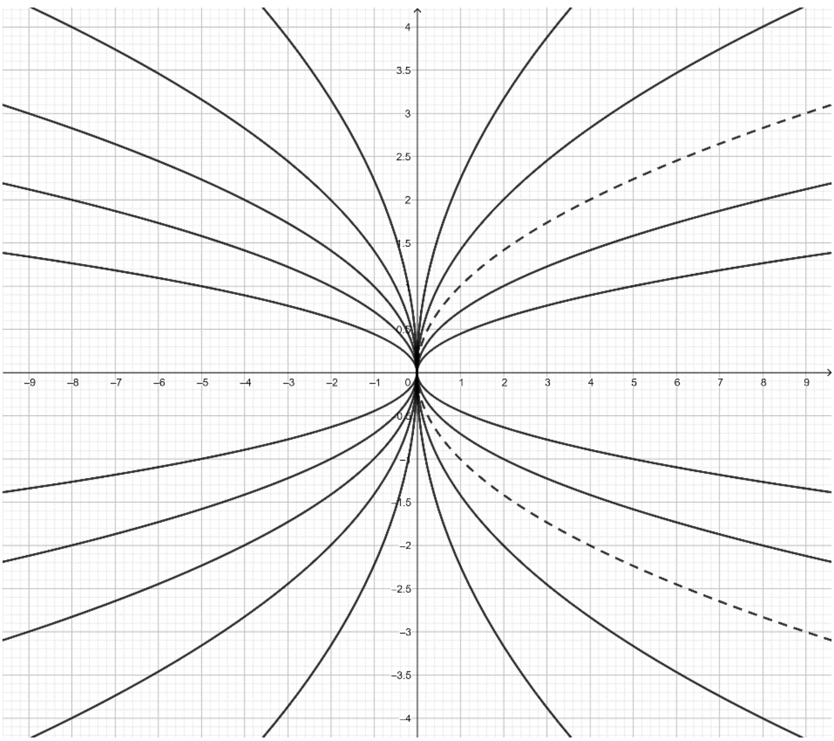
Algebra through Discovery and Experimentation



by Stephen Fratini

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Preface

This book covers the fundamentals of algebra, with an emphasis on discovery and exploration which is facilitated by online tools such as spreadsheets, symbolic computation engines and graphing tools. While I believe it is important for the student to learn the details of algebra, I also believe that the student should be aware of and able to effectively use the wealth of online (and mostly free) resources that can help in solving algebra problems.

In my view, algebra (to an extent) is arithmetic with variables. To be sure, the introduction of variables complicates things, but it is critical that the student have a good understanding of arithmetic before engaging in the study of algebra. For this reason, I have dedicated several sections of the book on numbers, arithmetic and sequences. This is followed by a short section describing the transition from arithmetic to algebra.

Dedications

I dedicate this book to all my mathematics teachers from grade school, high school, college and graduate school. I would not be in a position to write this book and my other mathematics books if it were not for them.

Stephen Fratini
Sole Proprietor of The Art of Managing Things
Eatontown, New Jersey (USA)
Email: sfratini@artofmanagingthings.com
LinkedIn: www.linkedin.com/in/stephenfratini

Other books by the author:

- *The Art of Managing Things (2nd edition)*, self-published on Amazon, https://www.amazon.com/Art-Managing-Things-Stephen-Fratini-ebook/dp/B07N4H4YWH/, January 2019.
- *Mathematical Thinking: Exercises for the Mind*, self-published on Amazon, https://www.amazon.com/Mathematical-Thinking-Exercises-Stephen-Fratini-ebook/dp/B08F75CDD6/, August 2020.
- *Financial Mathematics with Python*, self-published on Amazon, https://www.amazon.com/gp/product/B08VKQR141, February 2021.
- *Math in Art, and Art in Math*, self-published on Amazon, https://www.amazon.com/dp/B091D1F8MB, March 2021.

1 Introduction

1.1 Purpose

The purpose of the book is to introduce (or reintroduce) the reader to the subject of algebra, with an emphasis on online tools to facilitate discovery and experimentation.

1.2 Audience

The audience includes those learning algebra for the first time as well as folks looking to refresh their knowledge of algebra. Very little is assumed in terms of prerequisites.

1.3 Section Summary

- Section 1 is this introduction.
- Section 2 provides a classification of real numbers and brief introduction to number theory (e.g., primes, factorization, greatest common divisors and least common multiples).
- Section 3 is about number bases and conversion between bases.
- Section 4 provides a recap of arithmetic.
- Number sequences are discussed in Section 5.
- The transition from arithmetic to algebra is described in Section 6.
- Section 7 is the first section on algebra. It covers linear expressions.
- Polynomials of a single variable are covered in Section 8. There is a brief introduction to functions in Section 8.3.
- Polynomials with multiple variables are discussed in Section 9. The main focus is on conic sections, i.e., circles, ellipses, parabolas and hyperbolas.
- In Section 10, we cover functions in more detail. The focus is on the absolute value, logarithmic, exponential and trigonometric functions.
- Radical expressions are discussed in Section 11.
- Inequalities are covered in Section 12. The focus is on graphing the regions represented by inequalities.

1.4 Conventions

Multiplication is denoted several ways, e.g.,

- 2 times 3 can be represented as $2 \cdot 3$ or in some case using parenthesis such as $(2)(3)$.

- In some cases, juxtaposition is used, e.g., $3x$ means the number 3 times the variable x.

Division is represented by a slash, division sign or by one expression over another, e.g., 2/3, $2 \div 3$ and $\frac{2}{3}$ all represent 2 divided by 3.

Variables are usually chosen from x, y or z, with exceptions noted in the text.

Constants are usually chosen from a, b, c and d, with exceptions noted in the text.

In two-dimensional graphs, the horizontal axis is usually denoted as the x-axis, and the vertical axis is usually denoted as the y-axis. In three-dimensions, the additional axis is usually denoted as the z-axis.

The letters f, g and h are usually used for functions.

1.5 Online Tools

As noted in the preface, several online tools are used in this book.

Spreadsheets are used in the sections on numbers and arithmetic. Two online spreadsheets are referenced, i.e., Google Sheets (https://www.google.com/sheets/about/) and Microsoft Excel online (https://www.microsoft.com/en-us/microsoft-365/free-office-online-for-the-web).

Symbolic computation tools which can solve algebraic expressions are used in several places in this book. The two referenced tools are Wolfram Alpha (https://www.wolframalpha.com) and Symbolab (https://www.symbolab.com). Both of these tools offer some capabilities for free but charge a subscription fee if you want to see the intermediate steps that arrive at the final solution of a problem.

For graphing, we have made extensive use of GeoGebra (https://www.geogebra.org) and Desmos (https://www.desmos.com).

If you are aware of other tools that do a better job than those mentioned above (and are preferably free), please send an email to the author of this document (sfratini@artofmanagingthings.com).

2 Numbers

2.1 Overview

The numbers encountered in this book are all members of a large set known as the **real numbers**. In this section, various subsets of the real numbers are described, and a few essential properties are noted.

2.2 Natural Numbers

The **natural numbers** (or counting numbers) are $0, 1, 2, 3, 4, 5, \ldots$

The "…" means the pattern continues forever.

2.3 Integers (Whole Numbers)

The **integers** (or whole numbers) include all the natural numbers and the negatives of the natural numbers, i.e., the set

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

For example, the negative numbers can be used to represent debits on a bank account statement, money owed or temperatures below zero in the Celsius temperature scale.

2.4 Factorization

Some natural numbers can be factored into the product of several small natural numbers (not including 1), e.g., $60 = 2 \cdot 2 \cdot 3 \cdot 5$. Such numbers are known as **composite numbers**. Numbers that cannot be factored are known as **prime numbers**.

There is a theorem from mathematics that states every natural number can be factored into a unique product of prime numbers. Some examples,

$$21 = 3 \cdot 7$$

$$1463 = 7 \cdot 11 \cdot 19$$

$$120 = 2^3 \cdot 3 \cdot 5$$

Try factoring the following numbers:

$$18, \qquad 23, \qquad 70, \qquad 363, \qquad 7315$$

Prime numbers play a prominent role in the encryption of messages. One idea is to find two very large numbers and multiply them together. In order to break the encryption code, one needs to factor the product but this is a very difficult task. For example, consider the following two prime numbers (generated using the application at Big Primes, https://bigprimes.org):

453104128937253684286921782868435358242327

696264347926698025145345490076402432804133

When multiplied together, we get an extraordinarily large number which would take a great amount of computer processing time to factor. The article "Why are very large prime numbers important in cryptography?" [1] provides several examples of how primes numbers are used in the field of mathematics known as cryptography.

The Big Primes website also has an application to check if a number is prime, see https://bigprimes.org/primality-test. See if you can enter a number long enough that the application is not able to check. This site also has a prime number game to test your prime (and non-prime) identification skills, see https://bigprimes.org/prime-game.

2.4.1 Greatest Common Divisor

The **Greatest Common Divisor** (GCD) of two integers is the largest number that evenly divides both. For example, the GCD of 2 and 8 is 2 since 2 is the greatest number that divides both 2 and 8. This is written as $\gcd(2,8) = 2$.

Some additional examples:

$$\gcd(3,7) = 1, \qquad \gcd(6,21) = 3, \qquad \gcd(-5,30) = 5$$

In the example on the right, note that the GCD is a positive number even though one of the numbers is negative. The GCD is always a positive number even if both numbers are negative, e.g., $\gcd(-49,-7) = 7$. When the GCD of two numbers is 1 (such as for the example on the left), the two numbers are said to be **relatively prime**.

Try to develop a method for determining the GCD of two numbers. The "brute force" method is to test all combinations (multiples) of divisors of each number However, there is a way to eliminate some of the possibilities, i.e., factor each number into primes and then look for common factors between the two numbers. For example, take 24 and 112. We can write 24 as $2^3 \cdot 8$ and 112 as $2^4 \cdot 7$. Given the factorization, it is easy to see that $\gcd(24,56) = 2^3 = 8$.

Some examples to try:

$$\gcd(36,42)\,, \qquad \gcd(1155,1225)\,, \qquad \gcd(400,1100)$$

Answers can be checked using the Microsoft Excel GCD function or the Google Sheets GCD function.

One more example, which suggests a general approach:

$$\gcd(2^{17}5^{11}7^{47}, 5^{13}7^{43}17^{12})$$

The two numbers are already represented in their unique prime factors. The numbers are too big for most calculators or spreadsheets but you should be able to easily read off the answer, i.e., just take the smallest power of each factor present in both numbers. In this case, the answer is $5^{11}7^{43}$. Note that neither powers of 2 nor powers of 17 appear in both numbers and so, they are not included in the GCD.

2.4.2 Least Common Multiple

The **Least Common Multiple** (LCM) of two integers is the smallest number that is the product of both numbers. For example, the LCM of 5 and 11 is 55 since 55 is the smallest number which is a multiple of both numbers. This is written as $\mathrm{lcm}(11{,}55) = 55$.

Some additional examples:

$$\mathrm{lcm}(4{,}24) = 24, \qquad \mathrm{lcm}(6{,}21) = 42, \qquad \mathrm{lcm}(24{,}60) = 120$$

The GCD and LCM are related by the following formula,

$$\mathrm{lcm}(\mathrm{x}, \mathrm{y}) = \frac{\mathrm{x} \cdot \mathrm{y}}{\mathrm{gcd}(\mathrm{x}, \mathrm{y})}$$

For example, the $\mathrm{gcd}(24{,}60) = 12$ and $24 \cdot 60 = 1440$. So, from the equation above, we have $\mathrm{lcm}(24{,}60) = \frac{1440}{12} = 120$.

A key principle in mathematics is to reduce a problem to one or more simpler problems. For the task at hand, i.e., finding the LCM of two numbers, one approach is to reduce the problem to one we already know how to solve. The idea is to first compute the GCD and then divide into $x \cdot y$.

Another approach is to use a method similar to the one we developed for the GCD. Again, the idea is to first determine a prime factorization of the two numbers and then use this information to determine the LCM. Consider the problem of finding the LCM of $x = 2^{17}5^{11}7^{47}$ and $y = 5^{13}7^{43}17^{12}$. The answer is $2^{17}5^{13}7^{47}17^{12}$. If we included any of the prime factors at a smaller power (exponent), e.g., $2^{\mathbf{16}}5^{13}7^{47}17^{12}$, then we would no longer have a common multiple. The general answer is to take the largest power of each prime factor of either number. In the case of finding the GCD, the solution is to take the smallest power of each prime factor common to both numbers.

Try using the factorization method to solve the following:

$$\mathrm{lcm}(800, 560), \qquad \mathrm{lcm}(1265, 3025), \qquad \mathrm{lcm}(187, 51)$$

Check your answer with the LCM function in either Microsoft Excel or Google Sheets.

2.5 Rational Numbers

The set of all numbers that can be expressed as the ratio of two integers is known as the **rational numbers** (or fractions). For example, $\frac{1}{3}, -\frac{2}{5}$ and $\frac{11}{5}$ are rational numbers.

The **reciprocal** of a number is defined to be 1 divided by that number, e.g., $\frac{1}{19}$ is the reciprocal of 19 and vice versa. If a number is rational, so is its reciprocal.

2.6 Irrational Numbers

Any number that cannot be represented as the ratio of two integers is said to be an **irrational number**. For example, the ratio of the circumference of a circle to its diameter (regardless of the size of the circle) is the irrational number known as Pi (represented as π).

2.7 Real Numbers

The set of all rational and irrational numbers is known as the set of **real numbers**. When unqualified in this book, the term "number" or phrase "any number" refers to real numbers.

Surprisingly, the set of irrational numbers has measure 1 in the interval [0,1] while the set of rational numbers has measure 0 in the same interval. In other words, most numbers are irrational. This result comes from an area of mathematics known as measure theory. Even though the rational numbers are of measure 0 in the interval [0,1] or for that matter any interval, between every two real numbers there is a rational number. The rational numbers are said to be dense in the real numbers.

3 Number Bases (Radix)

Most of the numbers that we see in common everyday use are written in what is called “base 10.” This means that each digit in a number represents a progressively higher multiple of 10 as read from right to left. For example, the number 9237 means that we have

7 units (multiples of $10^0 = 1$)

3 tens (multiples of $10^1 = 10$)

2 hundreds (multiples of $10^2 = 10 \cdot 10 = 100$)

9 thousands (multiples of $10^3 = 10 \cdot 10 \cdot 10 = 100$)

This can be written as

$$9230 = 9 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10 + 7 \cdot 1$$

In the sphere of computer science, base 2 is more common. In base 2, all numbers are written in terms of powers of 2. For example,

$$110101_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$
$$= 32 + 16 + 0 + 4 + 0 + 1 = 53$$

In the above, the subscript on 110101 denotes the base. The second line in the calculations is a conversion to base 10. Typically, the subscript for base 10 is omitted.

