COMBINATORIC N-FOLD CATEGORIES AND N-FOLD OPERADS

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Abstract. Operads were originally defined as $\mathcal{V}$-operads, that is, enriched in a symmetric or braided monoidal category $\mathcal{V}$. The symmetry or braiding in $\mathcal{V}$ is required in order to describe the associativity axiom the operads must obey, as well as the associativity that must be a property of the action of an operad on any of its algebras. A sequence of categorical types that filter the category of monoidal categories and monoidal functors is given by Balteanu, Fiedorowicz, Schwanzl and Vogt in [Balteanu et.al, 2003]. These subcategories of **MonCat** are called n-fold monoidal categories. A k–fold monoidal category is n-fold monoidal for all $n \leq k$, and a symmetric monoidal category is n-fold monoidal for all n. After a review of the role of operads in loop space theory and higher categories we go over definitions of iterated monoidal categories and the beginnings of a large family of simple examples. Then we generalize the definition of operad by defining n-fold operads and their algebras in an iterated monoidal category. We discuss examples of these that live in the previously described categories. Finally we describe the $(n-2)$-fold monoidal category of n-fold $\mathcal{V}$-operads.

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Contents

1. Introduction

Operads in a category of topological spaces are the crystallization of several approaches to the recognition problem for iterated loop spaces. Beginning with Stasheff's associahedra and Boardman and Vaught's little n-cubes, and continuing with more general A_∞ and E_∞ operads described by May and others, that problem was largely solved. [Stasheff, 1963], [Boardman and Vogt, 1973], [May, 1972] Loop spaces are characterized by admitting an operad action of the appropriate kind.

Recently there has also been growing interest in the application of higher dimensional structured categories to the characterization of loop spaces. The program being advanced by many categorical homotopy theorists seeks to model the coherence laws governing homotopy types with the coherence axioms of stuctured n-categories. By modeling we mean a connection that will necessarily be in the form of a functorial equivalence between categories of special categories and categories of special spaces. The largest challenges currently are to find the most natural and efficient definition of (weak) n-category, and to find the right sort of connecting functor.

One major recent advance is the discovery of Balteanu, Fiedorowicz, Schwanzl and Vogt in [Balteanu et.al, 2003] that the nerve functor on categories gives a direct connection between iterated monoidal categories and iterated loop spaces. Stasheff [Stasheff, 1963] and MacLane [Mac Lane, 1965] showed that monoidal categories are precisely analogous to 1-fold loop spaces. There is a similar connection between symmetric monoidal categories and infinite loop spaces. The first step in filling in the gap between 1 and infinity was made in [Fiedorowicz] where it is shown that the group completion of the nerve of a braided monoidal category is a 2-fold loop space. In [Balteanu et.al, 2003] the authors finished this process by, in their words, "pursuing an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces." The first thing they focus on is the fact that a braided category is a special case of a carefully defined 2-fold monoidal category. Based on their observation of the correspondence between loop spaces and monoidal categories, they iteratively define the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. In [Balteanu et.al, 2003] a symmetric category is seen as a category that is n-fold monoidal for all n. The main result in that paper is that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. It is still open whether this is a complete characterization, that is, whether every n-fold loop space arises as the nerve of an n-fold category.

The connection between the n-fold monoidal categories of Fiedorowicz and the theory of higher categories is through Baez's periodic table.[Baez and Dolan, 1998] Here Baez organizes the k-tuply monoidal n-categories, by which terminology he refers to $(n+k)$-categories that are trivial below dimension k. The triviality of lower cells allows the higher ones to compose freely, and thus these special cases of $(n+k)$-categories are viewed as n-categories with k products. Of course a k-tuply monoidal n-category is a special k-fold monoidal n-category. The specialization results from the definition(s) of n-category, all of which seem to include the axiom that the interchange transformation between two ways of composing four higher morphisms along two different lower dimensions is required to be an isomorphism. As will be mentioned in the next section the property of having loop space nerves held by the k-fold categories relies on interchange transformations that are not isomorphisms. If those transformations are indeed isomorphisms then the k-fold 1-categories do reduce to the braided and symmetric 1-categories of the periodic table. Whether this continues for higher dimensions, yielding for example sylleptic monoidal 2-categories as 3-fold 2-categories with interchange isomorphisms, is yet to be determined.

A further refinement of higher categories is to require all morphisms to have inverses. These special cases are referred to as n-groupoids, and since their nerves are simpler to describe it has been long known that they model homotopy n-types. A homotopy n-type is a topological space X for which $\pi_k(X)$ is trivial for all $k > n$. Thus the homotopy n-types are clasified by π_n. It has been suggested that a key requirement for the eventual accepted definition of n-category is that a k-tuply monoidal n-groupoid be associated functorially (by a nerve) to a topological space which is a homotopy n-type k-fold loop space. [Baez and Dolan, 1998] The loop degree will be precise for $k < n+1$, but for $k > n$ the associated homotopy n-type will be an infinite loop space. This last statement is a consequence of the stabilization hypothesis , which states that there should be a left adjoint to forgetting monoidal structure that is an equivalence of $(n+k+2)$ categories between k-tuply monoidal n-categories and $(k+1)$-tuply monoidal n-categories for $k > n+1$. For the case of $n = 1$ if the interchange transformations are isomorphic then a k-fold (and k-tuply) monoidal 1-category is equivalent to a symmetric category for $k > 2$. With these facts in mind it is clear that if we wish to precisely model homotopy n-type k-fold loop spaces for $k > n$ then we need to consider k-fold as well as k-tuply monoidal n-categories. This paper is part of an embrionic effort in that direction.

Since a loop space can be efficiently described as an operad algebra, it is not surprising that there are several existing definitions of n-category that utilize operad actions. These definitions fall into two main classes: those that define an n-category as an algebra of a higher order operad, and those that achieve an inductive definition using classical operads in symmetric monoidal categories to parameterize iterated enrichment. The first class of definitions is typified by Batanin and Leinster. [Batanin,1998],[Leinster, 2004] The former author defines monoidal globular categories in which interchange transformations are isomorphisms and which thus resemble free strict n-categories. Globular operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string of objects as in the case of classical operads. The binary composition in an n-category derives from the action of a certain one of these globular operads. Leinster expands this concept to describe n-categories with unbiased composition of any number of cells. The second class of definitions is typified by the works of Trimble and May. [May, 2001], [Trimble, 1999] The former parameterizes iterated enrichment with a series of operads in $(n-1)$-Cat achieved by taking the fundamental $(n-1)$-groupoid of the kth component of the topological path composition operad E. The latter begins with an A_∞ operad in a symmetric monoidal category $\mathcal{V}$ and requires his enriched categories to be tensored over $\mathcal{V}$ so that the iterated enrichment always refers to the same original operad.

