Learning Math through Puzzles

by **Stephen Fratini**

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Preface

"Tell me and I forget, teach me and I may remember, involve me and I learn."
Benjamin Franklin

"I think the big mistake in schools is trying to teach children anything, and by using fear as the basic motivation. Fear of getting failing grades, fear of not staying with your class, etc. Interest can produce learning on a scale compared to fear as a nuclear explosion to a firecracker."
Stanley Kubrick

This book offers a collection of mathematical puzzles that aim to instruct and challenge readers interested in basic mathematics. The puzzles cover a wide range of topics, including number properties, counting, sequences and series, algebra, functions, and mathematical reasoning. There are a limited number of puzzles in the areas of probability and geometry (I hope to cover these topics more fully in a future puzzle book). While this is not intended as a textbook, I have added instructional background material for many of the puzzles. The goal is to both instruct the reader in various aspects of basic mathematics and reasoning, and to challenge, engage and delight the reader with interesting puzzles.

The intended audience includes people with a general interest in puzzles of the mathematical type. This book can be used as a supplement to a high school course in algebra or pre-calculus, or for self-study by those wanting to stay mentally sharp, and who like a challenge.

Many of the puzzles come from various national and international mathematics competitions, and as such, these puzzles are usually not easy. Even so, I strongly recommend trying to solve a puzzle before reading the solution. In any event, detailed solutions are provided for almost all the puzzles in the book.

"For me, I am driven by two main philosophies: know more today about the world than I knew yesterday and lessen the suffering of others. You'd be surprised how far that gets you."
Neil deGrasse Tyson

"What gets us into trouble is not what we don't know. It's what we know for sure that just ain't so."
Mark Twain

"Self-education is, I firmly believe, the only kind of education there is."
Isaac Asimov

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Other books by the author:

- *The Art of Managing Things (2nd edition)*, self-published on Amazon, https://www.amazon.com/Art-Managing-Things-Stephen-Fratini-ebook/dp/B07N4H4YWH/, January 2019.
- *Mathematical Thinking: Exercises for the Mind*, self-published on Amazon, https://www.amazon.com/Mathematical-Thinking-Exercises-Stephen-Fratini-ebook/dp/B08F75CDD6/, August 2020.
- *Financial Mathematics with Python*, self-published on Amazon, https://www.amazon.com/gp/product/B08VKQR141, February 2021.
- *Math in Art, and Art in Math*, self-published on Amazon, https://www.amazon.com/dp/B091D1F8MB, March 2021.
- *Algebra through Discovery and Experimentation*, self-published on Amazon, https://www.amazon.com/dp/B09B5L9WL5, July 2021.
- *The Struggle Against Chaos*, self-published on Amazon, https://www.amazon.com/dp/B09BLPQ86Q, July 2021.
- *Mathematical Vignettes*, self-published on Amazon, https://www.amazon.com/dp/B0BBQ9YKXW, August 2022.

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1 Introduction

"Without the element of enjoyment, it is not worth trying to excel at anything."

Magnus Carlsen (chess champion)

1.1 Overview

The mathematical puzzles in this book cover topics concerning number properties, counting (i.e., combinatorics), sequences and series, algebra, functions, and basic mathematical reasoning.

The following is a summary of the sections in this book:

- Section 1 is this introduction.
- Section 2 provides some background material concerning topics that arise in several places later in the book.
    - Section 2.1 covers a method of mathematical proof known as "finite induction". This technique is used in several places to prove results.
    - While the intention was to not include calculus puzzles, it was necessary to use the concept of a limit (from calculus) in a few of the puzzles. Section 2.2 provides a short introduction to limits.
- Section 3 contains puzzles concerning number properties (e.g., puzzles related to prime numbers), combinators, sequences, infinite series and infinite products.
- Section 4 contains algebra puzzles with a focus on word problems.
- Section 5 is about functions. Section 5.1 provides a brief overview of functions, and Section 5.2 has function puzzles.
- Section 6 has a large collection of puzzles that mostly require logical reasoning (and a limited amount of mathematics).

1.2 Intended Audience

The intended audience includes people with a general interest in puzzles of the mathematical type. This book can be used as a supplement to a high school course in algebra or pre-calculus, or for self-study by those wanting to stay mentally sharp, and who like a challenge.

1.3 Prerequisites

The main prerequisites are a knowledge of arithmetic and basic algebra. As noted, background material is provided in Section 2, and as needed, in other places throughout the book.

It would be helpful to have a knowledge of the various sets of numbers, i.e.,

- natural or counting numbers, i.e., the set $\{1,2,3\ldots\}$ where 0 is sometimes included
- rational numbers (fractions and mixed numbers), e.g., $\frac{7}{12}$ and $3\,\frac{2}{5}$
- irrational numbers are numbers that cannot be expressed as the quotient of two integers
- real numbers include all the rational and irrational numbers

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- complex numbers, i.e., numbers of the form $a + ib$ where $i = \sqrt{-1}$

1.4 Conventions

In places where I state an opinion, I start my comment with "**Author's Remark**"; otherwise, I've tried to stick to the facts.

Various notations are used for multiplication, e.g., "two times three" can be written as $2 \cdot 3$ or $2 * 3$ or $2(3)$ or 2×3. In other cases, juxtaposition is used to indicate multiplication, e.g., xy means "the variable x times the variable y."

Standard symbols are used to represent common sets of numbers, e.g., $\mathbb{R}$ for the real numbers and $\mathbb{N}$ for the natural (or counting) numbers. A list of symbols is provided on the Symbols page toward the back of the book.

The logarithm base e is known as the **natural logarithm** and is written as $ln\, x$ or (less commonly) as $log_e\, x$.

Proofs (of which there are not many in this book) are ended with the symbol ■

Sources are listed for many of the puzzles. If no source is listed, either the puzzle comes from the author or the puzzle is in many places on the Internet (or elsewhere) with no clear point of origination.

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2 Background Material

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

Carl Friedrich Gauss

It is not recommended to read this section until a particular item is referenced later in the document. The purpose of this section is to provide brief overviews of some technical items, the understanding of which is necessary for some of the puzzles that follow.

2.1 Principle of Finite Induction

The principle of finite induction is very important in mathematics as it is used to prove many theorems, including several theorems in this book. By way of analogy, finite induction is like dominoes. If you know (for a given configuration of dominoes)

- the dominoes are equally spaced so that if any given domino falls, the next will fall and so on, and
- the first domino has fallen,

then you can conclude eventually every domino in the configuration will fall.

In finite induction, we have a statement with positive integer variable n (rather than a domino). If the statement can be shown true for $n = 1$, and one can prove that "if the statement is true for $n = k$ then the statement is true for $n = k + 1$," then finite induction tells us the statement is true for all values of n.

Another analog comes from the book "Concrete Mathematics" [1]:

> Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).

Although its name may suggest otherwise, mathematical induction is not considered to be a form of inductive reasoning (i.e., the process of making inferences based upon observed patterns). Mathematical induction is, in fact, an example of deductive reasoning (i.e., reasoning from the general to the particular). The confusing terminology is unfortunate but the terms "inductive reasoning" and "mathematical induction" are firmly embedded in the literature and not likely to change.

The principle of finite induction is stated more formally in the following theorem. This may seem terribly obvious but the concept is very powerful.

***Theorem 1**. (First Principle of Finite Induction) Let S be a set of positive integers such that*

- $1 \in S$
- *whenever $k \in S$, it must be that $k + 1 \in S$*

then S is necessarily the set of all positive integers.

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Notes:

- If need be, we could start with an integer larger than 1 (say n_0) in the first condition of the theorem and keep the second condition as-is. In this case, the implication of the theorem would be that S is the set of all integers greater than or equal to n_0.
- For the second condition, we could (for example) assume $k-1 \in S$ and show $k \in S$, or assume $k+1 \in S$ and show $k+2 \in S$. The point is to show that if an arbitrary integer is in S then so is the next integer.

Proof: By way of contradiction, assume that the set T (of all positive integers not in S) is nonempty. By the well-ordering principle, T must have a least element (call it x). We are given that $1 \in S$ and so it must be that $x > 1$ and thus, $0 < x-1 < x$. Since x is the least element in T, $x-1 \notin T$ which implies that $x-1 \in S$. By hypothesis, S must contain $(x-1)+1 = x$ which contradicts the fact that $x \in T$. So, T must be empty and thus S is the set of all positive integers. ■

As an easy illustration of the first principle of finite induction, we prove that the sum of the first n odd numbers is n^2, i.e., $1+3+5+\cdots+(2n-1) = n^2$ (for all positive integer values of n).

Proof: Clearly, the formula holds for $n=1$. Assume the formula is true for $n=k$, i.e., $1+3+5+\cdots+(2k-1) = k^2$. Consider the case of $n=k+1$, i.e., $[1+3+5+\cdots+(2k-1)] + (2k+1) = k^2 + (2k+1) = (k+1)^2$ (which was to be proved). ■

There is an alternate version of the principle of finite induction that strengthens the second hypothesis. This is often referred to as **strong induction**.

Theorem 2. *(Second Principle of Finite Induction) Let S be a set of positive integers such that*

- $1 \in S$
- *whenever* $1,2,\ldots,k \in S$, *it must be that* $k+1 \in S$

then S is necessarily the set of all positive integers.

Both of the notes to ***Theorem 1*** apply here.

Proof: By way of contradiction, assume that set T (of all positive integers not in S) is nonempty. By the well-ordering principle, T must have a least element (call it x). By hypothesis, x must be greater than 1. Further, since x is the least element in T, it must be that $1,2,\ldots,(x-1)$ are not in T and are thus in S. But the second hypothesis implies that $(x-1)+1 = x \in S$ which contradicts the fact that $x \in T$. So, T must be empty and thus S is the set of all positive integers. ■

Some statements do require the second principle of finite induction (as opposed to the first principle of induction). For example, consider the Lucas sequence: 1,3,4,7,11,18,29,47,76, …

The general pattern (after the first two terms) is $x_n = x_{n-1} + x_{n-2}$ (basically add the previous two numbers to get the next number in the sequence). In the following proof, we use the second principle of finite induction to show that $x_n < \left(\frac{7}{4}\right)^n$.

Proof: For $n=1$, it is clear that $x_1 = 1 < \left(\frac{7}{4}\right)^1$ and $x_2 = 3 < \left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\,\frac{1}{16}$.

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Next, assume that the statement holds for $n = 1,2, \dots, k-1$. This implies that

$$x_{k-1} < \left(\frac{7}{4}\right)^{k-1}, \qquad x_{k-2} < \left(\frac{7}{4}\right)^{k-2}$$

It then follows that

$$x_k = x_{k-1} + x_{k-2} < \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2} = \left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}+1\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{11}{4}\right) < \left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}\right)^{2} = \left(\frac{7}{4}\right)^{k}$$

Thus, given that the statement is true for $n = 1,2, \dots, k-1$, we have proven the statement true for the case $n = k$, and thus, the statement must therefore hold true for all values of n by the second principle of finite induction. ■

2.2 Limits (from Calculus)

Simply put, a limit is the value that an expression (e.g., a function or sequence of numbers) approaches as the input to the expression approaches some value. In what follows, several basic examples of limits are given. For a good summary of the various properties of limits and a listing of limits for some common functions, see the Wikipedia article "List of limits" [2].

2.2.1 Example 1: f(x) = x/(x+10)

For example, the limit of $f(x) = \frac{x}{x+10}$ is 1 as x approaches infinity. In notation, this is written as

$$\lim_{x \to \infty} \frac{x}{x+10} = 1$$

For large values of x, the numerator and denominator are almost the same (with ratio slightly less than 1). This formula can be proven but such details will not be covered in this overview.

Continuing with this example, one can see that $f(x)$ approaches 1 (from above) as x approaches negative infinity. Thus, we can say $\lim_{x \to -\infty} \frac{x}{x+10} = 1$.

If we observe the graph of $f(x)$ (see Figure 1), there are some additional points of interest. While $f(x)$ is not defined at $x = -10$ (division by zero is not allowed), it is still possible to define left and right-hand limits. Approaching $x = -10$ from the right (i.e., values larger than -10), we can see that $f(x)$ takes on larger and larger negative values, as shown in Table 1. So, we have that $\lim_{x \to -10^+} \frac{x}{x+10} = -\infty$ where the notation $x \to -10^+$ means that x approaches -10 from the right. If we approach -10 from the left (i.e., values smaller than -10), we get $\lim_{x \to -10^-} \frac{x}{x+10} = \infty$ where the notation $x \to -10^-$ means that x approaches -10 from the left.

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Table 1. Approaching x = -10 from the left

x	-9	-9.9	-9.99	-9.999	…
$f(x)$	-9	-99	-999	-9999	…

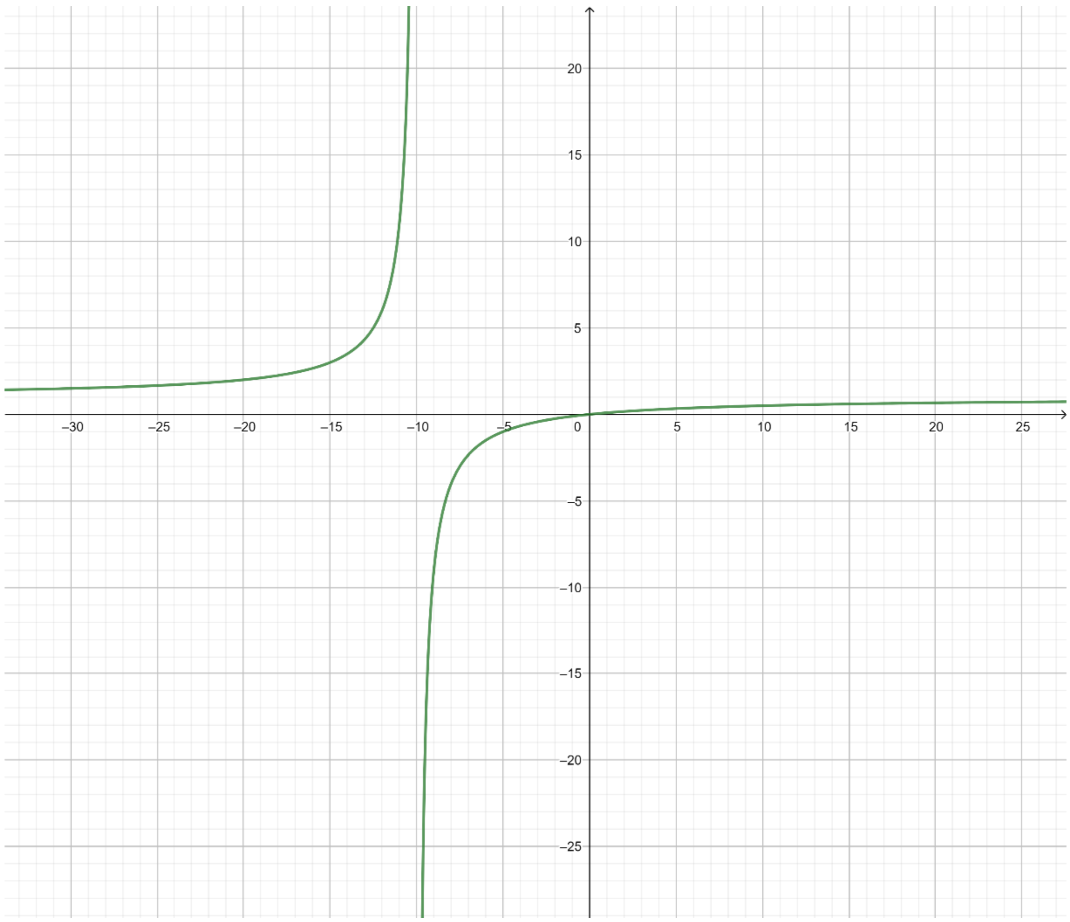

Figure 1. Limit example concerning f(x) = x/(x+10)

2.2.2 Example 2: Missing Point in Straight Line

As another example, consider the function $g(x) = \frac{x^2-1}{x-1}$ which is undefined at $x = 1$ since division by 0 is undefined. When $x \neq 0$, we can simplify the function as follows:

$$g(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1$$

The graph is basically that of $y = x + 1$ with a gap at the point $(1{,}2)$, shown as point A in Figure 2. Further, we have that $\lim\limits_{x \to 1} \frac{x^2-1}{x-1} = 2$. So, $g(1)$ is undefined but the $\lim\limits_{x \to 1} g(x) = 2$.

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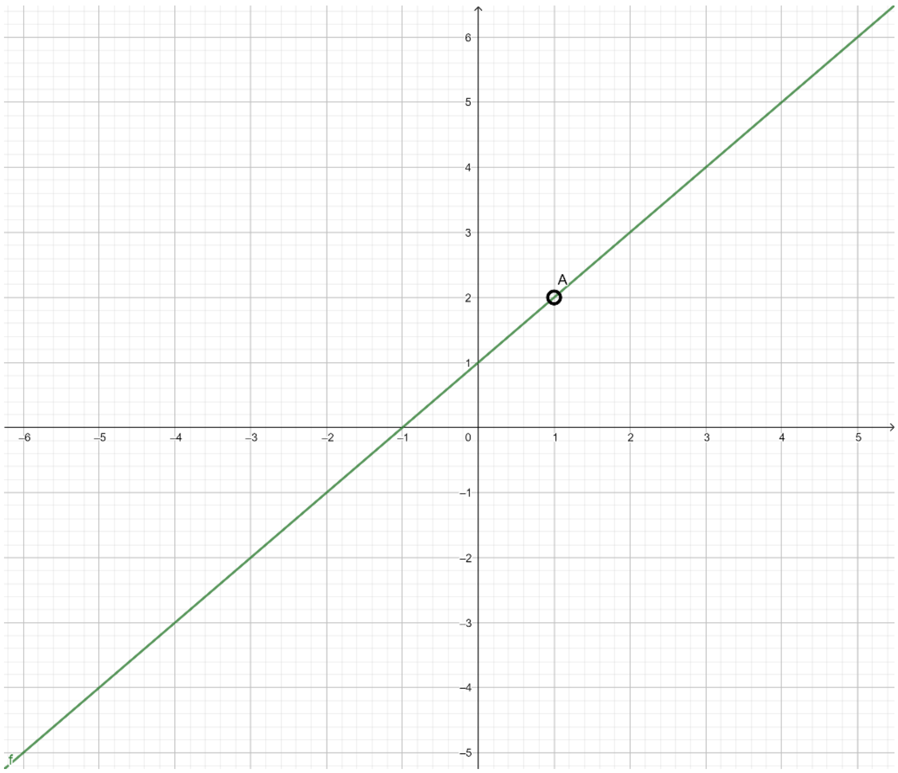

Figure 2. Limit example concerning g(x) = (x^2 - 1) / (x-1)

[**Author's Remark**: The equation in Figure 2 is that of $g(x) = \frac{x^2-1}{x-1}$. The alternate notation is a restriction imposed by the editor used to create the document.]

2.2.3 Example 3: Continuous Interest and Euler's Number

Consider an initial investment (P) in an interest bearing bank account. Let r be the nominal annual interest rate, n be the number of times per year interest is compounded, and t be the amount of time in years (or fraction of a year in increments of $\frac{1}{n}$) that interest is accrued. The principal grows as shown in Table 2.

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Table 2. Compound Interest Formula

Time Period	Principal
0	P
1	$P + P\left(\frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)$
2	$P\left(1 + \frac{r}{n}\right) + \left(\frac{r}{n}\right)P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$
3	$P\left(1 + \frac{r}{n}\right)^2 + \left(\frac{r}{n}\right)P\left(1 + \frac{r}{n}\right)^2 = P\left(1 + \frac{r}{n}\right)^3$
…	…
$\boldsymbol{nt}$	$P\left(1 + \frac{r}{n}\right)^{nt}$

What happens if interest is compounded an infinite number of times per year, i.e., $n \longrightarrow \infty$? It turns out that the formula actually converges, i.e.,

$$\lim_{n \longrightarrow \infty} P\left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$$

The term e in the above formula appears frequently in mathematics and is known as **Euler's number** [3]. It is an irrational (actually transcendental) number whose value is approximately 2.7182818284. There are many ways to define e, with one of the most common definitions being $e = \lim_{n \longrightarrow \infty} P\left(1 + \frac{1}{n}\right)^n$. Another representation of e is given in the following theorem (which will be needed later in this book).

Theorem 3. $e = \sum_{i=0}^{\infty} \frac{1}{i!}$

Proof: From the binomial theorem [4], we have that

$$\left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{n}\right)^i$$

where

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!} \text{ and } n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

Next, note that

$$\binom{n}{i}\left(\frac{1}{n}\right)^i = \frac{n!}{(n-i)!\,i!} \cdot \frac{1}{n^i} = \left(\frac{1}{i!}\right)\frac{n(n-1)\ldots(n-i+1)}{n^i} = \left(\frac{1}{i!}\right) \cdot 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots(1 - \frac{i-1}{n})$$

As $n \longrightarrow \infty$, the terms to the right of $\left(\frac{1}{i!}\right)$ in the above equation all converge to 1.

So, we have

$$e = \lim_{n \longrightarrow \infty}\left(1 + \frac{1}{n}\right)^n = \lim_{n \longrightarrow \infty} \sum_{i=0}^{n} \binom{n}{i}\left(\frac{1}{n}\right)^i = \sum_{i=0}^{\infty} \lim_{n \longrightarrow \infty} \binom{n}{i}\left(\frac{1}{n}\right)^i = \sum_{i=0}^{\infty} \frac{1}{i!}$$

which was to be proved. ■

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3 Numbers, Counting, Sequences and Series

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.

David Hilbert

3.1 Number Theory

3.1.1 Exact Change

Puzzle: A hypothetical country has 7 cent and 11 cent coins. Assuming an unlimited number of each type of coin, what is the largest amount for which exact change cannot be made?

Solution: Clearly, we cannot make exact change for 1,2,3,4,5, or 6 cents since the smallest coin in this currency is 7 cents. Table 3 shows the first few cases where it is possible to make exact change using 7 and 11 cent coins, e.g., the amount of 29 can be achieved with one 7-cent coin and two 11-cent coins.

Table 3. Exact change with 7 and 11 cent coins

7	11	14	18	21	22	25	28	29	32	33	35
$1 \cdot 7$	$1 \cdot 11$	$2 \cdot 7$	$7 + 11$	$3 \cdot 7$	$2 \cdot 11$	$2 \cdot 7 + 11$	$2 \cdot 14$	$7 + 2 \cdot 11$	$3 \cdot 7 + 11$	$3 \cdot 11$	$5 \cdot 7$

One cannot make exact change for 59, i.e., the equation $7x + 11y = 59$ has no solutions with both x and y being positive integers. This can be verified by trying $x = 0,1,2,3,4,5,6,7,8$ in the above equation and noting that no positive integer value for y will satisfy the equation. For example, if we take $x = 3$, the equation reduces to $11y = 38$ which has no positive integer solution.

After 59, the next 7 numbers can all be written as linear combinations of 7 and 11, as shown in Table 4. After that, we can get the next 7 numbers by just adding 7 to the numbers from 60 to 66. Clearly, this process can continue indefinitely. So, we can conclude that 59 is the largest number for which one cannot make exact change using 7 and 11 cent coins.

Table 4. Exact change for 60 to 66

60	61	62	63	64	65	66
$7 \cdot 7 + 11$	$4 \cdot 7 + 3 \cdot 11$	$7 + 5 \cdot 11$	$9 \cdot 7$	$6 \cdot 7 + 2 \cdot 11$	$3 \cdot 7 + 4 \cdot 11$	$6 \cdot 11$

. . .

Puzzle: Same question as the previous puzzle, but with an unlimited number of 5-cent and 8-cent coins.

Solution: In general, if we have coins of value a and b such that the greatest common divisor of a and b is 1, the largest number for which one cannot make exact change is $ab - a - b$. This formula was discovered by J.J. Sylvester [43].

For the problem at hand, let $a = 5$ and $b = 8$. The above formula yields the solution $5(8) - 5 - 8 = 27$. The reader may want to verify that $28, 29, 30, 31$ and 32 can all be represented as some combination of 5-cent and 8-cent coins.

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The problem can be extended to coins (basically positive integers) of n different denominations $a_1, a_2, \dots, a_n$ such that the greatest common divisor of the numbers is 1. In this most general case, there is no closed formula but rather many different algorithms that one can use to determine the largest number which cannot be written as a non-negative linear combination of $a_1, a_2, \dots, a_n$. This is known as the Diophantine Frobenius problem [44].

. . .

Puzzle: What if the denominations of the coins have a common divisors, e.g., 2-cent and 6-cent coins? Is there a largest number for which one cannot make change?

Solution: Clearly, there is no way to make change for any odd number since the sum of two even numbers (in this case combinations of 2 and 6) is always even.

3.1.2 Modular Arithmetic

For a given positive integer n, it is possible to divide all integers into n partitions known as **congruence** classes. Relative to the selection of n, two integers a and b are in the same congruence class if $a - b = nk$, where k is an integer. In this case, we say that a is congruent to b modulo n and write this as

$$a \equiv b \pmod{n}$$

If we take $n = 5$, there are five congruence classes, i.e.,

$$\overline{0} = \{\dots, -10, -5, 0, 5, 10, \dots\}$$

$$\overline{1} = \{\dots, -9, -4, 1, 6, 11, \dots\}$$

$$\overline{2} = \{\dots, -8, -3, 2, 7, 12, \dots\}$$

$$\overline{3} = \{\dots, -7, -2, 3, 8, 13, \dots\}$$

$$\overline{4} = \{\dots, -6, -1, 4, 9, 14, \dots\}$$

In the above example and in general (for any value of n), each integer is in one and only one congruence class.

The set of congruence class modulo n, i.e., $\{\overline{0}, \overline{1}, \overline{2}, \dots, \overline{n-1}\}$ is denoted by the symbol $\mathbb{Z}_n$.

Addition in $\mathbb{Z}_n$ is modulo n. For example, if $n = 9$, then $\overline{4} + \overline{7} = \overline{2}$ since

$$4 + 7 \equiv 11 \equiv 2 \pmod{9}$$

In the above equation and in general, we usually only write the modulo sign, i.e., $\equiv$, once at the end.

Subtraction works in an analogous way. For example,

$$17 - 24 \equiv -7 \equiv 3 \pmod{10}$$

For multiplication, just multiply the integers and then reduce relative to the associated modulo. For example,

$$6 \cdot 19 \equiv 114 \equiv 2 \pmod{7}$$

Note that the remainder of 114 divided by 7 is 2 and thus, the above result.

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Puzzle: Place 268 numbers around a circle, such that the sum of 20 consecutive numbers is always equal to 75. The number 3, 4 and 9 are written in positions 17, 83 and 144 respectively. Find the number in position 210.

Source: Problem #3 in the Pan African Mathematics Olympiad 2004, http://africamathunion.org/PAMO_2004_Problems_En.pdf.

Hint: The same numbers repeat several times.

Solution: Since we don't yet know the numbers around the circle, we will use the variables $a_0, a_1, a_2, \dots, a_{267}$. The indices are modulo 268. Clearly, $a_{n\,(\mathrm{mod}\,268)} = a_{n+268\,(\mathrm{mod}\,268)}$, i.e., if you move 268 places around the circle, you are back to the same place. For convenience, we will drop the $(\mathrm{mod}\ 268)$ in the indices.

Every twenty places, the same number necessarily repeats. This comes from the condition that the sum of every 20 consecutive numbers is 75. In particular, we have

$$a_n + a_{n+1} + \dots + a_{n+19} = 75$$

$$a_{n+1} + a_{n+2} + \dots + a_{n+20} = 75$$

Subtracting second equation from the first, gives us $a_n = a_{n+20}$. Thus, the same 20 numbers are repeated around the circle, which we can state more generally as $a_n = a_{n+20k}$ for any integer k.

[**Background**: The Greatest Common Divisor (GCD) of two numbers a and b is the largest number that exactly divides the two numbers. This is written as $\gcd(a, b)$. For example, $\gcd(5,30) = 5$ and $\gcd(10,32) = 2$.

From number theory, we have Bézout's identity [50]:

> Let a and b be integers with greatest common divisor d. Then there exist integers x and y such that $ax + by = d$.]

For the problem at hand, we know that the numbers repeat every 20 places and also every 268 places (one loop around the circle). Next, we apply Bézout's identity to 20 and 268. By inspection, we see that $\gcd(20,268) = 4$. Further, it is not too hard to experiment and determine that

$$27 \cdot 20 - 2 \cdot 268 = 4$$

which can be written as

$$27 \cdot 20 = 2 \cdot 268 + 4$$

So, we can write $a_n = a_{n+20\cdot 27} = a_{n+2\cdot 268+4} = a_{n+4}$, noting the $2 \cdot 268 \equiv 0\ (\mathrm{mod}\ 268)$. Thus, each number repeats every 4 places. This fact makes our job much easier.

We are given

$$a_{17} = 3, \qquad a_{83} = 4, \qquad a_{144} = 9$$

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Since we know the numbers repeat every 4 places, we can recast the above information modulo 4 as follows:

$$a_{17} = a_{1\ (\mathrm{mod}\ 4)} = 3$$

$$a_{83} = a_{3\ (\mathrm{mod}\ 4)} = 4$$

$$a_{144} = a_{0\ (\mathrm{mod}\ 4)} = 9$$

Using similar logic, the number in position 210 should be the same number as in position 2, i.e.,

$$a_{210} = a_{2\ (\mathrm{mod}\ 4)} = x$$

With the above knowledge, we can write down the first 20 numbers on the circle:

Position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Number	3	x	4	9	3	x	4	9	3	x	4	9	3	x	4	9	3	x	4	9

Since 20 consecutive numbers must add to 75, we have

$$5x + 5(3 + 4 + 9) = 75$$

$$5x + 80 = 75 \ \Rightarrow x = -1$$

So, the answer to the question is that -1 is in position 210.

. . .

Puzzle: Find positive integer solutions to the equation $1 + 2^a + 3^b = 6^c$.

Solution: Consider each of the terms in the equation modulo 8.

- $1 \equiv 1 \pmod 8$
- $2^1 \equiv 2 \pmod 8, 2^2 \equiv 4 \pmod 8$ and $2^a \equiv 0 \pmod 8$ for $a \geq 3$, since 2^a is a multiple of 8 for $a \geq 3$. So, $2^a \pmod 8$ can only be $0, 2$ or 4.
- $3^b \equiv 3 \pmod 8$ if b is an odd positive integer, and $3^b \equiv 1 \pmod 8$ if b is an even positive integer. So, $3^b \pmod 8$ can only be 1 or 3.
- $6^1 \equiv 6 \pmod 8, 6^2 \equiv 4 \pmod 8$ and $6^c \equiv 0 \pmod 8$ for $c \geq 3$, since 6^c is a multiple of 8 for $c \geq 3$. So, $6^c \pmod 8$ can only be $0, 4$ or 6.

This leaves us with only 3 cases to consider for c.

If $c \geq 3$, then $1 + 2^a + 3^b \equiv 0 \pmod 8$ which is only possible if $2^a = 4$ (i.e., $a = 2$) and $3^b = 3$ (i.e., b must be odd based on our analysis above). However, this implies that

$$5 + 3^b = 6^c \text{ for } c \geq 3 \text{ and } b \text{ odd}$$

which is impossible since 3 evenly divides 3^b and 6^c but does not divide 5.

If $c = 1$, then $1 + 2^a + 3^b = 6$. In this case, the only solution is $(a, b, c) = (1,1,1)$.

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If $c = 2$, then $1 + 2^a + 3^b = 36$ which implies b can only be 1,2 or 3.

- If $b = 1$, then $2^a = 32$ which implies $a = 5$.
- If $b = 2$, then $2^a = 23$ which is not possible for an integer value of a.
- If $b = 3$, then $2^a = 8$ which implies $a = 3$.

So, for $c = 2$, we get two more solutions, i.e., $(a, b, c) = (5,1,2)$ and $(a, b, c) = (3,3,2)$.

. . .

The next puzzle requires some additional background on modular arithmetic and the summation of series.

There is something called Faulhaber's formula [5] that expresses, in a closed form, the sum of whole numbers raised to a power. For example (and what we will need for the following puzzle), the sum of cubes of whole numbers is given by the formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Further, with respect to modular arithmetic, we need the following identities:

Given $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then

- $ac \equiv bd \pmod{n}$
- $a + c \equiv b + d \pmod{n}$
- $a^k \equiv b^k \pmod{n}$ for any non-negative integer k

For an extensive list of modular arithmetic identities see the Wikipedia article on modular arithmetic [6].

. . .

Puzzle: Find the sum of all positive integers n such that when $1^3 + 2^3 + 3^3 + \cdots + n^3$ is divided by $n + 5$, the remainder is17.

Source: Question #10 from the 2020 II American Invitational Mathematics Exam (AIME)

Solution: For $n + 5$ to leave a remainder of 17 when divided into

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

there must exist some integer q such that

$$\frac{n^2(n+1)^2}{4} = (n+5)q + 17$$

or equivalently,

$$n^2(n+1)^2 = 4(n+5)q + 4 \cdot 17$$

So, $n^2(n+1)^2 \equiv 68 \pmod{n+5}$.

Next, we need to develop some identities which we will use to simplify the above expression.

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First, we have that

$$n^2 - (-5)^2 = (n+5)(n-5) \equiv 0 \ (\mathrm{mod}\ n+5)$$

which implies

$$n^2 \equiv (-5)^2 \ (\mathrm{mod}\ n+5)$$

Secondly,

$$(n+1)^2 - (-4)^2 = (n+5)(n-3) \equiv 0 \ (\mathrm{mod}\ n+5)$$

which implies

$$(n+1)^2 \equiv (-4)^2 \ (\mathrm{mod}\ n+5)$$

Putting the above two results together, we have that

$$n^2(n+1)^2 \equiv (-5)^2(-4)^2 \equiv 400 \ (\mathrm{mod}\ n+5)$$

Thus, $n^2(n+1)^2$ is congruent, modulo $n+5$, to both 68 and 400, which implies that $n+5$ exactly divides $400 - 68 = 332 = 2^2 \cdot 83$. With this condition, the only possible values for $n+5$ are $83, 166$ and 332 or equivalently, $n = 78, 161$ or 327.

Finally, we test each of the possible values of n. As shown in the table below, only 78 and 161 are valid solutions. As requested in the statement of the puzzle, the sum of all valid solutions is $78 + 161 = 239$.

n	78	161	327
$\frac{n^2(n+1)^2}{4}$	$9{,}492{,}561$	$170{,}067{,}681$	$2{,}875{,}962{,}384$
Remainder when dividing $\frac{n^2(n+1)^2}{4}$ by $n+5$	17.00	17.00	100.00

. . .

For the next puzzle, we need a result concerning the remainder when a positive integer is divided by 9.

Theorem 4. *Let n be a positive integer and let s be the sum of the digits of n, then $n \equiv s\ (mod\ 9)$.*

For example, $n = 123732$ and $s = 1+2+3+7+3+2 = 18$, s is divisible by 9 and by the theorem, 123732 is also divisible by 9 (which is true if one checks).

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Proof: If the digits of n are $d_k d_{k-1} \dots d_1 d_0$, we can write n as

$$d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10 + d_0$$

Using the binomial theorem, powers of 10 can be written as

$$10^i = (9+1)^i = \sum_{j=0}^{i} \binom{i}{j} 9^k = 9m_i + 1$$

where m_i is an integer. The above equation holds for any non-negative integer i. When $i = 0$, the formula still holds if we take $m_0 = 0$.

Using the above result, we have

$$n = d_k + d_{k-1} + \cdots + d_1 + d_0 + 9(d_k m_k + d_{k-1} m_{k-1} + \cdots + d_1 m_1 + d_0 m_0)$$

$$n = s + 9(d_k m_k + d_{k-1} m_{k-1} + \cdots + d_1 m_1 + d_0 m_0)$$

which implies

$$n \equiv s \pmod 9$$

■

Puzzle: When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

Source: Problem #4 from The First International Mathematics Olympiad, 1975 [51].

Solution: We first find a bound on A by noting

$$4444^{4444} < 10{,}000^{4444} = (10^4)^{4444} = 10^{17{,}776}$$

which means that 4444^{4444} has fewer than 17,776 digits. So, $A \leq 9 \cdot 17{,}775 = 159{,}975$. The sum of the digits of A reaches a maximum value when $A = 99{,}999$ which implies $B \leq 45$. The maximum possible value for the sum of the digits of B is 12 which occurs when (if) $B = 39$.

Applying **Theorem 4** twice, we have

$$4444^{4444} \equiv A \equiv B \pmod 9$$

Next, we compute the first several powers of 4444 modulo 9.

$$4444 \equiv 7 \pmod 9$$

$$4444^2 \equiv 7 \cdot 4444 \equiv 4 \pmod 9$$

$$4444^3 \equiv 4 \cdot 4444 \equiv 1 \pmod 9$$

Using the above results, and noting that $4444 = 3 \cdot 1481 + 1$, we have

$$4444^{4444} \equiv 4444^{3 \cdot 1481 + 1} \equiv (4444^3)^{1481} \cdot 4444^1 \equiv 4444 \equiv 7 \pmod 9$$

Since we previously showed $4444^{4444} \equiv B \pmod 9$, it must be that $B = 7$.

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3.1.3 Divisibility

An integer a is said to divide another integer b, if there exists an integer k such that $b = ka$. This can be expressed with the shorthand notation $a|b$ which reads as "a divides b". For example, $5|35$ since $35 = 7 \cdot 5$.

Theorem 5. *For integers a, b and c, the following properties hold.*

- If $a|b$ and $b|c$, then $a|c$ (transitivity).
- If $a|b$ and $a|c$, then $a|(kb + mc)$ for any two integers k and m. The expression $kb + mc$ is a linear combination of b and c.

. . .

An integer x is defined to be even if it is divisible by 2, or equivalently, if x can be written in the form $2k$ where k is an integer. An integer is defined to be odd if it can be written in the form$2k + 1$ where k is an integer. Using the above representations for even and odd numbers, it is easy to see why the addition of an even and odd number is always odd, i.e.,

> Given even number $x = 2k$ and odd number $y = 2n + 1$, we have $x + y = 2(k + n) + 1$ which is in the form of an odd number.

We can further divide the integers based on the remainder when dividing by 4. This approach divides the integers into 4 groups, i.e., integers of the forms $4k, 4k + 1, 4k + 2$ and $4k + 3$. The even numbers are of the form $4k$ or $4k + 2$. The odd numbers are of the form $4k + 1$ or $4k + 3$.

. . .

Puzzle: Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n. ("Irreducible" means the numerator and denominator have no common factors.)

Source: Problem #1 from The First International Mathematics Olympiad, 1959 [51].

Solution: The natural numbers are $0,1,2,3, \ldots$

Let $d = \gcd\,(21n + 4{,}14n + 3)$. This implies that d exactly divides $21n + 4$ and d exactly divides $14n + 3$. By **Theorem 5**, we have that d divides the linear combination

$$3(14n + 3) - 2(21n + 4) = 1$$

Thus, $\gcd(21n + 4{,}14n + 3) = 1$ which implies $\frac{21n+4}{14n+3}$ cannot be reduced further, i.e., irreducible.

. . .

There are extensive divisibility rules (perhaps "shortcuts" is more descriptive) for determining whether a whole number is divisible by a one or two digit number, see "Divisibility rule" [52]. In **Theorem 4**, we already saw one such rule concerning divisibility by 9.

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For the next puzzle, we need a divisibility rule for 11. In particular, a non-negative integer is divisible by 11 if the alternating sum of its digits is also divisible by 11. For example, the alternating sum of the digits in 648,274 is

$$6 - 4 + 8 - 2 + 7 - 4 = 11$$

which is divisible by 11, and in fact, 648,274 is also divisible by 11, i.e., $648{,}274 = 11 \cdot 58934$.

The divisibility rule for 11 is proven as follows:

Proof: Let $N = a_k a_{k-1} \ldots a_1 a_0$ be a $k + 1$ digit number. In terms of powers of 10, we can write N as

$$N = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$$

From our previous discussion on modular arithmetic, we know that $10 \equiv -1 \pmod{11}, 10^2 \equiv 1 \pmod{11}, 10^3 \equiv -1 \pmod{11}$ and in general, $10^k \equiv (-1)^k \pmod{11}$. So, we have

$$10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0 \equiv (-1)^k a_k + (-1)^{k-1} a_{k-1} + \cdots - a_1 + a_0 \pmod{11}$$

Thus, if the alternating sum of the digits of N is divisible by 11, so is N itself. ■

. . .

Puzzle: Determine all three-digit numbers N having the properties that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

Source: Problem #1 from The 2nd International Mathematics Olympiad, 1960 [51].

Solution: Let the three digits of N be a, b and c. Thus, we can write $N = 100a + 10b + c$. By the divisibility rule for 11, we have $11|(a - b + c)$, and since a, b and c are single digits, it must that that either $a - b + c = 0$ or $a - b + c = 11$.

Further, we are given that

$$\frac{N}{11} = a^2 + b^2 + c^2$$

Putting this together with the other given information, we have

$$100a + 10b + c = 11a^2 + 11b^2 + 11c^2 \qquad \text{(Equation 1)}$$

Case 1: $b = a + c$

Note that a cannot be zero since we are told N is a 3-digit number.

Plugging $b = a + c$ into Equation 1, we get

$$100a + 10(a + c) + c = 11a^2 + 11(a + c)^2 + 11c^2$$

which reduces to

$$110a + 11c = 22a^2 + 22ac + 22c^2$$

Divide both sides by 11 to get

$$10a + c = 2a^2 + 2ac + 2c^2$$

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Since the right-side of the equation is even, so must be the left-side. Further, $10a$ is even and thus, c must be even for the sum on the left to result in an even number. Let $c = 2n$ where $n = 0,1,2,3,4$. The last equation above then becomes

$$10a + 2n = 2a^2 + 4an + 8n^2$$

Dividing by 2, we get

$$5a + n = a^2 + 2an + 4n^2$$

which can be written as

$$a^2 + (2n - 5)a + n(4n - 1) = 0$$

Using the quadratic formula to solve for a and with some simplification, we get

$$a = \frac{(-2n + 5) \pm \sqrt{-12n^2 - 16n + 25}}{2}$$

Next, we check to see what values of n yield a feasible value for a.

For $n = 0$, $a = \frac{5 \pm \sqrt{25}}{2}$ which gives us $a = 5$. So, one solution is $a = 5, b = 5, c = 0$ and $\boldsymbol{N = 550}$.

For $n = 1,2,3$ or 4, the quantity inside the square root is negative, making a a complex number. Thus, $n = 1,2,3,4$ do not produce feasible solutions for a.

Case 2: $b = a + c - 11$

Plugging $b = a + c - 11$ into Equation 1, we get

$$100a + 10a + 10c - 110 + c = 11a^2 + 11(a + c - 11)^2 + c^2$$

After some simplifications, the above equation reduces to

$$32a + 23c = 2a^2 + 2c^2 + 2ac + 131 \qquad \text{(Equation 2)}$$

The right side of the equation has three even numbers added to an odd number. Thus, the ride-side of the equation is odd. The left-side of the equation must also be odd. Since $32a$ is even, it must be that $23c$ is odd. Since 23 is odd, it must be that c is also odd (since the product of an even and odd number is even). Since c is a single digit, it can only be $1,3,5,7,9$.

Rearrange Equation 2 as

$$2a^2 + 2(c - 16)a + c(2c - 23) + 131 = 0$$

Using the quadratic formula to solve for a, we get

$$a = \frac{-2(c - 16) \pm \sqrt{4(c - 16)^2 - 8c(2c - 23) - 8(131)}}{4}$$

With some simplifications, we get

$$a = \frac{(16 - c) \pm \sqrt{-3c^2 + 14c - 6}}{2}$$

Next, we try the possible values of c to see which, if any, provide a valid value for a.

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For $c = 1$, we get

$$a = \frac{15 \pm \sqrt{5}}{2}$$

which is not an 1-digit number and thus, not a valid value for a.

For $c = 3$, we get

$$a = \frac{13 \pm \sqrt{9}}{2} = 5 \text{ or } 8$$

$a = 5, c = 3$ implies $b = 5 + 3 - 11 = -3$. So, $a = 5$ is not valid.

However, $a = 8, c = 3$ implies $b = 8 + 3 - 11 = 0$ which is valid. This gives us a second solution to the problem, i.e., $\boldsymbol{N = 803}$.

For $c = 5$, the number inside the square root in the formula for a is -11, and thus not valid.

For $c = 7$, the number inside the square root in the formula for a is -55, and thus not valid.

For $c = 9$, the number inside the square root in the formula for a is -123, and thus not valid.

. . .

Puzzle: Which integers have the following property? If the final digit is deleted, the integer is divisible by the new number.

Source: Problem #15 in “The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics” [79]

Solution: Represent an integer as xy where y is the final (least significant) digit and x represents the other digits. (Note that xy denotes the concatenation of x and y, and not the product of x and y.) For example, if the integer is 3408938, then $y = 8$ and $x = 340893$.

The condition given the puzzle can be translated to the equivalent conditions $x|(10x + y)$. Since $x|10x$ and $x|(10x + y)$, it must be that $x|y$. Keeping in mind that y is a single-digit number, we have only a few possibilities to try (as shown in the following table).

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y	x	xy
0	can be any integer	any integer ending in 0
1	1	11
2	1,2	12,22
3	1,3	13,33
4	1,2,4	14,24,44
5	1,5	15,55
6	1,2,3,6	16,26,36,66
7	1,7	17,77
8	1,2,4,8	18,28,48,88
9	1,3,9	19,39,99

. . .

Puzzle: Find all integers having the property that, when the third digit is deleted, the resulting number divides the original one.

Source: Problem #18a in "The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics" [79]

Solution: Let the original number be $N = a_n a_{n-1} \boldsymbol{a_{n-2}} \ldots a_2 a_1 a_0$. The number with the third digit removed is $D = a_n a_{n-1} a_{n-3} \ldots a_2 a_1 a_0$. Writing N and D as powers of 10, we have

$$N = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \cdots + 10^2 a_2 + 10 a_1 + a_0$$

$$D = 10^{n-1} a_n + 10^{n-2} a_{n-1} + 10^{n-3} a_{n-3} + \cdots + 10^2 a_2 + 10 a_1 + a_0$$

Consider $N - 10D$ which we can write as

$$10^{n-2}(a_{n-2} - a_{n-3}) + 10^{n-3}(a_{n-3} - a_{n-4}) + \cdots + 10^2(a_2 - a_1) + 10(a_1 - a_0) + a_0$$

Since $D|N$ (by the condition given in the puzzle statement) and $D|10N$, we have that $D|(N - 10D)$. However, D is an n digit number and $N - 10D$ is at most an $n - 1$ digit number. Thus, it must be that $N - 10D = 0$. This implies that $a_0 = 0$ and that $a_i - a_{i-1} = 0$ for $i = 1,2, \ldots, n - 2$. Solving these equations recursively, we get that $a_0 = a_1 = a_2 = \cdots = a_{n-2} = 0$. Thus, the set of integers that have the property stated in the puzzle must have all digits, other than the first two, equal to zero. For example, 8,700,000 satisfies the condition of the puzzle, but 8,710,000 does not.

. . .

Puzzle: Prove that for any integer n, the expression $n^2 + 4n + 2$ is not divisible by 11.

Solution: To prove the assertion, we only need compute $n^2 + 4n + 2 \pmod{11}$ for $n = 0,1,2, \ldots, 10$. As shown in the table below, the remainder is never 0.

n	0	1	2	3	4	5	6	7	8	9	10
$n^2 + 2n + 1$	2	7	3	1	1	3	7	2	10	9	10

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The pattern repeats since

$$(n+11)^2 + 4(n+11) + 2 \equiv n^2 + 4n + 2 \pmod{11}$$

. . .

Puzzle: Find all non-negative integers (i.e., numbers from the set 0,1,2,3, ...) such that $n^5 + 3n + 7$ is divisible by 13.

Solution: Similar to the last problem, we compute $n^5 + 3n + 7 \pmod{13}$ for $n = 0,1,2,\dots,12$. As can be seen from the table below, 7 and 8 are congruent to 0 modulo 13. This means that $n^5 + 3n + 7$ is divisible by 13 for all natural numbers of the forms $7 + 13j$ for $j = 0,1,2,\dots$ or $8 + 13k$ for $k = 0,1,2,\dots$

For example, take $j = 1$. In this case, $7 + 13j = 20$ and $20^5 + 3 \cdot 20 + 7 = 3200067 = 13 \cdot 246159$.

n	0	1	2	3	4	5	6	**7**	**8**	9	10	11	12
$n^5 + 3n + 7$	7	11	6	12	3	1	1	**0**	**0**	11	2	8	3

. . .

Puzzle: Given a set S consisting of 100 positive integers, prove that is always possible to find one or more numbers from S whose sum ends in two zeros, i.e., is divisible by 100.

Solution: Let the 100 numbers be represented as $n_1, n_2, \dots, n_{100}$. Form the sums

$$S_i = \sum_{j=1}^{j=i} n_i = n_1 + n_2 + \cdots + n_i, \qquad i = 1,2,\dots,100$$

If any of the sums are divisible by 100, we are done.

If none of the sums is divisible by 100, then the only remainders (when dividing by 100) can be $1,2,3,\dots,99$. Since there are 100 of the S_i sums and only 99 possible remainders in this case, at least two of the sums must have the same remainder (say S_k and S_l, with $k < l$). So, we have

$$S_l - S_k = n_{l+1} + n_{l+2} + \cdots + n_k$$

is divisible by 100, and we are done.

. . .

Puzzle: For what numbers $n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers, i.e., $\{1,2,3,\dots\}$) do we have $(n-7)|(n^2 + 2n - 11)$.

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Solution: We use polynomial long division to determine the remainder when $n-7$ is divided into $n^2+2n-11$.

		n	9	
n	-7	n^2	$2n$	-11
		n^2	$-7n$	
			$9n$	-11
			$9n$	-63
				52

The above calculation means that

$$\frac{n^2+2n-11}{n-7}=(n+9)+\frac{52}{n-7}$$

So, for $(n-7)|(n^2+2n-11)$ we must have $(n-7)|52$ or equivalently, $n-7$ equals one of the divisors of $52=2^2\cdot 13$, i.e.,

$$n-7=1,2,4,13,26,52$$

$$n=8,9,11,20,33,59$$

3.1.4 Prime Factorization

A key result in number theory is the prime factorization theorem, which states that every positive integer can be uniquely factored as the product of prime numbers, e.g., $472=2^3\cdot 59$. We state the result more formally below. A **prime number** is a positive integer which is only divisible by 1 and itself.

Theorem 6 *(Fundamental Theorem of Arithmetic) Every positive integer $n > 1$ can be represented uniquely as a product of prime powers, i.e.,*

$$n=p_1^{k_1}p_2^{k_2}\dots p_m^{k_m}$$

where $p_1,p_2,\dots,p_m$ are prime numbers.

It follows from the above theorem that the number of distinct divisors of a given number (as represented in the theorem) is given by the formula

$$(k_1+1)(k_2+1)\dots(k_m+1)$$

For example, take the simple case of a number whose prime factorization involves only one prime such as $16=2^4$. There are 5 divisors of 16, i.e., $1,2^1,2^2,2^3,2^4$. If the factorization of a number involves several primes, then we just multiply the number of factors associated with each prime factor to get the total number of combinations. For example, the distinct factors of $432=2^4 3^3$ consist of all combinations of members of the sets $\{2^0=1,2^1,2^2,2^3,2^4\}$ and $\{3^0=1,3^1,3^2,3^3\}$ for a total of $5\cdot 4=20$ factors.

. . .

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Puzzle: What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that the product is a perfect square?

Source: Problem #6 from the Indian Olympiad Qualifier Mathematics (IOQM) 2021, https://www.mtai.org.in/wp-content/uploads/2021/01/IOQM-question-paper.pdf

Solution: We first simplify the given integer, i.e.,

$$2^5 \cdot 3^6 \cdot (2^2)^3 \cdot 5^3 \cdot (2 \cdot 3)^7 = 2^{18}3^{13}5^3$$

To make the above number a perfect square, we multiple by $3 \cdot 5$ to get

$$2^{18}3^{14}5^4 = (2^9 3^7 5^2)^2$$

. . .

Puzzle: A proper divisor of a positive integer is a positive integral divisor other than 1 and the number itself. A natural number greater than 1 is called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

Source: Question #3 from the 1987 American Invitational Mathematics Exam (AIME)

Solution: A nice number n must have either two distinct factors $p_1 p_2$ (which has factors $1, p_1, p_2$ and n) or have a factorization of the form p_1^3 (which has prime factors $1, p_1, p_1^2$ and p_1^3). With this insight, we can list the first ten nice numbers, i.e.,

$$6 = 2 \cdot 3$$

$$8 = 2^3$$

$$10 = 2 \cdot 5$$

$$14 = 2 \cdot 7$$

$$15 = 3 \cdot 5$$

$$21 = 3 \cdot 7$$

$$22 = 2 \cdot 11$$

$$26 = 2 \cdot 13$$

$$27 = 3^3$$

$$33 = 3 \cdot 11$$

So, the sum of the first 10 nice number is $6 + 8 + 10 + 14 + 15 + 21 + 22 + 26 + 27 + 33 = 182$.

. . .

Puzzle: Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999)$$

What is the largest prime factor of S?

Source: Question #5 from the 1994 American Invitational Mathematics Exam (AIME)

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Solution: For example, $p(123) = 1 \cdot 2 \cdot 3 = 6$ and $p(702) = 7 \cdot 2 = 14$.

The expression $(0+1+2+3+4+5+6+7+8+9)^3$ when expanded out, gives all combinations of the product of three digits. This is almost what we need except for terms like 002 and 705. However, the expression $(1+1+2+3+4+5+6+7+8+9)^3 - 1$ does give all combinations of the product of three digits where zero-digits are effectively replaced by a 1 (which does not change the product of the non-zero digits). The "-1" term in the expression is needed to account for 000 being replaced by 111 (which we do not want to count).

So, $S = (1+1+2+3+4+5+6+7+8+9)^3 - 1 = (46-1)(46^2+46+1) = 3^3 \cdot 5 \cdot 7 \cdot 103$. Thus, the largest prime factor of S is 103.

Note: In expanding the expression for S we used the following formula from algebra with $a = 1+1+2+3+4+5+6+7+8+9 = 46$ and $b = 1$.

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

. . .

Puzzle: Find the number of ordered pairs of integers (m, n) such that $m^2 n = 20^{20}$.

Source: Question #1 from the 2020 II American Invitational Mathematics Exam (AIME)

Solution: For each value of m^2 there is a unique integer value of n, i.e., $n = 20^{20}/m^2$. So, the problem reduces to finding the number of perfect square factors of 20^{20}.

