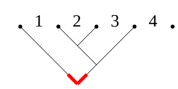
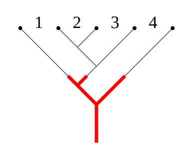
4. Using all of the non-0 vertices inside t, connect leaves as you would to construct a (non-leveled) tree from a path graph. These edges and vertices will NOT be painted.

5. Tube t indicates the location of the paint line. Edges and nodes below those constructed in the previous step will be painted.

6. Starting with the first tube containing t and working out, add appropriate edges to the tree to create leveled nodes below the existing nodes. Work down and connect new nodes to any existing adjacent nodes. These nodes are painted.

See table 4.1 for an example of a tubing $\rightarrow$ vertex tree mapping.

Table 4.1: Illustration of tubing → vertex tree mapping

Step	Description	Figure
1	Tubing to be mapped to a painted tree	
2	Edges and node for vertex 2	
3	Add edges and node for vertex 3	
4	Add paint line	
5	Add edge and node for vertex 1	
6	Final tree with added edges and node for vertex 4	

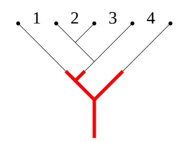
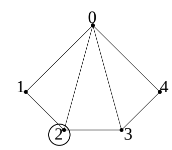
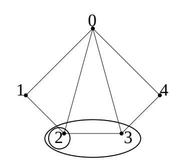
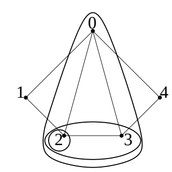
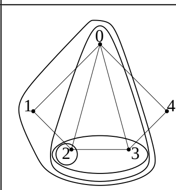
Tree $\rightarrow$ Tubing

1. Number the spaces between the tree leaves, 1 to n, from left to right.
2. Draw a fan graph and number the vertices, 0 to n, in a counter-clockwise fashion beginning with the top vertex.
3. Consider the unpainted portions of the tree. Draw tubes as you would to construct path-graph tubings from (non-leveled) binary trees.
4. To indicate the beginning of the paint, draw a tube containing all existing tubes and the 0-vertex.
5. Starting with the topmost painted node and working down, draw a tube for each leveled node of the tree containing all existing tubes and adding the vertex corresponding to the current node.

See table 4.2 for an example of a vertex tree $\rightarrow$ tubing mapping.

This vertex-tree/maximal-tubing bijection is instructive, but it does not provide us with enough information to prove that the poset of painted trees is isomorphic to the face poset of a polytope. Ultimately, we need a bijection between face trees and tubings with any number of tubes, which we describe in the next section.

Table 4.2: Illustration of vertex tree → tubing mapping

Step	Description	Figure
1	Tree to be mapped to a tubing	
2	Draw tube for node below tree space 2	
3	Add tube for node below tree space 3	
4	Add tube for paint line	
5	Final tubing w/ added tube for painted node below space 1 and implied all-inclusive tube	

4.2 Face Tree Bijection

The m-faces, $m > 0$, (e.g., edges and facets) of the associahedron and permutahedron correspond to trees that are no longer binary. Face tree nodes are not always trivalent, as they may have more than two edges on the leaf side. This is true with painted face trees as well. Additionally, the paint line may or may not run through a node. Thus the paint line can either be mid-edge, meaning it does not hit any nodes, or on node, meaning it falls on at least one node. We will call nodes that reside on the paint line *half-painted* nodes.

Figure 4.3: Additional face-tree nodes with a *half-painted* node on the far right

Recall that in vertex trees, painted portions are leveled and no two nodes may reside on the same level. In face trees, painted nodes can occur at the same level. This is indicated with a dotted line as illustrated in figure 4.4. Though structurally separated, these nodes are treated as one combined node.

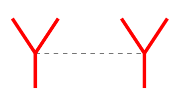


Figure 4.4: Painted face-tree nodes on the same level

See figure 4.5 for the face trees of the 2-dimensional pterahedron.

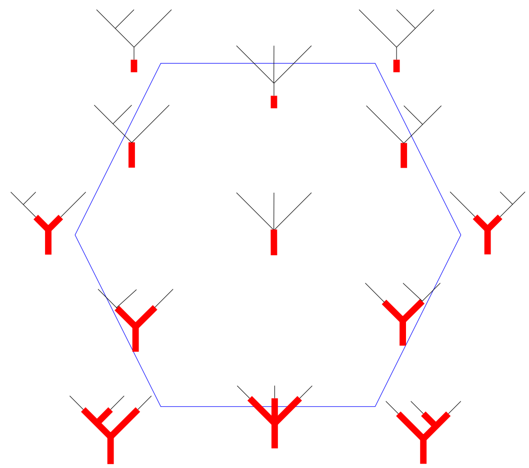


Figure 4.5: Vertex, edge, and facet trees of the 2-dimensional pterahedron

We describe a bijection between face trees and tubings that contain any number of tubes.

<u>Tubing $\rightarrow$ Tree</u>

1. Number the graph vertices, 0 to n, in a counter-clockwise fashion beginning with the top vertex. Identify the smallest tube containing the 0-vertex, t.

2. Draw $n+1$ dots for the tree leaves. Number spaces between the leaves 1 to n.

3. Note that the node under space i between tree leaves corresponds to vertex i of the graph.

4. Using all the non-0 vertices inside t, connect leaves as you would for a (non-leveled and possibly non-binary) tree from a path graph. These edges and vertices will NOT be painted. Note that it may be necessary to connect more than two adjacent leaves to one node.

5. If the only vertex in tube t that is not contained in another tube is the 0-vertex,

   - Paint line is mid-edge all across the tree.

   Otherwise,

   - Add half-painted nodes to the tree that correspond to the non-0-vertices contained in t but in no other tubes nested in t. The paint line is on these nodes of the tree.

6. Starting with the first tube containing t and working out, add appropriate edges to the tree to create leveled nodes below the existing nodes. Work down and connect new nodes to any existing adjacent nodes. Note that it may be necessary to connect more than two adjacent edges to one node as you must put all of the nodes corresponding to vertices in the new tube at the same level. These nodes are painted. (Note paint line lies on all edges immediately below the unpainted trees drawn earlier.)

7. If new nodes in a level are NOT adjacent, indicate they are on the same level with a dotted line.

Tree $\rightarrow$ Tubing

1. Number the spaces between the tree leaves, 1 to n, from left to right.

2. Draw a fan graph and number the vertices, 0 to n, in a counter-clockwise fashion beginning with the top vertex.

3. Consider the portion of the tree above the paint line. Draw tubes as you would to make a path-graph tubing from a non-leveled tree.

4. If the paint line is mid-edge,

   - To indicate the beginning of the paint, draw a tube containing all existing tubes and the 0-vertex.

   Otherwise,

   - To indicate the beginning of the paint, draw a tube containing the 0-vertex, all existing tubes and vertices associated with any nodes on the paint line.

5. Starting with the topmost node(s) below the paint line and working down, draw a tube for each painted level of the tree which contains all existing tubes and adding the vertex/vertices corresponding to the node(s) on the next level.

Observe that the vertex tree bijection is correctly described by the more complicated face tree bijection. See figure 4.6 for an illustration of the face tree bijection using four identical tree structures with different paint lines.

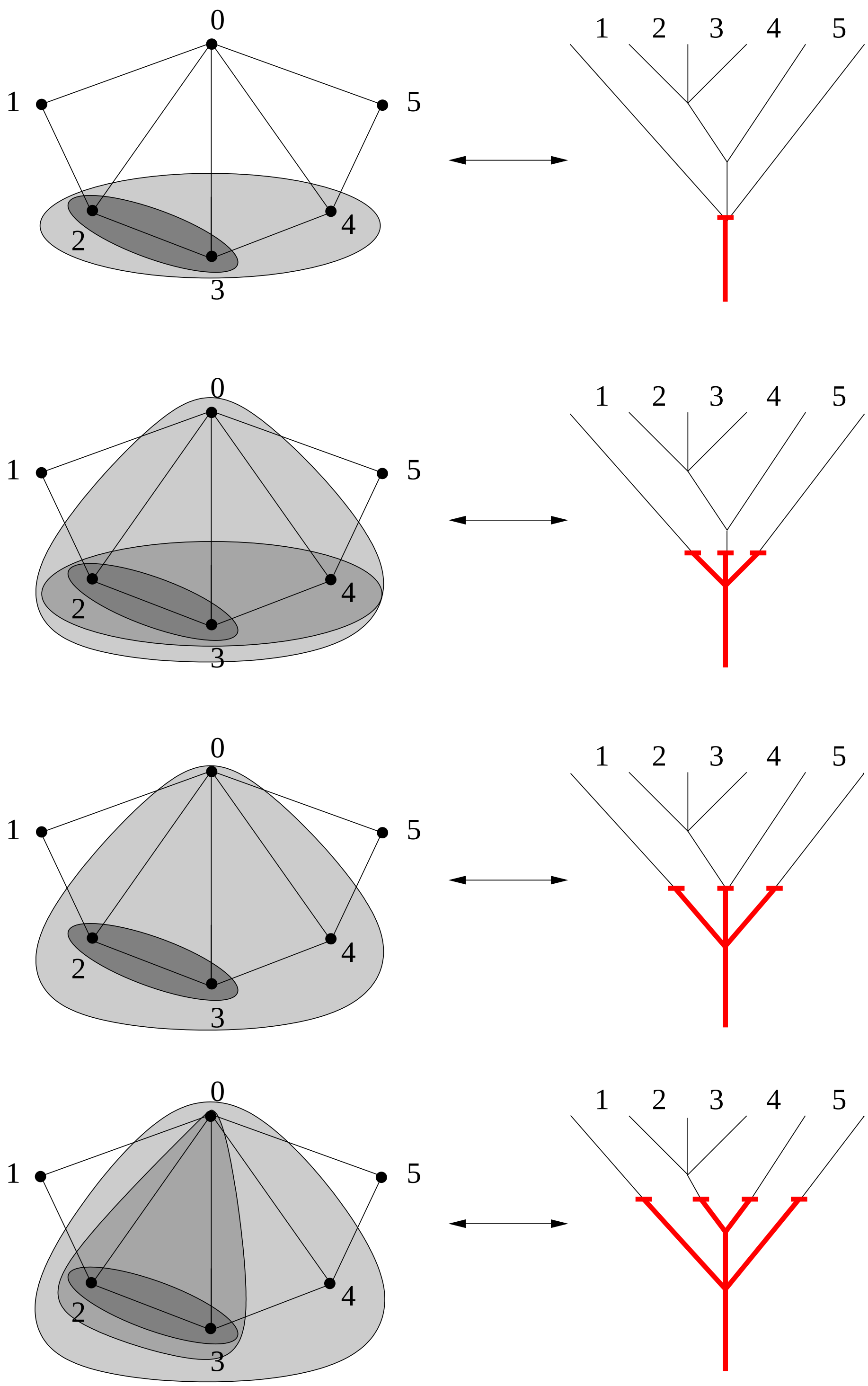


Figure 4.6: The painted tree/tubing bijection using trees with identical structures but different paint lines.

As proof of this bijection, observe the following. Clearly there is an injective and surjective mapping between half-painted nodes of a tree and vertices that are contained in tube t of a fan graph but are not in any additional tubes nested inside t. (This is illustrated by vertex/node 4 of the third tubing/tree in figure 4.6.) Next, if we consider only the tubes inside t, we have one or more path graphs. As Forcey and Springfield [12] discussed, there is a bijection between path-graph tubings and trees (faces of associahedra). Finally, if we consider the reconnected complement of a fan-graph tubing with t and everything contained in t removed we have a complete graph. Devadoss [11] proved that there is a bijection between complete-graph tubings and partially-leveled trees (faces of permutahedra). Thus because our painted trees are a composition of binary trees on leveled trees, there exists a bijection between fan-graph tubings and our painted trees.

CHAPTER V

A POSET ISOMORPHISM

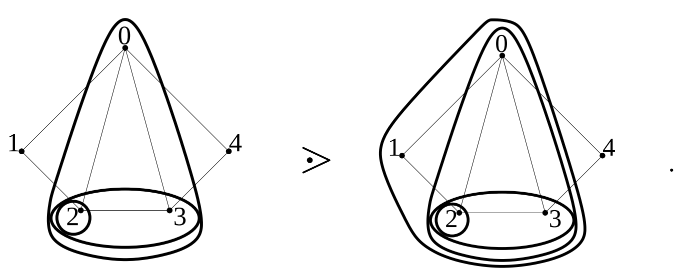
5.1 Poset Isomorphism Description

Next we show a poset isomorphism between face trees and tubings. Two posets are isomorphic if there exists an order-preserving bijection between them whose inverse is also order preserving. In chapter 4 we described a bijection, which we now call φ, from tubings on a fan graph with $n+1$ vertices to painted face trees with $n+1$ leaves. We now show that φ is order-preserving. Since any ordering can be broken into a chain of covering relations, we describe covering relations such that $\forall x, y \in$ tubings,

$$x \gtrdot y \;\Leftrightarrow\; \varphi(x) \gtrdot \varphi(y).$$

We define the covering relation on tubings as follows: $x \gtrdot y$ if tubing x is contained in tubing y and y has exactly one additional tube that is not in x.

For example,



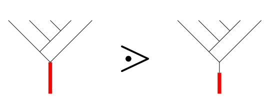
While there certainly may be many covering relations on face trees, we choose to define one as follows: $\varphi(x) \gtrdot \varphi(y)$ if one of the following is true:

- $\varphi(x)$ paint line goes through a node, $\varphi(y)$ paint line is mid-edge and is immediately above or below the same node, and $\varphi(x)$ and $\varphi(y)$ have identical tree structures and identical painting except at that one node.

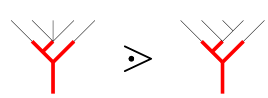
- $\varphi(x)$ has one node with a greater number of edges on the leaf side than the comparable node in $\varphi(y)$, these additional edges have been "slid down" from nodes immediately above, and other than this difference $\varphi(x)$ and $\varphi(y)$ have identical tree structures. Note that the paint line may move along with a "sliding" edge and recall that two painted nodes on the same level can be considered one (combined) node.

- $\varphi(x)$ has two painted nodes on the same level, in $\varphi(y)$ these two nodes' levels differ by one level, and other than this difference $\varphi(x)$ and $\varphi(y)$ have identical tree structures.

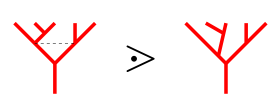
For example,



,



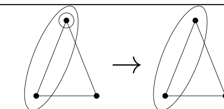
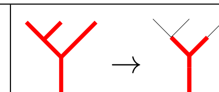
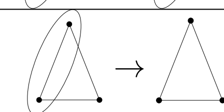
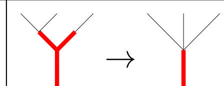
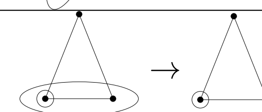
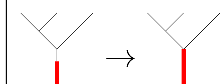
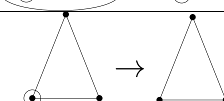
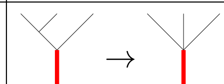
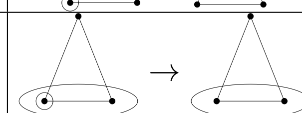
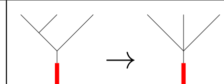
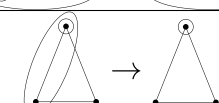
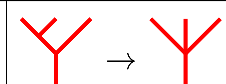
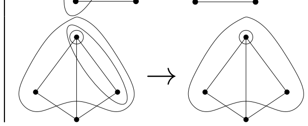
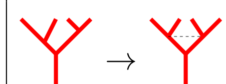
and



.

As proof our bijection is order-preserving, observe the following. Assume $x \gtrdot y$. Then x is contained in y and there is exactly one tube in y that is not in x. We list in table 5.1 the potential changes to tree $\varphi(y)$ that will result from removing the additional tube in y. In every case, the change results in a new tree, $\varphi(x)$, that covers $\varphi(y)$. Thus $\varphi(x) \gtrdot \varphi(y)$. Conversely, if $\varphi(x) \gtrdot \varphi(y)$ then clearly it must be true that $x \gtrdot y$.

Table 5.1: Consequences of removing one tube from a tubing

tube removed from y	$\varphi(y)$ paint line before change	change applied to $\varphi(y)$	Example $y \to x$	Example $\varphi(y) \to \varphi(x)$
t	mid-edge	drop paint line down to node		
	node	drop paint line down to lower node [1]		
just inside t	mid-edge	raise paint line up to one half-painted node		
	node	move unpainted edge(s) from mid-edge to node		
further inside t	any	move unpainted edge(s) from mid-edge to node		
outside t	any	move painted edge(s) from mid-edge to node		
		or bring two painted nodes to same level		

[1]Note if moving paint line up/down from node to node, any edges in the original half-painted node move up/down to the new half-painted node along with the paint line.

Since any ordering can be broken into a chain of covers and we have shown $\forall x, y \in$ tubings, $x \gtrdot y \iff \varphi(x) \gtrdot \varphi(y)$, then any change to a tubing that results in a new tubing of lower order (i.e., removing contained tubes) maps to a change to a tree resulting in a new tree of lower order. Clearly the reverse is also true. Thus we have a poset isomorphism between face trees and tubings.

