Math 223, 9:45 section

Final Exam

8/5/16

Name: ______________________________

Problem	1	2	3	4	5	6	Total
Score							
Possible	40	40	30	30	30	30	100

NOTE: I need to see all of your work for each problem. Unjustified work will receive little or no credit.

1. (40 points) Let $f(x,y) = x^2y - x^3y^2$.

(a) (10 points) Compute the tangent plane to $f(x,y)$ at the point $(1,1)$.

$$f_x = 2xy - 3x^2y^2 \qquad f_y = x^2 - 2x^3y \xrightarrow{(1,1)} \left.\begin{matrix} f_x = -1 \\ f_y = -1 \end{matrix}\right\} \vec{n} = \langle -1,-1,-1 \rangle$$

$$z_0 = f(1,1) = 0 \qquad \Rightarrow -1(x-1) - 1(y-1) - 1(z-0) = 0$$

$$\Rightarrow \boxed{x + y + z = 2}$$

(b) (10 points) Use linear approximation to estimate $f(1.1, -.97)$.

$$L(1.1, -.97) = -1(1.1-1) - 1(-.97-1) + 0$$

$$= -0.1 + 1.97$$

$$\boxed{= 1.87}$$

(c) (10 points) Compute the directional derivative of f at $(1,1)$ in the direction of the vector $\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$.

$$D_{\vec{u}} f(1,1) = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \frac{\langle -1,-1 \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle}{1} = \boxed{\frac{-3}{\sqrt{5}}}$$

(d) (10 points) What is the maximum rate of change of f at the point $(1,1)$?

$$|\nabla f| = \sqrt{(-1)^2 + (-1)^2} = \boxed{\sqrt{2}}$$

2. (40 points) Let $\mathbf{F}(x,y) = (x+2y)\mathbf{i} + (x-2y)\mathbf{j}$. $= \langle x+2y, x-2y, 0 \rangle$

(a) (10 points) Show that $\mathbf{F}$ is NOT conservative.

$$\nabla \times \vec{F} = \langle 0, 0, 1-2 \rangle = \boxed{\langle 0,0,-1 \rangle \neq 0}$$

(b) (20 points) Compute the work done by $\mathbf{F}$ on an object moving from $(0,0)$ to $(2,4)$ along a straight line.

$$C = \begin{cases} x = 0 + 2t \\ y = 0 + 4t \\ z = 0 \end{cases} \quad t \in [0,1] \qquad \begin{aligned} dx &= 2 \\ dy &= 4 \\ dz &= 0 \end{aligned}$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_0^1 (2t + 2(4t))2 + (2t - 8t)4 \, dt = \int_0^1 (-4t)\,dt$$

$4 + 16 + 8 - 32 = -4$

$$= \left[-2t^2\right]_0^1$$

$$= \boxed{-2}$$

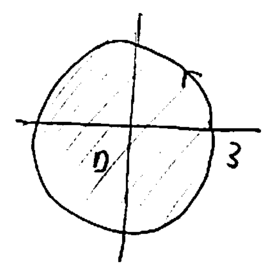
(c) (10 points) Compute the work done by $\mathbf{F}$ on an object moving along the circular path $x = 3\cos(t)$, $y = 3\sin(t)$, where $0 \le t \le 2\pi$.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot (-\vec{n})$$

$$= \int_0^{2\pi} \int_0^3 (-1)\, r\,dr\,d\theta$$

$$= -\pi 3^2$$

$$= \boxed{-9\pi}$$



xy plane

$-\vec{n} = \langle 0,0,1 \rangle$

3. (30 points) Find each of the following (10 points each)

(a) The plane that contains the points $(1,0,1)$, $(1,1,0)$, and $(0,1,,1)$.

$$\boxed{x+y+z=2}$$

(b) The point of intersection of the line $x = 2+t$, $y = -3+2t$, $z = 1-t$ with the plane $x+2y+3z = 4$,

$$2+t+2(-3+2t)+3(1-t)=4$$

$$2+t-6+4t+3-3t=4$$

$$2t-1=4$$

$$2t=5$$

$$t=5/2$$

point:

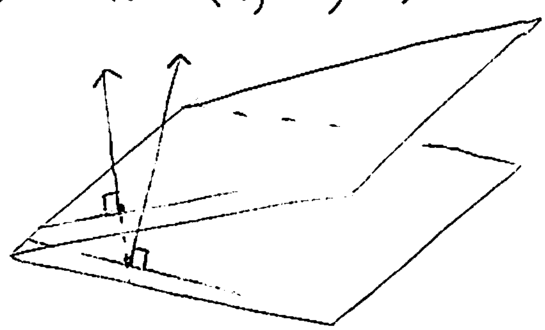
$$\left(2+5/2,\ -3+5,\ 1-\frac{5}{2}\right)$$

$$= \boxed{\left(\frac{9}{2},\ 2,\ \frac{-3}{2}\right)}$$

(c) The angle between the planes $x+2y+3z=4$ and $x-2y-3z=6$.

normals:

$$\langle 1, 2, 3\rangle \quad \text{and} \quad \langle 1, -2, -3\rangle$$



$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1-4-9}{14} = \frac{-12}{14} = -\frac{6}{7}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-6}{7}\right)}$$

4. (30 points) Let $\mathbf{r}(t) = t^2\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$.