The prime factorization of 20^{20} is

$$2^{40} \cdot 5^{20} = (2^2)^{20} \cdot (5^2)^{10}$$

So, m^2 can be $(2^2)^x \cdot (5^2)^y = (2^x)^2 \cdot (5^y)^2 = (2^x \cdot 5^y)^2$ where $x = 0,1,2,\ldots,20$ and $y = 0,1,2,\ldots 10$. Thus, there are $21 \cdot 11 = 231$ possible values for pairs of x and y.

Once x and y are chosen, m and n are determined. For example, if $x = 3$ and $y = 5$, then $m = 2^3 \cdot 5^5$ and

$$n = \frac{20^{20}}{m^2} = \frac{(2^2)^{20} \cdot (5^2)^{10}}{2^6 \cdot 5^{10}} = 2^{34} 5^{10}$$

(Note that we cannot have $m^2 = 2^r 5^s$ for r or s being odd integers.)

. . .

Puzzle: Consider the set of pairs of distinct integers n and m, such that the set of prime factors of n is the same as the set of prime factors of m, and such that the sets of prime factors of $n+1$ and $m+1$ are also equal. Is the set of such pairs finite or not?

Sources:

- This is one of the "Coffin problems" from the former Soviet Union, see the background on such problems at http://www.tanyakhovanova.com/coffins.html
- YouTube video entitled "Simple Math Problems To Fool The Best" [53]
- Unpublished paper entitled "MEKH-MET Entrance Examination Problems" by Ilan Vardi, see http://www.tanyakhovanova.com/Coffins/Vardi-solutions.pdf

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Solution: Let $m = 2^r - 2$ and $n = (m+1)^2 - 1$ for $r = 2,3,4, \ldots$

By **Theorem 6**,

$$m + 1 = p_1^{k_1} p_2^{k_2} \ldots p_j^{k_j}$$

where $p_1, p_2, \ldots, p_j$ are prime numbers. Further,

$$(m+1)^2 = p_1^{2k_1} p_2^{2k_2} \ldots p_j^{2k_j}$$

Since $n + 1 = (m+1)^2$, $n + 1$ and $m + 1$ have the same prime factors, i.e., $p_1, p_2, \ldots, p_j$.

Regarding the prime factors of n and m, note that $n = m(m+2)$ and $m + 2 = 2^r$ has 2 as its only prime factor. By **Theorem 6**, and noting that m is even, we have

$$m = 2q_2^{s_2} q_3^{s_3} \ldots q_i^{s_i}$$

where $2, q_1, q_2, \ldots, q_i$ are prime numbers. Thus, both n and m have the same prime factors, i.e., $2, q_1, q_2, \ldots, q_i$.

In summary, we have shown there are an infinite number of pairs meeting the conditions of the puzzle. In the paper by Vardi (noted in the sources for this puzzle), the author states that pairs (other than those in the format of the above solution) do exist, e.g., $m = 75 = 3^2 \cdot 25$ and $n = 1215 = 3^5 \cdot 5$ which implies $m + 1 = 76 = 2^2 \cdot 19$ and $n + 1 = 1216 = 2^6 \cdot 19$.

. . .

If natural number n has prime factorization

$$n = p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$$

Each factor of n is of the form

$$p_1^{l_1} p_2^{l_2} \ldots p_m^{l_m}$$

where $0 \le l_i \le k_i$ for $i = 1,2, \ldots, m$.

Thus, the number of possible factors of n is given by the following expression. The 1 in each term comes from the possibility of having 0 appearances of a given prime factor.

$$(k_1 + 1)(k_2 + 1) \ldots (k_m + 1)$$

The preceding concepts are a hint to the following puzzle.

Puzzle: Let $x_1, x_2, \ldots, x_{n-1}, x_n$ be the positive factors of the number

$$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$$

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Determine the value of

$$\sum_{i=1}^{n} \frac{1}{x_i + \sqrt{7!}}$$

Solution: Recall the formula for the number of factors of an integer (see the discussion following **Theorem 6**). Using said formula, we can compute the number of factors of 7!

$$(4+1)(2+1)(1+1)(1+1) = 5 \cdot 3 \cdot 2 \cdot 2 = 60$$

So, $n = 60$.

In general, the factors of an integer come in pairs. For example, $24 = 2^3 \cdot 3^1$ has eight factors, i.e., $1, 2, 3, 4, 6, 8, 12, 24$ which can be grouped as follows:

$$1 \cdot 24 = 24$$

$$2 \cdot 12 = 24$$

$$3 \cdot 8 = 24$$

$$4 \cdot 6 = 24$$

Applying this idea to the problem at hand, multiply pairs of factors to get $7! = 5040$, i.e.,

$$x_1 x_n = 7!$$

$$x_2 x_{n-1} = 7!$$

$$\cdots$$

$$x_{30} x_{31} = 7!$$

We can write the above more succinctly as

$$x_j x_{n+1-j} = 7!, \qquad 1 \le j \le \frac{n}{2}$$

Next, we expand and then rearrange the series in question as follows:

$$\sum_{i=1}^{n} \frac{1}{x_i + \sqrt{7!}} = \frac{1}{x_1 + \sqrt{7!}} + \frac{1}{x_2 + \sqrt{7!}} + \cdots + \frac{1}{x_{n-1} + \sqrt{7!}} + \frac{1}{x_n + \sqrt{7!}}$$

$$= \left(\frac{1}{x_1 + \sqrt{7!}} + \frac{1}{x_n + \sqrt{7!}}\right) + \left(\frac{1}{x_2 + \sqrt{7!}} + \frac{1}{x_{n-1} + \sqrt{7!}}\right) + \cdots + \left(\frac{1}{x_j + \sqrt{7!}} + \frac{1}{x_{n+1-j} + \sqrt{7!}}\right) + \cdots$$

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Working with the general term in the previous equation, we have the following simplification

$$\frac{1}{x_j+\sqrt{7!}}+\frac{1}{x_{n+1-j}+\sqrt{7!}}=\frac{x_j+x_{n+1-j}+2\sqrt{7!}}{\left(x_j+\sqrt{7!}\right)\left(x_{n+1-j}+\sqrt{7!}\right)}$$

$$=\frac{x_j+x_{n+1-j}+2\sqrt{7!}}{x_jx_{n+1-j}+\left(x_j+x_{n+1-j}\right)\sqrt{7!}+7!}$$

$$=\frac{x_j+x_{n+1-j}+2\sqrt{7!}}{7!+\left(x_j+x_{n+1-j}\right)\sqrt{7!}+7!}$$

$$=\frac{x_j+x_{n+1-j}+2\sqrt{7!}}{\left(x_j+x_{n+1-j}\right)\sqrt{7!}+2\cdot 7!}=\frac{x_j+x_{n+1-j}+2\sqrt{7!}}{\sqrt{7!}\ \left(x_j+x_{n+1-j}+2\sqrt{7!}\right)}=\frac{1}{\sqrt{7!}}$$

Putting the previous results together, and recalling that $n=60$, we get the final result

$$\sum_{i=1}^{n}\frac{1}{x_i+\sqrt{7!}}=\sum_{i=1}^{n/2}\left(\frac{1}{x_j+\sqrt{7!}}+\frac{1}{x_{n+1-j}+\sqrt{7!}}\right)=\sum_{i=1}^{n/2}\frac{1}{\sqrt{7!}}=\frac{n}{2}\cdot\frac{1}{\sqrt{7!}}=\frac{15}{\sqrt{7!}}$$

3.1.5 Miscellaneous

Puzzle: Demonstrate that for all $n \geq 6$ the equation

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\cdots+\frac{1}{x_n^2}=1$$

has integer solutions.

Source: Introductory Problem #20 from "101 Problems in Algebra: From the Training of the USA IMO Team" [28]

Solution: We find solutions for $n = 6,7,8$ and then show how to recursively determine subsequent solutions.

First, notice that for $n = 4$, we have the solution

$$\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}=1$$

This leads one to consider the following identity which, as we shall see, can be used to develop a recursive procedure.

$$\frac{1}{(2a)^2}+\frac{1}{(2a)^2}+\frac{1}{(2a)^2}+\frac{1}{(2a)^2}=\frac{4}{(2a)^2}=\frac{1}{a^2}$$

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If $(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$ is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

then

$$(x_1, x_2, \dots, x_n, x_{n+1}, x_{n+2}, x_{n+3}) = (a_1, a_2, \dots, a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n)$$

is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n+3}^2} = 1$$

because

$$\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{n-1}^2}\right) + \left(\frac{1}{(2a_n)^2} + \frac{1}{(2a_n)^2} + \frac{1}{(2a_n)^2} + \frac{1}{(2a_n)^2}\right) = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{n-1}^2} + \frac{1}{a_n^2} = 1$$

It is easy to check that (2,2,2,3,3,6), (2,2,2,4,4,4,4) and (2,2,2,3,4,4,12,12) are solutions for the cases $n = 6,7,8$, respectively. To find a solution for $n = 9$, we use the formula above and the solution for $n = 6$ to get (2,2,2,3,3,12,12,12,12). We can use the solution for $n = 7$ to determine a solution for $n = 10$, and so on.

. . .

Puzzle: Let S be the set $\{0, 1\}$. Given any subset of S we may add its arithmetic mean to S (provided it is not already included since S never includes duplicates).

- Show that by repeating this process we can include the number $\frac{1}{5}$ is in S.
- Show that we can eventually include any rational number between 0 and 1.

Source: Problem #4 from the All Soviet Union Mathematical Olympiad in 1979 (Tbilisi, Georgia)

Solution: We will do the second part first and then use the general approach to show $\frac{1}{5} \in S$.

To start, we know that $\frac{1}{2} \in S$ since the average of 0 and 1 is $\frac{1}{2}$.

Taking pairwise averages of the 3 elements in S so far, we expand S to $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

Taking pairwise averages of the 5 elements in S so far, we expand S to $\{0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1\}$.

Continuing in this manner, we see that S contains all rational numbers of the form $\frac{m}{2^n}$ for $0 \leq m \leq 2^n, n \geq 1$.

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We can use the above result to solve the more general problem of showing any rational number $\frac{s}{t} \in (0,1)$ (reduced to lowest terms) is an element of set S. In the following, we will show that there exists an integer n, and t distinct positive integers $a_1, a_2, \ldots, a_t$ (each less than or equal to 2^n) such that $a_1 + a_2 + \cdots + a_t = s2^n$. Since $\frac{a_i}{2^n} \in S$, for $i = 1,2, \ldots, t$, so is there average, i.e.,

$$\frac{\frac{a_1}{2^n} + \frac{a_2}{2^n} + \cdots + \frac{a_t}{2^n}}{t} = \frac{s}{t} \in S$$

Note that since $\frac{s}{t}$ is in the interval $(0,1)$, $t \geq 2$, $s \geq 1$ and $s < t$.

To prove the above result, we start by defining

$$N(a) = a + (a+1) + (a+2) + \cdots + \big(a + (t-1)\big)$$

$$= ta + \big(1 + 2 + \cdots + (t-1)\big) = ta + \frac{t(t-1)}{2}$$

Let n be the smallest integer such that $N(0) < 2^n$.

Next, note that

$$N(a) - N(a-1) = \left[a + (a+1) + (a+2) + \cdots + \big(a + (t-1)\big)\right]$$

$$-\left[(a-1) + a + (a+1) + \cdots + \big((a-1) + (t-1)\big)\right]$$

$$= \big(a + (t-1)\big) - (a-1) = t$$

By definition,

$$N\big(2^n - (t-1)\big) = \big(2^n - (t-1)\big) + \big(2^n - (t-2)\big) + \cdots + 2^n$$

$$= t2^n - \big(1 + 2 + \cdots + (t-1)\big) = t2^n - N(0)$$

$$> t2^n - 2^n = (t-1)2^n \geq s2^n$$

Since $N(0) < 2^n$ (by definition) and $s \geq 1$, we have that $N(0) < s2^n$. Combining with the above result, we have

$$N(0) < s2^n < N\big(2^n - (t-1)\big)$$

The above inequality plus that fact that $N(a)$ is an increasing function implies that there exists a unique integer b such that $1 \leq b \leq \big(2^n - (t-1)\big)$ and $N(b-1) < s2^n \leq N(b)$.

Let $r = N(b) - s2^n$. From the above inequality, we have that $r < N(b) - N(b-1) = t$.

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Further,

$$s2^n = N(b) - r = b + (b+1) + (b+2) + \cdots + \big(b + (t-1)\big) - r$$

Since $r < t$, we can subtract 1 from the first r terms in the above sum and leave the rest as-is to get the desired set, i.e.,

$$a_1 = b - 1$$
$$a_2 = b$$
$$\ldots$$
$$a_r = b + (r-1) - 1 = b + r - 2$$
$$a_{r+1} = b + r$$
$$\ldots$$
$$a_t = b + (t-1)$$

where each element of the above set less than or equal to 2^n and whose sum is $s2^n$.

We still have the specific part of the puzzle to solve, i.e., show that $\frac{1}{5} \in S$. Using the procedure developed above, let

$$\frac{s}{t} = \frac{1}{5} \Rightarrow s = 1, t = 5$$

So, $N(a) = a + (a+1) + (a+2) + (a+3) + (a+4) = 5a + 10$. Since $N(0) = 10 < 2^4 = 16$, it follows that $n = 4$.

Since $N(1) = 15 < s2^n = 16 < N(2) = 20$, the value of b that we seek is 2.

Next, we compute r, i.e.,

$$r = N(b) - s2^n = N(2) - 16 = 20 - 16 = 4$$

Thus, $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 6$.

We can check the result as follows:

$$\frac{\frac{a_1}{2^n} + \frac{a_2}{2^n} + \frac{a_3}{2^n} + \frac{a_4}{2^n} + \frac{a_5}{2^n}}{t} = \frac{1}{5}\left(\frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{6}{16}\right) = \frac{1}{5}\left(\frac{16}{16}\right) = \frac{1}{5}$$

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3.2 Counting (Combinatorics)

We start with stating several of the most fundamental rules in combinatorics (the theory of counting).

Sum Rule for Counting: If an event (e.g., rolling a die) can happen in m ways and another event (e.g., drawing a card from a deck of 52 cards) can happen in n ways, and **only one** of the two events can happen at a given instance, then there are $m + n$ ways for one of two events to happen. For the example at hand, there are 58 possible outcomes for the die roll or card draw. This rule can be extended to r events of which **only one** can occur, where the i^{th} event can occur m_i ways. In this case, the total number of possible outcomes is $m_1 + m_2 + \cdots + m_r$.

In the case of the sum rule, there is an implied dependence between the events, i.e., if one event happens, the other events cannot occur (at least not in the time frame under consideration).

Product Rule for Counting: If an event (e.g., rolling a die) can happen in m ways and another event (e.g., draw a card from a deck of 52 cards) can happen in n ways, and the two events are independent, then the number of possible outcomes for the two events is mn. Using the die roll and card drawing example again, but now assume both can happen simultaneously, we have a total of $6(52) = 312$ possible outcomes. This rule can be extended to r events, where the i^{th} event can occur in m_i ways. In this case, the total number of possible outcomes is $m_1 m_2 \ldots m_r$.

Pigeonhole Principle: If each item in a set of n items (call this Set #1) is to be associated with one item in another set (call this Set #2) where Set #2 has m items and $n > m$, then at least one item in Set #2 must have more than one item associated with it from Set #1.

This may seem trivial and obvious, e.g., if we put 10 pigeons (Set #1 in our definition) into 9 pigeonholes (Set #2 in our definition), then clearly one of the pigeonholes must have 2 or more pigeons. The principle is nevertheless very useful, and it is not always so obvious as to when or how it can be applied.

Puzzle: Suppose that you have a collection of many marbles in a box, and the marbles are of four different colors (red, green, blue and black). What is the least number of marbles that you need to retrieve from the box (while not looking) to be sure of getting two marbles of the same color?

Solution: At first glance, it is not clear how this problem relates to the pigeonhole principle. After some thought, one approach is to let the four colors be the pigeonholes (Set #2 in the definition). So, $m = 4$. We are being asked to find the smallest number n such that we are sure to draw two marbles of the same color. One can view Set #1 as the draws. The smallest value of n to ensure the desired result is 5. "Worst case" (assuming that two marbles of the same color are desired), is to draw one each of a red, blue, green and black marble (in no particular order) in the first four draws. The next draw will force a match since there are only four colors.

Alternate phasing of the puzzle: You have a box of marbles, which you remove one at a time. Each marble is then placed in one of four containers (numbered from 1 to 4). Each marble has instructions as to which container the marble should be placed (e.g., this could be done by etching a number from 1 to 4 on each marble). What is the least number of marbles that one needs to select from the box and place in a container to be sure that one of the four containers has 2 marbles? This is equivalent to the above problem but perhaps easier to map to the pigeonhole principle.

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Pigeonhole Principle – Generalization #1: If each item in a set of $k \cdot m + 1$ items (call this Set #1) is to be associated with one item in another set (call this Set #2) that has m items, then at least one item in Set #2 must have $k + 1$ items associated with it from Set #1. (Note that $k, m \in \mathbb{N}$.)

For example, let Set #1 consist of $3(10) + 1 = 31$ identical balls (i.e., $k = 3, m = 10$) and Set #2 consists of 10 containers. Further, assume all the balls are to be distributed among the 10 containers. The generalized pigeonhole principle tells us that at least one of the containers must have $k + 1 = 4$ balls. Figure 3 shows the best that one can do to avoid putting 4 balls into one container (given 31 balls). The 31st ball (at the top left of the figure) must go into one of the containers and thus, forcing 4 balls in one of the containers.

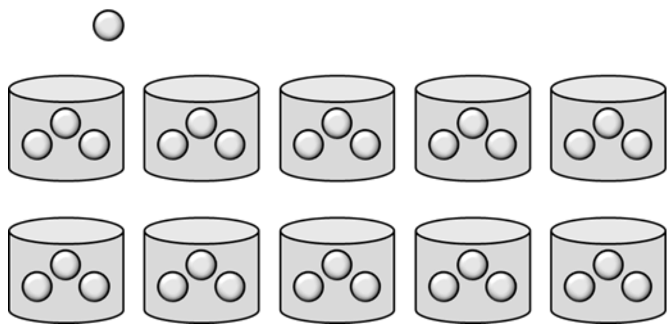

Figure 3. Pigeonhole Principle – Generalization #1 - Example

Puzzle: Consider a box with 4 types of marbles, determine the least number of draws that are needed to ensure that at least 5 marbles of the same type.

Solution: Using the Pigeonhole Principle – Generalization #1, we want $k + 1 = 5 \Rightarrow k = 4$. Further, let $m = 4$. So, we have $km + 1 = 17$, i.e., 17 draws are required to ensure at least 5 marbles of the same type. Think of it this way: if you selected only 16 marbles from the box, it is possible that you would get 4 of each type. The 17th selection forces 5 marbles of one of the types. As shown in Figure 4, each container represents marbles of a given type, and the type of a marble selected essentially provides instruction as to the container in which the marble should be placed. The best one can do without getting 5 marbles of the same color is by getting 4 of each type. The 17th draw will force at least 5 marbles of one type.

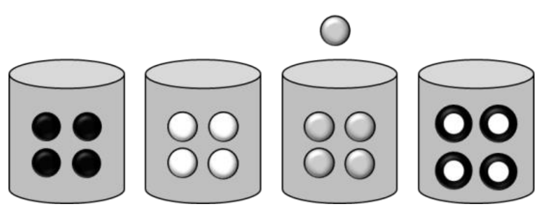

Figure 4. Pigeonhole Principle – Generalization #1 – Puzzle

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Pigeonhole Principle – Generalization #2: If there are $a_1 + a_2 + \cdots + a_n - n + 1$ or more items in a set (Set #1) to be associated with n items in another set (Set #2), then there exists a number $j \in \{1,2, \ldots n\}$ such that item (container) j (in Set #2) has a_j or more associations with items in Set #1.

Think of the principle this way, i.e., you have n containers with capacities as shown in Figure 5. Assume that each container is filled to capacity for a total of $a_1 + a_2 + \cdots + a_n - n$ items. If you are forced to add one more item to any container, then at least one containers (say j) has a_j items.

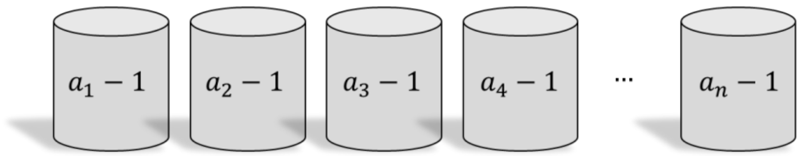

Figure 5. Pigeonhole Principle – Generalization #2

Proof: Assume to the contrary that $\forall j \in \{1,2, \ldots n\}$ item j (in Set #2) has at most $a_j - 1$ associations with items in Set #1. Summing the number of associations, we get at most $a_1 + a_2 + \cdots + a_n - n$ but this is one less than the number of items in Set #1, all of which were assumed to be associated with an item in Set #2. Thus, we have a contradiction and our contrary assumption is false. So, there must exist $j \in \{1,2, \ldots n\}$ such that item j (in Set #2) has a_j or more associations with items in Set #1. ■

Puzzle: If someone wants their pocket change to have at least 4 half-dollars or at least 8 quarters or at least 20 dimes or at least 40 nickels, what is the least number of coins that this person needs to select from a jar of coins (assume there are more than 100 of each type of coin in the jar). Another way to state the problem is to ask: "what is the least number of coins that needs to be (blindly) selected from a jar of half-dollars, quarter, dimes and nickels to ensure $2 in change **of a particular coinage** (half-dollars, quarters, dimes or nickels)?"

Solution: Using Pigeonhole Principle – Generalization #2, we have $n = 4$ and $a_1 = 4, a_2 = 8, a_3 = 20, a_4 = 40$. Thus, $a_1 + a_2 + a_3 + a_4 - n + 1 = 4 + 8 + 20 + 40 - 4 + 1 = 69$ is the minimum number of selections from the coin jar to ensure at least $2 in change of a particular coinage. For example (although improbable), one could draw 3 half-dollars, 7 quarters, 19 dimes and 39 nickels for a total of 68 coins and still not have $2 in any one coinage (failing one short in each case).

. . .

The solution to the next problem makes use of the pigeonhole principle.

Puzzle: Twenty-one girls and twenty-one boys took part in a mathematics contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

What is the least number of problems solved by all the contestants?

What is the largest number of problems solved by all the contestants?

Prove that there was a problem that was solved by at least three girls and at least three boys.

Source: With the exception of the two questions, the above puzzle is Problem #3 from The 42nd International Mathematics Olympiad, 2001 [51].

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Solution: The least number of problems that can be solved and still meet the conditions of the puzzle is 1. This would happen if all the girls and boys only solved 1 problem each, and it was the same problem.

For the second question, each of the girls could solve a set of 6 different problems (total of 6×21 =126). Each of the boys could solve a set of 6 different problems (total of 126). This gives us 252 problems, but we still need to deal with the condition of each girl-boy pair solving at least one common problem. Let's say the problems solved by each girl and by each boy are numbered 1 to 6. Let the common problem solved by each girl-boy pair be the #1 problem for the girl and the #1 problem for the boy, i.e., all 42 contestants solve the same problem with no other overlap. Thus, we need to remove $21 + 21 - 1 = 41$ from our initial calculation of 252. So, the maximum total number distinct problems solved by all the girls and boys is $252 - 41 = 211$. There could be more problems in the competition, but we cannot make a determination based on the information provided in the puzzle.

Now, for the hard part of the problem, we define an incident matrix (see Table 5). Each column represents a problem. Each row represents a boy or girl. If a given contestant gets a given problem correct, then a 1 is entered in the corresponding entry of the matrix. As an example, one of the columns in the matrix is populated with results, i.e., problem P_1 was solved correctly by B_1, B_2, G_1, G_2 and G_{21}.

Table 5. Incident matrix for math contest

	P_1	P_2	P_3	…	P_n
B_1	1				
B_2	1				
B_3	0				
…					
B_{21}	0				
G_1	1				
G_2	1				
G_3	0				
…					
G_{21}	1				

Let b_i be the number boys who solve problem P_i, and let g_i be the number of girls who solve P_i. The conditions of the puzzle imply that the number of ones in each row of the incident matrix is at most 6.

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If we let B_{Total} be the total number of problems solved by the boys, and G_{Total} be the total number of problems solved by the girls, the

$$B_{Total} = \sum_{i=1}^{n} b_i \leq 6 \cdot 21 = 126$$

$$G_{Total} = \sum_{i=1}^{n} g_i \leq 6 \cdot 21 = 126$$

Because of the condition that each girl-boy pair solves at least one common problem, the rows B_i and G_j have at least one pair of ones in the same column (this is true for any boy i and any girl j). We call such a pair a "one-pair." Since there are 21 boys and 21 girls, and each pair of girl and boy must have a one-pair, there are at least 21^2 one-pairs. (**Caution**: do not confuse one-pairs with ones in the incident matrix. For example, in the P_1 column of our example matrix, there are 5 ones but there are $2 \cdot 3 = 6$ one-pairs.) The number of one-pairs in column i of the incident matrix is $b_i g_i$ and so, the number of one-pairs is $\sum_{i=1}^{n} b_i g_i$. We already know that the number of one-pairs is at least 21^2 and so, $\sum_{i=1}^{n} b_i g_i \geq 21^2$.

Assume the conclusion of the proposition is false, i.e.,

- If $g_i \geq 3$, then it must be that $b_i \leq 2$.
- If $b_i \geq 3$, then it must be that $g_i \leq 2$.

Let G be the set of problems such that each is solved by at least 3 girls and at most 2 boys, and B be the set of problems such that each is solved by at least 3 boys and at most 2 girls. Let X be the set of problems such that each is solved by at most 2 boys and at most 2 girls. We have that

$$\sum_{i=1}^{n} b_i g_i = \left(\sum_{i \in B} b_i g_i + \sum_{i \in G \cup X} b_i g_i\right) \leq \left(2 \sum_{i \in B} b_i + 2 \sum_{i \in G \cup X} g_i\right) \qquad \text{(Equation 1)}$$

For any girl G_i, we construct the matrix M_i whose columns correspond to the problems solved by G_i and whose rows are the 21 boys. For each girl, M_i has 21 rows and at most 6 columns (one for each correct problem answered). Every row of M_i has at least one 1 (by the second condition of the puzzle). By the pigeonhole principle – generalization #1, there is at least one column of M_i that has 4 ones, since 21 ones need to be distributed over at most 6 columns. Thus, each girl solves at least one problem in the set B. This implies that

$$\sum_{i \in B} g_i \geq 21$$

or equivalently, since each girl can solve at most 6 problems and at least one solution is already taken based on the above equation, we have

$$\sum_{i \in G \cup X} g_i \leq 5 \cdot 21$$

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We can apply the same argument as above for any boy, and thus get

$$\sum_{i \in B \cup X} b_i \le 5 \cdot 21$$

Since B is a subset of $B \cup X$, we have

$$\sum_{i \in B} b_i \le \sum_{i \in B \cup X} b_i \le 5 \cdot 21$$

Plugging the above results into Equation 1, we get

$$441 = 21^2 \le \sum_{i=1}^{n} b_i g_i \le 2 \sum_{i \in B} b_i + 2 \sum_{i \in G \cup X} g_i \le 2 \cdot 5 \cdot 21 + 2 \cdot 5 \cdot 21 = 420$$

which is a contradiction. Thus, our assumption is incorrect and the proposition of the puzzle must be true, i.e., there must be a problem that was solved by at least three girls and at least three boys.

. . .

The next two puzzles are related to something called the **inclusion-exclusion principle**.

In general, for sets A and B, we have the identity $\#(A \cup B) = \#A + \#B - \#(A \cap B)$ where $\#S$ means "the number of elements in set S". This is a special case of the inclusion-exclusion principle [35] for the case of two sets. We can see this intuitively by examining the Venn diagram in Figure 6. If we add the number of elements in sets A and B, then the intersection of the two sets is added in twice. If we subtract the intersection from the sum of the number of elements in the two sets, we get the number of elements in the union.

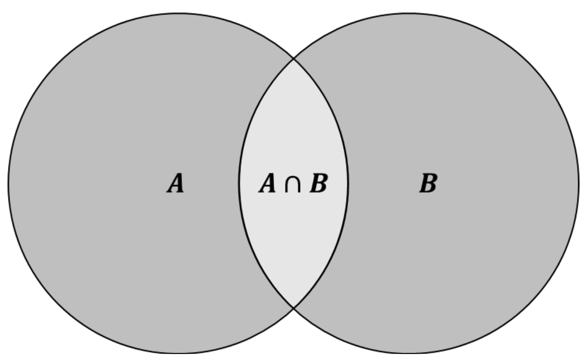

Figure 6. Inclusion-exclusion principle for two sets

Puzzle: If set A has 100 elements, set B has 120 elements and their union has 180 elements, how many elements are common to sets A and B?

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Solution: We first rearrange the general formula as follows:

$$\#(A \cap B) = \#A + \#B - \#(A \cup B)$$

We are given that $\#A = 100, \#B = 120$, and $\#(AUB) = 180$. Thus, we have

$$\#(A \cap B) = 100 + 120 - 180 = 40$$

. . .

Puzzle: A train has 18 passenger cars (attached in a serial manner to a locomotive car). Any five consecutive cars have exactly 199 passengers. If there are a total of 700 passengers on the train, how many passengers are in the middle two cars of the train, i.e., cars number 9 and 10?

Source: 2019 Math Kangaroo Levels 7-8 Problem #30, YouTube, https://youtu.be/7J2DKvlY0T0

Solution: By the rule that we are given, trains 1-5 must have 199 passengers, and trains 14-18 must also have 199 passengers. This leaves 302 passengers to be distributed among trains 6-13.

Let set A represent the passengers in trains 6-10, and let set B represent the passengers in trains 9-13. By the rule given in the puzzle, we know that $\#A = 199$ and $\#B = 199$. We seek the number of passengers in trains 9 and 10, but this is just $\#(A \cap B) = \#A + \#B - \#(A \cup B) = 199 + 199 - 302 = 96$.

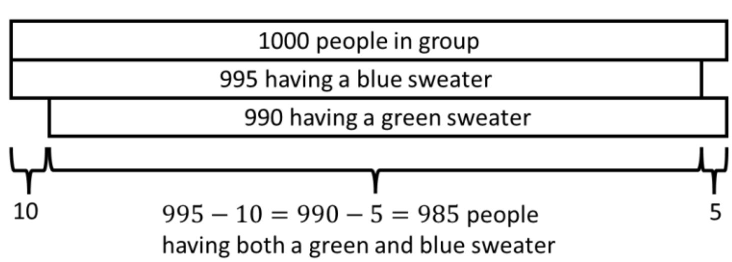
. . .

Puzzle: In a group of 1000 people,

- 995 have a blue sweater
- 990 have a green sweater
- 985 have a red sweater
- 980 have a black sweater
- 975 have a white sweater.

What is the least number of people in the group that have sweaters of all five of the noted colors?

Solution: We start by determining the least number of people that have both blue and green sweaters. Of the 995 people who have a blue sweater at most 10 could not have a green sweater. If more than 10 of the 995 blue sweater people did not have a green sweater, it would not be possible for 990 people to have green sweaters. So, the least number of people who have blue and green sweaters is $995 - 10 = 985$. The figure below may help visualize the situation.



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Next, we determine the least number of people that have a blue, green and red sweater (treating those with both a blue and green sweater as one group of size 985). Using the same logic as before, we want the 15 people that do not have both a blue and green sweater to have a red sweater. In this scenario, there are $985 - 15 = 970$ people who have a blue, green and red sweater (this is a minimum).

Treating the people with a blue, green and red sweater as one group of size 970, we next determine the minimum number of people who have a blue, green, red and black sweater. Using the same reasoning as before, there are $980 - 30 = 950$ who have a blue, green, red and black sweater (this is a minimum).

We play the same game once more, and reason that there is a minimum of $975 - 50 = 925$ people who have sweaters of all five colors.

Alternate solution: Another approach is to determine the number of people not having each color of sweater, and then subtract the sum of these numbers from 1000 to get the solution.

- 5 do not have a blue sweater
- 10 do not have a green sweater
- 15 do not have a red sweater
- 20 do not have a black sweater
- 25 do not have a white sweater.

So, $5 + 10 + 15 + 20 + 25 = 75$ people are lacking at least one sweater of a particular color. (“Worst” case is “no overlap in the 5 groups not having a sweater of a particular color.”) Thus, at least $1000 - 75 = 925$ have sweaters of all five colors.

. . .

Puzzle: “A positive integer is said to be bi-digital if it uses two different digits, with each digit used exactly twice. For example, 1331 is bi-digital, whereas 1113, 1111, 1333, and 303 are not. Determine the exact value of the integer b, the number of bi-digital positive integers.”

[**Author’s Remark**: The statement of the puzzle is slightly ambiguous, e.g., does 71133 bi-digital? I think the intention is that bi-digital numbers consist of exactly four digits and so, 71133 would be excluded.]

Source: Problem A4 from the 2013 Canadian Open Math Challenge, https://www2.cms.math.ca/Competitions/COMC/2013/

Solution: We consider two cases, i.e., when 0 is not one of the two digits, and when it is.

If 0 is not one of the two digits, there are $\binom{9}{2} = \frac{9!}{2!7!} = 36$ ways of choosing the 2 digits, and $\binom{4}{2} = \frac{4!}{2!2!} = 6$ ways of selecting 2 positions for one of the digits, and 2 positions for the other digit. So, in this case, we have $36 \cdot 6 = 216$ possibilities.

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If 0 is one of the two digits, it cannot be the first since that would leave us with a 3-digit number. There are $\binom{9}{1} = 9$ ways of choosing the other (non-zero) digit. The non-zero needs to go first, and then we are left with selecting one of the three remaining positions to locate the non-zero digit. So, in this case, we have $9 \cdot 3 = 27$ possibilities.

Taking both cases into consideration, we have $b = 216 + 27 = 243$ possibilities.

. . .

Puzzle: In a variation of the previous puzzle, find all 9-digit numbers consisting of 3 digits each used exactly three times each, e.g., 800,877,087. Further, the number cannot start with 0.

Solution: To solve this puzzle, we make use of the multinomial coefficient [9] which is defined as

$$\binom{n}{k_1 k_2 \dots k_m} = \frac{n!}{k_1!\, k_2! \dots k_m!}$$

The interpretation of the above formula is “the number of ways of putting n objects into m categories (bins, buckets, etc.) where k_i is the number of objects that go into category i. Further, it is stipulated that $k_1 + k_2 + \cdots + k_m = n$.

For the problem at hand, we consider two cases, i.e., when 0 is one of the selected digits and when it is not.

When 0 is not one of the selected digits, we have $\binom{9}{3} = \frac{9!}{3!6!} = 84$ ways of choosing the 3 digits to be used to construct the 9-digit number. Next, we use the multinomial coefficient formula to compute the number of ways of assigning each position of the 9-digit number to one of the 3 selected digits:

$$\binom{9}{3\ 3\ 3} = \frac{9!}{3!\,3!\,3!} = 1680$$

So, for this case, we have $84 \cdot 1680 = 141{,}120$ possibilities.

For the case where 0 is one of the digits, there are $\binom{9}{2} = 36$ way of choosing the other two (non-zero) digits. Further, we are told that 0 cannot be the first digit. So, one of the other 2 digits needs to be first, i.e., 2 possible choices.

Of the 8 remaining positions in the 9-digit number, 3 positions are assigned to 0, 2 positions are allocated to the digit used in the first position, and 3 positions go to the remaining digit. The number of possibilities is given by the following multinomial coefficient:

$$\binom{8}{3\ 3\ 2} = \frac{8!}{3!\,3!\,2!} = 560$$

So, for this case, we have $2 \cdot 36 \cdot 560 = 40{,}320$.

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Adding the number of possibilities from the two cases, we get the overall answer of $141{,}120 + 40{,}320 = 181{,}440$.

. . .

Puzzle: Figure 7 depicts two lines, with 6 points on one line segment and 4 points on the other line segment. By connecting 3 points, how many different triangles can be formed (one example triangle is shown in the figure).

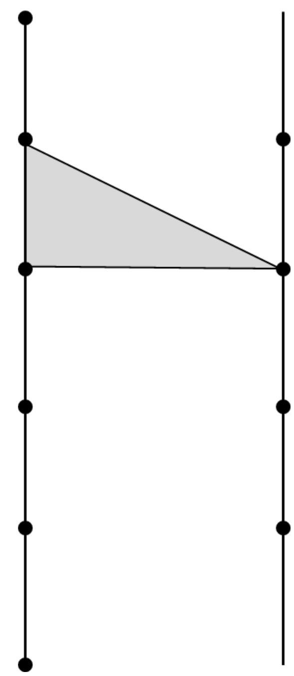

Figure 7. Triangle puzzle

Solution: To form a triangle, we need to select two points from one line segment and one from the other. We can select 2 points from the line segment on the left and 1 point from the line segment on the right in the following number of ways:

$$\binom{6}{2}\binom{4}{1} = 15 \cdot 4 = 60$$

Alternately, we can select 2 points from the line segment on the right and 1 point from the line segment on the left in the following number of ways:

$$\binom{4}{2}\binom{6}{1} = 6 \cdot 6 = 36$$

So, the total number of triangles that can be formed is $60 + 36 = 96$.

. . .

Puzzle: This is similar to the previous puzzle, except that the task is to determine the number of convex quadrilaterals using the points on the two line segments shown in Figure 8.

A polygon (including quadrilaterals) is convex if every point on every line segment between two points inside or on the boundary of the polygon remains inside or on the boundary.

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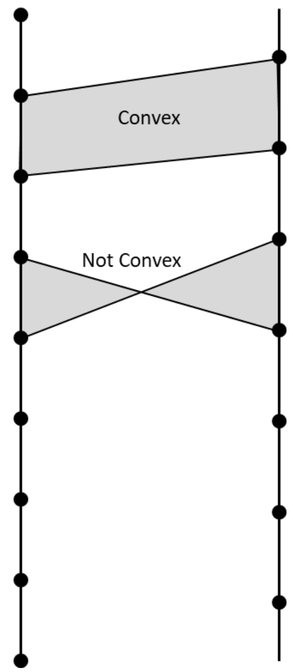

Figure 8. Quadrilateral puzzle

Solution: Starting with the line segment on the left, we first select 2 points out of the 9 points on the line segment. This can be done in $\binom{9}{2} = 36$ ways. We then select 2 points out of the 7 points from the line segment on the right, which can be done in $\binom{7}{2} = 21$ ways. Given the convexity constraint, the top point on the left must be connected to the top point on the right, and the lower point on the left must be connected to the lower point on the right. Thus, we have a total of $36 \cdot 21 = 756$ possible convex quadrilaterals. If we allowed for non-convex quadrilaterals, there would be twice as many possibilities.

. . .

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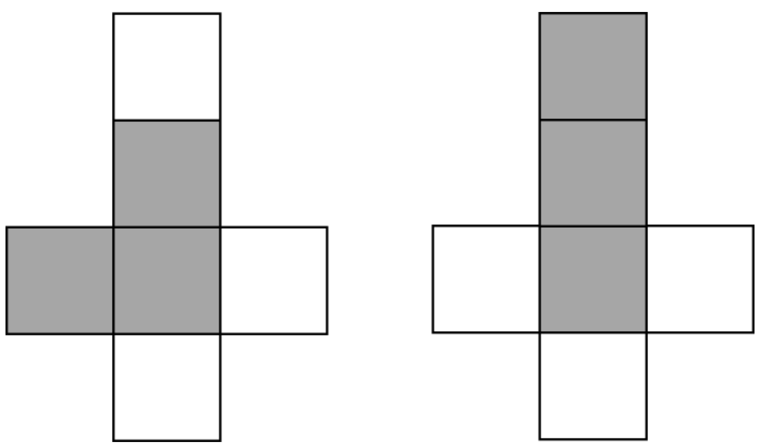
Puzzle: The 6 faces of a cube are colored either black or white, e.g., we could color one face white and the other faces black. How many different ways are there of coloring the cube? Rotations do not count as additional colorings, e.g., there is only one way to color one face white and the others black.

Solution:

Coloring	**Number of Ways**
All faces black	1
All faces white	1
One white face and 5 black	1
One black face and 5 white	1
Two white faces (on opposite sides of the cube) and 4 black faces	1
Two black faces (on opposite sides of the cube) and 4 white faces	1
Two adjacent white faces (sharing an edge) and 4 black faces	1
Two adjacent black faces (sharing an edge) and 4 white faces	1
Three white faces and three black faces	2
Total	**10**

The two cases with 3 white and 3 black faces are shown in the figure below. The cubes are flatted for ease of drawing.



. . .

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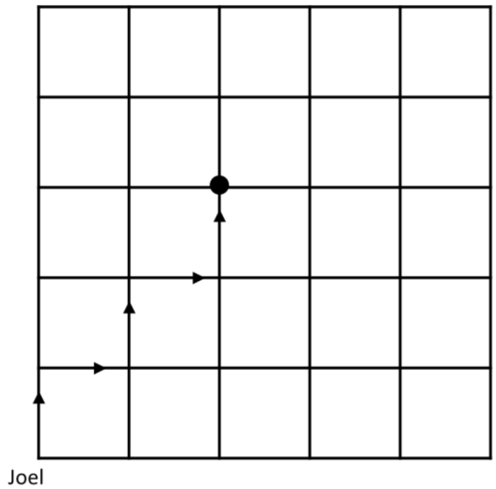
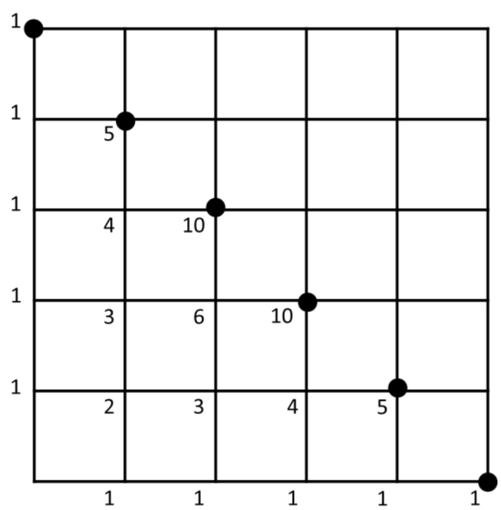
Puzzle: Starting at the bottom left intersection of the street grid below, Joel either goes right or up at each intersection (with equal probability .5). What are the possible locations for Joel after 5 steps? For each location, in how many ways can Joel get there?

The following figure shows the starting position for Joel, and one possible route entailing 5 steps.



Source: YouTube video entitled "Counter-Intuitive Probability Puzzle: Random Walkers Meeting on a Grid" [10]

Solution: The following figure shows the number of ways Joel can arrive at the various intersections. After 5 steps, the only possible locations for Joel are indicated by the black dots.



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The pattern in the above diagram is part of something known as Pascal's triangle [11], where each number in the triangle is the sum of the two numbers above (to the left and right). Several rows of Pascal's triangle are depicted below:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . .

Puzzle: Continuing with the previous puzzle, assume there is another walker (Giannis) who starts from the upper right of the grid. Giannis either goes left or down at each intersection (with equal probability .5). Assuming Joel and Giannis start at the same time and move at the same speed, in how many ways can they meet at the same intersection?

Solution: The path traced by Giannis is symmetric to that of Joel. The two can only meet after 5 steps and the possible meeting points are the black dots shown in the previous figure. Counting up the number of possible ways that the two can arrive at the same intersection at the same time, we have

$$1^2 + 5^2 + 10^2 + 10^2 + 5^2 + 1^2 = 252$$

Although not asked in the puzzle statement, we can also compute the probability that the two will meet. For each walker, there are 2^5 possible paths consisting of 5 steps. So, for the two, there are $2^5 \cdot 2^5 = 32^2 = 1024$ possible pairs of paths. As we know from the previous calculation, out of the 1024 paths, 252 paths result in the two walkers meeting at an intersection. Thus, the probability of the two meeting is

$$\frac{252}{1024} \cong .246$$

. . .

Puzzle (Ants Marching): One ant is stationed at each vertex of a cube for a total of 8 ants. Assuming the ants move continuously and at the same speed, in how many ways can the ants march around the edges of a cube without colliding?

Source: YouTube video entitled "Can You Solve The Ants On A Cube Puzzle?" [12]

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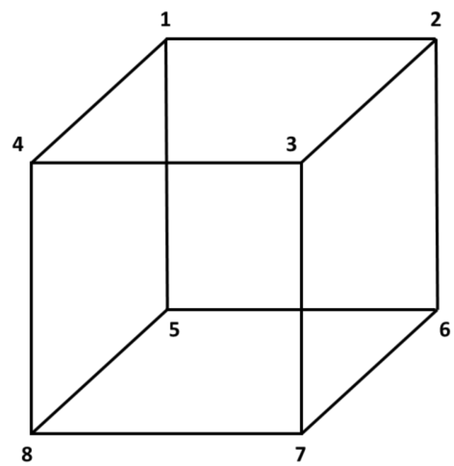
Solution: The ants can move in either 4 step or 8 step cycles. The figure below shows a cube with the ants represented with the numbers from 1 to 8.



For example, the ants on the top can move in a clockwise direction and the ants on the bottom can move in a clockwise direction, without any collisions. This is basically two cycles of length of 4, which we represent as

$$(1,2,3,4)(5,6,7,8)$$

The ants on the top could go counter-clockwise and the ants on the bottom could go clockwise for another possible collision-free march. This is represented as

$$(1,4,3,2)(5,6,7,8)$$

There are a total of four possible 4-cycles if we divide the ants into those on the top and those on the bottom of the cube. We could also divide the ants into left (i.e., ants 1,4,5,8) and right (i.e., ants 2,3,6,7) for another four 4-cycles. An example of one such 4-cycle is

$$(1,4,8,5)(3,2,6,7)$$

Further, we can divide the ants into front (i.e., ants 3,4,7,8) and back (i.e., ants 1,2,5,6) for yet another four 4-cycles, e.g.,

$$(3,4,8,7)(1,2,6,5)$$

So, we have a total of 12 pairs of 4-cycles that allow the ants to march without collision.

The ants can also move in 8-cycle paths without collision, e.g., see the path in the following figure (noting that the cube has been flattened to simplify the illustration). This can be represented more succinctly as $(1,2,3,4,8,7,6,5)$. Another 8-cycle can be obtained by reversing the flow of the path shown in the figure, i.e., $(1,5,6,7,8,4,3,2)$.

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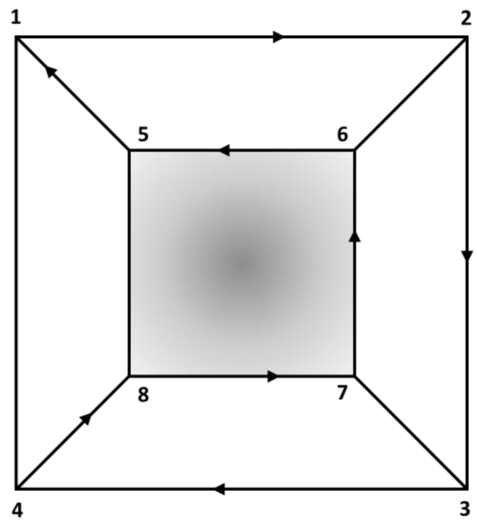
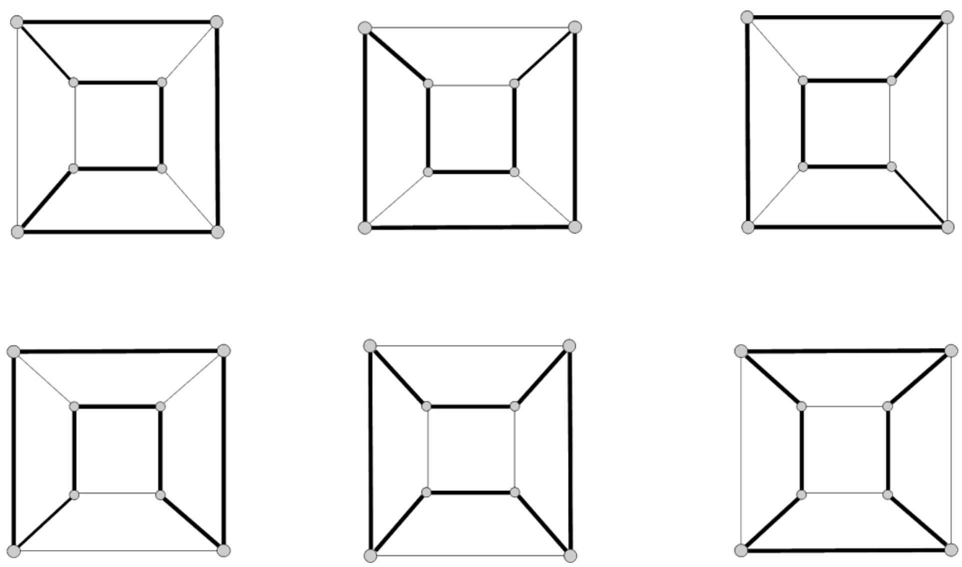


In all, there are 12 distinct 8-cycles (all of which are collision-free for the ants). In the following figure, each graph represents two 8-cycles. The graph in the upper left is the one that we have already discussed. These 8-cycles are examples of what are known as Hamiltonian cycles [13].



In summary, the ants can move along the edge of a cube, in a collision free manner, in 24 different ways, i.e., via 12 distinct pairs of 4-cycles, or 12 distinct 8-cycles.

. . .

Puzzle: Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the order. How many arrangements are possible?

Source: Problem #15 from the Indian Olympiad Qualifier Mathematics (IOQM) 2021, https://www.mtai.org.in/wp-content/uploads/2021/01/IOQM-question-paper.pdf

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Solution: Let the first couple be represented by A_1, A_2, the second by B_1, B_2 and the third by C_1, C_2. Let the possible positions be represented by the following table

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}

Position a_{11} can be filled in 6 ways, e.g., say that A_1 goes in that position.

A_1		

Once position a_{11} is filled, we can fill a_{12} in 4 ways, e.g., say that B_1 goes in that position.

A_1	B_1	

After a_{11} and a_{12}, we have only two choices for a_{13}, e.g., say that C_1 goes in that position.

A_1	B_1	C_1
B_2 or C_2		

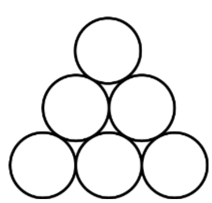
This leaves only 2 choices for a_{21} (say we choose B_2). At this point, the last two spot are determined, e.g.,

A_1	B_1	C_1
B_2	C_2	A_2

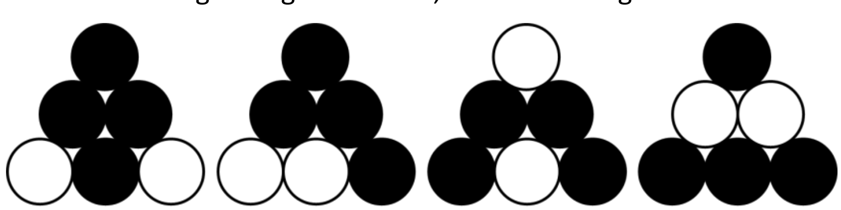
So, in total, we have $6 \cdot 4 \cdot 2 \cdot 2 = 96$ possible arrangements for the photograph.

. . .

Puzzle: In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered to be the same.



There are only four such colorings using two colors, as shown in figure below.



(i) (ii) (iii) (iv)

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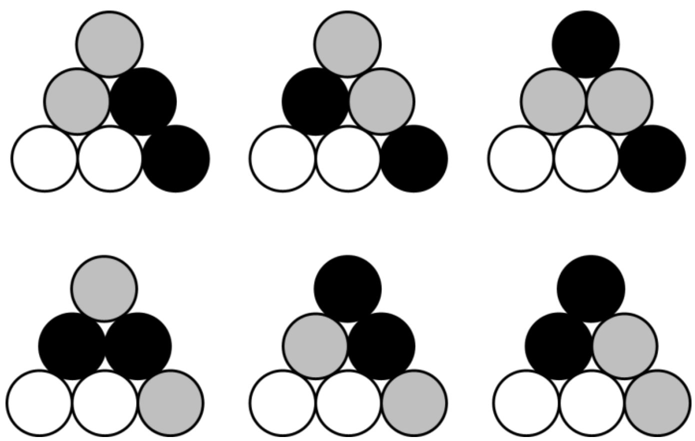
In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored gray with the condition that rotated or reflected configurations are considered to be the same, i.e., only count once?

Source: Problem #26 from the Indian Olympiad Qualifier Mathematics (IOQM) 2021, https://www.mtai.org.in/wp-content/uploads/2021/01/IOQM-question-paper.pdf

Solution: The problem is complicated by the condition that we consider rotated or reflected configurations of the disks as being the same. For the example given in the statement of the puzzle, there are $\binom{6}{2} = 15$ ways of selecting 2 white disks out of 6. However, configuration (ii) in the previous figure represents 6 equivalent configurations, and configurations (i), (ii) and (iv) each represent 3 equivalent configurations. Thus, we are left with just 4 distinct configurations (excluding rotations and reflections).

In any event, the example given in the statement of the puzzle can be used to solve the puzzle involving black, white and gray disks. For each of the four configurations in the black and white puzzle, we only need to determine the $\binom{4}{2} = 6$ ways of coloring two of the black disks gray (while subtracting any configurations that are equivalent under reflection or rotation).

We first consider configuration (ii). The six ways of selecting two of the black disks and coloring them gray are shown in the following figure. There are no equivalences due to reflections or rotations.