Iterated enrichment over n-fold categories is described in [Forcey, 2004] and [Forcey2, 2004]. We would like to define n-fold operads in n-fold monoidal categories in a way that is consistent with the spirit of Batanin's globular operads, and with the eventual goal of using them to weaken enrichment over n-fold categories in a way that is in the spirit of Trimble. This program carries with it the promise of characterizing k-fold loop spaces with homotopy n-type for all n, k.

This paper comprises a naive beginning, illustrated with a very basic set of examples that we hope will help clarify the definitions. First we present the definition of n-fold monoidal category and go over a collection of related examples from semigroup theory. The examples of most visual value are from the combinatorial theory of tableau shapes. Secondly we present constructive definitions of n-fold operads and their algebras in an iterated monoidal category. We discuss examples of these that live in the previously described categories. Finally we describe the $(n-2)$-fold monoidal category of n-fold $\mathcal{V}$-operads.

A more abstract approach for future consideration would begin by finding an equivalent definition of n-fold operad in terms of monoids in a certain category. Then the full abstraction would be to find an equivalent definition in the language of Weber, where an operad lives within a monoidal pseudo algebra of a 2-monad. [Weber, 2004] This latter is a general notion of operad which includes as instances both classical and higher operads.

2. k-fold monoidal categories

This sort of category was developed and defined in [Balteanu et.al, 2003]. The authors describe its structure as arising from its description as a monoid in the category of $(k-1)$-fold monoidal categories. Here is that definition altered only slightly to make visible the coherent associators as in [Forcey, 2004]. In that paper I describe its structure as arising from its description as a tensor object in the category of $(k-1)$-fold monoidal categories.

2.1. Definition. *An n-fold monoidal category is a category $\mathcal{V}$ with the following structure.*

1. *There are n distinct multiplications*

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

for each of which the associativity pentagon commutes

$$\begin{tikzcd}
& ((U \otimes_i V) \otimes_i W) \otimes_i X \arrow[r, "\alpha^i_{UVW} \otimes_i 1_X"] \arrow[dl, "\alpha^i_{(U\otimes_i V)WX}"'] & (U \otimes_i (V \otimes_i W)) \otimes_i X \arrow[dr, "\alpha^i_{U(V\otimes_i W)X}"] & \\
(U \otimes_i V) \otimes_i (W \otimes_i X) \arrow[drr, "\alpha^i_{UV(W\otimes_i X)}"] & & & U \otimes_i ((V \otimes_i W) \otimes_i X) \arrow[dl, "1_U \otimes_i \alpha^i_{VWX}"] \\
& & U \otimes_i (V \otimes_i (W \otimes_i X)) &
\end{tikzcd}$$

$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.

2. *For each pair (i, j) such that $1 \le i < j \le n$ there is a natural transformation*

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) *Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$*

(b) *External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$*

(c) *Internal associativity condition: The following diagram commutes*

$$\begin{tikzcd}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) \arrow[r, "\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}"] \arrow[d, "\alpha^i"] & \big((U \otimes_i W) \otimes_j (V \otimes_i X)\big) \otimes_i (Y \otimes_j Z) \arrow[d, "\eta^{ij}_{(U\otimes_i W)(V\otimes_i X)YZ}"] \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) \arrow[d, "1_{U\otimes_j V} \otimes_i \eta^{ij}_{WXYZ}"] & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \arrow[d, "\alpha^i \otimes_j \alpha^i"] \\
(U \otimes_j V) \otimes_i \big((W \otimes_i Y) \otimes_j (X \otimes_i Z)\big) \arrow[r, "\eta^{ij}_{UV(W\otimes_i Y)(X\otimes_i Z)}"] & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes

$$\begin{tikzcd}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) \arrow[r, "\eta^{ij}_{(U \otimes_j V)W(X \otimes_j Y)Z}"] \arrow[d, "\alpha^j \otimes_i \alpha^j"] & \big((U \otimes_j V) \otimes_i (X \otimes_j Y)\big) \otimes_j (W \otimes_i Z) \arrow[d, "\eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}"] \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) \arrow[d, "\eta^{ij}_{U(V \otimes_j W)X(Y \otimes_j Z)}"] & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\alpha^j"] \\
(U \otimes_i X) \otimes_j \big((V \otimes_j W) \otimes_i (Y \otimes_j Z)\big) \arrow[r, "1_{U \otimes_i X} \otimes_j \eta^{ij}_{VWYZ}"] & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{tikzcd}$$

(e) Finally it is required for each triple (i, j, k) satisfying $1 \leq i < j < k \leq n$ that the giant hexagonal interchange diagram commutes.

$$\begin{tikzcd}
 & ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}"'] \arrow[dr, "\eta^{ij}_{(A \otimes_k A')(B \otimes_k B')(C \otimes_k C')(D \otimes_k D')}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\eta^{ik}_{(A \otimes_j B)(A' \otimes_j B')(C \otimes_j D)(C' \otimes_j D')}"'] & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\eta^{jk}_{(A \otimes_i C)(A' \otimes_i C')(B \otimes_i D)(B' \otimes_i D')}"] \\
 & ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}$$

The authors of [Balteanu et.al, 2003] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \ldots = \otimes_n = \otimes$$

and defining (associators added by myself)

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation.

3. Examples of iterated monoidal categories

1. We start with an example of a symmetric monoidal category. Given a totally ordered set G with a smallest element $e \in G$, then we have an ordered semigroup structure where the semigroup product is max and the 2-sided unit is the least element e. The elements of G make up the objects of a strict monoidal category whose morphisms are given by the ordering; there is only an arrow $a \to b$ if $a \le b$. The ordered semigroup structure implies the monoidal structure on morphisms, since if $a \le b$ and $a' \le b'$ then $\max(a, a') \le \max(b, b')$. The category will also be denoted G. If G is such a set then by $\mathrm{Seq}(G)$ we denote the infinite sequences X_n of elements of G for which there exists a natural number $l(X)$ such that $k > l(X)$ implies $X_k = e$ and $X_{l(X)} \ne e$. Under lexicographic ordering $\mathrm{Seq}(G)$ is in turn a totally ordered set with a least element. The latter is the sequence 0 where $0_n = e$ for all n. We let $l(0) = 0$. The lexicographic order means that $A \le B$ if either $A_k = B_k$ for all k or there is a natural number $n = n_{AB}$ such that $A_k = B_k$ for all $k < n$, and such that $A_n < B_n$.

