Linear Algebra
chp. 1

Examples:

$3, 5, -1, \frac{1}{2}, 1$ scalar coefficients

① system of equations
$$\begin{cases} 3x + 5y - z = 2 \\ x - y = 0 \\ y + \frac{1}{2}z = 1 \end{cases}$$

x, y, z scalar variables

$2, 0, 1$ scalar constants

homogeneous linear equation

*affine linear equation

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②

homogeneous system of linear equations
$$\begin{cases} 3x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_3 - x_2 = 0 \end{cases}$$

(Alternate scalar variables $x_1, x_2, x_3, \ldots$)

Solve: simultaneous solution (x_1, x_2, x_3)

(1) Subtract equations:
$$\begin{array}{r} x_1 + x_3 = 0 \\ - (x_1 + x_3 - x_2 = 0) \\ \hline \Rightarrow \quad \boxed{x_2 = 0} \end{array}$$

(2) Substitute back: $3x_1 - 0 = 0$ $\quad$ $0 + x_3 = 0$

$\Rightarrow \boxed{x_1 = 0}$ $\qquad$ $\boxed{x_3 = 0}$

$(x_1, x_2, x_3) = (0,0,0)$ makes all 3 true.

Solving with a matrix of coefficients
→ same as combining (subtracting) equations to eliminate variables
→ Three allowed moves, to break it down:

① Switch 2 rows (just like reordering equations)
$$R_3 \leftrightarrow R_5$$

Row Reduction Moves

② Replace a row with a multiple of _itself_ $R_5 \leftarrow -\frac{2}{3} R_5$

③ Replace a _row_ with a combination of _itself_ with another row.
$$R_7 \leftarrow R_7 + 2R_5$$

Rows: Equations
R_1
R_2
R_3

Matrix A

Ex:
$$3x_1 - x_2 = 0$$
$$x_1 + x_3 = 2$$
$$x_1 + x_3 - x_2 = 1$$

$$\begin{bmatrix} 3 & -1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 2 \\ 1 & -1 & 1 & | & 1 \end{bmatrix}$$

Columns
coeffs of x_1, x_2, x_3 | augmented constants

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 1 & 0 & 1 & | & 2 \\ 3 & -1 & 0 & | & 0 \end{bmatrix}$$

$R_3 \leftarrow -\frac{1}{3} R_3$
$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 5/3 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - 3R_1$
$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 2 & -3 & | & -3 \end{bmatrix}$$

$R_1 \leftarrow R_1 + R_2$
$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 5/3 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 2R_2$
$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & -3 & | & -5 \end{bmatrix}$$

$R_1 \leftarrow R_1 - R_3$
$$\begin{bmatrix} 1 & 0 & 0 & | & 1/3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 5/3 \end{bmatrix}$$

$\cdots$ so $\begin{cases} x_1 = 1/3 \\ x_2 = 1 \\ x_3 = 5/3 \end{cases}$

Check: these make all three original
equations true

$$3\left(\tfrac{1}{3}\right) - 1 = 0 \quad \checkmark$$
$$1/3 + 5/3 = 2 \quad \checkmark$$
$$1/3 + 5/3 - 1 = 1 \quad \checkmark$$