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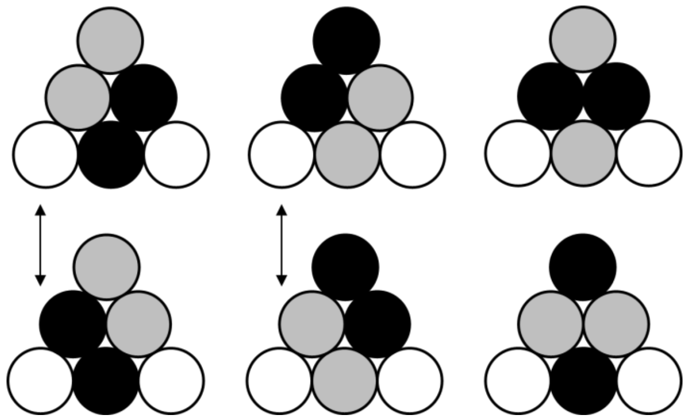
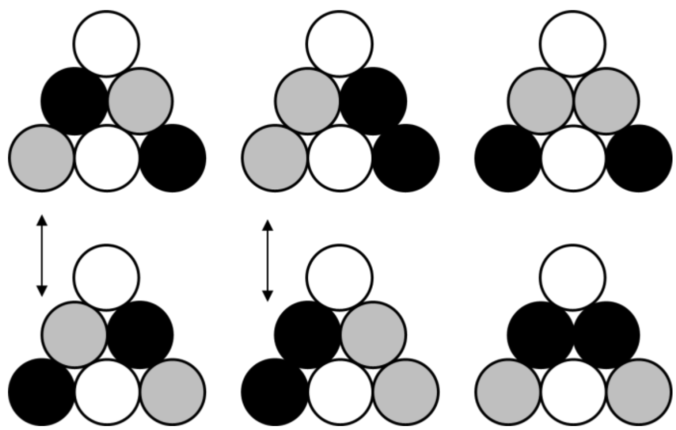
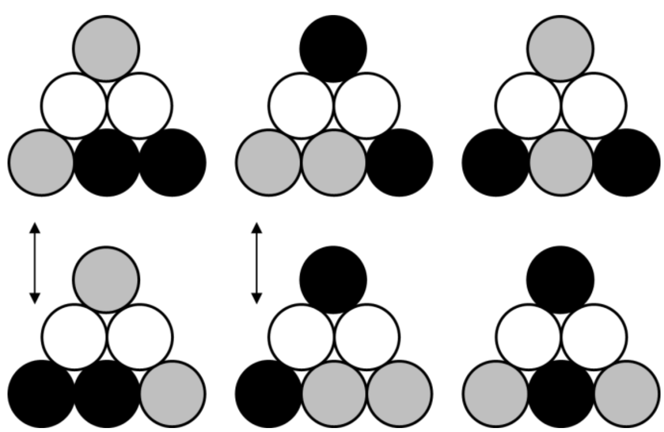
The white/black/gray configurations that we can derive from white/black configuration (i) are shown in the following figure. There are two sets of equivalent configurations (indicated by the arrow) and thus, we have only 4 distinct configurations in this case.



The white/black/gray configurations that we can derive from white/black configuration (iii) are shown in the following figure. There are only 4 distinct configurations in this case.



The white/black/gray configurations that we can derive from white/black configuration (iv) are shown in the following figure. There are only 4 distinct configurations in this case.



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Thus, there are $6 + 4 + 4 + 4 = 18$ possible configurations with 2 white, 2 black and 2 gray disks (excluding equivalent configurations based on reflections or rotations).

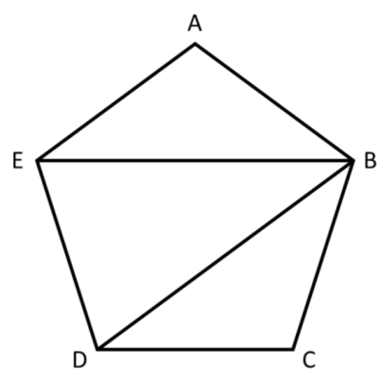
. . .

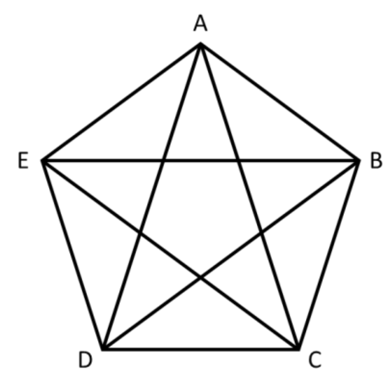
The next puzzle involves a bit of graph theory.

A **graph** (sometimes called an undirected graph to distinguish it from a directed graph, or a simple graph to distinguish it from a multigraph) is a pair $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set whose elements are called vertices (sometimes called nodes), and $\mathcal{E}$ is a set of paired vertices known as edges (sometimes called links). In a **complete graph**, all pairs of vertices are connected via an edge.

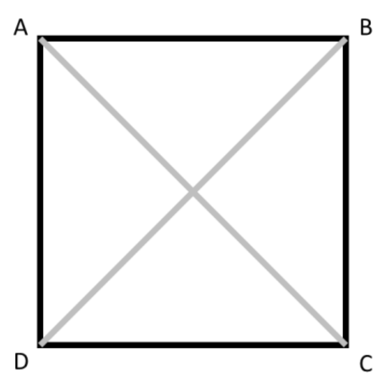
The **order** of a graph is its number of vertices, and the **size** of a graph is its number of edges.

In the figure below, the graph on the left has vertex set $V = \{A, B, C, D, E\}$ and edge set $\mathcal{E} = \{(A, B), (B, C), (C, D), (D, E), (E, A), (E, B), (D, B)\}$. The graph on the left is not complete since there is not an edge between each pairs of vertices, e.g., there is no edge between A and D. The graph on the right is a complete graph of order 5 and size 10. [**Warning**: The intersection of the edges inside the figure on the right do not constitute additional vertices.]





It is also possible to "label" the edges of a graph with different colors. The following graphs have edges of two different colors, i.e., black or gray.



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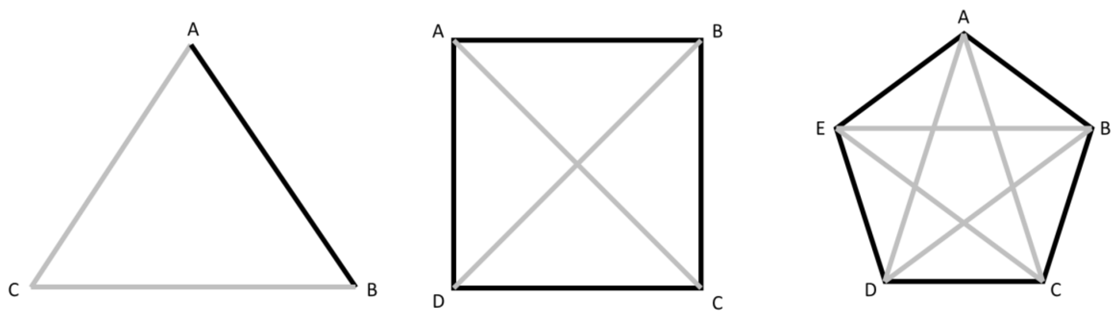
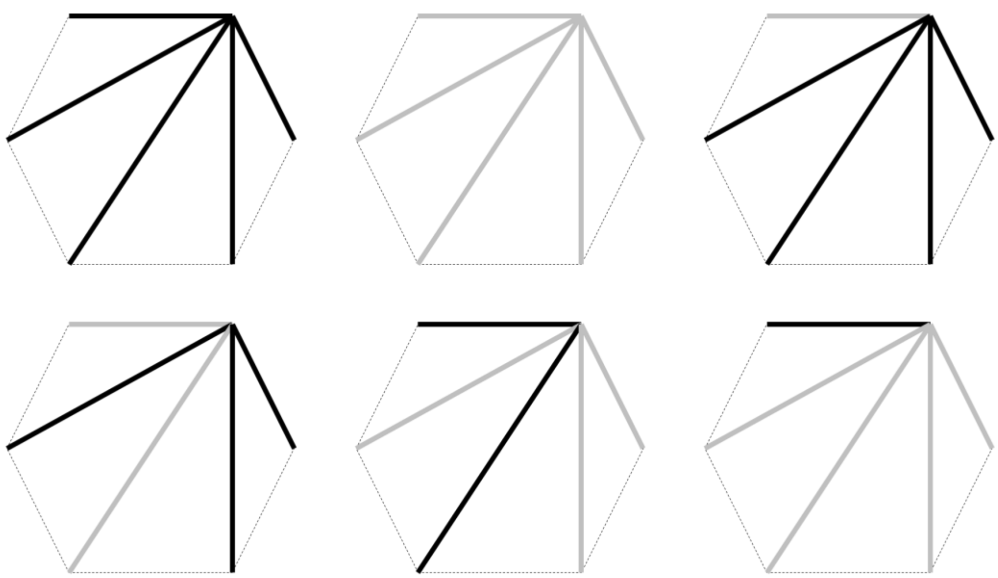
Puzzle: What is the smallest order graph such that no matter how the edges are labeled with 2 colors there will be a triangle of one color or the other.

Source: This problem is a special case of a sub-branch of graph theory known as Ramsey's theory [14]

Solution: For the complete graphs of order 3, 4 or 5, there are configurations that do not contain a triangle of a single color. In each of the three graphs in the following figure, there is no triangle of a single color whose vertices are a subset of the vertices of the encompassing graph.



In the case of a complete graph of order 6, there must be at least one triangle of a single color. To see this, we consider any vertex. There are 6 cases in terms of the number of black and gray edges associated with a given vertex (as shown in the following figure). The key point is that in each of the 6 cases there are always at least three edges of the same color.



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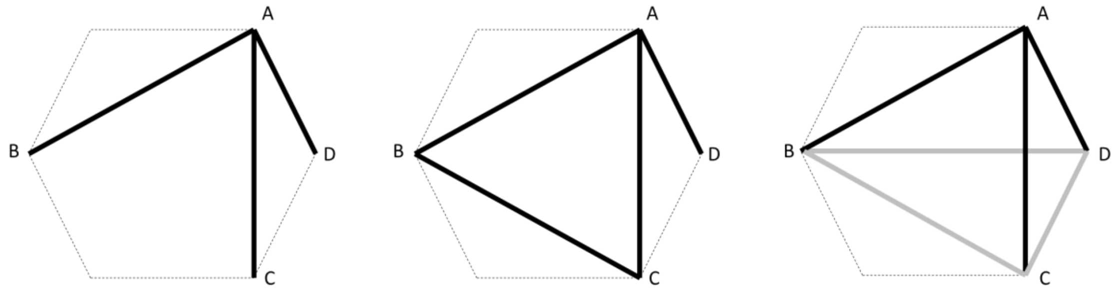
Without loss of generality, consider a vertex with three black edges (see vertex A on the left of the figure below). There are only two possibilities concerning the edges connecting vertices B, C and D.

- At least one of the edges is black, in which case we get a black triangle (see triangle ABC in the center of the figure below).
- All the edges connecting B, C and D are gray, in which case we get a gray triangle (see triangle BCD on the right of the figure below).



So, no matter how we color the edges of an order 6 complete graph, there must be a least one triangle, within the graph, all of whose edges are of the same color.

. . .

Puzzle: You are given two boxes of marbles. One box has 1000 red marbles and the other box has 1000 blue marbles. You are to distribute the marbles into 4 containers in any way that you choose just so that every marble is in one of the 4 containers. After you are done, the containers are closed and shaken, and numbered from 1 to 4 (without your knowledge of which container is assigned which number). You are then asked to pick a number from 1 to 4. At which point, another person (who is blindfolded) is asked to pick one marble from the container associated with the number that you picked. Your task is to distribute the marbles in such a way as to maximize the probability that a red marble is selected.

Solution: One thought is to distribute the marbles evenly among the containers, i.e., 250 red and 250 blue marbles in each container. In this case, no matter what container is chosen by the other person, he or she has a 50% chance of picking a red marble.

We can do better, however. Here is the scheme:

- In three of the containers, put a single red marble.
- Put the remaining 997 red and 1000 blue marbles in the 4^{th} container.

Each container has a .25 probability of being selected. If one of the three containers with a single red marble is selected, there is a 100% chance of a red marble being selected. If the container with the other 1997 marbles is selected, there is a $\frac{997}{2000} = .4985$ probability that a red marble will be selected. So, the probability of a red marble being selected is

$$.25(1) + .25(1) + .25(1) + .25(.4985) = .874625$$

What would be the best solution if you needed to distribute the 2000 marbles among 100 containers?

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3.3 Sequences

A sequence is a list of numbers that follow some pattern. A series usually refers to the sum of the numbers in a sequence. Unfortunately, the terms are not used consistently in the mathematical literature.

A simple example is an **arithmetic sequence** where the next number is determined by adding a given constant, e.g.,

$$-7, -3, 1, 5, 9, 13, \ldots$$

The pattern is to add 4 to the current term to get the next term. We can represent this sequence as $\{x_n\}$ where $x_0 = -7$ and $x_{n+1} = x_n + 4$.

In the case of **geometric sequences**, we multiple by a given constant to get the next term, e.g.,

$$2, 6, 18, 54, 162, \ldots$$

The pattern is to multiple the current term by 3 to get the next term. We can represent this sequence as $\{x_n\}$ where $x_0 =$ and $x_{n+1} = 3x_n$.

The two previous examples are generated by what are called **difference equations**. For example, the difference equation $x_{n+1} = x_n + x_{n-1}$ with initial conditions $x_0 = 1,\ x_1 = 1$ generates the famous **Fibonacci sequence**, i.e.,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

If we change the initial conditions in the above difference equation to $x_0 = 2,\ x_1 = 1$, we get the **Lucas series** i.e.,

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

The Lucas series is a sequence but is commonly referred to as a series (an example of the inconsistent usage of the two terms that was mentioned earlier).

Puzzle: Find the next term in the sequence

$$3, 13, 1113, 3113, 132113, 1113122113, 311311222113, \ldots$$

Hint: This sequence does not follow a mathematical pattern of the type described in the previous examples.

Solution: This is known as a **look-and-say sequence** [15]. The idea is to start with an single digit number (3 for our example) and then describe the number to get the next element in the list. The first element is described as "One instance of the number Three" or 13 in shorthand (this is the next element in the sequence). We then describe 13 as "One One and One Three", i.e., 1113. Next, we describe in words 1113, i.e., Three Ones and One Three (3113).

We can start with any single digit d. The associated sequence is

$$d, 1d, 111d, 311d, 13211d, 111312211d, 31131122211d, \ldots$$

. . .

Yet another interesting and unusual sequence is the **Kolakoski sequence** [16]. This sequence is comprised of only two numbers, i.e., 1 and 2. The sequence is based on runs (or blocks), which are subsequences consisting of the same number. For example, 1,1,1 is a block of length three. The i^{th} element in the sequence indicates the length of the i^{th} block. The first element is defined to be 1.

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The sequence is constructed as follows:

- We are given that the first element of the sequence is 1. This tells us that the first run is of length 1. Thus, the second element cannot be 1, and we are left with only one choice for the second element, i.e., 2.
- Since the second element is 2, we know that the 2-run (i.e., run consisting of the number 2) must be of length 2. This implies that the third element must be 2. So far, we have 1,2,2.
- Since the third element is 2, we know that the third run is of length 2, and it must be two ones; otherwise, the second run would not be of length 2. So, we have 1,2,2,1,1.
- The 4^{th} and 5^{th} runs are of length 1, and so we must have 1,2,2,1,1,2,1. The process continues in a similar manner.

We list a bit more of the sequence below:

$$1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 1, 2, 2, 1,$$

$$2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 1, 2,$$

$$2, 1, 2, 1, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 2, 2, \ldots$$

If we write down in order the length of each run, we get exactly the same sequence.

Variations of this sequence are possible using numbers other than 1 and 2. For example, we could use 1 and 3, and start with 1. This leads to the following sequence (with each run in parenthesis)

$$(1), (3,3,3), (1,1,1), (3,3,3), (1), (3), (1), (3,3,3), (1,1,1), (3,3,3), \ldots$$

The same idea can be applied to sets with more than two elements. For example, take the set $\{1,2,3\}$ and start with 1.

- The first element is given to be 1. The second element can be 2 or 3. Let's go with 2 which implies that we also need a 2 in the 3^{rd} position. So, we have 1,2,2.
- Since the 3^{rd} element is 2, we must have the same number for the 4^{th} and 5^{th} elements, and it could be 1 or 3. This type of decision point will occur repeatedly. So, let's define a rule, i.e., when two numbers are possible for the next element in the sequence, pick the number that was last used further back in the sequence. So, in this case, we choose 3 since it hasn't been used at all. Since the second element is 2, we must have a run of two threes. So far, we have $1,2,2,3,3$.
- We next need a run of length 3, and could use 1 or 2 but our rule says that we should use 1. Thus, we have $1,2,2,3,3,1,1,1$.
- Next, we need another run of length 3 and our rule says to use 2. We now have $1,2,2,3,3,1,1,1,2,2,2$.

Continuing in this manner, we can list a few more elements of the sequence (with each run in parenthesis)

$$1,2,2,3,3,1,1,1,2,2,2,3,1,2,3,3,1,1,2,2,3,3,3, \ldots$$

For more on this sequence, see the Online Encyclopedia of Integer Sequences (OEIS) at https://oeis.org/A079729.

. . .

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Puzzle: Write down the integer sequence $\{x_n\}$ satisfying the conditions

1. The sequence is non-decreasing
2. x_n is the number of times that n occurs in the sequence.
3. The first term is 1.
4. For $n > 1$, each x_n is the smallest unique integer which makes it possible to satisfy Condition #1.

Solution: This is known as the Golomb sequence, see the Wikipedia article on this topic [17]. The first few terms of the sequence are shown below:

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 10, 10, 10, 10, 10,$$
$$11, 11, 11, 11, 11, 12, 12, 12, 12, 12, 12, \ldots$$

. . .

Puzzle: Write down the self-describing integer sequence $\{x_n\}$ where each term indicates the maximum number of repeated blocks of numbers in the sequence immediately preceding that term. Further, the first element in the sequence is 1. The definition of "block" differs from the definition of "block" as defined for the Kolakoski sequence. For this sequence, a block is any pattern of integers, e.g., 1,1,2 is a block.

Hint: The first few elements are

$$(1,1,2), (1,1,2), \mathbf{2}, 2, 3, (1,1,2), (1,1,2), \mathbf{2}, 2,3,2,1, \ldots$$

The calculation gets a bit tricky. For example, the 7th element in the sequence is 2 because there are two blocks of 1,1,2 preceding it. A similar issue occurs with the 16th element.

Solution: This is known as Gijswijt's sequence, see the Wikipedia article on this topic [18].

. . .

Puzzle: Given the sequence defined as

$$x_0 = 3, \qquad x_1 = 4, \qquad x_{n+1} = x_{n-1}^2 - nx_n$$

determine a formula for x_n in terms of n.

Source: Introductory Problem #2 from "101 Problems in Algebra: From the Training of the USA IMO Team" [28]

Solution: We first compute several of the initial terms:

$$x_2 = x_{1+1} = x_{1-1}^2 - 1 \cdot x_1 = 3^2 - 4 = 5$$

$$x_3 = x_{2+1} = x_{2-1}^2 - 2 \cdot x_2 = 4^2 - 10 = 6$$

$$x_4 = x_{3+1} = x_{3-1}^2 - 3 \cdot x_3 = 5^2 - 18 = 7$$

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It appears that the general form of x_n is $n+3$. We can prove this using strong induction (see the background material on the second principle of finite induction in Section 2.1). Assume the result is true for $n = 0,1,2, \dots, k-1$. To complete the induction proof, we need to show $x_k = k+3$. By definition

$$x_k = x_{(k-1)+1} = x_{(k-1)-1}^2 - (k-1)x_{k-1}$$

$$x_k = x_{k-2}^2 - (k-1)x_{k-1}$$

From the induction hypothesis, we have

$$x_k = (k+1)^2 - (k-1)(k+2) = k^2 + 2k + 1 - (k^2 + k - 2) = k + 3$$

. . .

Rational numbers (i.e., numbers that can be represented as a fraction such as $\frac{25}{3}$) have either a finite decimal representation or a repeating pattern in their decimal representation. Irrational numbers are numbers that are not rational.

Puzzle: Consider the decimal formed by listed all positive integers in sequence, i.e.,

$$0.123456789\ 10\ 11\ 12\ 13 \dots$$

(Spaces are added for ease of reading only.)

Prove that this number is irrational and as such, its decimal representation is not periodic (repeating).

Solution: Assume that the number does have a period of size n. Further, assume the period starts with the single digit a. Consider the number $bb \dots b$ (when it appears in our sequence) where $b \neq a, b \neq 0$, and b is repeated $n+1$ times. This means we have a gap of $n+1$ digits (in our sequence) that does not have the digit a which contradicts our assumption.

. . .

Puzzle: Determine the next few terms and the general pattern behind the sequence

0

1 1 0

2 2 2 0

3 2 4 1 1 0

4 4 4 1 4 0

5 5 4 1 6 2 1 0

Sources: This is known as the inventory sequence.

- YouTube video entitled “A Number Sequence with Everything” [19]
- “Inventory sequence” [20]

Solution: Each line in the sequence is an inventory of the number of appearances of non-negative integers thus far in the sequence. When no appearances of an integer are found, one goes to the next line and restarts the inventory (starting from 0 each time).

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We start with 0 in the first row, indicating that before we started (empty sequence) there are zeros appearances of 0.

Next, we restart the inventory on the second row. The first element indicates there is 1 appearance of 0 thus far. The second element indicates one appearance of 1 (i.e., the first element in the row). The third element indicates 0 appearances of 2 so far. Since we hit a 0, we go to the next row and restart the inventory.

Using this logic, the next row (beyond the ones provided in the statement of the puzzle) is

6 7 5 1 6 3 3 1 0

. . .

Puzzle: Determine the next few terms and the general pattern behind the sequence

0,1,2,5,3,6,9,4,7,10,10,5,8,8,11,11,11, 6,14,9,9,12,12,12,15,12,7,18,15,10,10,10,13,13,13,13, …

Source: ”Traversing the Infinite Sidewalk” [21]

Solution: [**Author's Remark**: Without the following background, I think this sequence is close to impossible to phantom.]

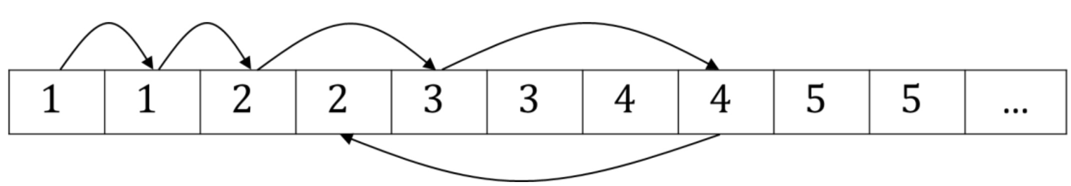
The sequence is based on the infinite sidewalk shown in the figure below:

1	1	2	2	3	3	4	4	5	5	…

The number in each cell indicates how many steps one is allowed to jump from the cell to another cell (either before it or after). One must jump to the exact number shown in the cell.

The n^{th} term in the sequence indicates the fewest number of jumps required to reach the n^{th} cell along the sidewalk. For example,

- Since we start in the first cell, it takes 0 jumps and thus, the first term in the sequence is 0.
- The second cell can be reached in 1 forward jump from cell 1.
- The third cell can be reached by two forward jumps of length 1, starting from cell 1.
- Reaching the 4^{th} is tricky and gives an inkling of the difficulty in computing the sequence. The following figure shows the minimum number of steps in going from the 1^{st} cell to the 4^{th} cell.



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Puzzle: Determine the next few terms and the general pattern behind the sequence

$$0, 0, 1, 0, 2, 0, 2, 2, 1, 6, 0, 5, 0, 2, 6, 5, 4, 0, 5, 3, 0, 3, 2, 9, 0, 4, 9, 3, 6, 14, \ldots$$

Sources:

- YouTube video entitled "Don't Know (the Van Eck Sequence)" [22]
- "Van Eck's Sequence" [23]

Solution: The n^th element in the sequence tells how recently the $n-1$ element in the sequence previously appeared. For example, consider the first 9 elements of the sequence

$$0, 0, 1, 0, 2, 0, 2, 2, 1$$

The 10^th element is 6 since the 9^th element (the number 1) last appeared 6 steps before (in cell #3). As explained in the referenced YouTube video, the sequence is highly irregular and much more is not known than known about the sequence.

3.4 Sums of Infinite Series

3.4.1 Telescoping Sums

The following is an example of a finite **telescoping sum**:

$$(1-2)+(2-3)+(3-4)+(4-5)+(5-6)$$

Most of the terms cancel each other and we are left with $1-6=-5$.

The general format for a finite telescoping sum is

$$\sum_{i=1}^{n-1}(a_i - a_{i+1}) = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) = a_1 - a_n$$

This technique is useful in reducing summations into simpler problems.

For example, assume we want to find the following sum for any possible value of n:

$$\sum_{i=2}^{n}\frac{1}{i^2-1}$$

We first reduce $\frac{1}{i^2-1}$ using the technique of partial fractions, i.e.,

$$\frac{1}{i^2-1} = \frac{1}{(i-1)(i+1)} = \frac{A}{i-1} + \frac{B}{i+1} = \frac{A(i+1)+B(i-1)}{(i-1)(i+1)} = \frac{(A+B)i+(A-B)}{i^2-1}$$

This gives us two equations to solve

$$A+B=0$$

$$A-B=1$$

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Applying some algebra, we have that $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Using this result, we can write our summation as

$$\sum_{i=2}^{n} \frac{1}{i^2-1} = \frac{1}{2}\sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i+1}$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right]$$

$$= \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right]$$

For example, if $n = 1000$, we can use the above formula to determine that

$$\sum_{i=2\cdot}^{1000} \frac{1}{i^2-1} = \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{1000}-\frac{1}{1001}\right] = \frac{1{,}499{,}499}{2{,}002{,}000} = 0.749\overline{0004995}$$

(The overline in the above result means that 0004995 repeats indefinitely.)

We can also use the formula to determine the sum for $n = \infty$, i.e.,

$$\sum_{i=2}^{\infty} \frac{1}{i^2-1} = \lim_{n\to\infty} \frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right] = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$$

(For the reader not familiar with the concept of a limit from calculus, see Section 2.2.)

. . .

There is also a shorthand notation for products. Some examples

$$\prod_{k=1}^{n} k = 1\cdot 2\cdot 3\cdot \ldots \cdot n$$

$$\prod_{k=1}^{5} \frac{1}{k^2+3k} = \frac{1}{4}\cdot\frac{1}{10}\cdot\frac{1}{18}\cdot\frac{1}{28}\cdot\frac{1}{40}$$

We use this notation in the following puzzle.

Puzzle: Here's a more complex example that also involves telescoping sums but not in a straightforward way. Determine the value of

$$\frac{1}{3}+\frac{3}{3\cdot 7}+\frac{5}{3\cdot 7\cdot 11}+\frac{7}{3\cdot 7\cdot 11\cdot 15}+\cdots+\frac{2n-1}{\prod_{k=1}^{n}(4k-1)}+\cdots$$

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Solution: The key observation is that

$$\frac{1}{3}-\frac{1}{3\cdot 7}=\frac{6}{3\cdot 7}=\frac{2\cdot 3}{3\cdot 7}$$

$$\frac{1}{3\cdot 7}-\frac{1}{3\cdot 7\cdot 11}=\frac{10}{3\cdot 7\cdot 11}=\frac{2\cdot 5}{3\cdot 7\cdot 11}$$

$$\frac{1}{3\cdot 7\cdot 11}-\frac{1}{3\cdot 7\cdot 11\cdot 15}=\frac{14}{3\cdot 7\cdot 11\cdot 15}=\frac{2\cdot 7}{3\cdot 7\cdot 11\cdot 15}$$

$$\dots$$

$$\frac{1}{\prod_{k=1}^{n-1}(4k-1)}-\frac{1}{\prod_{k=1}^{n}(4k-1)}=\frac{2(2n-1)}{\prod_{k=1}^{n}(4k-1)}$$

$$\dots$$

Thus, we can write the first n terms of the original summation as

$$S_n=\frac{1}{3}+\frac{1}{2}\left[\left(\frac{1}{3}-\frac{1}{3\cdot 7}\right)+\left(\frac{1}{3\cdot 7}-\frac{1}{3\cdot 7\cdot 11}\right)+\left(\frac{1}{3\cdot 7\cdot 11}-\frac{1}{3\cdot 7\cdot 11\cdot 15}\right)+\cdots\right.$$
$$\left.+\frac{1}{\prod_{k=1}^{n-1}(4k-1)}-\frac{1}{\prod_{k=1}^{n}(4k-1)}\right]$$

Most of the terms cancel out each other, and we have

$$S_n=\frac{1}{3}+\frac{1}{2}\left[\frac{1}{3}-\frac{1}{\prod_{k=1}^{n}(4k-1)}\right]$$

This leads us to our final result, i.e.,

$$\frac{1}{3}+\frac{3}{3\cdot 7}+\frac{5}{3\cdot 7\cdot 11}+\frac{7}{3\cdot 7\cdot 11\cdot 15}+\cdots+\frac{2n-1}{\prod_{k=1}^{n}(4k-1)}+\cdots=\lim_{n\to\infty}S_n=\frac{1}{3}+\frac{1}{2}\left[\frac{1}{3}-0\right]=\frac{1}{2}$$

Alternate solution:

Let

$$S=\frac{1}{3}+\frac{3}{3\cdot 7}+\frac{5}{3\cdot 7\cdot 11}+\frac{7}{3\cdot 7\cdot 11\cdot 15}+\cdots$$

and

$$A=\frac{1}{3}+\frac{1}{3\cdot 7}+\frac{1}{3\cdot 7\cdot 11}+\frac{1}{3\cdot 7\cdot 11\cdot 15}+\cdots$$

Note that

$$2S+A=1+\frac{1}{3}+\frac{1}{3\cdot 7}+\frac{1}{3\cdot 7\cdot 11}+\frac{1}{3\cdot 7\cdot 11\cdot 15}+\cdots=1+A$$

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Solving the equation $2S + A = 1 + A$, we get $S = \frac{1}{2}$.

. . .

Puzzle: Determine the following sum

$$\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{4k(k+1)-1}+\sqrt{4k(k+1)+1}}$$

Solution: We first write the expression in a more compact form

$$\sum_{n=1}^{2k(k+1)} \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}$$

Next, we multiple the expression within the summation by $\frac{\sqrt{2n-1}-\sqrt{2n+1}}{\sqrt{2n-1}-\sqrt{2n+1}} = 1$. This is a common technique to remove radicals from the denominator of an expression.

$$\sum_{n=1}^{2k(k+1)} \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}$$

$$= \sum_{n=1}^{2k(k+1)} \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}} \cdot \frac{\sqrt{2n-1}-\sqrt{2n+1}}{\sqrt{2n-1}-\sqrt{2n+1}}$$

$$= -\frac{1}{2} \sum_{n=1}^{2k(k+1)} \sqrt{2n-1} - \sqrt{2n+1}$$

$$= -\frac{1}{2}\left[\left(1-\sqrt{3}\right) + \left(\sqrt{3}-\sqrt{5}\right) + \left(\sqrt{5}-\sqrt{7}\right) + \cdots + \left(\sqrt{4k(k+1)-1} - \sqrt{4k(k+1)+1}\right)\right]$$

$$= -\frac{1}{2}\left(1 - \sqrt{4k(k+1)+1}\right) = -\frac{1}{2}\left(1 - \sqrt{(2k+1)^2}\right) = -\frac{1}{2}(1-2k-1) = k$$

3.4.2 Alternating Series

An alternating series is one in which the elements of the series alternate sign. The general format is

$$\sum_{n=n_0}^{m} (-1)^n a_n$$

The value of m in the above expression can be finite or m can be infinite.

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For example, the Mercator series (shown below) is equal to the natural logarithm (i.e., log base e) of $1 + x$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \ln(1 + x)$$

For $x = 1$, the Mecator series is the well-known **alternating harmonic series**, i.e.,

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots$$

If we take the absolute value of the elements in the above series, we get what is known as the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

Unlike the alternating harmonic series, the harmonic series does not converge, i.e., its terms add to infinity.

The order of the terms in an alternating series does matter. For example, consider the following rearrangement of the alternating harmonic series

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) + \cdots$$

The above series is composed of blocks of three in the following form

$$\frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k}, \qquad k = 1,2,3,\ldots$$

We can regroup the terms in the above series as follows

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\right) = \frac{1}{2}\ln 2$$

So, the same terms when rearranged sum to a different value!

The various rearrangements of the alternating series can be categorized if we limit ourselves to simple rearrangements, where a **simple rearrangement** of an alternating series is a rearrangement of the series in which the positive terms of the rearranged series occur in the same order as in the original series and the negative terms occur in the same order as in the original series.

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In a rearrangement of the alternating harmonic series, let p_k be the number of positive terms in the first k terms of a rearrangement, and let α be the ratio of positive terms to the total number of terms as $k \to \infty$, i.e.,

$$\alpha = \lim_{k\to\infty} \frac{p_k}{k}$$

The following theorem provides a way to determine the exact value of a simple rearrangement of the alternating harmonic series when α exists.

Theorem 7. *A simple rearrangement of the alternating harmonic series converges to*

$$ln\, 2 + \frac{1}{2} ln \left(\frac{\alpha}{1-\alpha}\right)$$

if and only if α exists.

Proof: see the article "Rearranging the Alternating Harmonic Series" [7].

Let's apply the above theorem to our previous example, i.e.,

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots$$

The pattern is 1 positive term every three terms, and so $\alpha = \frac{1}{3}$. Using the formula from the theorem, we have that the sum is

$$\ln 2 + \frac{1}{2}\ln\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right) = \ln 2 + \frac{1}{2}\ln\frac{1}{2} = \ln 2 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 2$$

which agrees with our previous analysis.

Exercises: Use **Theorem 7** to determine the sum of the following rearrangement of the alternating harmonic series

$$a)\ \ 1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots$$

$$b)\ \ \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2}\right) + \left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{4}\right) + \cdots$$

Answers: a) $\frac{3}{2}\ln 2$; b) $2\ln 2$

On this topic, there is also something called Riemann's rearrangement theorem [8] which says that for any conditionally convergent series, the series can be rearranged to achieve any sum that you wish, including infinity. (A series is conditionally convergent if $\sum_{n=n_0}^{\infty} a_n$ converges but $\sum_{n=n_0}^{\infty} |a_n|$ diverges.) Riemann's rearrangement theorem, however, is an existence theorem and gives no guidance on how to make the desired rearrangement.

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Puzzle: Using what you've learned in this section and the previous section, determine the value of the following expression

$$\sum_{x=1}^{\infty} \frac{1}{2x^2+7x-4}$$

Solution: The general approach here is to first use the partial fraction technique on the expression within the summation and then compare to some rearrangement of the alternating harmonic series.

$$\frac{1}{2x^2+7x-4} = \frac{1}{2}\left(\frac{1}{x^2+\frac{7}{2}x-2}\right) = \frac{1}{2}\left(\frac{1}{\left(x-\frac{1}{2}\right)(x+4)}\right)$$

We factored out the 2 from the denominator so that what is left can be represented as a partial fraction with one nominator being the negative of the other and thus forming an alternating series.

$$\frac{1}{\left(x-\frac{1}{2}\right)(x+4)} = \frac{A}{x-\frac{1}{2}} + \frac{B}{x+4} = \frac{A(x+4)+B\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)(x+4)} = \frac{(A+B)x+(4A-\frac{B}{2})}{\left(x-\frac{1}{2}\right)(x+4)}$$

This gives us two equations in A and B, i.e.,

$$A+B=0$$

$$4A-\frac{B}{2}=1$$

Solving the above equations, we get $A=\frac{2}{9}$ and $B=-\frac{2}{9}$. So, the problem is now reduced to an alternating series, i.e.,

$$\sum_{x=1}^{\infty} \frac{1}{2x^2+7x-4} = \frac{1}{2}\cdot\frac{2}{9}\sum_{x=1}^{\infty} \frac{1}{x-\frac{1}{2}} - \frac{1}{x+4} = \frac{1}{9}\sum_{x=1}^{\infty} \frac{2}{2x-1} - \frac{1}{x+4}$$

Next step is to write out some of the terms in the series and see if we can determine a pattern. We leave out the $\frac{1}{9}$ while we analyze the series.

$$2-\frac{1}{5}+\frac{2}{3}-\frac{1}{6}+\frac{2}{5}-\frac{1}{7}+\frac{2}{7}-\frac{1}{8}+\frac{2}{9}-\frac{1}{9}+\frac{2}{11}-\frac{1}{10}+\frac{2}{13}-\frac{1}{11}+\frac{2}{15}-\frac{1}{12}+\frac{1}{17}-\frac{1}{13}+\cdots$$

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Consider 2 times the alternating harmonic series

$$2\sum_{x=1}^{\infty}(-1)^{x+1}\frac{1}{x}$$

$$= 2 - 1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \frac{2}{9} - \frac{2}{10} + \frac{2}{11} - \frac{2}{12} + \frac{2}{13} - \frac{2}{14} + \frac{2}{15} - \frac{2}{16} + \frac{2}{17} - \frac{2}{18} + \cdots$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \frac{2}{15} - \frac{1}{8} + \frac{2}{17} - \frac{1}{9} + \cdots$$

After the first few terms, the above looks very close to our original problem but with some rearrangement, as shown in Figure 9.

$$2 - \frac{1}{5} + \frac{2}{3} - \frac{1}{6} + \frac{2}{5} - \frac{1}{7} + \frac{2}{7} - \frac{1}{8} + \frac{2}{9} - \frac{1}{9} + \frac{2}{11} - \frac{1}{10} + \frac{2}{13} - \frac{1}{11} + \frac{2}{15} - \frac{1}{12} + \frac{1}{17} - \frac{1}{13} + \cdots$$

$$2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \frac{2}{13} - \frac{1}{7} + \frac{2}{15} - \frac{1}{8} + \frac{2}{17} - \frac{1}{9} + \cdots$$

Figure 9. Comparison of series

In the original series, we can write the first term as

$$2 - \frac{1}{12} + \frac{1}{12} = \frac{25}{12} + (1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4})$$

So, except for a difference of $\frac{25}{12}$, our original series can be put into the form of series

$$2\sum_{x=1}^{\infty}(-1)^{x+1}\frac{1}{x}$$

via a simple rearrangement.

Using **Theorem 7** and noting that $\alpha = 1/2$ and letting y be the simple rearrangement of the alternating harmonic series as described above, we get

$$\sum\nolimits_{x=1}^{\infty}\frac{2}{2x-1} - \frac{1}{x+4} = \frac{25}{12} + 2y = \frac{25}{12} + 2\ln 2$$

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Thus, our final result is

$$\sum_{x=1}^{\infty} \frac{1}{2x^2+7x-4} = \frac{1}{9}\left(\frac{25}{12} + 2\ln 2\right)$$

[**Author's Remark**: I was impressed that the Wolfram Alpha website was able to symbolically solve this problem. Just type in "sum 1/(2x^2 + 7x -4) from x=1 to infinity" and the result $\frac{25}{108} + \frac{2\log 2}{9}$ is returned, noting Wolfram Alpha uses "log" to represent the natural log. If you'd like to try a similar problem have a go at

$$\sum_{x=1}^{\infty} \frac{1}{2x^2+5x-3}$$

The answer is $\frac{11}{42} + \frac{2\ln 2}{7}$.]

. . .

Puzzle: Show that for any integer $n \geq 1$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

Solution: In what follows, it should be emphasized that the rearrangement of the terms in a **finite** alternate series does not affect the sum.

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) - 1 - \frac{1}{2} - \cdots - \frac{1}{n}$$

$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

3.4.3 Miscellaneous

Puzzle: Determine the positive integer n that satisfies the following equation:

$$\frac{1}{2^{10}} + \frac{1}{2^8} + \frac{1}{2^6} + \frac{1}{2^4} + \frac{1}{2} = \frac{n}{2^{10}}$$

Source: Variation of problem A1 from the 2013 Canadian Open Math Challenge, https://www2.cms.math.ca/Competitions/COMC/2013/

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Solution:

$$\frac{1}{2^{10}} + \frac{1}{2^8} + \frac{1}{2^6} + \frac{1}{2^4} + \frac{1}{2} = \frac{1 + 2^2 + 2^4 + 2^6 + 2^8}{2^{10}} = \frac{1 + 4 + 16 + 64 + 256}{2^{10}} = \frac{341}{2^{10}}$$

So, $n = 341$.

. . .

For the next puzzle, we need the formula for the sum integers, i.e.,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

For example, $1 + 2 + 3 + 4 + 5 = \frac{5(6)}{2} = 15$.

With a small modification, we also have a formula for the integers going from $k > 1$ to $n > k$, i.e.,

$$\sum_{i=k}^{n} i = k + (k+1) + \cdots + (n-1) + n = \frac{(n+k)(n-k+1)}{2}$$

For example, $7 + 8 + 9 + 10 + 11 + 12 + 13 = \frac{(7+13)(13-7+1)}{2} = \frac{(20)(7)}{2} = 70$

Puzzle: Let set A be a 90-element subset of $\{1,2,3, \dots ,100\}$ and let S be the sum of the elements of A. Find the number of possible values of S.

Source: Question #2 from the 2006 American Invitational Mathematics Exam (AIME)

Solution: The smallest 90-element subset is

$$1 + 2 + \cdots + 90 = \frac{(90)(91)}{2} = 4095$$

and the largest 90-element subset is

$$11 + 12 + \cdots + 100 = \frac{(111)(90)}{2} = 4995$$

Subsets of all sizes (i.e., sum of elements in the subset) between 4095 and 4995 are possible. Start with the subset $\{1,2, \dots ,90\}$ and replace 90 with 91 to get a subset the sum of whose elements is 1 greater. At each step, replace one of the elements with an element that is 1 greater until the largest subset is reached. The number of elements in S is $4995 - 4095 + 1 = 901$.

. . .

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Puzzle: The following function is defined over the natural number, i.e., $n \in \mathbb{N}$. Further, $m \in \mathbb{N}$.

$$f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{m^n}$$

Determine a formula for

$$\sum_{i=2}^{\infty} f(i) = f(2) + f(3) + f(4) + \cdots$$

Solution:

We need the following formula for the sum of the powers of a number r where $0 < r < 1$.

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

For example, $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$.

Back to the problem at hand; list each of the terms in a separate line, as shown below.

$$f(2) = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{m^2}$$

$$f(3) = \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{m^3}$$

$$f(4) = \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots + \frac{1}{m^4}$$

...

If we add all the powers of $\frac{1}{2}$, all the powers of $\frac{1}{3}$, and so on, we get

$$\sum_{i=2}^{\infty} f(i) = f(2) + f(3) + f(4) + \cdots = \sum_{i=2}^{\infty}\left(\frac{1}{2}\right)^i + \sum_{i=2}^{\infty}\left(\frac{1}{3}\right)^i + \sum_{i=2}^{\infty}\left(\frac{1}{4}\right)^i + \cdots + \sum_{i=2}^{\infty}\left(\frac{1}{m}\right)^i$$

$$= \left(\frac{1}{2}\right)^2 \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i + \left(\frac{1}{3}\right)^2 \sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i + \left(\frac{1}{4}\right)^2 \sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i + \cdots + \left(\frac{1}{m}\right)^2 \sum_{i=0}^{\infty}\left(\frac{1}{m}\right)^i$$

$$= \left(\frac{1}{2}\right)^2 \left(\frac{1}{1-\frac{1}{2}}\right) + \left(\frac{1}{3}\right)^2 \left(\frac{1}{1-\frac{1}{3}}\right) + \left(\frac{1}{4}\right)^2 \left(\frac{1}{1-\frac{1}{4}}\right) + \cdots + \left(\frac{1}{m}\right)^2 \left(\frac{1}{1-\frac{1}{m}}\right)$$

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Each term in the above sum is of the form

$$\left(\frac{1}{k}\right)^2\left(\frac{1}{1-\frac{1}{k}}\right)=\frac{1}{k(k-1)}=\frac{1}{k-1}-\frac{1}{k}$$

Applying the above formula to the previous equation, we get

$$\sum_{i=2}^{\infty} f(i)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{m-1}-\frac{1}{m}\right)=1-\frac{1}{m}$$

3.5 Infinite Products

Similar to infinite sums, we can also have infinite product, e.g.,

$$\prod_{n=1}^{\infty}\frac{n}{n+1}=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots$$

To find the product, we write the above as a limit, i.e.,

$$\lim_{N\to\infty}\prod_{n=1}^{N}\frac{n}{n+1}=\lim_{N\to\infty}\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\ldots\cdot\frac{N}{N+1}=\lim_{N\to\infty}\frac{1}{N+1}=0$$

Puzzle: Evaluate the following infinite product

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$$

Source: Problem B.1 from The 38th William Lowell Putnam Mathematical Competition, 3 December 1977.

Solution: First, note the following identities from basic algebra

$$n^3-1=(n-1)(n^2+n+1)=(n-1)(n(n+1)+1)$$

$$n^3+1=(n+1)(n^2-n+1)=(n+1)(n(n-1)+1)$$

Using the above, we have

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\prod_{n=2}^{\infty}\frac{(n-1)(n(n+1)+1)}{(n+1)(n(n-1)+1)}=\lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)(n(n+1)+1)}{(n+1)(n(n-1)+1)}$$

$$\lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)}{(n+1)}\cdot\prod_{n=2}^{N}\frac{(n(n+1)+1)}{(n(n-1)+1)}$$

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Next, we simplify each of the two products via cancellation of terms.

$$\prod_{n=2}^{N}\frac{(n-1)}{(n+1)}=\frac{1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot (N-1)}{3\cdot 4\cdot 5\cdot 6\cdot \ldots \cdot (N+1)}=\frac{1\cdot 2}{N(N+1)}=\frac{2}{N^2+N}$$

$$\prod_{n=2}^{N}\frac{(n(n+1)+1)}{(n(n-1)+1)}=\frac{7}{3}\cdot\frac{13}{7}\cdot\frac{21}{13}\cdot\frac{31}{21}\cdot \ldots \cdot\frac{(N(N+1)+1)}{(N(N-1)+1)}=\frac{(N(N+1)+1)}{3}=\frac{N^2+N+1}{3}$$

Combining the above two results, we get

$$\prod_{n=2}^{N}\frac{(n-1)}{(n+1)}\cdot\prod_{n=2}^{N}\frac{(n(n+1)+1)}{(n(n-1)+1)}=\frac{2}{3}\cdot\frac{N^2+N+1}{N^2+N}=\frac{2}{3}\cdot\frac{1+\frac{1}{N}+\frac{1}{N^2}}{1+\frac{1}{N}}$$

So,

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\lim_{N\to\infty}\prod_{n=2}^{N}\frac{2}{3}\cdot\frac{1+\frac{1}{N}+\frac{1}{N^2}}{1+\frac{1}{N}}=\frac{2}{3}\cdot\frac{1+0+0}{1+0}=\frac{2}{3}$$

3.6 Miscellaneous

3.6.1 Rational solutions of y^x = x^y

Puzzle: Find a rational (i.e., fractional) solution to the equation $y^x = x^y$ where $y > x > 0$.

Source: See the article entitled “On the Rational Solution of $x^y = y^x$” [18] by Marta Sved

Solution:

We can write $y = ax$ for some $a > 1$, since we are told that $y > x > 0$. Plugging this into the given equation yields

$$x^{ax} = (ax)^x$$

Taking the natural log of both sides of the above equation, we have

$$ax\ln x = x\ln(ax)$$

$$a\ln x = \ln(ax)$$

$$\ln x^a = \ln(ax)$$

Thus, $x^a = ax$ which implies $x^{a-1} = a$ or equivalently, $x = a^{1/(a-1)}$. Also, $y = ax = a^{a/(a-1)}$.

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Letting $u = \frac{1}{a-1}$, which implies $a = 1 + \frac{1}{u}$. Substituting into the above equations for x and y, we get

$$x = \left(1 + \frac{1}{u}\right)^u, \qquad y = \left(1 + \frac{1}{u}\right)^{u+1}$$

The table below gives solutions (x, y) for various integer values for u.

u	(x, y)
2	$\left(\frac{9}{4}, \frac{27}{8}\right)$
3	$\left(\frac{64}{27}, \frac{256}{81}\right)$
4	$\left(\frac{625}{256}, \frac{3125}{1024}\right)$

The article mentioned in the source above proves that the only rational solutions are in the form indicated above. Further, from basic calculus, we know that

$$\lim_{u \to \infty} (x, y) = (e, e)$$

where $e \cong 2.71828$ (which is known as Euler's number).

3.6.2 The Monkey and the Coconuts Puzzle

Puzzle: Five men and a monkey were shipwrecked on an island. They spent the first day gathering coconuts. During the night, one man secretly decided to take his share of the coconuts. After giving one coconut to the monkey, he was able to divide the coconuts into five piles. He then took and hid one of the file piles of coconuts. In succession, the four other men each did the same thing. First thing the next morning, the men divided the remaining coconuts into five equal shares. What is the smallest number of coconuts that could have been in the original pile?

Sources:

- Wikipedia article entitled "The monkey and the coconuts" [30]
- "Monkey and Coconut Problem" in Wolfram MathWorld [31]

Solution:

Let N be the total number of coconuts collected by the 5 men.

The first man gives one coconut away and then takes one fifth of what remains. This leaves the following number of coconuts:

$$\frac{4}{5}(N-1) = \frac{4N}{5} - \frac{4}{5} = \frac{4N}{5} + \frac{16}{5} - \frac{16}{5} - \frac{4}{5} = \frac{4(N+4)}{5} - 4$$

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The starting point for the second man is $N_2 = \frac{4(N+4)}{5} - 4$ coconuts. Applying the above procedure to N_2 we get

$$\frac{4(N_2+4)}{5} - 4 = \frac{4}{5}\left(\frac{4(N+1)}{5} - 5\right) = \frac{16(N+4)}{25} - 4$$

Using the same procedure, the third, fourth and fifth men leave the following number of coconuts, respectively.

$$\frac{64(N+4)}{125} - 4, \qquad \frac{256(N+4)}{625} - 4, \qquad \frac{1024(N+4)}{3125} - 4$$

Further, we are told that after the fifth man has secretly taken his share, the number of remaining coconuts (which we now know is $\frac{1024(N+4)}{3125} - 4$) is divisible by 5.

To finish the analysis of this puzzle, we need a variation of Euclid's lemma.

Theorem 8 *(Euclid's lemma) Given integers a, b, and c. If c exactly divides the ab and $gcd(a, c) = 1$, then $a|b$.*

For example, since $3|30 = 5 \cdot 6$ and $\gcd(3,5) = 1$, then we know by the Euclid's lemma that $3|6$. Of course, this is easy to see without Euclid's lemma but we are just illustrating the concept here. On the other hand, $6|2 \cdot 3$ but 6 does not divide 2 or 3.

Now, back to our problem. We have that $\gcd(1024,3125) = 1$. Further, we know from the statement of the puzzle that, after the last division of coconuts, there is an integer number of coconuts remaining, and so, it must be that $3125|1024(N+4)$. Thus, by Euclid's lemma, $3125|(N+4)$ and the smallest positive value for this to occur is when $N + 4 = 3125$ or $N = 3121$. So, the five men started with a minimum of 3121 coconuts.

We can get that the final number of coconuts is divisible by 5, i.e.,

$$\frac{1024(N+4)}{3125} - 4 = \frac{1024(3121+4)}{3125} - 4 = 1020$$

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The following table summarizes the coconut distribution process.

Round	Removed	Monkey	Coconuts left
Start			3121
1	624	1	2496
2	499	1	1996
3	399	1	1596
4	319	1	1276
5	255	1	1020
Final Division	$5 \times 204 = 1020$	0	0

. . .

Puzzle: Same as the previous puzzle, except the number of coconuts left by the 5th man is divisible by 5 only after 1 coconut is given to the monkey.

Solution: As in the previous puzzle, the number of coconuts left by the 5th man is

$$\frac{1024(N+4)}{3125} - 4$$

However, in this case, we are told that 1 minus the above number is divisible by 5, i.e.,

$$5|x = \left(\frac{1024(N+4)}{3125} - 5\right)$$

Since 5 divides x, and 5 clearly divides 5, we have that 5 divides

$$x + 5 = \frac{1024(N+4)}{3125}$$

Thus,

$$\frac{1}{5} \cdot \frac{1024(N+4)}{3125} = \frac{1024(N+4)}{15625}$$

is a whole number.

Since $\gcd(1024{,}15625) = 1$, we know from Euclid's lemma that $15625|(N+4)$. The smallest positive value for N when $15625|(N+4)$ occurs when $N + 4 = 15625$ which implies $N = 15621$.

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The following table summarizes the coconut distribution process.

Round	Removed	Monkey	Coconuts left
Start			15621
1	3124	1	12496
2	2499	1	9996
3	1999	1	7996
4	1599	1	6396
5	1279	1	5116
Final Division	$5 \times 1023 = 5115$	1	0

3.6.3 Positive and Negative Runs of Numbers in the Same Sequence

Puzzle: In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in such a sequence.

Source: Problem #2 from the 19th International Mathematical Olympiad (1977) [51]

Solution: We first show that such a sequence cannot have 17 (or more) terms. To that end, assume that we have a sequence $a_1, a_2, \dots, a_{17}$ meeting the conditions of the puzzle. Arrange the elements of the sequence as shown in the following table:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}

Our assumption that the sum of every seven successive terms is negative implies that the sum of the elements in each column of the table is negative and thus, the sum of all elements in the table is negative. On the other hand, our assumption also implies that the sum of the elements in each row is positive and thus, the sum of all elements in the table is positive. So, by assuming a sequence satisfying the conditions of the puzzle has 17 or more terms, we have created a contradiction. Thus, such a sequence must have 16 or less terms.

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Next, we will determine that there is a sequence of 16 terms that does satisfy the conditions of the puzzle. Assume that $a_1, a_2, \dots, a_{16}$ is such a sequence. Further, let $b_k = a_1 + a_2 + \cdots + a_k$ for $k = 1,2, \dots, 16$. We have sufficient information to order the b_k terms which will give us a way to determine the desired sequence. We start by noting that $b_8 - b_1 = a_2 + a_3 + \cdots + a_8$ is negative since it is the sum of seven successive terms, and so, $b_8 < b_1$. Similarly, we have

$$b_9 < b_2$$

$$b_{10} < b_3$$

$$b_{11} < b_4$$

$$b_{12} < b_5$$

$$b_{13} < b_6$$

$$b_{14} < b_7$$

$$b_{15} < b_8$$

$$b_{16} < b_9$$

We get the following inequalities taking the difference of b_k terms whose subscripts differ by 11.

$$b_1 < b_{12}$$

$$b_2 < b_{13}$$

$$b_3 < b_{14}$$

$$b_4 < b_{15}$$

$$b_5 < b_{16}$$

Further, we have the $b_7 < 0$ and $b_{11} > 0$. Putting the above results together, we have a complete ordering for the b_k terms.

$$b_{10} < b_3 < b_{14} < b_7 < 0 < b_{11} < b_4 < b_{15} < b_8 < b_1 < b_{12} < b_5 < b_{16} < b_9 < b_2 < b_{13} < b_6$$

To get our example, we just choose values for the b_k terms that satisfy the above sting of inequalities and then compute the a_k terms. For the b_k terms, we choose

$$-4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

which yields the following values for a_k terms

$$5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5$$

. . .

In general, if x is the number of successive terms whose sum is positive, and y is the number of successive terms whose sum is negative, then the maximum length sequence is given by

$$x + y - 1 - \gcd(x, y)$$

With this knowledge in hand, the reader is invited to try the following puzzle.

Puzzle: Determine the maximum number of terms in a sequence of numbers such that the sum of any 7 successive terms is positive and the sum of any 5 successive terms is negative. Given an example sequence that illustrates your answer.