   The ordering is easily shown to be reflexive, transitive, and antisymmetric. See for instance [Schröder, 2003] where the case of lexicographic ordering of n-tuples of natural numbers is dicussed. In our case we will need to modify the proof by always making comparisons of $\max(l(A), l(B))$-tuples.

2. As a category $\mathrm{Seq}(G)$ is 2-fold monoidal since we can demonstrate two interchanging products: max using the lexicographic order– $A \otimes_1 B = \max(A, B)$; and the piecewise max– $(A \otimes_2 B)_n = \max(A_n, B_n)$. The latter is more generally described as the piecewise application of the ordered semigroup product from G. Such a piecewise application will always be in turn an ordered semigroup product since $A_i \le B_i$ and $C_i \le D_i$ implies that $A_i C_i \le B_i D_i$. $\mathrm{Seq}(G)$ is thus an example of a 2-fold monoidal category that is formed from any totally ordered semigroup .

   3.1. Theorem. *Given a totally ordered semigroup $\{H \le\}$ such that the identity element $0 \in H$ is less than every other element then we have a 2-fold monoidal category whose objects are elements of H.*

   Proof: Morphisms are again given by the ordering. The products are given by max and the semigroup operation: $a \otimes_1 b = \max(a, b)$ and $a \otimes_2 b = ab$. The shared two-sided unit for these products is the identity element 0. The products are both strictly associative and functorial since if $a \le b$ and $a' \le b'$ then $aa' \le bb'$ and $\max(a, a') \le \max(b, b')$. The interchange natural tranformations exist since $\max(ab, cd) \le \max(a, c) \max(b, d)$. This last theorem is easily seen by checking the four possible cases: $\{c \le a, d \le b\}$; $\{c \le a, b \le d\}$; $\{a \le c, d \le b\}$; $\{a \le c, b \le d\}$; or by the quick argument that

   $a \le max(a, c)$ and $b \le max(b, d)$ so

   $ab \le max(a, c)max(b, d)$ and similarly

$cd \leq max(a,c)max(b,d)$.

The internal and external unit and associativity conditions of Definition 2.1 are all satisfied due to the fact that there is only one morphism between two objects. Besides Seq(G) examples of such semigroups as in Theorem 3.1 are the nonnegative integers under addition and the braids on n strands with only right-handed crossings. Further examples are found in the papers on semirings and idempotent mathematics, such as [Litvinov and Sobolevskii, 2001] and its references as well as on the related concept of tropical mathematics, such as [Speyer and Sturmfels, 2004] and its references. Semirings that arise in these two areas of study usually require some translation to yield 2-fold monoidal categories since they typically have a zero and a unity, $0 \neq 1$.

3. There is also an alternative to the piecewise max on the objects of Seq(G)– concatenation of sequences. This is a valid semigroup operation and thus monoidal product since it clearly preserves the lexicographic order. Notice that concatenation, like braid composition, is a non-symmetric example. A non-example, however, is revealed in an attempt to have both the piecewise max and the concatenation at once in an iterated monoidal category. The required interchange morphisms between concatenation and piecewise max do not exist.

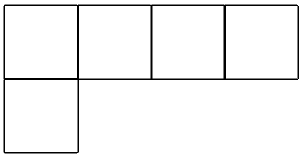
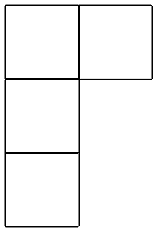
4. The step taken to create Seq(G) can be iterated to create n-fold monoidal categories for all n. In general, if we have a k-fold monoidal category C with morphisms ordering of objects and the common unit and smallest element I such that $A \in C \Rightarrow I \leq A$ then:

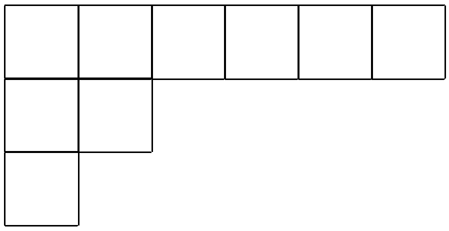
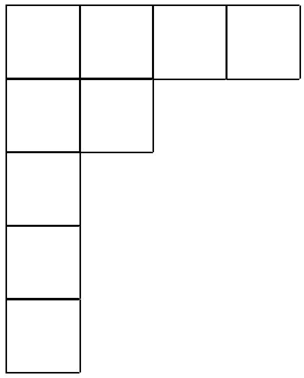
    1. we can form a $(k+1)$-fold monoidal category $\hat{C}$ with $\hat{\otimes}_1 = \max$ and $\hat{\otimes}_i = \otimes_{i-1}$
    2. we can form the $(k+2)$-fold monoidal Seq(C) with objects ordered lexicographically.

   Now the new $\otimes_1$ of Seq(C) is the maximum of sequences with respect to the lexicographic order, and the new $\otimes_2 ... \otimes_{k+2}$ are the piecewise products based respectively on $\hat{\otimes}_1 ... \hat{\otimes}_{k+1}$ in $\hat{C}$.

5. Even more structure is found in examples with a natural geometric representation which allows the use of addition in each product. One such category is that whose objects are Ferrers diagrams, by which we mean the shapes of Young tableaux. These can be presented by a decreasing sequence of nonnegative integers in two ways: the sequence that gives the heights of the columns or the sequence that gives the lengths of the rows. We let $\otimes_2$ be the product which adds the heights of columns

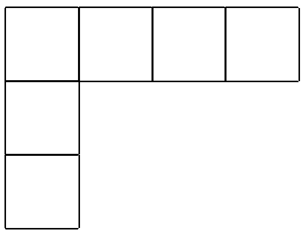
of two tableaux, $\otimes_1$ adds the length of rows.

