Chp2 **Linear combinations**

Given several vectors $\vec{x}, \vec{y}, \vec{z}, \vec{w} \ldots$

a linear combination is : choosing scalar multipliers $c_1, c_2, c_3 \ldots$ for each, and then adding them up like this:

$$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} + c_4\vec{w} + \ldots$$

→ We have seen this already: a system of linear equations (with constant term) can be described as a lin. comb. of the coefficient vectors with variable multipliers.

ex)

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 5 \\ 2x_1 \qquad + 4x_3 = 2 \end{cases}$$

equals

$$\left.\underbrace{x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\0\end{pmatrix} + x_3\begin{pmatrix}-3\\4\end{pmatrix}}_{\text{lin. comb.}} = \begin{pmatrix}5\\2\end{pmatrix}\right\} \text{affine vector equation}$$

→ And we saw it as a way to write the general solution with free variables.

$$\left[\begin{array}{ccc|c}1 & 2 & -3 & 5\\2 & 0 & 4 & 2\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & -3 & 5\\0 & -4 & 10 & -8\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\0 & 1 & -5/2 & 2\end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 - 5/2\, x_3 &= 2 \\ x_3 &= x_3 \end{aligned}\right\} \quad \left.\begin{aligned} x_1 &= -2x_3 + 1 \\ x_2 &= \tfrac{5}{2}x_3 + 2 \\ x_3 &= x_3 \end{aligned}\right\} \quad \vec{x} = \underbrace{x_3\begin{pmatrix}-2\\5/2\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix}}_{\text{lin. comb.}}$$

Linear Dependence & Independence

$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

→ a set of vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_n$ is $\boxed{\text{linearly dependent}}$ when there exists a set of scalars $c_1, c_2, \dots, c_n$ (which are not all equal to 0) such that $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$.

→ that same set of vectors is $\boxed{\text{linearly independent}}$ if there is no such set of scalars, that is, $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$ only when $c_i = 0$ for all $i = 1, \dots, n$.

Ex) Are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ lin. dep. or lin. indep.?

Solve $c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Same as solving this system:

$$\left.\begin{aligned} c_1 + 4c_2 - 2c_3 &= 0 \\ 2c_1 \quad\;\; - 4c_3 &= 0 \\ 3c_1 + 7c_2 - 6c_3 &= 0 \end{aligned}\right\} \text{homogeneous}$$

Same as solving: $A\vec{c} = \vec{0}$ with $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} \overbrace{1 \quad 4 \quad -2}^{A} & & & 0 \\ 2 & 0 & -4 & 0 \\ 3 & 7 & -6 & 0 \end{array}\right]$$

Same as finding intersection of 3 homogeneous planes. Note $\vec{c} = \vec{0}$ is definitely a solution!

$\rightarrow$ Check that: $0\begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}4\\0\\7\end{pmatrix} + 0\begin{pmatrix}-2\\-4\\-6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

(This is always true: $A\vec{x} = \vec{0}$ always has at least one solution, $\vec{x} = \vec{0}$)

But: there could still be either 1 solution or ∞ solutions.

* Lin. dep. is another term for ∞ solutions to the "lin. comb. $= \vec{0}$" equation. Lin. indep. is a term for 1 unique solution, $\vec{0}$.

For practice, solve it!

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 2 & 0 & -4 & 0 \\ 3 & 7 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} c_1 - 2c_3 = 0 \\ c_2 = 0 \\ c_3 = c_3 \text{ free} \end{cases} \quad \begin{cases} c_1 = 2c_3 \\ c_2 = 0 \\ c_3 = c_3 \end{cases}$$

General solution $\vec{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = c_3\begin{pmatrix}2\\0\\1\end{pmatrix}$

Note: homogeneous systems never have a non-$\vec{0}$ constant vector added to their solution.

Specific solutions: pick any c_3.

$c_3 = 1 \Rightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix}$, so

$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\0\\7\end{pmatrix}, \begin{pmatrix}-2\\-4\\-6\end{pmatrix}$ are lin. dep.

So to decide lin. dep. or lin. indep,
we can always solve the vector equation
Unique solution $\vec{0}$ $\Longrightarrow$ lin. indep.
∞ solution (free variables) $\Rightarrow$ lin. dep.

Short cuts!
For $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ all vectors in $\mathbb{R}^m$
there are several shortcuts:

- if one of them (or more) is $\vec{x}_i = \vec{0}$, then lin. dep.

- if one of them (or more) is a scalar times another
$\vec{x}_i = c\vec{x}_j$, then lin dep.
[see previous example: $\vec{x}_3 = -2\vec{x}_1$]

- if one of them can be found as a lin. comb. of the others
$\vec{x}_i = c_j\vec{x}_j + \dots + c_k\vec{x}_k$, then lin. dep.
[here, the converse is also true.]

- if the number of vectors is larger than the number of components (dimension) of each $(n > m)$, then lin. dep.

- if $n = m$ and $\det\left[\vec{x}_1\ \vec{x}_2 \dots \vec{x}_n\right] = 0$
then lin. dep.
and if that $\det. \neq 0$, then lin. indep.

ex) "{ ... }" means "the set of"

1) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $n = 4$, $m = 3$

→ $\boxed{\text{lin. dep.}}$ since $4 > 3$.

→ ∞ solutions to $A\vec{x} = \vec{0}$ if A is the matrix with these columns

→ at least one of these can be made as a lin. comb. of the others

2) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right\}$

→ $\boxed{\text{lin. indep.}}$ since $\det \overbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -1 \end{bmatrix}}^{A} = -3 \neq 0$

→ only one solution $\vec{x} = \vec{0}$ to $A\vec{x} = \vec{0}$.

→ none of these can be made as a lin. comb. of the other two.

3) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right\}$

→ $\boxed{\text{lin. indep.}}$ since if $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ is made as a lin. comb. of $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ that would mean

$\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, so $\left.\begin{matrix} 2 = c1 \\ 4 = c2 \end{matrix}\right\} c = 2$

$3 = c0 \rightarrow c = 2$ fails. $(3 \neq 0)$

→ two vectors are lin. dep. only when parallel; $\vec{x}_2 = c\vec{x}_1$.

ex 4) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} \right\}$

$\to$ lin. dep. since $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$

5) $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\to$ columns are lin. dep. $4 > 3$

$\to$ rows are lin. dep., since one row is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$.

6) $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = A$

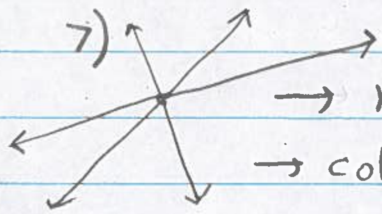
$\to$ columns are lin. dep. since one is $\vec{0}$

$\to$ det $A = 0$

$\to$ det $A^t = 0$

$\to$ rows are lin. dep.

$\to$ for square matrix $n \times n$ the columns and rows are either both lin. dep. or both lin. indep.

7) 

$\to$ system $\to$ $A\vec{x} = \vec{b}$; A 3×2

$\to$ rows of A are lin. indep. ($3 > 2$)

$\to$ columns of A are lin. indep. (one solution)