Other bases or possible, including bases higher than 10. For bases higher than 10, one needs to create symbols for 10, 11, 12, etc. For example, in base 16 (known as hexadecimal), we have the digits $0,1,2,3,4,5,6,7,8,9, A, B, C, D, E, F$. In base 16, $A = 10, B = 11, C = 12, D = 13, E = 14$ and $F = 15$.

For example,

$$A9F70_{16} = A \cdot 16^4 + 9 \cdot 16^3 + F \cdot 16^2 + 7 \cdot 16^1 + 0 \cdot 16^0$$
$$= 10 \cdot 16^4 + 9 \cdot 16^3 + 15 \cdot 16^2 + 7 \cdot 16^1 = 696{,}176$$

The above answer can be checked using the BASE function in Microsoft Excel or Google Sheets. Type in the following “=BASE(696176,16)” and the spreadsheet will return the result $A9F70$.

Converting directly between bases (e.g., between bases 3 and 7) is a bit complex. See the YouTube video “Converting between bases DIRECTLY (no need to go to base 10)” [2] for a detailed explanation of how to convert between bases. It is also possible to do the conversion between bases a and b (where neither a nor b is 10) by converting from base a to base 10 and then converting from base 10 to base b. This approach is conceptually simpler but takes more steps than the approach explained in the referenced YouTube video.

Some examples to try:

- Write 32 in base 2.
- Write 79 in base 3.
- What is $(1010)_2$ in base 10?
- What is $(12012)_3$ in base 10?
- Convert $(12012)_3$ to base 5.

Check your answers with the online application at https://www.unitconverters.net/numbers-converter.html. Using the online converter, try something wild like converting 1123498 to base 33.

4 Arithmetic

4.1 Motivation

With the advent of calculators (as stand-alone devices or as cell phone apps) and computer-based computation applications (e.g., spreadsheets), few people still do manual arithmetic calculations. Nevertheless, this section plays a role in the study of algebra. Several purposes are covered in this section:

- Discover and understand the various algorithms for addition, subtraction, multiplication and division of numbers
- Learn the various properties of numbers (e.g., distributive laws) that carry over to algebra
- Exposure to the use of variables
- Learn various "mental math" techniques for doing calculations in your head. For a lively introduction to mental math, see the YouTube video entitled "Faster than a calculator" [8].

4.2 Addition

The algorithm taught in most grade schools for the addition of two whole numbers is based on an expansion of each number as a summation of multiples of 10. For example, consider the expansions of 472 and 513

$$472 = 400 + 70 + 2 = 4 \cdot 100 + 7 \cdot 10 + 2 \cdot 1$$

$$513 = 500 + 10 + 3 = 5 \cdot 100 + 1 \cdot 10 + 3 \cdot 1$$

The addition of two numbers proceeds from right to left, i.e., add the units, then multiples of 10, followed by multiples of 100 and so on. For the example at hand,

$$472 + 513 = 9 \cdot 100 + 8 \cdot 10 + 5 \cdot 1 = 985$$

There is a complication when the sum of any multiple of 10 is greater than 9. For example, consider the addition of 493 and 971. First, expand the two numbers:

$$493 = 4 \cdot 100 + 9 \cdot 10 + 3 \cdot 1$$

$$971 = 9 \cdot 100 + 7 \cdot 10 + 1 \cdot 1$$

Adding the two numbers gives

$$13 \cdot 100 + 16 \cdot 10 + 4 \cdot 1$$

The middle term can be written as $1 \cdot 100 + 6 \cdot 10$ which leads to

$$13 \cdot 100 + 1 \cdot 100 + 6 \cdot 10 + 4 \cdot 1$$

and by adding multiples of 100, we get

$$14 \cdot 100 + 6 \cdot 10 + 4 \cdot 1$$

Finally, write $14 \cdot 100 = 1 \cdot 1000 + 4 \cdot 100$ and then plug into the above expression to get

$$1 \cdot 1000 + 4 \cdot 100 + 6 \cdot 10 + 4 \cdot 1 = 1464$$

Agreed that this is a bit painful, which is why the algorithm is put in a shorthand notation (as shown in Table 1). The top row is used for the carry-over and the bottom row is the sum.

Table 1. Addition of two numbers (right-left approach)

1	*1*		
	4	9	3
	9	7	1
1	4	6	4

The same method can be extended to the addition of more than two numbers. The only difference is that the carryover can be more than 1, e.g.,

Table 2. Addition of several numbers (right-left approach)

2	*2*	*2*	*1*	
		5	9	3
	6	7	0	1
	3	0	5	2
	1	8	5	4
	9	7	1	3
2	1	9	1	3

Another approach (which works better if you are doing the math in your head) is to do the addition from left to right. Consider the addition of 516 and 223. In the left-right method, the smaller number is written in terms multiples of 10s, e.g., write 223 as $200 + 20 + 3$, then each term of the small number is progressively added to the larger number. For the example at hand,

$$516 + (200 + 20 + 3) =^{+\,200} 716 + 20 + 3 =^{+\,20} 736 + 3 =^{+\,3} 739$$

The notation $=^{+\,200}$ just means that 200 was added to the first term (i.e., 516) when going to the next step in the calculation.

This method also works with carry-overs, e.g.,

$$729 + 383 = 729 + (300 + 80 + 3) =^{+300} 1029 + 80 + 3$$

$$=^{+80} 1109 + 3 =^{+3} 1112$$

When doing the calculations in your head, the idea is to forget about 729 once you've added 300 to it. In the second step, you should only have 1029, 80 and 3 in your head. Once you add 80, you only need to remember 1109 and 3.

This method is based on ideas from Chapter 1 of the book "Secrets of mental math" [7].

Try the following additions in your head using the left-right method:

$$45+12, \qquad 17+25, \qquad 79+23, \qquad 254+35$$

$$153+214, \qquad 723+264, \qquad 472+398, \qquad 847+685$$

Try a few of the above exercises using the right-left method, and decide for yourself which method is easier to do in your head.

4.3 Subtraction

The usual algorithm for the subtraction of one number from another also goes from right to left. The details of the algorithm can be discovered by considering the expansion of each number in terms of multiples of 10 and then doing the subtraction. For example, consider $524-335$. First, represent both numbers as multiples of 10:

$$524 = 5\cdot 100 + 2\cdot 10 + 4\cdot 1$$

$$335 = 3\cdot 100 + 3\cdot 10 + 5\cdot 1$$

Next, do the subtraction of the various multiples of 10:

$$524-335 = 2\cdot 200 - 1\cdot 10 - 1\cdot 1$$

However, the multiple of 10 and 1 are both negative numbers, which we need to correct. To correct the negative number in the multiples of 10, we take 100 from the 200 terms, and write it as $10\cdot 10$ to get

$$= 1\cdot 100 + 10\cdot 10 - 1\cdot 10 - 1\cdot 1$$

$$= 1\cdot 100 + 9\cdot 10 - 1\cdot 1$$

This corrects the multiple of 10 term but we still have a problem with the multiple of 1 term, but we can do the same procedure again, i.e., take one of the multiples of 10 and add it to the multiple of 1 term:

$$= 1\cdot 100 + 8\cdot 10 + 10\cdot 1 - 1\cdot 1$$

$$= 1\cdot 100 + 8\cdot 10 + 9\cdot 1 = 189$$

The point in going through the cumbersome process above is to discover the algorithm. Figure 1 depicts the typical shorthand notation for doing subtraction. In the first step of the subtraction (table on the left of the figure), 10 is taken from the multiple of 10s and added to the multiples of 1 to get 14. At this point, the rightmost digit in the solution can be determined. In the next step (table on the right), 100 is taken from the multiples of 100 and added to the multiples of 10 to get 11 (tens). The remaining two digits of the solution can be determined at this point.

	1	*14*
5	~~2~~	~~4~~
3	3	5
		9

4	*11*	*14*
~~5~~	~~2~~	~~4~~
3	3	5
1	8	9

Figure 1. Shorthand notation for subtraction

The basic idea is to take a higher multiple of 10 and add it to a lower multiple of 10 in order to avoid a negative digit in the solution to a subtraction problem.

There is an alternative method for subtraction that goes from left to right. The approach is illustrated in the example below. First decompose the smaller number into multiples of 10. Subtract the highest multiple of ten (in this case $300 = 3 \cdot 10^2$) and then work down to progressively lower powers of ten.

$$524 - 335 = 524 - 300 - 30 - 5 =^{-300} 224 - 30 - 5 =^{-30} 194 - 5 =^{-5} 189$$

The notation $=^{-300}$ indicates that 300 was subtracted from the first term (i.e., 524) in going to the next step in the calculation. As with the alternative method for addition, the alternative approach for subtraction is designed for mental computations.

Try the following exercises in your head using the left-right approach for subtraction:

$$59 - 31, \qquad 83 - 35, \qquad 99 - 39$$

$$736 - 524, \qquad 625 - 465, \qquad 738 - 347$$

The calculations can sometimes be simplified (regarding mental computations) by modifying the problem to an equivalent problem. For example, $738 - 347$ has the same answer as $740 - 349$ but the latter problem is easier solve mentally, as show below:

$$738 - 347 = 740 - 349 = 740 - 300 - 40 - 9 =^{-300} 440 - 40 - 9$$

$$=^{-40} 400 - 9 =^{-9} 391$$

Here's another example that is much easier to do if the problem adjusted to an equivalent problem:

$$787 - 299 = 788 - 300 = 488$$

In some cases, a subtraction problem can be simplified by subtracting the same number from both terms. For example, consider $303 - 199$. If we subtract 3 from both terms, we get a simpler problem:

$$303 - 199 = 300 - 196 = 104$$

Alternately, we could have added one to both terms:

$$303 - 199 = 304 - 200 = 104$$

In a 3-digit subtraction problem when neither number is close to a multiple of 100, it may still be helpful to adjust one of the two numbers so that it is an exact multiple of 10. For example, consider $456 - 367$. We can add 4 to both numbers so that 456 becomes 460 (an exact multiple of 10). The calculation goes as follows:

$$456 - 367 = 460 - 371 = 460 - (300 + 70 + 1)$$

$$=^{-300} 160 - (70 + 1) =^{-70} 90 - 1 =^{-1} 89$$

This is easier to compute in your head than the original problem without any modification. Try a few for yourself:

$$697 - 188, \qquad 703 - 567, \qquad 957 - 678, \qquad 572 - 483$$

The same ideas can be used for the subtract of 4-digits numbers. If one of the numbers is close to a multiple of a 1000, adjust the problem by adding or subtracting the same number, e.g.,

$$6135 - 4999 = 6136 - 5000 = 1136$$

Algebraic notation can be used to express the above methods. Let the two numbers in the subtraction be represented by x and y, and let z be the number that is added or subtracted from x and y, then we have

$$x - y = (x + z) - (y + z)$$

or

$$x - y = (x - z) - (y - z)$$

In both cases, the z terms cancel out, and thus, the adjustment leads to the same solution.

4.4 Multiplication

What does it mean to multiply 43 times 56? The problem is basically an addition problem, i.e., it means $56 + 56 + \cdots + 56$ for a total of 43 times or conversely, add $43 + 43 + \cdots + 43$ for a total of 56 times. The usual approach is to first multiple 3 times 56 (third row in Table 3), then multiple 40 times 56 (fourth row in Table 3) and finally, add the two numbers to get the final result (bottom row in Table 3).

Table 3. Multiplication – "Usual Approach"

		5	6
		4	3
	1	6	8
2	2	4	0
2	4	0	8

Another way to view the multiplication is to decompose both numbers into powers of 10, i.e., $56 = 50 + 6$ and $43 = 40 + 3$, and then multiple as follows:

$$56 \cdot 43 = (50 + 6) \cdot (40 + 3) =^{Dist\ Law} 50 \cdot 40 + 6 \cdot 40 + 50 \cdot 3 + 6 \cdot 3$$

$$= 2000 + 240 + 150 + 18 = 2000 + 390 + 18 = 2390 + 18 = 2408$$

The distributive law was used on the first line above. Basically, if x, y and z are any three numbers then $z \cdot (x + y) = z \cdot x + z \cdot y$. This also works for four variables as follows:

$$(z + w) \cdot (x + y) = (z + w) \cdot x + (z + w) \cdot y = z \cdot x + w \cdot x + z \cdot y + w \cdot y$$

The above pattern will be seen many times when we get to the algebra part of this book.

This idea can be used to formulate another approach for the multiplication of two numbers. For 2-digit numbers, the alternate approach is as follows:

1. multiple the power of 10 terms first (50 times 40 in the above example)
2. cross multiple the single digit times the power of 10 (6 times 40, and 3 times 50 in the example)
    - add the two numbers together, and then add to the result from step 1 (in the example, add 240 to 150 to get 390, and then add to 2000 to get 2390)
3. multiple the single digits (6 times 3 in the example)
    - add to the running total to get the final result (add 18 to 2390 in the example)

This approach goes from left to right, unlike the usual approach which goes from right to left. The alternative approach is better suited for mental math.

Let's try another example

$$38 \cdot 89 = 30 \cdot 80 + 8 \cdot 80 + 9 \cdot 30 + 8 \cdot 9 = 2400 + 640 + 270 + 72$$

$$= 2400 + 910 + 72 = 3310 + 72 = 3382$$

Okay, maybe not so easy to do in your head but the technique can be mastered with some practice.

Some additional examples to try using the alternative method:

$$11 \cdot 17, \qquad 12 \cdot 19, \qquad 32 \cdot 41, \qquad 44 \cdot 57, \qquad 71 \cdot 93$$

4.5 Division

Division is the process of determining how many times one number "fits" into another. For example, 37 fits into 18,093 exactly 489 times. In other words, $18{,}093 = 37 \cdot 489$. The typical algorithm for solving division problems entails a series of steps, with each step leading to a simpler problem (a common theme in mathematics). Let's try to discover the general algorithm by solving the problem at hand, i.e., $18{,}093 \div 37$.