$A =$  and $B =$ 

then $A \otimes_1 B =$  and $A \otimes_2 B =$ 

There are several possibilities for morphisms. We can create a category equivalent to the non-negative integers in the previous example by pre-ordering the tableaux by height. Here the height $h(A)$ of the tableau is the number of boxes in its leftmost column, and we say $A \leq B$ if $h(A) \leq h(B)$. Two tableaux with the same height are isomorphic objects, and the one-column stacks form both a full subcategory and a skeleton of the height preordered category. Everything works as for the previous example of natural numbers since $h(A \otimes_2 B) = h(A) + h(B)$ and $h(A \otimes_1 B) = \max(h(A), h(B))$.

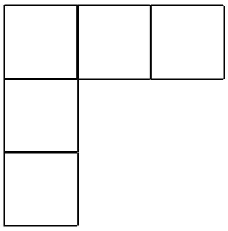
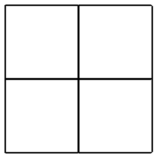
Since we are working with sequences there are also inherited max products as dicussed in the first example. The new max with respect to the height preordering is defined as $\max(A, B) = A$ if $B \leq A$ and $= B$ otherwise. The piecewise application of max with respect to the ordering of natural numbers results in taking a union of the two tableau shapes. Above, $\max(A, B) = B$ and the piecewise max becomes

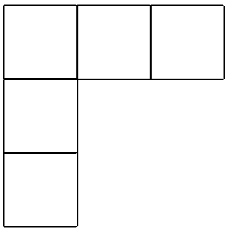
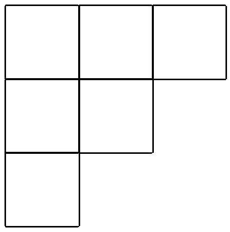


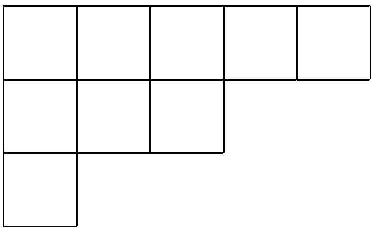
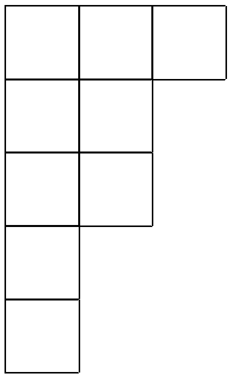
In the height preordered category both of these latter products are equivalent to the horizontal composition $\otimes_1$.

6. Alternatively we can work with the totally ordered structure of the tableau shapes given by lexicographic ordering. For now we consider the presentation in terms of a

monotone decreasing sequence of columns, and regard an empty column as a height zero column. Every tableau A considered as an infinite sequence has all zero height columns after some finite natural number $l(A)$. Now the four products just described in the second example above still exist, but are truly distinct. The overall max with respect to the lexicographic order is thus $\otimes_1$. Piecewise max is $\otimes_2$. Horizontal addition is $\otimes_3$ and vertical addition (piecewise addition of sequence terms) is $\otimes_4$. For comparison purposes here is an example of the four possible products. For objects

$A =$  and $B =$ 

then $A \otimes_1 B =$  and $A \otimes_2 B =$ 

$A \otimes_3 B =$  and $A \otimes_4 B =$ 

3.2. Theorem. *Tableau shapes, lexicographic ordering, and $\otimes_1, \otimes_2, \otimes_3, \otimes_4$ as described above form a 4-fold monoidal category.*

Proof: By previous discussion then the tableau shapes with $\otimes_1, \otimes_2, \otimes_4$ form a 3-fold monoidal category called Seq($\mathbf{N}$). To see that with the additional $\otimes_3$ this becomes a valid 4-fold monoidal category we need to check first that horizontal addition is functorial with respect to morphisms (defined as the $\leq$ relations of the lexicographic ordering.) Note that the horizontal product of tableau shapes A and C can be described as a reorganization of all the columns of both A and C into a new tableau shape made up of those columns in descending order of height. Especially when speaking of general sequences we'll call this product sorting. The cases where $A = B$ or $C = D$ are easy. For example let $A_k = B_k$ for all k and $C_k = D_k$ for all

$k < n_{CD}$ Thus the columns from the copies of, for instance A in $A \otimes_1 C$ and $A \otimes_1 D$ fall into the same final spot under the sortings right up to the critical location, so if $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D$. Similarly, it is clear that $A \leq B$ implies $(A \otimes_1 D) \leq (B \otimes_1 D)$. Hence if $A \leq B$ and $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D \leq B \otimes_1 D$ which by transitivity gives us our desired property.

Next we check that our interchange transformations will always exist. η^{1j} exists by the proof of Theorem 3.1 for $j = 2, 3, 4$ since the higher products all respect morphisms(ordering) and are thus ordered semigroup operations. We need to check for existence of η^{23}, η^{24}, and η^{34}.

First for existence of η^{23} we need to show that $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$. We actually show more generally that for any two sequences a and b that $\max_{piecewise}(\mathrm{sort}(a), \mathrm{sort}(b)) \leq \mathrm{sort}(\max_{piecewise}(a, b))$. where we are sorting greater to smaller and taking piecewise maximums. To prove the original question we consider the special case of two sequences formed by letting a be A followed by C and letting b be B followed by D. By "followed by" we mean padded by zeroes so that piecewise addition of a and b results in piecewise addition of A and B, and respectively C and D. Thus in the original question $l(a) = \max(l(A), l(B)) + l(C)$ and $l(b) = \max(l(A), l(B)) + l(D)$. Recall that $a_k = 0$ for $k > l(a)$.