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Solution: Using the given formula, we know that the maximum length sequence is $7+5-1-1=10$. Using the same procedure as the previous problem, we get the following sequence of minimum length that meets the conditions of the puzzle:

$$-5,7,-5,7,-5,-5,7,-5,7,-5$$

3.6.4 Recursive Structures

Puzzle: Determine y, if $y > 0$ and

$$y = \frac{3}{2} - 2\left(\frac{1}{4} - \left(\frac{1}{4} - \left(\frac{1}{4} - \cdots\right)^2\right)^2\right)^2$$

Solution: The nested part of the above expression can be represented as

$$x = \left(\frac{1}{4} - x\right)^2$$

$$x = \frac{1}{16} - \frac{x}{2} + x^2$$

$$16x^2 - 24x + 1 = 0$$

Using the quadratic formula, we get

$$x = \frac{3 \pm 2\sqrt{2}}{4}$$

Given the condition that $y > 0$, we choose the $x = \frac{3-2\sqrt{2}}{4}$.

Plugging x into the original equation, we get

$$y = \frac{3}{2} - 2\left(\frac{3-2\sqrt{2}}{4}\right) = \frac{3}{2} - \frac{3}{2} + \sqrt{2} = \sqrt{2}$$

. . .

Puzzle: Determine x, if

$$x = 2 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cdots}}}}}$$

Solution: The above expression is known as a **continued fraction**. We can solve the puzzle by taking advantage of the implied recursion in the structure of the expression. To that end, let

$$y = 6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cdots}}}}$$

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Thus, we have

$$x = 2 + \frac{1}{y}, \qquad y = 6 + \cfrac{1}{3 + \cfrac{1}{y}}$$

Working with the latter equation, gives us

$$y = 6 + \frac{y}{3y + 1}$$

$$3y^2 + y = 6(3y + 1) + y$$

$$y^2 - 6y - 2 = 0$$

Using the quadratic formula and choosing the positive alternative, we have that

$$y = \frac{6 + \sqrt{44}}{2} = \frac{6 + 2\sqrt{11}}{2} = 3 + \sqrt{11}$$

Plugging the value of y into the equation for x yields

$$x = 2 + \frac{1}{3 + \sqrt{11}} = 2 + \frac{1}{3 + \sqrt{11}} \cdot \frac{3 - \sqrt{11}}{3 - \sqrt{11}} = 2 + \frac{3 - \sqrt{11}}{-2} = 2 - \frac{3}{2} + \frac{\sqrt{11}}{2} = \frac{1 + \sqrt{11}}{2}$$

. . .

Puzzle: Determine the value of

$$x = 6 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{\cdots}}}}}}}}}$$

Solution: The pattern repeats after every 4-levels of the continued fraction, and so, we can write

$$x = 6 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{x}}}} = 6 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{x}{x + 1}}} = 6 + \cfrac{1}{1 + \cfrac{x + 1}{3x + 2}} = 6 + \frac{3x + 2}{4x + 3} = \frac{27x + 20}{4x + 3}$$

Thus,

$$x = \frac{27x + 20}{4x + 3}$$

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Multiplying $4x + 3$ on both sides of the above equation, we get

$$4x^2 + 3x = 27x + 20$$

$$4x^2 - 24x - 20 = 0$$

$$x^2 - 6x - 5 = 0$$

Determine x using the quadratic formula, i.e.,

$$x = \frac{6 \pm \sqrt{36 - 20}}{2} = 3 \pm \sqrt{14}$$

Since the definition of x in the puzzle statement implies that $x > 0$, we select the positive option, i.e., $x = 3 + \sqrt{14}$.

3.6.5 Digit Rearrangement/Removal Puzzles

Puzzle: The number below is created by concatenating the first 60 positive integers (space added for readability)

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ \ldots 59\ 60$$

What is the small number that can be created from the above number by deleting 100 of the digits (without any rearrangements)?

Solution: First, note that the number has $9 + 2(51) = 111$ digits. Thus, after 100 digits are removed, we are left with an 11-digit number.

We want as many leading zeros as possible to minimize the resulting number. So, we delete all the non-zero digits between 1 and 50 which gives us 00000 and the remaining digits $51\ 52 \ldots 60$. We then make deletions from the remaining digits to create the smallest 11-digit number possible under the conditions of the puzzle, i.e.,

$$00000123450$$

. . .

Puzzle: Same as previous puzzle, except find the largest number.

Solution: In this case, we want as many leading nines as possible. So, we delete all the non-nine digits between 1 and 49 to get 99999. This entails the removal of $9 + 80 = 89$ digits, leaving 22 digits of which we need to delete 16 more to arrive at our 11 digit solution. In particular, we remove $50, 51, 52, 53, 54, 55, 56$, and the 5 from both 57 and 58 to get the final solution:

$$99999785960$$

. . .

Puzzle: Find the smallest integer whose first digit *(i.e., most significant digit)* is seven, and which is reduced to $\frac{1}{3}$ its original value when its first digit is transferred to the end *(i.e., least significant digit)*.

Source: Problem #23 in "The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics" [79]

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Solution: Represent the integer in question as

$$7a_n a_{n-1} \dots a_1 a_0$$

where 7 and the a_i terms are the digits of the said number.

We want to determine the a_i terms such that

$$7a_n a_{n-1} \dots a_1 a_0 \cdot \frac{1}{3} = a_n a_{n-1} \dots a_1 a_0 7$$

or equivalently,

$$7a_n a_{n-1} \dots a_1 a_0 = a_n a_{n-1} \dots a_1 a_0 7 \cdot 3$$

Starting the multiplication on the right of the above equation (i.e., $3 \cdot 7 = 21$), we see that the last digit must be 1. Thus, $a_0 = 1$. The problem reduces to

$$7a_n a_{n-1} \dots a_1 1 = a_n a_{n-1} \dots a_1 17 \cdot 3$$

Since $3 \cdot 17 = 51$, it must be that $a_1 = 5$.

We continue the process until we find i such that the most significant digit in $a_{i-1} a_{i-2} \dots a_1 a_0 7 \cdot 3$ is 7. The calculations are shown in the table below. In the table, the solution is highlighted in bold, i.e.,

$$(2{,}413{,}793{,}103{,}448{,}275{,}862{,}068{,}965{,}517) \cdot 3 = 7{,}241{,}379{,}310{,}344{,}827{,}586{,}206{,}896{,}551$$

or in the form specified in the puzzle statement

$$2{,}413{,}793{,}103{,}448{,}275{,}862{,}068{,}965{,}517 = \frac{1}{3}(7{,}241{,}379{,}310{,}344{,}827{,}586{,}206{,}896{,}551)$$

	$\boldsymbol{a_{i-1} a_{i-2} \dots a_1 a_0 7 \cdot 3}$	$\boldsymbol{a_i a_{i-1} \dots a_1 a_0 7}$
		7
0	21	17
1	51	$a_1 a_0 7 = 517$
2	$a_1 a_0 7 \cdot 3 = 1{,}551$	$a_2 a_1 a_0 7 = 5{,}517$
3	$a_2 a_1 a_0 7 \cdot 3 = 16{,}551$	65,517
4	196,551	965,517
5	2,896,551	8,965,517
6	26,896,551	68,965,517
7	206,896,551	068,965,517
8	206,896,551	2,068,965,517
9	6,206,896,551	62,068,965,517
10	186,206,896,551	862,068,965,517
11	2,586,206,896,551	5,862,068,965,517

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	$a_{i-1}a_{i-2} \dots a_1 a_0 7 \cdot 3$	$a_i a_{i-1} \dots a_1 a_0 7$
12	17,586,206,896,551	75,862,068,965,517
13	227,586,206,896,551	275,862,068,965,517
14	827,586,206,896,551	8,275,862,068,965,517
15	24,827,586,206,896,551	48,275,862,068,965,517
16	144,827,586,206,896,551	448,275,862,068,965,517
17	1,344,827,586,206,896,551	3,448,275,862,068,965,517
18	10,344,827,586,206,896,551	03,448,275,862,068,965,517
19	310,344,827,586,206,896,551	103,448,275,862,068,965,517
20	310,344,827,586,206,896,551	3,103,448,275,862,068,965,517
21	9,310,344,827,586,206,896,551	93,103,448,275,862,068,965,517
22	279,310,344,827,586,206,896,551	793,103,448,275,862,068,965,517
23	2,379,310,344,827,586,206,896,551	3,793,103,448,275,862,068,965,517
24	11,379,310,344,827,586,206,896,551	13,793,103,448,275,862,068,965,517
25	41,379,310,344,827,586,206,896,551	413,793,103,448,275,862,068,965,517
26	1,241,379,310,344,827,586,206,896,551	**2,413,793,103,448,275,862,068,965,517**
27	**7,241,379,310,344,827,586,206,896,551**	**7**2,413,793,103,448,275,862,068,965,517
28	217,241,379,310,344,827,586,206,896,551	**17**2,413,793,103,448,275,862,068,965,517
29	517,241,379,310,344,827,586,206,896,551	**5,17**2,413,793,103,448,275,862,068,965,517

A few more calculations are shown after the smallest number satisfying the condition of the puzzle is found. This is to illustrate that the pattern repeats. All numbers meeting the condition of the puzzle are multiple copies of the smallest solution concatenated together, e.g., the second smallest number meeting the conditions of the puzzle is $7{,}241{,}379{,}310{,}344{,}827{,}586{,}206{,}896{,}551$ concatenated with itself. The third solution consists of three concatenated copies of $7{,}241{,}379{,}310{,}344{,}827{,}586{,}206{,}896{,}551$. There are an infinite number of solutions.

[Author's Remark: Keep in mind that problems stated in "The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics" [68] are from the 1940s and 1950s (long before the advent of generally available electronic calculators). I find it hard to believe this problem was given to grade school or high school students unless they were only asked to describe a process (algorithm) for solving the problem.

Conceptually, the solution is simple, and it is easy to start the necessary calculations by hand. As one can see, the numbers quickly become very large. I initially tried to do the calculations with a spreadsheet (Microsoft Excel) but could only go so far since Excel only supports the first 15 significant digits of a number. After some searching, I found an online application that handles very large integers and was then able to complete the calculations. (See the Big Number Calculator at https://www.calculator.net/big-number-calculator.html).**]**

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3.6.6 Problems with Integer Solutions

Puzzle: Find a four-digit number equal to the square of the sum of the two two-digit numbers formed by taking the first two digits and the last two digits of the original number.

In other words, find a four-digit number $N = 100a + b$ (where a and b are two-digit numbers) such that

$$(a + b)^2 = N = 100a + b$$

Source: Problem #111 in "The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics" [79]

Solution: One approach is to test all possibilities using a computer program. The following Python program just does that.

```
for i in range(1,101):
    for j in range(1,101):
        k = (i+j)**2
        n = 100*i + j
        if n == k:
            print (i,j)
```

The only solutions are 2025, 3025 and 9801.

[Author's Remark: There are analytic ways of solving this problem and the next several problems, see the solutions in "The USSR Olympiad Problem Book" [79]. However, I want to emphasize the point here that one also needs to consider practical (computer-based) solutions when feasible.**]**

. . .

Puzzle: Find all three-digit numbers that are equal to the sum of the factorials of each digit.

Source: Problem #113a in "The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics" [79]

Solution: If the three-digit number is xyz, we are being asked to check whether

$$100x + 10y + z = x! + y! + z!$$

for $0 \leq x \leq 9, 0 \leq y \leq 9, 0 \leq z \leq 9$ where x, y and z are integers.

The following Python program checks each possibility. The only solution is 145.

```
import math
for i in range(0,10):
    for j in range(0,10):
        for k in range(0,10):
            a = 100*i + 10*j + k
            b = math.factorial(i) + math.factorial(j) + math.factorial(k)
            if a == b:
                print (a)
```

. . .

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Puzzle: Find all six-digit numbers $1000a + b$ (where a and b are 3-digit numbers) such that

$$(a + b)^2 = 10^3 a + b$$

Solution: Using the following Python program, we find the only three solutions, i.e., 088209 (if we allow for leading zero), 494209 and 998001.

```
for i in range(1,1001):
   for j in range(1,1001):
      k = (i+j)**2
      n = 1000*i + j
      if n == k:
         print (i,j)
```

. . .

Puzzle: Modify the program from the previous puzzle to find all the eight-digit numbers $10{,}000a + b$ (where a and b are 4-digit numbers) such that

$$(a + b)^2 = 10^4 a + b$$

Answer: There are 7 solutions if we allow for leading zeros (the first two solutions below); otherwise, there are only 5 solutions.

0494 1729

0744 1984

2450 2500

2550 2500

5288 1984

6048 1729

9998 0001

. . .

[Author's Remark: If you extend the previous puzzle once more, i.e., 10-digit numbers who equal the square of the sum of its first five and last five digits, the program will run for well over an hour (depending on the speed of your computer). The result, including solutions with leading zeros, is

00238 04641

03008 14336

04938 17284

60494 17284

68320 14336

90480 04641

99998 00001

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Unless you have access to a very powerful computer, solving the problem for a 10-digit number is about as far as one can go with a computer-based solution. The general problem is to describe all the solutions to the problem for $2n$ digits. More formally, find all integers $2n$ with first n digits a and last (least significant) digits b such that

$$(a+b)^2 = 10^n a + b$$

One solution is $a = 10^n - 2, b = 1$ (as we have seen for the first few values of n). Determination of other solutions is an open problem posed to the reader.**]**

. . .

Puzzle: Describe all positive integer solutions to the equation $x^2 + y^2 + z^2 = 3xyz$.

Solution: This equation is known as Markov's equation [32]. It is easy to see that $(x, y, z) = (1,1,1)$ is a solution. Next, we prove that if (x, y, z) is a solution, so is $(x, y, 3xy - z)$.

The proof is straightforward. Assume (x, y, z) is a solution and proceed as follows:

$$\begin{aligned} x^2 + y^2 + (3xy - z)^2 &= x^2 + y^2 + 9x^2y^2 - 6xyz + z^2 \\ &= (x^2 + y^2 + z^2) + 9x^2y^2 - 6xyz \\ &= 3xyz + 9x^2y^2 - 6xyz \\ &= 9x^2y^2 - 3xyz \\ &= 3xy(3xy - z) \end{aligned}$$

Thus, $(x, y, 3xy - z)$ is a solution.

Given the symmetry of Markov's equation, $(3yz - x, y, z)$ and $(x, 3xz - y, z)$ are also solutions if (x, y, z) is a solution.

Apply any of the three transformations to original solution $(1,1,1)$, and we get the new solution $(1,1,2)$ or a permutation thereof, i.e., $(2,1,1)$ or $(1,2,1)$.

From $(1,2,1)$, we can generate the solution $(1,2,5)$.

Next, apply the transformation to $(1,5,2)$ to get $(1,5,13)$. We can continue to generate more solutions (all with $x = 1$) by switching the z and y elements, and applying the transformation. We refer to the following as Tree 1.

$$(1, 2, 5)$$

$$(1, 5, 13)$$

$$(1, 13, 34)$$

$$(1, 34, 89)$$

$$(1, 89, 233)$$

$$(1, 233, 610)$$

$$(1, 610, 1597)$$

...

The sequence 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, ... is known as the Markov numbers [33].

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We can start another branch of the solution tree starting with $(2,5,1)$ while maintaining $x = 2$.

$$(2, 5, 1)$$
$$(2, 5, 29)$$
$$(2, 29, 169)$$
$$(2, 169, 986)$$
$$(2, 986, 5741)$$
$$\ldots$$

Another branch of the solution tree can be started with $(5,13,1)$ while maintaining $x = 5$.

$$(5, 13, 1)$$
$$(5, 13, 194)$$
$$(5, 194, 2897)$$
$$(5, 2897, 43261)$$
$$(5, 43261, 646018)$$
$$\ldots$$

We can continue branching off of Tree 1 fixing $x = 13$, followed by $x = 29$, then $x = 34$ and so on. The idea is to select the smallest number appearing in any solution (not yet used to anchor a branch of the solution tree) as the next number to hold fixed when generating another branch of the solution tree. An alternate but similar view of the solution tree is depicted in the Wikipedia article entitled "Markov number" [34].

The more general equation $x^2 + y^2 + z^2 = kxyz$ only has integer solutions for $k = 1$ or $k = 3$, see Problem #118 in "The USSR Olympiad Problem Book" [79].

. . .

Puzzle: Find all solutions in non-negative integers a, b to $\sqrt{a} + \sqrt{b} = \sqrt{2009}$.

Source: Problem #1 from the British Mathematical Olympiad, Round 2, 29 January 2009, https://bmos.ukmt.org.uk/home/bmolot.pdf

Solution: Note that the prime factorization of 2009 is $7^2 \cdot 41$.

(In what follows, we use the following notation:

- $A \Leftrightarrow B$ to means that statements A and B are equivalent.
- $x \in \mathbb{N}$ means that x is an element of the Natural Numbers, i.e., the set $\{0,1,2,3, \ldots\}$. The natural numbers and non-negative integers are the same set, if we include zero in the natural numbers.
- $\ni$ means "such that".)

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We start by finding conditions on the variable b.

$$\sqrt{a}+\sqrt{b}=\sqrt{2009} \text{ has solutions in the natural numbers}$$

$$\Leftrightarrow \ \sqrt{a}=\sqrt{2009}-\sqrt{b} \text{ has solutions in the natural numbers}$$

$$\Leftrightarrow a=b-2\sqrt{2009b}+2009 \text{ has solutions in the natural numbers}$$

$$\Leftrightarrow 2\sqrt{2009b}=b-a+2009 \text{ has solutions in the natural numbers}$$

$$\Leftrightarrow \sqrt{2009b}=\sqrt{7^2\cdot 41b}=7\sqrt{41b}\in\mathbb{N}$$

$$\Leftrightarrow b=41x^2 \text{ for } x\in\mathbb{N}$$

A similar set of steps to the above implies we must also have $a=41y^2$ for $y\in\mathbb{N}$.

Putting both results into the original equation, we have

$$\sqrt{41x^2}+\sqrt{41y^2}=\sqrt{2009}=7\sqrt{41}$$

$$\Leftrightarrow x+y=7$$

So, we get a unique positive integer solution to the original equation for each pair

$$(a=41y^2, b=41x^2) \ \ni x+y=7$$

The following table shows the eight solutions to the problem.

x	0	1	2	3	4	5	6	7
y	7	6	5	4	3	2	1	0
a	2009	1476	1025	656	369	164	41	0
b	0	41	164	369	656	1025	1476	2009

3.6.7 Sum of Three Cubes

Puzzle: Represent the following numbers as the sum of three cubes: $8, 11, 12, 29$ and 33. The numbers being cubed can be any integer, and repeats are allowed, e.g., $3=1^3+1^3+1^3$ and $7=2^3+(-1)^3+0^3=8-1+0$.

Sources:

- Wikipedia article entitled “Sums of three cubes” [24]
- YouTube video entitled “The Uncracked Problem with 33” [25]
- YouTube video entitled “42 is the new 33” [26]
- Article entitled “Sum of three cubes for 42 finally solved—using real life planetary computer” [27]

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Solution: The solution for 8 and 29 are fairly easy, i.e.,

$$8 = 2^3 + 0^3 + 0^3$$

$$29 = 3^3 + 1^3 + 1^3$$

The solution for 11 takes a bit more thought, i.e.,

$$11 = 3^3 + (-2)^3 + (-2)^3 = 27 - 8 - 8$$

The solution for 12 is yet harder to figure, i.e.,

$$12 = 7^3 + 10^3 + (-11)^3 = 343 + 1000 - 1331$$

There is an answer for 33 but it is very hard to find, and it took a considerable amount of computer time to determine a solution (see the YouTube videos [25] and [26]).

$$33 = 8866128975287528^3 + (-8778405442862239)^3 + (-2736111468807040)^3$$

A necessary condition for a natural number n to be represented as the sum of three integer cubes is that n not be congruent to 4 or 5 modulo 9, i.e., n is not of the form $4 + 9k$ or $5 + 9k$. For example, $13 = 4 + 9 \cdot 1$ is not representable as the sum of three integer cubes. It is not known whether this is a sufficient condition for the problem.

For all numbers less than 100 which satisfy the necessary condition mentioned above, representations as the sum of three integer cubes are known. Some of the representations involve very large numbers, as we saw for 33. Another example is 42. The following solution took over a million hours of computer calculations

$$42 = (-80538738812075974)^3 + 80435758145817515^3 + \ 12602123297335631^3$$

An article on this topic from Phys.Org [27] described the solution for 42 as follows:

> Professors Booker and Sutherland's solution for 42 would be found by using Charity Engine; a "worldwide computer" that harnesses idle, unused computing power from over 500,000 home PCs to create a crowd-sourced, super-green platform made entirely from otherwise wasted capacity.

At time of this writing, there were still several number less than 1000 for which a solution to the problem is not yet known, i.e., $114, 390, 627, 633, 732, 921$ and 975.

[Author's Remark: I have not seen any suggestions concerning a practical use for this type of problem. One thought is to use the solution for a previously unsolved number as a key in message encryption / decryption. The idea would be for an organization to solve the problem using an amount of computer calculations that could not easily be matched by any other organization and then use the solution as a key.**]**

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4 Algebra Puzzles

If there is a problem you can't solve, then there is an easier problem you can solve: find it.

George Polya

4.1 Number Word Puzzles

Puzzle: If the sum of four consecutive even numbers is 92, what are the four numbers?

Solution: Let x be the smallest of the four even numbers, then the other three numbers are $x + 2$, $x + 4$ and $x + 6$. Since we are told the sum of the four numbers is 96, we have the following equation:

$$x + (x + 2) + (x + 4) + (x + 6) = 92$$

This reduces to

$$4x + 12 = 92$$

$$4x + 12 - 12 = 92 - 12$$

$$4x = 80$$

$$x = 20$$

So, the four numbers are $20, 22, 24$ and 26.

. . .

Puzzle: If the sum of three consecutive numbers is 96, what are the three numbers?

. . .

Puzzle: If one number is 25 more than another, and the sum of the two numbers is 175. What are the two numbers?

Solution: Let x be the smaller number, then the other number is $x + 25$. We are also given that

$$x + (x + 25) = 175$$

$$2x + 25 = 175$$

$$2x = 150$$

So, $x = 75$ and the other number is 100.

. . .

Puzzle: If one number is 10 times another, and their difference is 81. What are the two numbers?

Solution: Let x be the smaller number, then the other number is $10x$. We are also given that

$$10x - x = 81$$

$$9x = 81$$

So, the smaller number is 9 and the larger number is 90.

. . .

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Puzzle: The difference between two numbers is 11. If the larger number is subtracted from three times the smaller number, the difference is 99. Find the numbers.

Solution: If the difference between two numbers is 11, that means we can represent the two numbers as x and $x + 11$.

Further, we are given that

$$3x - (x + 11) = 99$$

$$2x = 110 \Rightarrow x = 55$$

So, the smaller number is 55 and the larger number is 66.

. . .

Puzzle: Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $\left(a + \frac{b}{2}\right)$ and $\left(b - \frac{a}{2}\right)$ instead. Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

Source: Introductory Problem #6 from "101 Problems in Algebra: From the Training of the USA IMO Team" [28]

Solution: Consider the squares of the two replacement numbers, i.e.,

$$\left(a + \frac{b}{2}\right)^2 + \left(b - \frac{a}{2}\right)^2 = a^2 + ab + \frac{b^2}{4} + b^2 - ab + \frac{a^2}{4} = a^2 + b^2 + \frac{b^2}{4} + \frac{a^2}{4} > a^2 + b^2$$

At each step, the overall sum of the squares of the numbers in the set increases and so, the modified set of numbers can never coincide with the initial set.

4.2 Time, Speed and Distance Problems

Puzzle: An escalator is moving downward at a constant speed. Abner walks down and takes 50 steps to reach the bottom. In the same amount of time that Abner takes 10 steps, Trudy runs down and takes 90 steps to reach the bottom. How many steps are visible when the escalator is not moving?

Solution:

Let T be the total number of visible steps when the escalator is not moving.

The information that we are given about Abner implies that $50 + x = T$ where x is the number of steps covered by the escalator while Abner descends. One way to view this is to envision Abner going down 50 steps while the escalator is not moving, followed by the escalator going down in a time (measured in number of steps) equal to what it took Abner to cover 50 down steps. These two events should put Abner at the bottom of the escalator.

The information that we are given about Trudy can be represented as $90 + \frac{x}{5} = T$. The $\frac{x}{5}$ term comes from the fact that Trudy takes only $\frac{1}{5}$ of the time that it takes Abner to descend the escalator.

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So, we have

$$50 + x = 90 + \frac{x}{5}$$

$$\frac{4}{5}x = 40$$

$$x = 100$$

$$\ldots$$

Puzzle: Bertrand can walk up an escalator (which is also going up) in 30 seconds. He can walk down the same moving escalator in 90 seconds (basically going against the flow of the escalator) and successfully reach the bottom. Bertrand's walking speed is the same, going upwards and downwards. How much time will he take to walk up (or down) the escalator when it's not moving?

Solution: Let x be the speed of the escalator in steps/second, and y be Bertrand's speed in steps/second. Let T be the length of the escalator measured in steps.

Since we are told Bertrand was able to walk down the escalator in 90 seconds and successfully reach the bottom, we know that $y > x$.

We are given

$$\frac{T\ steps}{30\ seconds} = \frac{x + y\ steps}{second}$$

and

$$\frac{T\ steps}{90\ seconds} = \frac{y - x\ steps}{second}$$

The above equations can be hard to visualize. In words, the first equation says that Bertrand with assistance from the escalator is traveling at $x + y$ steps per second, and will traverse the length of the escalator (T steps) in 30 seconds.

So, $T = 30(x + y)$ and $T = 90(y - x)$ which implies

$$30(x + y) = 90(y - x)$$

$$x + y = 3(y - x)$$

$$y = 2x$$

This means that Bertrand's stepping rate is $\frac{2x\ steps}{second}$.

We can now solve for T in terms of x. Substituting $y = 2x$ into the equation $\frac{T\ steps}{30\ seconds} = \frac{x+y\ steps}{second}$, we get $T = 90x$ steps. We could have used the second equation, with the same result.

The problem reduces to determining how long it takes to cover $90x$ steps at a rate of $\frac{2x\ steps}{second}$, which we can easily calculate to be 45 seconds.

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. . .

Puzzle: A battery operated toy car can traverse a moving sidewalk of length L in 30 seconds, when going in the same direction as the sidewalk. When going in the opposite direction of the motion of the moving sidewalk, the car takes 90 seconds to traverse the sidewalk. How long will it take the toy car to traverse the sidewalk when it's not moving?

Solution: The solution is exactly the same as the previous puzzle. The point in reformulating the puzzle is to emphasize that the incline of the escalator and associated steps are a distraction from the underlying structure of the puzzle.

. . .

Puzzle: Two elevators are at the same height (at the top of a skyscraper). One elevator can reach the bottom floor in 4 minutes, and the second elevator can reach the bottom floor in 5 minutes. If both elevators start to descend at the same time, when will the first elevator be at half the height from the bottom as the second elevator?

Solution: Let L be the height at the top of the skyscraper in meters. The position of the first elevator at time t in terms of distance from the bottom is given by $h_1 = L - \frac{Lt}{4}$ where t is time in minutes and $\frac{L}{4}$ meters/minute is the rate at which the first elevator descends. Similarly, the position of the second elevator at time t is given by $h_2 = L - \frac{Lt}{5}$. We want to find t such that

$$h_1 = \frac{1}{2} h_2$$

$$L - \frac{Lt}{4} = \frac{1}{2}\left(L - \frac{Lt}{5}\right)$$

$$\frac{L}{2} = \frac{Lt}{4} - \frac{Lt}{10} = t\left(\frac{3L}{20}\right)$$

$$t = \frac{L}{2} \cdot \frac{20}{3L} = \frac{10}{3} = 3\frac{1}{3} \text{ minutes}$$

. . .

Puzzle: How long after $1{:}00$ do the minute and hour hands of a clock overlap?

Solution: At $1{:}00$, the minute hand is directly pointing to 12 and the hour hand is pointed at 1. So, we know the overlap will happen between 1 and 2, given that the minute hand is moving faster than the hour hand.

In particular, the tip of the minute hand covers 60 ticks (i.e., the little hash marks on the clock) every hour while the tip of the hour hand only traverses 5 ticks per hour. The "speed" of the minute hand is 60 ticks/hour and that of the hour hand is 5 ticks/hour. Further, the hour hand has a 5 tick lead.

The distance from 12 on the clock (measured in tick marks on the clock) by the tip of the minute hand in t hours is $60t$, and the distance from 12 on the clock by the tip of the hour hand in t hours is $5 + 5t$. [Why is t in hours?]

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We need to solve the following equation to determine the solution for the problem:

$$60t = 5 + 5t$$

$$t = \frac{1}{11} \text{ hours}$$

So, the tips of both hands overlap at $\frac{60}{11}$ or $5\frac{5}{11}$ ticks past $12:00$, or equivalently $\frac{5}{11}$ of a tick past $1:00$. In terms of time, the two hands overlap at about 5 minutes and 27.27 seconds after $1:00$.

4.3 Age Puzzles

Author's Remark: The first three puzzles below are from my book "Algebra through Discovery and Experimentation" [45].

Puzzle: Aaron is 3 years older than his brother William. If the product of their ages is three more than five times the sum of their ages, how old are Aaron and William?

Solution:

If we let x be the age of Aaron, then William is $x - 3$.

We are given that

$$x(x-3) = 5\big(x + (x-3)\big) + 3$$

which is equivalent to

$$x^2 - 3x = 5(2x-3) + 3$$
$$x^2 - 3x = 10x - 12$$
$$x^2 - 13x + 12 = 0$$

We can factor $x^2 - 13x + 12$ as $(x-1)(x-12) = 0$. Aaron is either 1 or 12, but William is three years younger and so, Aaron must be 12 and not 1. William must be 9.

. . .

Puzzle: Three times Cupcake's age plus 7 is equal to Max's age. The product of their ages is 110. How old is each cat?

Hint: If we let x be Cupcake's age, then Max's age is $3x + 7$. Further, we are given the $x(3x+7) = 110$. Of the two solutions, select the positive one.

Answer: $x = 5$. So, Cupcake is 5 and Max is an ancient cat at 22. For your information, the current record for the oldest cat is 38 years 3 days, see https://en.wikipedia.org/wiki/List_of_oldest_cats.

. . .

Puzzle: Eric is 7 less than four times the age of Erica. The product of their ages is 47 more than the sum of their ages. How old are Eric and Erica?

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Hint: Let x be Erica's age, then Eric is $4x-7$. The other piece of information translates to

$$x(4x-7)=(x+4x-7)+47$$

which simplifies to

$$4x^2-12x-40=0.$$

. . .

The above three puzzles are fairly straightforward. The following puzzle is very difficult to translate from words to equations that can be solved.

Puzzle: Yolanda is as old as Zion will be when Yolanda will be twice as old as Zion was when Yolanda's age was half the sum of their present ages.

Further, Zion is (now) as old as Yolanda was when Zion was half the age he will be in 10 years.

How old are Yolanda and Zion now?

Source: YouTube video entitled "Hard Puzzle - Most Difficult Age Puzzle" [46]

Solution:

[Author's Remark: The solution provided here is different from the approach taken in the referenced YouTube video.**]**

Let y and z represent the current ages of Yolanda and Zion. Further, let $d=y-z$, i.e., d is the difference between the ages of Yolanda and Zion. In what follows, it is critical to remember that d is a constant.

The second sentence of the puzzle can be restated as "At the point in time when Zion was half the age he will be in 10 years, Zion's current age z was the same as Yolanda's." This can be written more succinctly as:

When the age of Yolanda was z, the age of Zion was $(z+10)/2$.

The above can be used to determine d, i.e., $d=z-\frac{z+10}{2}=\frac{z-10}{2}$.

The phrase "when Yolanda's age was half the sum of their present ages" translates to

When the age of Yolanda was $\frac{y+z}{2}$, the age of Zion was x (unknown at this point).

The above give us the following (keeping in mind that the age difference of the two is constant):

$$\frac{y+z}{2}-x=d$$

Rearranging the above and the substituting the previous value we determined for d, gives us

$$x=\frac{y+z}{2}-d=\frac{y+z}{2}-\left(\frac{z-10}{2}\right)=\frac{y+10}{2}$$

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The phrase "Yolanda is as old as Zion will be when Yolanda will be twice as old as Zion was …" translates to

When the age of Yolanda will be $2x$, the age of Zion will be y.

The above implies that

$$d = 2x - y = 2\left(\frac{y+10}{2}\right) - y = 10$$

Using the earlier equation for z and substituting $d = 10$, we have

$$d = 10 = \frac{z-10}{2} \Rightarrow z = 30$$

Since we know the difference of the two ages $d = 10 = y - z = y - 30$, we have that $y = 40$.

So, Yolanda is 40 and Zion is 30.

4.4 Work-related Puzzles

Puzzle: Ursula can mow a given lawn in 7 minutes, Vern can mow the same lawn in 13 minutes and Willie can mow the same lawn in 19 minutes. How long would it take for all three to collaborate and mow the given lawn?

Solution:

The respective lawn mowing rates for Ursula, Vern and Willie, are $\frac{1}{7}$ lawn/min, $\frac{1}{13}$ lawn/min and $\frac{1}{19}$ lawn/min, respectively. Summing the rates, we get $\frac{1}{7}+\frac{1}{13}+\frac{1}{19}=\frac{471}{1729}$ lawn/min. So, if the three work together, they can mow 1 lawn in $\frac{1729}{471} \cong 3.67$ minutes.

. . .

In the general work problem, one is given the time that it takes different entities to complete a task (call these times $t_1, t_2, \dots, t_n$). It is assumed that the entities can work together to complete the same task, without interfering with each other, in time t. Each time can be converted into a rate. For example, if the first entity can complete a given task in time t_1, we can say that $\frac{1}{t_1}$ is the rate at which the first entity completes the given task. Further, the various rates are related as follows:

$$\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_n}=\frac{1}{t}$$

If all but one rate is given, we can solve for that unknown.

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Consider the example of 10 stone masons who can complete a boundary wall in the following times: 6,7,8,9,10,7,8,6,11 and 9 minutes, respectively. How long will it take all ten of the stone masons to complete one boundary wall? From the above discussion, we have

$$\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{7}+\frac{1}{8}+\frac{1}{6}+\frac{1}{11}+\frac{1}{9}=\frac{1}{x}$$

To add the above fractions, we multiple the above equation by the Least Common Multiple (LCM) of the distinct denominators, i.e., $\mathrm{lcm}(6,7,8,9,10,11) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 27720$, to get

$$4620 + 3960 + 3465 + 3080 + 2772 + 3960 + 3465 + 4620 + 2520 + 3080$$

$$= 35542 = \frac{27720}{x}$$

Solving for x, we get $x = \frac{27720}{35542} \cong .78$ minutes.

. . .

Puzzle: Al can paint a given room in 40 minutes and Betty can paint the same room in 60 minutes. If by working together Al, Betty and Carl can paint the room in 15 minutes, how long would it take Carl to paint the room by himself?

Solution:

Let x be the number of minutes it takes Carl to paint the room. Thus, Carl paints the room at a rate of 1 room/x minutes. As discussed previously, the rates of the individuals must equal the rate of work for the combined effort, i.e.,

$$\frac{1}{40}+\frac{1}{60}+\frac{1}{x}=\frac{1}{15}$$

Multiply both sides of the equation by 120, we get

$$3+2+\frac{120}{x}=8$$

Thus, $\frac{120}{x} = 3$ which implies $x = \frac{120}{3} = 40$, i.e., Carl can paint the room in 40 minutes when working alone.

4.5 Mixture Problems

The following problems are taken from the author's algebra book [45].

Puzzle: Given 20 liquid ounces of a 20% vinegar and 80% water solution. How much vinegar should be added to make the solution 25% vinegar and 75% water?

Answer:

At first glance, it may seem that we don't need algebra to solve this problem, since 20% of 20 is 4 and if you just add 1 ounce of vinegar, then the solution is 25% vinegar. However, this is wrong, since the overall amount of liquid increases when you add 1 ounce of vinegar.

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Let x be the amount of vinegar added to the solution, and note that we already have $.2(20) = 4$ ounces of vinegar. We want the amount of vinegar (i.e., $x + 4$) to be .25 of the total volume, which is now $20 + x$. This can be represented by the equation

$$x + 4 = .25(20 + x)$$

$$x + 4 = .25x + 5$$

$$.75x = 1$$

$$x = 1\frac{1}{3} = 1.\overline{33}$$

. . .

Puzzle: Do the previous puzzle in reverse, i.e., assume you have a $21\frac{1}{3}$ (i.e., $\frac{64}{3}$) ounce solution of 25% vinegar in water, and want to know how much vinegar to extract to get the mixture down to 20% vinegar.

Answer: Let x be the amount of vinegar to be extracted from the solution, and note that we start with $.25\left(\frac{64}{3}\right) = \frac{1}{4} \cdot \frac{64}{3} = \frac{16}{3}$ ounces of vinegar. We are being asked to determine how much vinegar to extract such that the resulting solution is 20% vinegar. This can be represented by the equation

$$\left(\frac{16}{3} - x\right) = .2\left(\frac{64}{3} - x\right) = \frac{1}{5}\left(\frac{64}{3} - x\right)$$

Multiple both sides of the above equation by 15 to get

$$80 - 15x = 64 - 3x$$

$$16 = 12x$$

$$x = \frac{16}{12} = \frac{4}{3} = 1\frac{1}{3}$$

which is verified by the previous problem.

. . .

Puzzle: A car's fuel tank has a capacity of 20 gallons. The completely filled tank currently contains a mixture of 85% gasoline and 15% ethanol. How many gallons must be replaced by a 55% gasoline and 45% ethanol solution to result in a full tank (i.e., 20 gallons) of a 70% gasoline and 30% ethanol solution?

Answer:

Let x be the amount of fuel removed from the starting mixture. The problem is to determine x from the given information. If we remove x amount of the original fuel, then $.15x$ of the ethanol has been removed. By adding an amount x of the new mixture, we add $.45x$ ethanol. We want the resulting mixture to have $.3(20) = 6$ gallons of ethanol. The situation is summarized in Table 6.

Table 6. Gasoline mixture problem

	Original	**Removed**	**Added**	**Result**
Concentration of ethanol	$.15$	$.15$	$.45$	$.3$
Amount of ethanol	$.15(20) = 3$	$-.15x$	$.45x$	$.3(20) = 6$

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To determine the value of x, we need to solve the following equation (derived from the bottom row in the above table)

$$3 - .15x + .45x = 6$$

$$3 + .3x = 6$$

$$.3x = 3 \text{ which implies } x = 10$$

Let's check the answer

Amount of ethanol from the 10 gallons of the original mixture	Amount of ethanol from the 10 gallons added to the tank	Total amount of ethanol after change	Percentage of ethanol after the change
$.15(10) = 1.5$	$.45(10) = 4.5$	$(3 - 1.5) + 4.5 = 6$	$\frac{6}{20} = .3$ or 30%

. . .

Puzzle: Coffee mixture A is 10% cream, and coffee mixture B is 20% cream. How many liters of each solution should be used to make 20 liters of a coffee mixture which is 15% cream?

Hint: Let x be the amount of coffee from mixture A.

Solution: If x is the amount of coffee from mixture A, then $20 - x$ is the amount of coffee used from mixture B. We want

$$.1\,x + .2\,(20 - x) = .15(20) = 3$$

$$-.1\,x + 4 = 3$$

$$x = 10$$

As a check, 10 liters of coffee mixture A and 10 liters of coffee mixture B yields $10(.1) + 10(.2) = 1 + 2 = 3$ liters of cream in a 20 liter mixture of coffee which is the required 15%.

4.6 Solving Equations

Puzzle: Find all real and complex solutions to

$$x^3 - 6x + \frac{1}{x^3} - \frac{6}{x} = 0$$

By "complex", we mean solutions of the form $a + bi$ where $i = \sqrt{-1}$.

Solution: This problem illustrates a useful algebra formula, and the technique of "completing the square."

Regrouping the terms, we have

$$\left(x^3 + \frac{1}{x^3}\right) - 6\left(x + \frac{1}{x}\right) = 0$$

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The sum of cubes formula is $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. We apply this formula to the left term in the above, with $a = x$ and $b = \frac{1}{x}$, to get

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 1\right) - 6\left(x + \frac{1}{x}\right) = 0$$

which implies

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 1 - 6\right) = 0$$

So, either $x + \frac{1}{x} = 0$ or $x^2 - 7 + \frac{1}{x^2} = 0$.

In the former case, we have

$$x + \frac{1}{x} = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow x = \pm i$$

where $i = \sqrt{-1}$ is an imaginary number.

In the second case, we first multiply by x^2 to get

$$x^4 - 7x^2 + 1 = 0, \qquad x \neq 0$$

The above expression can be factored to get

$$(x^2 - 3x + 1)(x^2 + 3x + 1) = 0$$

Applying the quadratic formula to each of the terms on the left, we get

$$x = \frac{3 \pm \sqrt{5}}{2}, \frac{-3 \pm \sqrt{5}}{2}$$

Thus, we have two complex and four real solutions, i.e.,

$$i, \qquad -i, \qquad \frac{3+\sqrt{5}}{2}, \qquad \frac{3-\sqrt{5}}{2}, \qquad \frac{-3+\sqrt{5}}{2}, \qquad \frac{-3-\sqrt{5}}{2}$$

. . .

Puzzle: Given that $a + b = 3$ and $ab = 1$, find the value of $a^5 + b^5$.

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Solution: We first note that

$$(a^3+b^3)(a^2+b^2)=a^5+b^5+a^2b^3+a^3b^2=(a^5+b^5)+(ab)^2(a+b)$$

which implies

$$a^5+b^5=(a^3+b^3)(a^2+b^2)-(ab)^2(a+b)$$

Using the information given in the puzzle, the above equation reduces to

$$a^5+b^5=(a^3+b^3)(a^2+b^2)-3 \quad \text{(Equation 1)}$$

The above expression reduces the problem to finding a^2+b^2 and a^3+b^3.

We can easily solve for a^2+b^2 as follows:

$$(a+b)^2=a^2+b^2+2ab$$

$$a^2+b^2=(a+b)^2-2ab=3^2-2(1)=7$$

We take a similar approach to evaluate a^3+b^3

$$(a+b)^3=a^3+b^3+3ab(a+b)$$

$$a^3+b^3=(a+b)^3-3ab(a+b)=3^3-3(3)=18$$

Plugging the above results into Equation 1, we get

$$a^5+b^5=18(7)-3=123$$

. . .

The above puzzle and many other problems can be solved using something known as the Newton's identities (aka the Girard–Newton formulae) [47]. Before we can state the Newton identities, we need several definitions.

For a set of variable $x_1, x_2, \dots, x_n$, the k^th^ **power sum** is defined as

$$p_k={x_1}^k+{x_2}^k+\cdots+{x_n}^k$$

For example, the 3^rd^ power sum of the variables x, y, z is $x^3+y^3+z^3$.

The **elementary symmetric polynomials** for the set of variables $x_1, x_2, \dots, x_n$ are defined as

$e_0=1$

$e_1=x_1+x_2+\cdots+x_n$

$e_2=\sum_{1\le i<j\le n} x_i x_j$

e_3 is the sum of all possible products of the variables taken 3 at time

e_4 is the sum of all possible products of the variables taken 4 at time and so on

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…

$e_n = x_1 x_2 \dots x_n$

$e_k = 0,\ k > n$

Newton's identities relate the power sums to the elementary symmetric polynomials as follows:

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

In terms of power sums and elementary symmetric polynomials, we can restate the previous puzzle as follows:

Given $e_0 = 1, e_1 = p_1 = a + b = 3$ and $e_2 = ab = 1$, find p_5

Applying Newton's identity for $k = 2$, we have $2e_2 = e_1 p_1 - p_2$. Plugging in the known values, we get $2(1) = 3(3)\ - p_2$ which implies $p_2 = 7$.

Next, we consider the case for $k = 3$ and note that (by definition of the elementary symmetric polynomials $e_3 = e_4 = \cdots = 0$). For this case, $3e_3 = e_2 p_1 - e_1 p_2 + p_3$. Plugging in the known values thus far, we get $0 = 1(3) - 3(7) + p_3$ which implies $p_3 = 18$.

For $k = 4$, we have $0 = e_3 p_1 - e_2 p_2 + e_1 p_3 - p_4$. Inserting known values, we get $0 = 0 - 1(7) + 3(18) - p_4$ which implies $p_4 = 47$.

Finally, for $k = 5$, we have $0 = e_4 p_1 - e_3 p_2 + e_2 p_3 - e_1 p_4 + p_5$. Inserting know values, we get $0 = 0 - 0 + 1(18) - 3(47) + p_5$ which implies $p_5 = a^5 + b^5 = 123$.

. . .

Puzzle: Using Newton's identities, find $x^5 + y^5 + z^5$ given that

$$x + y + z = 1$$

$$x^2 + y^2 + z^2 = 2$$

$$x^3 + y^3 + z^3 = 1$$

Solution: Relative to Newton's identities, we are given $p_1 = e_1 = 1, p_2 = 2$ and $p_3 = 1$. Further, by definition, we have $e_k = 0, k > 3$.

We first compute e_2:

$$2e_2 = e_1 p_1 - p_2$$

$$2e_2 = 1 \cdot 1 - 2 \ \Rightarrow\ e_2 = -\frac{1}{2}$$

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Next, compute e_3:

$$3e_3 = e_2p_1 - e_1p_2 + p_3$$

$$3e_3 = -\frac{1}{2} - 2 + 1 \ \Rightarrow\ e_3 = -\frac{1}{2}$$

We can determine p_4 from the following identity

$$4e_4 = e_3p_1 - e_2p_2 + e_1p_3 - p_4$$

$$0 = -\frac{1}{2} + 1 + 1 - p_4 \ \Rightarrow\ p_4 = \frac{3}{2}$$

Finally, we are able to determine p_5 which, by definition, is $x^5 + y^5 + z^5$.

$$5e_5 = e_4p_1 - e_3p_2 + e_2p_3 - e_1p_4 + p_5$$

$$0 = 0 + \frac{1}{2} \cdot 2 - \frac{1}{2} \cdot 1 - \frac{3}{2} + p_5 \ \Rightarrow\ p_5 = 1$$

. . .

Puzzle: Using Newton's identities to find $x^4 + y^4 + z^4$ given

$$x + y + z = -5$$

$$x^2 + y^2 + z^2 = 2$$

$$x^3 + y^3 + z^3 = 1$$

Answer: 73

. . .

To solve the following puzzle, we don't need Newton's identities.

Puzzle: Given $x + y = 3$ and $x^2 + y^2 = 7$, determine $x^6 + y^6$.

Solution: We first note that

$$9 = (x + y)^2 = (x^2 + y^2) + 2xy = 7 + 2xy$$

which implies that $xy = 1$.

In general, it is true that

$$x^n + y^n = (x + y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2})$$

For the problem at hand, the above equation reduces to

$$x^n + y^n = 3(x^{n-1} + y^{n-1}) - (x^{n-2} + y^{n-2})$$

Substituting $n = 3$ in the above equation, we get

$$x^3 + y^3 = 3(x^2 + y^2) - (x + y) = 3(7) - 3 = 18$$

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We now have sufficient information to determine $x^4 + y^4$, i.e.,

$$x^4 + y^4 = 3(x^3 + y^3) - (x^2 + y^2) = 3(18) - 7 = 47$$

Continuing in this manner, we have

$$x^5 + y^5 = 3(47) - 18 = 123$$

$$x^6 + y^6 = 3(123) - 47 = 322$$

. . .

Puzzle: Find three integers a, b and c such that

$$a + b + c = 29$$

$$a^2 + b^2 + c^2 = 315$$

$$abc = 715$$

Solution: To solve this problem, we need to make use of a result from the theory of equations. Consider the equation $x^3 + dx^2 + ex + f = 0$ with roots x_1, x_2 and x_3. The equation can be written as

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

which expands as

$$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_1x_3)x - x_1x_2x_3 = 0$$

Equating the coefficients of the two forms of the equation, we have that

$$-d = x_1 + x_2 + x_3$$

$$e = x_1x_2 + x_2x_3 + x_1x_3$$

$$-f = x_1x_2x_3$$

We can apply this result to the problem at hand, but we first need to determine $ab + bc + ac$. To that end, consider

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac)$$

Substituting the values that are provided in the statement of the puzzle, we get

$$29^2 = 315 + 2(ab + bc + ac)$$

$$ab + bc + ac = 263$$

Using the result that we developed above, we have that a, b and c are the roots of the equation

$$x^3 - 29x^2 + 263x - 715 = 0$$

By the rational root theorem [36], we know that the rational roots of the above equation must be divisors of $715 = 5 \cdot 11 \cdot 13$. If we test 5,11 and 13, each is a root of the above equation. Thus,

$$a = 5, \qquad b = 11, \qquad c = 13$$

. . .

Puzzle: Find all integers a, b, c for which

$$(x - a)(x - 10) + 1 = (x + b)(x + c) \text{ for all } x$$

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Source: Problem #1 from the British Mathematical Olympiad, 1988, https://bmos.ukmt.org.uk/home/bmo-1989.pdf.

Solution: Expanding the equation, we get

$$x^2 - (10 + a)x + 10a + 1 = x^2 + (b + c)x + bc$$

Equating the coefficients of the x terms on both sides of the equation gives us

$$-(10 + a) = b + c$$

which implies

$$a = -(10 + b + c)$$

Equating the constant terms, we have

$$10a + 1 = bc$$

Substituting our equation for a into the above equation, we get

$$-10(10 + b + c) + 1 = bc$$

Solving for b in terms of c

$$-100 - 10b - 10c + 1 = bc$$

$$-100 - 10c + 1 = bc + 10b = b(c + 10)$$

$$-10(c + 10) + 1 = b(c + 10)$$

$$b = \frac{-10(c + 10) + 1}{c + 10} = -10 + \frac{1}{c + 10}$$

b is only an integer when $\frac{1}{c+10}$ is an integer, and that is true only when $c + 10 = \pm 1$ or $c = -11, -9$.

- When $c = -9$, we get one solution, i.e., $a = 8,\ b = -9, c = -9$.
- When $c = -11$, we get a second solution, i.e., $a = 12, b = -11, c = -11$.

. . .

Puzzle: Given $\sqrt[x]{36} + \sqrt[x]{24} = \sqrt[x]{16}$. Determine the value of x.

Solution: In general, note that $\sqrt[b]{a}$ is the same as $a^{\frac{1}{b}}$. Also, we will need to use the fact that $a^{xy} = (a^x)^y = (a^y)^x$.

Getting back to the problem at hand, divide both sides of the equation by $\sqrt[x]{36}$ to get

$$1 + \sqrt[x]{\frac{24}{36}} = \sqrt[x]{\frac{12}{36}} \Rightarrow 1 + \sqrt[x]{\frac{2}{3}} = \sqrt[x]{\frac{4}{9}} \Rightarrow 1 + \sqrt[x]{\frac{2}{3}} = \sqrt[x]{\left(\frac{2}{3}\right)^2} \Rightarrow 1 + \sqrt[x]{\frac{2}{3}} = \left(\sqrt[x]{\frac{2}{3}}\right)^2$$

(Note that $\Rightarrow$ is a shorthand notation for “which implies”.)

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Let $y = \sqrt[x]{\frac{2}{3}}$ in the above equation to get

$$y^2 - y - 1 = 0$$

Using the quadratic formula, we can determine y, i.e.,

$$y = \frac{1 \pm \sqrt{5}}{2}$$

Of the two solutions, we choose the positive one since $\sqrt[x]{\frac{2}{3}}$ is a positive number. Substituting the expression for y into the above, we get $\sqrt[x]{\frac{2}{3}} = \frac{1+\sqrt{5}}{2}$, or equivalently, $\left(\frac{2}{3}\right)^{\frac{1}{x}} = \frac{1+\sqrt{5}}{2}$.

Take the natural logarithm on both sides of the above equation to get

$$\left(\frac{1}{x}\right)\left(\ln\frac{2}{3}\right) = \ln(1+\sqrt{5}) - \ln 2$$

$$x = \frac{\ln 2 - \ln 3}{\ln(1+\sqrt{5}) - \ln 2} \cong -0.84259$$

. . .

Puzzle: Find all real valued solutions to the equation

$$16^{x^2+y} + 16^{x+y^2} = 1$$

Source: Problem from the Mathematics topic on StackExchange [48]

Solution: To solve the problem, we make use of the arithmetic mean – geometric mean (AM-GM) inequality which states that for any two non-negative real numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab}$ (or equivalently, $a + b \geq 2\sqrt{ab}$) with equality if and only if $a = b$.

Applying the AM-GM inequality to the problem at hand, we have

$$1 = 16^{x^2+y} + 16^{x+y^2} \geq 2\sqrt{16^{x^2+y}16^{x+y^2}}$$

Next, we simplify the above inequality as follows

$$1 = 16^{x^2+y} + 16^{x+y^2} \geq 2\sqrt{16^{x^2+y}16^{x+y^2}} = 2\left((4^2)^{x^2+y}(4^2)^{x+y^2}\right)^{\frac{1}{2}} = 2\left(4^{2(x^2+y)}4^{2(x+y^2)}\right)^{\frac{1}{2}}$$

$$= 2\left(4^{x^2+y}4^{x+y^2}\right) = 4^{\frac{1}{2}}\left(4^{x^2+x}4^{y^2+y}\right) = \left(4^{x^2+x+\frac{1}{4}}\right)\left(4^{y^2+y+\frac{1}{4}}\right) = 4^{\left(x+\frac{1}{2}\right)^2}4^{\left(y+\frac{1}{2}\right)^2} \geq 1$$

The last inequality above holds true since 4 raised to any positive number is greater than 1.

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We have shown that $1 \geq 1$, which implies that we must have equality at each point in the above string of expressions. In particular, it must be that

$$4^{\left(x+\frac{1}{2}\right)^2} 4^{\left(y+\frac{1}{2}\right)^2} = 1$$

but this can only have if $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. Thus, we have but one solution to your equation.

. . .

Although not clear from the statement of the problem, the solution to the following puzzle involves something known as the Fibonacci numbers [37], i.e., the sequence

$$0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597, \dots$$

The first two elements are 1, and each following element is the sum of the previous two elements. The sequence can be defined more formally as

$$F_0 = 0, \qquad x_1 = 1$$

$$F_{n+2} = F_{n+1} + F_n, \qquad n = 0,1,2, \dots$$

Puzzle: Find all pairs of integers (a, b) such that the polynomial

$$ax^{17} + bx^{16} + 1$$

is divisible by $x^2 - x - 1$. ("Divisible" means "without any remainder.")