1. It is easily observed that the solution is between 100 and 1000 since $100 \cdot 37 = 3{,}700 < 18{,}093$ and $1000 \cdot 37 = 37{,}000 > 18{,}093$.
2. By trial and error, we can determine that 400 is the highest multiple of 100 that when multiplied by 37 is less than 18, 093, i.e., $400 \cdot 37 = 14{,}800 < 18{,}093$ and $500 \cdot 37 = 18{,}500 > 18{,}093$. This means the first digit in the solution is 4.
3. Next, subtract 14,800 from 18,093 to get 3,293. This leaves us with a simpler problem, i.e., determining how many times 37 fits into 3,293. This will give us the second digit in the solution.
4. By trial and error, we can determine that 80 is the highest multiple of 10 such that $80 \cdot 37 = 2{,}960 < 3{,}293$. So, the second digit in the solution is 8.
5. Subtract 2,960 from 3,293 to get 333, and then determine how many times 37 fits into 333.
6. By trial and error, we determine that 37 exactly divides 333 by 9, which gives us the final digit in the solution.
7. Putting all the preliminary results together, we get the solution 489.

The above steps can be summarized as shown in Table 4. This is the typical format used to teach long-division to grade-school students. The point here was to discover the rationale behind the process.

Table 4. Long-division example with exact solution

				4	8	9
3	7	1	8	0	9	3
		1	4	8	0	0
			3	2	9	3
			2	9	6	0
				3	3	3
				3	3	3
						0

The number that is divided into is called the **dividend** (18093 in our example). The number which divides the given number is known as the **divisor** (37 in our example). The resulting solution is called the **quotient** (489 in our example).

It is also possible to use the same procedure when the solution to a division is not a whole number. For example, consider a slight modification to the previous example, $18{,}111 \div 37$. The solution to this problem starts out as the previous division problem but at the last step shown in Table 5, there is a remainder of 18. One way to view this is that a little more of 37 (a fraction) can still be fit into 18,111 beyond 489. The fraction is exactly $\frac{18}{37}$ and thus, the solution is $489 + \frac{18}{37}$ which can be written in shorthand notation as $489\frac{18}{37}$. The latter notation is known as a **mixed number**, i.e., a whole number and a fraction.

In general, when the divisor does not fit exactly into the dividend, the discrepancy is referred to as the **remainder**. In terms of equations, we have

$$divisor \cdot quotient + remainder = dividend$$

If we divide both sides of the above equation by the divisor, we get

$$\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$$

Going back to our example, we have $\frac{18111}{37} = 489 + \frac{18}{37}$.

Table 5. Long-division example with remainder – Part 1

				4	8	9
3	7	1	8	1	1	1
		1	4	8	0	0
			3	3	1	1
			2	9	6	0
				3	5	1
				3	3	3
					1	8

Alternatively, the remainder can be expressed as a decimal, as shown in Table 6. Up to this point, the procedure was to see how many multiples of 100, 10 and 1 times 37 fit into the remainder at each step. We could continue by determining how many multiples of $\frac{1}{10} = .1$ times 37 (i.e., 3.7) fit into 18. The answer is 4. In other words, we can fit another .4 of 37 into 18,111. However, we are still left with a remainder of 3.2. So, next we determine how many times $(.01)(37) = .37$ fits into the remainder 3.2. The answer is 8 but again, we have a remainder (i.e., .24). We go on to determine the number of times $(.001)(37) = .037$ fits into the remainder .24, which turns out to be 6. The process continues indefinitely but the pattern 486 repeats. The solution is $489.486486486\ \ldots$ which can also be written as $489.\overline{486}$ (where the … and the line over 486 have the same meaning, i.e., 486 repeats forever).

Table 6. Long-division example with remainder – Part 2

				4	8	9	.4	8	6
3	7	1	8	1	1	1	.0	0	0
		1	4	8	0	0	.0	0	0
			3	3	1	1	.0	0	0
			2	9	6	0	.0	0	0
				3	5	1	.0	0	0
				3	3	3	.0	0	0
					1	8	.0	0	0
					1	4	.8	0	0
						3	.2	0	0
						2	.9	6	0
							.2	4	0
							.2	2	2

4.6 Fractions (Rational Numbers)

As noted earlier, a **fraction** (or **rational number**) is a whole number (integer) divide by another, e.g., $\frac{1}{3}, \frac{19}{7}, -\frac{12}{13}$. The bottom number in a fraction is called the **denominator** and the top number is called the **numerator**.

The general form of a fraction is $\frac{a}{b}$ where a and b are integers (can be negative). This can be viewed as a instances of $\frac{1}{b}$ of a whole. For example, $\frac{5}{8}$ can be viewed as 5 instances of $\frac{1}{8}$ of a whole. This can be visualized as a pie divided into 8 equal sections of which 5 are chosen (the 5 darker sections in Figure 2).

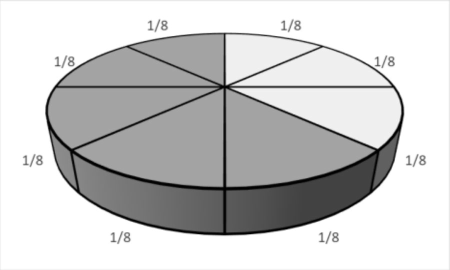

Figure 2. Fractions of a pie

4.6.1 Simplification

There are multiple (actually an infinite) number of ways to represent the same fractional quantity. For example, $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}$ and we could follow this pattern indefinitely.

As a general rule, we can cancel like terms in the numerator and denominator, i.e.,

$$\frac{a\cdot b}{a\cdot c}=\frac{a}{a}\cdot\frac{b}{c}=1\cdot\frac{b}{c}=\frac{b}{c}$$

For example, $\frac{34}{51}=\frac{2\cdot 17}{3\cdot 17}=\frac{2}{3}$.

The basic idea is to factor the numerator and denominator as much as possible, and then cancel like terms. This leads to the simplest form of a given fraction.

Here's a more complex example:

$$\frac{1330}{1615}=\frac{2\cdot 5\cdot 7\cdot 19}{5\cdot 17\cdot 19}=\frac{2\cdot 7}{17}=\frac{14}{17}$$

Reduce the following fractions into their simplest form:

$$\frac{12}{18},\qquad \frac{57}{133},\qquad \frac{14630}{17765},\qquad \frac{115}{56},\qquad \frac{a\cdot a\cdot b\cdot c}{a\cdot b\cdot b\cdot c\cdot c}$$

4.6.2 Comparison

If two fractions have the same denominator, they are easy to compare. For example, it is clear the $\frac{1}{4}$ is less than $\frac{3}{4}$.

If one denominator is a multiple of the other, it is also easy to compare after a slight modification is made. Consider the fractions $\frac{5}{7}$ and $\frac{9}{14}$. We can write $\frac{5}{7}$ as the equivalent fraction $\frac{10}{14}$ which is then seen as being the large of the two.

The problem of fraction comparison becomes more difficult when the denominators are not a multiple of each other. For example, which is larger $\frac{8}{15}$ or $\frac{15}{31}$? One approach is to multiple both fractions by 1, but with the twist of writing 1 as a fraction in two different ways, i.e.,

$$\frac{8}{15}=\frac{8}{15}\cdot\frac{31}{31}=\frac{248}{465}$$
$$\frac{15}{31}=\frac{15}{31}\cdot\frac{15}{15}=\frac{225}{465}$$

This approach yields a common denominator and thus allows for comparison. Thus, we can conclude that $\frac{8}{15}$ is the larger of the two fractions.

While the above approach always works, it does not give the most efficient solution in every case. For example, consider $\frac{3}{14}$ and $\frac{5}{18}$. Using the above approach, we get

$$\frac{3}{14} = \frac{3}{14} \cdot \frac{18}{18} = \frac{54}{252}$$
$$\frac{5}{18} = \frac{5}{18} \cdot \frac{14}{14} = \frac{70}{252}$$

So, we see that $\frac{5}{18}$ is greater than $\frac{3}{14}$. This is all fine but there is a smaller common denominator, i.e., 126 which happens to be the least common multiple of 14 and 18. This is no coincidence. The least common multiple of the denominators of two fractions is always the smallest common denominator for the two fractions. Using the LCM approach in the above problem, we have

$$\frac{3}{14} = \frac{3}{14} \cdot \frac{9}{9} = \frac{27}{126}$$
$$\frac{5}{18} = \frac{5}{18} \cdot \frac{7}{7} = \frac{35}{126}$$

Note to get the 9 in the top equation, we needed to divide 14 into 126, and similar, we needed to divide 18 into 126 to get the 7 used in the bottom equation.

Determine the larger fraction in each of the following pairs:

$$\frac{1}{2}, \frac{5}{12} \qquad \frac{5}{6}, \frac{7}{8} \qquad \frac{9}{17}, \frac{10}{19} \qquad \frac{17}{33}, \frac{13}{22} \qquad \frac{77}{154}, \frac{189}{385}$$

4.6.3 Addition

To add fractions, it is necessary to put the fractions in a form such that their denominators are the same. Look at this this way, $\frac{3}{4}$ is just a way of saying you have three quarters $(\frac{1}{4} + \frac{1}{4} + \frac{1}{4})$ of something and $\frac{7}{8}$ can be seen as being seven eights $(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8})$ of something. We cannot add quarters and eights directly – need to normalize the two fractions, i.e.,

$$\frac{3}{4} + \frac{7}{8} = \frac{6}{8} + \frac{7}{8} = \frac{13}{8}$$

Let's try a more complex example: $\frac{77}{154} + \frac{189}{385}$

First, find the least common multiple of the denominators

$$lcm(154, 385) = lcm(2 \cdot 7 \cdot 11, 5 \cdot 7 \cdot 11) = 2 \cdot 5 \cdot 7 \cdot 11 = 770$$

and then recast the two fractions in terms of their least common denominator

$$\frac{77}{154} + \frac{189}{385} = \frac{77}{154} \cdot \frac{5}{5} + \frac{189}{385} \cdot \frac{2}{2} = \frac{385}{770} + \frac{378}{770} = \frac{763}{770}$$

Try the following exercises (you already found the common denominator if you did the exercises in the previous subsection):

$$\frac{1}{2}+\frac{5}{12};\quad \frac{5}{6}+\frac{7}{8};\quad \frac{9}{17}+\frac{10}{19};\quad \frac{17}{33}+\frac{13}{22};\quad \frac{77}{154}+\frac{189}{385}$$

4.6.4 Subtraction

Subtraction of fractions is almost the same as addition in the sense that a common denominator needs to be found before doing the subtraction. Consider the problem of finding $\frac{77}{154}-\frac{189}{385}$. We already did the hard part of finding the common denominator in the previous section. Thus, we have

$$\frac{77}{154}-\frac{189}{385}=\frac{385}{770}-\frac{378}{770}=\frac{7}{770}=\frac{1}{110}$$

A few subtraction exercise to try:

$$\frac{9}{17}-\frac{10}{19};\quad \frac{77}{154}-\frac{189}{385};\quad \frac{5}{6}-\frac{7}{8}$$

4.6.5 Multiplication

Multiplication of fractions is the process of taking a fraction of a fraction. For example, taking $\frac{1}{2}$ of $\frac{1}{8}$ is equivalent to multiplying $\frac{1}{2}$ times $\frac{1}{8}$ to get $\frac{1}{16}$.

Multiplication of fractions is easier than addition and subtraction since it is not necessary to find a common denominator.

In general, to get the product of fraction $\frac{a}{b}$ times the fraction $\frac{c}{d}$, we simply multiply the numerators and the denominators to get $\frac{a\cdot c}{b\cdot d}$.

Let's try the procedure with actual numbers:

$$\frac{3}{5}\cdot\frac{7}{11}=\frac{3\cdot 7}{5\cdot 11}=\frac{21}{55}$$

Cancellation is also allowed and encouraged, e.g.,

$$\frac{8}{21}\cdot\frac{7}{13}=\frac{8}{3}\cdot\frac{1}{13}=\frac{8}{39}$$

In the above, we effectively reduced $\frac{7}{21}$ to $\frac{1}{3}$.

A few more to try:

$$\frac{3}{11}\cdot\frac{33}{47};\quad \frac{121}{190}\cdot\frac{10}{11};\quad \frac{17}{35}\cdot\frac{7}{23}$$

4.6.6 Division

What does it mean to divide one fraction by another, e.g., $\frac{1}{2}\div\frac{1}{4}$? For the particular example, we are asking how many times does $\frac{1}{4}$ fit into $\frac{1}{2}$? In other words, by what number to we need to multiple $\frac{1}{4}$ to get $\frac{1}{2}$? This can be written as an equation with a variable (call it x):

$$x \cdot \frac{1}{4} = \frac{1}{2}$$

The solution is 2 since $2 \cdot \frac{1}{4} = \frac{1}{2}$.

Let's consider a more complicated example and see if a pattern can be determined. What is $\frac{6}{11} \div \frac{5}{21}$? Again, we write down the associated equation

$$x \cdot \frac{5}{21} = \frac{6}{11}$$

In general, if both sides of an equation are multiplied by the same number, the equation still holds true. For the problem at hand, multiple both sides of the equation by $\frac{21}{5}$ to get

$$x \cdot \frac{5}{21} \cdot \frac{21}{5} = \frac{6}{11} \cdot \frac{21}{5}$$

which reduces to

$$x = \frac{6}{11} \cdot \frac{21}{5} = \frac{126}{55}$$

In general, $\frac{a}{b} \div \frac{c}{d}$ (which can also be written as $\frac{\frac{a}{b}}{\frac{c}{d}}$) is just $\frac{a}{b} \cdot \frac{d}{c}$.

Some additional examples:

$$\frac{1}{4} \div \frac{1}{16} = \frac{1}{4} \cdot \frac{16}{1} = 4, \qquad \frac{7}{8} \div \frac{9}{16} = \frac{7}{8} \cdot \frac{16}{9} = \frac{14}{9}, \qquad \frac{3}{13} \div 7 = \frac{3}{13} \cdot \frac{7}{1} = \frac{21}{13}$$

In the example to the right, note that the 7 can be viewed as $\frac{7}{1}$. Also, we did some cancellation in the first two examples above.

Some examples to try for yourself:

$$\frac{2}{3} \div \frac{1}{3}, \qquad \frac{3}{13} \div 9, \qquad 7 \div \frac{4}{7}, \qquad \frac{5}{11} \div \frac{6}{17}, \qquad \frac{6}{19} \div \frac{3}{38}, \qquad \frac{a}{bc} \div \frac{c}{ad}$$

4.6.7 Converting between Decimal and Fraction

To convert from a fraction to a decimal, we use the long division algorithm from Section 4.5. Table 7 shows the conversion of $\frac{5}{11}$ to a decimal. In this case, we get the repeating decimal $.\overline{45}$.