Consider the right hand side of the inequality, where we start with the two sequences and take the piecewise maximum of the corresponding pairs of elements before sorting. We can metamorphose this into the left hand side in stages by using an algorithm to sort a and b. Note that if a and b are already sorted, the inequality becomes an equality. For our algorithm we choose parallel bubble sorts. These consist of a series of passes through the sequences comparing a_n and a_{n+1} and comparing b_n and b_{n+1} simultaneously. If the two elements of a given sequence are not already in decreasing order we switch their places. We claim that such an operation always results in a lexicographically smaller sequence after taking piecewise max and sorting. If both the elements of a and of b are switched, or if neither, then the result is unaltered. Therefore without loss of generality we assume that $a_n < a_{n+1}$ and that $b_{n+1} \leq b_n$. Then we compare the original result of sorting after taking the piecewise max and the same but after the switching of a_n and a_{n+1}. If $b_n \leq a_n$ and $b_{n+1} \leq a_{n+1}$ then $\max(a_n, b_n) = \max(a_n, b_{n+1})$ and $\max(a_{n+1}, b_{n+1}) = \max(a_{n+1}, b_n)$. If $a_n \leq b_n$ and $a_{n+1} \leq b_{n+1}$ then $\max(a_n, b_n) = \max(a_{n+1}, b_n)$ and $\max(a_{n+1}, b_{n+1}) = \max(a_n, b_{n+1})$. Since neither of these cases lead to any change in the final result, we may assume $a_n \leq b_n$ and $b_{n+1} \leq a_{n+1}$, and check two sub-cases: First $a_{n+1} \leq b_n$ implies that $\max(a_n, b_n) = \max(a_{n+1}, b_n)$ and $\max(a_n, b_{n+1}) \leq \max(a_{n+1}, b_{n+1})$. Secondly $b_n \leq a_{n+1}$ implies that $\max(a_n, b_{n+1}) \leq max(a_n, b_n)$ and $\max(a_{n+1}, b_{n+1}) = \max(a_{n+1}, b_n)$.

Thus after taking piecewise max and sorting the new result is indeed smaller lexicographically.

Thus since each move of the parallel bubble sort results in a smaller expression and the moves eventually result in $(A \otimes_3 B) \otimes_2 (C \otimes_3 D)$, we have $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$.

Secondly the existence of η^{24} is clear since the interchange between max and $+$ implies the interchange between their piecewise application. Thus we have $(A \otimes_4 B) \otimes_2 (C \otimes_4 D) \leq (A \otimes_2 C) \otimes_4 (B \otimes_2 D)$.

Finally we need to check that $(A \otimes_4 B) \otimes_3 (C \otimes_4 D) \leq (A \otimes_3 C) \otimes_4 (B \otimes_3 D)$. We actually show more generally that for any two sequences a and b of total length that $Y = \mathrm{sort}(a+b) \leq \mathrm{sort}(a)+\mathrm{sort}(b) = X$ where $+$ is the piecewise addition. To prove the original question we make the two sequences by letting a be A followed by C and letting b be B followed by D as in the previous proof.

Consider Y, where we start with the two sequences and add them piecewise before sorting. We can metamorphose this into X in stages by using an algorithm to sort a and b. Note that if a and b are already sorted, the inequality becomes an equality. For our algorithm we again use parallel bubble sorts. We claim that switching consecutive sequence elements into order always results in a lexicographically larger sequence after adding piecewise and sorting. If both the elements of a and of b are switched, or if neither, then the result is unaltered. Therefore without loss of generality we assume that $a_n < a_{n+1}$ and that $b_{n+1} < b_n$. Then we compare the original result of sorting after adding and the same but after the switching of a_n and a_{n+1}. It is simplest to note that the new result includes $a_{n+1}+b_n$, which is larger than both $a_n + b_n$ and $a_{n+1} + b_{n+1}$.

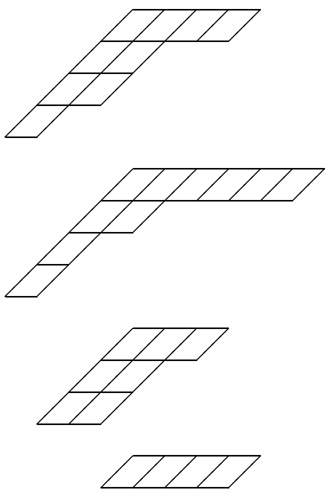
So after adding and sorting the new result is indeed larger lexicographically. Thus since each move of the parallel bubble sort results in a larger expression and the moves eventually result in $(A \otimes_3 C) \otimes_4 (B \otimes_3 D)$, we have $(A \otimes_4 B) \otimes_3 (C \otimes_4 D) \leq (A \otimes_3 C) \otimes_4 (B \otimes_3 D)$.

7. Notice that in the last two sections of the proof of Theorem 3.2, the product of two sequences given by sorting their elements behaves differently with respect to piecewise max and piecewise $+$. In general for a category of decreasing sequences the sorting given by $\otimes_k$ will have an interchange with a piecewise operation $(A \otimes_j B)_n = A_n * B_n, j < k$ if $A_n \leq B_n$ implies $A_n * B_n = B_n$. Also $\otimes_k$ will have an interchange with a piecewise operation $(A \otimes_j B)_n = A_n * B_n, j > k$ if $A_n > B_n$ implies $A_n * C_n > B_n * C_n$. This is important when attempting to iterate the construction. We can start with any totally ordered semigroup $\{G, \leq, +\}$ such that the identity 0 is less than any other element and such that $a > b$ implies $a + c > b + c$ for all $a, b, c \in G$. We create a 4-fold monoidal category $\mathrm{ModSeq}(G)$ with objects monotone decreasing finitely non-zero sequences of elements of G and morphisms given by the lexicographic ordering. The products are as described for natural numbers in the previous example. The common unit is the zero sequence. The proofs we have given in the previous example for $G = \mathbf{N}$ are all still valid. The objects of $\mathrm{ModSeq}(G)$, sequences under lexicographic order, form four totally ordered semigroups using each

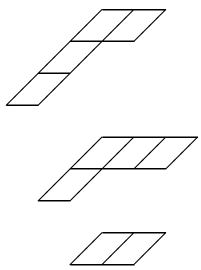
of the products respectively as the group operation. Thus we can form four 3-fold monoidal categories Seq(ModSeq($G, *$)) based on each of these semigroup structures. Forming ModSeq(ModSeq(G)) is only possible for the semigroup structures that use $\otimes_3$ or $\otimes_4$ from Theorem 3.2, It would seem that we also could potentially use the objects of ModSeq(ModSeq(G)) in a 6-fold category. However, we must choose between two possible 5-fold monoidal categories of decreasing sequences (of terms which are themselves decreasing sequences). The reason is that the product given by sorting two sequences of sequences by lexicographic order of their element sequences does not interchange at all with the product given by taking piecewise–piecewise max, by which we mean entrywise max, at the level of original semigroup elements. Either of these two as $\otimes_3$ however, along with overall (lexicographic ordering of sequence of sequences) max as $\otimes_1$, piecewise lexicographic max as $\otimes_2$, piecewise sorting as $\otimes_4$, and piecewise–piecewise addition as $\otimes_5$, do form 5-fold monoidal categories.