5.2 Consequence of the Poset Isomorphism

Devadoss [11] stated that for a graph G, the *graph associahedron* $\mathcal{K}G$ is a simple, convex polytope whose face poset is isomorphic to the set of tubings of G. Because we have shown that a face poset isomorphism exists between painted trees with $n+1$ leaves and fan-graph tubings with $n+1$ vertices, we know that the poset of painted face trees with $n+1$ leaves is isomorphic to the face poset of the graph-associahedron, $\mathcal{K}F_{1,n}$ for all values of n.

5.3 Realizing the Pterahedron with Painted Trees

To use Devadoss's theorem, we must use tubings with $n+1$ vertices to realize the n-dimensional pterahedron as a convex hull of points in $\mathbf{R}^{n+1}$. Working in $\mathbf{R}^{n+1}$ can present some difficulties. We now describe a method to associate vertices of the n-dimensional pterahedron to points in $\mathbf{R}^n$ using painted trees with $n+1$ leaves. Taking the convex hull of these vertices, we have realized the pterahedron in dimensions n=1 thru 4.

5.3.1 Painted Tree Coordinates

Recall in chapter 3 we realized polytopes by deriving coordinates from binary and leveled trees. We combine those methods already discussed with a multiplier to indicate the unpainted portion of the tree. For a painted vertex tree with n nodes, derive the point $(a_1, a_2, ..., a_n)$ as follows. This is illustrated in figure 5.1.

1. Number the nodes from left to right, 1 to n, according to the spaces above the nodes.

2. Number the levels of the painted portion of the tree starting with n at the bottom and working up, n, $n-1$, $n-2$, ...

3. Consider node i. If the node is painted,

   $a_i =$ the node's level

   Otherwise

   $a_i = \frac{1}{2}$ (the number of leaves to the left) (the number of leaves to the right)

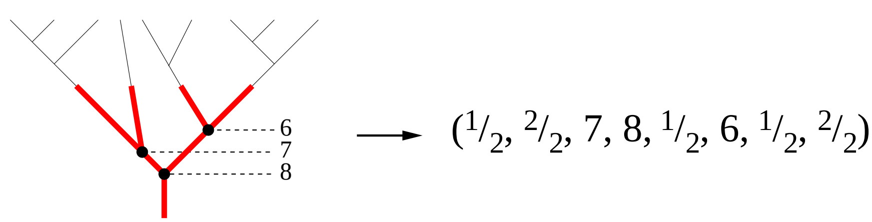
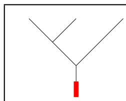
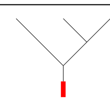
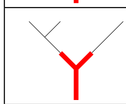
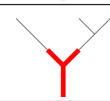
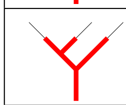
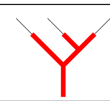


Figure 5.1: Example of coordinates derived from a painted tree

5.3.2 Pterahedra in $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$

Finally, we answer the question: What is the polytope realized from the set of painted trees $(T \circ Y)_n$? Because of the composition, the cardinality of $(T \circ Y)_n$ is greater than that of T_n or Y_n. As a result, we can look at the points derived from $(T \circ Y)_2$ in $\mathbf{R}^2$ and $(T \circ Y)_3$ in $\mathbf{R}^3$, as opposed to working in $\mathbf{R}^{n+1}$ as we did with binary and leveled trees. The coordinates for the sets $(T \circ Y)_2$ and $(T \circ Y)_3$ can be seen in tables 5.2 and 5.3. Plots of the convex hulls are shown in figures 5.2 and 5.3. In both cases the number of points matches the number of polytope vertices and the tool Polymake [13] confirms this. Examination proves that, for $n = 2$ and 3, the polytope derived from painted trees is combinatorially equivalent to the pterahedron.

Table 5.2: Coordinates for painted trees with 2 nodes

Tree	Coordinates	Tree	Coordinates
	$(\frac{1}{2}, \frac{2}{2})$		$(\frac{2}{2}, \frac{1}{2})$
	$(\frac{1}{2}, 2)$		$(2, \frac{1}{2})$
	$(1, 2)$		$(2, 1)$

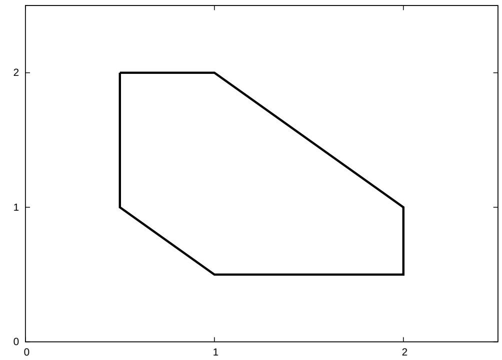
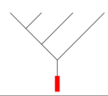
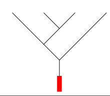
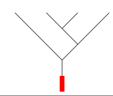
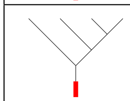
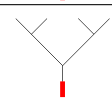
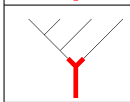
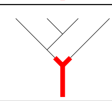
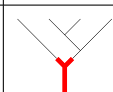
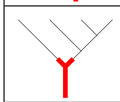
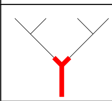
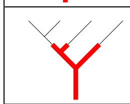
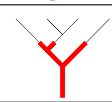
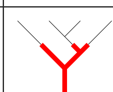
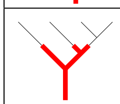
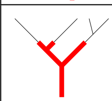
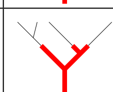
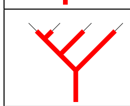
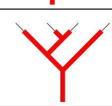
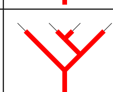
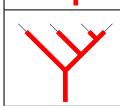
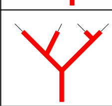
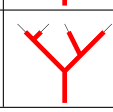


Figure 5.2: Plot of coordinates from table 5.2

Table 5.3: Coordinates for painted trees with 3 nodes

Tree	Coordinates	Tree	Coordinates	Tree	Coordinates
	$(\frac{1}{2}, \frac{2}{2}, \frac{3}{2})$		$(\frac{2}{2}, \frac{1}{2}, \frac{3}{2})$		$(\frac{3}{2}, \frac{1}{2}, \frac{2}{2})$
	$(\frac{3}{2}, \frac{2}{2}, \frac{1}{2})$		$(\frac{1}{2}, \frac{4}{2}, \frac{1}{2})$		
	$(\frac{1}{2}, \frac{2}{2}, 3)$		$(\frac{2}{2}, \frac{1}{2}, 3)$		$(3, \frac{1}{2}, \frac{2}{2})$
	$(3, \frac{2}{2}, \frac{1}{2})$		$(\frac{1}{2}, 3, \frac{1}{2})$		
	$(\frac{1}{2}, 2, 3)$		$(2, \frac{1}{2}, 3)$		$(3, \frac{1}{2}, 2)$
	$(3, 2, \frac{1}{2})$		$(2, 3, \frac{1}{2})$		$(\frac{1}{2}, 3, 2)$
	$(1, 2, 3)$		$(2, 1, 3)$		$(3, 1, 2)$
	$(3, 2, 1)$		$(2, 3, 1)$		$(1, 3, 2)$

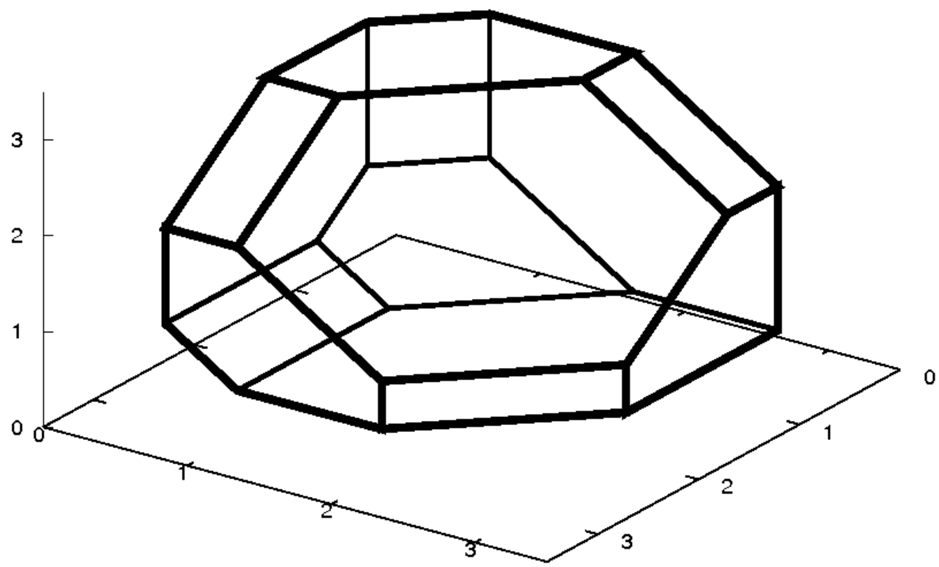


Figure 5.3: Plot of convex hull of coordinates from table 5.3

Further work in Polymake confirms that when we take the convex hull of the points of the trees in $(T \circ Y)_4$, every point is a vertex of the resulting polytope. See appendix B for Polymake output. Polymake shows that the following 3-dimensional polyhedra make up the 25 facets of the pterahedron in $\mathbf{R}^4$:

- 6 pterahedra
- 1 permutahedron
- 1 associahedron

- 2 pairahedra, as described by Tradler [14]
- 10 hexagonal prisms
- 5 pentagonal prisms

Examination shows that the facets of the pterahedron in $\mathbf{R}^4$ correspond to the painted facet trees with $n+1$ leaves. This is described in appendix C. Ziegler [15] states that two polytopes of equal dimension, P and Q, are combinatorially equivalent if there exists a bijection between vertices of P and vertices of Q such that the vertex sets of the facets of P correspond (under this bijection) to the vertex sets of the facets of Q. Since this holds true for the pterahedron in $\mathbf{R}^4$ and $\mathcal{K}F_{1,4}$, we have shown that the polytope in $\mathbf{R}^4$ derived from painted trees is combinatorially equivalent to the pterahedron. We conjecture this is true for all n and leave the proof for further study.

CHAPTER VI

THE HASSE DIAGRAM

6.1 Tamari and Bruhat Lattices

For reference, we include the Tamari and Bruhat lattices. The Tamari lattice (combinatorially equivalent to the associahedron) illustrates the ordering of binary trees, while the Bruhat lattice (combinatorially equivalent to the permutahedron) orders leveled trees.

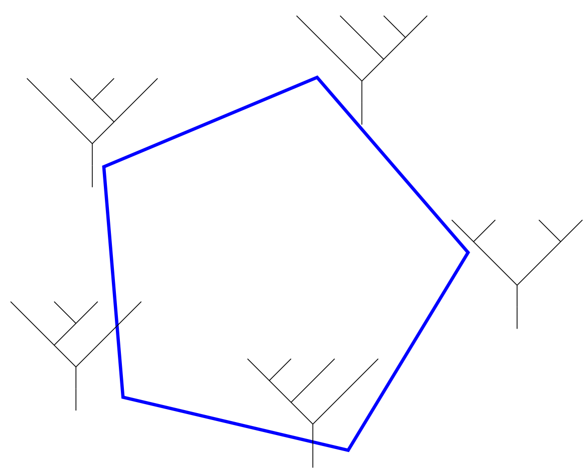


Figure 6.1: Tamari lattice of binary trees with 3 nodes

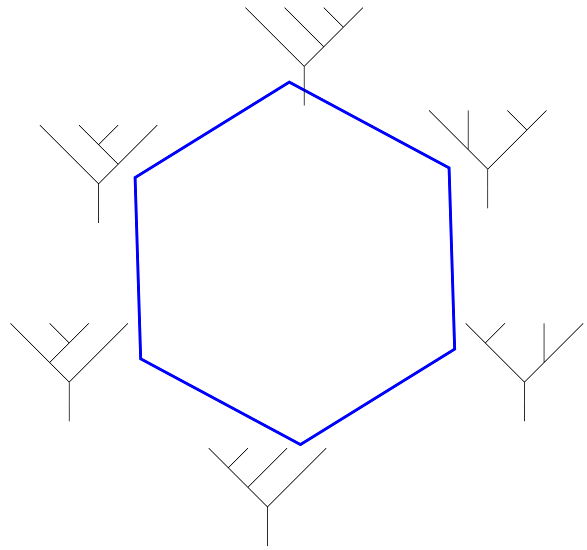


Figure 6.2: Bruhat lattice of leveled trees with 3 nodes

6.2 An Ordering of Painted Trees

Recall that painted-tree nodes are identified by the order of the spaces $(1, ..., n)$ directly above them and that painted node level numbers increase going down the tree.

Let (t, T) denote a painted tree where $t \in Y_n$ is the tree structure forgetting levels and T is a list of the painted nodes in the painted tree. Let $\hat{t}$ be just the painted portion of (t, T). We define an order on painted trees and conjecture that it is a partial order. This order is defined by $(s, S) \leq (t, T)$ if the following are all true.

- $s \leq t$ in the Tamari lattice (See figure 6.1.)
- $T \subseteq S$

- If $T = S$ then $\hat{s} \leq \hat{t}$ in the Bruhat lattice (See figure 6.2.)

Observe in figure 6.3 that with this ordering the Hasse diagram of painted trees with 3 nodes is the 1-skeleton of the 3-dimensional pterahedron.

Ronco [16] proved that there exists a poset on tubings for any graph. We conjecture that a bijection exists proving that the Hasse diagram for painted trees (for all values of n) is actually a poset and leave the proof for future study.

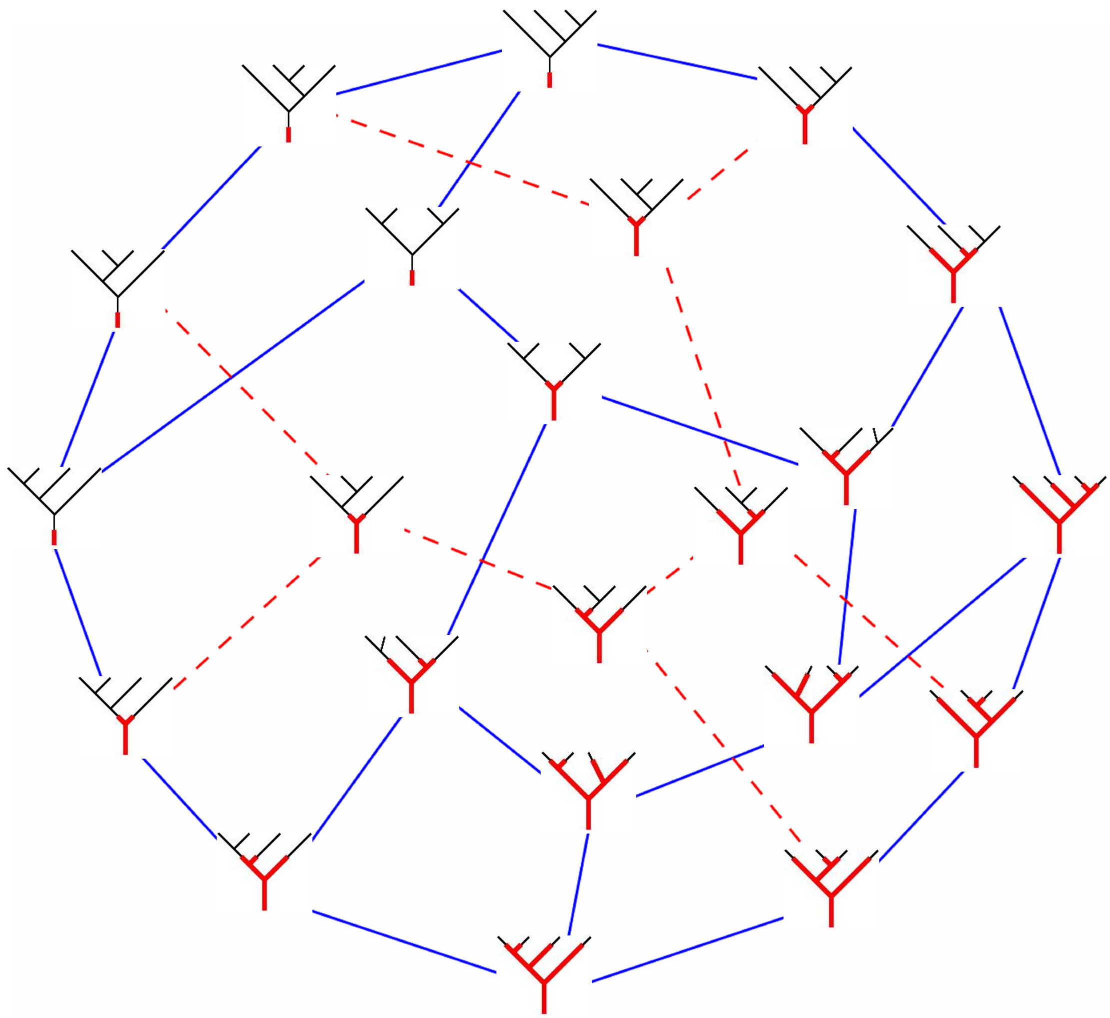


Figure 6.3: Hasse diagram of painted trees with 3 nodes

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APPENDICES

APPENDIX A

PAINTED TREE SET SIZE COMPUTATIONS

$\boxed{n=0}$

$$
\begin{aligned}
|(T \circ Y)_0| &= 0!\ [C_0] \\
&= 1(1) \\
&= 1
\end{aligned}
$$

