(1) Prove $\forall a, b \in \mathbb{Z}$, if ($a \mod 6 = 5$ and $b \mod 4 = 3$) then $4a + 6b \mod 8 = 6$. Use a Direct proof.

a) Write the assumption, translated to algebraic equations.

$\boxed{a = 6m + 5 \quad \text{and} \quad b = 4k+3}$

b) Write what we want to show, translated to algebraic equations.

$\boxed{4a + 6b = 8p + 6}$

c) Write the proof steps.

$$\begin{aligned} 4a + 6b &= 4(6m+5) + 6(4k+3) \\ &= 24m + 20 + 24k + 18 \\ &= 24m + 24k + 32 + 6 \\ &= \boxed{8(3m + 3k + 4) + 6} \end{aligned}$$

(2) Suppose we were to prove the statement "$\forall y \in \mathbb{Z}$, y is even $\Rightarrow (y^3 - 1)$ is odd." (Answer using alegebraic equations, without using the word "not" or the symbol "$\sim$.")

a)For a direct proof we assume $y = 2k$ and show $y^3 - 1 = 2m+1$.

b) For proof using the contrapositive we assume $y^3 - 1 = 2p$ and show $y = 2q + 1$.

c) For proof by contradiction we assume $y = 2k$ and $y^3 - 1 = 2m$ and show that we reach a false conclusion.

(3) Use contradiction to prove: $\forall a, b \in \mathbb{Z}$, if a is even and b is odd then 4 does not divide $(a^2 + 2b^2)$.

a) Negate the statement.

$\boxed{\exists a, b \in \mathbb{Z} \text{ s.t. } a \text{ is even and } b \text{ is odd and } 4 \mid (a^2 + 2b^2).}$

b) What do we assume? Translate to algebraic equations.

$\boxed{a = 2k}$ and $\boxed{b = 2m+1}$ and $\boxed{a^2 + 2b^2 = 4p}$

c) Use the assumptions to prove that $4|2$, as an algebraic equation.

$$\begin{aligned} & a^2 + 2b^2 = 4p \\ \Rightarrow\ & (2k)^2 + 2(2m+1)^2 = 4p \\ \Rightarrow\ & 4k^2 + 2(4m^2 + 4m + 1) = 4p \end{aligned}$$

$$\begin{aligned} & 4(k^2 + 2m^2 + 2m) + 2 = 4p \\ \Rightarrow\ & 4p - 4(k^2 + 2m^2 + 2m) = 2 \\ \Rightarrow\ & \boxed{4(p - k^2 - 2m^2 - 2m) = 2} \end{aligned}$$

$\square$

(4) Prove by induction that: $\forall n \in \mathbb{N}$, if $n \geq 2$ then $3|(2^{(4n-4)} + 2^{(2n-3)})$.

a) Show the base case.

Base case: $n = 2$: $\boxed{2^4 + 2^1 = 18 = 3(6).}$

b) State the induction assumption, translate to algebraic equations.

$$\boxed{2^{(4k-4)} + 2^{(2k-3)} = 3m.}$$

c) State what we need to show, translate to algebraic equations.

$$\boxed{2^{(4(k+1)-4)} + 2^{(2(k+1)-3)} = 3q}$$

d) Do the proof steps.

Proof.

$$2^{(4(k+1)-4)} + 2^{(2(k+1)-3)} = 16(2^{(4k-4)}) + 4(2^{(2k-3)})$$

$$= 15(2^{(4k-4)}) + 3(2^{(2k-3)}) + 2^{(4k-4)} + 2^{(2k-3)}$$

$$= 15(2^{(4k-4)}) + 3(2^{(2k-3)}) + 3m$$

$$\boxed{= 3(5(2^{(4k-4)}) + 2^{(2k-3)} + m).}$$

$\square$

(5) Use a Direct proof to prove: $\forall z \in \mathbb{Z}, 3|(z+1) \Rightarrow z^2 \mod 3 = 1$.

a) Write the assumption, translated to algebraic equations.

$$\boxed{z+1 = 3k}$$

b) Write what to show, translated to algebraic equations.

$$\boxed{z^2 = 3m + 1}$$

c) Do the proof steps.

$$z^2 = (3k-1)^2$$

$$= 9k^2 - 6k + 1$$

$$\boxed{= 3(3k^2 - 2k) + 1}$$

$\square$