Source: Question #13 from the 1988 American Invitational Mathematics Exam (AIME)

Solution: If $x^2 - x - 1$ divides $ax^{17} + bx^{16} + 1$, then the other factor is a polynomial of degree 15, i.e.,

$$(c_{15}x^{15} - c_{14}x^{14} + c_{13}x^{13} \dots - c_2x^2 + c_1x - c_0)(x^2 - x - 1) = ax^{17} + bx^{16} + 1$$

Equating the coefficients of like powers of x on either side of the above equation, we get

$$c_0 = 1$$

$$c_0 - c_1 = 0 \;\Rightarrow\; c_1 = 1$$

$$-c_0 - c_1 + c_2 = 0 \;\Rightarrow\; c_2 = 2$$

$$c_1 + c_2 - c_3 = 0 \;\Rightarrow\; c_3 = 3$$

$$-c_2 - c_3 + c_4 = 0 \;\Rightarrow\; c_4 = 5$$

The pattern continues and we have the general formula for the c_n terms, i.e.,

$$c_{n+2} = c_{n+1} + c_n$$

So, $c_n = F_{n+1}$ (i.e., the n+1st Fibonacci number).

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Equating the coefficients of x^{15}

$$-c_{15} + c_{14} + c_{13} = 0$$

$$c_{15} = c_{14} + c_{13} = F_{15} + F_{14} = 610 + 377 = 987$$

Equating the coefficients of x^{16}

$$b = -c_{14} - c_{15} = -F_{15} - F_{16} = -610 - 987 = -1579$$

Equating the coefficients of x^{16}

$$a = c_{15} = F_{16} = 987$$

. . .

Puzzle: Find all real-valued solutions to the equation

$$3^x - 54x + 135 = 0$$

Solution: Rearrange the equation as follows:

$$3^x = 54x - 135$$

$$3^x = 3^3(2x - 5)$$

$$3^{x-3} = 2x - 5$$

Letting $t = x - 3$ in the above, we have

$$3^t = 2t + 1$$

On the right-side of the above equation, $2t + 1$ is a straight line with slope 2 and vertical-axis intercept 1. The line $2t + 1$ intersects 3^t at $t = 0$ and $t = 1$, and so, x is either 3 or 4. The graphs of 3^t and $2t + 1$, and their intersection points are shown in Figure 10.

The figure was created using GeoGebra, see https://www.geogebra.org/m/gxswbfjz.

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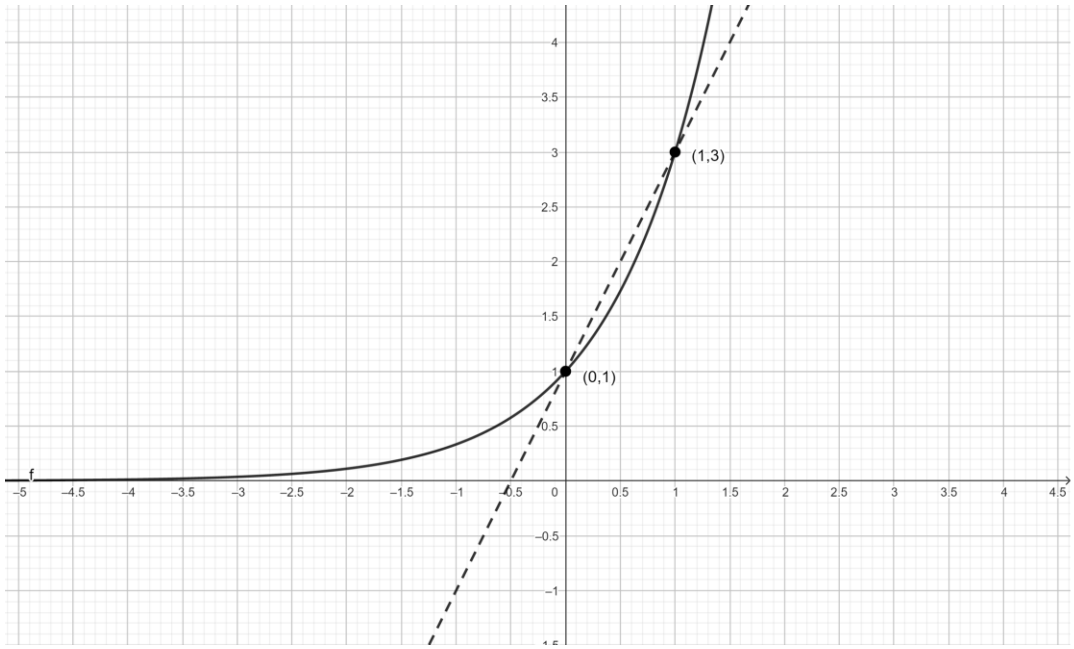

Figure 10. Graphs of 3^t and 2t+1

. . .

Puzzle: Given

$$\frac{a}{b}+\frac{b}{a}=7$$

$$\frac{a^2}{b}+\frac{b^2}{a}=18$$

find

$$\frac{1}{a}+\frac{1}{b}$$

Solution: Note that the structure of the given equations implies that $a \neq 0$ and $b \neq 0$.

The first given equation can be written as

$$a^2+b^2=7ab$$

Completing the square on the left side of the equation, we get

$$(a+b)^2-2ab=7ab \qquad \text{(Equation 1)}$$

The second given equation can be written as

$$a^3+b^3=18ab$$

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Next, note that (in general)

$$(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3 = a^3 + b^3 + 3ab(a+b)$$

which implies

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

Substituting the above into the modified version of the second given equation, we get

$$(a+b)^3 - 3ab(a+b) = 18ab \qquad \text{(Equation 2)}$$

Let $x = a + b$ and $y = ab$, and then substitute into Equations 1 and 2 to get

$$x^2 - 2y = 7y \Rightarrow x^2 = 9y$$

$$x^3 - 3xy = 18y$$

Substitute the result of the first equation above into the second equation to get

$$(9y)x - 3xy = 18y$$

$$6xy = 18y$$

We can cancel y on both sides of the above equation, since $y \neq 0$ (this follows since $a \neq 0$ and $b \neq 0$ which implies $ab \neq 0$). So, $x = 3$ and $9 = 9y \Rightarrow y = 1$.

Thus,

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{x}{y} = \frac{3}{1} = 3$$

4.7 Radicals

Puzzle: Determine the value of

$$\sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \ldots}}}}}$$

To be clear, the nesting goes on indefinitely, while repeating the same pattern.

Solution: If we let

$$x = \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \ldots}}}}}$$

then we also have $x = \sqrt{12 + x}$.

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Squaring both sides of the above equation, we get

$$x^2 = 12 + x$$
$$x^2 - x - 12 = 0$$
$$(x+3)(x-4) = 0$$

Since we implicitly defined x to be positive, only one of the two solutions in the above equation is valid, i.e., $x = 4$.

...

Continued fractions follow a similar recursive pattern to that of nested radicals. For example, consider

$$x = \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cdots}}}}}$$

Given the repeating pattern, we can write

$$x = \frac{12}{-1 + x}$$

which can be rearranged as

$$x^2 - x = 12$$
$$x^2 - x - 12 = (x+3)(x-4) = 0$$

The two possible solutions are $x = -3, 4$. To decide between the two possibilities for x, we look at the convergents of the continued fractions. ("Convergents" are truncated portions of the continued fraction.)

$$0, \frac{12}{-1}, \cfrac{12}{-1 + \cfrac{12}{-1}}, \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1}}}, \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1}}}}, \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1 + \cfrac{12}{-1}}}}}, \ldots$$

Doing the computations, we get

$$-12, -\frac{12}{13}, -\frac{156}{25}, -\frac{300}{181}, -\frac{2172}{481}, \ldots$$

All the convergents are negative, and eventually approach -3.

...

Puzzle: Is $\sqrt{4 - 2\sqrt{3}} + \sqrt{97 - 56\sqrt{3}}$ an integer?

Source: Problem #2 in the Pan African Mathematics Olympiad 2004, http://africamathunion.org/PAMO_2004_Problems_En.pdf

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Solution: The solution (which if far from obvious) is to de-nest each of the two terms. ("De-nest" means to remove at least one layer of the radicals. For more background on nested radicals, see the Wikipedia article entitled "Nested radical" [49].)

We first de-nest a more general expression, i.e., $\sqrt{a-b\sqrt{c}}$ and then apply the solution to the problem at hand. Assume $\sqrt{a-b\sqrt{c}}$ can be represented in the form $\sqrt{d}-\sqrt{e}$, i.e., set

$$\sqrt{a-b\sqrt{c}}=\sqrt{d}-\sqrt{e}$$

and solve for d and e in terms of a, b and c.

Squaring both sides of the equation, we get

$$a-b\sqrt{c}=d+e-2\sqrt{de}$$

Equating the non-radical terms, we have

$$a=d+e\ \Rightarrow d=a-e$$

Equating the radical terms, we have

$$-b\sqrt{c}=-2\sqrt{de}$$

Squaring both side of the above equation yields

$$b^2c=4de$$

Substituting $d=a-e$ into the above equation yields

$$b^2c=4(a-e)e=4ae-4e^2$$

$$4e^2-4ae+b^2c=0$$

Solve for e using the quadratic formula. After some simplification, we get

$$e=\frac{a\pm\sqrt{a^2-b^2c}}{2}$$

Given the symmetry of the problem,

$$d=\frac{a\mp\sqrt{a^2-b^2c}}{2}$$

If we choose the positive sign for e, then we choose the negative sign for d in the above formulas. This gives us two possible solutions which need to be checked against the original nested radical.

For the problem at hand, we have $a=97, b=56$ and $c=3$. Plugging these numbers into the above equations for e and d, gives two solutions, i.e., $e=49, d=48$ and $e=48, d=49$. The first solution is invalid since in this case, $\sqrt{d}-\sqrt{e}=\sqrt{48}-\sqrt{49}<0$, but $\sqrt{97+56\sqrt{3}}>0$. However, the second solution is valid, i.e.,

$$\sqrt{97+56\sqrt{3}}=\sqrt{49}-\sqrt{48}=7-4\sqrt{3}$$

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For other term in the original expression, we have $a = 4, b = 2$ and $c = 3$, which gives us two possible solutions, i.e., $e = 3, d = 1$ and $e = 1, d = 3$. In the first case, we get $\sqrt{d} - \sqrt{e} = 1 - \sqrt{3} < 0$ which is invalid since $\sqrt{4 - 2\sqrt{3}} > 0$. However, the other solution is valid, i.e.,

$$\sqrt{4 - 2\sqrt{3}} = \sqrt{d} - \sqrt{e} = \sqrt{3} - 1$$

Putting the two pieces of the puzzle together, we have

$$4\sqrt{4 - 2\sqrt{3}} + \sqrt{97 - 56\sqrt{3}} = 4\left(-1 + \sqrt{3}\right) + \left(7 - 4\sqrt{3}\right) = 3$$

So, the answer to the original question is “yes”.

4.8 Miscellaneous

Puzzle: Given that a is a real number chosen randomly from the interval $[-5,3]$, find the probability that the equation $x^3 - ax^2 - ax + 1 = 0$ has only real roots (as opposed to one real root and a pair of roots that are complex numbers).

Solution: By observation, one can see that $x = -1$ is one root of the equation. This means that $(x + 1)$ is a factor of $x^3 - ax^2 - ax + 1$.

Next, we divide $x + 1$ into $x^3 - ax^2 - ax + 1$ as follows:

$$\begin{array}{r|llll} & x^2 & -(a+1)x & 1 & \\ \hline x+1 & x^3 & -ax^2 & -ax & 1 \\ & x^3 & x^2 & & \\ \hline & & -(a+1)x^2 & -ax & 1 \\ & & -(a+1)x^2 & -(a+1)x & \\ \hline & & & x & 1 \\ & & & x & 1 \end{array}$$

So, we have that $x^3 - ax^2 - ax + 1 = (x + 1)(x^2 - (a + 1)x + 1) = 0$

Applying the quadratic formula to the factor $x^2 - (a + 1)x + 1$, we see that

$$x = \frac{(a + 1) \pm \sqrt{(a + 1)^2 - 4}}{2}$$

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The equation will have real roots if and only if the expression within the square root above is positive, i.e.,

$$(a+1)^2 - 4 \geq 0$$

$$(a+1)^2 \geq 4$$

$$-2 \leq a+1 \geq 2$$

$$a \leq -3, \qquad a \geq 1$$

$$a \in (-\infty, -3] \text{ or } a \in [1, \infty)$$

Given that $a \in [-5,3]$ which is an interval of length 8, and the intersection of $[-5,3]$ with $(-\infty, -3]$ or $[1, \infty)$ is of length 4, the probability that the equation has all real roots is $\frac{4}{8} = .5$.

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5 Functions

We learn...
10% of what we read
20% of what we hear
30% of what we see
50% of what we both hear and see
70% of what is discussed
80% of what we experience personally
95% of what we teach to someone else

William Glasser (American psychiatrist)

5.1 Overview

A **function** f is a mapping from one set A to another set B under the condition that each element of A is mapped to only one element of B. In notation, we write $f: A \to B$. Further, if $x \in A$ (i.e., x is an element of set A) and f maps x to $y \in B$, we write $f(x) = y$ to indicate the mapping. The mapping can also be represented as the ordered pair (x, y). Further, x is referred to as the argument of the function f.

The set A is called the **domain** of the function and the set B is known as the **codomain**. The mapping of all elements A by the function f is known as the **range** of f. The range of f is represented as $f(A)$, and it is a subset of B, i.e., $f(A) \subseteq B$.

If $f(A) = B$, the f is said to be **surjective** (or onto).

If for any $x, y \in A$, $f(x) = f(y)$ implies that $x = y$, then f is said to be **injective** (or one-to-one). In other words, a function is injective if it maps distinct elements of its domain to distinct elements in its range.

A **bijective** function is both surjective and injective.

For example, the function $f(x) = x^3$ maps each element of $\mathbb{R}$ (i.e., the real numbers) to a unique element of $\mathbb{R}$. We can represent the mapping with a graph, where the horizontal axis represents x and the vertical axis represent $f(x)$. Figure 11 shows the graph of $f(x) = x^3$ along with the two example points $(-1, -1)$ and $(2,8)$. $f(x)$ is bijective.

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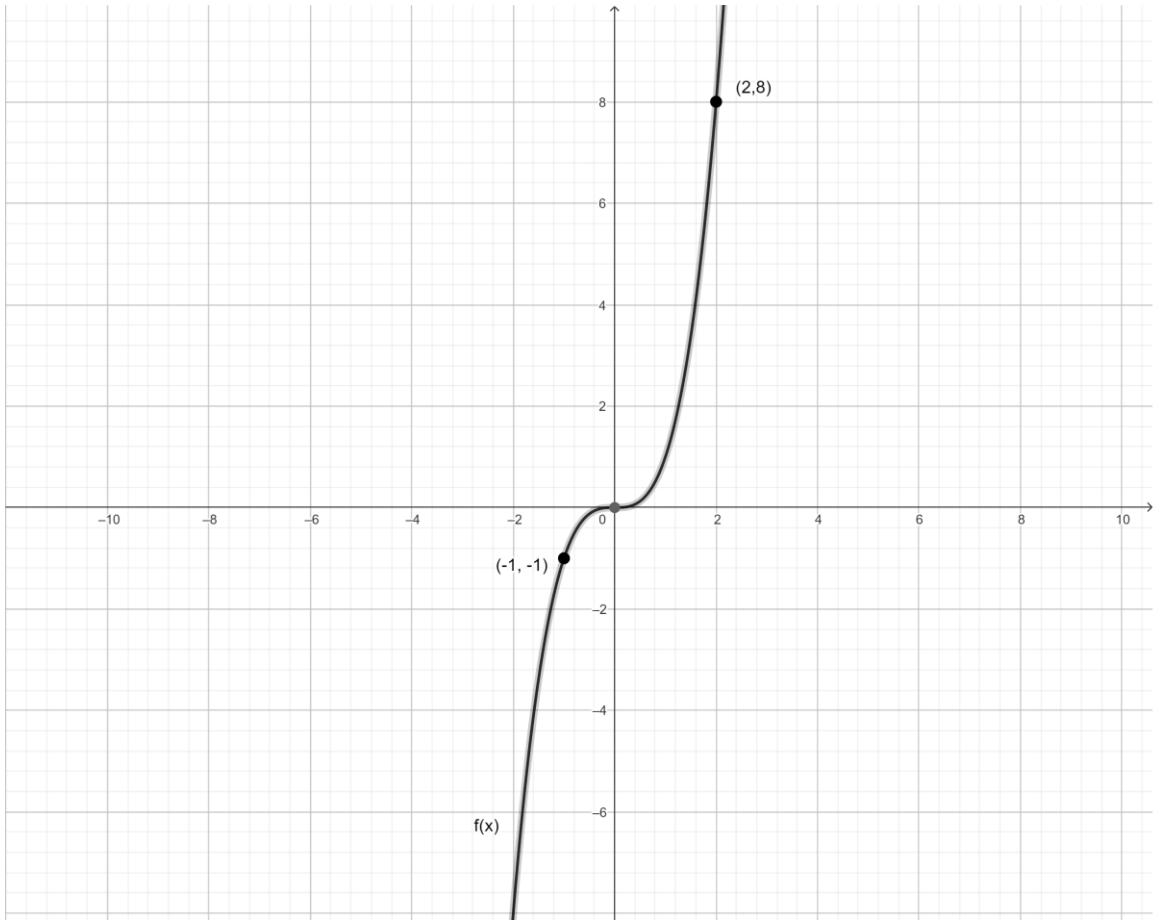

Figure 11. Graph of f(x)=x^3

If we define the function $g(x) = x^2$ from $\mathbb{R}$ (real numbers) to $\mathbb{R}^+$ (positive real numbers), then g is surjective but not injective. g is not injective since (for example) $g(-1) = g(1)$ but $-1 \neq 1$.

Continuity is an important concept associated with functions. The Wikipedia article on continuous functions [39] provides the following informal definition:

> A continuous function is a function such that a continuous variation (that is a change without jump) of the argument induces a continuous variation of the value of the function. This means that there are no abrupt changes in value, known as discontinuities. More precisely, a function is continuous if arbitrarily small changes in its value can be assured by restricting to sufficiently small changes of its argument.

Continuity of a function f at x is defined more formally using limits, i.e.,

> The function f is continuous at x_0 if the limit of $f(x)$, as x approaches x_0, exists and is equal to$f(x_0)$, i.e.,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

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For example, consider the following equation

$$f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

The graph of $f(x)$ is shown in Figure 12. A discontinuity was intentionally created at $x = 1$. Using the formal definition of continuity, we see that

$$\lim_{x \to 1} f(x) = 1 \ \neq f(1) = 2$$

and thus, the function is not continuous at $x = 2$. However, the function is continuous at all other points.

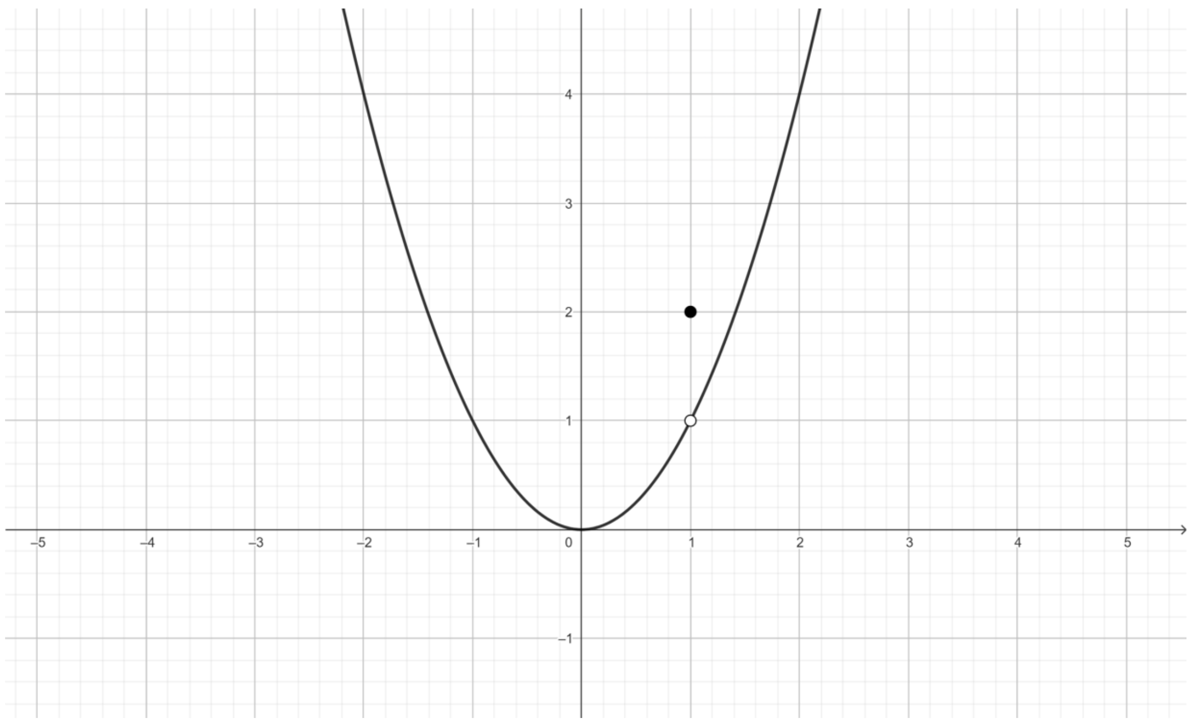

Figure 12. f(x)=x^2 with discontinuity at x=1

As another example, consider the function

$$f(x) = \begin{cases} \dfrac{1}{1-x}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

The function is an hyperbola (as shown in Figure 13), with a point added at $x = 1$. In this example, the limit as $x \to 1$ is undefined since the function goes to positive infinity from the left of $x = 1$ and to negative infinity on the right of $x = 1$. The function is defined at $x = 1$ but is also discontinuous at $x = 1$ since the limit as $x \to 1$ does not exist.

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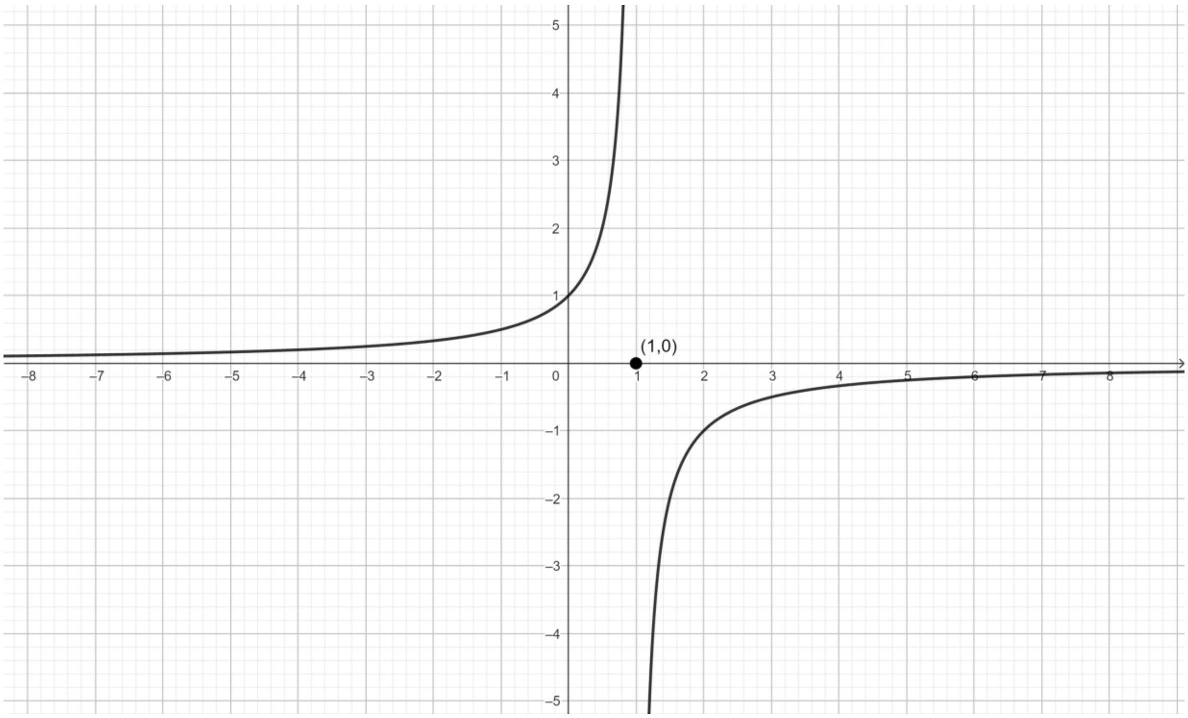

Figure 13. Graph of f(x)=1/(1-x)

5.2 Puzzles

A popular theme in the various mathematics contests is to provide a condition on a function, and then ask the contestant to determine the function (or set of functions) that satisfy the condition. The puzzles in this section are of this type.

Puzzle: Find $f(x)$ if $3f(x) - 4f\left(\frac{1}{x}\right) = x^2$.

Solution: Let $x = t$ and then $x = 1/t$ to get the following two equations

$$3f(t) - 4f\left(\frac{1}{t}\right) = t^2$$

$$3f\left(\frac{1}{t}\right) - 4f(t) = \left(\frac{1}{t}\right)^2$$

Multiply the first equation by 3 and the second by 4, and then add the two equations to get

$$-7f(t) = 3t^2 + \frac{4}{t^2}$$

$$f(t) = -\frac{3}{7}t^2 - \frac{4}{7t^2}$$

. . .

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Puzzle: Find all functions f from the real numbers to the real numbers which satisfy

$$f(x^3) + f(y^3) = (x + y)\big(f(x^2) + f(y^2) - f(xy)\big)$$

for all real numbers x and y.

Source: Problem #3 from the British Mathematical Olympiad, Round 2, 29 January 2009, https://bmos.ukmt.org.uk/home/bmolot.pdf

Solution: Let's see if setting $x = y = 0$ provides any information on the function in question.

$$f(0) + f(0) = (0 + 0)\big(f(0) + f(0) - f(0)\big)$$

$$2f(0) = 0 \Rightarrow \boldsymbol{f(0) = 0}$$

Next, try $x = \sqrt[3]{t}$ and $y = -\sqrt[3]{t}$ to get

$$f(t) + f(-t) = 0$$

$$f(t) = -f(-t)$$

For uniformity's sake, we can replace t by x in the above to get $\boldsymbol{f(x) = -f(-x)}$ for all real numbers x.

Letting $y = x$ in the original equation, we get

$$f(x^3) + f(x^3) = 2x\big(f(x^2) + f(x^2) - f(x^2)\big)$$

$$f(x^3) = xf(x^2)$$

Using the above in the original equation, we have that

$$xf(x^2) + yf(y^2) = (x + y)\big(f(x^2) + f(y^2) - f(xy)\big)$$

Expanding the right-side of the above yields

$$xf(x^2) + yf(y^2) = xf(x^2) + xf(y^2) - xf(xy) + yf(x^2) + yf(y^2) - yf(xy)$$

Canceling like terms, we get

$$0 = xf(y^2) - xf(xy) + yf(x^2) - yf(xy)$$

which implies

$$\boldsymbol{xf(y^2) + yf(x^2) = (x + y)f(xy)}$$

Letting $y = 1$ in the above equation, we get

$$xf(1) + f(x^2) = (x + 1)f(x)$$

We then let $y = -1$ and make use of the previous result $f(-x) = -f(x)$ to get

$$xf(1) - f(x^2) = (x - 1)f(-x) = (x - 1)(-f(x))$$

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Adding the two equations above, we have

$$2xf(1) = 2f(x)$$

$$f(x) = f(1)x$$

Since $f(1)$ is just an unknown constant, let's replace it with the generic constant C. So, we have that solutions satisfying the original equation must be of the form $f(x) = Cx$. As a cross-check, we can substitute $f(x) = Cx$ into the original equation.

When set $f(x) = Cx$, the left-side of the original equation becomes

$$f(x^3) + f(x^3) = Cx^3 + Cy^3 = C(x^3 + y^3) = C(x + y)(x^2 - 2xy + y^2)$$

and the right-side of the original equation becomes

$$(x + y)\big(f(x^2) + f(y^2) - f(xy)\big) = (x + y)(Cx^2 + Cy^2 - Cxy) = C(x + y)(x^2 - 2xy + y^2)$$

Thus, both sides of the expression are the same and we have verified that $f(x) = Cx$ satisfies the conditions of the problem.

. . .

For the next puzzle, we need the following theorem concerning continuous functions.

Theorem 9*. If f and g are continuous functions, and $f(x) = g(x)$ for all $x \in \mathbb{Q}$ (i.e., for all rational values), then $(x) = g(x)$ for all $x \in \mathbb{R}$ (i.e., for all real values).*

Proof: For any $x \in \mathbb{R} - \mathbb{Q}$ (i.e., for any x that is irrational), we can construct a sequence $x_n \in \mathbb{Q}$ that converges to x, i.e.,

$$\lim_{n\to\infty} x_n = x$$

Since we are told that f and g are continuous, it must be that

$$\lim_{n\to\infty} f(x_n) = x$$

$$\lim_{n\to\infty} g(x_n) = x$$

We are given in the statement of the theorem that $f(x) = g(x)$ for all x ∈ $\mathbb{Q}$, and thus

$$f(x_n) = g(x_n)\ \forall x \in \mathbb{N}$$

Therefore, the two functions converge to the same value for all values of x, i.e., $f(x) = g(x)$ for all x ∈ $\mathbb{R}$. ∎

Puzzle: Determine all continuous functions that satisfy the condition $f(x + y) = f(x) + f(y)$ for all real numbers x and y.

Source: This problem is a special case of the Cauchy functional equation [40].

Solution: We first note that

$$f(x) = f(x + 0) = f(x) + f(0)$$

which implies

$$f(0) = 0$$

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Further, we have

$$0 = f(0) = f(x - x) = f(x) + f(-x)$$

which implies

$$f(-x) = -f(x)$$

Next, consider the case for a positive integer n times x.

$$f(nx) = f\big((n-1)x + x\big) = f\big((n-1)x\big) + f(x)$$

Repeating the same idea as above, we have that $f\big((n-1)x\big) = f\big((n-2)\big) + f(x)$.

Repeating this process n times, we get $f(nx) = nf(x)$ for any positive integer n. Since $f(-x) = -f(x)$, we can say that $f(-nx) = -f(nx) = -nf(x)$. Thus, $f(nx) = nf(x)$ for any integer n.

Eventually, we want to show $f(rx) = rx$ for any rational number $r = \frac{n}{m}$ with n and m being integers. We first consider the case of $\frac{x}{m}$, with m being an integer. In the following equation, the term $\frac{1}{m}$ is repeated m times.

$$f(x) = f\left(\frac{x}{m} + \frac{x}{m} + \cdots + \frac{x}{m}\right) = f\left(\frac{x}{m}\right) + f\left(\frac{x}{m}\right) + \cdots + f\left(\frac{x}{m}\right) = mf\left(\frac{x}{m}\right)$$

which implies

$$f\left(\frac{x}{m}\right) = \frac{1}{m} f(x)$$

Now, take any rational number $r = \frac{n}{m}$. Using the results that have already proved, we have

$$f(rx) = f\left(n\left(\frac{x}{m}\right)\right) = nf\left(\frac{x}{m}\right) = \frac{n}{m} f(x) = rf(x)$$

If we let $x = 1$ in the above equation, we have that $f(r) = f(r \cdot 1) = rf(1)$. Noting that $f(1)$ is a constant, we let $C = f(1)$. So, for every rational number r, $f(r) = Cr$.

By **Theorem 9**, there cannot be different (distinct) function $g(x)$ that has the same values as $f(x)$ for all rational values of x. Thus, $f(x) = Cx$, such that $C = f(1)$, is the unique continuous function satisfying the condition stated in the puzzle.

. . .

Puzzle: Determine all continuous functions f that map from the positive real numbers to the real numbers, i.e., $f\colon \mathbb{R}^+ \to \mathbb{R}$, such that $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$.

Hint: For a function f that satisfies the condition of the puzzle, define $g(x) = f(e^x)$ and use the result from the previous puzzle.

Solution: Since the exponential function e^x is continuous and $f(x)$ is continuous, then the composition of $f(x)$ with e^x is continuous, i.e., $g(x)$ is continuous. (In general, the composition of a continuous function with another continuous function is continuous, see [41].)

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Consider

$$g(x+y) = f(e^{x+y}) = f(e^x e^y) = f(e^x) + f(e^y) = g(x) + g(y)$$

where $f(e^x e^y) = f(e^x) + f(e^y)$ follows from the condition stated in the puzzle.

From the previous puzzle, $g(x)$ must be a linear function of the form Cx where C is a constant. Thus, $f(e^x) = g(x) = Cx$. Noting the $e^{\ln x} = x$, we have the following

$$f(x) = f\left(e^{\ln x}\right) = g(\ln x) = C \ln x$$

So, any function f satisfying the conditions of the puzzle must be of the form $C \ln x$.

. . .

Puzzle: Find all functions f, defined on the real numbers and taking real values, which satisfy the equation $f(x)f(y) = f(x + y) + xy$ for all real numbers x and y.

Source: Problem #5 from the British Mathematical Olympiad, Round 1, 3 December 2009, https://bmos.ukmt.org.uk/home/bmolot.pdf

Solution: First, we try $y = 0$ and leave x free to get

$$f(x)f(0) = f(x) + 0$$

$$f(x)f(0) - f(x) = 0$$

$$f(x)[f(0) - 1] = 0$$

which implies either $f(x) = 0$ for all values of x, or $f(0) = 1$.

Case 1: $f(x) = 0$ for all values of x

Let $x = y = 1$ to get

$$0 = f(1)f(1) = f(2) + 1 = 0 + 1 = 1$$

This is clearly a contradiction, and thus, we exclude this case.

Case 2: $f(0) = 1$

Let $x = 1, y = -1$ to get

$$f(1)f(-1) = f(0) - 1$$

$$f(1)f(-1) = 0$$

The above equation implies that either $f(1) = 0$ or $f(-1) = 0$.

Case 2a: $f(1) = 0$

In this case, replace x by $x - 1$ in the original equation, and let $y = 1$ to get

$$f(x-1)f(1) = f(x - 1 + 1) + x - 1$$

$$0 = f(x) + x - 1$$

$$f(x) = 1 - x$$

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Case 2b: $f(-1) = 0$

In this case, replace x by $x + 1$ in the original equation, and let $y = -1$ to get

$$f(x+1)f(-1) = f(x+1-1) - x - 1$$

$$0 = f(x) - x - 1$$

$$f(x) = 1 + x$$

So, the only two functions that satisfy the conditions of the puzzle are $f(x) = 1 - x$ and $f(x) = 1 + x$.

. . .

Puzzle: Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x, y > 0$,

$$x^2\big(f(x) + f(y)\big) = (x+y)f\big(yf(x)\big)$$

($\mathbb{R}^+$ is the notation for positive real numbers.)

Source: Slovenian Team Selection Tests 2005 (First Test), https://imomath.com/othercomp/Slo/SloTST05.pdf

Solution: We start by trying $y = x$ in the given equation:

$$x^2\big(f(x) + f(x)\big) = (x+x)f\big(xf(x)\big)$$

$$2x^2f(x) = 2xf\big(xf(x)\big)$$

Since $x > 0$ (and thus $x \neq 0$), we can cancel $2x$ on both sides of the above equation to get

$$xf(x) = f\big(xf(x)\big)$$

Next, we let $x = 1$ in the above equation:

$$f(1) = f\big(f(1)\big)$$

So, $f(1)$ is a fixed point of our yet unknown function. In what follows, we let $a = f(1)$ and so, restating the above equation, $f(a) = a$.

In the original equation, let $x = a$ and $y = 1$ to get

$$a^2\big(f(a) + f(1)\big) = (a+1)f\big(f(a)\big)$$

$$a^2(2a) = (a+1)a$$

$$2a^2 = a + 1$$

$$2a^2 - a - 1 = 0$$

which factors as

$$(2a+1)(a-1) = 0$$

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Since $f(x)$ is positive for all values x, only $a = 1$ is a valid solution to the above equation. Thus, $f(1) = 1$.

Going back to the original equation, let $x = 1$ and leave y free:

$$f(1) + f(y) = (1 + y)f\big(yf(1)\big)$$

$$1 + f(y) = (1 + y)f(y)$$

$$f(y) = \frac{1}{y}$$

As a check, we show that $f(x) = \frac{1}{x}$ yields the same result for each side of the original equation.

Left side:

$$x^2\left(\frac{1}{x} + \frac{1}{y}\right) = x + \frac{x^2}{y}$$

Right side:

$$(x + y)f\big(yf(x)\big) = (x + y) \cdot \left(\frac{1}{yf(x)}\right) = (x + y)\left(\frac{x}{y}\right) = x + \frac{x^2}{y}$$

. . .

Puzzle: Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(n) + 2f\big(f(n)\big) = 3n + 5$$

Note that $\mathbb{N} = \{1,2,3, \dots\}$.

Solution: If we start with $n = 1$, we get

$$f(1) + 2f\big(f(1)\big) = 8$$

The term $2f\big(f(1)\big)$ is even, and so must be $f(1)$ to sum up to 8. Since $f(1)$ and $f(f(1))$ are natural numbers, the only possible choices for $f(1)$ are 2,4 or 6. Let's consider each possibility in turn.

Case 1: If $f(1) = 6$, then $2f\big(f(1)\big) = 2$ (as the two terms must sum to 8). Further, $2f\big(f(1)\big) = 2$ implies $f(6) = 1$. Now, put $n = 6$ into the original equation:

$$f(6) + 2f\big(f(6)\big) = 3(6) + 5 = 23$$

$$1 + 2f(1) = 23$$

$$f(1) = 11$$

which is impossible. So, we can exclude this case.

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Case 2: If $f(1) = 4$, then $2f\big(f(1)\big) = 4$ which implies $f(4) = 2$. Putting $n = 4$ in the original equation:

$$f(4) + 2f\big(f(4)\big) = 3(4) + 5 = 17$$

$$2 + 2f(2) = 17$$

$$f(2) = \frac{15}{2}$$

which contradicts $f(x)$ being a mapping to $\mathbb{N}$. So, we can also exclude this case.

Case 3: If $f(1) = 2$, then $2f\big(f(1)\big) = 6$ which implies $f(2) = 3$. Putting $n = 2$ in the original equation:

$$f(2) + 2f\big(f(2)\big) = 3(2) + 5 = 11$$

$$3 + 2f(3) = 11$$

$$f(3) = 4$$

This case is valid. Thus, we have $f(1) = 2, f(2) = 3$ and $f(3) = 4$.

We can finish the problem via an inductive proof on the hypothesis that $\boldsymbol{f(n) = n + 1}$. We have already shown the result to be true for the cases $n = 1,2,3$. Assume the hypothesis is true for $n = k$, i.e., $f(k) = k + 1$. Let $n = k$ into the original equation to get

$$f(k) + 2f\big(f(k)\big) = 3k + 5$$

$$(k + 1) + 2f(k + 1) = 3k + 5$$

$$f(k + 1) = k + 2$$

Thus, we have shown the induction hypothesis is true for $n = k + 1$.

. . .

Some of the more advanced function determination problems require one to determine whether the function is a surjection or an injection (or both). Such is the case for the following puzzle.

Puzzle: Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(x) + f(y)\big) = [f(x)]^2 + y$$

for all $x, y \in \mathbb{R}$.

Source: Problem #3 from the finals of the 2004 Japanese Mathematics Olympiad.

Solution: We first try $x = 0$ and let y be free, which gives us the equation

$$f\big(f(y)\big) = [f(0)]^2 + y \qquad \text{(Equation 1)}$$

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Next, we show that $f(x)$ is surjective. Recall that surjective means that for every element b in the codomain there is an element a that maps to b, i.e., $f(a) = b$. If we let $a = f(b - [f(0)]^2)$, and let $y = b = [f(0)]^2$ in Equation 1, then

$$f(a) = f(f(b - [f(0)]^2\ ) = \ [f(0)]^2 + (b - [f(0)]^2\ ) = b$$

We have shown that for every element b in the codomain there is an element $a = b - [f(0)]^2$ in the domain that f maps to b. Thus, f is surjective.

Using that fact that f is surjective, we know there exists an element z in the domain of f (i.e., $z \in \mathbb{R}$) such that $f(z) = 0$ (noting that 0 is an element in the codomain).

In the original equation, let $x = z$ and let y be free:

$$f\big(zf(z) + f(y)\big) = [f(z)]^2 + y$$

$$f\big(f(y)\big) = y \qquad\qquad \text{(Equation 2)}$$

Substituting Equation 2 into Equation 1, we get

$$y = [f(0)]^2 + y$$

which implies $f(0) = 0$.

[For the record, we can show that f is injective but this fact is not needed to solve the puzzle. Assume there exists $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Applying the function f to both sides of the equation $f(a) = f(b)$ and using Equation 2, we have

$$a = f\big(f(a)\big) = f\big(f(b)\big) = b$$

$f(a) = f(b)$ implies $a = b$ and thus, f is injective (by the definition of injective).]

Next, let $y = 0$ and let x remain free in the original equation to get

$$f\big(xf(x) + f(0)\big) = [f(x)]^2 + 0$$

and since $f(0) = 0$, we have

$$f\big(xf(x)\big) = [f(x)]^2 \qquad\qquad \text{(Equation 3)}$$

In the original equation, again let $y = 0$ and replace x with $f(x)$

$$f\Big(f(x)f\big(f(x)\big) + 0\Big) = [f(f(x))]^2 + 0$$

Applying Equation 2 to the above equation gives us

$$f\big(xf(x)\big) = x^2 \qquad\qquad \text{(Equation 4)}$$

From Equations 3 and 4, we have that

$$[f(x)]^2 = x^2$$

$$f(x) = \pm x$$

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This means that $f(x) = x$ for all $x \in \mathbb{R}$, or $f(x) = -x$ for all $x \in \mathbb{R}$. It is not possible (under the constraints of the puzzle) for $f(x) = x$ for some values of x and $f(x) = -x$ for other values of x. To show this, assume $f(a) = a$ and $f(b) = -b$ with $a \neq b$ and then set $x = a$ and $y = b$ in the original equation of the puzzle to get

$$f\big(af(a) + f(b)\big) = [f(a)]^2 + b$$

$$f(a^2 - b) = a^2 + b$$

However, $f(a^2 - b)$ is either $a^2 - b$ or $-a^2 + b$, and neither value equals the right hand side of the above equation, and so, we have a contradiction.

Finally, and we leave this as an exercise for the reader, one should check that both $f(x) = x$ and $f(x) = -x$ satisfy the original equation of the puzzle.

. . .

Puzzle: Find all polynomials $f: \mathbb{R} \to \mathbb{R}^+$ that satisfy the equation

$$f\big(f(x)\big) - 2 = x^2(f(x) + 1)$$

Solution: We first try linear polynomials, i.e., $f(x) = ax + b$.

In this case, the left side of the original equation becomes

$$f\big(f(x)\big) - 2 = f(ax + b) - 2 = a(ax + b) + b - 2 = a^2x + (ab + b - 2)$$

The right side of the original equation becomes

$$x^2(f(x) + 1) = x^2(ax + b + 1) = ax^3 + (b + 1)x^2$$

If we equate the coefficients of the x^3, x^2, x and constant terms from both equations, we get

$$a = 0$$

$$b + 1 = 0 \Rightarrow b = -1$$

$$a^2 = 0$$

$$ab + b - 2 = 0$$

Putting $a = 0$ and $b = -1$ into the last equation above, we get $-3 = 0$. Thus, assuming a linear solution to the problem results in a contradiction.

Next, we try a quadratic equation. Let $f(x) = ax^2 + bx + c, a \neq 0$. Compute the right and left hand sides of the original equation, and then compare coefficients.

Right side:

$$x^2(f(x) + 1) = x^2(ax^2 + bx + c + 1) = ax^4 + bx^3 + (c + 1)x^2$$

Left side (after some simplification):

$$f\big(f(x)\big) - 2 = a^3x^4 + 2a^2bx^3 + (2a^2c + ab^2 + ab)x^2 + (2abc + b^2)x + (ac^2 + bc + c - 2)$$

Equating the coefficients of the x^4 coefficients:

$$a^3 = a \Rightarrow a = \pm 1$$

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Equating the coefficients of the x^3 coefficients:

$$b = 2a^2b \Rightarrow b = 2b \Rightarrow b = 0$$

Equating the coefficients of the x^2 coefficients:

$$2a^2c + ab^2 + ab = c + 1$$

and since $b = 0$ and $a \pm 1$, the above becomes $2c = c + 1 \Rightarrow c = 1$.

Equating the coefficients of the x term, we get

$$2abc + b^2 = 0$$

which reduces to $0 = 0$ since $b = 0$. So, we obtain no new information in this case.

By equating the coefficients of the constant terms, we get

$$ac^2 + bc + c - 2 = 0$$

and since $b = 0$ and $a = \pm 1$, we have $\pm c^2 + c - 2$. There are two cases

- $-c^2 + c = 2$ which is impossible if c is a real number
- $c^2 + c = 2$ which implies $c = 1$.

So, when $f(x)$ is a second degree polynomial, we do have a solution, i.e., $f(x) = x^2 + 1$.

For higher order polynomials $f(x) = ax^n + \cdots, a \neq 0, n \geq 3$, the right side of the original equation is of order $n + 2$ and the left hand side is of order n^2. For $n \geq 3$, $n^2 > n + 2$ and so, there are no higher order solutions, and we have found the only polynomial solution to the original equation.

. . .

Puzzle: For each positive real number x, we define $\{x\}$ to be the greater of x and $\frac{1}{x}$, with $\{1\} = 1$.

Find, with proof, all positive real numbers y such that

$$5y\{8y\}\{25y\} = 1$$

Source: Problem #2 from the British Mathematical Olympiad, Round 1, 2 December 2016, https://bmos.ukmt.org.uk/home/bmolot.pdf

Solution: Let's first get an understanding of the function $f(x) = \{x\}$. It is basically a function defined in two parts as follows:

$$f(x) = \begin{cases} \frac{1}{x}, & 0 < x < 1 \\ x, & x \geq 1 \end{cases}$$

The graph of $f(x) = \{x\}$ is depicted in Figure 14.

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For the problem at hand, let $g(y) = 5y\{8y\}\{25y\}$. The function $g(y)$ can be defined over three intervals as follows:

$$g(y) = \begin{cases} 5y\left(\frac{1}{8y}\right)\left(\frac{1}{25y}\right) = \frac{1}{40y}, & 0 < y \le \frac{1}{25} \\ 5y\left(\frac{1}{8y}\right)(25y) = \frac{125y}{8}, & \frac{1}{25} < y \le \frac{1}{8} \\ 5y(8y)(25y) = 1000y^3, & y > \frac{1}{8} \end{cases}$$

For $g(y) = 1$ in the case $0 < y \le \frac{1}{25}$, we must have $\frac{1}{40y} = 1 \Rightarrow y = \frac{1}{40}$. This is a valid solution since $\frac{1}{40}$ falls into the interval $(0, \frac{1}{25}]$ and as the reader can check, it satisfies the original equation.

For $g(y) = 1$ in the case $\frac{1}{25} < y \le \frac{1}{8}$, it must be that $\frac{125y}{8} = 1 \Rightarrow y = \frac{8}{125}$. This is a valid solution since $\frac{8}{125}$ falls into the interval $(\frac{1}{25}, \frac{1}{8}]$ and it satisfies the original equation.

For $g(y) = 1$ when $y > \frac{1}{8}$, we must have $1000y^3 = 1 \Rightarrow y = \frac{1}{10}$ but $\frac{1}{10} < \frac{1}{8}$, and so, $\frac{1}{10}$ is not a valid solution.

Thus, the only valid solutions are $\frac{1}{40}$ and $\frac{8}{125}$.

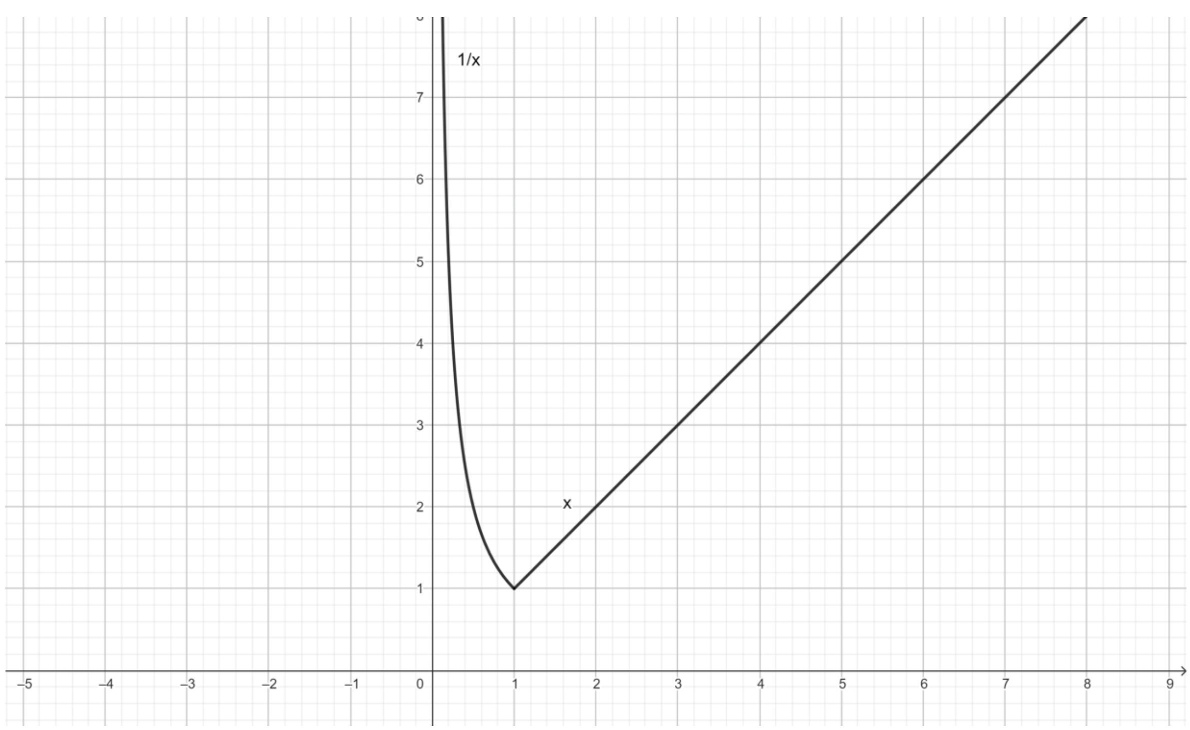

Figure 14. f(x)={x}

. . .

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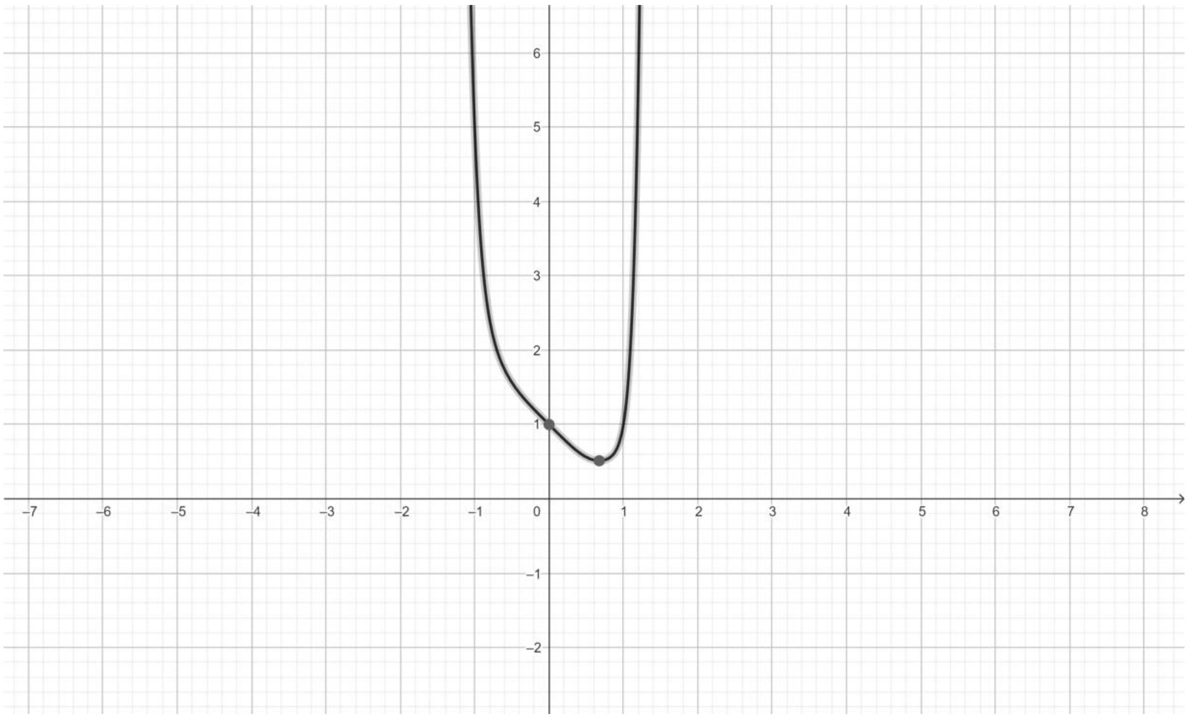
Puzzle: Show that $f(x) = x^{12} - x^9 + x^4 - x + 1 > 0$ for all $x \in \mathbb{R}$.

Solution: While not a proof, a good start is to visualize the problem by graphing the given function (using an online application such as https://www.geogebra.org/), see the following figure.



We can divide the problem into the following three cases:

Case 1: If $x \leq 0$, then each term in $x^{12} - x^9 + x^4 - x + 1$ is positive and so, $f(x) > 0$ in this case.

Case 2: $0 < x < 1$

In this case, we rearrange the function as follows

$$x^{12} - x^9 + x^4 - x + 1 = (1 - x) + x^4(1 - x^5) + x^{12}$$

It is clear that each of the three terms on the right is greater than zero, and so, $f(x) > 0$ in this case.

Case 3: $x \geq 1$

In this case, we rearrange the function as follows

$$x^{12} - x^9 + x^4 - x + 1 = x^9(x^3 - 1) + x(x^3 - 1) + 1$$

Again, it is clear that each of the three terms on the right is greater than zero, and so, $f(x) > 0$ in this case.

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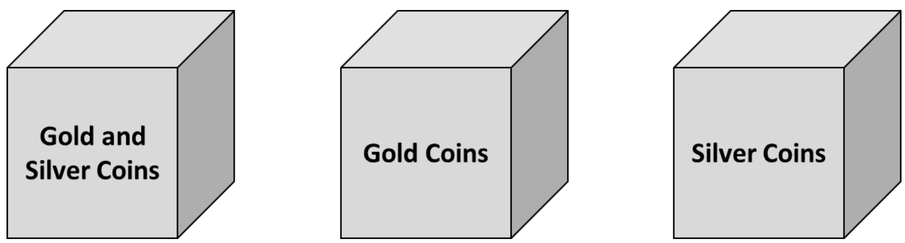
6 General Reasoning Puzzles

Success in solving the problem depends on choosing the right aspect, on attacking the fortress from its accessible side.