$\boxed{n=1}$

$$
\begin{aligned}
|(T \circ Y)_1| &= 0!\ [C_1] + 1!\ [C_0 C_0] \\
&= 1(1) + 1(1) \\
&= 1 + 1 \\
&= 2
\end{aligned}
$$

$\boxed{n=2}$

$$
\begin{aligned}
|(T \circ Y)_2| &= 0!\ [C_2] + 1!\ [C_1 C_0 + C_0 C_1] + 2!\ [C_0 C_0 C_0] \\
&= 1(2) + 1(2) + 2(1) \\
&= 2 + 2 + 2 \\
&= 6
\end{aligned}
$$

$\boxed{n=3}$

$$\begin{aligned}
|(T\circ Y)_3| &= 0!\ [C_3] \\
&+ 1!\ [C_2C_0 + C_1C_1 + C_2C_0] \\
&+ 2!\ [C_1C_0C_0 + C_0C_1C_0 + C_0C_0C_1] \\
&+ 3!\ [C_0C_0C_0C_0] \\
&= 1(5) + 1(5) + 2(3) + 6(1) \\
&= 5 + 5 + 6 + 6 \\
&= 22
\end{aligned}$$

$\boxed{n=4}$

$$\begin{aligned}
|(T\circ Y)_4| &= 0!\ [C_4] \\
&+ 1!\ [C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3] \\
&+ 2!\ [C_2C_0C_0 + C_1C_1C_0 + C_1C_0C_1 + C_0C_2C_0 + C_0C_1C_1 + C_0C_0C_2] \\
&+ 3!\ [C_1C_0C_0C_0 + C_0C_1C_0C_0 + C_0C_0C_1C_0 + C_0C_0C_0C_1] \\
&+ 4!\ [C_0C_0C_0C_0C_0] \\
&= 1(14) + 1(14) + 2(9) + 6(4) + 24(1) \\
&= 14 + 14 + 18 + 24 + 24 \\
&= 94
\end{aligned}$$

$\boxed{n = 5}$

$$
\begin{aligned}
|(T \circ Y)_5| &= 0!\ [C_5] \\
&+ 1!\ [C_4C_0 + C_3C_1 + C_2C_2 + C_1C_3 + C_0C_4] \\
&+ 2!\ [C_3C_0C_0 + C_2C_1C_0 + C_2C_0C_1 + C_1C_2C_0 + C_1C_1C_1 + C_1C_0C_2 \\
&\qquad + C_0C_3C_0 + C_0C_2C_1 + C_0C_1C_2 + C_0C_0C_3] \\
&+ 3!\ [C_2C_0C_0C_0 + C_1C_1C_0C_0 + C_1C_0C_1C_0 + C_1C_0C_0C_1 + C_0C_2C_0C_0 \\
&\qquad + C_0C_1C_1C_0 + C_0C_1C_0C_1 + C_0C_0C_2C_0 + C_0C_0C_1C_1 + C_0C_0C_0C_2] \\
&+ 4!\ [C_1C_0C_0C_0C_0 + C_0C_1C_0C_0C_0 + C_0C_0C_1C_0C_0 + C_0C_0C_0C_1C_0 \\
&\qquad + C_0C_0C_0C_0C_1] \\
&+ 5!\ [C_0C_0C_0C_0C_0C_0] \\
&= 1(42) + 1(42) + 2(28) + 6(14) + 24(5) + 120(1) \\
&= 42 + 42 + 56 + 84 + 120 + 120 \\
&= 464
\end{aligned}
$$

$\boxed{n=6}$

$$
\begin{aligned}
|(T\circ Y)_6| &= 0!\ [C_6] \\
&+ 1!\ [C_5C_0 + C_4C_1 + C_3C_2 + C_2C_3 + C_1C_4 + C_0C_5] \\
&+ 2!\ [C_4C_0C_0 + C_3C_1C_0 + C_3C_0C_1 + C_2C_2C_0 + C_2C_1C_1 + C_2C_0C_2 \\
&\qquad + C_1C_3C_0 + C_1C_2C_1 + C_1C_1C_2 + C_1C_0C_3 + C_0C_4C_0 \\
&\qquad + C_0C_3C_1 + C_0C_2C_2 + C_0C_1C_3 + C_0C_0C_4] \\
&+ 3!\ [C_3C_0C_0C_0 + C_2C_1C_0C_0 + C_2C_0C_1C_0 + C_2C_0C_0C_1 + C_1C_2C_0C_0 \\
&\qquad + C_1C_1C_1C_0 + C_1C_1C_0C_1 + C_1C_0C_2C_0 + C_1C_0C_1C_1 + C_1C_0C_0C_2 \\
&\qquad + C_0C_3C_0C_0 + C_0C_2C_1C_0 + C_0C_2C_0C_1 + C_0C_1C_2C_0 + C_0C_1C_1C_1 \\
&\qquad + C_0C_1C_0C_2 + C_0C_0C_3C_0 + C_0C_0C_2C_1 + C_0C_0C_1C_2 + C_0C_0C_0C_3] \\
&+ 4!\ [C_2C_0C_0C_0C_0 + C_1C_1C_0C_0C_0 + C_1C_0C_1C_0C_0 + C_1C_0C_0C_1C_0 \\
&\qquad + C_1C_0C_0C_0C_1 + C_0C_2C_0C_0C_0 + C_0C_1C_1C_0C_0 + C_0C_1C_0C_1C_0 \\
&\qquad + C_0C_1C_0C_0C_1 + C_0C_0C_2C_0C_0 + C_0C_0C_1C_1C_0 + C_0C_0C_1C_0C_1 \\
&\qquad + C_0C_0C_0C_2C_0 + C_0C_0C_0C_1C_1 + C_0C_0C_0C_0C_2] \\
&+ 5!\ [C_1C_0C_0C_0C_0C_0 + C_0C_1C_0C_0C_0C_0 + C_0C_0C_1C_0C_0C_0 \\
&\qquad + C_0C_0C_0C_1C_0C_0 + C_0C_0C_0C_0C_1C_0 + C_0C_0C_0C_0C_0C_1] \\
&+ 6!\ [C_0C_0C_0C_0C_0C_0C_0] \\
&= 1(132) + 1(132) + 2(90) + 6(48) + 24(20) + 120(6) + 720(1) \\
&= 132 + 132 + 180 + 288 + 480 + 720 + 720 \\
&= 2652
\end{aligned}
$$

$\boxed{n=7}$

$$
\begin{aligned}
|(T\circ Y)_7| = 0!\ & [C_7] \\
& + 1!\ [C_6C_0 + C_5C_1 + C_4C_2 + C_3C_3 + C_2C_4 + C_1C_5 + C_0C_6] \\
& + 2!\ [C_5C_0C_0 + C_4C_1C_0 + C_4C_0C_1 + C_3C_2C_0 + C_3C_1C_1 + C_3C_0C_2 \\
& \qquad + C_2C_3C_0 + C_2C_2C_1 + C_2C_1C_2 + C_2C_0C_3 + C_1C_4C_0 \\
& \qquad + C_1C_3C_1 + C_1C_2C_2 + C_1C_1C_3 + C_1C_0C_4 + C_0C_5C_0 \\
& \qquad + C_0C_4C_1 + C_0C_3C_2 + C_0C_2C_3 + C_0C_1C_4 + C_0C_0C_5] \\
& + 3!\ [C_4C_0C_0C_0 + C_3C_1C_0C_0 + C_3C_0C_1C_0 + C_3C_0C_0C_1 + C_2C_2C_0C_0 \\
& \qquad + C_2C_1C_1C_0 + C_2C_1C_0C_1 + C_2C_0C_2C_0 + C_2C_0C_1C_1 + C_2C_0C_0C_2 \\
& \qquad + C_1C_3C_0C_0 + C_1C_2C_1C_0 + C_1C_2C_0C_1 + C_1C_1C_2C_0 + C_1C_1C_1C_1 \\
& \qquad + C_1C_1C_0C_2 + C_1C_0C_3C_0 + C_1C_0C_2C_1 + C_1C_0C_1C_2 + C_1C_0C_0C_3 \\
& \qquad + C_0C_4C_0C_0 + C_0C_3C_1C_0 + C_0C_3C_0C_1 + C_0C_2C_2C_0 + C_0C_2C_1C_1 \\
& \qquad + C_0C_2C_0C_2 + C_0C_1C_3C_0 + C_0C_1C_2C_1 + C_0C_1C_1C_2 + C_0C_1C_0C_3 \\
& \qquad + C_0C_0C_4C_0 + C_0C_0C_3C_1 + C_0C_0C_2C_2 + C_0C_0C_1C_3 + C_0C_0C_0C_4] \\
& + 4!\ [C_3C_0C_0C_0C_0 + C_2C_1C_0C_0C_0 + C_2C_0C_1C_0C_0 + C_2C_0C_0C_1C_0 \\
& \qquad + C_2C_0C_0C_0C_1 + C_1C_2C_0C_0C_0 + C_1C_1C_1C_0C_0 + C_1C_1C_0C_1C_0 \\
& \qquad + C_1C_1C_0C_0C_1 + C_1C_0C_2C_0C_0 + C_1C_0C_1C_1C_0 + C_1C_0C_1C_0C_1 \\
& \qquad + C_1C_0C_0C_2C_0 + C_1C_0C_0C_1C_1 + C_1C_0C_0C_0C_2 + C_0C_3C_0C_0C_0 \\
& \qquad + C_0C_2C_1C_0C_0 + C_0C_2C_0C_1C_0 + C_0C_2C_0C_0C_1 + C_0C_1C_2C_0C_0 \\
& \qquad + C_0C_1C_1C_1C_0 + C_0C_1C_1C_0C_1 + C_0C_1C_0C_2C_0 + C_0C_1C_0C_1C_1
\end{aligned}
$$

$|(T \circ Y)_7| \ldots$

$$+ C_0C_1C_0C_0C_2 + C_0C_0C_3C_0C_0 + C_0C_0C_2C_1C_0 + C_0C_0C_2C_0C_1$$

$$+ C_0C_0C_1C_2C_0 + C_0C_0C_1C_1C_1 + C_0C_0C_1C_0C_2 + C_0C_0C_0C_3C_0$$

$$+ C_0C_0C_0C_2C_1 + C_0C_0C_0C_1C_2 + C_0C_0C_0C_0C_3]$$

$$+ 5!\ [C_2C_0C_0C_0C_0C_0 + C_1C_1C_0C_0C_0C_0 + C_1C_0C_1C_0C_0C_0$$

$$+ C_1C_0C_0C_1C_0C_0 + C_1C_0C_0C_0C_1C_0 + C_1C_0C_0C_0C_0C_1$$

$$+ C_0C_2C_0C_0C_0C_0 + C_0C_1C_1C_0C_0C_0 + C_0C_1C_0C_1C_0C_0$$

$$+ C_0C_1C_0C_0C_1C_0 + C_0C_1C_0C_0C_0C_1 + C_0C_0C_2C_0C_0C_0$$

$$+ C_0C_0C_1C_1C_0C_0 + C_0C_0C_1C_0C_1C_0 + C_0C_0C_1C_0C_0C_1$$

$$+ C_0C_0C_0C_2C_0C_0 + C_0C_0C_0C_1C_1C_0 + C_0C_0C_0C_1C_0C_1$$

$$+ C_0C_0C_0C_0C_2C_0 + C_0C_0C_0C_0C_1C_1 + C_0C_0C_0C_0C_0C_2]$$

$$+ 6!\ [C_1C_0C_0C_0C_0C_0C_0 + C_0C_1C_0C_0C_0C_0C_0 + C_0C_0C_1C_0C_0C_0C_0$$

$$+ C_0C_0C_0C_1C_0C_0C_0 + C_0C_0C_0C_0C_1C_0C_0 + C_0C_0C_0C_0C_0C_1C_0$$

$$+ C_0C_0C_0C_0C_0C_0C_1]$$

$$+ 7!\ [C_0C_0C_0C_0C_0C_0C_0C_0]$$

$$= 1(429) + 1(429) + 2(297) + 6(165) + 24(75) + 120(27) + 720(7)$$

$$+ 5040(1)$$

$$= 429 + 429 + 594 + 990 + 1800 + 3240 + 5040 + 5040$$

$$= 17{,}562$$

$\boxed{n=8}$

$$
\begin{aligned}
|(T\circ Y)_8| = 0!\ &[C_8]\\
&+ 1!\ [C_7C_0 + C_6C_1 + C_5C_2 + C_4C_3 + C_3C_4 + C_2C_5 + C_1C_6 + C_0C_7]\\
&+ 2!\ [C_6C_0C_0 + C_5C_1C_0 + C_5C_0C_1 + C_4C_2C_0 + C_4C_1C_1 + C_4C_0C_2\\
&\qquad + C_3C_3C_0 + C_3C_2C_1 + C_3C_1C_2 + C_3C_0C_3 + C_2C_4C_0\\
&\qquad + C_2C_3C_1 + C_2C_2C_2 + C_2C_1C_3 + C_2C_0C_4 + C_1C_5C_0\\
&\qquad + C_1C_4C_1 + C_1C_3C_2 + C_1C_2C_3 + C_1C_1C_4 + C_1C_0C_5\\
&\qquad + C_0C_6C_0 + C_0C_5C_1 + C_0C_4C_2 + C_0C_3C_3 + C_0C_2C_4\\
&\qquad + C_0C_1C_5 + C_0C_0C_6]\\
&+ 3!\ [C_5C_0C_0C_0 + C_4C_1C_0C_0 + C_4C_0C_1C_0 + C_4C_0C_0C_1 + C_3C_2C_0C_0\\
&\qquad + C_3C_1C_1C_0 + C_3C_1C_0C_1 + C_3C_0C_2C_0 + C_3C_0C_1C_1 + C_3C_0C_0C_2\\
&\qquad + C_2C_3C_0C_0 + C_2C_2C_1C_0 + C_2C_2C_0C_1 + C_2C_1C_2C_0 + C_2C_1C_1C_1\\
&\qquad + C_2C_1C_0C_2 + C_2C_0C_3C_0 + C_2C_0C_2C_1 + C_2C_0C_1C_2 + C_2C_0C_0C_3\\
&\qquad + C_1C_4C_0C_0 + C_1C_3C_1C_0 + C_1C_3C_0C_1 + C_1C_2C_2C_0 + C_1C_2C_1C_1\\
&\qquad + C_1C_2C_0C_2 + C_1C_1C_3C_0 + C_1C_1C_2C_1 + C_1C_1C_1C_2 + C_1C_1C_0C_3\\
&\qquad + C_1C_0C_4C_0 + C_1C_0C_3C_1 + C_1C_0C_2C_2 + C_1C_0C_1C_3 + C_1C_0C_0C_4\\
&\qquad + C_0C_5C_0C_0 + C_0C_4C_1C_0 + C_0C_4C_0C_1 + C_0C_3C_2C_0 + C_0C_3C_1C_1\\
&\qquad + C_0C_3C_0C_2 + C_0C_2C_3C_0 + C_0C_2C_2C_1 + C_0C_2C_1C_2 + C_0C_2C_0C_3\\
&\qquad + C_0C_1C_4C_0 + C_0C_1C_3C_1 + C_0C_1C_2C_2 + C_0C_1C_1C_3 + C_0C_1C_0C_4
\end{aligned}
$$