For instance if ModSeq($\mathbf{N}$) is our category of tableau shapes then ModSeq(ModSeq($\mathbf{N}$)) has objects monotone decreasing sequences of tableau, which we can visualize along the z-axis. For example, choosing the lexicographic sorting as $\otimes_3$, if:

$A =$

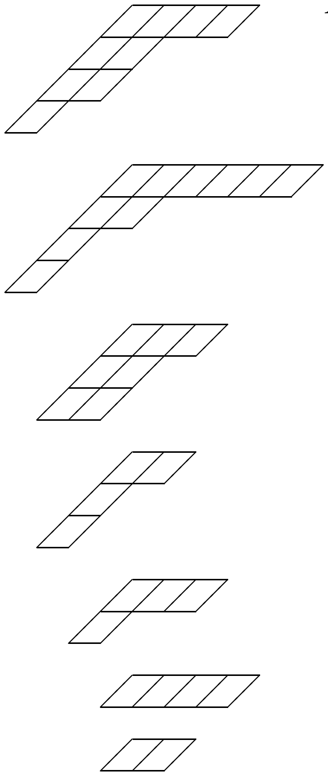


and $B =$

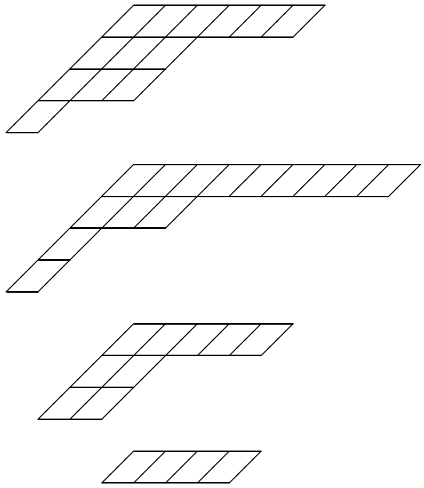


then $A \otimes_1 B = A$, $A \otimes_2 B = A$ and

$A \otimes_3 B =$

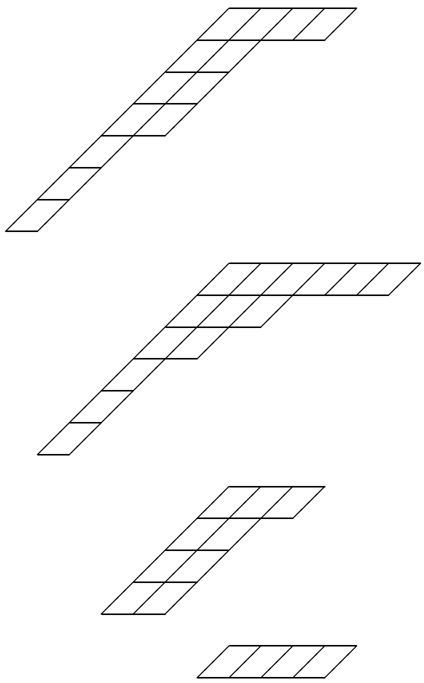


$A \otimes_4 B =$



and

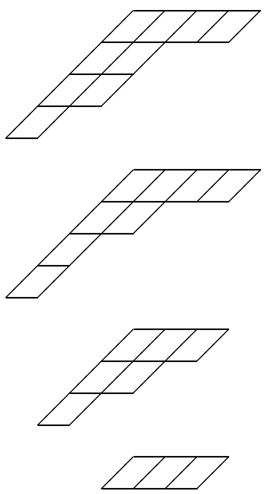
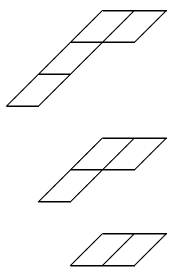
$A \otimes_5 B =$



Here the alternative $\otimes_3'$ = piecewise–piecewise max also turns oute to give $A \otimes_3' B = A$.

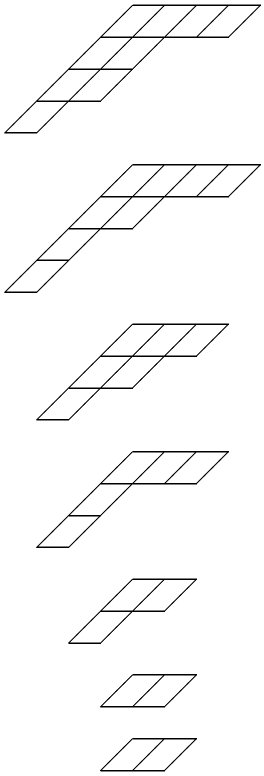
8. It might be nice to retain the geometric picture of the products of tableau shapes in terms of vertical and horizontal addition, and addition in other directions as dimension increases. This is not found in the current version of iteration. This "tableau stacking" point of view is restored if we restrict to sequences of tableau shapes that are decreasing in columns as well as rows. These are lexicographically decreasing sequences of decreasing sequences so already obey the requirement that A_{n1} is decreasing in n. We expand this to require that A_{nk} be decreasing in n for constant k, which implies decreasing in k for constant n. We can represent these objects as infinite matrices with finitely nonzero natural number entries, and with monotone decreasing columns and rows. We choose the sequence of rows to be the sequence of sequences, i.e. each row represents a tableau which we draw as being parallel to the xy plane. This choice is important because it determines the total ordering of matrices and thus the morphisms of the category. Thus horizontal composition is horizontal concatenation (diregarding zeroes) of matrices followed by sorting the new longer rows. Piecewise addition is addition of matrices. Now we define vertical (z-axis) addition as vertical concatentation of matrices followed by sorting the new long columns.

Here is a visual example of the vertical addition, labeled $\otimes_1$: if

$A =$  and $B =$ 

then we let

$$A \otimes_1 B =$$



Note that in this restricted setting of decreasing matrices the lexicographic sorting of sequences (rows) of two operands in a product does not preserve the decreasing property. Thus we consider whether the new product is incorporated via interchanges into the 5-fold monoidal category described above as containing the entrywise max as a product of matrices.