George Polya

6.1 Mislabeled Boxes Puzzle

Puzzle: You are presented with 3 boxes, all of which are mislabeled regarding their contents. One box contains gold coins, one box contains silver coins, and the third box contains a mixture of gold and silver coins. You may choose a box and request that one sample be withdrawn and shown to you. The coin is returned to the box, and you may sample another box if you still cannot determine the correct contents for all three boxes. What is the least number of boxes that you need to sample in order to determine the correct contents for each box?



Solution: The solution is to request that one coin be sampled from the box with the labeled “Gold and Silver Coins”.

If the coin is gold, we know the selected box cannot be the one containing only silver coins, and since the box is mislabeled, it also cannot be the box containing gold and silver coins. So, in this case, the selected box must be the one containing only gold coins. The box labeled “Silver Coins” cannot be the box containing only silver coins (since it is mislabeled) and it cannot be the box containing only gold coins. Thus, the box labeled “Silver Coins” must be the box containing gold and silver coins, and by process of elimination, the box labeled “Gold Coins” must be the box containing only silver coins.

If the sampled coin from the box labeled “Gold and Silver Coins” was silver, then using similar logic to the previous case, we can draw the following conclusions:

- box labeled “Gold and Silver Coins” has only silver coins
- box labeled "Gold Coins” has gold and silver coins
- box labeled “Silver Coins” has only gold coins.

6.2 Marble Selection Game

Puzzle: In a two-person game, players take turns selecting marbles from either of two containers. Each container has at least 1 marble. When it is a player’s turn, he or she may select as many marbles as they want from a container. The player who takes the last marble wins. If you can decide to go first or let your opponent go first, what is a strategy that guarantees victory?

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Solution: If the two containers have a different number of marbles, decide to go first and select marbles from the container with the larger number of marbles such that the two containers have the same number of marbles after your selection. After your opponent draws marbles from a container, you again make a selection that leaves the two containers with the same number of marbles.

If the two containers initially have the same number of marbles, let your opponent go first and then follow the above strategy.

6.3 Playing Cards Puzzles

Puzzle: Two people (Don and Coke) play cards for pennies such that the winner of each game gets one penny from the loser. When the two decided to call it quits for the night, Coke had won three games, and Don had a profit of three pennies. How many games did they play?

Solution: Don would need to have won three games against Coke to offset the 3 wins by Coke, and then win 3 more games to finish with a profit of 3 pennies. So, they played a total of 9 games.

. . .

Puzzle: Given n combined decks of playing cards where $n \geq 1$, what is the least number of cards that you need to draw to be guaranteed to get 4 cards of the same rank, e.g., 4 sevens?

Solution: The solution entails the pigeonhole principle that we saw earlier. The first 39 draws could have 3 of each card of card, but the 40th draw forces 4 of the same rank (usually called "four of kind" in poker).

The number of decks of cards is irrelevant to the puzzle.

. . .

Puzzle: Someone hands you a deck of 52 playing cards with 13 of the cards facing up and the rest facing down. You are blindfolded and cannot see (or detect in any other way) whether a card is facing up or down. How can you divide the deck in two piles (not necessarily of the same size) such that each of the two piles of cards has the same number of cards facing up?

Solution: Take the top 13 cards from the deck to form pile #1, with the rest of the cards forming pile #2. Flip pile #1. Now both piles must have the same number of cards facing up. For example, say that pile #1 had 5 cards facing up (before the flip), then pile #2 would have 8 cards facing up. When we flip pile #1, it will also have 8 cards facing up.

6.4 The 100 Prisoners Problem

Sources:

- Book entitled "Analytic Combinatorics" [38] by Philippe Flajolet and Robert Sedgewick
- YouTube video "The Riddle That Seems Impossible Even If You Know The Answer" [42]

The warden of a prison has decided to give 100 prisoners a chance to be set free. Each prisoner is assigned a different number from 1 to 100. In a room, there is a chest with 100 small drawers, numbered from 1 to 100. Within each drawer is a card with the number of one prisoner. The card numbers are randomly distributed among the drawers (one per drawer). For example, card #33 could be put in drawer #59. Each prisoner is to enter the room, and open 50 drawers of their choice. The warden keeps track of whether a prisoner opens a drawer with his or her number. Once

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a prisoner is done, all of the drawers are closed and the next prisoner gives the task a try. If all 100 prisoners find their number, then the entire group of prisoners will be set free; otherwise, they must serve the rest of their sentence. The prisoners may discuss strategy before the process starts, but have no ability to transfer information to the other prisoners once the process starts.

If a prisoner randomly opens drawers, he or she has a 50% chance of finding a drawer with his or her number. For all 100 prisoners to find their number, the probability is virtually zero, i.e., $(.5)^{100} \cong 7.9 \times 10^{-31}$.

Puzzle: Is there a strategy that can improve the prisoners' chance of collective success?

Solution: Each prisoner selects the drawer with their number on the exterior. If there is a match with the prisoner's number, success; otherwise, the prisoner selects a 2nd drawer corresponding to the number on the card in the first drawer. The prison continues in this manner until he or she finds a drawer with his or her number on the card within, or until fifty drawers have been opened with no match. This strategy is decided by the prisoners before the drawer opening process starts.

For example, prisoner #45 would first open drawer #45. Let's say drawer #45 has the card #52. Since this is not the prisoner's number, the prisoner next opens drawer #52. The sequence could eventually lead to the drawer with #45 or just end after 50 drawers have been opened (with none having card #45). Two possible (example) sequences of drawers being opened:

- $45,52,33,75,91,100,1,5,6,8,87$, with drawer #87 having card #45. So, a success for prisoner #45.
- $45,96,35, \dots ,73$, where not one of the 50 drawers opened has card #45. Failure for prisoner #45, and failure for the entire process since the deal is for all prisoners to find their number.

The key here is that the sequence of numbers, when a prisoner starts with the drawer corresponding to his or her number, eventually cycles back to that starting drawer (although it may take more than the allotted 50 tries). This is always true. To see this, let's look at an example with only 7 prisoners. Table 7 shows the number of each card within each drawer. In the example, an attempt is made not to cycle back to the original drawer but as can be seen this is impossible. For example, if prisoner #1 picks drawer #1, he or she goes next to drawer #2, followed by drawers #3, #4, #5, #6, #7 and finally back to #1. Of course, the cycle could be short, but the point here is that if a prisoner starts with the drawer corresponding to their number, they must eventually return to that drawer (using the agreed strategy).

Table 7. Example cycle with 7 drawers and 7 prisoners

Drawer number	1	2	3	4	5	6	7
Prisoner number	2	3	4	5	6	7	1

Using the given strategy effectively imposes a set of cycles that cover all 100 of the drawers. If none of the cycles are greater than 50 in length, then no prisoner will need to open more than 50 drawers to be guaranteed to find a drawer with their number on the card within the drawer! So, the problem reduces to determining the probability of a cycle with length 51 or greater.

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The number of possible arrangements of drawers and associated cards that generate a cycle of length k is as follows. The formula works for any value of k from 1 to 100 but we are only concerned with cycles greater than or equal to 51.

Number of ways of selecting k numbers from 100	$\times$	Number of ways of creating a cycle from the k numbers	$\times$	Number of ways of arranging the $100 - k$ numbers not selected

$$\binom{100}{k}(k-1)!\,(100-k)!$$

$$= \frac{100!}{k!\,(100-k)!}(k-1)!\,(100-k)!$$

$$= \frac{100!}{k}$$

To get the probability of a cycle of length k occurring, we need to divide the above by the total number of assignments of card numbers to drawers (which is $100!$). So, the probability of a cycle of length k occurring is simply $\frac{1}{k}$.

So, the probability of a cycle of length 51, 52,…, or 100 is

$$\frac{1}{51}+\frac{1}{52}+\cdots+\frac{1}{100}$$

(Note these events are mutually exclusive, i.e., there can be at most one cycle of length greater than 50.)

Now, for all the prisoners to find their number and thus, all to be released, we want the probability of no cycles of length greater than 50, i.e.,

$$1-\left(\frac{1}{51}+\frac{1}{52}+\cdots+\frac{1}{100}\right)\cong 0.31183$$

So, with this strategy, the prisoners have over a 31% chance of all being released!

If we increase the number of prisoners to 1000 and allow for the opening of 500 boxes, the cycle strategy has the following probability of success

$$1-\left(\frac{1}{501}+\frac{1}{502}+\cdots+\frac{1}{1000}\right)\cong .305373$$

In the limit (as the number of prisoners goes to infinity), the probability of success is $1-\ln 2 \cong .306853$.

6.5 Rabbit in Hats Puzzle

Puzzle: A rabbit is placed (hidden) in one of 7 hats that are lined-up in a row. At the end of each day, the rabbit can move either one hat to the left or right, except when the rabbit is in hat #1 or #7. When the rabbit is in hat #1, it can only move to hat #2 on the next day. When the rabbit is in hat #7, it can only move to hat #6 on the next day. In any event, the rabbit must make one move per day based on the rules noted above. You are allowed to make one guess each day as to the location of the rabbit. Is there any way that you can be sure to find the rabbit?

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Source: The ultimate source of this puzzle is not known. However, variations of this puzzle are known to be used in the interview process of some high-tech companies.

Solution: The situation is depicted in Figure 15.

Figure 15. Rabbit in hat puzzle

Surprisingly, it is possible to eventually determine the location of the rabbit.

First, consider the case where the rabbit is in an even numbered hat. Proceed as follows:

- On Day 1, guess that the rabbit is under hat #2. If correct, you win. If not (and given our assumption), the rabbit is under hat #4 or #6.
- At the end of Day 1 and after the rabbit makes its move, the rabbit must be under hat #3, #5, or #7.
- On Day 2, guess that the rabbit is under hat #3. If correct, you win. If not, the rabbit is under hat #5 or #7.
- At the end of Day 2 and after the rabbit makes its move, the rabbit must be under hat #4, or #6.
- On Day 3, guess that the rabbit is under hat #4. If correct, you win. If not, the rabbit is under hat #6.
- At the end of Day 3 and after the rabbit makes its move, the rabbit must be under hat #5 or #7.
- On Day 4, guess that the rabbit is under hat #5. If correct, you win. If not, the rabbit is under hat #7.
- At the end of Day 4 and after the rabbit makes its move, the rabbit must be under hat #6.
- On Day 5, guess that the rabbit is under hat #6. If correct, you win. If not, the only possibility is that your initial assumption was incorrect.

So, if the rabbit started under an even numbered hat, you will catch the rabbit by Day 5. If the rabbit started under an odd numbered hat and given that the rabbit must alternate between even and odd numbered hats, we know that the rabbit will be under an even number hat on Day 6, but this reduces to the even start problem. This leads us to a general solution, i.e., using the guess sequence 2,3,4,5,6; 2,3,4,5,6 guarantees that you will catch the rabbit. You will catch the rabbit by Day 5 if the rabbit started in an even numbered hat, and if not, you know that the rabbit will be in an even numbered hat on Day 6, thus allowing for the even numbered hat procedure to be applied.

Further, the problem is symmetric. We could do 6,5,4,3,2; 6,5,4,3,2, or 6,5,4,3,2; 2,3,4,5,6, or 2,3,4,5,6; 6,5,4,3,2.

Puzzle: Same puzzle as before but this time you have only 6 hats.

Puzzle: Same puzzle as before but with n hats.

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6.6 1000 Lights Puzzle

Puzzle: There are 1000 lights in an array. All the lights are initially switched-off. There are a series of 1000 rounds where some of the lights are toggled (either from On to Off, or Off to On). In round n, every n lights are toggled. Some examples,

- In round #1, all the lights are toggle On.
- In round #2, all the even numbered lights are toggled Off.
- In round #20, all lights numbers a multiple of 20 are toggled.

What numbered lights are On at the end of the 1000 rounds?

Source: Multiple versions of this puzzle can be found on the Internet. Usually, the puzzle involves 100 doors which are either opened or shut. The ultimate source of the puzzle is not known.

Solution: Let's first look at a particular light (say #36) and see if we can solve the puzzle for that light, and then extend the solution to the entire set of lights.

The state of light #36 is as follows:

- In state On after first round
- Turned Off after second round, since 36 is a multiple of 2
- Turned On after round 3, since 36 is a multiple of 3
- Turned Off after round 3, since 36 is a multiple of 4
- No change in round 5 since 36 is not a multiple 5
- Turned On after round 6, since 36 is a multiple 6
- No change in rounds 7 or 8 since 36 is not a multiple of either 7 or 8
- Turned Off after round 9
- Turned On after round 12
- Turned Off after round 18
- Turned On after round 36
- Remains On for the rest of the rounds

So, light #36 is Off at the end of the rounds. The general idea is that we count the number of divisors for a given light position and conclude the given light will be Off at the end of the rounds if the number of divisors is odd, and On if the number of divisors is even. For 36, we have the following divisors: 1,2,3,4,6,9,12,18,36. So, we have 9 divisors in this case, and light #36 is Off at the end of the rounds.

The problem reduces to whether the number of divisors of a given number has an even or odd number of divisors. However, only perfect squares (numbers of the form x^2 such as $36 = 6 \cdot 6$) have an odd number of factors. All integers which are not perfect squares have an even number of divisors. To see this, note that if $j < \frac{n}{2}$ is a divisor integer n, then so is $\frac{n}{j}$ since $j \cdot \frac{n}{j} = n.$ So, divisors come in pairs. The exception is with perfect squares (i.e., $n = x^2$), since x is paired with itself which leaves us with an odd number of divisors.

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Back to the question asked in the puzzle, the lights that remain On at the end of the rounds are the perfect squares less than 1000, i.e.,

$$1,4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 186, 225, 156, 289, 324,$$
$$361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961$$

6.7 Using Links of a Gold Chain for Payment

Puzzle: Zelda has a gold chain consisting of 23 links that she wants to use as collateral for a loan from a pawnbroker. The loan will be given to Zelda in 23 installments over the course of 23 days. Zelda will provide the associated collateral of one gold link from the chain per day.

What is the least number of cuts Zelda needs to make to her chain?

Source: A version of this puzzle appears in the book The Stargazer Talks [54] where the following quote is provided concerning the origin of the puzzle:

> I believe, though, that the “gold chain puzzle” which I mentioned made its first appearance in John O’London’s Weekly for the 16^{th} March 1935.

Hint: The pawnbroker is willing to “make change” with Zelda in terms of gold links from the chain. For example, for a given day’s payment, Zelda could give the pawnbroker 2 connected links and receive one link back in exchange. To be clear, the broker may return change, as long as he can return exact change using previous payments, without cutting any link.

Solution: Let’s first look at the problem with only 7 links. Zelda could make just one cut (ring #3) and be able to pay the pawnbroker one link per day, as shown in Figure 16.

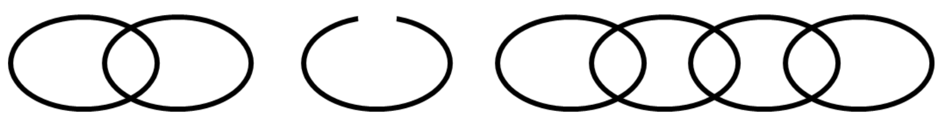

Figure 16. Gold link puzzle with 7 links and only one cut link at #3

In this case, the transactions proceed as follows:

- On Day 1, Zelda gives the single link to the pawnbroker.
- On Day 2, Zelda gives the 2-link segment to the pawnbroker and gets back the single link.
- On Day 3, Zelda gives the single link to the pawnbroker.
- On Day 4, Zelda gives the 4-link segment to the pawnbroker, and gets back the single link and the 2-link segment in return.
- At this point, Zelda and the pawnbroker repeat the process from Days 1-3.

Let’s study the case of a chain with 23 links. With one cut, the best we can do is make payments for 3 days. For example, if we cut the 3^{rd} link (or 21^{st} link), we have chains of size 1, 2 and 20, which only allows for 3 days of payments (without further cuts in the chain). If we cut a link between the 4^{th} and 20^{th} position, the situation is even worse since we would only be able to pay for one day. So, in the case of 23 links, we need at least two cuts.

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If we make two cuts (resulting in two single link chains and three other chains), we can pay for all 23 days as follows:

- We pay for the first two days with the single links.
- After that, we need a link of length 3 for the third day (getting the two single links in return). So, the first cut should be at the 4th link. Using the two single links and the 3-link, we can pay up to and including the 5th day.
- After that, we need a link of length 6 for the sixth day (getting the two single links and the 3-link in return). So, the second cut should be at the 11th link. Using the two single links, the 3-link and 6-link, we can pay up to the 11th day.
- The remaining link is of length 12 and so, we can use that to pay for the 12th day (getting the two single links, the 3-link and 6-link in return which allows us to pay for the remaining 11 days). Success!

Going beyond what is asked in the puzzle, we consider the general case of determining the maximum number of days that can be covered with n cuts and a chain of minimum length. For n cuts, the chain would obviously need to be a least of length n.

- If the chain is of length $\boldsymbol{n}$ and we make n cuts, then we can cover payments for n days.
- To go beyond this, we need a segment of length $n+1$ that remains after making n cuts to a chain of total length $n+(n+1)=\mathbf{2}\boldsymbol{n}+\mathbf{1}$. We can pay for the first n days using the n single links. On Day number $n+1$, we can exchange the segment of length $n+1$ for the n single links. With the n single links and the link of length $n+1$, we can arrange to pay for $n+(n+1)=2n+1$ days.
- To go beyond this, we need segments of length $2n+2$ and $n+1$ to remain after making n cuts to a chain of total length $n+(n+1)+(2n+2)=\mathbf{4}\boldsymbol{n}+\mathbf{3}$. Using the procedure from the previous step, we can pay for $2n+1$ days. On Day number $2n+2$, we can exchange the segment of length $2n+2$ for the n single links and the link of length $n+1$. With the n single links and the links of length $n+1$ and $2n+2$, we can arrange to pay for $n+(n+1)+(2n+2)=4n+3$ days.
- To go beyond this, we need segments of length $4n+4, 2n+2$ and $n+1$ to remain after making n cuts to a chain of total length $n+(n+1)+(2n+2)+(4n+4)=\mathbf{8}\boldsymbol{n}+\mathbf{7}$. Using the procedure from the previous step, we can pay for $4n+3$ days. On Day number $4n+4$, we can exchange the segment of length $4n+4$ for the n single links and the links of length $n+1$ and $2n+2$. With the n single links and the links of length $n+1, 2n+2$ and $4n+4$ we can arrange to pay for $n+(n+1)+(2n+2)+(4n+4)=8n+7$ days.

The pattern is further elaborated in Table 8.

- The pattern of the cuts is shown in the left-hand column. Square brackets are used to represent a segment of a given size. For example, $[n+1],[1],[2n+2],(n-1)[1]$ should be interpreted as a segment of length $n+1$, followed by a single link, followed by a segment of length $2n+2$, and then followed by $n-1$ single links. The arrangements are not unique.
- The middle column shows the additional segment needed to facilitate the next jump in the number of days that can be covered.

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- In the right-hand column, note that the length of the chain is also equal to the maximum number of days in which payment can be covered.

Table 8. Gold link puzzle with n cuts

Cut pattern	**Additional segment**	**Length of chain = Max days**
$n[1]$	n single links	n
$[n+1], n[1]$	$n+1$ link	$2n+1$
$[n+1], [1], [2n+2], (n-1)[1]$	$2n+2$ link	$4n+3$
$[n+1], [1], [2n+2],$ $[1], [4n+4], (n-2)[1]$	$4n+4$ link	$8n+7$
$[n+1], [1], [2n+2],$ $[1], [4n+4], [1], [8n+8],$ $(n-3)[1]$	$8n+8$ link	$16n+15$
...	...	...
$[n+1], [1], \dots, [2^{n-2}(n+1)], 2[1]$	$2^{n-2}(n+1)$	$2^{n-1}n + (2^{n-1}-1)$
$[n+1], [1], \dots, [2^{n-2}(n+1)], [1],$ $[2^{n-1}(n+1)], [1], [2^n(n+1)]$	$2^{n-1}(n+1)$ $2^n(n+1)$	S

The total number of days that can be covered with n cuts is

$$S = n + (n+1) + 2(n+2) + 2^2(n+1) + 2^3(n+1) + \cdots + 2^{n-1}(n+1) + 2^n(n+1)$$

$$= n + (n+1)(1 + 2 + 2^2 + 2^3 + \cdots + 2^{n+1})$$

Using the formula for the sum of a geometric series, we have

$$S = n + (n+1)\left(\frac{2^{n+1}-1}{2-1}\right) = n + (n+1)(2^{n+1}-1)$$

If we make one cut, the above formula tells us that a maximum of $1 + (2)(4-1) = 7$ days can be covered with a chain of length 7, which agrees with our previous analysis. Table 9 shows the maximum number of days that can be covered corresponding to the number of cuts. If the gold chain has a number of links different from the various maximums, just choose the cuts associated with next higher chain length. For example, if the gold chain has 13 links, then we use 2 cuts to cover 13 days. The cut pattern is [3], [1], [6], [1], [2].

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Table 9. Maximum number of days covered with n cuts

Cuts	**Maximum number of days / Chain length**
1	7
2	23
3	63
4	159
5	383
6	895
7	2047
…	…

The above sequence is known as the Woodall (or Riesel) numbers [55].

6.8 Circuit Breaker Puzzle

Puzzle: The electrician for a large factory has the task of sorting out the relationship between the switches on a circuit breaker box and the various electrical outlets (e.g., lights and electrical receptacles). She knows that the relationship between switches on the circuit-breaker box and the electrical outlets is one-to-one. In addition, there are 64 outlets. The electrician has several people from the factory maintenance staff distributed about the factory who can check whether an outlet is turned On or Off by its associated circuit breaker. Each time the electrician changes the status of one or more switches (which we will refer to as a "round"), she will inform her assistants (via the factory public address system) to check the outlets under their watch. **This is the only communication used in the process.** What is the least number of rounds in which the relationship between the circuit breaker switches and outlets can be determined?

Source: This puzzle is a variation of the "100 Switches and Lights Puzzle" [56].

Solution: In the brute force approach, one switch is turned On per round (with the other switches turned Off). This approach would take 63 rounds. (By process of elimination, the 64th round is not needed.)

The following approach can be accomplished in just 6 rounds. In the first round, half of the switches are turned On, and the other half turned Off. The circuit breaker box is depicted in Table 10, where 1 represents a switch being turned On, and 0 represents a switch that is turned Off. Once the switches are set, the electrician requests (via the public address system) that the staff members record whether each of the outlets they are monitoring is turned On or remain Off. This info (0 or 1) is recorded on a sticky note (or similar) next to the outlet. When the process is complete, the sticky notes can be converted to a more permanent solution, e.g., a small plaque mounted next to the electrical outlet. In addition, the circuit breaker / outlet numbers could be recorded on the electrical plan for the building. So, when the process is complete, the binary sequence on the post-it next to the outlet identifies the circuit breaker that controls it. Also, assume there is some numbering scheme used on the circuit breakers (e.g., the circuit breakers are numbered 1 to 100). So, it would be necessary to have a mapping table between the circuit breaker numbers and the generated identifier for each of the outlets.

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It should be noted that the electrician does not need to know the assignment of maintenance staff to electrical outlets.

Table 10. Round 1 of circuit breaker determination

1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	0

For round 2, the electrician divides the switches a second time (as depicted in Table 11). On the left-side of the table, we have two subsections. The switches are turned On in the top subsection and Off in the bottom subsection. A similar approach is used on the right-side of the table. The second digit in the number associated with each switch indicates whether the switch is turned On or Off in the second round. Once the switches are set for the second round, the electrician requests that the staff members check whether the outlets that they monitor are On or Off, and record the result (using the same notation convention that is being used for the circuit breaker).

Table 11. Round 2 of circuit breaker determination

11	11	11	11	01	01	01	01
11	11	11	11	01	01	01	01
11	11	11	11	01	01	01	01
11	11	11	11	01	01	01	01
10	10	10	10	00	00	00	00
10	10	10	10	00	00	00	00
10	10	10	10	00	00	00	00
10	10	10	10	00	00	00	00

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We still do not have unique numbers for the switches and associated outlets, but we are making progress. In round 3, the 4 subsets of switches are divided in half, with each half being turned On or Off (as shown in Table 12).

Table 12. Round 3 of circuit breaker determination

111	111	110	110	011	011	010	010
111	111	110	110	011	011	010	010
111	111	110	110	011	011	010	010
111	111	110	110	011	011	010	010
101	101	100	100	001	001	000	000
101	101	100	100	001	001	000	000
101	101	100	100	001	001	000	000
101	101	100	100	001	001	000	000

In round 4, each subset of switches is divided in half, with each half being turned On or Off (as shown in Table 13). The selection of 64 may seem advantageous but as we shall see in a subsequent puzzle, this process works even when the number of switches is not a power of 2.

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Table 13. Round 4 of circuit breaker determination

1111	1111	1101	1101	0111	0111	0101	0101
1111	1111	1101	1101	0111	0111	0101	0101
1110	1110	1100	1100	0110	0110	0100	0100
1110	1110	1100	1100	0110	0110	0100	0100
1011	1011	1001	1001	0011	0011	0001	0001
1011	1011	1001	1001	0011	0011	0001	0001
1010	1010	1000	1000	0010	0010	0000	0000
1010	1010	1000	1000	0010	0010	0000	0000

The process continues in round 5 (as shown in Table 14) but we need one more round to uniquely determine the relationship between the circuit breaker switches and the outlets.

Table 14. Round 5 of circuit breaker determination

11111	11110	11011	11010	01111	01110	01011	01010
11111	11110	11011	11010	01111	01110	01011	01010
11101	11100	11001	11000	01101	01100	01001	01000
11101	11100	11001	11000	01101	01100	01001	01000
10111	10110	10011	10010	00111	00110	00011	00010
10111	10110	10011	10010	00111	00110	00011	00010
10101	10100	10001	10000	00101	00100	00001	00000
10101	10100	10001	10000	00101	00100	00001	00000

Finally, once round 6 is complete, we have a unique number for each switch in the circuit breaker and this number is associated with a unique outlet. For example, 011010 (shown in bold in Table 15) controls outlet 011010.

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Table 15. Round 6 of circuit breaker determination

111111	111101	110111	110101	011111	011101	010111	010101
111110	111100	110110	110100	011110	011100	010110	010100
111011	111001	110011	110001	011011	011001	010011	010001
111010	111000	110010	110000	**011010**	011000	010010	010000
101111	101101	100111	100101	001111	001101	000111	000101
101110	101100	100110	100100	001110	001100	000110	000100
101011	101001	100011	100001	001011	001001	000011	000001
101010	101000	100010	100000	001010	001000	000010	000000

. . .

Puzzle: What if the number of switches (and associated outlets) is not a power of 2, e.g., 52?

Solution: The same process works but the divisions are not always in half. Since 52 is between $2^5 = 32$ and $2^6 = 64$, we still need 6 rounds. The divisions in terms of number of electrical outlets is as follows:

- We start with dividing 52 into two groups of 26 which we represent with the notation (26,26)
- In the second round each of the groups of 26 are divided into two groups of 13, which we represent with the notation $(13,13),(13,13)$.
- For round three, we have the groupings $\big((7,6),(7,6)\big),\big((7,6),(7,6)\big)$.
- For round four, we have the groupings $\big((4,3,3,3),(4,3,3,3)\big),\big((4,3,3,3),(4,3,3,3)\big)$
- Four round five, we have the groupings $\big((2,2,2,1,2,1,2,1),(2,2,2,1,2,1,2,1),(2,2,2,1,2,1,2,1),(2,2,2,1,2,1,2,1)\big)$
- In round six, each of the electrical outlets are in a separate group.

. . .

Puzzle: What if each switch on the circuit breaker box controls several outlets?

Solution: The same process as above works. All the outlets associated with a given circuit breaker will be assigned the same number which is exactly what we want. Further, even if the outlets associated with a given circuit breaker are being observed by different staff members, the process still works as there is no need for the staff members to communicate among themselves.

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6.9 Tossing Coin onto a Board

Puzzle: A coin is randomly tossed onto a board as shown in Figure 17. The board consists of 16 squares of dimension 1 cm by 1 cm. The coin is a circle with radius $\frac{1}{6}$ cm. What is the probability that the coin will land completely within one of the squares (without touching any edge)?

Also, note that the toss does not count unless the center of the coin lands within the boundary of the board.

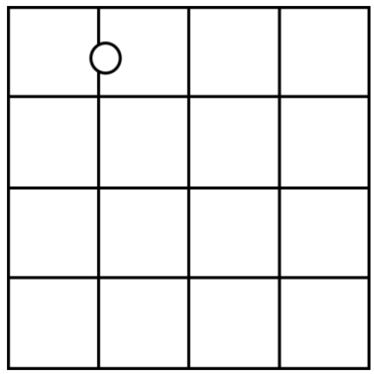
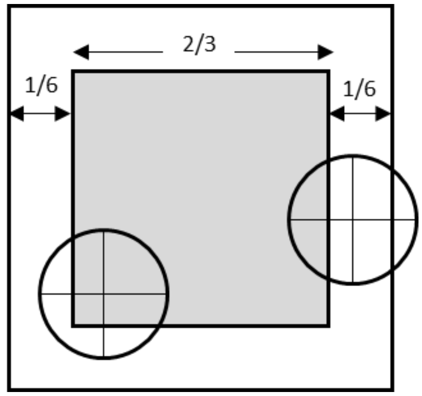


Figure 17. Coin randomly tossed onto board

Solution: The key to this puzzle is to realize that the location of the center of the circle determines whether or not the coin falls completely within one of the squares. Since the coin is tossed randomly, we can assume that all locations for the landing point of the center of the coin are equally probable. To solve the puzzle, we only need to consider one of the squares, as shown in the figure below. If the center of the coin lands within the gray region, then the circle will be entirely within the particular square. The gray region has area $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$ and the area of the surrounding square is 1.



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In the context of the entire 16×16 board, Figure 18 shows the areas (in gray) where the coin's center can land without the coin touching any of the edges of the board. The area of the entire board is $4 \times 4 = 16$ and the area of the gray areas is $16 \cdot \frac{4}{9} = \frac{64}{9}$. The probability of the coin's center landing on a gray area (and thus not touching an edge) is

$$\frac{\left(\frac{64}{9}\right)}{16} = \frac{4}{9}$$

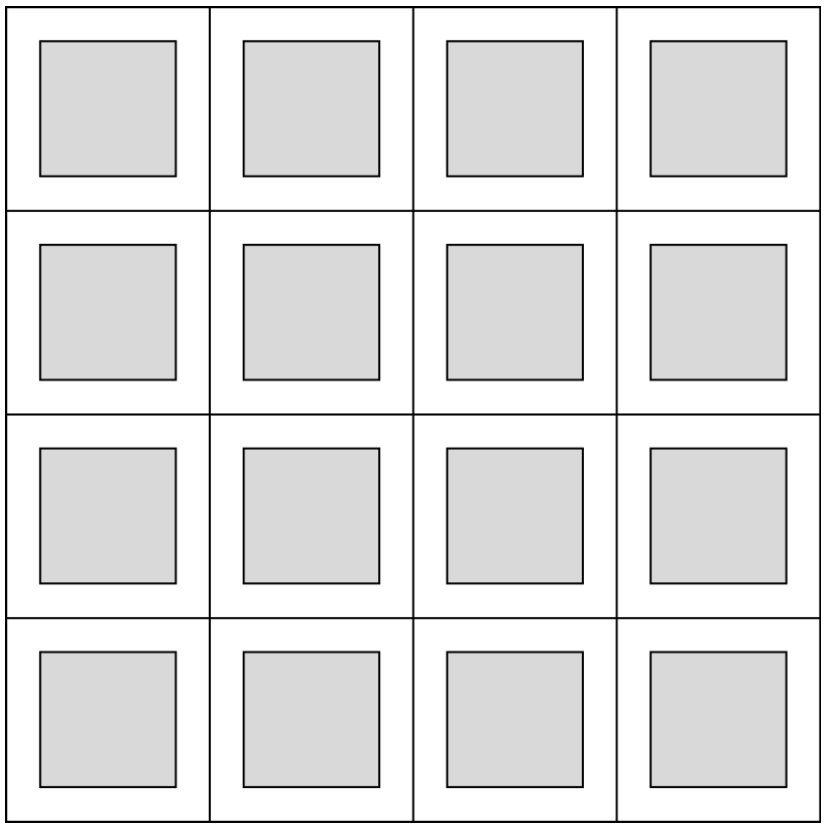

Figure 18. Coin toss onto board – view of entire board

. . .

Puzzle: Compute the probability that the coin will land completely within a square if we modify the definition of a valid coin toss to include tosses where any part of the coin touches the board.

Hint: The diagram on the left of the figure below shows a coin at the limit of where it can be considered as a valid toss, based on the definition in the previous puzzle. The diagram on the right shows a coin at the limit of where it can be considered as a valid toss based on the definition in this puzzle. We are effectively adding a band of thickness $\frac{1}{6}$ cm around the entire board when using the modified definition of a valid coin toss.

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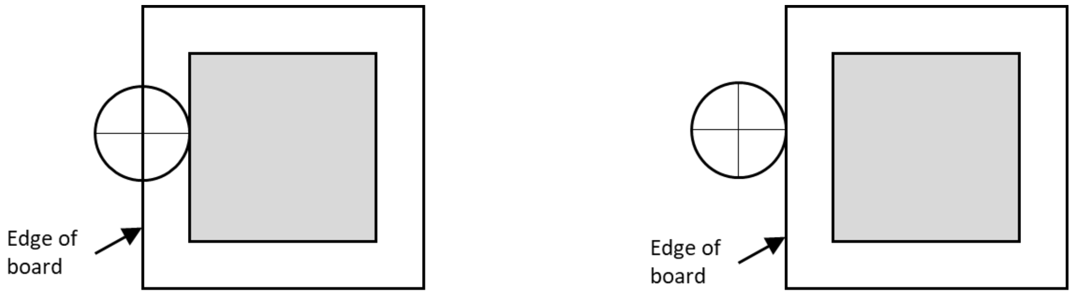

6.10 Counterfeit Coin Puzzles

There is a rather large collection of puzzles that involve the determination of an object of a different weight from a set of other objects all of which are of the same weight. The objects all look the same and one can only determine the counterfeit (different weight) object by use of a 2-arm balancing scale (Figure 19). The problems typically only allow a maximum number of weighings, or ask one to demonstrate the minimum number of weighings required to determine the counterfeit object.

Figure 19. Balancing scale

In the article “Counterfeit Coin” [60], the author states that such puzzles originate from the following problem that appeared in the American Mathematical Monthly (January 1945), contributed by E. D. Schell:

> You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you detect whether there is an underweight coin, and if so, which one, using the balance only twice?

Sources:

- “Seeking Mathematical Truth in Counterfeit Coin Puzzles” in Quanta Magazine [57]
- “Balance puzzle” in Wikipedia [58]
- “Parallel Weighings” by Tanya Khovanova [59]
- “Counterfeit Coin” Problem by Bennet Manvel [60].

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Puzzle: You are given a collection of nine coins which look identical. However, eight of the coins have the same weight, and one is lighter. Find the different coin in the least number of weighings using only a 2-arm balancing scale.

Solution: One weighing is not going to work. If we do an even / odd combination on the scale (e.g., 5 on one side and 4 on the other), the larger number of coins will bring its side of the scale lower, but gives no definitive information which coin is the lighter coin. If we use an same number of coins on both sides of the scale (e.g., 4 and 4, with one coin not on the scale), we could get lucky if the one coin not on the scale is the one with the different weight but the puzzle does not involve luck – we want a guaranteed method to find the lighter coin.

So, we next try with 2 weighings. Divide the coins into 3 groups of 3.

- Weight two of the groups. If the scale balances, the lighter coin is in the other group of three. If the scale does not balance, the group of coins that lifts higher on the scale has the lighter coin. In either event, we have isolated the lighter coin to one of the groups of three.
- From the group with the lighter coin, select any two coins and put them on the scale. If the scale balances, the other coin is the lighter coin. If the scale does not balance, the coin that lifts higher on the scale is the lighter coin.

. . .

Puzzle: Same puzzle as the previous, except that you have n coins with one lighter coin.

Solution: First, we consider a specific case with 27 coins, where one of the coins is lighter than the others. We can reduce this to the problem that we already solved by dividing the 27 coins into 3 groups of 9. Pick two of the groups of 9, and compare them on the scale. If they balance, the lighter coin is in the other group of 9. If they don't balance, the lighter coin is in the group that lifts higher on the scale. Either way, we know what group of 9 contains the lighter coin, and this reduces to the 9 coin problem which requires two more weighings to determine the lighter coins. The reader is invited to verify that we can handle between 10 and 26 coins using an approach similar that for 27 coins.

In general, if we are given n coins (identical in appearance) with one coin being lighter and all the other coins be of the same weight, we first find the least power of 3 greater than is greater than or equal to n (say it is 3^m). In the case, it takes m weighings to be sure of determining the lighter coin. We divide the 3^m coins into three groups of 3^{m-1} (or less) coins, pick two of the groups of size 3^{m-1} (or less) and put them on either side of the balancing scale. If they balance, the lighter coin is in the other group of size 3^{m-1} (or less). If they don't balance, the lighter coin is in the group that is lifted higher on the scale. Either way, now know in which group of size 3^{m-1} the lighter coin resides. Thus, the problem is reduced to the 3^{m-1} coin problem. We continue in this manner until we reduce the problem to the 27 and then 9 coin problem, using one weighing at each step.

It may help to see the groupings if we start with a number of coins that is not a power of 3, e.g., let's say we start with 59 coins. The next higher power of 3 is $3^4 = 81$. So, we divide the 59 coins into three groups of sizes 20, 20 and 19. For the sake of agreement, assume the lighter coin is determined to be in the group of 19 after the first weighing. We then divide the group of 19 into groups of 7, 6 and 6. Let's say the coin is isolated to the group of 7. From here we need just two more weighings (for a total of 4 weighings) to determine the lighter coin.

. . .

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Puzzle: Given 12 coins that are identical in appearance, and all of the same weight w, except for one coin which is either lighter or heavier than the others, find the different coin in the fewest number of weighings and determine whether it is lighter or heavier than the other coins.

Solution 1:

Divide the coins into three groups of 4 coins each.

Compare groups 1 and 2 by weighing on the scale.

- Case 1: If groups 1 and 2 balance, then the different coin is in group 3. Further, we know all the coins in groups 1 and 2 are of weight w. Take 3 of the coins from group 3 and compare them to 3 coins of normal weight w.
    - Case 1a: If the coins balance, this means the 4th coin in group 3 is the different coin. To determine if this coins weights less or more than w, compare it to one of the other normal weight coins.
    - Case 1b: The three coins from group 3 tip the balance down, then we know that the different coin is heavier and it is one of the coins selected from group 3. Take two of the coins from group 3 (out of the selected 3) and compare them. If they balance, the 3rd coin is the heavier coin. If they don't balance, the coin that tips the scale down is the heavier coin.
    - Case 1c: The three coins from group 3 tip the balance up. The analysis here is similar to Case 1b and left as an exercise to the reader.
- Case 2: Groups 1 and 2 do not balance. Without loss of generality, assume group 1 tips the scale downward. In this case, we know all the coins in group 1 are greater than or equal to w (call them type h), all the coins in group 2 are less than or equal to w (call them type l) and all the coins in group 3 are of normal weight w. Select all 4 of the type h coins, one of the type l coins and one coin of normal weight w. On the scale, compare 2 type h coins plus normal weight w coin, against 2 type h coins plus an type l coin (as shown in the abstract schematic below).

h	h	w		h	h	l

  - Case 2a: If this weighing balances, then one of the other three type l coins is different (and it is lighter). Compare two of the three coins in questions. If they balance, then the third coin is the lighter coin. If they don't balance, the coin that lifts higher on the scale is the lighter coin.
  - Case 2b: The right-side goes down. In this case, the type l coin on the right could not possibly be the lighter coin and so must be of normal weight. So, one of two type h coins on the right must be the heavier coin. We compare these two to determine the heavy coin.
  - Case 2c: The right-side goes up. In this case, either the type l coin on the right is actually lighter, or one of the type h coins on the left is actually heavier. Weight the two type h coins on the left.
    - If they balance, then the type l is actually lighter.
    - If they don't balance, the heavier coin has been identified.

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Solution 2:

In a very different approach, we label the coins with numbers from 1 to 12, and do the following weighings:

First weighing shown below, with 9,10,11 and 12 not on the scale

1	2	6	8		3	4	5	7

Second weighing shown below, with 3,4,5 and 12 not on the scale

1	2	7	9		6	8	10	11

Third weighing shown below, with 1,4,7 and 10 not on the scale

5	6	9	11		2	3	8	12

The distribution of the numbers in the weighings have been carefully chosen so that the set of outcomes from the three weighings implies a particular coin is different, and whether that coin is lighter or heavier than the other coins. For example, if the scale dips left on the 1st and 2nd weighings, and balances on the third, then the odd coin must be 1 and it must be heavier (see the first non-header row in Table 16). For each combination of results for the three weighings, the following table shows which coin is different and whether it is heavier or lighter than the other coins. L stands for “scale dips left”, R stands for “scale dips rights”, and B for “balanced”. The combinations BBB, LLL and RRR are not possible if there is a different coin.

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Table 16. Alternate solution to 12-coin puzzle

Weighing 1	Weighing 2	Weighing 3	Different Coin	Heavier (Hv) or Lighter (Lt)
L	L	B	1	Hv
L	L	R	2	Hv
L	B	L	3	Lt
L	B	B	4	Lt
L	B	R	5	Lt
L	R	L	6	Hv
L	R	B	7	Lt
L	R	R	8	Hv
B	L	L	9	Hv
B	L	B	10	Lt
B	L	R	11	Lt
B	B	L	12	Lt
B	B	R	12	Hv
B	R	L	11	Hv
B	R	B	10	Hv
B	R	R	9	Lt
R	L	L	8	Lt
R	L	B	7	Hv
R	L	R	6	Lt
R	B	L	5	Hv
R	B	B	4	Hv
R	B	R	3	Hv
R	R	L	2	Lt
R	R	B	1	Lt

. . .

Puzzle: What if you have between 13 coins in the previous puzzle?

Solution: In the case of 13 coins, we put one coin aside and execute the previously mentioned procedure for 12 coins. If the procedure on the 12 coins results in finding a different coin, we will also know whether it is lighter or heavier than the other coins. If the procedure on the 12 coins shows that all the coins are of the same weight, then it must be that the 13th coin is different but we will need a 4th weighing to determine if the 13th coin is lighter or heavier than the others.

. . .

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Puzzle: Given 36 coins that are identical in appearance, and all of the same weight w, except for one coin which is either lighter or heavier than the others, find the different coin in the fewest number of weighings and determine whether it is lighter or heavier than the other coins.

Solution: Divide the coins into three groups of 12. Compare two of the groups.

If the two groups balance, the third group of 12 contains the different coin, and this reduces to the 12 coin puzzle which we solved previously.

If the two groups do not balance, then we have still gained some information. The 12 coins on the side of the balance that dips down are of type h (i.e., cannot be lighter than w) and the other 12 coins must be of type l (i.e., cannot be heavier than w). Further, all 12 coins in the third group must be normal, i.e., of weight w.

Divide the remaining 24 coins into 3 groups (call them A, B and C) such that each group has 4 coins of type h and 4 coins of type l. Compare groups A and B on the scale.

- Case 2a: If A and B balance, then the different coin is in group C. All the coins in groups A and B are of normal weight w.
- Case 2b: When put on the scale, the coins of A are heavier than the coins of B. This implies that the 4 type l coins in A must be normal (of weight w) and the 4 type h coins in B must also be normal. Thus, between A and B, we know there are 8 normal coins, 4 coins of type h and 4 coins of type l.
- Case 2c: The coins of A are lighter than the coins of B. Similar to Case 2b, we are left with the knowledge that between groups A and B, there are 8 normal coins, 4 coins of type h and 4 coins of type l.

After the second weighing (in all three cases), we are left with 8 undecided coins (4 of type h and 4 of type l). So, we have reduced the problem to determining the different coin in a set of 8 coins, 4 of which are of type h and the other four of type l. Further, we know the other 28 coins are normal.

For the third weighing, we form three groups from the remaining 8 undecided coins and one normal coin:

- Group D consisting of 2 type h coins and 1 type l coin
- Group E consisting of 2 type h coins and a normal coin
- Group F consisting of the other 3 type l coins.

Next, we compare group D and E on the scale.

- Case 3a: The scale balances. This means the different coin must be in group F. We then weigh two of the type l coins from group F. If they balance, the third coin in group F is the different coin. If they don't balance, the coin that goes up on the scale is the different coin and so, the different coin is lighter in this case.
- Case 3b: Group D goes down on the scale. This means that the type l coin in group D and the two type h coins in group E must be normal. This leaves us with the two type h coins in Group D, which we then put either side of the scale. The h coin on the lower side of the scale must be the different coin and it must be heavier than the other coins.
- Case 3c: Group E goes down on the scale. This means that the two type h coins in group D must be normal. This leaves us with the type l coin in group D and the two type h coins in

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group E. We then compare the two type h coins. If they balance, the type l coin is the different coin and it is lighter than the other coins. If they don't balance, the coin on the lower side of the scale is the different coin and it is heavier than the other coins.

. . .

Assume we have a set S of m coins that are identical in appearance, with one coin being of different weight from the rest (all of which are the same weight). The different coin could have weight less than or greater than the other coins. For this situation, we have the following result (see Theorem 3 of "Counterfeit Coin Problems" [60] for a proof):

The least number of weighings in which the different coin can be found is the unique n satisfying the inequality

$$\frac{3^{n-1}-3}{2} < m \leq \frac{3^n - 3}{2}$$

For example, if $m = 37$, then $n = 4$ since

$$\frac{3^{4-1}-3}{2} = 12 < m \leq 39 = \frac{3^4 - 3}{2}$$

. . .

Puzzle: From the above result, we know that for 39 coins, it only takes 4 weighings to determine the different coin, and also whether the coin is lighter or heavier than the others. However, the procedure that we used for 36 coins will not work here. See if you can find a solution.

[Author's Remark: The solution that follows is at the level of complexity that one would find in moderately difficult proof in a junior or senior level mathematics course in college.]

Solution: Divide the 39 coins into 3 sets of 13 coins (call these sets A, B and C). Further, divide each of the three sets into three subsets of sizes 9, 3, and 1. For example, A has subsets A_1 (consisting of 9 coins), A_2 (consisting of 3 coins) and A_1 (consisting of 1 coin). We do the same for sets B and C, using the same notation for the respective subsets.

In the first weighing, compare the coins in set A against those in set B (as shown in the abstracted diagram below).

A_1	A_2	A_3		B_1	B_2	B_3

Regardless of the outcome of the first weighing, rotate the groups of 9 (i.e., replace A_1 with C_1, replace B_1 with A_1, and remove B_1 from the scale) and do a second weighing as shown in the figure below.

C_1	A_2	A_3		A_1	B_2	B_3

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Case 1: If the outcome of the two weighings is different, then one of the sets of 9 has the different coin (without loss of generality, say it is C_1). Further, the first weighing must be balanced (given our assumption that C_1 has the different coin), and thus, we know all the coins in sets A and B are of normal weight. So, in the second weighing, if the side with C_1 goes down, then we know that C_1 has the different coin and it is heavier. Similarly, if the side with C_1 goes up, then we know that C_1 has the different coin and it is lighter.

Next, we divide C_1 into three sets of 3, and use the third weighing to compare two of the sets.

- If they balance, the different coin (which we already know to be heavier or lighter) is in the third subset of C_1.
    - For the fourth weighing, we compare two of the coins from the third subset of C_1. If they balance, the other coin is the different coin. If they don't balance, we can tell which is the different coin since we already know whether it is heavier or lighter.
- If the two subsets of 3 in C_1 don't balance, and different coin in C_1 is lighter, then we know the set of 3 that lifted higher on the scale has as the lighter coin. Similarly, if the different coin in C_1 were heavier, then the set of 3 that moved lower has the heavier coin. In either case, we will have isolated the location of the different coin. As before, one additional weighing is necessary to find the different coin in the remaining set of 3.

Case 2: If the outcome of the first two weighings is the same (balanced, down left or down right), then we know that all the coins in sets A_1, B_1 and C_1 are of normal weight. (If group A_1, B_1 or C_1 had the different coin, it would not be possible for the first two weighings to have the same outcome.)

For this case, we rotate the groups of 3 and do the **third weighing** (see the third row in the figure below). The coins on either side of the scale for the first three weighings are shown in the table below (under the assumption the first two weighings have the same result).

A_1	A_2	A_3		B_1	B_2	B_3
C_1	A_2	A_3		A_1	B_2	B_3
A_1	C_2	A_3		B_1	A_2	B_3

Case 2a: If the 3rd weighing has a different outcome than the first two, then we can pinpoint one of the groups of 3 as having the different coin, and whether it is lighter or heavier.

- First two weighings are balanced implies the coins in A_2, B_2, A_3 and B_3 are normal
    - Scale dips to the right on 3rd weighing implies C_2 has the lighter coin
    - Scale dips to the left on 3rd weighing implies C_2 has the heavier coin

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- First two weighings dipped to the left
  - Balanced on 3rd weighing implies B_2 has the lighter coin
  - Scale dips to the right on 3rd weighing implies A_2 has the heavier coin
- First two weighings dipped to the right
  - Balanced on 3rd weighing implies B_2 has the heavier coin
  - Scale dips to the left on 3rd weighing implies A_2 has the lighter coin

In the 4th weighing, we compare two of the coins from the set of three that has been identified as having the different coin in the 3rd weighing.

Case 2b: If the 3rd weighing has the same outcome as the first two, then we know that all the coins in A_2, B_2 and C_2 must be of normal weight (recall that for Case 2, we already know that A_1, B_1, and C_1 are of normal weight).

For the 4th weighing, we rotate the groups of 1 (as shown in the figure below).

A_1	A_2	C_3		B_1	B_2	A_3

- If the first three weighings were balanced, then the coins in A_3 and B_3 must also be of normal weight.
  - Scale dips to the left on 4th weighing, then C_3 is the heavier coin
  - Scale dips to the right on 4th weighing, then C_3 is the lighter coin
  - The scale cannot balance is if there is, in fact, a different coin
- If the first three weighings dipped to the left, then either A_3 is the heavier coin or B_3 is the lighter coin. By the processes of elimination, C_3 must be of normal weight.
  - Scale balances on 4th weighing, then B_3 is the lighter coin.
  - Scale dips to the right on the 4th weighing, then A_3 is the heavier coin.
  - Scale dipping to the left is not possible on the 4th weighing.
- If the first three weighings dipped to the right, then either A_3 is the lighter coin or B_3 is the heavier coin. By the processes of elimination, C_3 must be of normal weight.
  - Scale balances on 4th weighing, then B_3 is the heavier coin.
  - Scale dips to the left on the 4th weighing, then A_3 is the lighter coin.
  - Scale dipping to the right is not possible on the 4th weighing.

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6.11 Bridge Crossing Puzzle

Puzzle: Four people, on a nighttime military mission, are at the South-side of a rickety, unstable bridge and need to get to the North-side of the bridge in one hour or less. Given the condition of the bridge, at most two people can cross at a time. Further, use of a flashlight is necessary for each crossing. The four travelers have but one flashlight. The travelers (some of whom are injured) can cross the bridge in 5, 10, 20 and 25 minutes, respectively. When a pair crosses the bridge, they must stay close together because they both need the light from the flashlight. Thus, when two cross, the crossing time is limited by the slower walker. There are no tricks here, such as throwing the flashing light from one side to the other.

Sources:

- A version of this puzzle appears in a TED-Ed YouTube video [61].
- The problem for N travelers, whose respective walking paces are just one second different, is covered in item A078476 [62] in the OEIS.
- The article entitled "The Bridge Crossing Problem" [63] provides a technical analysis of the puzzle.

Solution: The key is to have the two slower walkers cross the bridge together, with neither having a need to return back to the South-side. This can be done as follows:

- The two fastest walkers cross first (taking 10 minutes).
- The second fastest walker remains on the North-side and the fastest walker returns to the South-side (taking 5 minutes).
- The two slowest walkers cross together (taking 25 minutes).
- The second fastest walker returns to the South-side (10 minutes).
- The two fastest walkers cross over to the North-side (10 minutes).

This adds to a total of 60 minutes. Note that the speed of the 3rd fastest walker does not appear in the calculations. The 3rd fastest walker (i.e., the one crossing in 20 minutes) could be anywhere from 10 to 25 minutes with the same result for the overall crossing of the quartet.

There is a slight variation in the above process that yields the same result, i.e.,

- The two fastest walkers cross first (taking 10 minutes).
- The fastest walker remains on the North-side and the second fastest walker returns to the South-side (taking 10 minutes).
- The two slowest walkers cross together (taking 25 minutes).
- The fastest walker returns to the South-side (5 minutes).
- The two fastest walkers cross over to the North-side (10 minutes).

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Puzzle: Same as previous puzzle, except that the crossing times are a, b, c, d where $a < b < c < d$. We can try the same approach as the previous puzzle, i.e.,

- The two fastest walkers cross first (taking b minutes).
- The second fastest walker remains on the North-side and the fastest walker returns to the South-side (taking a minutes).
- The two slowest walkers cross together (taking d minutes).
- The second fastest walker returns to the South-side (b minutes).
- The two fastest walkers cross over to the North-side (b minutes).

This gives us a total of $a + 3b + d$.

Another approach is as follows:

- The fastest and slowest walkers cross (d minutes).
- The fastest walker returns to the South-side of the bridge (a minutes).
- The fastest and second fastest walkers cross (b minutes).
- The fastest walker returns to the South-side of the bridge (a minutes).
- The fastest and 3rd fastest walker cross (c minutes).

This yields a total of $2a + b + c + d$.

If $2b < a + c$, the former approach is faster. If $2b > a + c$, the latter approach is faster. The article entitled "The Bridge Crossing Problem" [63] proves that these are the only two optimal solutions (depending on the noted inequality). The same article considers the more general case of N people crossing the bridge.

6.12 Flashlight Battery Puzzles

Puzzle: You have 8 batteries, 4 of which are uncharged and 4 of which are charged. Which batteries are charged or uncharged are unknown to you. You need two charged batteries for a flashlight ("torch" in the UK) to work. Putting one charged and one uncharged battery in the flashlight will provide you with no information. What is the minimum number of tries (inserting two batteries into the flashlight) that will **ensure** that that the flashlight is on?