$|(T \circ Y)_8|...$

$$+ C_0C_0C_5C_0 + C_0C_0C_4C_1 + C_0C_0C_3C_2 + C_0C_0C_2C_3 + C_0C_0C_1C_4$$

$$+ C_0C_0C_0C_5]$$

$$+ 4!\ [C_4C_0C_0C_0C_0 + C_3C_1C_0C_0C_0 + C_3C_0C_1C_0C_0 + C_3C_0C_0C_1C_0$$

$$+ C_3C_0C_0C_0C_1 + C_2C_2C_0C_0C_0 + C_2C_1C_1C_0C_0 + C_2C_1C_0C_1C_0$$

$$+ C_2C_1C_0C_0C_1 + C_2C_0C_2C_0C_0 + C_2C_0C_1C_1C_0 + C_2C_0C_1C_0C_1$$

$$+ C_2C_0C_0C_2C_0 + C_2C_0C_0C_1C_1 + C_2C_0C_0C_0C_2 + C_1C_3C_0C_0C_0$$

$$+ C_1C_2C_1C_0C_0 + C_1C_2C_0C_1C_0 + C_1C_2C_0C_0C_1 + C_1C_1C_2C_0C_0$$

$$+ C_1C_1C_1C_1C_0 + C_1C_1C_1C_0C_1 + C_1C_1C_0C_2C_0 + C_1C_1C_0C_1C_1$$

$$+ C_1C_1C_0C_0C_2 + C_1C_0C_3C_0C_0 + C_1C_0C_2C_1C_0 + C_1C_0C_2C_0C_1$$

$$+ C_1C_0C_1C_2C_0 + C_1C_0C_1C_1C_1 + C_1C_0C_1C_0C_2 + C_1C_0C_0C_3C_0$$

$$+ C_1C_0C_0C_2C_1 + C_1C_0C_0C_1C_2 + C_1C_0C_0C_0C_3 + C_0C_4C_0C_0C_0$$

$$+ C_0C_3C_1C_0C_0 + C_0C_3C_0C_1C_0 + C_0C_3C_0C_0C_1 + C_0C_2C_2C_0C_0$$

$$+ C_0C_2C_1C_1C_0 + C_0C_2C_1C_0C_1 + C_0C_2C_0C_2C_0 + C_0C_2C_0C_1C_1$$

$$+ C_0C_2C_0C_0C_2 + C_0C_1C_3C_0C_0 + C_0C_1C_2C_1C_0 + C_0C_1C_2C_0C_1$$

$$+ C_0C_1C_1C_2C_0 + C_0C_1C_1C_1C_1 + C_0C_1C_1C_0C_2 + C_0C_1C_0C_3C_0$$

$$+ C_0C_1C_0C_2C_1 + C_0C_1C_0C_1C_2 + C_0C_1C_0C_0C_3 + C_0C_0C_4C_0C_0$$

$$+ C_0C_0C_3C_1C_0 + C_0C_0C_3C_0C_1 + C_0C_0C_2C_2C_0 + C_0C_0C_2C_1C_1$$

$|(T \circ Y)_8| \ldots$

$$+ C_0C_0C_2C_0C_2 + C_0C_0C_1C_3C_0 + C_0C_0C_1C_2C_1 + C_0C_0C_1C_1C_2$$

$$+ C_0C_0C_1C_0C_3 + C_0C_0C_0C_4C_0 + C_0C_0C_0C_3C_1 + C_0C_0C_0C_2C_2$$

$$+ C_0C_0C_0C_1C_3 + C_0C_0C_0C_0C_4]$$

$$+ 5!\ [C_3C_0C_0C_0C_0C_0 + C_2C_1C_0C_0C_0C_0 + C_2C_0C_1C_0C_0C_0$$

$$+ C_2C_0C_0C_1C_0C_0 + C_2C_0C_0C_0C_1C_0 + C_2C_0C_0C_0C_0C_1$$

$$+ C_1C_2C_0C_0C_0C_0 + C_1C_1C_1C_0C_0C_0 + C_1C_1C_0C_1C_0C_0$$

$$+ C_1C_1C_0C_0C_1C_0 + C_1C_1C_0C_0C_0C_1 + C_1C_0C_2C_0C_0C_0$$

$$+ C_1C_0C_1C_1C_0C_0 + C_1C_0C_1C_0C_1C_0 + C_1C_0C_1C_0C_0C_1$$

$$+ C_1C_0C_0C_2C_0C_0 + C_1C_0C_0C_1C_1C_0 + C_1C_0C_0C_1C_0C_1$$

$$+ C_1C_0C_0C_0C_2C_0 + C_1C_0C_0C_0C_1C_1 + C_1C_0C_0C_0C_0C_2$$

$$+ C_0C_3C_0C_0C_0C_0 + C_0C_2C_1C_0C_0C_0 + C_0C_2C_0C_1C_0C_0$$

$$+ C_0C_2C_0C_0C_1C_0 + C_0C_2C_0C_0C_0C_1 + C_0C_1C_2C_0C_0C_0$$

$$+ C_0C_1C_1C_1C_0C_0 + C_0C_1C_1C_0C_1C_0 + C_0C_1C_1C_0C_0C_1$$

$$+ C_0C_1C_0C_2C_0C_0 + C_0C_1C_0C_1C_1C_0 + C_0C_1C_0C_1C_0C_1$$

$$+ C_0C_1C_0C_0C_2C_0 + C_0C_1C_0C_0C_1C_1 + C_0C_1C_0C_0C_0C_2$$

$$+ C_0C_0C_3C_0C_0C_0 + C_0C_0C_2C_1C_0C_0 + C_0C_0C_2C_0C_1C_0$$

$$+ C_0C_0C_2C_0C_0C_1 + C_0C_0C_1C_2C_0C_0 + C_0C_0C_1C_1C_1C_0$$

$|(T \circ Y)_8| \ldots$

$$+ C_0C_0C_1C_1C_0C_1 + C_0C_0C_1C_0C_2C_0 + C_0C_0C_1C_0C_1C_1$$

$$+ C_0C_0C_1C_0C_0C_2 + C_0C_0C_0C_3C_0C_0 + C_0C_0C_0C_2C_1C_0$$

$$+ C_0C_0C_0C_2C_0C_1 + C_0C_0C_0C_1C_2C_0 + C_0C_0C_0C_1C_1C_1$$

$$+ C_0C_0C_0C_1C_0C_2 + C_0C_0C_0C_0C_3C_0 + C_0C_0C_0C_0C_2C_1$$

$$+ C_0C_0C_0C_0C_1C_2 + C_0C_0C_0C_0C_0C_3]$$

$$+ 6!\ [C_2C_0C_0C_0C_0C_0C_0 + C_1C_1C_0C_0C_0C_0C_0 + C_1C_0C_1C_0C_0C_0C_0$$

$$+ C_1C_0C_0C_1C_0C_0C_0 + C_1C_0C_0C_0C_1C_0C_0 + C_1C_0C_0C_0C_0C_1C_0$$

$$+ C_1C_0C_0C_0C_0C_0C_1 + C_0C_2C_0C_0C_0C_0C_0 + C_0C_1C_1C_0C_0C_0C_0$$

$$+ C_0C_1C_0C_1C_0C_0C_0 + C_0C_1C_0C_0C_1C_0C_0 + C_0C_1C_0C_0C_0C_1C_0$$

$$+ C_0C_1C_0C_0C_0C_0C_1 + C_0C_0C_2C_0C_0C_0C_0 + C_0C_0C_1C_1C_0C_0C_0$$

$$+ C_0C_0C_1C_0C_1C_0C_0 + C_0C_0C_1C_0C_0C_1C_0 + C_0C_0C_1C_0C_0C_0C_1$$

$$+ C_0C_0C_0C_2C_0C_0C_0 + C_0C_0C_0C_1C_1C_0C_0 + C_0C_0C_0C_1C_0C_1C_0$$

$$+ C_0C_0C_0C_1C_0C_0C_1 + C_0C_0C_0C_0C_2C_0C_0 + C_0C_0C_0C_0C_1C_1C_0$$

$$+ C_0C_0C_0C_0C_1C_0C_1 + C_0C_0C_0C_0C_0C_2C_0 + C_0C_0C_0C_0C_0C_1C_1$$

$$+ C_0C_0C_0C_0C_0C_0C_2]$$

$$+ 7!\ [C_1C_0C_0C_0C_0C_0C_0C_0 + C_0C_1C_0C_0C_0C_0C_0C_0 + C_0C_0C_1C_0C_0C_0C_0C_0$$

$$+ C_0C_0C_0C_1C_0C_0C_0C_0 + C_0C_0C_0C_0C_1C_0C_0C_0$$

$|(T \circ Y)_8| \ldots$

$$+ C_0C_0C_0C_0C_0C_1C_0C_0 + C_0C_0C_0C_0C_0C_0C_1C_0$$

$$+ C_0C_0C_0C_0C_0C_0C_0C_1]$$

$$+ 8!\ [C_0C_0C_0C_0C_0C_0C_0C_0C_0]$$

$$= 1(1430) + 1(1430) + 2(1001) + 6(572) + 24(275) + 120(110) + 720(35)$$

$$+ 5040(8) + 40{,}320(1)$$

$$= 1430 + 1430 + 2002 + 3432 + 6600 + 13{,}200 + 25{,}200 + 40{,}320$$

$$+ 40{,}320$$

$$= 133{,}934$$

$\boxed{n = 9}$

$$
\begin{aligned}
|(T \circ Y)_9| = 0!\ [C_9] & \\
& + 1!\ [C_8C_0 + C_7C_1 + C_6C_2 + C_5C_3 + C_4C_4 + C_3C_5 + C_2C_6 + C_1C_7 \\
& \qquad + C_0C_8] \\
& + 2!\ [C_7C_0C_0 + C_6C_1C_0 + C_6C_0C_1 + C_5C_2C_0 + C_5C_1C_1 + C_5C_0C_2 \\
& \qquad + C_4C_3C_0 + C_4C_2C_1 + C_4C_1C_2 + C_4C_0C_3 + C_3C_4C_0 \\
& \qquad + C_3C_3C_1 + C_3C_2C_2 + C_3C_1C_3 + C_3C_0C_4 + C_2C_5C_0 \\
& \qquad + C_2C_4C_1 + C_2C_3C_2 + C_2C_2C_3 + C_2C_1C_4 + C_2C_0C_5 \\
& \qquad + C_1C_6C_0 + C_1C_5C_1 + C_1C_4C_2 + C_1C_3C_3 + C_1C_2C_4 \\
& \qquad + C_1C_1C_5 + C_1C_0C_6 + C_0C_7C_0 + C_0C_6C_1 + C_0C_5C_2 \\
& \qquad + C_0C_4C_3 + C_0C_3C_4 + C_0C_2C_5 + C_0C_1C_6 + C_0C_0C_7] \\
& + 3!\ [C_6C_0C_0C_0 + C_5C_1C_0C_0 + C_5C_0C_1C_0 + C_5C_0C_0C_1 + C_4C_2C_0C_0 \\
& \qquad + C_4C_1C_1C_0 + C_4C_1C_0C_1 + C_4C_0C_2C_0 + C_4C_0C_1C_1 + C_4C_0C_0C_2 \\
& \qquad + C_3C_3C_0C_0 + C_3C_2C_1C_0 + C_3C_2C_0C_1 + C_3C_1C_2C_0 + C_3C_1C_1C_1 \\
& \qquad + C_3C_1C_0C_2 + C_3C_0C_3C_0 + C_3C_0C_2C_1 + C_3C_0C_1C_2 + C_3C_0C_0C_3 \\
& \qquad + C_2C_4C_0C_0 + C_2C_3C_1C_0 + C_2C_3C_0C_1 + C_2C_2C_2C_0 + C_2C_2C_1C_1 \\
& \qquad + C_2C_2C_0C_2 + C_2C_1C_3C_0 + C_2C_1C_2C_1 + C_2C_1C_1C_2 + C_2C_1C_0C_3 \\
& \qquad + C_2C_0C_4C_0 + C_2C_0C_3C_1 + C_2C_0C_2C_2 + C_2C_0C_1C_3 + C_2C_0C_0C_4 \\
& \qquad + C_1C_5C_0C_0 + C_1C_4C_1C_0 + C_1C_4C_0C_1 + C_1C_3C_2C_0 + C_1C_3C_1C_1
\end{aligned}
$$