Then note that the natural place for this new product is in a orthogonal category to the previously described 5-fold monoidal category of matrices (sequence of rows) that has the same objects (but seen as sequences of columns.) The ordering is lex order on the columns, and there is also a piecewise lex max on the columns and a piecewise sorting of the columns. Integrating the two orthogaonal categories is nontrivial. A nonexample is easy to realize when we try to find an interchange between for instance the overall max on matrices ordered lexicographically by rows and the overall max when ordered by columns

It is true however that at least the three products that preserve the total sum of the entries in both matrices do interact via interchanges. Renumbered they are: $\otimes_1$ is the vertical concatentation of matrices followed by sorting the new longer columns, $\otimes_2$ is horizontal concatenation of matrices followed by sorting the new longer rows and $\otimes_3$ is the addition of matrices. We already have existence of η^{13} and η^{23} by previous arguments, we need only check for η^{12}. This existence is clearly the case since we are ordering the matrices by giving precedence to the rows. Thus sorting columns first and then rows is guaranteed to give something larger lexicographically than sorting horizontally first.

4. n-fold operads

Let $\mathcal{V}$ be an n-fold monoidal category as defined in Section 2.

4.1. Definition. *For $2 \leq m \leq n$ an m-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 1$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$*

$$\gamma^{pq} : \mathcal{C}(k) \otimes_p (\mathcal{C}(j_1) \otimes_q \ldots \otimes_q \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $m \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $m \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s = \sum_{u=1}^{s} j_u$ and let $h_s = \sum_{u=1+g_{s-1}}^{g_s} i_u$.*

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k}{}_q \mathcal{C}(j_s)\right) \otimes_p \left(\bigotimes_{t=1}^{j}{}_q \mathcal{C}(i_t)\right) \arrow[r, "\gamma^{pq}\otimes_p \mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes_p \eta^{pq}"'] & \mathcal{C}(j) \otimes_p \left(\bigotimes_{t=1}^{j}{}_q \mathcal{C}(i_t)\right) \arrow[d, "\gamma^{pq}"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k}{}_q \mathcal{C}(j_s) \otimes_p \left(\bigotimes_{u=1}^{j_s}{}_q \mathcal{C}(i_{u+g_{s-1}})\right)\right) \arrow[r, "\mathrm{id}\otimes_p (\otimes_q^k \gamma^{pq})"'] & \mathcal{C}(k) \otimes_p \left(\bigotimes_{s=1}^{k}{}_q \mathcal{C}(h_s)\right) \arrow[u, "\gamma^{pq}"']
\end{tikzcd}$$

where the η^{pq} on the left actually stands for a variety of equivalent maps which factor into instances of the pq interchange.

Respect of units is required just as in the symmetric case. The following unit diagrams commute for all $m \geq q > p \geq 1$.

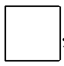
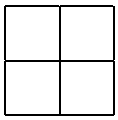
$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\otimes_q^k I) \arrow[r, equal] \arrow[d, "1\otimes_p (\otimes_q^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_p (\otimes_q^k \mathcal{C}(1)) \arrow[ur, "\gamma^{pq}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_p \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J}\otimes_p 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_p \mathcal{C}(k) \arrow[ur, "\gamma^{pq}"] &
\end{tikzcd}$$

4.2. Example.

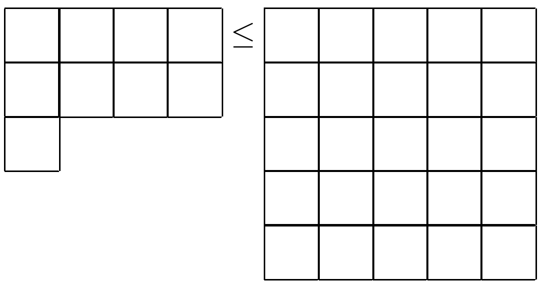
One large family of operads in ModSeq($\mathbf{N}$) is that of natural number indexed collections of tableau shapes $\mathcal{C}(n), n \in \mathbf{N}$, such that $h(\mathcal{C}(n)) = f(n)$ where $f : \mathbf{N} \to \mathbf{N}$ is a function such that $f(1) = 0$ and $f(i+j) \geq f(i) + f(j)$. These conditions are not necessary, but they are sufficient since the first implies that $\mathcal{C}(1) = 0$ which shows that the unit conditions are satisfied; and the second implies that the maps γ exist. We see existence of γ^{12} since $h(\mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 ... \otimes_2 \mathcal{C}(j_k))) = \max(f(k), \max(f(j_i))) \leq f(j) = h(\mathcal{C}(j))$. We also have existence of γ^{13} and γ^{23} since $\max(f(k), \max(f(j_i))) \leq f(j)$. We have existence of γ^{14} ,γ^{24} and γ^{34} since $\max(f(k), \sum f(j_i)) \leq f(j)$.

Examples of f include $(x-1)P(x)$ where P is a polynomial with coefficients in $\mathbf{N}$. This is easy to show since then P will be monotone increasing for $x \geq 1$ and thus $(i+j-1)P(i+j) = (i-1)P(i+j) + jP(i+j) \geq (i-1)P(i) + jP(j) - P(j)$. By this argument we can also use any $f = (x-1)g(x)$ where $g : \mathbf{N} \to \mathbf{N}$ is monotone increasing for $x \geq 1$.