Source: The origin of this puzzle is unclear. There are several appearances of this puzzle on the Internet with claims that it is sometimes used as a job interview question.

Solution: One approach (which turns out not to be optimal) is to divide the batteries into two sets of 4 and check all the batteries in one set. Selecting 2 out of 4 can be done in six ways. This comes from the formula for combinations. The general formula for the number of ways of selecting k objects out of n is

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

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The worst case is that there is one charged battery out of the 4, and so all 6 tries fail. This means that the other 4 batteries must have 3 charged and 1 uncharged (which we represent with the shorthand notation 3c, 1u). At this point, choose two of the batteries. The worst case is 1c, 1u. If that happens, choose the other two batteries which must be 2c. So, using this approach, the minimum to ensure finding two batteries that light the torch is 8.

However, there is a better approach that ensures finding two charged batteries in just 7 tries. In this approach, we divide the batteries into groups of 3, 3 and 2. It takes 3 tries to test all possible pairs in each of the groups of three. The worst case here is 1c, 2u for each of the groups of three. This would leave us with 2c in the group of 2. Thus, the 7th try would be sure to light the flashlight.

. . .

[Author's Remark: Try doing the same puzzle with 6,10,12,14,16 or 18 batteries (half are charged and half are uncharged). It is also possible to vary the problem with a flashlight that requires more than two batteries. Yet another variation is to not have an equal number of charged and uncharged batteries. The general solution appears not to have been studied. My best solution for the case of 16 batteries (8 charged and 8 uncharged) is given below.**]**

We could try the approach of dividing the group of 16 into four groups of 3 and one group of 4. The worst case for each of the groups of 3 is 2u, 1c which results in 12 failed tries. Given the worst case results for the groups of 3, the group of 4 must have 4c, and thus we have success on the 13th try.

However, we can ensure success in 11 tries using a different approach. We first take two groups of 3 and test all subsets in each group of 3 (total of 6 tries), and assume the worst case, i.e., 1c,2u in each group. After this and assuming the worst case, we have 6c,4u remaining batteries. Next, we test two batteries at a time (as summarized in the table below).

Try number	Worst case result	Configuration of remaining batteries
7	c,u	5c,3u
8	c,u	4c,2u
9	c,u	3c,u
10	c,u	2c
11	success	

[Author's Remark: Below are listed the best known (to me) solutions for several cases. In each case, we start with half the batteries being charged and the other half uncharged. The flashlight requires 2 charged batteries.

- 6 batteries: using 2 groups of 3, we can ensure that 2 charged batteries are found in 6 tries
- 8 batteries: as we saw, the best solution is 7 tries
- 10 batteries: using 2 groups of three, and then trying 2 batteries at a time, we can ensure success in 8 tries.
- 12 batteries: using 2 groups of three, and then trying 2 batteries at a time, we can ensure success in 9 tries
- 14 batteries: using 2 groups of three, and then trying 2 batteries at a time, we can ensure success in 10 tries

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- 16 batteries: as we saw, the best solution is in 11 tries
- 18 batteries: using 2 groups of three, and then trying 2 batteries at a time, we can ensure success in 12 tries.

We can use the “using 2 groups of three, followed by 2 at a time” approach for the general case of $2N$ batteries (half charged and half uncharged). With this approach, we are ensured of success on try number $6 + (N - 3) = N + 3$.

Starting with a group of 4 or more is less efficient than starting with 2 groups of three.**]**

6.13 Muddy Children Puzzle

There is a class of puzzles that concern the application of **common knowledge logic** [64] among a group of people, where the common knowledge increases at each round of the puzzle. One such puzzle is known as the “Muddy Children Puzzle”.

Puzzle: After playing outside, several girls return home. Their mother says to them, at least one of you has mud on your forehead. Each child can see the mud on others but cannot see her own forehead. She then asks the following question several times, with a chance for each child to answer: “Can you tell for sure (i.e., logically deduce) whether or not you have mud on your head?” The allowed responses are “yes”, “no” or “cannot determine”. The rules of the game are that the children are not allowed to communicate during the game. It is assumed all three girls are highly skilled in logical reasoning.

- Puzzle 1. Analyze the problem if there is exactly one child with mud on her forehead.
- Puzzle 2. Analyze the problem if there are exactly two children with mud on their forehead.
- Problem 3. Analyze the case where all three children have mud on their foreheads.

To avoid timing issues, it is assumed all the children answer at the same time (perhaps in writing) and are then informed of the answers from their two sisters.

Source: Section 8.1 of “Distributed Computing: Principles, Algorithms, and Systems” [65]

Solution: For all three puzzles, assume identify the children by the letters A, B and C. All three are female.

In Puzzle 1, assume A has mud on her forehead. A will see that the other two do not have any mud, and will answer “yes” to the question. B and C will not know for sure before they hear the response from A.

After the first round, B knows that A could not have deduced that she had mud on her forehead unless she (B) had a clean forehead. So, B knows that she has a clean forehead. Similarly, C knows that her forehead is without mud. When their mother asks the question a second time, both B and C can say that they know for sure (logically) that their foreheads have no mud.

For Puzzle 2, assume B and C have mud on their heads, and A does not. When the question is first asked, not one of the three can tell for sure if she has mud on her head.

After the three girls have answered “cannot determine” to the first question from their mother, B knows that C would have answered “yes” if she (B) had a clear forehead, since in that case A and B would not have muddy foreheads. But C did not answer “yes” and so, B can deduce that she does have a muddy forehead. Similarly, C can deduce that she has a muddy forehead. When their

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mother asks the question a second time, both B and C can say “yes”. A still cannot determine concerning her forehead.

After the second round, A knows that B and C could not have deduced they have mud on their foreheads unless she (A) had a clear forehead. So, when the question is asked a third time, A will say “no”.

For Puzzle 3, all three girls will answer “cannot determine” the first time the question is asked. When the question is asked a second time, all three again say “cannot determine”. However, all three understand that if they had a clean head, the other two would have said “yes” after the second time the question was asked (using the logic from Puzzle 2). Thus, when the question is asked the third time, all three girls say “yes”.

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Puzzle: Same as the previous puzzle set, except that there are now 10 children.

Solution: Let’s number the children from 1 to 10, assume they are all male. The key here is that the additional children with clean foreheads do not change the reasoning from the previous puzzle.

In Puzzle 1, assume child 1 has mud on his forehead. He will see that the others do not have any mud, and will answer “yes” to the question. The others will not know for sure before they hear the response from child 1.

After the first round, child 2 knows that child 1 could not have deduced that he had mud on his forehead unless he had a clean forehead. So, child 2 knows that he has a clean forehead. Similarly, children 3-10 know that their foreheads are without mud.

For Puzzle 2, assume children 1 and 2 have mud on their heads, and the other children do not. When the question is first asked, not one of the children can tell for sure if he has mud on his head.

At this point, child 1 knows that child 2 would have answered “yes” if he (child 1) had a clear forehead, since in that case child 2 would see 9 muddy foreheads. But child 2 did not answer “yes” and so, child 1 can deduce that he does have a muddy forehead. Similarly, child 2 can deduce that he has a muddy forehead. When their mother asks the question a second time, both 1 and 2 can say “yes”.

After the second round, children 3-9 know that 1 and 2 could not have deduced they have mud on their foreheads unless each of them (3-9) had clear foreheads. So, when the question is asked a third time, children 3-9 will say “no”.

For the third puzzle, children 1-3 have mud on their forehead, and the other children do not. All 10 children will answer “cannot determine” the first two times the question is asked, but this allows children 1-3 to deduce that they must all have mud on their foreheads. So, they all say “yes” when the question is asked the third time.

The pattern continues with 4, 5, … , 10 children having muddy foreheads and the others being clean.

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In general, if we have n children, with a subset of children of size k having muddy foreheads and the others are clean, then all children will answer "cannot determine" the first $k - 1$ times the question is asked, but on the next repetition of the question, all the children with muddy foreheads will answer "yes". We can prove this using mathematical induction. (For a general overview of mathematical induction, see Section 2.1.)

For the case $k = 1$, the one child with a muddy forehead will say "yes" since he or she can see that all the other children have muddy foreheads.

For case $k = 2$, no child will be able to deduce whether they have a muddy forehead when the question is asked the first time. After all the children say "cannot determine" in response to the first question, the two children with muddy foreheads can deduce they have muddy foreheads (using the logic described previously).

For the induction step, we assume the hypothesis is true for $k = 1,2, \dots, m$ and show that the hypothesis must be true for the case $k = m + 1$.

For the case $k = m + 1$, each child with a muddy forehead knows (by the induction hypothesis) that if there were exactly m children with muddy foreheads, then they would all have answered "yes" when the question is asked for time number m. Since that did not happen, there must be more than m muddy children. Since each child with a muddy forehead can see only m other children with muddy foreheads, each concludes that he or she must also have a muddy forehead and thus, answers "yes" when the question is asked for time number $m + 1$.

6.14 Egg Drop Puzzle

Puzzle: A new shock absorbing material is being tested at a 98 story building. A set of three specially designed eggs will be used for the initial test of the new material. Each egg has the same shell strength. Each egg will break when dropped onto the shock absorbing material from a yet to be determined height (this could be at the height of any floor of the building, or the eggs could even survive a drop from the 98th floor). Your task is to determine the least number of egg drops (from various floors) that can guarantee sufficient information to classify all the floors of the building as either "egg will break if dropped from this floor" or "egg will not break if not dropped from this floor".

Source:

- The YouTube video entitled "Can you solve the egg drop riddle?" [66]
- Egg dropping essay by J. Tanton [67]

[**Note**: In what follows, the words "drop" and "try" are used interchangeably.]

Before we get to the general solution, it is instructive to try some experimental ideas. One plausible solution is as follows:

- Drop the first egg from floor 33, then (if not broken) from floor 65, then (if still not broken) from floor 98.
    - If the egg does not break after a drop from these 3 floors, we know that this type of egg can survive a drop from at least 98 floors (or equivalent height) onto the shock absorbing material.

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- If the egg breaks when dropped from floor 98 but not from the other 2 floors, we can use the second egg to test the floors between 66 and 97.
  - Drop the second egg from floor 81.
    - If the egg breaks at floor 81, use the 3rd egg to test all the floors from 66 to 80 (at most 15 more tests for a total of 19 tests).
    - If the egg does not break at floor 81), use the 3rd egg to test all the floors from 82 to 97 (at most 16 more tests for a total of 20 tests).
- If the egg breaks when dropped from floor 65 but not at floor 33, use the second egg to test the floors between 34 and 64.
  - Drop the second egg from floor 49.
    - If the egg breaks at floor 49, use the 3rd egg to test all the floors from floors 35 to 48 (at most 14 more tests for a total of 18 tests).
    - If the egg does not break at floor 49, use the 3rd egg to test all the floors from 50 to 64 (at most 15 more tests for a total of 19 tests).
- If the egg breaks when dropped from floor 33, use the second egg to test the floors between 1 and 32.
  - Drop the second egg from floor 16.
    - If the egg breaks at floor 16, use the 3rd egg to test all the floors from floors 1 to 15 (at most 15 more tests for a total of 19 tests).
    - If the egg does not break at floor 16, use the 3rd egg to test all the floors from 17 to 32 (at most 16 more tests for a total of 20 tests).

So, in two of the above scenarios, we achieve the worst case scenario of 20 tests needed to guarantee sufficient information to classify all the floors. One can experiment with trying more drops with eggs #1 and #2, and as we shall see, it is possible to reduce the number of required steps to just 9.

. . .

We now show how to solve the puzzle for any number of eggs and experiments. The general solution is based on the essay by Tanton [67].

Let $x_k(n)$ be the maximum number of floors that one can guarantee to classify (**starting from the first floor**) with k eggs and the ability to do up to n experiments (i.e., egg drops). In what follows, we assume the building has n or more floors. The goal here is to develop a recursive formula for $x_k(n)$. To that end, we need some initial values for $x_k(n)$.

- $x_k(1) = 1$ for any number of eggs, i.e., $k \geq 1$. If we can do only one test, we can only guarantee the ability to classify floor 1. If we did the test from a floor other than 1, we could not guarantee the classification of the 1st floor (if the egg cracked). Further, the fact that we have $k \geq 1$ doesn't make any difference since we only get one try in this case.
- $x_1(n) = n, n \geq 1$ since (with n drops) we can either break the egg before try #n (which allows as to classify all the floors) or the egg does not break when being dropped for any floor from 1 to n (which also allows us to classify floor n).

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For the case where we have k eggs and are allowed n tries (i.e., drops), we can guarantee (by definition) the classification of $x_k(n)$ floors. Consider dropping an egg from floor f.

- If the egg breaks, we can classify floor f and all floors above, but cannot classify any of the $f-1$ floors below without additional tries. By definition, the maximum number of floors that we can guarantee to classify after the first try is $x_{k-1}(n-1)$. So, to ensure an optimal solution, we should choose $f-1 = x_{k-1}(n-1)$ or equivalently, $f = x_{k-1}(n-1)+1$. With this selection of f, we can guarantee to classify all the floors in the building if the egg breaks when being dropped from floor f.
- Assuming we choose $f = x_{k-1}(n-1)+1$ and the egg does not break, then we can classify all the floors from 1 to f. By definition, we can guarantee the classification of another $x_k(n-1)$ floors. Thus, in this case, we can guarantee the classification of $f + x_k(n-1) = x_{k-1}(n-1) + x_k(n-1) + 1$ floors.

The second case is the limiting case when we start by dropping the egg from floor f. Thus, we have the following recursive formula

$$x_k(n) = x_{k-1}(n-1) + x_k(n-1) + 1$$

The above equation and the values we determined for $x_k(1)$ and $x_1(n)$ allow us to calculate $x_k(n)$ for any values of k and n.

The computation of $x_k(n)$ for several values of k and n are shown in Table 17.

Table 17. Guaranteed number of floors that can be classified for egg drop puzzle

		Number of Eggs (k)				
		1	**2**	**3**	**4**	**5**
Number of Tries (n)	**1**	1	1	1	1	1
	2	2	3	3	3	3
	3	3	6	7	7	7
	4	4	10	14	15	15
	5	5	15	25	30	31
	6	6	21	41	56	62
	7	7	28	63	98	119
	8	8	36	92	162	218
	9	9	45	129	255	381
	10	10	55	175	385	637
	11	11	66	231	561	1023

For the problem at hand, we see from Table 17 that for $k=3$ eggs, we need 8 tries to guarantee the classification of 92 floors, which falls short of our 98 floor building. So, we need to go to the next number of tries to cover 98 (we could actually do 129 floors with $n=9$ tries as the table indicates). The solution entails repeated use of the recursive formula, and the progressive classification of floors. The idea is to reduce the problem into a smaller number of floors to be

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classified after each try. At various points in the process, we will encounter essentially the same problem, e.g., the need to classify 10 floors in 4 tires with 2 eggs. Such repeating patterns are defined once (see the following text) and then reused in the solution tree for the problem at hand.

. . .

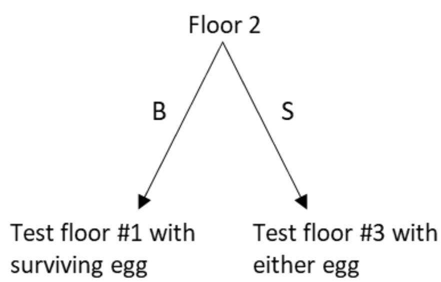
The following approaches allow for the classification of buildings with 1 to 10 floors. Some of these approaches are reused as part of the solution to the 98-floor building puzzle.

For 1 floor, we need just one egg and one try.

For 2 floors, the most efficient approach is to use one egg. Test floor #1 first and if the egg does not break, test floor #2.

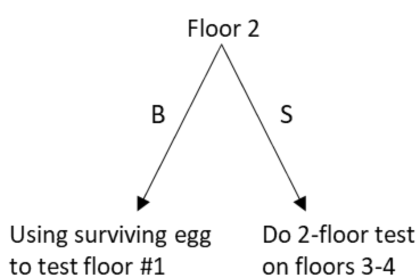
For 3 floors, there are two cases:

- If we have only one egg, then test in the order $1 \rightarrow 2 \rightarrow 3$.
- If we have 2 eggs (or more), do the first try at floor #2 and then proceed as shown in the diagram below, where B stands for “egg breaks” and S stands for “egg survived try”. This requires at most 2 tries.



For 4 floors, there are two cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$.
- If we have 2 eggs (or more), do the first try at floor #2 and then proceed as shown in the diagram below. This requires at most 3 tries.



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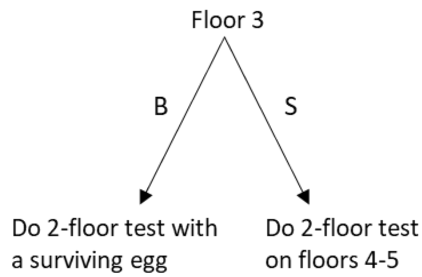
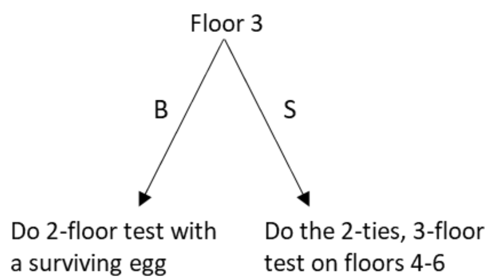
For 5 floors, there are two cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$.
- If we have 2 eggs (or more), do the first try at floor #3 and then proceed as shown in the diagram below. This requires at most 3 tries.



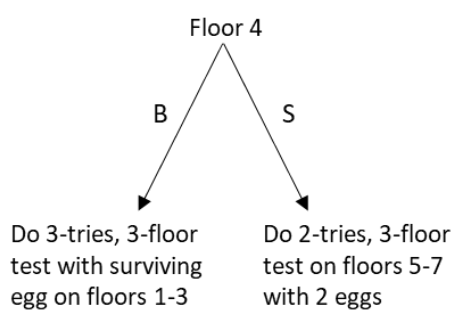
For 6 floors, there are two cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6$.
- If we have 2 eggs (or more), do the first try at floor #3 and then proceed as shown in the diagram below. This requires at most 3 tries.



For 7 floors, there are three cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$.
- If we have 2 eggs, do the first try at floor #4 and then proceed as shown in the diagram below. This requires at most 4 tries.



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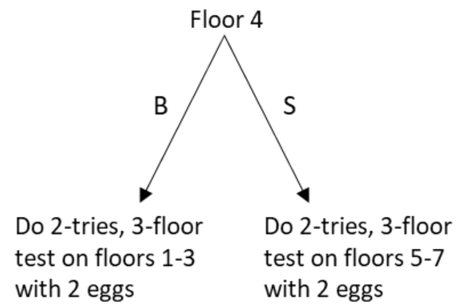
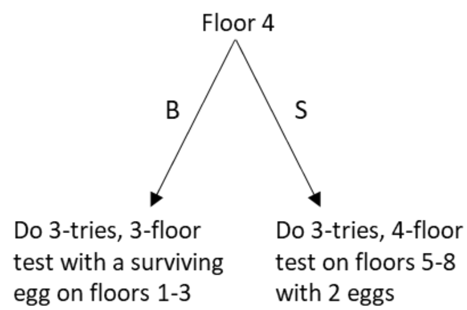
- If we have 3 eggs, do the first try at floor #4 and then proceed as shown in the diagram below. This requires at most 3 tries.



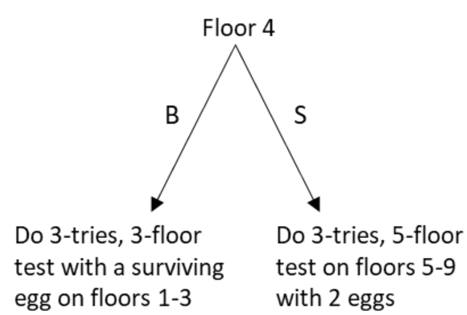
For 8 floors, there are two cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8$.
- If we have 2 or 3 eggs, do the first try at floor #4 and then proceed as shown in the diagram below. This requires at most 4 tries.



For 9 floors, there are two cases:

- If we have only 1 egg, then test in the order $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9$.
- If we have 2 or 3 eggs, do the first try at floor #4 and then proceed as shown in the diagram below. This requires at most 4 tries.



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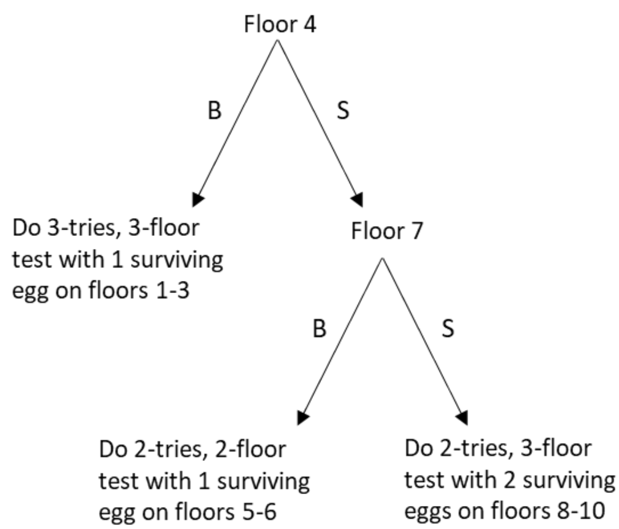
For 10 floors, there are 2 cases:

- If we have only 1 egg, then test in the order $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10$.
- If we have 2 eggs, do the first try at floor #4 and then proceed as shown in the diagram below. This requires at most 4 tries.



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Finally, we are ready to solve the problem for 98 floors, with 9 tries at most and 3 eggs. The solution is shown in Figure 20 and Figure 21. At each step in the process, the computation $f = x_{k-1}(n-1) + b + 1$ is provided. (Note the adjustment in the formula f, i.e., we need add the base floor b.) At several points, reference is made to the previous stated solutions for 1-10 floors.

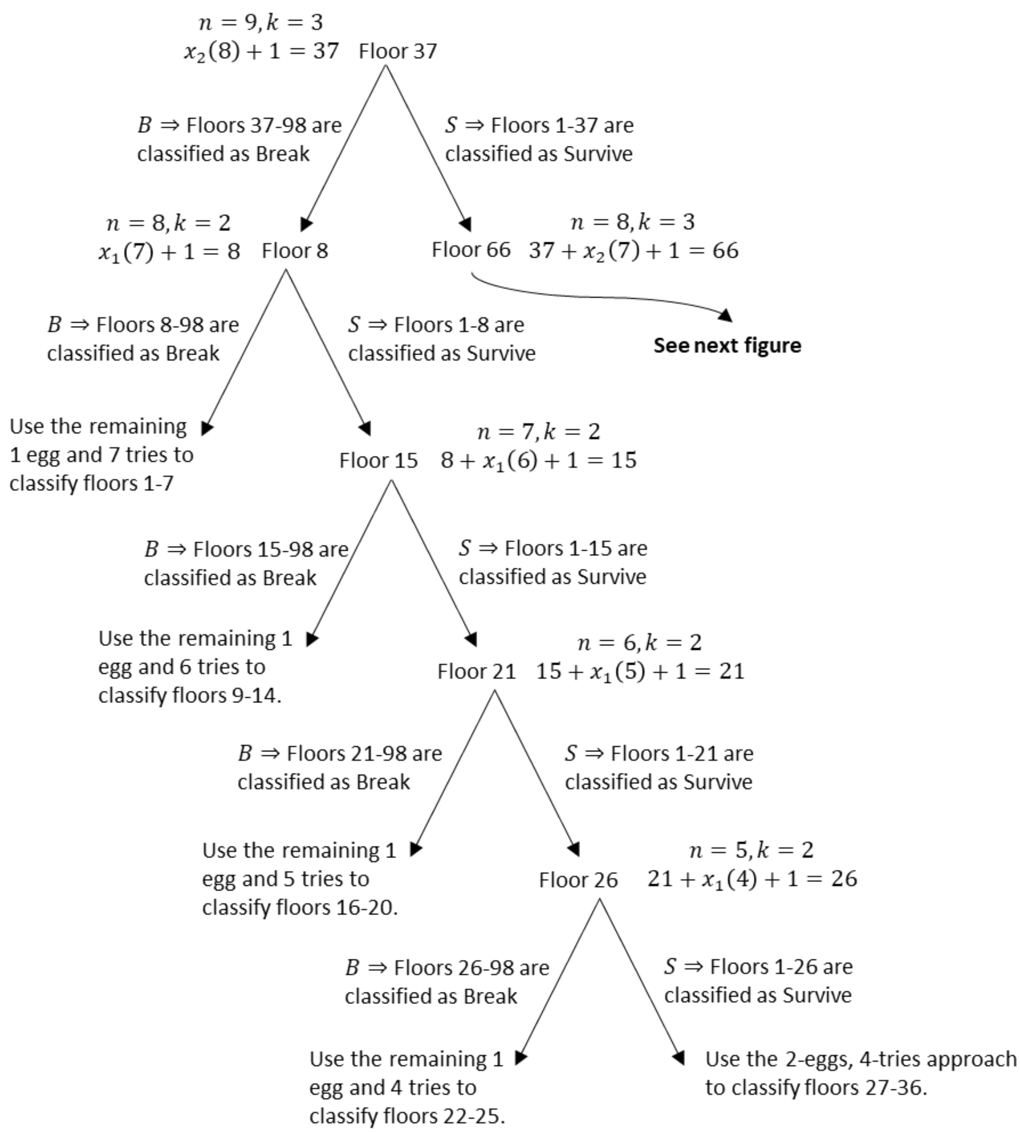

Figure 20. Egg drop analysis, 98 floors, Part 1

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In Figure 21, part of the solution is left as an exercise for the reader (see the node labeled "Floor 44" in the figure). Notice that at floors 77 and 88, we have more tries and eggs than are needed. This is because with 9 tries and 3 eggs, we can classify up to 129 floors (per Table 17).

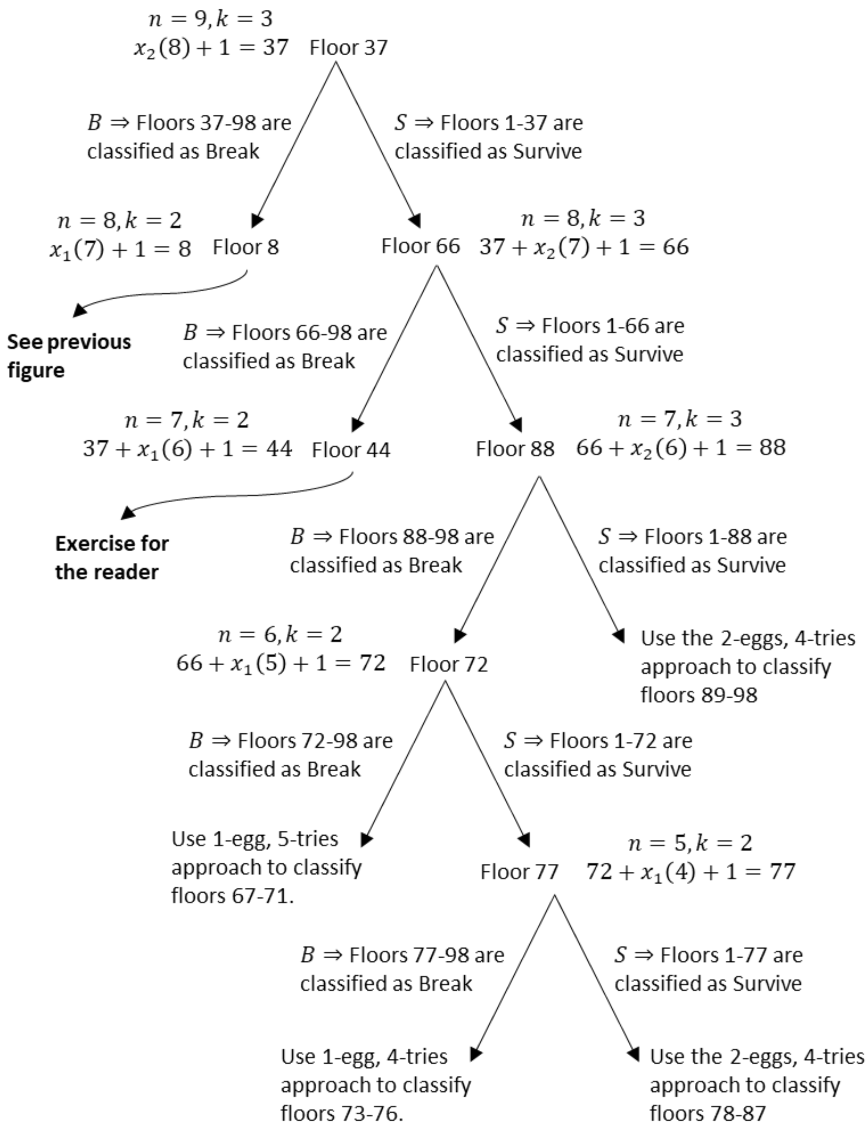

Figure 21. Egg drop analysis, 98 floors, Part 2

6.15 Liquid Distribution Puzzle

Puzzle: You are given three containers that can hold 3, 5 and 8 units of liquid, respectively. The 8-unit container is full of water, and the other two containers are empty. Your task is to distribute the water so that both the 5-unit and 8-unit containers have exactly 4 units of water. When a transfer of liquid is made, one container must be completely emptied or the other completely filled. For example, under the rules of this puzzle, one cannot estimate the pouring of 4 units of water from the 8-unit container into the 5-unit container.

Find the least number of transfers to complete the given task.

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Sources:

- Journal article entitled "A Graphical Method of Solving Tartaglian Measuring Puzzles" [68]
- Wikipedia article entitled "Water pouring puzzle" [69]

Solution: Let the notation (x, y, z) represent the state of the three container system where the 3-unit container has x units of water, the 5-unit container has y units of water and the 8-unit container has z units of water. We start in state $(0,0,8)$ and we want to finish in state $(0,4,4)$, while making the least number of transfers.

One approach is to work backward using only reversible transfers of liquid. There are two types of reversible transfers, i.e.,

- Pouring water from a full container to any other container
- Pouring water from any container to an empty container.

Pouring water from one partially full container to another partially full container is not reversible.

Working backwards from the desired end result, we find the following 7-step solution:

$$(0,4,4) \leftrightarrow (3,4,1) \leftrightarrow (2,5,1) \leftrightarrow (2,0,6) \leftrightarrow (0,2,6) \leftrightarrow (3,2,3) \leftrightarrow (0,5,3) \leftrightarrow (0,0,8)$$

There are longer solutions, e.g., the following solutions takes 8 steps (again, working the problem in reverse):

$$(0,4,4) \leftrightarrow (3,1,4) \leftrightarrow (0,1,7) \leftrightarrow (1,0,7) \leftrightarrow (1,5,2) \leftrightarrow (3,3,2) \leftrightarrow (0,3,5) \leftrightarrow (3,0,5) \leftrightarrow (0,0,8)$$

The article by Tweedie [68] describes another approach to the problem that makes use of trilinear coordinates [70]. For the problem at hand, we use the three sides of a triangle to represent the possible values for the amount of water in each of the three containers (as shown in Figure 22). The intersection points within the gray area represent the possible valid values for the puzzle. We start at the bottom left, i.e., (0,0,8), and want to finish at (0,4,4).

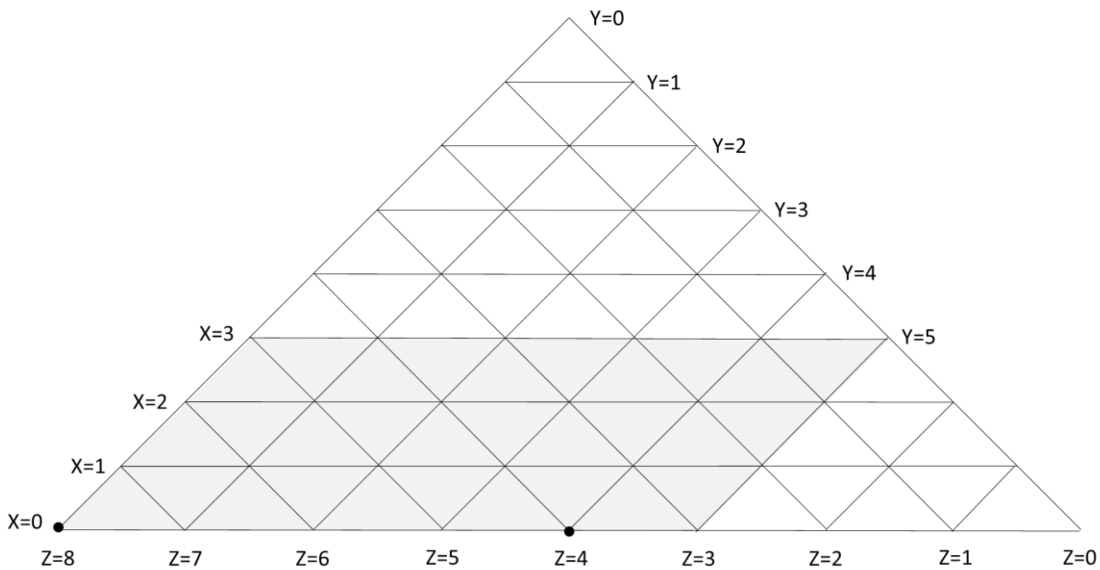

Figure 22. Trilinear representation of liquid distribution puzzle

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Given that one container must be completely emptied or the other completely filled when a transfer is made implies that each transfer results in a point on the boundary of the gray parallelogram in the figure. Figure 23 depicts the transitions that comprise the minimal solution.

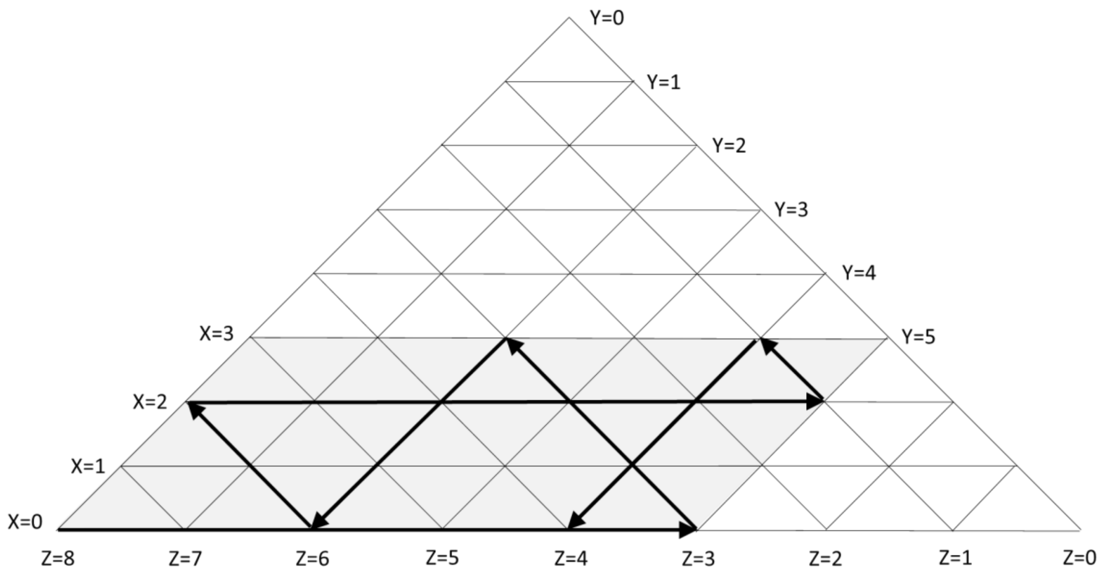

Figure 23. Minimal solution to liquid distribution puzzle

6.16 Rope (Fuse) Burning Puzzles

This set of puzzles concerns ropes (or fuses) that take a given length of time to burn. The goal of each puzzle is to measure a given amount of time. For example, if we have a rope that burns in 10 minutes, we can measure 10 minutes by lighting one end of the rope, or measure 5 minutes by lighting both ends of the rope at the same time. The puzzles become more interesting with 2 or more ropes of different burn-out times.

In these puzzles, the only option is to light a given rope at one or both ends. No estimation is allowed since the ropes burn unevenly.

Sources:

- Journal article entitled “Fusible numbers and Peano Arithmetic” [71]
- Wikipedia article entitled “Rope-burning puzzle” [72].

Puzzle: Given a rope that burns-out in 100 minutes and another rope that burns-out in 74 minutes, measure out 13 minutes using the two ropes.

Solution:

- Light both ropes at one end at the same time.
- When the 74-minute rope burns-out, the other rope has 26 minutes left if it continues to burn at just one end. If we light the other end, it will burn-out in 13 minutes.

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Puzzle: You are given 3 ropes, i.e., Rope 1 burns-out in 132 seconds, Rope 2 burns-out in 72 seconds and Rope 3 burns-out in 48 seconds. Your task is to measure out 63 seconds.

Solution:

- Light Rope 2 on both ends and light Rope 3 on one end. When Rope 2 burns-out (36 seconds), Rope 3 will have 12 seconds left.
- At this point, light the other end of Rope 3 (will burn out in 6 seconds), and light one end of Rope 1.
- When Rope 3 burns out, Rope 1 will have 126 seconds left to burn (if left burning at one end).
- At this point, light the other end of Rope 1. Rope 1 will then burn-out in 63 seconds.

6.17 Hiker Puzzles

Puzzle: Five hikers start from a base camp in the Antarctic, with the goal of one hiker from the group going as far as possible and with all hikers returning to base camp safely. Each hiker can carry a maximum of 7 days of supplies. All five hikers leave at the same time with a full supply, but can return at different times. What strategy maximizes the hikers' goal?

Source: This is a variant of "The Expedition" puzzle on page 124 of "Mathematical Puzzles and Perplexities" [73]. This is also similar to the motorcycle puzzle [74], except that the motorcycle puzzle does not involve a return trip.

Solution: The best strategy is for one of the hikers (Hiker 1) to go as far as possible while (1) having enough supplies to return to base camp and (2) at the same time, replenish the other hikers to full capacity, i.e., 7 days' supply. Let t_1 be the time that meets the conditions above. At this time, each of the hikers has $7 - t_1$ supplies remaining. Hiker 1 needs t_1 supplies to return to base camp (condition 1) and from the remaining $7 - 2t_1$, Hiker 1 needs to replenish each of the other 4 hikers. For each of the remaining 4 hikers, this translates to the following equation, i.e., the contribution from Hiker 1 plus remaining supplies for a given hiker should add to 7 if t_1 satisfies the following equation.

$$\frac{1}{4}\big((7 - t_1) - t_1\big) + (7 - t_1) = 7$$

Solving the above equation, we get $t_1 = \frac{7}{6}$. So, Hiker 1 should return to base camp after $\frac{7}{6}$ days, and equally distribute all but t_1 of his or her supplies to the other 4 hikers (returning each to full capacity of 7 days supplies).

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Next, Hiker 2 does essentially the same as Hiker 1 but starting from point t_1. Hiker 2 will turn back at time $t_1 + t_2$, using $t_1 + t_2$ of his or her remaining supplies to return to base camp, and dividing the remaining supplies among the other three hikers (returning each to full capacity). This translates to the following equation

$$\frac{1}{3}\Big(\big(7 - (t_1 + t_2)\big) - t_2\Big) + (7 - t_2) = 7$$

$$\frac{1}{3}\left(\frac{35}{6} - 2t_2\right) + (7 - t_2) = 7$$

$$35 - 12t_2 - 18t_2 = 0$$

$$t_2 = \frac{7}{6}$$

So, Hiker 2 turns back to base camp after $t_1 + t_2 = \frac{14}{6}$ days.

Next, Hiker 3 takes a similar approach to Hikers 1 and 2. Doing a similar analysis to the previous two cases, we get the following each involving t_3 (the turn-around time for Hiker 3):

$$\frac{1}{2}\left(\frac{14}{3} - 2t_3\right) + (7 - t_3) = 7$$

Solving for t_3, we again get $\frac{7}{6}$. So, Hiker 3 turns back to base camp after $t_1 + t_2 + t_3 = \frac{21}{6}$ days.

For Hiker 4, the equation for his or her turn-around point is

$$\left(\frac{7}{2} - 2t_4\right) + (7 - t_4) = 7$$

which implies $t_4 = \frac{7}{6}$. Hiker 4 turns back to base camp after $\frac{28}{6}$ days.

Hiker 5 now has a full 7-day supply, and is $4\left(\frac{7}{6}\right) = \frac{28}{6}$ days away from base camp. The turn-around time t_5 for Hiker 5 is constrained by

$$\frac{28}{6} + t_5 = 7 - t_5$$

which implies $t_5 = \frac{7}{6}$.

In summary, Hiker 5 travels $5\left(\frac{7}{6}\right) = \frac{35}{6}$ days away from base camp, and takes the same amount of time to return back to base camp. So, the solution to the puzzle is $\frac{70}{6} = 11\frac{2}{3}$ days.

. . .

Puzzle: Same as the previous puzzle but in generic form, i.e., n hikers such that each hiker is able to carry a maximum of d days supply. All hikers start at the same time with a full supply, but can turn back to base camp at different times.

Solution: We use the same strategy as before.

Hiker 1 turns around at time t_1, requiring the use of $2t_1$ supplies (going out and back). This leaves $d - 2t_1$ to be divided among the remaining $n - 1$ hikers. The time t_1 is such that the remaining supplies from Hiker 1 are sufficient to resupply the other $n - 1$ hikers to their maximum carrying capacity of d.

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Thus, we want

$$\frac{1}{n-1}(d-2t_1)+(d-t_1)=d$$

Solving the above equation for t_1 in terms of n and d, we get $t_1=\frac{d}{n+1}$.

Define the turn back time for Hiker i to be $t_1+t_2+\cdots+t_i$. Next, we use mathematical induction to prove that $t_i=\frac{d}{n+1}$ for $i=1,2,\ldots,n$. We have already proved the result for $i=1$. Assume that the result holds for $i=1,2,\ldots,k-1$. We need to show that the result holds for $i=k$ to complete the induction proof.

From the induction assumption, $k-1$ of the hikers have turned back to base camp at respective times, $t_1, t_1+t_2, \ldots, t_1+t_2+\cdots+t_{k-1}$. At time $t_1+t_2+\cdots+t_{k-1}$, all of the remaining $n-k+1$ hikers (i.e., hikers who have not yet headed back to base camp or who are already back at base camp) have a full supply of d days. Hiker k travels another t_k days and then turns back to base camp. While reserving supplies to return back to base camp, Hiker k has the following amount of supplies available to share with the other hikers who will go further:

$$d-(t_1+t_2+\cdots+t_k)-t_k=d-\frac{(k-1)d}{n+1}-2t_k=\frac{(n-k+2)d}{n+1}-2t_k$$

To replenish the other $n-k$ hikers (i.e., Hikers $k+1, k+2, \ldots, n$) to full capacity d, Hiker k must share $\frac{1}{n-k}$ of his or her available supplies. Thus, t_k must satisfy the following equation

$$\frac{1}{n-k}\left[\frac{(n-k+2)d}{n+1}-2t_k\right]+(d-t_k)=d$$

$$\frac{(n-k+2)d}{n+1}-2t_k-(n-k)t_k=0$$

$$\frac{(n-k+2)d}{n+1}-(n-k+2)t_k=0$$

$$t_k=\frac{d}{n+1}$$

which completes the induction proof.

So, the last hiker takes $\frac{2dn}{n+1}$ days to complete the round trip.

As a test, consider the previous puzzle where $n=5$ and $d=7$. Using the generic formula, we get $\frac{2(7)(5)}{6}=\frac{70}{6}$ days which agrees with our previous result.

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Puzzle: Same as the previous puzzle with n hikers and each with d days of supplies to start, except that supplies can be deposited at various stopping points for later use. Again, the maximum carrying capacity of a hiker is d days of supplies.

Solution: For an optimal solution, Hiker 1 should stop at time t_1, resupply the other $n-1$ hikers, retain t_1 supplies for his or her return trip to base camp **and** deposit supplies for each of the $n-1$ hikers to return to base camp on their way back. When Hiker 1 returns to base camp, he or she will have 0 supplies left. These conditions can be expressed in the following equation:

$$2t_1 + (n-1)t_1 + (n-1)t_1 = 0$$

which implies that $t_1 = \frac{d}{2n}$.

Hiker 2 behaves in a similar manner to Hiker 1. Hiker 2 will stop at time $t_1 + t_2$, resupply the other $n-2$ hikers, retain t_2 supplies for his or her return to the previous stopping point (at t_1), and deposit supplies for each of the other $n-2$ hikers to return to the previous stopping point (at t_1). These conditions are represented as follows:

$$2t_2 + (n-2)t_2 + (n-2)t_2 = 0$$

which gives us $t_2 = \frac{d}{2(n-1)}$.

Continuing in this manner, we get that $t_i = \frac{d}{2(n-i+1)}$ for $i = 1,2,\ldots,n$.

Thus, the total number of days traveled by Hiker n (out and back) is

$$2(t_n + t_{n-1} + t_{n-2} + \cdots + t_2 + t_{1\_} =$$

$$2\left(\frac{d}{2} + \frac{d}{4} + \frac{d}{6} + \cdots + \frac{d}{2(n-1)} + \frac{d}{2n}\right) = d\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right)$$

[Author's Remark: We could have used the same line of reasoning to solve the two previous problems, but I decided to show both approaches.**]**

The possibility of depositing supplies does allow for a longer maximum trip. For example, consider our example with 5 hikers and 7 days of supplies, i.e., $n = 5$ and $d = 7$.

Without the ability to deposit supplies, we previously calculated the maximum trip duration to be $11\frac{2}{3}$ days.

With the ability to deposit supplies, the maximum trip length duration is $7\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = 7\left(\frac{137}{60}\right) = 15\frac{59}{60}$ days.

6.18 The Windmill Puzzle

Puzzle: Let S be a finite set of at least two points in a geometric plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line ℓ going through a single point P ∈ S. The line rotates clockwise around the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise around Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

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Sources:

- Problem #2 from The 52nd International Mathematics Olympiad, 2011 [51].
- The solution provided in the YouTube video entitled "The unexpectedly hard windmill question" [75] is excellent. The following is a brief summary of that solution.

Solution:

One key to the solution of this problem is how one defines the line going through the initial pivot point. The line should be drawn so that the number of points on either side differs by at most one.

The left-side of Figure 24 shows an example set of points, with an oriented (directed) line through the initial pivot point (i.e., the point with the heavy black outline and white center). (Orienting the line allows one to define the left and right side of the line.) As the line rotates about the pivot point, it touches another point which then becomes the new pivot (shown on the right-side of Figure 24). It is critical to note that for this example and in general, the number of points on the left and right side of the line remains constant.

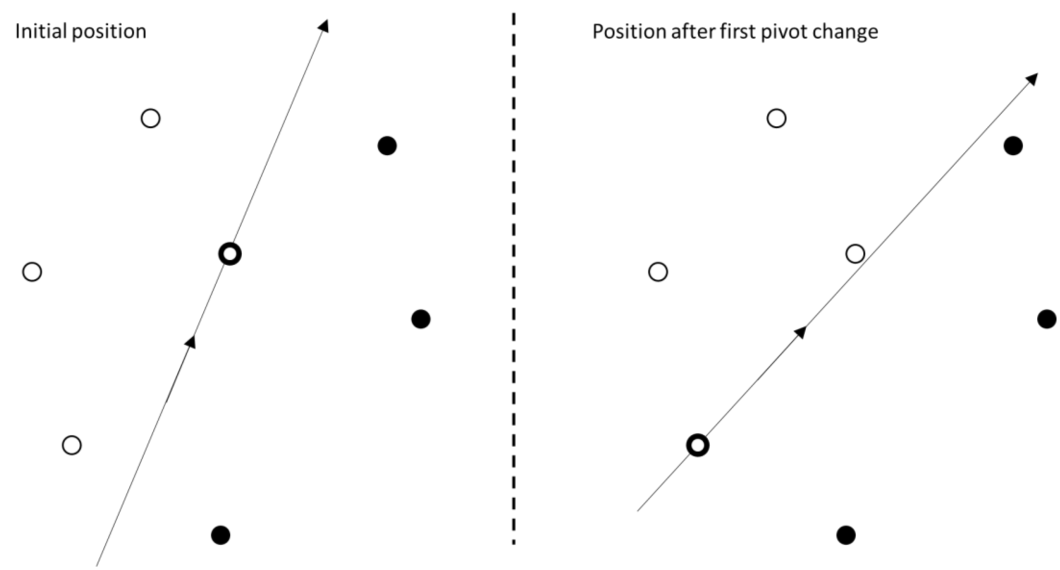

Figure 24. Windmill – initial position and first pivot change

Figure 25 shows the configuration of our example after the line has been rotated 180 degrees. All the points that were initially on the left side of the line are now on the right side, and vice versa. A point can only change from left to right of the line after it is touched by the line as part of a pivot change. Thus, all the points have been touched by the line after 180 degrees rotation, and the pattern continues indefinitely.

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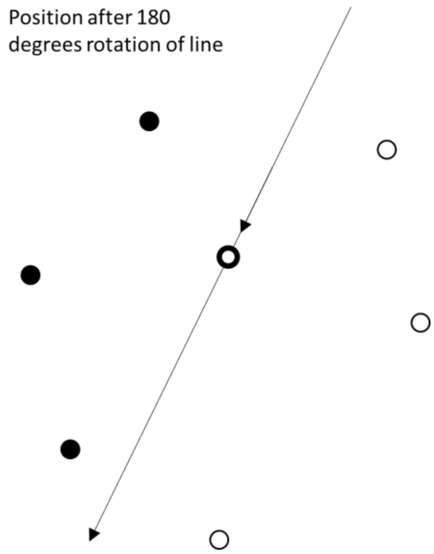


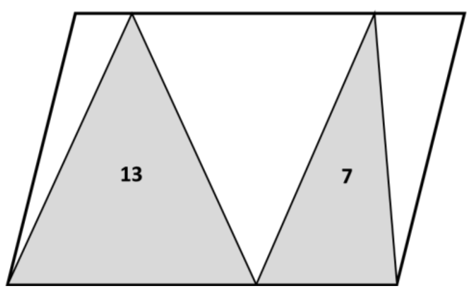
Figure 25. Windmill – after 180 degree rotation of line

The above statements are true in general, if one starts with an odd number of points.

If one starts with an even number of points, a 360 degree rotation of the line is required before the line touches all the points and returns to its initial position (but pointing in the opposite direction). Both cases are nicely illustrated in the YouTube video noted in the sources for this puzzle.

6.19 Parallelogram / Triangles Puzzles

Puzzle: Find the area of the parallelogram shown in the following figure.



Solution: Let b be the length of the base of the parallelogram, and h be its height. The area of the parallelogram is given by bh. Let the length of the base of the larger triangle be b_1 and that of the smaller triangle be b_2. So, $b = b_1 + b_2$.

The area of the larger triangle is $\frac{1}{2}b_1h = 13$ and the area of the smaller triangle is $\frac{1}{2}b_2h = 7$. By adding the areas of the two triangle, we get half the area of the parallelogram, i.e.,

$$20 = 13 + 7 = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}bh$$

Thus, the parallelogram has area 40.

. . .

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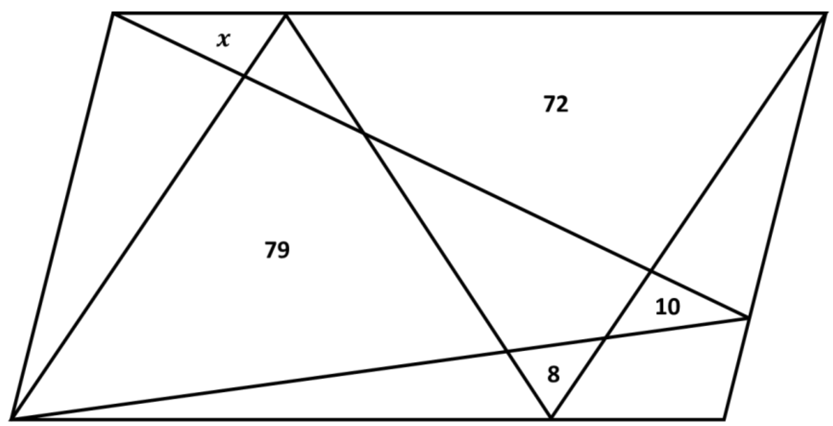
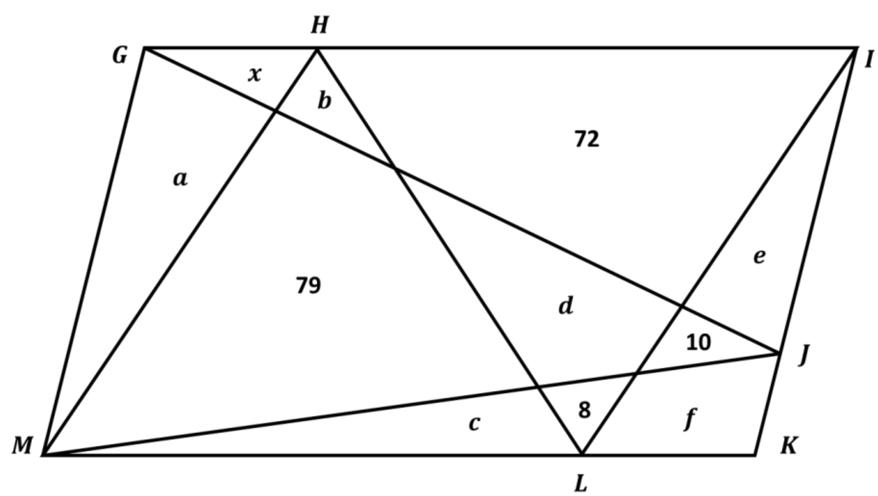
The following problem appears in several places on the Internet. The claim is that the puzzle was given to 5th grade students in China with the goal of determining students who were gifted in mathematics. Some of the students solved the puzzle in less than a minute. Keep the previous puzzle in mind when attempting to solve this puzzle.

Puzzle: Find the area of the triangular region marked with an x in the following figure (not drawn to scale). The numbers in the diagram indicate the area of the associated regions, e.g., 72 is the area of the surrounding quadrilateral.



Solution: Label the other regions in the parallelogram (lower case letters) and the boundary vertices (upper case letters) as shown in the figure below.



Using a similar analysis to that employed in the previous puzzle, we know that the sum of the areas of triangles GHM and HIL is equal to half the area of the entire parallelogram (call it y), i.e.,

$$(a + x) + (72 + d + 8) = y$$

Similarly, the area of triangle GJM is also have the area of the parallelogram, i.e.,

$$a + 79 + d + 10 = y$$

So, we have

$$(a + x) + (72 + d + 8) = a + 79 + d + 10$$

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Canceling like terms, we get

$$x = 79 + 10 - 72 - 8 = 9$$

. . .