$|(T \circ Y)_9|...$

$$+ C_1C_3C_0C_2 + C_1C_2C_3C_0 + C_1C_2C_2C_1 + C_1C_2C_1C_2 + C_1C_2C_0C_3$$

$$+ C_1C_1C_4C_0 + C_1C_1C_3C_1 + C_1C_1C_2C_2 + C_1C_1C_1C_3 + C_1C_1C_0C_4$$

$$+ C_1C_0C_5C_0 + C_1C_0C_4C_1 + C_1C_0C_3C_2 + C_1C_0C_2C_3 + C_1C_0C_1C_4$$

$$+ C_1C_0C_0C_5 + C_0C_6C_0C_0 + C_0C_5C_1C_0 + C_0C_5C_0C_1 + C_0C_4C_2C_0$$

$$+ C_0C_4C_1C_1 + C_0C_4C_0C_2 + C_0C_3C_3C_0 + C_0C_3C_2C_1 + C_0C_3C_1C_2$$

$$+ C_0C_3C_0C_3 + C_0C_2C_4C_0 + C_0C_2C_3C_1 + C_0C_2C_2C_2 + C_0C_2C_1C_3$$

$$+ C_0C_2C_0C_4 + C_0C_1C_5C_0 + C_0C_1C_4C_1 + C_0C_1C_3C_2 + C_0C_1C_2C_3$$

$$+ C_0C_1C_1C_4 + C_0C_1C_0C_5 + C_0C_0C_6C_0 + C_0C_0C_5C_1 + C_0C_0C_4C_2$$

$$+ C_0C_0C_3C_3 + C_0C_0C_2C_4 + C_0C_0C_1C_5 + C_0C_0C_0C_6]$$

$$+ 4!\ [C_5C_0C_0C_0C_0 + C_4C_1C_0C_0C_0 + C_4C_0C_1C_0C_0 + C_4C_0C_0C_1C_0$$

$$+ C_4C_0C_0C_0C_1 + C_3C_2C_0C_0C_0 + C_3C_1C_1C_0C_0 + C_3C_1C_0C_1C_0$$

$$+ C_3C_1C_0C_0C_1 + C_3C_0C_2C_0C_0 + C_3C_0C_1C_1C_0 + C_3C_0C_1C_0C_1$$

$$+ C_3C_0C_0C_2C_0 + C_3C_0C_0C_1C_1 + C_3C_0C_0C_0C_2 + C_2C_3C_0C_0C_0$$

$$+ C_2C_2C_1C_0C_0 + C_2C_2C_0C_1C_0 + C_2C_2C_0C_0C_1 + C_2C_1C_2C_0C_0$$

$$+ C_2C_1C_1C_1C_0 + C_2C_1C_1C_0C_1 + C_2C_1C_0C_2C_0 + C_2C_1C_0C_1C_1$$

$$+ C_2C_1C_0C_0C_2 + C_2C_0C_3C_0C_0 + C_2C_0C_2C_1C_0 + C_2C_0C_2C_0C_1$$

$$+ C_2C_0C_1C_2C_0 + C_2C_0C_1C_1C_1 + C_2C_0C_1C_0C_2 + C_2C_0C_0C_3C_0$$

$|(T \circ Y)_9|...$

$$+ C_2C_0C_0C_2C_1 + C_2C_0C_0C_1C_2 + C_2C_0C_0C_0C_3 + C_1C_4C_0C_0C_0$$

$$+ C_1C_3C_1C_0C_0 + C_1C_3C_0C_1C_0 + C_1C_3C_0C_0C_1 + C_1C_2C_2C_0C_0$$

$$+ C_1C_2C_1C_1C_0 + C_1C_2C_1C_0C_1 + C_1C_2C_0C_2C_0 + C_1C_2C_0C_1C_1$$

$$+ C_1C_2C_0C_0C_2 + C_1C_1C_3C_0C_0 + C_1C_1C_2C_1C_0 + C_1C_1C_2C_0C_1$$

$$+ C_1C_1C_1C_2C_0 + C_1C_1C_1C_1C_1 + C_1C_1C_1C_0C_2 + C_1C_1C_0C_3C_0$$

$$+ C_1C_1C_0C_2C_1 + C_1C_1C_0C_1C_2 + C_1C_1C_0C_0C_3 + C_1C_0C_4C_0C_0$$

$$+ C_1C_0C_3C_1C_0 + C_1C_0C_3C_0C_1 + C_1C_0C_2C_2C_0 + C_1C_0C_2C_1C_1$$

$$+ C_1C_0C_2C_0C_2 + C_1C_0C_1C_3C_0 + C_1C_0C_1C_2C_1 + C_1C_0C_1C_1C_2$$

$$+ C_1C_0C_1C_0C_3 + C_1C_0C_0C_4C_0 + C_1C_0C_0C_3C_1 + C_1C_0C_0C_2C_2$$

$$+ C_1C_0C_0C_1C_3 + C_1C_0C_0C_0C_4 + C_0C_5C_0C_0C_0 + C_0C_4C_1C_0C_0$$

$$+ C_0C_4C_0C_1C_0 + C_0C_4C_0C_0C_1 + C_0C_3C_2C_0C_0 + C_0C_3C_1C_1C_0$$

$$+ C_0C_3C_1C_0C_1 + C_0C_3C_0C_2C_0 + C_0C_3C_0C_1C_1 + C_0C_3C_0C_0C_2$$

$$+ C_0C_2C_3C_0C_0 + C_0C_2C_2C_1C_0 + C_0C_2C_2C_0C_1 + C_0C_2C_1C_2C_0$$

$$+ C_0C_2C_1C_1C_1 + C_0C_2C_1C_0C_2 + C_0C_2C_0C_3C_0 + C_0C_2C_0C_2C_1$$

$$+ C_0C_2C_0C_1C_2 + C_0C_2C_0C_0C_3 + C_0C_1C_4C_0C_0 + C_0C_1C_3C_1C_0$$

$$+ C_0C_1C_3C_0C_1 + C_0C_1C_2C_2C_0 + C_0C_1C_2C_1C_1 + C_0C_1C_2C_0C_2$$

$$+ C_0C_1C_1C_3C_0 + C_0C_1C_1C_2C_1 + C_0C_1C_1C_1C_2 + C_0C_1C_1C_0C_3$$

$|(T \circ Y)_9| ...$

$$+ C_0C_1C_0C_4C_0 + C_0C_1C_0C_3C_1 + C_0C_1C_0C_2C_2 + C_0C_1C_0C_1C_3$$

$$+ C_0C_1C_0C_0C_4 + C_0C_0C_5C_0C_0 + C_0C_0C_4C_1C_0 + C_0C_0C_4C_0C_1$$

$$+ C_0C_0C_3C_2C_0 + C_0C_0C_3C_1C_1 + C_0C_0C_3C_0C_2 + C_0C_0C_2C_3C_0$$

$$+ C_0C_0C_2C_2C_1 + C_0C_0C_2C_1C_2 + C_0C_0C_2C_0C_3 + C_0C_0C_1C_4C_0$$

$$+ C_0C_0C_1C_3C_1 + C_0C_0C_1C_2C_2 + C_0C_0C_1C_1C_3 + C_0C_0C_1C_0C_4$$

$$+ C_0C_0C_0C_5C_0 + C_0C_0C_0C_4C_1 + C_0C_0C_0C_3C_2 + C_0C_0C_0C_2C_3$$

$$+ C_0C_0C_0C_1C_4 + C_0C_0C_0C_0C_5]$$

$$+ 5!\ [C_4C_0C_0C_0C_0C_0 + C_3C_1C_0C_0C_0C_0 + C_3C_0C_1C_0C_0C_0$$

$$+ C_3C_0C_0C_1C_0C_0 + C_3C_0C_0C_0C_1C_0 + C_3C_0C_0C_0C_0C_1$$

$$+ C_2C_2C_0C_0C_0C_0 + C_2C_1C_1C_0C_0C_0 + C_2C_1C_0C_1C_0C_0$$

$$+ C_2C_1C_0C_0C_1C_0 + C_2C_1C_0C_0C_0C_1 + C_2C_0C_2C_0C_0C_0$$

$$+ C_2C_0C_1C_1C_0C_0 + C_2C_0C_1C_0C_1C_0 + C_2C_0C_1C_0C_0C_1$$

$$+ C_2C_0C_0C_2C_0C_0 + C_2C_0C_0C_1C_1C_0 + C_2C_0C_0C_1C_0C_1$$

$$+ C_2C_0C_0C_0C_2C_0 + C_2C_0C_0C_0C_1C_1 + C_2C_0C_0C_0C_0C_2$$

$$+ C_1C_3C_0C_0C_0C_0 + C_1C_2C_1C_0C_0C_0 + C_1C_2C_0C_1C_0C_0$$

$$+ C_1C_2C_0C_0C_1C_0 + C_1C_2C_0C_0C_0C_1 + C_1C_1C_2C_0C_0C_0$$

$$+ C_1C_1C_1C_1C_0C_0 + C_1C_1C_1C_0C_1C_0 + C_1C_1C_1C_0C_0C_1$$

$|(T \circ Y)_9| \ldots$

$$+ C_1C_1C_0C_2C_0C_0 + C_1C_1C_0C_1C_1C_0 + C_1C_1C_0C_1C_0C_1$$

$$+ C_1C_1C_0C_0C_2C_0 + C_1C_1C_0C_0C_1C_1 + C_1C_1C_0C_0C_0C_2$$

$$+ C_1C_0C_3C_0C_0C_0 + C_1C_0C_2C_1C_0C_0 + C_1C_0C_2C_0C_1C_0$$

$$+ C_1C_0C_2C_0C_0C_1 + C_1C_0C_1C_2C_0C_0 + C_1C_0C_1C_1C_1C_0$$

$$+ C_1C_0C_1C_1C_0C_1 + C_1C_0C_1C_0C_2C_0 + C_1C_0C_1C_0C_1C_1$$

$$+ C_1C_0C_1C_0C_0C_2 + C_1C_0C_0C_3C_0C_0 + C_1C_0C_0C_2C_1C_0$$

$$+ C_1C_0C_0C_2C_0C_1 + C_1C_0C_0C_1C_2C_0 + C_1C_0C_0C_1C_1C_1$$

$$+ C_1C_0C_0C_1C_0C_2 + C_1C_0C_0C_0C_3C_0 + C_1C_0C_0C_0C_2C_1$$

$$+ C_1C_0C_0C_0C_1C_2 + C_1C_0C_0C_0C_0C_3 + C_0C_4C_0C_0C_0C_0$$

$$+ C_0C_3C_1C_0C_0C_0 + C_0C_3C_0C_1C_0C_0 + C_0C_3C_0C_0C_1C_0$$

$$+ C_0C_3C_0C_0C_0C_1 + C_0C_2C_2C_0C_0C_0 + C_0C_2C_1C_1C_0C_0$$

$$+ C_0C_2C_1C_0C_1C_0 + C_0C_2C_1C_0C_0C_1 + C_0C_2C_0C_2C_0C_0$$

$$+ C_0C_2C_0C_1C_1C_0 + C_0C_2C_0C_1C_0C_1 + C_0C_2C_0C_0C_2C_0$$

$$+ C_0C_2C_0C_0C_1C_1 + C_0C_2C_0C_0C_0C_2 + C_0C_1C_3C_0C_0C_0$$

$$+ C_0C_1C_2C_1C_0C_0 + C_0C_1C_2C_0C_1C_0 + C_0C_1C_2C_0C_0C_1$$

$$+ C_0C_1C_1C_2C_0C_0 + C_0C_1C_1C_1C_1C_0 + C_0C_1C_1C_1C_0C_1$$

$$+ C_0C_1C_1C_0C_2C_0 + C_0C_1C_1C_0C_1C_1 + C_0C_1C_1C_0C_0C_2$$

$|(T \circ Y)_9| \ldots$

$$+ C_0C_1C_0C_3C_0C_0 + C_0C_1C_0C_2C_1C_0 + C_0C_1C_0C_2C_0C_1$$

$$+ C_0C_1C_0C_1C_2C_0 + C_0C_1C_0C_1C_1C_1 + C_0C_1C_0C_1C_0C_2$$

$$+ C_0C_1C_0C_0C_3C_0 + C_0C_1C_0C_0C_2C_1 + C_0C_1C_0C_0C_1C_2$$

$$+ C_0C_1C_0C_0C_0C_3 + C_0C_0C_4C_0C_0C_0 + C_0C_0C_3C_1C_0C_0$$

$$+ C_0C_0C_3C_0C_1C_0 + C_0C_0C_3C_0C_0C_1 + C_0C_0C_2C_2C_0C_0$$

$$+ C_0C_0C_2C_1C_1C_0 + C_0C_0C_2C_1C_0C_1 + C_0C_0C_2C_0C_2C_0$$

$$+ C_0C_0C_2C_0C_1C_1 + C_0C_0C_2C_0C_0C_2 + C_0C_0C_1C_3C_0C_0$$

$$+ C_0C_0C_1C_2C_1C_0 + C_0C_0C_1C_2C_0C_1 + C_0C_0C_1C_1C_2C_0$$

$$+ C_0C_0C_1C_1C_1C_1 + C_0C_0C_1C_1C_0C_2 + C_0C_0C_1C_0C_3C_0$$

$$+ C_0C_0C_1C_0C_2C_1 + C_0C_0C_1C_0C_1C_2 + C_0C_0C_1C_0C_0C_3$$

$$+ C_0C_0C_0C_4C_0C_0 + C_0C_0C_0C_3C_1C_0 + C_0C_0C_0C_3C_0C_1$$

$$+ C_0C_0C_0C_2C_2C_0 + C_0C_0C_0C_2C_1C_1 + C_0C_0C_0C_2C_0C_2$$

$$+ C_0C_0C_0C_1C_3C_0 + C_0C_0C_0C_1C_2C_1 + C_0C_0C_0C_1C_1C_2$$

$$+ C_0C_0C_0C_1C_0C_3 + C_0C_0C_0C_0C_4C_0 + C_0C_0C_0C_0C_3C_1$$

$$+ C_0C_0C_0C_0C_2C_2 + C_0C_0C_0C_0C_1C_3 + C_0C_0C_0C_0C_0C_4]$$

$$+ 6!\,[C_3C_0C_0C_0C_0C_0C_0 + C_2C_1C_0C_0C_0C_0C_0 + C_2C_0C_1C_0C_0C_0C_0$$

$$+ C_2C_0C_0C_1C_0C_0C_0 + C_2C_0C_0C_0C_1C_0C_0 + C_2C_0C_0C_0C_0C_1C_0$$

$|(T \circ Y)_9|...$

$$+ C_2C_0C_0C_0C_0C_0C_1 + C_1C_2C_0C_0C_0C_0C_0 + C_1C_1C_1C_0C_0C_0C_0$$

$$+ C_1C_1C_0C_1C_0C_0C_0 + C_1C_1C_0C_0C_1C_0C_0 + C_1C_1C_0C_0C_0C_1C_0$$

$$+ C_1C_1C_0C_0C_0C_0C_1 + C_1C_0C_2C_0C_0C_0C_0 + C_1C_0C_1C_1C_0C_0C_0$$

$$+ C_1C_0C_1C_0C_1C_0C_0 + C_1C_0C_1C_0C_0C_1C_0 + C_1C_0C_1C_0C_0C_0C_1$$

$$+ C_1C_0C_0C_2C_0C_0C_0 + C_1C_0C_0C_1C_1C_0C_0 + C_1C_0C_0C_1C_0C_1C_0$$

$$+ C_1C_0C_0C_1C_0C_0C_1 + C_1C_0C_0C_0C_2C_0C_0 + C_1C_0C_0C_0C_1C_1C_0$$

$$+ C_1C_0C_0C_0C_1C_0C_1 + C_1C_0C_0C_0C_0C_2C_0 + C_1C_0C_0C_0C_0C_1C_1$$

$$+ C_1C_0C_0C_0C_0C_0C_2 + C_0C_3C_0C_0C_0C_0C_0 + C_0C_2C_1C_0C_0C_0C_0$$

$$+ C_0C_2C_0C_1C_0C_0C_0 + C_0C_2C_0C_0C_1C_0C_0 + C_0C_2C_0C_0C_0C_1C_0$$

$$+ C_0C_2C_0C_0C_0C_0C_1 + C_0C_1C_2C_0C_0C_0C_0 + C_0C_1C_1C_1C_0C_0C_0$$

$$+ C_0C_1C_1C_0C_1C_0C_0 + C_0C_1C_1C_0C_0C_1C_0 + C_0C_1C_1C_0C_0C_0C_1$$

$$+ C_0C_1C_0C_2C_0C_0C_0 + C_0C_1C_0C_1C_1C_0C_0 + C_0C_1C_0C_1C_0C_1C_0$$

$$+ C_0C_1C_0C_1C_0C_0C_1 + C_0C_1C_0C_0C_2C_0C_0 + C_0C_1C_0C_0C_1C_1C_0$$

$$+ C_0C_1C_0C_0C_1C_0C_1 + C_0C_1C_0C_0C_0C_2C_0 + C_0C_1C_0C_0C_0C_1C_1$$

$$+ C_0C_1C_0C_0C_0C_0C_2 + C_0C_0C_3C_0C_0C_0C_0 + C_0C_0C_2C_1C_0C_0C_0$$

$$+ C_0C_0C_2C_0C_1C_0C_0 + C_0C_0C_2C_0C_0C_1C_0 + C_0C_0C_2C_0C_0C_0C_1$$

$$+ C_0C_0C_1C_2C_0C_0C_0 + C_0C_0C_1C_1C_1C_0C_0 + C_0C_0C_1C_1C_0C_1C_0$$

$|(T \circ Y)_9| \ldots$

$$+ C_0C_0C_1C_1C_0C_0C_1 + C_0C_0C_1C_0C_2C_0C_0 + C_0C_0C_1C_0C_1C_1C_0$$

$$+ C_0C_0C_1C_0C_1C_0C_1 + C_0C_0C_1C_0C_0C_2C_0 + C_0C_0C_1C_0C_0C_1C_1$$

$$+ C_0C_0C_1C_0C_0C_0C_2 + C_0C_0C_0C_3C_0C_0C_0 + C_0C_0C_0C_2C_1C_0C_0$$

$$+ C_0C_0C_0C_2C_0C_1C_0 + C_0C_0C_0C_2C_0C_0C_1 + C_0C_0C_0C_1C_2C_0C_0$$

$$+ C_0C_0C_0C_1C_1C_1C_0 + C_0C_0C_0C_1C_1C_0C_1 + C_0C_0C_0C_1C_0C_2C_0$$

$$+ C_0C_0C_0C_1C_0C_1C_1 + C_0C_0C_0C_1C_0C_0C_2 + C_0C_0C_0C_0C_3C_0C_0$$

$$+ C_0C_0C_0C_0C_2C_1C_0 + C_0C_0C_0C_0C_2C_0C_1 + C_0C_0C_0C_0C_1C_2C_0$$

$$+ C_0C_0C_0C_0C_1C_1C_1 + C_0C_0C_0C_0C_1C_0C_2 + C_0C_0C_0C_0C_0C_3C_0$$

$$+ C_0C_0C_0C_0C_0C_2C_1 + C_0C_0C_0C_0C_0C_1C_2 + C_0C_0C_0C_0C_0C_0C_3]$$

$$+ 7!\ [C_2C_0C_0C_0C_0C_0C_0C_0 + C_10C_1C_0C_0C_0C_0C_0C_0 + C_1C_0C_1C_0C_0C_0C_0C_0$$

$$+ C_1C_0C_0C_1C_0C_0C_0C_0 + C_1C_0C_0C_0C_1C_0C_0C_0$$

$$+ C_1C_0C_0C_0C_0C_1C_0C_0 + C_1C_0C_0C_0C_0C_0C_1C_0$$

$$+ C_1C_0C_0C_0C_0C_0C_0C_1 + C_0C_2C_0C_0C_0C_0C_0C_0$$

$$+ C_0C_1C_1C_0C_0C_0C_0C_0 + C_0C_1C_0C_1C_0C_0C_0C_0$$

$$+ C_0C_1C_0C_0C_1C_0C_0C_0 + C_0C_1C_0C_0C_0C_1C_0C_0$$

$$+ C_0C_1C_0C_0C_0C_0C_1C_0 + C_0C_1C_0C_0C_0C_0C_0C_1$$

$$+ C_0C_0C_2C_0C_0C_0C_0C_0 + C_0C_0C_1C_1C_0C_0C_0C_0$$

$|(T \circ Y)_9|$...

$$+ C_0C_0C_1C_0C_1C_0C_0C_0 + C_0C_0C_1C_0C_0C_1C_0C_0$$

$$+ C_0C_0C_1C_0C_0C_0C_1C_0 + C_0C_0C_1C_0C_0C_0C_0C_1$$

$$+ C_0C_0C_0C_2C_0C_0C_0C_0 + C_0C_0C_0C_1C_1C_0C_0C_0$$

$$+ C_0C_0C_0C_1C_0C_1C_0C_0 + C_0C_0C_0C_1C_0C_0C_1C_0$$

$$+ C_0C_0C_0C_1C_0C_0C_0C_1 + C_0C_0C_0C_0C_2C_0C_0C_0$$

$$+ C_0C_0C_0C_0C_1C_1C_0C_0 + C_0C_0C_0C_0C_1C_0C_1C_0$$

$$+ C_0C_0C_0C_0C_1C_0C_0C_1 + C_0C_0C_0C_0C_0C_2C_0C_0$$

$$+ C_0C_0C_0C_0C_0C_1C_1C_0 + C_0C_0C_0C_0C_0C_1C_0C_1$$

$$+ C_0C_0C_0C_0C_0C_0C_2C_0 + C_0C_0C_0C_0C_0C_0C_1C_1$$

$$+ C_0C_0C_0C_0C_0C_0C_0C_2]$$

$$+ 8!\ [C_1C_0C_0C_0C_0C_0C_0C_0C_0 + C_0C_1C_0C_0C_0C_0C_0C_0C_0$$

$$+ C_0C_0C_1C_0C_0C_0C_0C_0C_0 + C_0C_0C_0C_1C_0C_0C_0C_0C_0$$

$$+ C_0C_0C_0C_0C_1C_0C_0C_0C_0 + C_0C_0C_0C_0C_0C_1C_0C_0C_0$$

$$+ C_0C_0C_0C_0C_0C_0C_1C_0C_0 + C_0C_0C_0C_0C_0C_0C_0C_1C_0$$

$$+ C_0C_0C_0C_0C_0C_0C_0C_0C_1]$$

$$+ 9!\ [C_0C_0C_0C_0C_0C_0C_0C_0C_0C_0]$$

$|(T \circ Y)_9|$...

$$\begin{aligned}
&= 1(4862) + 1(4862) + 2(3432) + 6(2002) + 24(1001) + 120(429) \\
&\qquad + 720(154) + 5040(44) + 40{,}320(9) + 362{,}880(1) \\
&= 4862 + 4862 + 6864 + 12{,}012 + 24{,}024 + 51{,}480 + 110{,}880 + 221{,}760 \\
&\qquad + 362{,}880 + 362{,}880 \\
&= 1{,}162{,}504
\end{aligned}$$

APPENDIX B

POLYMAKE OUTPUT

Polymake was used to verify the convex hull of the points derived from 3- and 4-node painted trees. What follows is the output from Polymake. Be advised that Polymake prepends coordinates with a "1". Thus the point (2,2,2) in $\mathbf{R}^3$ is referred to as (1,2,2,2). To interpret the VERTICES_IN_FACETS data, be aware that the integers listed in each facet are indices into the POINTS list, beginning with 0.

n=3

```
polytope >  $points=new Matrix<Rational >([[1,1,1/2,3/2],[1,1/2,1,3/2],
[1,1/2,2,1/2],[1,3/2,1,1/2],[1,3/2,1/2,1],[1,1,2,3],[1,2,1,3],[1,3,1,2],[1,3,2,1],[1,2,3,1],
[1,1,3,2],[1,1/2,1,3],[1,1/2,2,3],[1,1/2,3,2],[1,1/2,3,1/2],[1,2,3,1/2],[1,3,2,1/2],[1,3,1,1/2],
[1,3,1/2,1],[1,3,1/2,2],[1,2,1/2,3],[1,1,1/2,3]]);
polytope >  $p=new Polytope<Rational >(POINTS=>$points);
polytope >  print $p->F_VECTOR;
22 33 13
polytope >  print $p->POINTS;
1 1 1/2 3/2
1 1/2 1 3/2
```