Table 7. Converting a fraction to a decimal using long division

			.4	5	4	5
1	1	5	.0	0	0	0
		4	4	0	0	0
			6	0	0	0
			5	5	0	0
				5	0	0
				4	4	0
					6	0
					5	5

Going in the other direction (decimal to fraction) is easy for non-repeating decimals. For example, .342 is $\frac{342}{1000}$ and .9731 is $\frac{9731}{10000}$. The idea is to remove the decimal and divide by a power of 10 that is one more than the number of digits in the decimal. This can be seen by the following string of equivalent expressions:

$$.342 = 3 \cdot \frac{1}{10} + 4 \cdot \frac{1}{100} + 2 \cdot \frac{1}{1000} = \frac{300}{1000} + \frac{40}{1000} + \frac{2}{1000} = \frac{342}{1000}$$

The conversion is more complex in the case of repeating decimals. The key to the conversion is to make use of the following information:

$$\frac{1}{9} = .11111\ldots$$

$$\frac{1}{99} = .010101\ldots$$

$$\frac{1}{999} = .001001001\ldots$$

This gives us a way to convert any repeating decimal. For example, .484848… can be represented as

$$\frac{48}{99} = 48 \cdot \frac{1}{99} = 48 \cdot (.010101\ldots) = .484848\ldots$$

Working backward from our previous example concerning $\frac{5}{11}$

$$.454545\ldots = 45 \cdot (.010101\ldots) = 45 \cdot \frac{1}{99} = \frac{45}{99} = \frac{5}{11}$$

In some cases, there are leading zeros before the repeating part of a decimal, e.g., .00231231231… To do this conversion, note that

$$\frac{1}{999000} =.\mathbf{00}001001001\ldots$$

and so,

$$\frac{231}{99900} = .00231231231\ldots$$

There are two steps here, i.e., first determine the number of digits in the repeating part of the decimal, and second determine the number of zeros before the repeating part begins.

Let’s try another example: .000486348634863…

In this case, we have 4 digits in the repeating part of the decimal and the repeat doesn’t start until after four zeros. So, we need to divide the repeating part (4863) by 9999000 to get

$$\frac{4863}{9999000} = 4863 \cdot (.\mathbf{000}000100010001\ldots) = .\mathbf{000}486348634863\ldots$$

There is one further complication, i.e., the non-repeating part before the repeating part may not be all zeros, e.g., .**927**232323… In such cases, we divide the decimal as the sum of a repeating and non-repeating part, and thus reduce the problem to the previous solutions. For the example at hand, we have

$$.927232323\ldots = .927 + .000232323\ldots = \frac{927}{1000} + \frac{23}{99000}$$
$$= \frac{927 \cdot 99}{99000} + \frac{23}{99000} = \frac{91796}{99000}$$

This can be simplified to $\frac{22949}{24750}$ since there is a common factor of 4 in the above.

4.7 Exponents and Powers

When raising a number to a power, such as 10^4, the power is referred to as an **exponent**. Most of our examples, so far, have involved powers of 10 but any number can be raised to a power. For example, 2^5 is just a shorthand way of representing 2 times itself five times, i.e., $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

Note that a number to the 1 power is itself and by definition, a number to the zero power is 1. For example, $5^1 = 5$ and $5^0 = 1$.

Taking powers of negative numbers is slightly complicated by the fact that a negative number times a negative number is a positive. The following examples illustrate the point:

$$(-3)^0 = 1$$
$$(-3)^1 = -3$$
$$(-3)^2 = 9$$
$$(-3)^3 = -27$$

The parenthesis in the above example is important. If we just wrote -3^2, this would be interpreted as $(-1) \cdot (3^2) = -9$.

Raising fractions to a power is defined in the same way as for the integers. We also need to make use of the rules for multiplying fractions (as stated in Section 4.6.5). For example, $\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{16}{81}$. We can also write $\left(\frac{2}{3}\right)^4$ as $\frac{2^4}{3^4}$. However, we need to be careful about the notation here. The placement of the parenthesis in the expression $\left(\frac{2}{3}\right)^4$ is important. If we had $\frac{2^4}{3}$, this would give a different result, i.e., $\frac{16}{3}$ (the exponent only applies to the numerator in this case).

It is also possible to take the power of something raised to a power, e.g.,

$$(7^3)^4 = 7^3 \cdot 7^3 \cdot 7^3 \cdot 7^3 = 7 \cdot 7 \cdot 7 \ \cdot \ 7 \cdot 7 \cdot 7 \ \cdot \ 7 \cdot 7 \cdot 7 \ \cdot \ 7 \cdot 7 \cdot 7 = 7^{12}$$

The general rule (as can be deduced from the above example) is to multiple the powers when raising a power of a number to a power, i.e., $(a^b)^c = a^{b \cdot c}$ for any numbers a, b and c.

Raising a number to a negative power is also possible, e.g.,

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

The negative power can be interpreted as the reciprocal. In general, $a^{-b} = \frac{1}{a^b}$.

Find the value of the following expressions:

$$2^5, \quad \left(\frac{1}{3}\right)^3, \quad 4^{-3}, \quad \left(\frac{5}{7}\right)^2, \quad \frac{5^2}{7}, \quad \frac{5}{7^2}, \quad \left(\frac{1}{11}\right)^{-2}, \quad \left(\frac{3}{5}\right)^{-3}$$

4.8 Roots

The **root** of a number x is another number, which when multiplied by itself a given number of times, equals x. For example, the 2nd root (referred to as the **square root**) of 25 is 5 since $5 \cdot 5 = 25$. The 3rd or **cube root** of 8 is 2 since $2 \cdot 2 \cdot 2 = 8$.

There are two different notations used to represent roots, i.e., $\sqrt[3]{8}$ and $8^{\frac{1}{3}}$ represent the same quantity. Taking the root of a number is equivalent to raising that number to a fractional power, e.g., the 5th root of 243 is written as $243^{\frac{1}{5}}$ which evaluates to 3.

The root of a number can be raised to a power, e.g.,

$$\left(243^{\frac{1}{5}}\right)^2 = 3^2 = 9$$

This can also be written as $243^{\frac{2}{5}}$.

A number raised to a fractional power can be seen as taking a root followed by raising to an integer power, or as a number raised to an integer power and then taking a root. The same answer will result either way. Consider the following example:

$$343^{\frac{2}{3}} = \left(343^{\frac{1}{3}}\right)^2 = 7^2 = 49$$

$$343^{\frac{2}{3}} = (343^2)^{\frac{1}{3}} = 117649^{\frac{1}{3}} = 49$$

We can also raise a number to a negative fraction, e.g.,

$$243^{-\frac{2}{5}} = \left(243^{\frac{1}{5}}\right)^{-2} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Find the value of the following expressions :

$$144^{\frac{1}{2}}, \quad 125^{\frac{1}{3}}, \quad 32^{\frac{1}{5}}, \quad 2197^{\frac{2}{3}}, \quad 64^{-\frac{1}{3}}$$

4.9 Laws

The following laws apply to all real numbers. These laws will be used both explicitly and implicitly throughout this book.

A surprising number of deep results can be derived under the assumption of just a few such laws. Such results are covered in an area of mathematics known as abstract algebra.

The **commutative law of addition** says that the order of addition does not matter, e.g., $7 + 24 = 24 + 7 = 31$. This holds true even for negative numbers but we need to be a bit careful. It is true that $17 + (-4) = (-4) + 17 = 13$ but it is not true that $17 - 4 = 4 - 17$. The commutative law does not hold for subtraction. Subtraction is not even considered to be an operation but rather addition of a negative number.

The **commutative law of multiplication** says that the order of multiplication does not matter, e.g., $23 \cdot 19 = 19 \cdot 23 = 437$. This also holds for fractions, i.e.,

$$\frac{1}{8} \cdot 34 = 34 \cdot \frac{1}{8} = \frac{34}{8} = \frac{17}{4}$$

However, the commutative law does not hold for division, e.g., $34 \div 8 \neq 8 \div 34$. Division is not considered to be an operation but rather multiplication by the reciprocal of a number, e.g., $34 \div 8$ can be viewed as $34 \cdot \frac{1}{8}$.

The **associative law of addition** states that the grouping of terms with regard to addition does not affect the result. For example, $(2 + 7) + 9 = 2 + (7 + 9) = 18$. In words, we can add 2 to 7 first and then add 9, or first add 7 to 9 and then add 2, and get the same result in both cases.

The **associative law of multiplication** states that the grouping of terms with regard to multiplication does not affect the result, e.g.,

$$(4 \cdot 19) \cdot 11 = 4 \cdot (19 \cdot 11) = 836$$

The **distributive law** combines both addition and multiplication, e.g.,

$$(5 + 7) \cdot 12 = 5 \cdot 12 + 7 \cdot 12$$

Recall that we already had a need to use the distributive law in Section 4.4.

The **additive identity** is 0, since 0 added to any number is the number itself.

Every number x has an **additive inverse** $-x$, e.g., the additive inverse of 8 is -8.

The **multiplicative identity** is 1, since 1 times any number is the number itself.

Every number x (except 0) has a **multiplicative inverse** $\frac{1}{x}$, e.g., the multiplicative inverse of -19 is $-\frac{1}{19}$. Zero does not have a multiplicative inverse since $\frac{1}{0}$ is undefined. In fact, any number divided by zero is undefined.

. . .

For exponents, the following rules apply. This is a formalization of what we saw in Sections 4.7 and 4.8. The letters a, b and c represent real numbers.

- $a^b a^c = a^{b+c}$ (product rule for exponents)
- $\frac{a^b}{a^c} = a^{b-c}, a \neq 0$ (quotient rule for exponents)
- $(a^b)^c = a^{bc}$ (power rule for exponents)
- $(a \cdot b)^c = a^c \cdot b^c$ (this follows from the commutative law of multiplication)

5 Sequences

A **sequence** is an ordered list of numbers following a pattern. A sequence may be finite or infinite.

5.1 Arithmetic Sequences

In an arithmetic sequence, there is an initial number followied by a series of numbers, each an equal distance away. For example, the following arithmetic sequence has an initial term of 5 and increment 3:

$$5, 8, 11, 14, 17, \ldots$$

Arithmetic sequence can decrease. In such cases, the increment is a negative number, e.g.,

$$14, 10, 6, 2, -2, -6, \ldots$$

The increment as well as the initial number in an arithmetic sequence can be a fraction, e.g.,

$$\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \ldots$$

What are the increments in the above sequence? What is the 120th term in the above sequence? Try to determine a formula rather than writing out 120 terms. Check your formula to see if it works for say the 6th term.

5.2 Geometric Sequences

Geometric sequences have an initial number followed by a series of numbers where each term is a multiple of the previous term. For example, the following geometric sequence has an initial term of 2 and a multiplicative increment of 3:

$$2, 6, 18, 54, 162, \ldots$$

If the multiplicative increment is negative, the geometric sequence will alternate between positive and negative terms, because a negative number times a negative number is a positive number. The following geometric sequence has an initial term of 3 and a multiplicative increment of -2:

$$3, -6, 12, -24, 28, \ldots$$

Determine the multiplicative increment of the following geometric sequence:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

What is the 10th term in this sequence? What is the n^{th} term?

5.3 Sequence Notation

Sequences are represented using a notation involving subscripts. For example, the arithmetic sequence $-3, 0, 3, 6, 9, \ldots$ can be represented as follows:

$$a_0 = -3$$

$$a_{n+1} = a_n + 3 \text{ for } n = 1,2,3, \ldots$$

The name of the sequence is a, the first term is a_0 (which we are told is -3) and the $n + 1$ term is gotten by adding 3 to the n^{th} term (as indicated by the given formula).

We can represent the geometric series $7,14,28,56,112, \ldots$ as

$$b_0 = 7$$

$$b_{n+1} = 2 \cdot b_n \text{ for } n = 1,2,3, \ldots$$

5.4 Recursively Defined Sequences

In addition to the arithmetic and geometric types of sequences, there are other sequences that are defined recursively. One of the best known recursive sequences is the Fibonacci sequence. The first two terms are 1 and 1. All following terms are the sum of the two previous terms. These rules generate the following sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

In terms of notation, the Fibonacci sequence is defined as follows:

$$f_0 = 1$$

$$f_1 = 1$$

$$f_{n+2} = f_{n+1} + f_n \text{ for } n = 0,1,2,3, \ldots$$

The Lucas numbers are defined in a manner similar to the Fibonacci sequence, i.e., the first two numbers in the sequence are given and the following numbers are defined recursively. The first few Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$$

Write the Lucas numbers in sequence notation.

A comprehensive list of sequences is available from The On-Line Encyclopedia of Integer Sequences® (OEIS®) [3]. Go to the website for the OEIS and try typing the first few members of the Lucas numbers sequence into the search bar.

5.5 Sequences Following Complex Patterns

Not all sequences are defined recursively. For example, consider the following sequence:

$$9378\ (9873, 3789), 6084\ (8640, 468), 8172\ (8721, 1278),$$

$$7443\ (7443, 3447), 3996\ (9963, 3699), 6264\ (6642, 2466),$$

$$4176\ (7641, 1467), 6174\ (7641, 1467), 6174\ (7641, 1467), \ldots$$

The pattern is different from what we have seen in several respects. First, the numbers come in groups of three. Secondly, the sequence starts to repeat after a finite number of steps. Before reading on, try to identify the pattern.

The process behind the sequence is as follows:

- Select a 4-digit number (for the example, we used 9378)
- Take the digits of the selected number and create two more 4-digits numbers – one with descending digits and the other with ascending digits (for the example, 9873 and 3789)
- Subtract the two numbers and repeat the process.

It turns out that if at least two of the digits in the four-digit number are different, the process will always converge to the number 6174. Try a few starting numbers for yourself, e.g., 3974 and 1137. Also, see the Wikipedia article on Kaprekar's constant [4] and the article entitled “The Puzzling Power of Simple Arithmetic” [5].

. . .

As a final example in this section, see if you can figure out the pattern in the following sequence:

$$1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, \ldots$$

The answer is not at all obvious but give it a try. This is known as the Kolakoski sequence [6].

6 Transition to Algebra

6.1 Algebra as an Extension of Arithmetic

With some modification, the operations that we covered concerning arithmetic in Section 4 apply to algebra. The main difference is that with algebra we have some unknowns (variables) in the various expressions.

First, we consider several operations on the following arithmetic equation, and then show similar operations on an algebraic equation (in the following paragraph):

$$247 + 123 = 370$$

If we subtract a number from both sides (say 70), the equation still holds true:

$$247 + 123 - 70 = 370 - 70 = 300$$

We can also multiple both sides of the equation (say 3) and the equation will still hold true:

$$3 \cdot (247 + 123) = 3 \cdot 370$$

Note that we can apply the distributive law on the left-side of the above equation to get

$$3 \cdot (247 + 123) = 3 \cdot 247 + 3 \cdot 123 = 741 + 369 = 1100$$

which is equal to $3 \cdot 370$.