(a) Set up, but DO NOT EVALUATE, the integral to compute the length of $\mathbf{r}(t)$ from $t = 0$ to $t = 2\pi$.

$$\mathcal{L} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt \qquad \vec{r}\,' = \langle 2t, -\sin t, \cos t \rangle$$

$$\boxed{= \int_0^{2\pi} \sqrt{4t^2 + \sin^2 t + \cos^2 t}\; dt}$$

(b) Compute $\mathbf{T}(t)$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{4t^2 + \underbrace{\sin^2 t + \cos^2 t}_{1}}} \cdot \vec{r}\,'(t) = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{-\sin t}{\sqrt{4t^2+1}}, \frac{\cos t}{\sqrt{4t^2+1}} \right\rangle$$

(c) Compute the curvature $\kappa(t)$. Don't bother to simplify your answer.

$$\kappa = \frac{\vec{T}\,'}{\vec{r}\,'} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$= \frac{\left| \langle 2t, -\sin t, \cos t \rangle \times \langle 2, -\cos t, -\sin t \rangle \right|}{\sqrt{4t^2+1}^{\,3}}$$

$$= \frac{|\langle \sin^2 t + \cos^2 t, -(-2t\sin t - 2\cos t), -2t\cos t + 2\sin t \rangle|}{\sqrt{4t^2+1}^{\,3}}$$

$$= \frac{\sqrt{1 + (2t\sin t + 2\cos t)^2 + (-2t\cos t + 2\sin t)^2}}{\sqrt{4t^2+1}^{\,3}}$$

5. (30 points) Set up, but DO NOT EVALUATE, each of the following integrals.

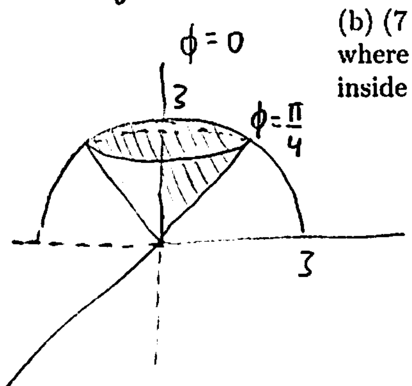
(a) (7 points) The integral(s) to find the x corrdinate of the center of mass of the top half of the circle $x^2 + y^2 = 4$ with density function $\rho(x, y) = \cos(x^2 + y^2)$. Your answer should be in polar coordinates.

SKIP

$z^2 = x^2 + y^2$

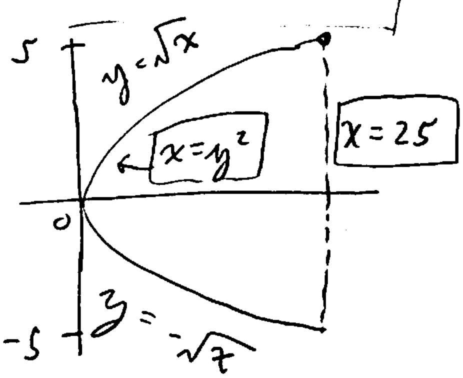
$\boxed{x = 0}$

$z = \pm y$



(b) (7 points) The integral of $f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$ over the region E, where E is the region within the top half of the sphere $x^2 + y^2 + z^2 = 9$ and inside the cone $z^2 = x^2 + y^2$. Your answer should be in spherical coordinates.

$$\boxed{\int_0^{\pi/4} \int_0^{2\pi} \int_0^{3} (\rho^2)^{3/2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$



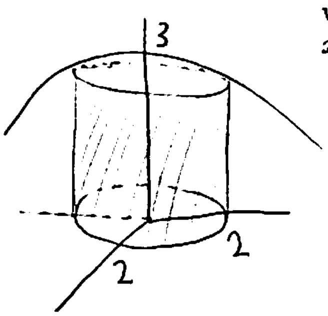
(c) (8 points) Swtich

$$\int_{x=0}^{x=25} \int_{y=-\sqrt{x}}^{\sqrt{x}} (x + y^2) dy dx$$

to $dxdy$.

$$\boxed{\int_{-5}^{5} \int_{y^2}^{25} (x + y^2) \, dx \, dy}$$

(d) (8 points) The integral of $f(x, y, z) = z(x^2 + y^2)$ over the region E, where E is within the cylinder $x^2 + y^2 = 4$ and the top half of the sphere $x^2 + y^2 + z^2 = 9$, in cyclindrical coordinates.



$$\boxed{\int_0^{2\pi} \int_0^{2} \int_0^{z = \sqrt{9 - x^2 - y^2}} (z r^2) r \, dz \, dr \, d\theta}$$

$$\left(\frac{r^4}{4}\right]_{r^2}^{4} \qquad \left[r^3 z\right]_{r^2}^{4}$$

$$\frac{4^4}{4} - \frac{r^8}{4} \qquad 4r^3 - r^6$$

$$\frac{4r^4}{4} - \frac{r^7}{7}$$

6. (30 points)

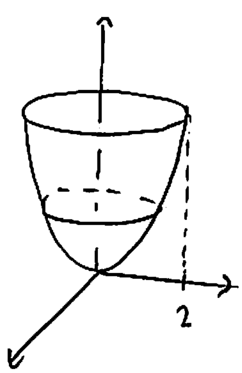
(a) (15 points) Compute $\int_S y dS$, where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.

SKIP

(b) (15 points) Use the Divergence Theorem to compute the surface integral $\mathbf{F} \cdot d\mathbf{S}$ (i.e. the flux of $\mathbf{F}$ across the surface S) if

$$\mathbf{F}(x, y, z) = (\cos(z) + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin(y) + x^2 z)\mathbf{k}$$

and S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.



$$\boxed{\nabla \cdot \vec{F} = y^2 + 0 + x^2}$$

$$\text{flux} = \int_0^{2\pi} \int_0^2 \int_{x^2+y^2}^4 (y^2 + x^2)\, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 r\, dz\, dr\, d\theta$$

Also study 3 midterm tests.

Enjoy the rest of your summer!