1 1/2 2 1/2

1 3/2 1 1/2

1 3/2 1/2 1

1 1 2 3

1 2 1 3

1 3 1 2

1 3 2 1

1 2 3 1

1 1 3 2

1 1/2 1 3

1 1/2 2 3

1 1/2 3 2

1 1/2 3 1/2

1 2 3 1/2

1 3 2 1/2

1 3 1 1/2

1 3 1/2 1

1 3 1/2 2

1 2 1/2 3

1 1 1/2 3

polytope > print $p->VERTICES_IN_FACETS;

{2 3 14 15 16 17}

{3 4 17 18} {1 2 11 12 13 14} {0 1 11 21}

{5 10 12 13}

{0 1 2 3 4}

{5 6 11 12 20 21}

{9 10 13 14 15}

{0 4 18 19 20 21}

{6 7 19 20}

{5 6 7 8 9 10}

{8 9 15 16}

{7 8 16 17 18 19}

n=4

polytope > $points=new Matrix<Rational >([[1,1/2,1,3/2,2],[1,1,1/2,3/2,2],
[1,3/2,1/2,1,2],[1,3/2,1,1/2,2],[1,2,1,1/2,3/2],[1,2,1/2,1,3/2],[1,2,3/2,1/2,1],
[1,2,3/2,1,1/2],[1,2,1/2,2,1/2],[1,1/2,2,1/2,2],[1,1,1/2,3,1/2],[1,1/2,3,1/2,1],
[1,1/2,1,3,1/2],[1,1/2,3,1,1/2],[1,4,3/2,1,1/2],[1,4,1/2,2,1/2],[1,4,3/2,1/2,1],
[1,4,1,1/2,3/2],[1,4,1/2,1,3/2],[1,1/2,4,1/2,1],[1,1/2,4,1,1/2],[1,1/2,1,4,1/2],
[1,1,1/2,4,1/2],[1,1/2,1,3/2,4],[1,1,1/2,3/2,4],[1,1/2,2,1/2,4],[1,3/2,1/2,1,4],
[1,3/2,1,1/2,4],[1,4,3,1,1/2],[1,4,3,1/2,1],[1,4,1/2,3,1/2],[1,4,1/2,1,3],[1,4,1,1/2,3],
[1,1/2,4,3,1/2],[1,1/2,4,1/2,3],[1,1/2,1,4,3],[1,1,1/2,4,3],[1,3,4,1/2,1],[1,3,4,1,1/2],
[1,3,1/2,4,1/2],[1,3,1,1/2,4],[1,3,1/2,1,4],[1,1/2,3,4,1/2],[1,1/2,3,1/2,4],[1,1/2,1,3,4],
[1,1,1/2,3,4],[1,1/2,4,3,2],[1,4,1/2,3,2],[1,4,3,1/2,2],[1,4,3,2,1/2],[1,1/2,3,4,2],

[1,3,1/2,4,2],[1,3,4,1/2,2],[1,3,4,2,1/2],[1,1/2,3,2,4],[1,3,1/2,2,4],[1,3,2,1/2,4],

[1,3,2,4,1/2],[1,1/2,4,2,3],[1,4,1/2,2,3],[1,4,2,1/2,3],[1,4,2,3,1/2],[1,1/2,2,4,3],

[1,2,1/2,4,3],[1,2,4,1/2,3],[1,2,4,3,1/2],[1,1/2,2,3,4],[1,2,1/2,3,4],[1,2,3,1/2,4],

[1,2,3,4,1/2],[1,1,2,3,4],[1,2,1,3,4],[1,3,1,2,4],[1,3,2,1,4],[1,4,2,1,3],[1,4,1,2,3],

[1,4,3,1,2],[1,4,3,2,1],[1,4,1,3,2],[1,4,2,3,1],[1,2,3,1,4],[1,1,3,2,4],[1,3,2,4,1],

[1,3,1,4,2],[1,2,1,4,3],[1,1,4,2,3],[1,2,4,1,3],[1,3,4,1,2],[1,2,3,4,1],[1,1,3,4,2],

[1,1,2,4,3],[1,1,4,3,2],[1,2,4,3,1],[1,3,4,2,1]]);

polytope > $p=new Polytope<Rational >(POINTS=>$points);

polytope > print $p->F_VECTOR;

94 188 119 25

polytope > print $p->POINTS;

1 1/2 1 3/2 2

1 1 1/2 3/2 2

1 3/2 1/2 1 2

1 3/2 1 1/2 2

1 2 1 1/2 3/2

1 2 1/2 1 3/2

1 2 3/2 1/2 1

1 2 3/2 1 1/2

1 2 1/2 2 1/2

1 1/2 2 1/2 2

1 1 1/2 3 1/2

1 1/2 3 1/2 1

1 1/2 1 3 1/2

1 1/2 3 1 1/2

1 4 3/2 1 1/2

1 4 1/2 2 1/2

1 4 3/2 1/2 1

1 4 1 1/2 3/2

1 4 1/2 1 3/2

1 1/2 4 1/2 1

1 1/2 4 1 1/2

1 1/2 1 4 1/2

1 1 1/2 4 1/2

1 1/2 1 3/2 4

1 1 1/2 3/2 4

1 1/2 2 1/2 4

1 3/2 1/2 1 4

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1 4 3 1 1/2

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1 2 4 1 3

1 3 4 1 2

1 2 3 4 1

1 1 3 4 2

1 1 2 4 3

1 1 4 3 2

1 2 4 3 1

1 3 4 2 1

polytope > print $p->VERTICES_IN_FACETS;

{7 8 10 12 13 14 15 20 21 22 28 30 33 38 39 42 49 53 57 61 65 69}

{6 7 11 13 14 16 19 20 28 29 37 38}

{33 42 46 50 65 69 88 89 91 92}

{34 43 54 58 64 68 80 81 85 86}

{0 1 2 3 4 5 6 7 8 9 10 11 12 13}

{35 36 44 45 62 63 66 67 70 71 84 90}

{0 9 11 12 13 19 20 21 23 25 33 34 35 42 43 44 46 50 54 58 62 66}

{0 1 10 12 21 22 23 24 35 36 44 45}

{46 50 54 58 62 66 70 81 85 89 90 91}

{0 1 2 3 9 23 24 25 26 27}

{23 24 25 26 27 40 41 43 44 45 54 55 56 66 67 68 70 71 72 73 80 81}

{21 22 35 36 39 42 50 51 57 62 63 69 82 83 84 88 89 90}

{19 20 33 34 37 38 46 52 53 58 64 65 85 86 87 91 92 93}

{1 2 5 8 10 15 18 22 24 26 30 31 36 39 41 45 47 51 55 59 63 67}

{2 3 4 5 17 18 26 27 31 32 40 41}

{47 51 55 59 63 67 71 72 75 78 83 84}

{4 5 6 7 8 14 15 16 17 18}

{31 32 40 41 55 56 59 60 72 73 74 75}

{30 39 47 51 57 61 78 79 82 83}

{3 4 6 9 11 16 17 19 25 27 29 32 34 37 40 43 48 52 56 60 64 68}

{48 52 56 60 64 68 73 74 76 80 86 87}

{70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93}

{49 53 57 61 65 69 77 79 82 88 92 93}

{28 29 37 38 48 49 52 53 76 77 87 93}

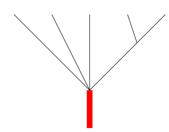
{14 15 16 17 18 28 29 30 31 32 47 48 49 59 60 61 74 75 76 77 78 79}

APPENDIX C

PTERAHEDRON FACETS IN $\mathbf{R}^4$

The following pages describe the facets of the pterahedron in $\mathbf{R}^4$ using data generated in Polymake. The Polymake output has been formatted for readability. We describe the polygonal faces for each facet. As in appendix B, the integers listed for each polygon (enclosed in curly braces) are indices into the VERTICES list for that facet. Facets are represented graphically with the integers (from the VERTICES list) identifying each vertex. Additionally, the facet trees are given.

FACET 0: Pterahedron

FACET TREE: 

VERTICES:

0: 2 3/2 1 1/2

1: 2 1/2 2 1/2

2: 1 1/2 3 1/2

3: 1/2 1 3 1/2

4: 1/2 3 1 1/2

5: 4 3/2 1 1/2

6: 4 1/2 2 1/2

7: 1/2 4 1 1/2

8: 1/2 1 4 1/2

9: 1 1/2 4 1/2

10: 4 3 1 1/2

11: 4 1/2 3 1/2

12: 1/2 4 3 1/2

13: 3 4 1 1/2

14: 3 1/2 4 1/2

15: 1/2 3 4 1/2

16: 4 3 2 1/2

17: 3 4 2 1/2

18: 3 2 4 1/2

19: 4 2 3 1/2

20: 2 4 3 1/2

21: 2 3 4 1/2

FACES:

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{0 1 2 3 4}

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{2 3 8 9}

{8 9 14 15 18 21}

{7 12 13 17 20}

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{16 17 18 19 20 21}

{10 13 16 17}

{5 6 10 11 16 19}

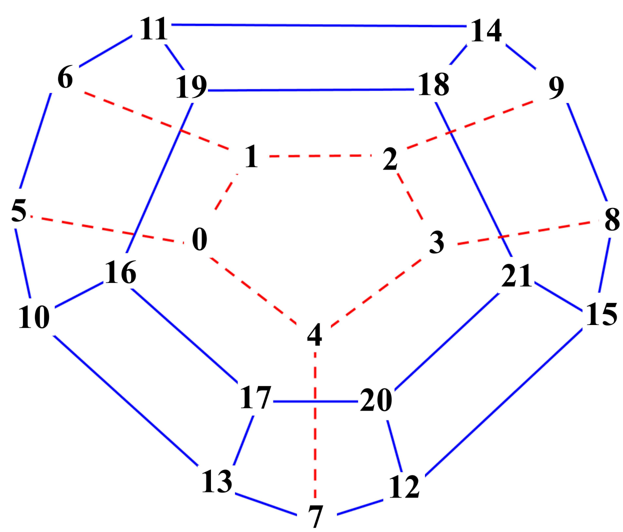
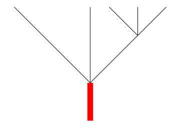


Figure C.1: Facet 0 of the pterahedron in $\mathbf{R}^4$.

FACET 1: Hexagonal Prism

FACET TREE: 

VERTICES:

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1: 2 3/2 1 1/2

2: 1/2 3 1/2 1

3: 1/2 3 1 1/2

4: 4 3/2 1 1/2

5: 4 3/2 1/2 1

6: 1/2 4 1/2 1

7: 1/2 4 1 1/2

8: 4 3 1 1/2

9: 4 3 1/2 1

10: 3 4 1/2 1

11: 3 4 1 1/2

FACES:

{1 3 4 7 8 11}

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{2 3 6 7}

{6 7 10 11}

{0 1 4 5}

{0 2 5 6 9 10}

{8 9 10 11}

{4 5 8 9}

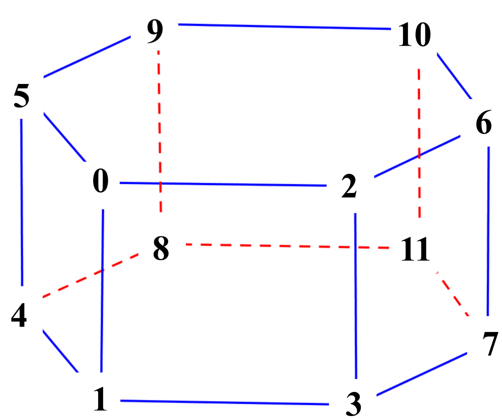
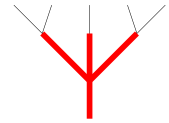


Figure C.2: Facet 1 of the pterahedron in $\mathbf{R}^4$.

FACET 2: Pentagonal Prism

FACET TREE: 

VERTICES:

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1: 1/2 3 4 1/2

2: 1/2 4 3 2

3: 1/2 3 4 2

4: 2 4 3 1/2

5: 2 3 4 1/2

6: 2 3 4 1

7: 1 3 4 2

8: 1 4 3 2

9: 2 4 3 1

FACES:

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{1 3 5 6 7}

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{6 7 8 9}

{4 5 6 9}

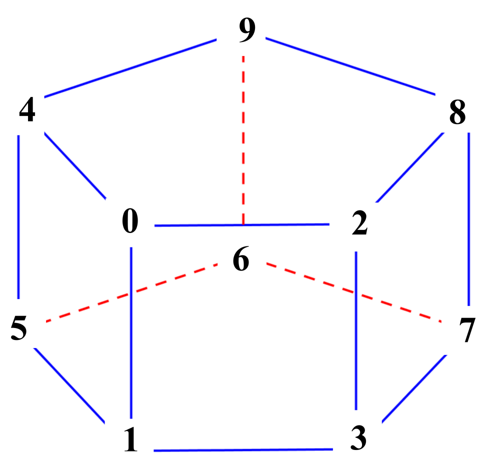
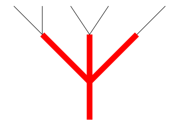


Figure C.3: Facet 2 of the pterahedron in $\mathbf{R}^4$.

FACET 3: Pentagonal Prism

FACET TREE: 

VERTICES:

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1: 1/2 3 1/2 4

2: 1/2 3 2 4

3: 1/2 4 2 3

4: 2 4 1/2 3

5: 2 3 1/2 4

6: 2 3 1 4

7: 1 3 2 4

8: 1 4 2 3

9: 2 4 1 3

FACES:

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{2 3 7 8}

{1 2 5 6 7}

{0 3 4 8 9}

{0 1 4 5}

{4 5 6 9}

{6 7 8 9}

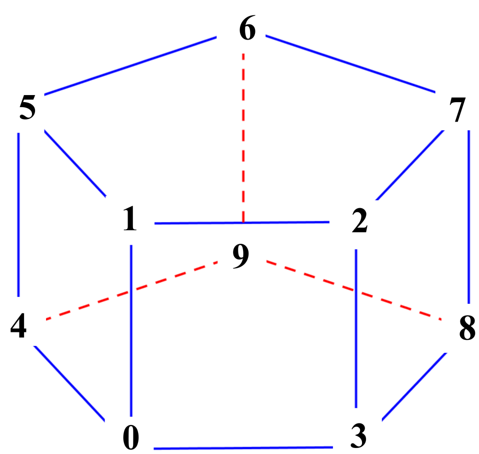
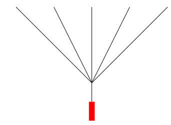


Figure C.4: Facet 3 of the pterahedron in $\mathbf{R}^4$.

FACET 4: Associahedron

FACET TREE: 

VERTICES:

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1: 1 1/2 3/2 2

2: 3/2 1/2 1 2

3: 3/2 1 1/2 2

4: 2 1 1/2 3/2

5: 2 1/2 1 3/2

6: 2 3/2 1/2 1

7: 2 3/2 1 1/2

8: 2 1/2 2 1/2

9: 1/2 2 1/2 2

10: 1 1/2 3 1/2

11: 1/2 3 1/2 1

12: 1/2 1 3 1/2

13: 1/2 3 1 1/2

FACES:

{7 8 10 12 13}

{6 7 11 13}

{0 9 11 12 13}

{0 1 10 12}

{0 1 2 3 9}

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{2 3 4 5}

{4 5 6 7 8}

{3 4 6 9 11}

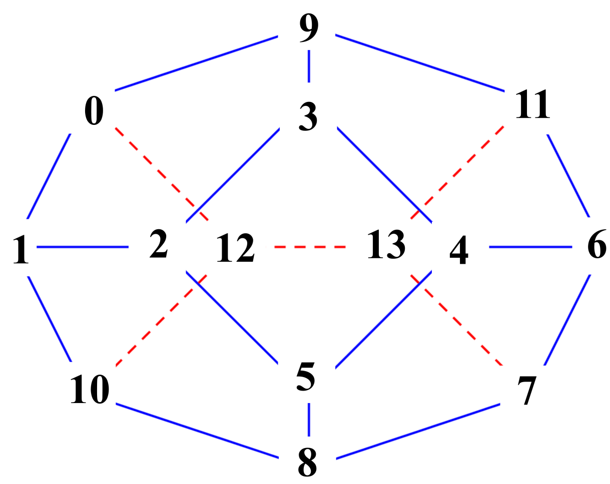
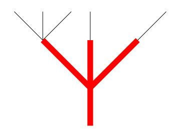


Figure C.5: Facet 4 of the pterahedron in $\mathbf{R}^4$.