When an equation has an unknown (call it x), the same type of operations (as those above) can be performed. For example, consider the algebraic expression

$$x + 348 = 154$$

We can determine the value of the unknown x by subtracting 348 from both sides of the equation

$$x + 348 - 348 = 154 - 348$$

which implies (noting that cancellation of 348 by -348)

$$x = -194$$

All the rules for exponents (Section 4.7) and roots (Section 4.8) hold true for variables, e.g.,

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

All of the laws stated in Section 4.9 hold true when the equations have one or more unknowns. In fact, these laws are typically stated with unknowns. For example, the distributed law is typically stated as follows:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ for any real numbers } x, y, z$$

The details will be explained in the following sections. The key points to remember:

- algebra is arithmetic with unknowns
- the laws of arithmetic carry over to algebra.

If a particular algebraic expression is difficult to fathom, try replacing the unknowns with actual numbers to gain a better understanding of the expression. For example, take $\frac{x^a}{x^b}$. If you don't remember how to reduce the expression, just try with actual numbers, e.g.,

$$\frac{10^5}{10^3} = \frac{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}{10 \cdot 10 \cdot 10} = 10 \cdot 10 = 10^2$$

So, it looks like we subtract exponents when dividing (which is correct). We can then abstract to the general solution, based on our analysis, i.e.,

$$\frac{x^a}{x^b} = x^{a-b}$$

The general formula holds even if a or b are not integers.

6.2 Notation

For arithmetic, a dot was used to represent multiplication, e.g., $52 \cdot 17$. In algebra, juxtaposition or parentheses are used to indicate multiplication, e.g.,

xy means the unknown x times the unknown y

$5x$ means 5 times the unknown x

$(x+3)(x-8)$ means $(x+3)$ times $(x-8)$

However, to avoid ambiguity, we still need to write the product of two numbers using the dot notation or with parenthesis. For example, we cannot write 5217 and hope this to be understood as $52 \cdot 17$.

6.3 Constants versus Variables

In the expression $y = x + 4$, x and y are variables and 4 is a constant. For $x = 3$, the expression allows us to compute the value of y which, in this case, is 7.

It is also possible to have an expression with unknown constants. For example, $y = x + a$ is an expression with two variables (x and y) and a constant a. What the expression says is "y is equal to x plus some constant (fixed) number a which we don't know at this point."

7 Linear Expressions

7.1 Single Variable

7.1.1 Overview

We start our study of algebra with the simplest case, i.e., one variable raised to the power of 1. The general form of a linear, single variable expression is

$$ax + b$$

where a and b are constants, and x is the sole variable in the expression. For example,

$$3x + 2$$

is a linear, single variable expression. Each item in the expression is called a **term**. In the expression above, $3x$ and 2 are terms.

We can also have linear equations with a single variable. The general format is

$$ax + b = c$$

where a, b and c are constants, and x is the sole variable. For example,

$$3x + 2 = 7$$

is a linear equation, with a single variable.

From what has been discussed so far in this book, you should be able to solve for x in the above equation.

7.1.2 Operations

7.1.2.1 Addition and Subtraction

Addition of linear equations with a single variable is a matter of adding like terms, e.g.,

$$(3x + 7) + (2x + 3) = 5x + 10$$

This is analogous to the way we added whole numbers by breaking the number into powers of 10 and then adding like terms.

Subtraction pretty much follows the same pattern as addition, e.g.,

$$(3x + 7) - (2x + 3) = x + 4$$

Some additions and subtraction examples to try:

- $(x + 7) + (x - 2)$
- $(7x + 3) + (-2x + 4)$
- $\left(3x + \frac{5}{6}\right) + \left(5x + \frac{7}{8}\right)$

- $\left(\frac{1}{3}x - \frac{1}{7}\right) + \left(\frac{1}{4}x + 9\right)$
- $(22x + 7) - (14x + 3)$
- $(17x - 2) - (-3x - 5)$
- $\left(\frac{1}{7}x + \frac{5}{6}\right) - \left(\frac{2}{5}x + \frac{3}{8}\right)$

7.1.2.2 Multiplication

Multiplication of single variable linear expressions depends on the distributive law, and the result is a quadratic expression (i.e., highest power of the variable is 2). In the following computation, the distributive law is applied twice (as indicated next to the equal signs):

$$(2x - 5)(7x + 2)$$
$$=^{dist.\ law} (2x - 5)(7x) + (2x - 5)(2)$$
$$=^{dist.\ law} (2x)(7x) - (5)(7x) + (2)(2x) - (5)(2)$$
$$= 14x^2 - 35x + 4x - 10$$
$$= 14x^2 - 31x - 10$$

In the above, notice that each term on the left is multiplied times each term on the right, and then we add the four items to get the final result (which is not a linear expression, however).

The above approach is sometimes referred to as the **First Outer Inner Last** (FOIL) method. This means to

- multiply the first terms in each expression; $(2x)(7x)$ in the example above
- multiple the outer terms; $(2)(2x)$
- multiple the inner terms; $-(5)(7x)$ in the example
- multiple the last terms; $-(5)(2)$ in the example
- and then add all four of the items together.

[**Author's Remark**: If the FOIL method helps you remember, fine. However, I think it is better to understand why based on the distributive law, as described above.]

Some multiplication examples to try:

- $(5)(2x + 3)$
- $(11x)(4x - 7)$
- $(x + 1)(x - 1)$
- $(7x + 8)(2x - 3)$

- $\left(2x - \frac{1}{2}\right)(4x + 2)$
- $\left(\frac{1}{3}x + 8\right)(x + 12)$
- $\left(\frac{5}{6}x + 4\right)\left(\frac{2}{3}x - 6\right)$

You can check your answer on Wolfram Alpha (https://www.wolframalpha.com) or Symbolab (https://www.symbolab.com). It takes a little practice to enter fractions, e.g., to enter $\frac{5}{6}x$ correctly, you need to enter 5 / 6 then a space, and the x.

7.1.2.3 Division

It is possible to divide one linear expression into another. For example, the expression

$$\frac{3x + 2}{x + 7}$$

can be solved using long division as shown in Table 8.

Table 8. Long-division of linear expressions with a single variable

			3
x	7	$3x$	2
		$3x$	21
			-19

In the above division, powers of x are treated in a manner similar to how we treated powers of 10 when we did long division of numbers in Section 4.6.6.

This result can be interpreted as $x + 7$ divides $3x + 1$ three times with a remainder of -19 which can be written as

$$\frac{3x + 2}{x + 7} = 3 - \frac{19}{x + 7}$$

We can check our answers as follows:

$$3 - \frac{19}{x + 7} = \frac{3(x + 7)}{x + 7} - \frac{19}{x + 7} = \frac{3x + 21 - 19}{x + 7} = \frac{3x + 2}{x + 7}$$

In the above calculation, we multiplied 3 by $\frac{x+7}{x+7} = 1$ in order to get a common denominator, and we then added the numerators.

Some division problems to try:

- $\frac{2x+2}{x+1}$ (note that $2x + 2 = 2(x + 1)$ and so we can avoid using long-division for this problem)
- $\frac{3x-1}{x+4}$

- $\frac{7x+4}{3x-1}$

All of the above exercises have a restriction since division by zero is undefined. For example, in the first exercise, we have the restriction that $x \neq -1$. What are the restrictions on the second and third exercises?

Try experimenting with Wolfram Alpha or Symbolab concerning the above exercises. Wolfram Alpha provides an alternate form of your input (basically the result of long division), a graph and a lot of advanced information related to calculus. Symbolab only provides the result of the long division in this case.

7.1.3 Solving

The task of determining the value of the single variable in a linear equation is referred to as solving the linear equation. For example, find the value of x given

$$5x + 43 = 2x - 16$$

The general approach is to isolate the terms with the variable on one side of the equation and the constants on the other side of the equation. This is done by adding or subtracting the same term on both sides of the equation.

For the problem at hand, we first subtract $2x$ from both sides of the equation

$$5x - 2x + 43 = 2x - 2x - 16$$

which reduces to

$$3x + 43 = -16$$

Next, we subtract 43 from both sides of the equation

$$3x + 43 - 43 = -16 - 43$$

which reduces to

$$3x = -59$$

Finally, we divide both sides of the equation by 3 to get the solution, i.e.,

$$x = -\frac{59}{3}$$

You can check the solution by substitution into each side of the original equation:

$$5\left(-\frac{59}{3}\right) + 43 = \frac{-5 \cdot 59 + 3 \cdot 43}{3} = -\frac{166}{3}$$
$$2\left(-\frac{59}{3}\right) - 16 = \frac{-2 \cdot 59 - 3 \cdot 16}{3} = -\frac{166}{3}$$

Some linear equations to solve:

- $3x - 7 = 0$
- $7x - 8 = 17$

- $5x + 10 = 14x + 7$
- $\frac{x}{4} + 6 = 3x - 4$ ($\frac{x}{4}$ is a shorthand notation for $\frac{1}{4}x$)
- $\frac{5}{6}x - \frac{1}{3} = \frac{2}{5}x + \frac{3}{4}$ (***Hint***: first multiply both sides of the equation by 60)

Check your answers with Wolfram Alpha and Symbolab.

7.1.4 Word Problems

7.1.4.1 Mobile Phone Service Bill

Your mobile telecommunications service provider charges a flat-rate of \$20/month for unlimited domestic phone calls, and \$10 for each 1 Gigabyte of data (measured in increments of 10 Megabytes or .01 Gigabytes). If your bill this month was \$43.50, how many Gigabytes of data did you use?

Answer:

Let x be the amount of wireless data that you used on your cellphone. For the given information, we have the following equation:

$$20 + 10x = 43.50$$

which we can solve to get

$$10x = 23.50$$

$$x = 2.35$$

So, 2.35 Gigabytes of data was used.

7.1.4.2 Saving Siblings

Abe and Beth (siblings) are having a contest to see who can save the most money during summer vacation from school. At the start, Beth has \$100 saved and Abe has \$0 saved. Abe is earning \$120/week on odd jobs, and Beth is earning \$100/week. They each save all that they earn. How long before Abe surpasses his sister in savings, given that it is agreed that Beth can count her \$100 starting amount?

Answer:

Let x be the time in weeks from the beginning of the contest.

At time x, Abe has earned $120x$ and Beth has earned $100x + 1000$.

We want to find the value of x such that

$$120x = 100x + 100$$

The above equation can be solved for x as follows:

$$120x - 100x = 100$$

$$20x = 100$$

$$x = 5$$

Thus, Abe will have caught up to his sister in 5 weeks and surpass her savings in the next week.

7.1.4.3 Age Problems

Seven years ago, Bert's age was half of the age that he will be in 3 years. How old is Bert now?

Answer: Let x be Bert's current age. We are given that

$$(x - 7) = \frac{1}{2}(x + 3)$$

$$2(x - 7) = x + 3$$

$$2x - 14 = x + 3$$

$$x = 17$$

. . .

Abby is twice as old as Bud. Bud is 7 years older than Carla. In 5 years, Abby will be three times as old as Carla. How old is Bud now?

Answer: Let x be Bud's current age.

The age of all three now and five years hence is summarized in Table 9.

Table 9. Age table for Abby, Bud and Carla

	Current Age	Five Years Hence
Abby	$2x$	$2x + 5$
Bud	x	$x + 5$
Carla	$x - 7$	$x - 2$

Further, we are given that Abby will be three times as old as Carla in 5 years, which can be written as

$$2x + 5 = 3(x - 2)$$

$$2x + 5 = 3x - 6$$

$$5 + 6 = 3x - 2x$$

$$x = 11$$

So, Bud is currently 11, Abby is 22 and Carla is 4. In five years, Abby will be 27 and Carla will be 9.

. . .

Solve the previous problem, but this time let x be Abby's current age. The final answer should be the same.

. . .

Luis is 3 less than twice Angelo's age. Four years hence, Sidney will be 2 more than twice Angelo's age. Five years ago, Sidney was three times Angelo's age. How old was Luis 5 years ago?

Answer:

Let Angelo's current age be x. The various information is summarized in Table 10.

Table 10. Age table for Luis, Angelo and Sidney

	Five Years Ago	Current Age	Four Years Hence
Luis	$2x-8$	$2x-3$	$2x+1$
Angelo	$x-5$	x	$x+4$
Sidney	$3(x-5)$	$3(x-5)+5$	$2(x+4)+2$

The last row gives us sufficient information to solve for x, since Sidney's age five years ago plus 9 should be equal to Sidney's age four years from now, i.e.,

$$3(x-5)+9=2(x+4)+2$$

$$3x-15+9=2x+8+2$$

$$3x-6=2x+10$$

$$3x-2x=10+6$$

$$x=16$$

So, Angelo is currently 16, Luis is 29 and Sidney is 38. Five years ago, Luis was 24.

. . .

In three years, Vladimir's father (Victor) will be 6 times the age of Vladimir from one year ago. When Vladimir's current age is added to Victors' current age, the total is 61. How old is each one now?

Hint: Let x be Vladimir's current age, then his father's current age is $6(x-1)-3$.

7.1.4.4 Ratio Problems

In a solution (i.e., a mixture), the ratio of coffee to cream is 5:2 (i.e., 5 parts coffee to 2 parts cream). If you have 4 ounces of cream in such a solution, how much coffee do you have?

Answer:

Let x be the amount of coffee. In general, the ratio of coffee to cream is $\frac{5}{2}$ and for the specific example, the ratio is $\frac{x}{4}$. So, we have

$$\frac{5}{2} = \frac{x}{4}$$

$$x = 4 \cdot \frac{5}{2} = 10$$

. . .

Alaine has 40 marbles, 24 are white and 16 are black. Bertrand has 30 marbles but we don't know how many are white or black. If the ratio of the white to black marbles is the same for Alaine and Bertrand, how many white marbles does Bertrand have?

Answer:

The ratio of white to black marbles is $\frac{24}{16} = \frac{3}{2}$.

Let x be the number of white marbles that Bertrand has, which implies Bertrand has $30 - x$ black marbles. Thus, it must be true that

$$\frac{x}{30 - x} = \frac{3}{2}$$

$$x = \frac{3}{2}(30 - x)$$

$$2x = 3(30 - x) = 90 - 3x$$

$$5x = 90$$

$$x = 18$$

So, Bertrand has 18 white marbles and 12 black marbles.

. . .

A special vitamin mixture contains supplements A, B and C in the ratio of 2:3:5 (i.e., 2 parts of A to 3 parts of B and 5 parts of C). If a single pill contains 150 mg of supplement A, how many milligrams of B and C are in the pill?