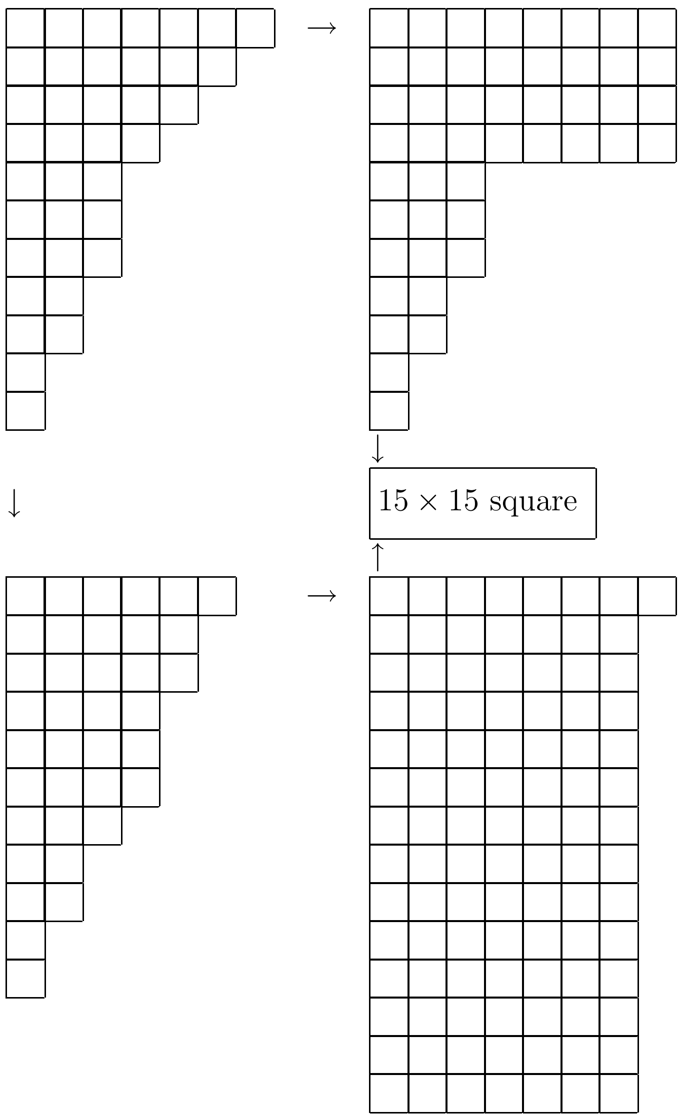
For a specific example with a handy picture that also illustrates again the nontrivial use of the interchange η we simply let $f = x - 1$. Then we have to actually describe the elements of ModSeq($\mathbf{N}$) that make up the operad. One nice choice is the operad $\mathcal{C}$ where $\mathcal{C}(n) = \{n-1, n-1, ..., n-1\}$, the $(n-1) \times (n-1)$ square tableau shape.

$\mathcal{C}(1) = 0$, $\mathcal{C}(2) =$ , $\mathcal{C}(3) =$  ...

For instance $\gamma^{34} : \mathcal{C}(3) \otimes_3 (\mathcal{C}(1) \otimes_4 \mathcal{C}(3) \otimes_4 \mathcal{C}(2)) \to \mathcal{C}(6)$ appears as the relation



An instance of the associativity diagram with upper left position $\mathcal{C}(2) \otimes_3 (\mathcal{C}(3) \otimes_4 \mathcal{C}(2)) \otimes_3 (\mathcal{C}(2) \otimes_4 \mathcal{C}(2) \otimes_4 \mathcal{C}(4) \otimes_4 \mathcal{C}(5) \otimes_4 \mathcal{C}(3))$ is as follows:



We conclude with a description of the concepts of n-fold operad algebra and of the tensor products of operads and algebras.

4.3. Definition. *Let $\mathcal{C}$ be an n-fold operad in $\mathcal{V}$. A $\mathcal{C}$-algebra is an object $A \in \mathcal{V}$ and maps*

$$\theta^{pq} : \mathcal{C}(j) \otimes_p (\otimes_q^j A) \to A$$

for $n \geq q > p \geq 1$, $j \geq 0$.

1. *Associativity: The following diagram is required to commute for all $n \geq q > p \geq 1$,*

$k \geq 1,\ j_s \geq 0$, and where $\sum_{s=1}^{k} j_s = j$.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\mathcal{C}(j_1) \otimes_q \dots \otimes_q \mathcal{C}(j_k)) \otimes_p (\otimes_q^j A) \arrow[r, "\gamma^{pq} \otimes_p \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_p \eta^{pq}"'] & \mathcal{C}(j) \otimes_p (\otimes_q^j A) \arrow[d, "\theta^{pq}"'] \\
& A \\
\mathcal{C}(k) \otimes_p ((\mathcal{C}(j_1) \otimes_p (\otimes_q^{j_1} A)) \otimes_q \dots \otimes_q (\mathcal{C}(j_k) \otimes_p (\otimes_q^{j_k} A))) \arrow[r, "\mathrm{id} \otimes_p (\otimes_q^k \theta^{pq})"] & \mathcal{C}(k) \otimes_p (\otimes_q^k A) \arrow[u, "\theta^{pq}"]
\end{tikzcd}$$

2. *Units: The following diagram is required to commute for all $n \geq q > p \geq 1$.*

$$\begin{tikzcd}
I \otimes_p A \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_p 1"'] & A \\
\mathcal{C}(1) \otimes_p A \arrow[ur, "\theta^{pq}"] &
\end{tikzcd}$$

4.4. Definition. *Let $\mathcal{C}$,$\mathcal{D}$ be n-fold operads. For $1 \leq i \leq (n-2)$ and using a $\otimes_k'$ to denote the product of two n-fold operads, we define that product to be given by:*

$$(\mathcal{C} \otimes_i' \mathcal{D})(j) = \mathcal{C}(j) \otimes_{i+2} \mathcal{D}(j).$$

That the product of n-fold operads is itself an n-fold operad is easy to verify once we note that the new γ is in terms of the two old ones:

$$\gamma^{pq}_{\mathcal{C} \otimes_i' \mathcal{D}} = (\gamma^{pq}_{\mathcal{C}} \otimes_{i+2} \gamma^{pq}_{\mathcal{D}}) \circ \eta^{p(i+2)} \circ (1 \otimes_p \eta^{q(i+2)})$$

where the subscripts denote the n-fold operad the γ belongs to and the η's actually stand for any of the equivalent maps which factor into them.

If A is an algebra of $\mathcal{C}$ and B is an algebra of $\mathcal{D}$ then $A \otimes_{i+2} B$ is an algebra for $\mathcal{C} \otimes_i' \mathcal{D}$.

That the product of n-fold operad algebras is itself an n-fold operad algebra is easy to verify once we note that the new θ is in terms of the two old ones:

$$\theta^{pq}_{A \otimes_{i+2} B} = (\theta^{pq}_{A} \otimes_{i+2} \theta^{pq}_{B}) \circ \eta^{p(i+2)} \circ (1 \otimes_p \eta^{q(i+2)})$$

Maps of operads and operad algebras are straightforward to describe–they are required to preserve all the structure in sight; that is to commute with γ and $\mathcal{J}$ and respectively with θ. It is left as an exercise for the reader to verify the assertion that the n-fold operads of a given category form themselves into an $(n-2)$-category, with interchange laws obeying all the axioms of Section 2 above. Basically the result follows almost immediately from the coherence of iterated monoidal categories.

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