FACET 5: Hexagonal Prism

FACET TREE: 

VERTICES:

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1: 1 1/2 4 3

2: 1/2 1 3 4

3: 1 1/2 3 4

4: 1/2 2 4 3

5: 2 1/2 4 3

6: 1/2 2 3 4

7: 2 1/2 3 4

8: 1 2 3 4

9: 2 1 3 4

10: 2 1 4 3

11: 1 2 4 3

FACES:

{0 2 4 6}

{0 1 2 3}

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{1 3 5 7}

{8 9 10 11}

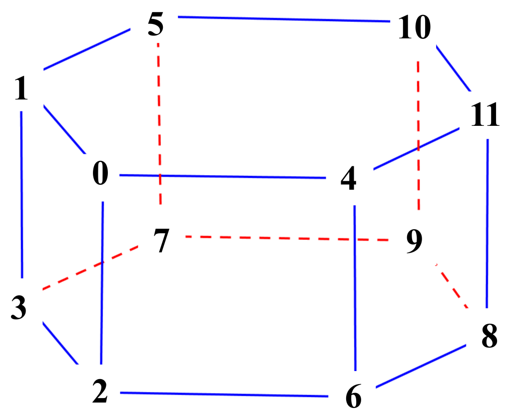
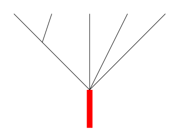


Figure C.6: Facet 5 of the pterahedron in $\mathbf{R}^4$.

FACET 6: Pterahedron

FACET TREE: 

VERTICES:

0: 1/2 1 3/2 2

1: 1/2 2 1/2 2

2: 1/2 3 1/2 1

3: 1/2 1 3 1/2

4: 1/2 3 1 1/2

5: 1/2 4 1/2 1

6: 1/2 4 1 1/2

7: 1/2 1 4 1/2

8: 1/2 1 3/2 4

9: 1/2 2 1/2 4

10: 1/2 4 3 1/2

11: 1/2 4 1/2 3

12: 1/2 1 4 3

13: 1/2 3 4 1/2

14: 1/2 3 1/2 4

15: 1/2 1 3 4

16: 1/2 4 3 2

17: 1/2 3 4 2

18: 1/2 3 2 4

19: 1/2 4 2 3

20: 1/2 2 4 3

21: 1/2 2 3 4

FACES:

{3 4 6 7 10 13}

{2 4 5 6}

{10 13 16 17}

{11 14 18 19}

{0 1 2 3 4}

{12 15 20 21}

{0 3 7 8 12 15}

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{1 2 5 9 11 14}

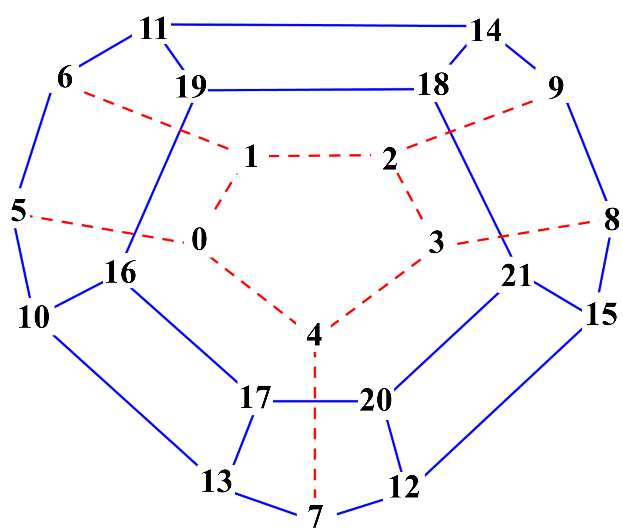
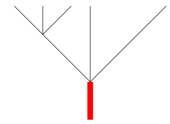


Figure C.7: Facet 6 of the pterahedron in $\mathbf{R}^4$.

FACET 7: Hexagonal Prism

FACET TREE: 

VERTICES:

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1: 1 1/2 3/2 2

2: 1 1/2 3 1/2

3: 1/2 1 3 1/2

4: 1/2 1 4 1/2

5: 1 1/2 4 1/2

6: 1/2 1 3/2 4

7: 1 1/2 3/2 4

8: 1/2 1 4 3

9: 1 1/2 4 3

10: 1/2 1 3 4

11: 1 1/2 3 4

FACES:

{2 3 4 5}

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{0 3 4 6 8 10}

{0 1 6 7}

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{4 5 8 9}

{1 2 5 7 9 11}

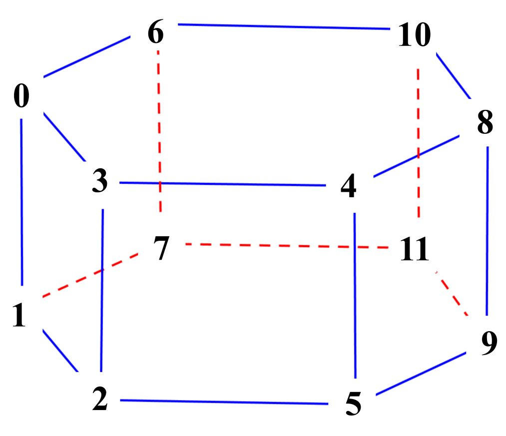
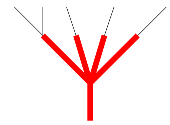


Figure C.8: Facet 7 of the pterahedron in $\mathbf{R}^4$.

FACET 8: Hexagonal Prism

FACET TREE: 

VERTICES:

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1: 1/2 3 4 2

2: 1/2 3 2 4

3: 1/2 4 2 3

4: 1/2 2 4 3

5: 1/2 2 3 4

6: 1 2 3 4

7: 1 3 2 4

8: 1 4 2 3

9: 1 3 4 2

10: 1 2 4 3

11: 1 4 3 2

FACES:

{0 1 9 11}

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{6 7 8 9 10 11}

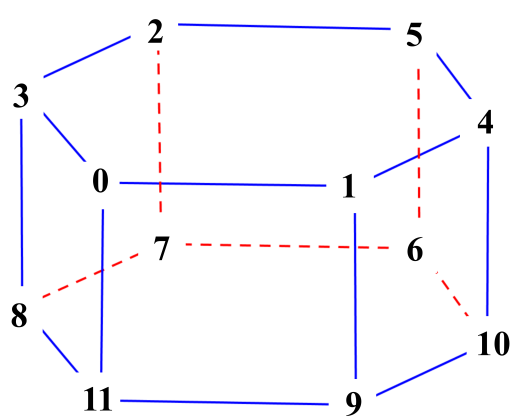
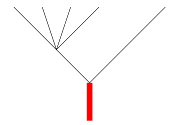


Figure C.9: Facet 8 of the pterahedron in $\mathbf{R}^4$.

FACET 9: Pentagonal Prism

FACET TREE: 

VERTICES:

0: 1/2 1 3/2 2

1: 1 1/2 3/2 2

2: 3/2 1/2 1 2

3: 3/2 1 1/2 2

4: 1/2 2 1/2 2

5: 1/2 1 3/2 4

6: 1 1/2 3/2 4

7: 1/2 2 1/2 4

8: 3/2 1/2 1 4

9: 3/2 1 1/2 4

FACES:

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{0 4 5 7}

{0 1 5 6}

{5 6 7 8 9}

{1 2 6 8}

{2 3 8 9}

{3 4 7 9}

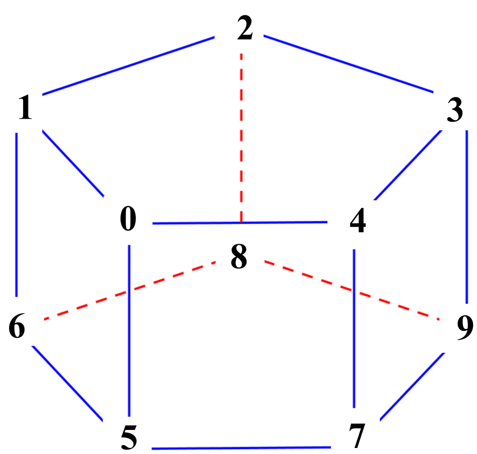
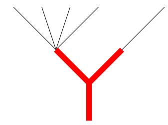


Figure C.10: Facet 9 of the pterahedron in $\mathbf{R}^4$.

FACET 10: Pterahedron

FACET TREE: 

VERTICES:

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2: 1/2 2 1/2 4

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4: 3/2 1 1/2 4

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6: 3 1/2 1 4

7: 1/2 3 1/2 4

8: 1/2 1 3 4

9: 1 1/2 3 4

10: 1/2 3 2 4

11: 3 1/2 2 4

12: 3 2 1/2 4

13: 1/2 2 3 4

14: 2 1/2 3 4

15: 2 3 1/2 4

16: 1 2 3 4

17: 2 1 3 4

18: 3 1 2 4

19: 3 2 1 4

20: 2 3 1 4

21: 1 3 2 4

FACES:

{7 10 15 20 21}

{8 9 13 14 16 17}

{0 2 7 8 10 13}

{0 1 8 9}

{10 13 16 21}

{0 1 2 3 4}

{1 3 6 9 11 14}

{3 4 5 6}

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{16 17 18 19 20 21}

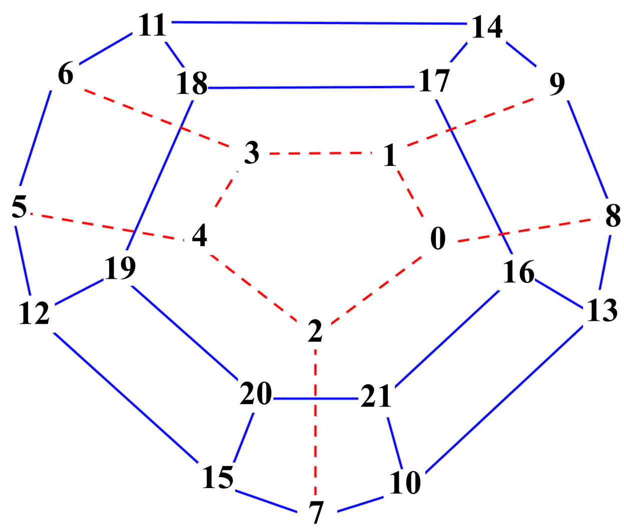
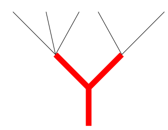


Figure C.11: Facet 10 of the pterahedron in $\mathbf{R}^4$.

FACET 11: Pairahedron

FACET TREE: 

VERTICES:

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2: 1/2 1 4 3

3: 1 1/2 4 3

4: 3 1/2 4 1/2

5: 1/2 3 4 1/2

6: 1/2 3 4 2

7: 3 1/2 4 2

8: 3 2 4 1/2

9: 1/2 2 4 3

10: 2 1/2 4 3

11: 2 3 4 1/2

12: 3 2 4 1

13: 3 1 4 2

14: 2 1 4 3

15: 2 3 4 1

16: 1 3 4 2

17: 1 2 4 3

FACES:

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{1 3 4 7 10}

{5 6 11 15 16}

{7 10 13 14}

{2 3 9 10 14 17}

{4 7 8 12 13}

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{8 11 12 15}

{6 9 16 17}

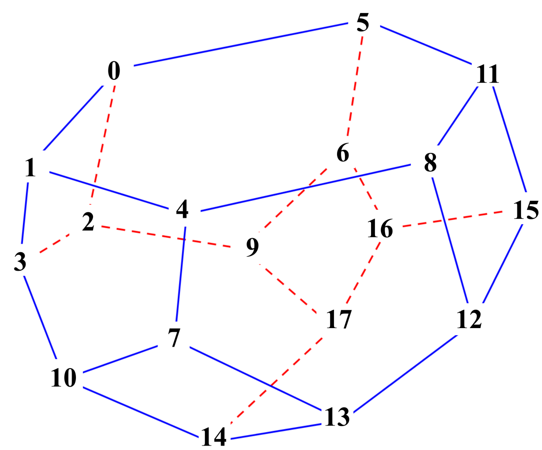
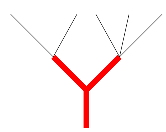


Figure C.12: Facet 11 of the pterahedron in $\mathbf{R}^4$.

FACET 12: Pairahedron

FACET TREE:



VERTICES:

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1: 1/2 4 1 1/2

2: 1/2 4 3 1/2

3: 1/2 4 1/2 3

4: 3 4 1/2 1

5: 3 4 1 1/2

6: 1/2 4 3 2

7: 3 4 1/2 2

8: 3 4 2 1/2

9: 1/2 4 2 3

10: 2 4 1/2 3

11: 2 4 3 1/2

12: 1 4 2 3

13: 2 4 1 3

14: 3 4 1 2

15: 1 4 3 2

16: 2 4 3 1

17: 3 4 2 1

FACES:

{1 2 5 8 11}

{0 1 4 5}

{2 6 11 15 16}

{3 9 10 12 13}

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{4 5 7 8 14 17}

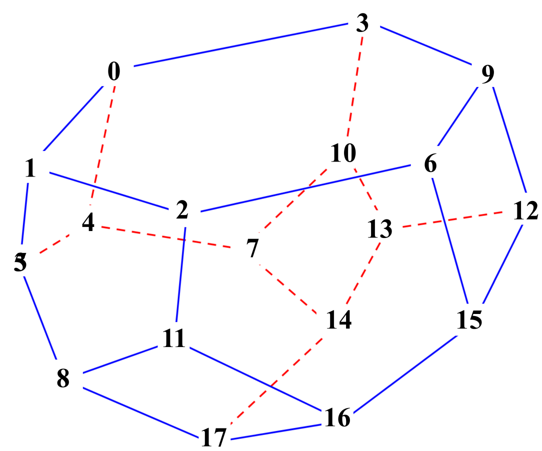
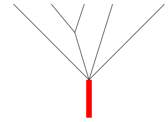


Figure C.13: Facet 12 of the pterahedron in $\mathbf{R}^4$.

FACET 13: Pterahedron

FACET TREE: 

VERTICES:

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1: 3/2 1/2 1 2

2: 2 1/2 1 3/2

3: 2 1/2 2 1/2

4: 1 1/2 3 1/2

5: 4 1/2 2 1/2

6: 4 1/2 1 3/2

7: 1 1/2 4 1/2

8: 1 1/2 3/2 4

9: 3/2 1/2 1 4

10: 4 1/2 3 1/2

11: 4 1/2 1 3

12: 1 1/2 4 3

13: 3 1/2 4 1/2

14: 3 1/2 1 4

15: 1 1/2 3 4

16: 4 1/2 3 2

17: 3 1/2 4 2

18: 3 1/2 2 4

19: 4 1/2 2 3

20: 2 1/2 4 3

21: 2 1/2 3 4

FACES:

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{0 1 2 3 4}

{12 15 20 21}

{0 4 7 8 12 15}

{0 1 8 9}

{8 9 14 15 18 21}

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{10 13 16 17}

{5 6 10 11 16 19}

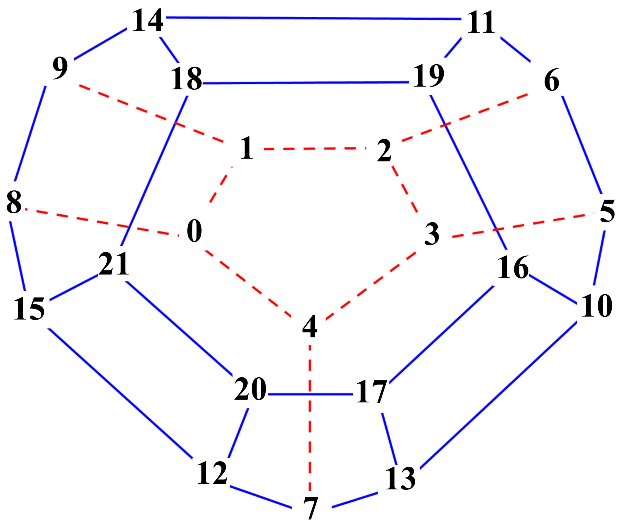
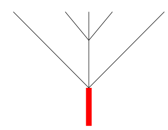


Figure C.14: Facet 13 of the pterahedron in $\mathbf{R}^4$.

FACET 14: Hexagonal Prism

FACET TREE: 

VERTICES:

0: 3/2 1/2 1 2

1: 3/2 1 1/2 2

2: 2 1 1/2 3/2

3: 2 1/2 1 3/2

4: 4 1 1/2 3/2

5: 4 1/2 1 3/2

6: 3/2 1/2 1 4

7: 3/2 1 1/2 4

8: 4 1/2 1 3

9: 4 1 1/2 3

10: 3 1 1/2 4

11: 3 1/2 1 4

FACES:

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{6 7 10 11}

{0 3 5 6 8 11}

{2 3 4 5} {4 5 8 9}

{8 9 10 11}

{1 2 4 7 9 10}

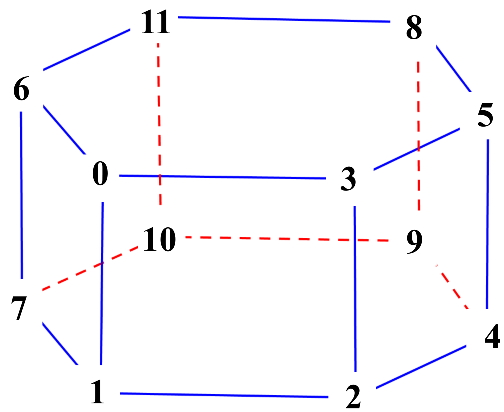
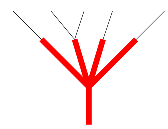


Figure C.15: Facet 14 of the pterahedron in $\mathbf{R}^4$.