7.1.4.5 Foot Race

Alice and Bob are to have an 800 meter foot race. Since Alice is faster than Bob, she gives Bob a 25 meter head start. Alice runs at a 4 meters/second place, and

Bob runs at 3.5 meters/second. How long will it be before Alice catches up to Bob?

Answer:

The distance covered by Alice (in meters) after t seconds is $4t$.

The distance covered by Bob (in meters) after t seconds is $3.5t + 25$.

So, our task is to determine the value for t such that $4t = 3.5t + 25$.

Subtract $3.5t$ from both sides of the equation to get $.5t = 25$ which implies that $t = 50$. So, in 50 seconds, both Alice and Bob will be 200 meters from the start of the race. Alice will remain ahead of Bob after that point.

7.1.4.6 Trains Heading Towards Each Other

Two trains (on separate tracks) are 280 miles apart and heading toward each other at rates of 60 mph and 80 mph, respectively. How long will it be before they meet?

Answer:

The gap between the trains is closing at a rate of $60 + 80 = 140$ mph. So, the question is how long does it take to close a gap of 280 miles if the gap is closing at 140 mph? This can be written as the equation $140t = 280$ where t is time. The solution is $t = 2$ hours.

We can check this as follows:

- In 2 hours, the train going 60 mph has traveled 120 miles.
- In 2 hours, the train going 80 mph has traveled 160 miles.
- The sum is 280, which was exactly the distance between the two trains when they started.

7.1.4.7 Clock Hands

How long after 1:00 do the minute and hour hands of a clock overlap?

Answer:

At 1:00, the minute hand is directly pointing to 12 and the hour hand is pointed at 1. So, we know the overlap will happen between 1 and 2, given that the minute hand is moving faster than the hour hand.

In particular, the tip of the minute hand covers 60 ticks (i.e., the little hash marks on the clock) every hour while the tip of the hour hand only traverses 5 ticks per hour. The "speed" of the minute hand is 60 ticks/hour and that of the hour hand is 5 ticks/hour. Further, the hour hand has a 5 tick lead.

The distance from 12 on the clock (measured in tick marks on the clock) by the tip of the minute hand in t hours is $60t$, and the distance from 12 on the clock by the tip of the hour hand in t hours is $5 + 5t$. We need to solve the following equation to determine the solution for the problem:

$$60t = 5 + 5t$$

which implies $55t = 5$ or $t = \frac{1}{11}$ hour.

In terms of ticks from 12:00 on the clock, the tips of both hands are at $\frac{60}{11}$ or $5\frac{5}{11}$ which is to say $\frac{5}{11}$ of a tick past 1:00. Each tick represented a minute. So, we can convert $\frac{5}{11}$ ticks to $\frac{5}{11} \cdot 60 \cong 27.27$ seconds, i.e., the minute and hour hand will overlap in about 27.27 seconds after 1.

. . .

What is the solution to the previous problem if we start at 2:00 rather than 1:00?

7.1.4.8 Rowing in a River

In the absence of any current, Rex can row his boat at a rate equal to 4 times that of the current in the Winding River. He takes a 12-mile trip up the Winding River and then returns, with a total trip time of 5 hours. Find the rate of the current in the Winding River.

Answer: Let x be the rate of the current in the Winding River. We are given that Rex's rowing rate is $4x$ (in the absence of any current).

When going up the river, Rex's rate is $4x - x = 3x$, which means that it takes Rex $\frac{12}{3x}$ hours to row up the river.

When Rex comes back down the river (with the current), his rate is $4x + x = 5x$, which means it takes Rex $\frac{12}{5x}$ hours to row back down the river.

Since we are told that the total trip takes 5 hours, we have that

$$\frac{12}{3x} + \frac{12}{5x} = 5$$

Multiple both sides of the above equation by $15x$ to get

$$60 + 36 = 96 = 75x$$

$$x = \frac{96}{75} = 1.28 \text{ mph.}$$

7.1.4.9 Insect Collecting for a Biology Project

Pat collected insects for a biology project at school. Pat collected 10 more worms than bees, and twice as many flies than bees.

If we let x represent the number of bees collected, then the number of worms is $x + 10$ and the number of flies is $2x$.

The teacher of the class awards 5 points for each worm collected, 2 points for each bee collected, and 3 points for each fly collected. Further, Pat earned 700 points. Determine the value of x?

Answer: From the information given, we have the following equation

$$5(x + 10) + 2x + 3(2x) = 700$$

which can be written as

$$5x + 50 + 2x + 6x = 700$$

$$13x + 50 = 700$$

$$13x + 50 - 50 = 700 - 50$$

$$13x = 650$$

$$x = 50$$

Thus, Pat collected 50 bees, 60 worms and 100 flies.

7.1.4.10 Sum of Four Consecutive Even Numbers

If the sum of four consecutive even numbers is 92, what are the four numbers?

Answer:

If we let x be the smallest of the four even numbers, then the other three numbers are $x + 2$, $x + 4$ and $x + 6$. Since we are told the sum of the four numbers is 96, we have the following equation:

$$x + (x + 2) + (x + 4) + (x + 6) = 92$$

This reduces to

$$4x + 12 = 92$$

$$4x + 12 - 12 = 92 - 12$$

$$4x = 80$$

$$x = 20$$

So, the four numbers are 20, 22, 24 and 26.

7.1.4.11 Sum of Three Consecutive Numbers

If the sum of three consecutive numbers is 96, what are the three numbers?

7.1.4.12 A Few Abstract Number Problems

One number is 25 more than another, and their sum is 175. What are the two numbers?

Answer: Let x be the smaller number, then the other number is $x + 25$. We are also given that

$$x + (x + 25) = 175$$

$$2x + 25 = 175$$

$$2x = 150$$

So, $x = 75$ and the other number is 100.

. . .

One number is 10 times another, and their difference is 81. What are the two numbers?

Answer: Let x be the smaller number, then the other number is $10x$. We are also given that

$$10x - x = 81$$

$$9x = 81$$

So, the smaller number is 9 and the larger number is 90.

. . .

The difference between two numbers is 11. If the larger is subtracted from three times the smaller, the difference is 99. Find the numbers.

Hint: If the difference between two numbers is 11, that means we can represent the two numbers as x and $x + 11$.

7.1.4.13 Coin Problems

There is a jar with pennies, nickels and dimes. You know that the total number of coins is 500, the number of pennies is 4 times the number of nickels, and the total value of the coins is $19.25. Determine the number of each type of coin.

Answer:

If we let x be the number of nickels, then the number of pennies is $4x$.

The number of dimes is 500 minus the number of nickels and pennies, i.e.,

$$500 - (x + 4x) = 500 - 5x$$

Adding the value of the coins in terms of x, we have

$$.01(4x) + .05(x) + .1(500 - 5x) = 19.25$$

$$(.04 + .05 - .5)x + 50 = 19.25$$

$$-.41x = 19.25 - 50 = -30.75$$

$$x = 75$$

Thus, we have 75 nickels, 300 pennies and 125 dimes.

. . .

Avik has collected change worth $130. He has 100 more dimes than nickels and 220 more quarters than dimes. How many coins of each does he have?

Answer:

Let x be the number of nickels.

The number of dimes is $100 + x$, and the number of quarters is $(100 + x) + 220 = 320 + x$.

Since the total value of the coins is \$130, we have

$$.05x + .1(100 + x) + .25(320 + x) = 130$$

Multiply both sides of the above equation by 100, we get

$$5x + 10(100 + x) + 25(320 + x) = 13000$$

$$5x + 10x + 25x + 1000 + 8000 = 13000$$

$$40x = 13000 - 9000 = 4000$$

$$x = 100$$

So, Avik has 100 nickels, 200 dimes and 420 quarters.

As a check, we see that

$$.05(100) + .1(200) + .25(420) = 5 + 20 + 105 = 130$$

. . .

Cookie has \$2.30 in her purse, which contains 36 coins in nickels and dimes. How many nickels does Cookie have in her purse?

7.1.4.14 Mixture Problems

Given 20 liquid ounces of a 20% vinegar solution (80% water). How much vinegar should be added to make it a 25% vinegar solution?

Answer:

At first glance, it may seem that we don't need algebra to solve this problem, since 20% of 20 is 4 and if you just add 1 ounce of vinegar, then the solution is 25% vinegar. However, this is wrong, since the overall amount of liquid increases when you add 1 ounce of vinegar.

Let x be the amount of vinegar added to the solution, and note that we already have $.2(20) = 4$ ounces of vinegar. We want the amount of vinegar (i.e., $x + 4$) to be .25 of the total volume, which is now $20 + x$. This can be represented by the equation

$$x + 4 = .25(20 + x)$$

$$x + 4 = .25x + 5$$

$$.75x = 1$$

$$x = 1\frac{1}{3} = 1.\overline{33}$$

. . .

We can also do the above problem in reverse, i.e., assume you have a $21\frac{1}{3}$ (i.e., $\frac{64}{3}$) ounce solution of 25% vinegar in water, and want to know how much vinegar to extract to get the mixture down to 20% vinegar.

Answer: Let x be the amount of vinegar to be extracted from the solution, and note that we start with $.25\left(\frac{64}{3}\right) = \frac{1}{4}\cdot\frac{64}{3} = \frac{16}{3}$ ounces of vinegar. We are being asked to determine how much vinegar to extract such that the resulting solution is 20% vinegar. This can be represented by the equation

$$\left(\frac{16}{3} - x\right) = .2\left(\frac{64}{3} - x\right) = \frac{1}{5}\left(\frac{64}{3} - x\right)$$

Multiple both sides of the above equation by 15 to get

$$80 - 15x = 64 - 3x$$

$$16 = 12x$$

$$x = \frac{16}{12} = \frac{4}{3} = 1\frac{1}{3}$$

which is verified by the previous problem.

. . .

A car's fuel tank has a capacity of 20 gallons. The completely full tank current contains a mixture of 85% gasoline and 15% ethanol. How many gallons must be replaced by a 55% gasoline and 45% ethanol solution to result in a full tank (i.e., 20 gallons) of a 70% gasoline and 30% ethanol solution?

Answer:

Let x be the amount of fuel removed from the starting mixture. The problem is to determine x from the given information. If we remove x amount of the original fuel, then $.15x$ of the ethanol has been removed. By adding an amount x of the new mixture, we add $.45x$ ethanol. We want the resulting mixture to have $.3(20) = 6$ gallons of ethanol. The situation is summarized in Table 11.

Table 11. Gasoline mixture problem

	Original	**Replaced**	**Added**	**Result**
Concentration of ethanol	$.15$	$.15$	$.5$	$.3$
Amount of ethanol	$.15(20) = 3$	$-.15x$	$.45x$	$.3(20) = 6$

To determine the value of x, we need to solve the following equation:

$$3 - .15x + .45x = 6$$

$$3 + .3x = 6$$

$$.3x = 3 \text{ which implies } x = 10$$

Let's check the answer

Amount of ethanol from the 10 gallons of the original mixture	Amount of ethanol from the 10 gallons added to the tank	Total amount of ethanol after change	Percentage of ethanol after the change
$.15(10) = 1.5$	$.45(10) = 4.5$	$1.5 + 4.5 = 6$	$\frac{6}{20} = .3$ or 30%

. . .

Coffee mixture A is 10% cream, and coffee mixture B is 20% cream. How many liters of each solution should be used to make 20 liters of a coffee mixture which is 15% cream?

Hint: Let x be the amount of coffee mixture A.

7.1.4.15 Averages

Ernesto's average score on the first 3 tests in a math class was 75. On the next 4 tests, his average score was 95. What was his average score on all 7 tests?

(Just an easy first averages problem to get us started – no algebra required to solve this.)

Answer:

The first bit of information tells us that the sum of Ernesto's first 3 tests was $3 \cdot 75 = 225$. The second piece of information implies the sum of his next 4 tests was $4 \cdot 95 = 380$. So, the sum of all 7 tests was $225 + 380 = 605$ and his overall test score average is $\frac{605}{7} = 86\frac{3}{7}$.

. . .

The average of 7 numbers 140. If one number is removed, the average of the remaining numbers is 107. What was the removed number?

Answer:

We are given that the average of the original 7 numbers is 140, which implies that the sum of the original 7 numbers is $7 \cdot 140 = 980$.

Let x be the number removed, then we have

$$\frac{980 - x}{6} = 107$$

$$980 - x = 6 \cdot 107 = 642$$

$$x = 980 - 642 = 338$$

. . .

Khalid's average score on the first 5 tests in a class was 82. What must Khalid's average be on the remaining 5 tests to increase his average to 87?

7.1.4.16 Work-related Problems

Al can mow a lawn of a given size in 40 minutes, Betty can mow the lawn in 60 minutes and Carl can mow the lawn in 60 minutes. How long will it take for the three of them to mow the lawn together?

Answer:

Al's rate of work regarding lawn mowing is 1 lawn/40 minutes, and Betty's and Carl's rate of work is 1 lawn/60 minutes. If they all work together on the same lawn, what is their combined rate of work? It is the sum of their individual rates of work, i.e.,

$$\frac{1}{40}+\frac{1}{60}+\frac{1}{60}=\frac{3+2+2}{120}=\frac{7}{120}$$

If we let x be how long it takes for all three to mow the lawn together, then

$\frac{1}{x}=\frac{1}{\frac{7}{120}}$ which implies $x=\frac{120}{7}=17\frac{1}{7}$ minutes.

Another way to view the problem is to recognize that the sum of the rates of work of the individuals must equal the overall (combined) rate of work, i.e.,

$$\frac{1}{40}+\frac{1}{60}+\frac{1}{60}=\frac{3+2+2}{120}=\frac{7}{120}=\frac{1}{x}$$

and the solve for x.

. . .

In a variation of the previous problem, assume the rates of work for Al and Betty are the same as before, and we know that all three can collectively mow the lawn in 15 minutes. How long does it take Carl to mow alone?

Answer:

Let x be the number of minutes it takes Carl to mow the lawn. Thus, Carl mows the lawn at a rate of 1 lawn/x minutes. As in the previous problem, the rates of the individual must equal the rate of work for the combined effort, i.e.,

$$\frac{1}{40}+\frac{1}{60}+\frac{1}{x}=\frac{1}{15}$$

Multiply both sides of the equation by 120 to get

$$3+2+\frac{120}{x}=8$$

Thus, $\frac{120}{x}=3$ which implies $x=\frac{120}{3}=40$.

. . .