Puzzle: Solve the previous puzzle using a different set of triangles.

Hint: Equate the areas of the two triangles MHL and LIK with the areas of the triangles GIJ and MKJ, and then solve the equation for x.

Solution: The equation noted in the hint is

$$(b + 79 + c) + (f + 10 + e) = (x + b + 72 + e) + (c + 8 + f)$$

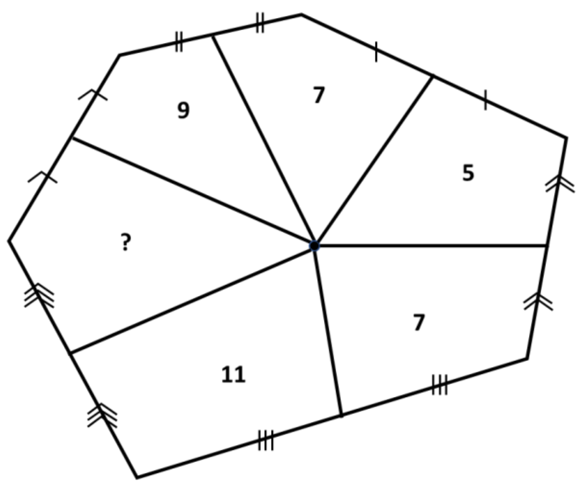
Canceling like terms, we get

$$79 + 10 = x + 72 + 8$$

$$x = 9$$

6.20 Polygon / Triangles Puzzle

Puzzle: In the diagram below, each line segment from the point within the polygon leads to the center point in one of the edges of the polygon. The little hatch-marks indicate segments of the same length. The areas of each region are shown, except for one region. Your task is to determine the area of the remaining region. The diagram is not drawn to scale with respect to the areas of the regions.



Solution: The solution involves a similar approach to that taken in the previous puzzles, i.e., take advantage of the fact that triangles with the same base and height have the same area.

In the following figure, we have added line segments from the point within the polygon to each of the vertices of the polygon. The critical idea here is that we now have a series of triangle pairs with the same area. The letters a, b, c, d, e and f indicate areas.

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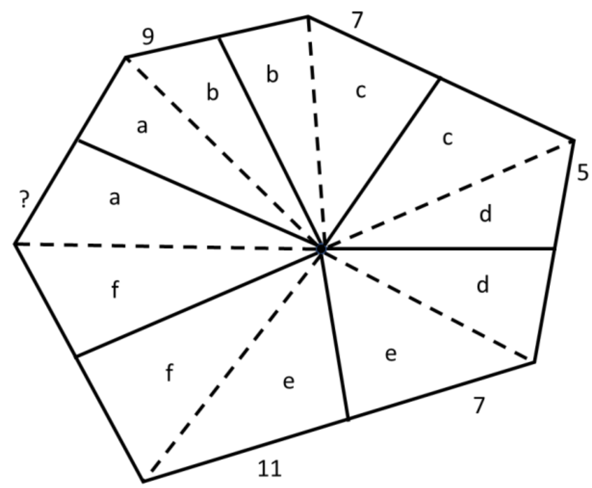

Making use of the area information given in the puzzle, we have the following equations:

$$a + b = 9$$

$$b + c = 7$$

$$c + d = 5$$

$$d + e = 7$$

$$e + f = 11$$

Adding the first, third and fifth equation above, and subtracting the second and fourth equation, we get

$$(a + b) - (b + c) + (c + d) - (d + e) + (e + f) = 9 - 7 + 5 - 7 + 11$$

$$a + f = 11$$

We are done since $a + f$ is the area of the region in question.

6.21 Block Stacking Puzzle

Puzzle: Given a collection of blocks each of 1 unit length, with evenly distributed mass of 1 mass unit, is it possible to stack them one above another until the top block extends entirely outside the bottom block by 2 units? An example stacking of blocks is shown in the figure below.

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Sources:

- Wikipedia article "Block-stacking problem" [76]
- Journal articles "Fun with stacking blocks" [77] and "Maximum overhang" [78]

Solution:

We assume it is sufficient for balancing to place the horizontal center of mass of the stacked blocks at the edge of the supporting table. Admittedly, this is barely a stable configuration but is the assumption typically made in various solutions to this type of puzzle.

The formula for the center of mass in one dimension is the sum of the products of each mass times its distance from the origin divided by the sum of the masses. An example is provided in Figure 26. In the example, we have five masses (labeled as m_1, m_2, m_3, m_4, m_5) and the distance of each mass from the origin (the various x_i terms).

$$\text{Center of Mass} = \frac{-m_5x_5 - m_4x_4 + m_1x_1 + m_2x_2 + m_2x_2}{m_5 + m_4 + m_1 + m_2 + m_3}$$

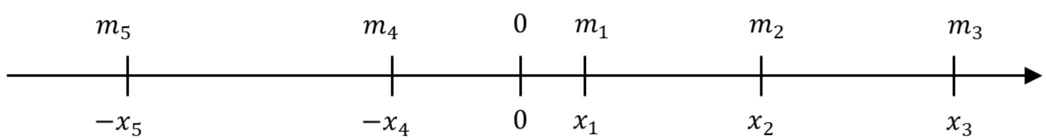
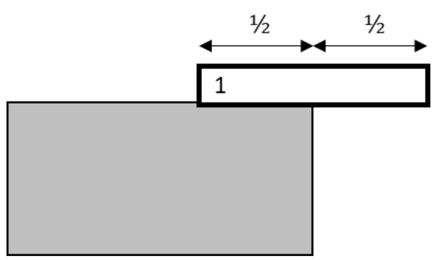


Figure 26. One dimensional center of mass example

For the problem at hand, we work from the top-down which is opposite of how one would actually stack the blocks. This approach makes computations easier, however.

Placement of the first (top) block on the table (or platform) is easy, i.e., we place the block such that half of the block extends beyond the table, see the figure below.



When adding a second block, there are two requirements, i.e., Block #1 needs to balance on Block #2, and the configuration of two blocks needs to balance at the edge of the table. We already know how to balance Block #1 on #2 (i.e., place Block #1 so that it extends halfway beyond the edge of Block #2, as shown in the following figure). Next, we need to compute the center of mass for the configuration (which we designate as the point 0 in the following figure). The center of mass for Block #1 is at x and the center of mass of Block #2 is at $x - \frac{1}{2}$. Applying the center of mass formula, we have

$$\frac{1 \cdot x + 1 \cdot \left(x - \frac{1}{2}\right)}{1 + 1} = 0$$

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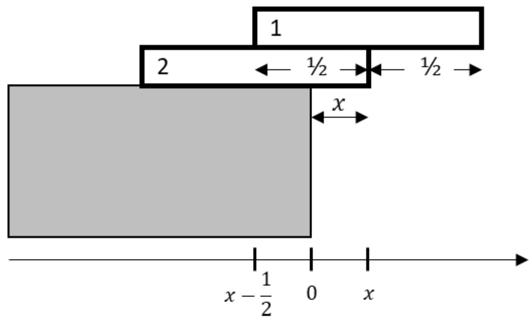
Solving for x, we get $x = \frac{1}{4}$.

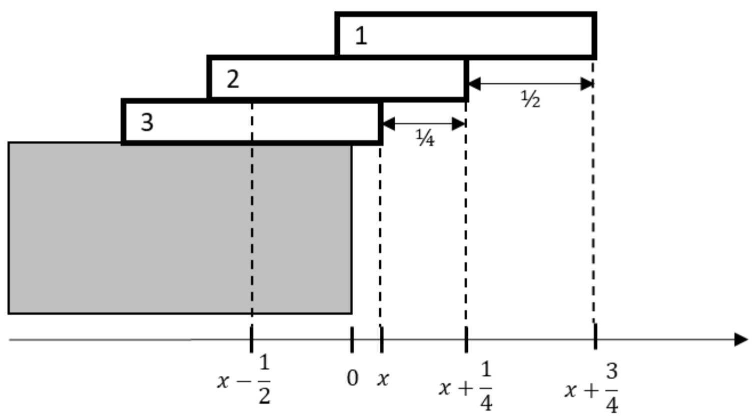
So, we extend Block #2 ¼ units beyond the edge of the table, as shown in the figure below.



Next, we insert a third block. We now have 3 requirements, i.e., balance Block #1 on Block #2, balance Blocks #1 and #2 on Block #3, and balance Blocks #1, #2 and #3 on the table. The first two requirements were solved in the 2-block scenario. To fulfill the third requirement, we need to place the configuration of Blocks #1, #2 and #3 such that their center of mass is a point 0 on the horizontal axis, as depicted in the figure below. We apply the center of mass formula, treating Block #1 and #2 as a unit with center of mass at x.

$$\frac{2x + \left(x - \frac{1}{2}\right)}{2 + 1} = 0$$

Solving for x, we get $x = \frac{1}{6}$. So, Block #3 is placed $\frac{1}{6}$ units beyond the edge of the table.



The pattern appears to be that Block n can be extended $\frac{1}{2n}$ units beyond Block $n+1$ while all the bricks balance on the table. We can prove this by mathematical induction. We've already proven the conjecture for $k = 1,2,3$. Assume the conjecture is true for $k = n$ and then prove the conjecture is true for $k = n + 1$. The situation is shown in Figure 27.

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By assumption the center of mass of the Blocks 1 through n (treated as one mass) is at x. The center of mass of Block $n+1$ is at $x-\frac{1}{2}$ relative to the origin at the right edge of the table. Applying the center of mass formula, we have

$$\frac{nx+\left(x-\frac{1}{2}\right)}{n+1}=0$$

Solving for x, we get $x=\frac{1}{2(n+1)}$ which is the desired result.

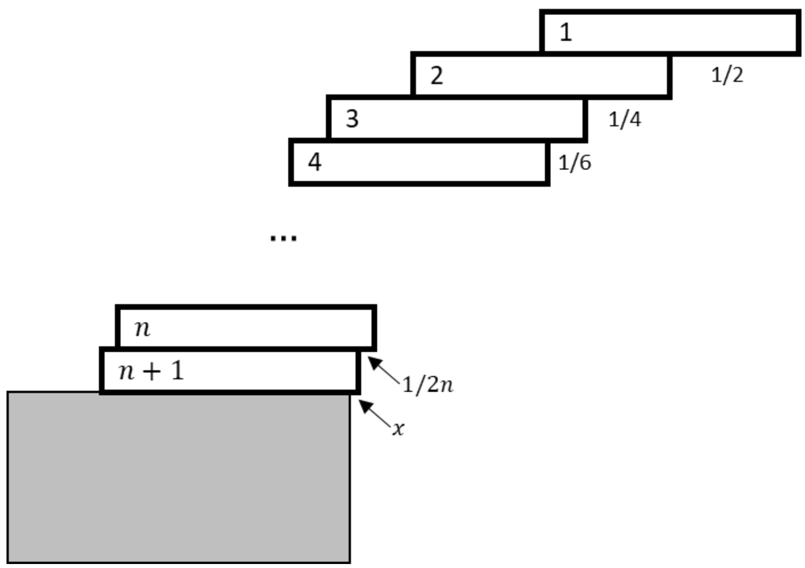

Figure 27. General block stacking problem

In summary, the extension beyond the table in the case of n blocks is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}=\frac{1}{2}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})$$

For $n=31$, the above summation exceeds 2 for the first time. So, we need 31 blocks to extend the configuration of stacked blocks 2 units beyond the edge of the table.

The infinite series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots$$

is known as the harmonic series, and it diverges to infinity. Thus, in theory, we could stack blocks such that they extend infinitely beyond the edge of the table.

In terms of stability, the above approach is tenuous since we assume balancing at the center of mass at each step. A small perturbation will cause the stack to collapse. Another approach is to reduce each extension by $\frac{1}{2}$. In this case, the extension beyond the table in the case of n blocks is

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\cdots+\frac{1}{4n}$$

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For $n = 31$, the configuration would only extend 1 unit beyond the end of the table but we would have a more stable situation.

Another issue is swaying from side to side which will eventually happen in practice as we add more blocks. This can be partially mitigated by wider and thinner blocks.

6.22 Min-max Puzzle

Puzzle: Two hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed, the shortest student is selected, and the tallest of these 20 (short) students is tagged A. These students now return to their initial places. Next, the tallest student in each row is selected, and from these 10 (tall) students, the shortest is tagged B. Which of the two tagged students is the taller (if they are different people)?

Source: USSR Mathematics Olympiad XV, 1952 [79]

[**Author's Remark**: This problem is listed as being given to 7th and 8th graders in the USSR Mathematics Olympiad. The problem is an example of a famous theorem first proven and published by the mathematician John von Neumann in 1928 as a key part of his foundational work on game theory. This leads me to ask "What were those in charge of the questions for the competition expecting to see from the 7th and 8th graders (even taking into consideration these children were likely gifted math students)?" Certainly, a proof of the general theorem was not expected.]

Solution: We present the solution more generally for an $n \times m$ matrix A whose entry in the row i and column j is denoted a_{ij}.

Theorem 10. *For an $n \times m$ matrix $A = [a_{ij}]$, the following inequality holds true*

$$\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} \ge \max_{1\le i\le n} \min_{1\le j\le m} a_{ij}$$

Equality holds if and only if there exist an element of A that is both the maximum in its column and the minimum in its row.

Before we get to the proof, an explanation of the notation used in the theorem is in order.

The expression $\min\limits_{1\le j\le m} \max\limits_{1\le i\le n} a_{ij}$ means that one should first determine the maximum element in each column and then select the minimum out of the column maximums. For example,

$$\min_{1\le j\le m} \max_{1\le i\le n} \begin{bmatrix} 7 & -1 & 5 \\ -2 & 0 & 3 \\ 4 & 3 & 1 \end{bmatrix} = \min_{1\le j\le m} [7 \quad 3 \quad 5] = 3$$

The expression $\max\limits_{1\le i\le n} \min\limits_{1\le j\le m} a_{ij}$ means that one should first determine the minimum element in each row and then select the maximum out of the row minimums. For example,

$$\max_{1\le i\le n} \min_{1\le j\le m} \begin{bmatrix} 7 & -1 & 5 \\ -2 & 0 & 3 \\ 4 & 3 & 1 \end{bmatrix} = \max_{1\le i\le n} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = 1$$

In words, the theorem states:

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> The minimum of the column maximums is greater than or equal to the maximum of the row minimums for a given matrix.

Since we apply the theorem to the **transpose of a matrix** [80], the following statement is also true.

> The minimum of the row maximums is greater than or equal to the maximum of the column minimums for a given matrix.

It is the latter version of the theorem that we need to solve the puzzle.

Proof:

Assume that $\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} = a_{xy}$ for some indices x and y. This implies that a_{xy} is the maximum element in column y of A. Every element in column y (i.e., $a_{iy}, i = 1,2 \dots, n$) may or may not be the smallest number in its row. However, the smallest element in a row cannot exceed any other element in that row, i.e.,

$$a_{iy} \ge \min_{1\le k\le m} a_{ik}, \qquad i = 1,2 \dots, n$$

Since, as noted, a_{xy} is the maximum element in column y of A, we have that

$$a_{xy} = \max_{1\le k\le n} a_{ky} \ge a_{iy}, \qquad i = 1,2, \dots, n$$

Combining the two previous results, we get

$$a_{xy} \ge \min_{1\le j\le m} a_{ik}, \qquad i = 1,2 \dots, n$$

Since the above inequality holds true for all values of i, we have

$$a_{xy} \ge \max_{1\le i\le n} \min_{1\le j\le m} a_{ij}$$

which proves the first part of the theorem.

The second part of the theorem is an “if and only if” statement, and so we need to prove the result in both directions. An example matrix with this property is

$$\begin{bmatrix} 7 & -1 & 5 \\ -2 & 0 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

Regarding the proof, we first assume that $\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} = \max_{1\le i\le n} \min_{1\le j\le m} a_{ij}$.

Let c_j be the maximum value in column j, and r_i be the minimum in row i.

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Next, define c_y and r_x as follows:

$$\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} = \min_{1\le j\le m} c_j = c_y$$

$$\max_{1\le i\le n} \min_{1\le j\le m} a_{ij} = \max_{1\le i\le n} r_i = r_x$$

Consider the element a_{xy} in A. By definition, we have

$$\max_{1\le i\le n} \min_{1\le j\le m} a_{ij} = r_x \le a_{xy} \le c_y = \min_{1\le j\le m} \max_{1\le i\le n} a_{ij}$$

Since we assumed $\min\limits_{1\le j\le m} \max\limits_{1\le i\le n} a_{ij} = \max\limits_{1\le i\le n} \min\limits_{1\le j\le m} a_{ij}$, it must be that $r_x = a_{xy} = c_y$, i.e., there is an element in A that is the maximum in its column and the minimum in its row.

Going in the other direction, assume there exists an element of A (say a_{xy}) that is the maximum in its column and the minimum in its row, i.e.,

$$c_y = \max_{1\le i\le n} a_{iy} = a_{xy} = \min_{1\le j\le m} a_{xj} = r_x$$

As before, let c_j be the maximum value in column j, and r_i be the minimum in row i.

By definition,

$$\max_{1\le i\le n} \min_{1\le j\le m} a_{ij} = \max_{1\le i\le n} r_i \ge r_x = c_y \ge \min_{1\le j\le m} c_j = \min_{1\le j\le m} \max_{1\le i\le n} a_{ij}$$

From the first part of the theorem, we know that

$$\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} \ge \max_{1\le i\le n} \min_{1\le j\le m} a_{ij}$$

So, we can conclude that

$$\min_{1\le j\le m} \max_{1\le i\le n} a_{ij} = \max_{1\le i\le n} \min_{1\le j\le m} a_{ij}$$

which completes the proof. ■

Long story, but we can now answer the puzzle with certainty, i.e., the student tagged B is taller.

Some examples may help to visualize the above concepts.

In Table 18, the value of the element in row 4 and column 1 is the largest number in its column and the smallest in its row. Further, as guaranteed by the theorem, the MinMax equals the MaxMin.

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Table 18. Min-Max example (equality)

					Min in Row
	1	2	3	4	1
	8	7	6	5	5
	9	10	11	12	9
	16	15	14	13	13
	17	18	19	20	17
Max in Column	17	18	19	20	

In Table 19, the MaxMin is 15 and the MinMax is 17. As expected, the MinMax is greater than the MaxMin, and there is no element in the matrix that is both the maximum in its row and minimum in its column.

Table 19. Min-Max example (inequality)

					Min in Row
	1	2	3	23	1
	8	7	6	5	5
	9	10	11	12	9
	16	15	14	13	13
	17	18	19	15	15
Max in Column	17	18	19	23	

. . .

There are more general statements of the max-min inequality, e.g., the following theorem applies to functions of two variables. Rather than max and min, the following theorem makes use of the concepts of supremum and infimum [82].

Theorem 11. *If X and Y are non-empty sets and $f: X \times Y \to \mathbb{R}$ then*

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

The notation $\mathrm{f}: \mathrm{X} \times \mathrm{Y} \to \mathbb{R}$ means that f is a function that maps each ordered pairs (x, y) to a real number where $x \in X$ and $y \in Y$.

Supremum (or just sup) means “least upper bound” and $\sup_{x \in X} f(x, y)$ means the least upper bound of $f(x, y)$ over all values of $x \in X$ and fixed y.

Infimum (or just inf) means “greatest lower bound”, and $\inf_{y \in Y} f(x, y)$ means the greatest lower bound of $f(x, y)$ over all values of $y \in Y$ and fixed x.

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Proof: We assume that the result is not true, i.e.,

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) > \inf_{y\in Y}\sup_{x\in X} f(x,y)\,, \qquad \text{(Expression 1)}$$

and then derive a contradiction.

By the definition of supremum (as applied to Expression 1), there must exist some $x' \in X$ such that

$$\inf_{y\in Y} f(x',y) > \inf_{y\in Y}\sup_{x\in X} f(x,y)\,, \qquad \text{(Expression 2)}$$

By the definition of infimum (as applied to Expression 2), there must exist some $y' \in Y$ such that

$$\inf_{y\in Y} f(x',y) > \sup_{x\in X} f(x,y')\,, \qquad \text{(Expression 3)}$$

Also, by the definitions of supremum and infimum, we have

$$f(x',y') \ge \inf_{y\in Y} f(x',y)\,, \qquad \text{(Expression 4)}$$

$$\sup_{x\in X} f(x,y') \ge f(x',y'), \qquad \text{(Expression 5)}$$

Combining Expressions 3, 4 and 5, we get the following contradiction

$$f(x',y') \ge \inf_{y\in Y} f(x',y) > \sup_{x\in X} f(x,y') \ge f(x',y')$$

Thus, our initial assumption is false and the theorem is proved. ■

For the conditions under which inequality in **Theorem 11** becomes an equality, see the Wikipedia article on the minimax theorem [81].

For example, consider the function $f(x,y) = x^2 - y^2$, for $-1 \le x \le 1$ and $-1 \le y \le 1$.

(Note that $\inf(-A) = -\sup(A)$ and $\sup(-A) = -\inf(A)$. Also, taking the sup with respect to x of a function in variable y treats the function as a constant, i.e.,

$$\sup_{x\in X} f(y) = f(y)$$

A similar statement can be about infimums.)

For the example at hand, we have

$$\sup_{x\in X}\inf_{y\in Y}(x^2 - y^2) = \sup_{x\in X}(\inf_{y\in Y} x^2 + \inf_{y\in Y}(-y^2)) = \sup_{x\in X} x^2 - \sup_{x\in X}\inf_{y\in Y}(-y^2)$$

$$= 1 + \inf_{y\in Y}(-y^2) = 1 - \sup_{y\in Y} y^2 = 1 - 1 = 0$$

$$\inf_{y\in Y}\sup_{x\in X}(x^2 - y^2) = \inf_{y\in Y}(\sup_{x\in X} x^2 + \sup_{x\in X}(-y^2)) = \sup_{x\in X} x^2 + \inf_{y\in Y}\sup_{x\in X}(-y^2)$$

$$= 1 + \inf_{y\in Y}(-y^2) = 1 - \sup_{y\in Y} y^2 = 1 - 1 = 0$$

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The point $(x, y, z) = (0,0,0)$ is known as a **saddle point**. The graph for $z = f(x, y) = x^2 - y^2$ is shown in Figure 28. The point $(0,0,0)$ is the minimum of $z = x^2$ and the maximum of $z = -y^2$. Figure 28 is available online at https://www.geogebra.org/3d/hvr44nxb where one can rotate the figure to get a better view of its shape.

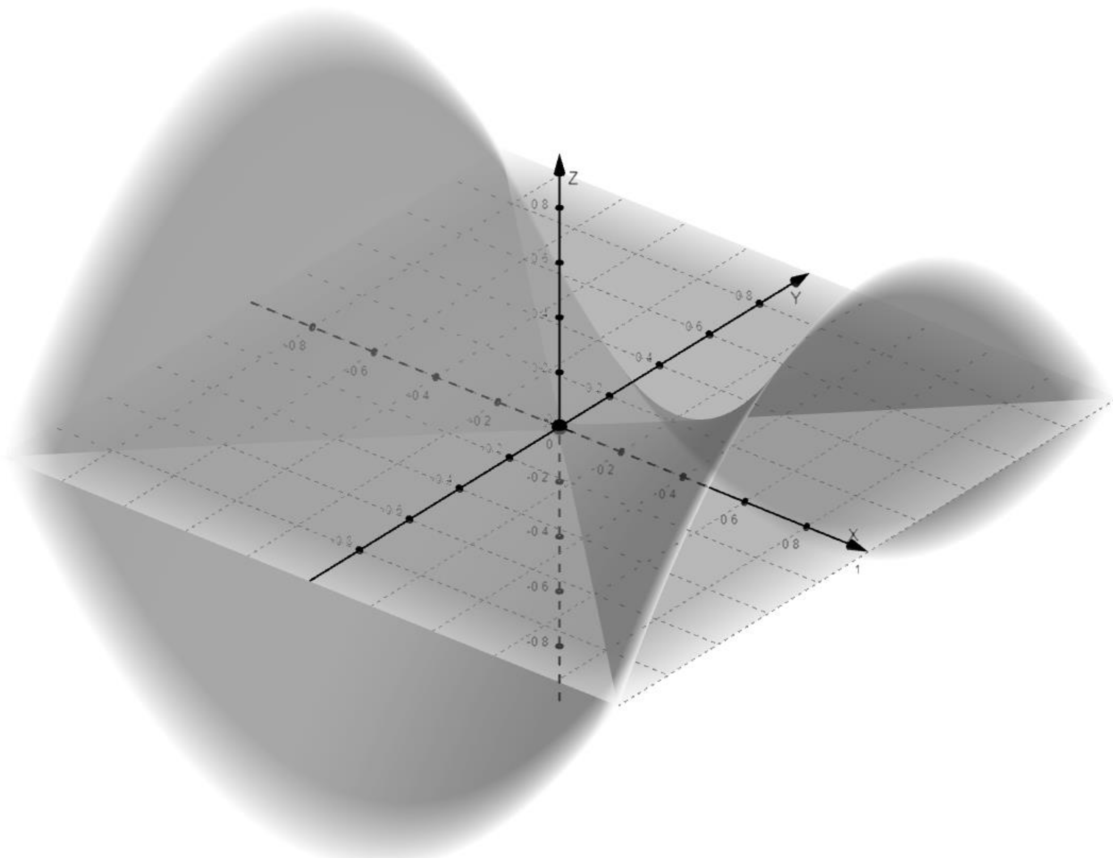

Figure 28. Saddle point

In general, the condition for equality for the expression in **Theorem 11** is the existence of a saddle point. This is similar to the matrix minimax problem for matrices where equality for the expression in **Theorem 10** requires that an element of the matrix be the maximum in its row and the minimum in its column. In fact, **Theorem 10** is a special case of **Theorem 11**, where X is a discrete variable representing the rows of matrix A, and Y represents the columns.

6.23 Probabilistic Coin

Puzzle: If a is an irrational number, $0 < a < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is a? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. By definition, a game is said to be finite if with probability 1 it must end in a finite number of moves.)

Sources:

- Problem A-4 from The 50th William Lowell Putnam Mathematical Competition, 2 December 1989.

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Solution: The difficulty in the puzzle arises from the fact that the decimal representation of an irrational number continues indefinitely with no repeating pattern.

Let the two players be A and B. We want a scheme where player A has probability a of winning, with player B having probability $1 - a$ of winning.

The key to solving the problem is to represent a in binary notation, i.e.,

$$a = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots = .b_1 b_2 b_3 \ldots$$

For each flip of the coin, we represent heads by 1 and tails by 0.

The coin continues to be flipped as long as the result of the flip matches the binary representation of a. When the result of the coin flip differs from the binary representation of a, player A wins if the result is 0 (i.e., tails) and loses if the result is 1 (i.e., heads). So, whenever the number constructed by the stated procedure is less than a, player A wins, and whenever the constructed number is greater than a, player B wins.

This solution works for all real values of $a \in [0,1]$.

6.24 Slappy Senators

Puzzle: 100 senators are initially seated peacefully at a round table (all facing towards the center). At the sound of a bell, each senator slaps the senator to his or her left with probability p or slaps the senator to his or her right with probability $1 - p$. What is the expected number of senators who have not been slapped?

Solution: A given senator (say senator A) is not slapped if the senator on the right of senator A slaps the senator on his or her the right (with probability $1 - p$), and the senator on the left of senator A slaps the senator on his or her the left (with probability p). So, senator A is un-slapped with probability $p(1 - p)$ and the expected number of un-slapped senators is $100p(1 - p)$.

For example, if $p = .5$, the expected number of un-slapped senators is $100(.5)^2 = 25$. If $p = 0$ or $p = 1$, all senators get slapped.

More formally (for those having some familiarity with probability theory), we let X_n be a random variable which equals 1 if senator n remains un-slapped and 0 is senator n gets slapped, for $n = 1,2, \ldots ,100$. So, $P(X_n = 1) = p(1 - p)$ and $P(X_n = 0) = 1 - p(1 - p)$.

The quantity that we seek is

$$E\left(\sum_{i=1}^{n} X_n\right) = \sum_{i=1}^{n} E(X_n) = \sum_{i=1}^{n} 1 \cdot p(1 - p) + 0 \cdot (1 - p(1 - p)) = \sum_{i=1}^{n} p(1 - p) = np(1 - p)$$

where $E(X)$ represents the expected value of random variable X.

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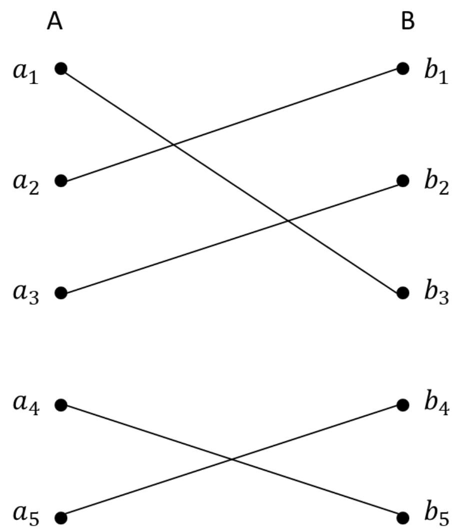
6.25 Matching Elements in Two Sets

Puzzle: Consider two sets A and B, each with n elements, such that each element of A is uniquely paired with an element in B. Without any knowledge of the pairings, you are given one guess at all the pairings between the two sets. What is the expected number of correct pairings in your guess?

The figure below shows an example pairing between the elements in sets A and B. In this example, $n = 5$.



Solution: To solve the problem, we need to determine the total number of possible pairings when sets A and B each have n elements.

Start with any element of set A and pair it with an element of set B. This can be done in n ways. Take another element of A and pair it with one of the $n - 1$ unpaired elements of B. This can be done in $n - 1$ ways. Continuing in this manner, we see that the number of possible pairings between the elements of A and B is $n!$.

Out of the total number of pairs between the two sets, how many of the pairings have the correct mapping for a given $a \in A$? If the element a is mapped to its proper pair in B, that leaves $n - 1$ elements in A and $n - 1$ elements in B which we can be paired in a total of $(n - 1)!$ ways. So, of the $n!$ total pairings, there are $(n - 1)!$ pairings that have the correct mapping for a given element in A. Thus, the probability of a given element being guessed correctly is

$$\frac{n!}{(n-1)!} = \frac{1}{n}$$

Let X_i be a random variable that equals 1 if $a_i \in A$ is mapped properly in our guess, and 0 if $a_i \in A$ is not mapped properly. So, $P(X_i = 1) = \frac{1}{n}$ and $P(X_i = 0) = 1 - \frac{1}{n}$. The expected value for X_i is

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{1}{n}$$

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The expected number of correct pairings in our guess is given by

$$E\left(\sum_{i=1}^{n} X_n\right) = \sum_{i=1}^{n} E(X_n) = n\left(\frac{1}{n}\right) = 1$$

So, independent of the size of n, the expected number of correct guesses is 1.

6.26 Dice Game

Puzzle: What is the best strategy for the following game?

You get three rolls of 2 dice at a time, with the option of stopping after any one of the three rolls and collecting the number of monetary units (e.g., US dollars) corresponding to the roll at which you choose to stop. To be clear, you only get the amount on the roll where you choose to stop.

Based on the strategy that you determine; would it make sense to play the game for 7.5 monetary units?

Solution: This is essentially an expected value problem. Let's start with the potential third roll and work backward.

If you have decided not to stop at the first or second roll, what is the expected value on the third roll of the 2 dice? To compute the expected value, we need to know the probability of rolling 2,3,4 … ,12 (shown in the following table).

Outcome of roll	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The expected value of the third roll is

$$\frac{1\cdot 2}{36}+\frac{2\cdot 3}{36}+\frac{3\cdot 4}{36}+\frac{4\cdot 5}{36}+\frac{5\cdot 6}{36}+\frac{6\cdot 7}{36}+\frac{5\cdot 8}{36}+\frac{4\cdot 9}{36}+\frac{3\cdot 10}{36}+\frac{2\cdot 11}{36}+\frac{1\cdot 12}{36}=\frac{252}{36}=7$$

Knowing that the expected value of the third roll is 7, the strategy for the second roll should be

- stop if your roll is 7,8,9,10,11 or 12
- do a third roll if your roll is 1,2,3,4,5 or 6

The probability of rolling a 2,3,4,5 or 6 is $\frac{15}{36}$ (using the table above). The expected value of the game if one rolls a 2,3,4,5 or 6 on the second roll is $\frac{15}{36}$ times the expected value of the third roll, i.e., $\frac{15}{36}\cdot$ 7.

The expected value for the second roll is

$$\left(\frac{15}{36}\cdot 7\right)+\left(\frac{6\cdot 7}{36}+\frac{5\cdot 8}{36}+\frac{4\cdot 9}{36}+\frac{3\cdot 10}{36}+\frac{2\cdot 11}{36}+\frac{1\cdot 12}{36}\right)=\frac{267}{36}\cong 7.4167$$

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For the first roll (knowing that the expected value of the second roll is about 7.4167), you should stop if you roll 8,9,10,11 or 12, and roll again if you roll 2,3,4,5,6 or 7 (which happens with probability $\frac{21}{36}$).

The expected value of the first roll (and for the game as a whole) is

$$\left(\frac{21}{36}\cdot\frac{267}{36}\right)+\left(\frac{5\cdot 8}{36}+\frac{4\cdot 9}{36}+\frac{3\cdot 10}{36}+\frac{2\cdot 11}{36}+\frac{1\cdot 12}{36}\right)=\left(\frac{21}{36}\cdot\frac{267}{36}\right)+\frac{140}{36}\cong 8.2153$$

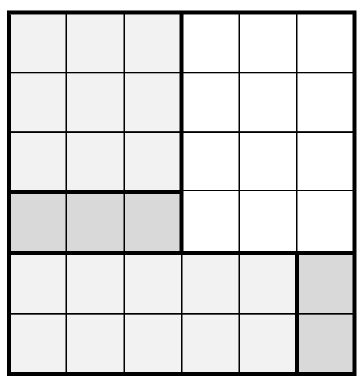
So, it would make sense to play the game for 7.5 monetary units.

6.27 Mondrian Art Puzzles

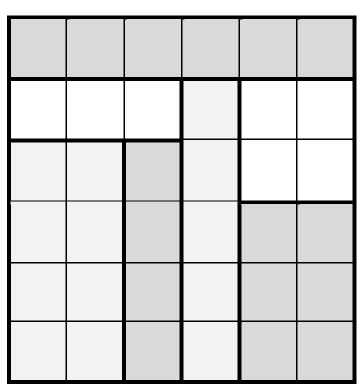
There is a collection of geometric-related puzzles based loosely on the art of Piet Mondrian. These are known as the Mondrian Art Puzzles [83][84][85]. The puzzle is defined as follows

> Partition an $n \times n$ square into multiple non-congruent integer-sided rectangles. Find the least possible difference between the largest and smallest area.

The figure below is an example partitioning of a 6×6 square. The difference between the area of the largest rectangle ($3 \cdot 4 = 12$) and the smallest rectangle ($1 \cdot 2 = 2$) is 10.



The minimal solution for a 6×6 square is 5. The associated partitioning is shown in the following figure. The largest rectangle has area $2 \cdot 4 = 8$ and the smallest rectangle has area $1 \cdot 3 = 3$.



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Clearly, the above arrangement is not unique since one can move the rectangles within the same partitioning, e.g., we could move the 1×5 to the extreme right or extreme left.

Puzzle: The reader may want to determine the optimal solution for a 7×7 square (known to be 5) or an 8×8 square (known to be 6).

Optimal solutions for $n = 3 - 57$ can be found at https://oeis.org/A276523/a276523.txt. Optimal solutions for $n = 58 - 65$ are available at https://oeis.org/A276523/a276523_1.txt. Both sources use letters of the alphabet to effectively color the partitions of the square.

Currently, there is no known formula that gives the optimal value for a given size square nor is there a general procedure for finding the optimal partitioning for a given size square (other than an exhaustive brute force approach). See the OEIS article on this topic [85] for the current state of knowledge concerning this problem.

6.28 Making the Row and Column Sums of a Matrix Non-negative

Puzzle: Given any $n \times m$ matrix of real numbers, consider the row sums and column sums (some of which may be negative). Assuming you are allowed to change the sign of all elements in a row or column (any number of times), show that it is possible to make all the row and column sums non-negative.

For example, consider the following 4×4 matrix (also showing its row and column sums in the right column and bottom row, respectively). The sum of all the elements in the matrix is shown in the bottom right cell (-4 in the table below).

-2	1	3	-4	-2
3	-4	4	-3	0
-1	5	-2	-3	-1
-4	2	7	-6	-1
-4	4	12	-16	-4

Flipping the signs of the elements in the 4th column, and recalculating the row and column sums, we get

-2	1	3	4	6
3	-4	4	3	6
-1	5	-2	3	5
-4	2	7	6	11
-4	4	12	16	32

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The first column still adds to a negative sum. So, let's flip the signs of that column.

2	1	3	4	10
-3	-4	4	3	0
1	5	-2	3	7
4	2	7	6	19
4	4	12	16	36

We are done since all the row and column sums are non-negative.

Solution: For the general problem, the issue is that when flipping the signs of all elements in a row (or column) one may cause some of the column sums (or row sums) to become negative. So, how do we know the process will stop at some point with all the row and column sums being non-negative? The key is to consider the sum of all the elements in the matrix. Whenever we flip a row or column sum from negative to positive (say from $-x$ to x), the overall sum of all elements in the matrix increases by $2x$. This is true since the sum of the row sums equals the sum of all the elements in the matrix (same is true for the sum of the column sums).

However, this process will stop at some point, since at most, the sum of all the elements of the matrix (after any sequence of flips) is the sum of the absolute value of each element in the matrix.

6.29 Infected Grid

Puzzle: Initially, some of the cells of an $n \times n$ grid are infected with a malady that spreads. In particular, if an uninfected cell shares at least two edges with infected cells, it will also become infected. The infection does not spread via "diagonal" contact. Once a cell is infected, it remains infected. It may help to think of the infection as spreading in unit time intervals (steps). Prove that it is not possible to infect the entire grid if less than n cells are initially infected.

For example, the grid below shows one possible initial configuration (infected cells are in gray).

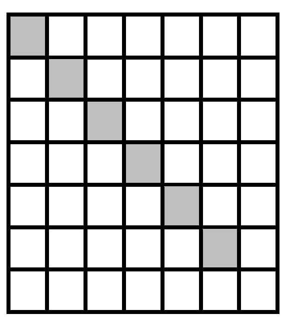

Figure 29. Infected grid – Example 1

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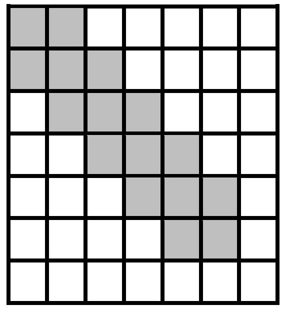
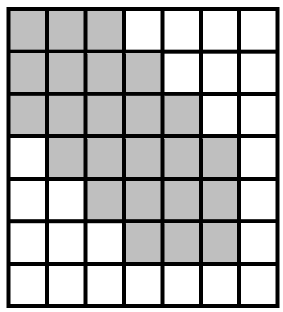
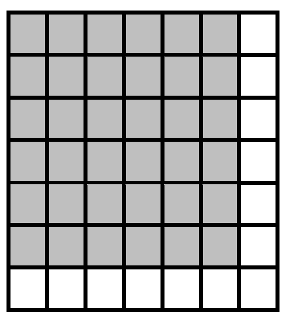
After one step, the following figure shows that more cells are infected.



After two steps, we have



After three more steps (figure below), the infection stops short of infecting the entire grid.



[Author's Remark: This puzzle can be found on several Internet sites. There is a claim (which I have not been able to verify) that the puzzle originated in the Kvant journal from 1986. Back issues of this journal are available online in Russian but without any search capability, see https://archive.org/details/kvant-journal/Kvant1986/ or http://kvant.mccme.ru/1986/index.htm).**]**

Solution: Consider the perimeter of the infected region at each step in our example. Initially, the perimeter is 24. After step 1, the perimeter of the infected region is still 24. When the infection stops spreading, the perimeter is still 24. However, it is clear that an infection of the entire grid would result in a perimeter of 28.

In any case, the perimeter of the infected area cannot increase from one step to the next. To see this, consider a newly infected cell.

- At least two of its edges become internal to the infected region.
- At most two of its edges become part of the new boundary for the infected region.

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Thus, any increase in the perimeter of the infected region is not possible.

For an $n \times n$ grid to be fully infected, the perimeter of the infected region must be $4n$. If the initial infected region has less than n cells (i.e., $n - 1$ cells or less), then the initial perimeter of the infected region is $\leq 4(n - 1)$ and since the perimeter of the infected region cannot increase, there is no way to infect the entire grid if we start with less than n infected cells. Further, even if we start with n or more infected cell, their perimeter must be at least $4n$ for the entire grid to eventually be infected. As we shall see, this condition (i.e., $4n$ infected perimeter) is not sufficient.

For example, consider a 7×7 grid ($n = 7$) with the initial configuration shown in the following figure. There are initially $9 > n$ infected cells but no further infection is possible. Also, note that the perimeter of the initially infected region is $12 < 4 \cdot 7 = 28$.

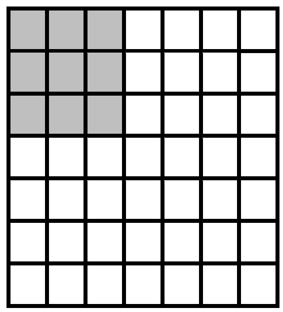

Figure 30. Infected grid – Example 2

In the following figure, there are 16 infected cells, and an infected region with a perimeter of 32, but the infection will stop at 36 infected cells.

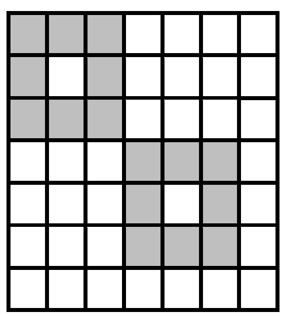

Figure 31. Infected grid – Example 3

After one step, we have the configuration in the following figure. At this point, the infected region has a perimeter 24 which is less than the necessary condition of 28. Further, we know that the perimeter cannot increase, and so, the infection will stop before the entire grid is infected. Thus, in general (since we have found counterexample), having a starting perimeter at least $4n$ is not a sufficient condition for the entire grid eventually becoming infected.

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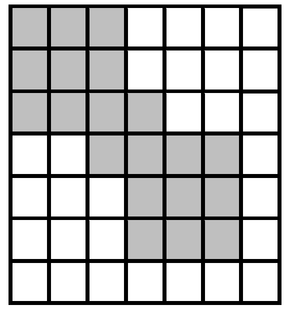

Next, we use another approach based on compartments, where a compartment of a grid is defined to have the following characteristics:

- It is a rectangular region that contains infected cells.
- The cells within a compartment are not isolated, meaning that each cell in a compartment either shares an edge or vertex with another infected cell within the compartment. This means the compartment will eventually become completely infected.
- There are no cells (external to the compartment) that touch the border of the compartment.

For example, the grid below has two compartments, i.e., the two heavy rectangles.

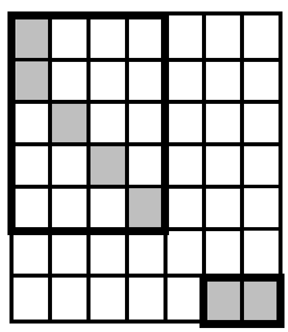

Figure 32. Infected grid with two compartments

If a grid contains one compartment, either it covers the entire grid (as in Figure 33) and the entire grid will eventually become infected, or it doesn't (as for the grids in Figure 29, Figure 30 and Figure 31) and the grid will not become completely infected.

The grid in Figure 33, is completely covered by one compartment and based on our rule, it will become completely infected.

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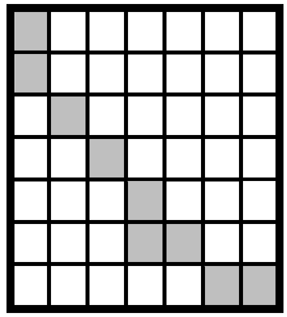
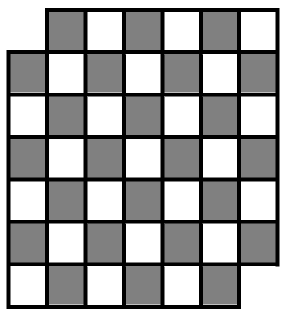


Figure 33. Infected grid with one compartment

6.30 Dominoes on Chessboard

Puzzle: If two diagonally opposite corners are removed from an 8×8 chessboard (as shown in the following figure), is it possible to place 31 dominoes of size 2×1 to cover all of these squares of the altered chessboard? To be clear, each domino is placed on the board to cover exactly two cells.



Source: "The mutilated chessboard problem" is attributed to Max Black [86].

Solution: A standard chessboard has 64 cells, with 32 white and 32 black cells. The altered chessboard has 62 cells, with 30 white and 32 black cells. No matter how you place a domino on the board, it will cover one black and one white cell. So, when any number of dominoes are placed on the board, the same number of white and black cells are covered, and thus, there is no way to cover the altered board with dominoes.

6.31 Self-referential Sentences

Puzzle: Fill in the blanks in the following sentence to make the sentence true:

> In this sentence, the number of occurrences of 0 is __, of 1 is __, of 2 is __, of 3 is __, of 4 is __, of 5 is _, of 6 is __, of 7 is __, of 8 is __, and of 9 is _.

Source: Douglas Hofstadter includes this puzzle in Section 2 of his book "Metamagical Themas" [87]. Hofstadter notes that the puzzle originated from logician Raphael Robinson.

Solution: The puzzle can be solved by trying various entries in the blanks and by making adjustments until one eventually finds a combination that makes the sentence true (basically, trial and error). This is the approach taken in the YouTube video entitled "Mind-bending Logic Puzzle – The Self-Counting Sentence" [88].

You can also use an iterative approach which is what we will do here. The approach is as follows:

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- Make an initial attempt at a solution, e.g., $(1,1,1,1,1,1,1,1,1,1)$ where the first position in the array represents the number of zeros, the second position represents the number of ones, and so on. To be clear, the initial attempt is not expected to be correct in the sense of making the sentence true.
- In the second step, one counts the number of occurrences of each digit with the initial guess substituted into the sentence.
- In the third step, one counts the number of occurrences of each digit with the result of the second step substituted into the sentence.
- One continues until the resulting array repeats. At which point, a solution has been found.

In the table below, the top row represents each of the digits, and the second row is the initial array. The third row is the number of digits in the sentence with the initial array substituted into the sentence. The iteration continues until the rows repeat. When the repeated row $(1,11,2,1,1,1,1,1,1,1)$ is substituted into the sentence, the sentence is true.

0	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
1	11	1	1	1	1	1	1	1	1
1	12	1	1	1	1	1	1	1	1
1	11	2	1	1	1	1	1	1	1
1	11	2	1	1	1	1	1	1	1

There is a second solution which we can obtain by starting with a different initial array (as shown in the following table). The repeated row $(1,7,3,2,1,1,1,2,1,1)$ does, in fact, result in a true sentence.

0	1	2	3	4	5	6	7	8	9
5	1	9	7	9	4	2	2	2	3
1	2	4	2	2	2	1	2	1	3
1	4	6	2	2	1	1	1	1	1
1	7	3	1	2	1	2	1	1	1
1	7	3	2	1	1	1	2	1	1
1	7	3	2	1	1	1	2	1	1

The reader may want to experiment with various initial arrays to see which ones converge to one of the two solutions. For some initial arrays, the iteration does not converge but rather oscillates between two arrays, e.g., see the example below.

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0	1	2	3	4	5	6	7	8	9
1	12	123	1234	12345	12345	1234	123	12	1
1	11	9	7	5	3	1	1	1	1
1	8	1	2	1	2	1	2	1	2
1	6	5	1	1	1	1	1	2	1
1	8	2	1	1	2	2	1	1	1
1	7	4	1	1	1	1	1	2	1
1	8	2	1	2	1	1	2	1	1
1	7	4	1	1	1	1	1	2	1
1	8	2	1	2	1	1	2	1	1

While the above example does not converge to a single solution for the original problem, it does provide an example of a pair of sentences that makes true statements about each other, i.e.,

> In the following sentence, the number of occurrences of 0 is 1, of 1 is 7, of 2 is 4, of 3 is 1, of 4 is 1, of 5 is 1, of 6 is 1, of 7 is 1, of 8 is 2, and of 9 is 1.
>
> In the previous sentence, the number of occurrences of 0 is 1, of 1 is 8, of 2 is 2, of 3 is 1, of 4 is 2, of 5 is 1, of 6 is 1, of 7 is 2, of 8 is 1, and of 9 is 1.

6.32 Party of Truth-tellers and Liars

Puzzle: There are N people at a party. Each person is either a liar (always tells lies) or a truth teller (always tells the truth). After the party is over, each person is asked the question:

> "How many truth tellers did you shake hands with?"

Each person gave a different answer to the question, ranging from 0 to $N-1$ (i.e., the answers were $0, 1, 2, \dots, N-1$). How many liars were at the party?

Solution: For easy of discussion, we label the people at the party as follows:

> Person $N-1$ gave the answer $N-1$
>
> Person $N-2$ gave the answer $N-2$
>
> …
>
> Person 0 gave the answer 0

We start by assuming that Person $N-1$ is a truth-teller. Consider Person 0. There are two cases, i.e., Person 0 is a truth-teller or a liar.

- Person 0 is a liar: In this case, Person $N-1$ is also lying since he or she could have shaken hands with at most $N-2$ truth-tellers if Person 0 is a liar.
- Person 0 is a truth teller: In this case, Person $N-1$ did not shake hands with Person 0 and thus, could not have shaken hands with $N-1$ truth-tellers. Again, Person $N-1$ is a liar.

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So, in either case, Person $N-1$ is lying and since liars lie all the time, it must be that Person $N-1$ is a liar.

Next, we assume Person $N-2$ is a truth-teller and again, consider the two cases associated with Person 0.

- Person 0 is a liar: In this case, Person $N-2$ is also lying since he or she could have shaken hands with at most $N-3$ truth-tellers if Person 0 is a liar (and we have already established that Person $N-1$ is a liar).
- Person 0 is a truth teller: In this case, Person $N-2$ did not shake hands with Person 0 and thus, could not have shaken hands with $N-2$ truth-tellers. Again, Person $N-2$ is a liar.

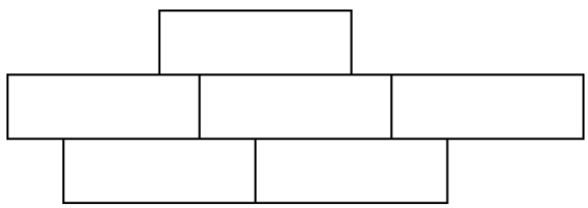
So, we now know that Person $N-2$ must also be a liar.

We can continue this process, showing that Persons $N-3, \dots 2,1$ are liars.

This means there is no way that Person 0 could have shaken hands with a truth-teller. Thus, Person 0's statement is true, and we can conclude that Person 0 was the only truth-teller at the party.

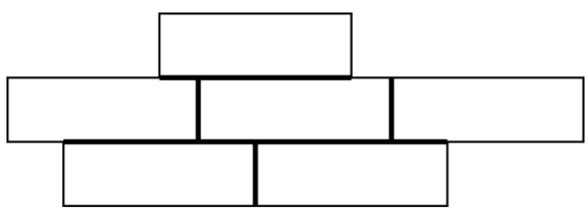
6.33 Tricky Perimeter Puzzle

Puzzle: The figure below consists of six rectangles of the same dimensions. Each rectangle has a perimeter 24 inches. What is the perimeter of the exterior of the configuration?



Solution: Let the length of each rectangle be l and the height be h.

One approach is to count all the lengths and heights, and then subtract the length and heights of the interior overlapping segments (see the heavy lines in the figure below).



The total of the lengths is $12l$ and the total of the heights is $12h$.

Noting that we need to double count the overlapping segments, the sum of the lengths of the interior segments is $6l$ and the sum of the heights of the interior segments is $6h$.

Subtracting the interior segments from the total, we are left with the measurements for the exterior of the configuration, i.e., $6l$ by $6h$, but this is exactly the same as 3 rectangles. Thus, the exterior perimeter of the configuration is $3 \times 24 = 72$ inches.

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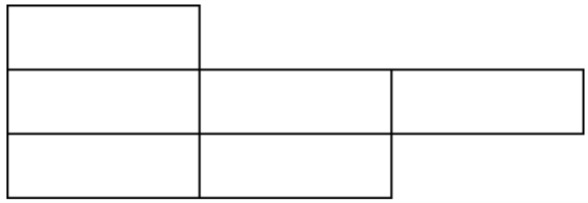
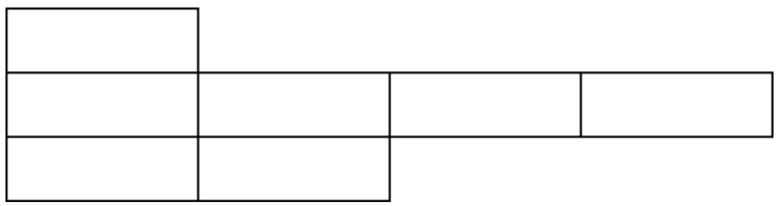
There is another approach. If we left-justify the rectangles, the exterior perimeter does not change (see the figure below). For the adjusted figure, it is easy to see that the exterior perimeter is $6l + 6h$ which is the same as the perimeter of 3 rectangles, i.e., 72 inches.



If the number of the interior overlapping lengths is not equal to the number of interior overlapping heights (e.g., see the figure below), then there does not appear to be any way of finding the perimeter of the overall figure with the given information.



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Acronyms

aka – also known as

AM-GM – Arithmetic Mean – Geometric Mean

GCD – Greatest Common Divisor

LCM – Least Common Multiple

USSR – Union of Soviet Socialist Republics

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Symbols

$\forall$ - "for every"

$\exists$ - "there exists"

$\in$ - "is an element of", e.g., $x \in A$ means that x is an element of set A

$\ni$ - "such that"

$\Rightarrow$ - "implies that"

$A \subset B - A$ is a proper subset of B where "proper" means A cannot equal B

$A \subseteq B - A$ is a subset of B where A can possibly equal B

$a|b$ – this means that integer a divides integer b exactly, e.g., $3|15$.

$\mathbb{N}$ - Natural numbers, i.e., the set {1,2,3,…}. In some definitions, 0 is included.

$\mathbb{Q}$ - Rational numbers, i.e., fractions (both positive and negative)

$\mathbb{R}$ - Real numbers

$\mathbb{Z}$ - Integers, i.e., the set $\{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Z}_n$ – Set of congruence class modulo n

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