FACET 15: Hexagonal Prism

FACET TREE: 

VERTICES:

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1: 3 1/2 4 2

2: 3 1/2 2 4

3: 4 1/2 2 3

4: 2 1/2 4 3

5: 2 1/2 3 4

6: 2 1 3 4

7: 3 1 2 4

8: 4 1 2 3

9: 4 1 3 2

10: 3 1 4 2

11: 2 1 4 3

FACES:

{4 5 6 11}

{2 3 7 8}

{2 5 6 7}

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{1 4 10 11}

{6 7 8 9 10 11}

{0 1 2 3 4 5}

{0 3 8 9}

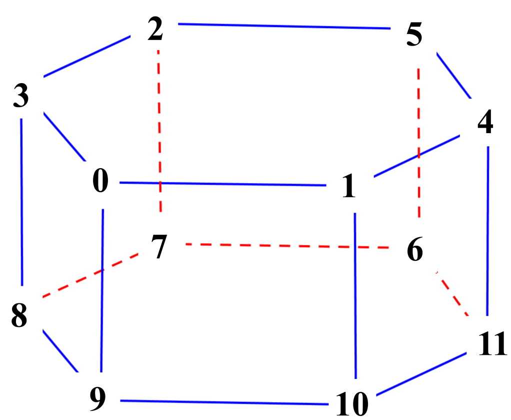
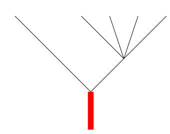


Figure C.16: Facet 15 of the pterahedron in $\mathbf{R}^4$.

FACET 16: Pentagonal Prism

FACET TREE: 

VERTICES:

0: 2 1 1/2 3/2

5: 4 3/2 1 1/2

1: 2 1/2 1 3/2

6: 4 1/2 2 1/2

2: 2 3/2 1/2 1

7: 4 3/2 1/2 1

3: 2 3/2 1 1/2

8: 4 1 1/2 3/2

4: 2 1/2 2 1/2

9: 4 1/2 1 3/2

FACES:

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{0 2 7 8}

{5 6 7 8 9}

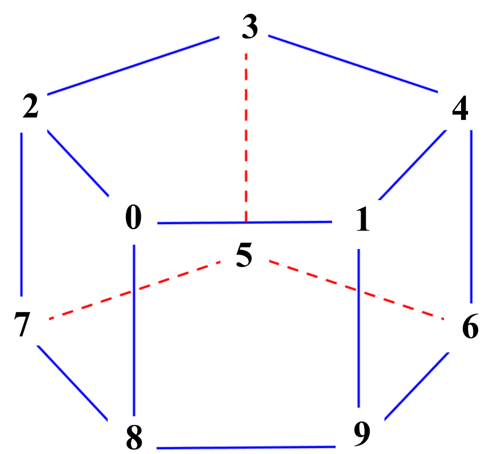
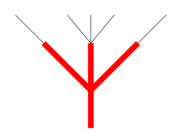


Figure C.17: Facet 16 of the pterahedron in $\mathbf{R}^4$.

FACET 17: Hexagonal Prism

FACET TREE: 

VERTICES:

0: 4 1/2 1 3

1: 4 1 1/2 3

2: 3 1 1/2 4

3: 3 1/2 1 4

4: 3 1/2 2 4

5: 3 2 1/2 4

6: 4 1/2 2 3

7: 4 2 1/2 3

8: 3 1 2 4

9: 3 2 1 4

10: 4 2 1 3

11: 4 1 2 3

FACES:

{2 3 4 5 8 9}

{0 3 4 6}

{0 1 2 3}

{4 6 8 11}

{1 2 5 7}

{5 7 9 10}

{8 9 10 11}

{0 1 6 7 10 11}

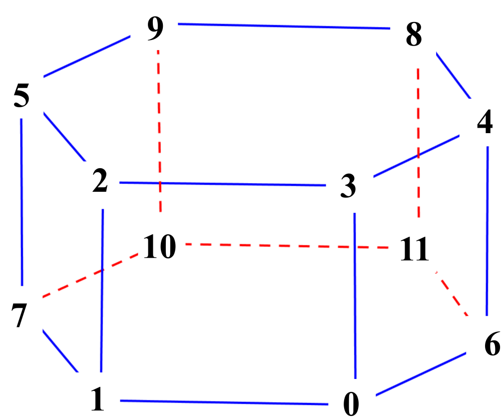
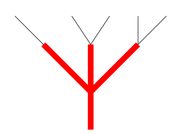


Figure C.18: Facet 17 of the pterahedron in $\mathbf{R}^4$.

FACET 18: Pentagonal Prism

FACET TREE: 

VERTICES:

0: 4 1/2 3 1/2

1: 3 1/2 4 1/2

2: 4 1/2 3 2

3: 3 1/2 4 2

4: 3 2 4 1/2

5: 4 2 3 1/2

6: 4 1 3 2

7: 4 2 3 1

8: 3 2 4 1

9: 3 1 4 2

FACES:

{0 1 4 5}

{1 3 4 8 9}

{0 1 2 3}

{2 3 6 9}

{6 7 8 9}

{4 5 7 8}

{0 2 5 6 7}

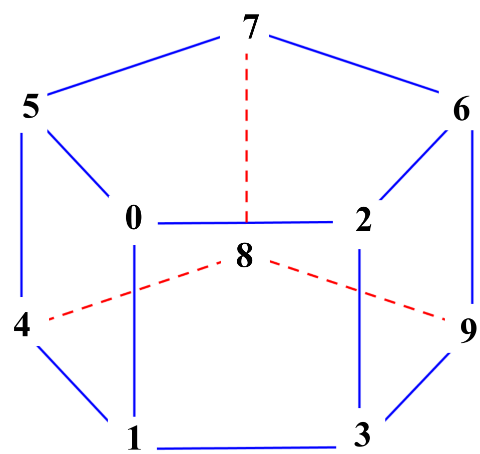
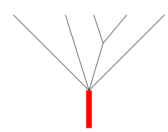


Figure C.19: Facet 18 of the pterahedron in $\mathbf{R}^4$.

FACET 19: Pterahedron

FACET TREE:



VERTICES:

0: 3/2 1 1/2 2

1: 2 1 1/2 3/2

2: 2 3/2 1/2 1

3: 1/2 2 1/2 2

4: 1/2 3 1/2 1

5: 4 3/2 1/2 1

6: 4 1 1/2 3/2

7: 1/2 4 1/2 1

8: 1/2 2 1/2 4

9: 3/2 1 1/2 4

10: 4 3 1/2 1

11: 4 1 1/2 3

12: 1/2 4 1/2 3

13: 3 4 1/2 1

14: 3 1 1/2 4

15: 1/2 3 1/2 4

16: 4 3 1/2 2

17: 3 4 1/2 2

18: 3 2 1/2 4

19: 4 2 1/2 3

20: 2 4 1/2 3

21: 2 3 1/2 4

FACES:

{2 4 5 7 10 13}

{12 15 20 21}

{0 1 2 3 4}

{3 4 7 8 12 15}

{0 3 8 9}

{8 9 14 15 18 21}

{7 12 13 17 20}

{0 1 6 9 11 14}

{1 2 5 6}

{11 14 18 19}

{16 17 18 19 20 21}

{10 13 16 17}

{5 6 10 11 16 19}

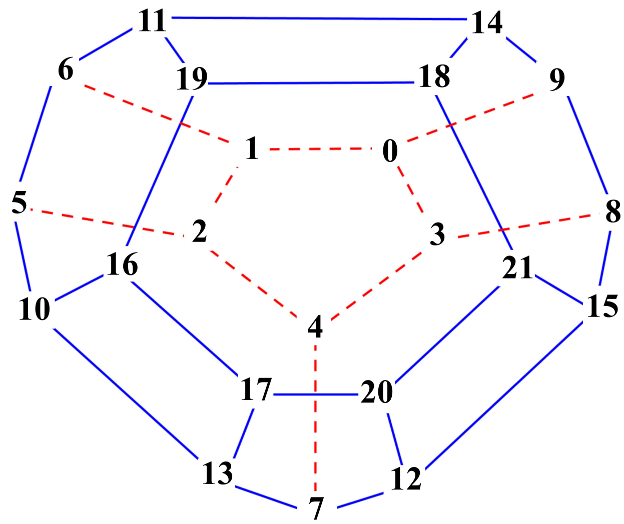
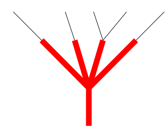


Figure C.20: Facet 19 of the pterahedron in $\mathbf{R}^4$.

FACET 20: Hexagonal Prism

FACET TREE: 

VERTICES:

0: 4 3 1/2 2

1: 3 4 1/2 2

2: 3 2 1/2 4

3: 4 2 1/2 3

4: 2 4 1/2 3

5: 2 3 1/2 4

6: 3 2 1 4

7: 4 2 1 3

8: 4 3 1 2

9: 2 3 1 4

10: 2 4 1 3

11: 3 4 1 2

FACES:

{4 5 9 10}

{1 4 10 11}

{2 5 6 9}

{2 3 6 7}

{0 1 2 3 4 5}

{0 3 7 8}

{6 7 8 9 10 11}

{0 1 8 11}

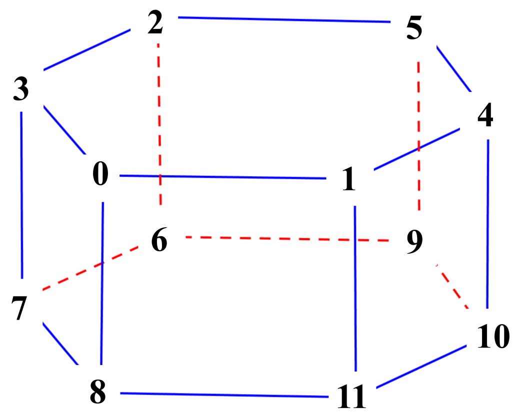
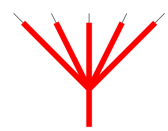


Figure C.21: Facet 20 of the pterahedron in $\mathbf{R}^4$.

FACET 21: Permutahedron

FACET TREE: 

VERTICES:

0: 1 2 3 4

1: 2 1 3 4

2: 3 1 2 4

3: 3 2 1 4

4: 4 2 1 3

5: 4 1 2 3

6: 4 3 1 2

7: 4 3 2 1

8: 4 1 3 2

9: 4 2 3 1

10: 2 3 1 4

11: 1 3 2 4

12: 3 2 4 1

13: 3 1 4 2

14: 2 1 4 3

15: 1 4 2 3

16: 2 4 1 3

17: 3 4 1 2

18: 2 3 4 1

19: 1 3 4 2

20: 1 2 4 3

21: 1 4 3 2

22: 2 4 3 1

23: 3 4 2 1

FACES:

{18 19 21 22}

{10 11 15 16}

{0 1 14 20}

{0 11 15 19 20 21}

{0 1 2 3 10 11}

{12 13 14 18 19 20}

{15 16 17 21 22 23}

{1 2 5 8 13 14}

{2 3 4 5}

{8 9 12 13}

{3 4 6 10 16 17}

{7 9 12 18 22 23}

{6 7 17 23}

{4 5 6 7 8 9}

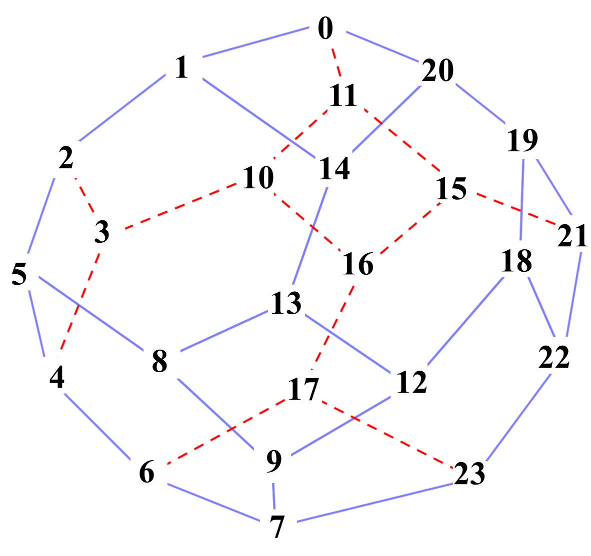
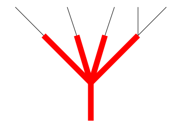


Figure C.22: Facet 21 of the pterahedron in $\mathbf{R}^4$.

FACET 22: Hexagonal Prism

FACET TREE: 

VERTICES:

0: 4 3 2 1/2

1: 3 4 2 1/2

2: 3 2 4 1/2

3: 4 2 3 1/2

4: 2 4 3 1/2

5: 2 3 4 1/2

6: 4 3 2 1

7: 4 2 3 1

8: 3 2 4 1

9: 2 3 4 1

10: 2 4 3 1

11: 3 4 2 1

FACES:

{0 1 2 3 4 5}

{4 5 9 10}

{2 5 8 9}

{1 4 10 11}

{2 3 7 8}

{6 7 8 9 10 11}

{0 1 6 11}

{0 3 6 7}

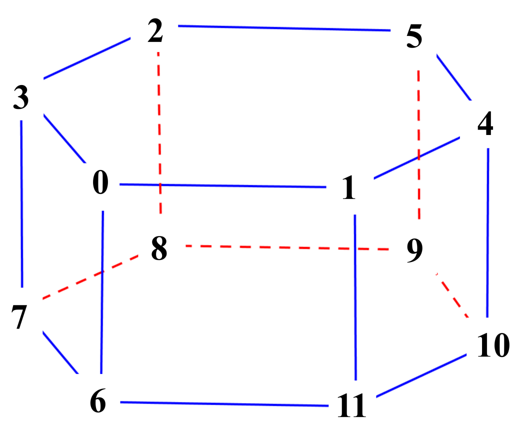
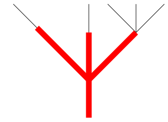


Figure C.23: Facet 22 of the pterahedron in $\mathbf{R}^4$.

FACET 23: Hexagonal Prism

FACET TREE: 

VERTICES:

0: 4 3 1 1/2

1: 4 3 1/2 1

2: 3 4 1/2 1

3: 3 4 1 1/2

4: 4 3 1/2 2

5: 4 3 2 1/2

6: 3 4 1/2 2

7: 3 4 2 1/2

8: 4 3 1 2

9: 4 3 2 1

10: 3 4 1 2

11: 3 4 2 1

FACES:

{0 3 5 7}

{0 1 2 3}

{2 3 6 7 10 11}

{1 2 4 6}

{4 6 8 10}

{8 9 10 11}

{5 7 9 11}

{0 1 4 5 8 9}

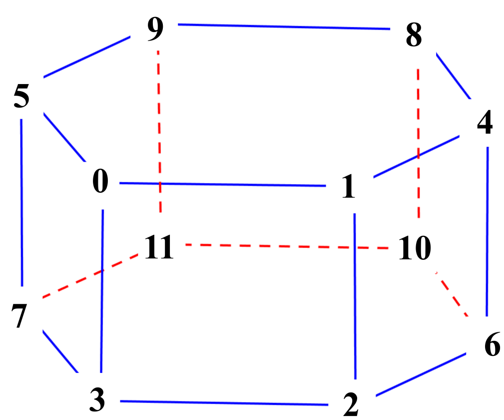
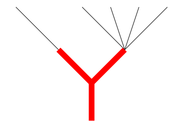


Figure C.24: Facet 23 of the pterahedron in $\mathbf{R}^4$.

FACET 24: Pterahedon

FACET TREE: 

VERTICES:

0: 4 3/2 1 1/2

1: 4 1/2 2 1/2

2: 4 3/2 1/2 1

3: 4 1 1/2 3/2

4: 4 1/2 1 3/2

5: 4 3 1 1/2

6: 4 3 1/2 1

7: 4 1/2 3 1/2

8: 4 1/2 1 3

9: 4 1 1/2 3

10: 4 1/2 3 2

11: 4 3 1/2 2

12: 4 3 2 1/2

13: 4 1/2 2 3

14: 4 2 1/2 3

15: 4 2 3 1/2

16: 4 2 1 3

17: 4 1 2 3

18: 4 3 1 2

19: 4 3 2 1

20: 4 1 3 2

21: 4 2 3 1

FACES:

{0 1 5 7 12 15}

{0 2 5 6}

{1 4 7 8 10 13}

{3 4 8 9}

{10 13 17 20}

{0 1 2 3 4}

{8 9 13 14 16 17}

{7 10 15 20 21}

{2 3 6 9 11 14}

{11 14 16 18}

{16 17 18 19 20 21}

{12 15 19 21}

{5 6 11 12 18 19}

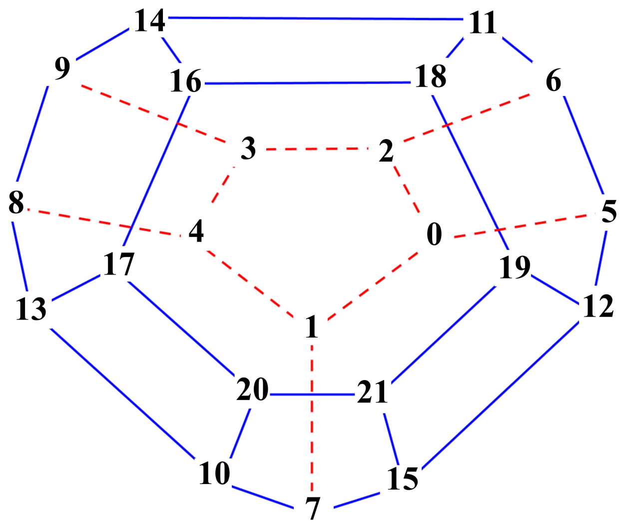


Figure C.25: Facet 24 of the pterahedron in $\mathbf{R}^4$.