For problems like the previous two, one is given the time that it takes different entities to complete a task (call these times $t_1, t_2, \dots, t_n$). It is assumed that the entities can work together to complete the same task, without interfering with each other, in time t. Each time can be converted into a rate. For example, if the first entity can complete a given task in time t_1, we can say that $\frac{1}{t_1}$ is the rate at which the first entity completes the given task. Further, the various rates are related as follows:

$$\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_n}=\frac{1}{t}$$

If all but one rate is given, we can solve for that one unknown.

Consider the example of 10 stone masons who can complete a boundary wall in the following times: $6,7,8,9,10,7,8,6,11$ and 9 minutes, respectively. How long will it take all ten of the stone masons to complete one boundary wall? From the above discussion, we have

$$\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{7}+\frac{1}{8}+\frac{1}{6}+\frac{1}{11}+\frac{1}{9}=\frac{1}{x}$$

Multiple both sides by the $lcm(6,7,8,9,10,11)=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11=27720$ to get

$$4620+3960+3465+3080+2772+3960+3465+4620+2520+3080$$
$$=35542=\frac{27720}{x}$$

Solving for x, we get $x=\frac{27720}{35542}\cong .78$ minutes.

Try repeating the above calculations using a spreadsheet. Microsoft Excel and Google Sheets both have an LCM function that works for a list of numbers.

. . .

Jack and Diane can each gather all the strawberries from a one acre lot at the same rate. Working alone Luigi can gather all the strawberries from a one acre lot in 180 minutes. Working together, Jack, Jill and Luigi can gather all the strawberries from a one acre lot in 90 minutes. How long does it take Jack (or Diane) to gather all the strawberries alone?

7.1.4.17 Balancing Problems

Figure 3 depicts a balance lever. The general rule is that the weight of the object on one side of the fulcrum times its distance from the fulcrum must equal the weight of the object on the other side times its distance from the fulcrum in order for the level to be balanced. Distance is measured from the center of the object to the fulcrum point. In the example, we have $2\cdot 2=4\cdot 1$ and so the lever is balanced.

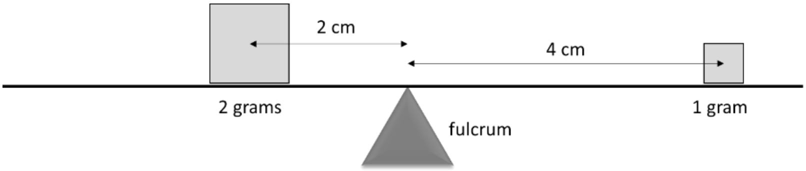

Figure 3. Balanced lever with one object on each side

The general rule can be extended to several objects on either side of the fulcrum. In such cases, the sum of the products of the weights times distances on one side of the fulcrum must each that of the other side for the lever to be balanced. In Figure 4, the lever is balance since $1 \cdot 4 + 2 \cdot 2 = 8 = 2 \cdot 4$.

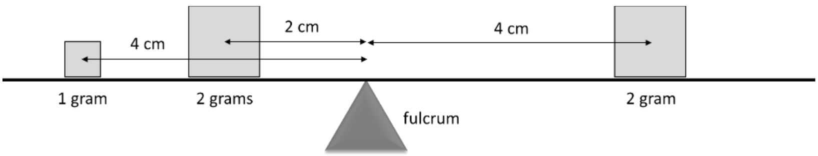

Figure 4. Balanced lever with several objects on one side

. . .

If on one side of a fulcrum there is a 3 gram weight 4 cm from the fulcrum and a 13 gram weight 1 cm from the fulcrum, where must a 5 gram weight be placed on the other side of the fulcrum in order to balance the lever?

Answer: The general rule tells us that we must satisfy the following equation in order to balance the lever, where x is the distance of the 5 gram weight from the fulcrum:

$$3 \cdot 4 + 1 \cdot 13 = 5x$$

$$25 = 5x \text{ which implies } x = 5$$

. . .

Where should one place 3 and 5 gram weights (on either side of a fulcrum) to balance a lever?

Answer: There are an infinite number of solutions. For example, if we choose to put the 3 gram weight at a distance 5 cm from the fulcrum, then the level can be balanced by putting the 5 gram weight at a distance 3 cm from the fulcrum. In general, if the 3 gram weight is placed x cm from the fulcrum, then we must have

$$3x = 5y$$

for the lever to be balanced. In other words, the 5 gram weight must be placed $y = \frac{3}{5}x$ cm from the fulcrum for the level to be balanced.

If the 3 gram weight is put 10 cm from the fulcrum, where should the 5 gram weight be placed to balance the level?

. . .

Arnold and Beatrice are sitting at either end of a 6 meter seesaw. Arnold weighs 120 kg and Beatrice weighs 60 kg. Where should the fulcrum be placed to balance the seesaw?

Hint: Let x be the distance that Arnold is sitting from the fulcrum.

. . .

The sum of two weights of two objects is 200 lbs. One weight is placed 36 inches from the fulcrum, and the other is placed 24 inches on the other side of the fulcrum. For the lever to be in balance, determine the weight of each object.

Answer:

Let x be the weight of the object that is 36 inches from the fulcrum.

We are given

$$36x = 24(200 - x)$$

$$36x = 4800 - 24x$$

$$60x = 4800$$

$$x = 80$$

So, the object 36 inches from the fulcrum weighs 80 lbs. and the object on the other side of the fulcrum weighs 120 lbs.

7.2 Multiple Variables

7.2.1 Overview

The next level of complexity, beyond the single linear variable case, is the case of multiple linear variables. In this case, there can be any number of variables where each variable is raised to the power 1. For example, the following equation has two linear variables:

$$3x + 4y = 10$$

The above equation is that of a straight line.

The general form of an equation with multiple linear variables is as follows:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

The terms $a_1, a_2, \ldots, a_n$ and c are all constants. The subscripts on the constants are only used for naming purposes and are not to be confused with exponents. The terms $x_1, x_2, \ldots, x_n$ are variables. Similarly, the subscripts on the variables are only used for naming purposes.

7.2.2 Operations

Operations in the case of multiple linear variables (or for that matter multiple non-linear variables) following the same basic laws as the single linear variable case.

7.2.2.1 Addition and Subtraction

The main rule here is that we add (or subtract) like terms. By "like terms", we mean relating to the same variable. For example, consider the two expressions

$$7x + 3y - 4z + 2$$

$$11x - 3y + 7z + 8$$

Adding the two expressions, we have

$$(7x + 3y - 4z + 2) + (11x - 3y + 7z + 8)$$

$$= (7 + 11)x + (3 - 3)y + (-4 + 7)z + (2 + 8)$$

$$= 18x + 3z + 10$$

Subtracting the second expression from the first, we get

$$(7x + 3y - 4z + 2) - (11x - 3y + 7z + 8)$$

$$= (7 - 11)x + (3 + 3)y + (-4 - 7)z + (2 - 8)$$

$$= -4x + 6y - 11z - 6$$

Note that – (-3) is just 3.

The pattern is exactly the same if we have more variables.

Also, the expressions don't need to have the same variables to be added or subtracted, e.g.,

$$(8x - z) + (4y + 3z) = 8x + 4y + 2z$$

or even something like

$$(7x + 3y) - (4w - 3z) = 7x + 3y - 4w + 3z$$

7.2.2.2 Multiplication

Multiplication of expressions with several linear variables is based on the distributed law (similar to what we saw for arithmetic and the single linear variable case). The complication is that the distribution law needs to be applied many times. The general rule (which can be proven using the distributive law) is

> When multiplying one linear expression by another, each term in one expression is to be multiplied times each term in the other expression, and then the terms are to be added.

For example, to multiple $3x + 2y + 5z + 1$ times $x + 3y + z + 2$, we need to multiply each of the 4 terms in the first expression times each of the 4 terms in the second expression, add and combine like terms. Here it goes

$$3x^2 + 9xy + 3xz + 6x + 2xy + 6y^2 + 2yz + 4y + 5xz + 15yz + 5z^2 + 10z + x + 3y + z + 2$$

Because of the commutative law, the order of the multiplication within a given term does not matter, e.g., $xy = yx$. Secondly, the result is no longer a linear expression but rather a second order expression (several terms are raised to the second power). Finally, we can simplify the expression by combining like terms as follows:

$$3x^2 + 6y^2 + 5z^2 + 11xy + 8xz + 17yz + 7x + 7y + 11z + 2$$

There are symbolic math applications that can do such calculations. For example, if one enters $(3x + 2y + 5z + 1)(x + 3y + z + 2)$ into Wolfram Alpha (at https://www.wolframalpha.com), the following answer is returned (along with a lot of other information):

$$3x^2 + 11xy + 8xz + 7x + 6y^2 + 17yz + 7y + 5z^2 + 11z + 2$$

which is the same as the answer provided previously but with a different order (which does not matter because of the commutative law of addition). Symbolab (www.symbolab.com) will also give you the same answer with yet another equivalent ordering of the terms.

When doing such multiplications by hand, another way to view the problem is with a table. Put the terms from one expression in the cells of the top row of a table, and put the terms from the other expression in the left-hand column. Next, multiply each item in the left column times each item in the top row, and record the results in the table (as shown in Table 12 for the example at hand). At this point, you need to add all the terms in the interior of the table and then collect like terms where possible.

Table 12. Multiplying linear expressions using a table

	$\boldsymbol{x}$	$\boldsymbol{3y}$	$\boldsymbol{z}$	$\boldsymbol{2}$
$\boldsymbol{3x}$	$3x^2$	$9xy$	$3xz$	$6x$
$\boldsymbol{2y}$	$2xy$	$6y^2$	$2yz$	$4y$
$\boldsymbol{5z}$	$5xz$	$15yz$	$5z^2$	$10z$
$\boldsymbol{1}$	x	$3y$	z	2

In terms of notation, juxtaposition is used to indicate multiplication, e.g., $2x + y - z$ times $4x - y + 3z - 6$ is written as

$$(2x + y - z)(4x - y + 3z - 6)$$

Try the following multiplications and check your answer with Wolfram Alpha or Symbolab:

- $(3x - y)(x + 2y)$
- $(2x + 3y + 1)(x + 7y)$
- $(x + y)(x - z)$
- $(5x - y + 3)(x + 2y - z)$
- $(x + y + z + 1)(x + y + z + 1)$

7.2.2.3 Division

Long division is usually done only when one variable is involved. However, we can still work with fractions involving several variables. For example,

$$\frac{1}{x+y} - \frac{4}{x+z} = \frac{x+z}{(x+y)(x+z)} - \frac{4(x+y)}{(x+y)(x+z)} = \frac{-3x+z-4y}{(x+y)(x+z)}$$

The process is the same as that for arithmetic, i.e., find a common denominator and then add or subtract. Since we are dealing with unknowns, the common denominator is typically just the product of the denominator (as was the case in the example above). There are some exceptions, however. Consider the following fraction addition problem:

$$\frac{7}{(x+2y)^3(7x-y)^5} + \frac{3}{(x+2y)^7(7x-y)^2}$$

In this case, we can do a little better than multiplying the denominators. We only need the highest power of like terms for the common denominator, i.e., $(x + 2y)^7(7x - y)^5$. Adding terms and using the common denominator, we get

$$\frac{7(x+2y)^4 + 3(7x-y)^3}{(x+2y)^7(7x-y)^5}$$

7.2.3 Solving

In the case of a linear equation with a single variable, we saw that only one solution exists. For example, the equation $3x + 4 = 7$ has the solution $x = 1$. The situation is more complex for a system of linear equations with several variables. In fact, there is an entire branch of mathematics (i.e., Linear Algebra) devoted to the study of systems of linear equations. For our purposes in this book, we will concentrate on the basic concepts. For a summary of Linear Algebra, see Section 15 of the book Mathematical Thinking [9].

The equation $-15x + 3y = 12$ has an infinite number of solutions. We can see this by solving for y in terms of x

$$3y = 15x + 12$$

$$y = 5x + 12$$

For each value of x, we have a different solution to the equation. For example, all of the following (x, y) pairs are solutions:

$$(1,17)$$

$$(-1,7)$$

$$(0,12)$$

$$(\frac{1}{5}, 13)$$

The general form of each solution (ordered pair) is $(x, 5x + 12)$ which can be decomposed as follows:

$$(x, 5x + 12) = x(1,5) + (0,12)$$

All the solutions can be viewed as the pair $(0,12)$ plus a multiple of $(1,5)$.

. . .

In the case of two variables (raised to the first power), it takes two equations to get a unique solution. For example, consider the two equations

$$x - 3y = 1$$

$$2x + 4y = 12$$

If we multiple the first equation by -2, to get $-2x + 6y = -2$, and then add to the second equation, we get

$$10y = 10$$

$$y = 1$$

Substituting back into either equation, we get that $x = 4$.

. . .

As a slightly more complex example, consider the equations

$$2x + 3y = 9$$

$$3x + 7y = 14$$

As before, we want to eliminate one of the variables. To that end, multiple the first equation by $\frac{1}{2}$ to get the following equivalent system of equations

$$x + \frac{3}{2}y = \frac{9}{2}$$

$$3x + 7y = 14$$

Now add -3 times the first equation to the second equation to get

$$x + \frac{3}{2}y = \frac{9}{2}$$

$$(3x - 3x) + \left(7 - \frac{9}{2}\right)y = 14 - \frac{27}{2}$$

which simplifies to

$$x + \frac{3}{2}y = \frac{9}{2}$$

$$\frac{5}{2}y = \frac{1}{2}$$

We can now solve the second equation to get the result $y = \frac{1}{5}$. Plugging this back into the first equation gives $x = \frac{9}{2} - \frac{3}{10} = \frac{45-3}{10} = \frac{42}{10} = \frac{21}{5}$.

. . .

In case of 2 variables, it is a necessary but not sufficient condition to have two equations for a unique solution. For example, the following two equations are multiples of each other and we effectively have only one equation. This leads to an infinite number of solutions which can be derived from either equation.

$$3x - 4y = 5$$

$$6x - 8y = 10$$

In yet other cases, we have two equations but there are no solutions at all, e.g.,

$$3x + 11y = 6$$

$$3x + 11y = 9$$

The two equations contradict each other, implying that $6 = 9$ which we know is false.

For linear systems with two variables, there are three cases, i.e., an infinite number of solutions, a unique solution or no solution.

. . .

In the case of 3 variables, we use a method of solution similar to that for the case of 2 variables, i.e., progressive elimination of variables by adding (or subtracting) multiples of one equation to (or from) another equation. Consider the following system of three linear equations:

$$x - y + z = 4$$

$$x - y - z = 4$$

$$2x + 2y + 4z = 6$$

We first eliminate x from the second and third equations. To accomplish this task, we subtract the first equation from the second, and add -2 times the first equation to the third. This results in the following equivalent system of equations:

$$x - y + z = 4$$

$$-2z = 0$$

$$4y + 2z = -2$$

From the second equation, we see that $z = 0$. Putting $z = 0$ into the third equations, gives us $y = -\frac{1}{2}$. Finally, we can use the first equation to determine that $x = 7/2$.

The solution can be checked at Wolfram Alpha or Symbolab by typing in the following:

x-y+z=4; x-y-z=4; 2x+2y+4z=6

. . .

Our first example with 3 variables was designed to resolve quickly. Normally, more steps are required to solve a linear system of 3 equations. Consider the following linear system:

$$x - y + z = 4$$

$$x - 2y - z = 4$$

$$2x + 2y + 4z = 6$$

In order to solve the problem, we first put the system in the form of a table (this is formally known as the **augmented matrix** representing the system of equations), see table below. The first column represents the x terms, the second column represents the y terms, the third column represents the z terms and the fourth column represents the constants.

1	−1	1	4
1	−2	−1	4
2	2	4	6

We first eliminate the x term from the second and third equations. This is done by subtracting the first equation (top row) from the second equation (2nd row), and then subtracting twice the first equation from the third equation (3rd row).

1	−1	1	4
0	−1	−2	0
0	4	2	−2

Next, eliminate the y term from the third equation, by adding 4 times the second equation to the third equation.

1	−1	1	4
0	−1	−2	0
0	0	−6	−2

Divide the third equation by -6 and we have determined the value of z, i.e., $\frac{1}{3}$.

1	−1	1	4
0	−1	−2	0
0	0	1	1/3

Now we can use the third equation to eliminate z from the second equation and then solve for y. Add twice the third equation to the second equation.

1	−1	1	4
0	−1	0	2/3
0	0	1	1/3

Multiple the second equation by -1 and we see that $y = -\frac{2}{3}$.

1	−1	1	4
0	1	0	−2/3
0	0	1	1/3

Subtract the third equation from the first to eliminate the z term.

1	-1	0	$11/3$
0	1	0	$-2/3$
0	0	1	$1/3$

Finally, we add the second equation to the first, which gives us the value for x.

1	0	0	3
0	1	0	$-2/3$
0	0	1	$1/3$

The above process is an example of the **Gauss-Jordan Elimination** [10]. The idea is to get the left-hand side of the table to have ones along the diagonal and zeros elsewhere. This allows you to just read-off the values of the variables in the right-most column of the table.

. . .

If there is only one equation with three variables, then there will be an infinite number of solutions. For example, consider the equation

$$x + y + z = 17$$

There are an infinite number of points (x, y, z) that satisfy the above equation, e.g., $(8,9,0)$, $(-5,10,12)$ and $(\frac{1}{2}, 10, \frac{13}{2})$. If we solve for x in terms of y and z (i.e., $x = 17 - y - z$), we can write the general form of the solution as

$$(17 - y - z, y, z) = (17,0,0) + y(-1,1,0) + z(-1,0,1)$$

Pick any two values for y and z, and the above expression will give a solution to the original equation. For example, try $y = \frac{4}{5}$ and $z = \frac{1}{5}$ to get

$$(17,0,0) + \left(-\frac{4}{5}, \frac{4}{5}, 0\right) + \left(-\frac{4}{5}, 0, \frac{4}{5}\right) = \left(17 - \frac{8}{5}, \frac{4}{5}, \frac{4}{5}\right) = (\frac{77}{5}, \frac{4}{5}, \frac{4}{5})$$

Plugging this solution back into the original equation, we get

$$\frac{77}{5} + \frac{4}{5} + \frac{4}{5} = \frac{85}{5} = 17$$

. . .

If you have two equations and three variables, there is either an infinite number of solutions or no solution. It is not possible to have a unique solution in this case.

For an example of no solutions, consider the contradictory equations

$$2x - 3y + 7z = 11$$

$$2x - 3y + 7z = -3$$

For an example of an infinite number of solutions, consider the following two equations

$$x + 2y + 3z = 7$$

$$2x + 5y + 7z = 19$$

The corresponding augmented matrix is

1	2	3	7
2	5	7	19

Multiple the first row by -2 and add to the second row to get

1	2	3	7
0	1	1	5

Subtract twice the second row from the first to get

1	0	1	-3
0	1	1	5

This is as far as we can reduce the equations. From the above matrix, we see that both x and y can be written in terms of z as follows:

$$x = -z - 3$$

$$y = -z + 5$$

The general solution can be written as

$$(x, y, z) = (-z - 3, -z + 5, z) = z(-1, -1, 1) + (-3, 5, 0)$$

Each value of z in the above gives another solution to the original system of equations. For example, take $z = 5$, which gives the solution

$$5(-1, -1, 1) + (-3, 5, 0) = (-8, 0, 5)$$

. . .

To get a unique solution when dealing with three variables in a linear system, you need at least three equations. However, this is only a necessary condition. This is not a sufficient condition, since the equations could result in a contradiction (leading to no solutions) or some of the equations could cancel after several row operations (leading to an infinite number of solutions).

In general, at least n equations are necessary to get a unique solution to a linear system with n variables. This condition is not sufficient, however.

- Some of the equations may cancel each other, leading to less than n equations. There are an infinite number of solutions in this case, assuming there are no contradictions among the remaining equations.

- Some of the equations (perhaps after several row operations) may lead to a contradiction. There are no solutions in this case.

. . .

Use the Gauss-Jordan Elimination to find unique solutions to the following systems of linear equations. You can check your answer using Wolfram Alpha or Symbolab (remember to separate the equations by a semicolon when entering into these applications). The order that one enters the equations into the augmented matrix does not matter. This fact can sometimes be helpful. For example, in the fourth problem below, it would be easier to put the second equation first since it already has a 1 for the coefficient of the x-term.

1. $x + y = 2$
   $3x - y = 4$
2. $x + 3y = 5$
   $3x - y = 3$
3. $4x + 2y = 5$
   $3x + 2y = 4$
4. $3x - 2y + 9z = 6$
   $x - y + z = 2$
   $2x + 4y - z = 4$
5. $x + 2y + 3z = 4$
   $4x + 3y + 2z = 1$
   $x + 3y + z = 5$

Use the Gauss-Jordan Elimination to find a contradiction in the following system of equations and thus conclude there is no solution.

$$3x + 4y + 2z = 8$$

$$-x + 2y - 2z = 4$$

$$2x + y + 2z = 3$$

Hint: Eventually, you will get the third row in the form of zeros for the x, y and z terms and 1 on the right, meaning that $0 = 1$ (a contradiction and thus there is no solution to the system of equations).

Use the Gauss-Jordan Elimination to show that the following systems of equations reduces to just two equivalent equations. Write x and y in terms of z. Show there is a different solution to the problem for each value of z. Thus, there are an infinite number of solutions.

$$x + 4y - 2z = 1$$
$$2x - 3y + 4z = 6$$
$$3x + y + 2z = 7$$

7.2.4 Graphs

7.2.4.1 Coordinate Systems, Points and Vectors

Thus far, we have focused on the symbolic aspects of algebraic equations. There is also a visual aspect (at least for two and three dimensions). As a first step, we need to talk about coordinate systems and points.

One of the more common (perhaps most common) coordinate systems is known as the **Cartesian coordinate system**. This system is named after René Descartes, the famous French philosopher, mathematician, and scientist who lived from 1596 to 1650. In the Cartesian coordinate system, points are located by two coordinates, i.e., a vertical coordinate and a horizontal coordinate.

- The vertical coordinate indicates how far (up or down) a point is from the **origin** (denoted as 0). The down direction is marked by negative numbers. The vertical axis is often referred to as the y-axis.
- The horizontal coordinate indicates how far (left or right) a point is from the origin. The left direction is marked by negative numbers. The horizontal axis is often referred to as the x-axis.
- There are four quadrants (as labeled in Figure 5) defined by the four possible combinations of positive and negative x and y coordinates.

Figure 5 depicts an example Cartesian coordinate system with four points. For example, point (1,2) is one unit to the right and 2 units up, and the point (-3,-2) is two units to the left and three units down.

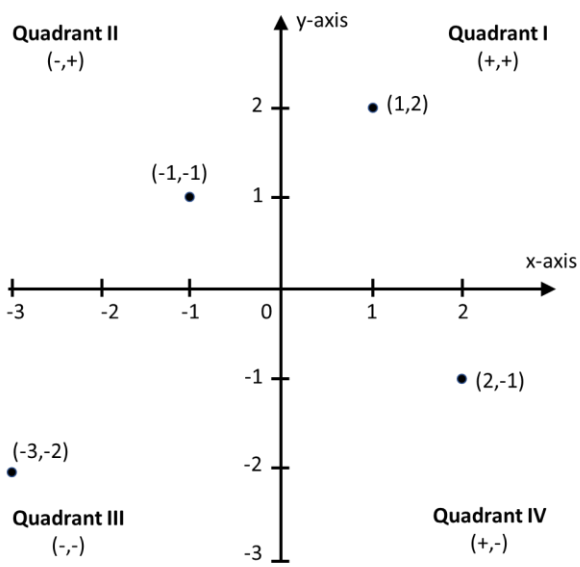

Figure 5. Cartesian coordinate system with several points

Slightly more complex than the concept of a point is that of a **vector**, which is defined by two points and a direction between the two points. Vectors are used, for example, in Physics to indicate the direction and magnitude of forces. Figure 6 shows two vectors:

- The vector $\overrightarrow{AB}$ from point $A(-1,-1)$ to point $B(1,2)$. The order is important, $\overrightarrow{BA}$ is a different vector from $\overrightarrow{AB}$. In the vector $\overrightarrow{AB}$, A is the initial point and B is the terminal point.
- The vector $\overrightarrow{OC}$ is from the origin $O(0,0)$ to the point $C(0,2)$.

Vectors $\overrightarrow{AB}$ and $\overrightarrow{OC}$ are equivalent in the sense that they have the same magnitude (length) and direction (slope). A vector with fixed initial and terminal points is called a **bound vector**. If you are only concerned about the magnitude and direction of the vector, then you have what is called a **free vector**. In the case of free vectors, only the termination point is specified (with the origin assumed as the initial point). In the example at hand, the vector $(0,2)$ is a vector with initial point at the origin and termination point at $(0,2)$ or any other vector with the same length and slope (e.g., $\overrightarrow{AB}$).

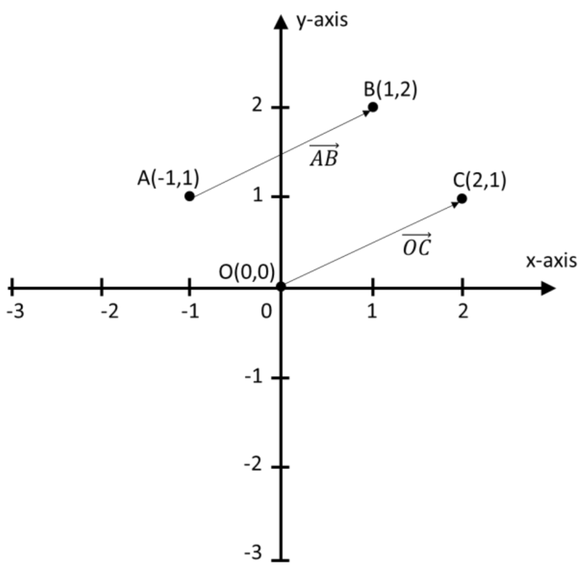

Figure 6. Equivalent vectors

Adding vectors is very easy but you first need to orient the vectors so they both have their initial point at the origin. As an example, let’s add the two vectors shown in Figure 7.

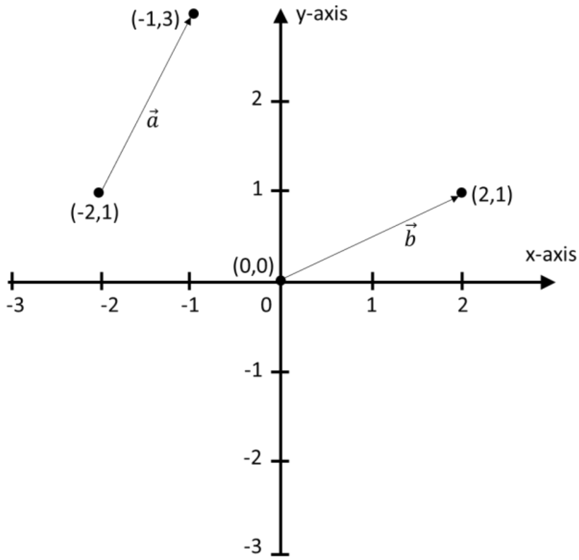

Figure 7. Vector addition – initial state

The first step is to translate vector $\vec{a}$ to have its initial point at the origin. To do this, we add 2 to the x-coordinates of the initial and terminal points of $\vec{a}$, and subtract 1 from the y-coordinates of the initial and terminal points of $\vec{a}$. The situation after the translation is shown in Figure 8.

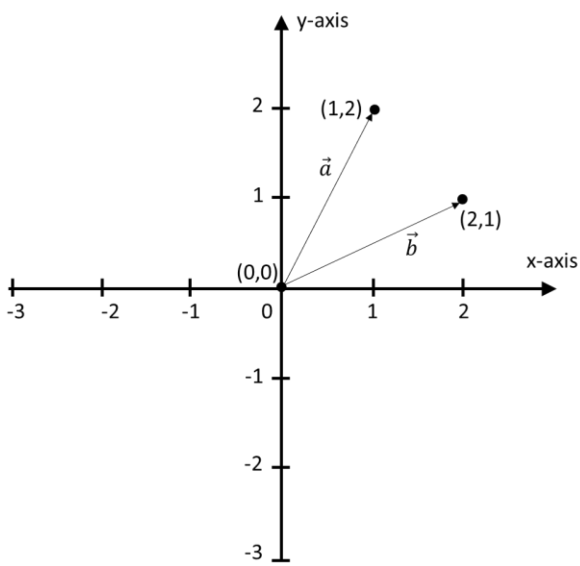

Figure 8. Vector translation (first step in addition)

With both vectors now having their initial point at the origin, the sum of the vectors is gotten by adding the x and y coordinates of each vector, as shown in Figure 9. So, $\overrightarrow{a+b}$ is the vector with initial point at the origin and terminal point (3,3).

There is also a geometric interpretation, i.e., the sum of two vectors (with a common initial point) is the diagonal of the parallelogram formed by the two vectors. The sum of the two vectors is the diagonal that starts with the initial point of $\vec{a}$ and ends with the terminal point of $\vec{b}$. The other diagonal is $\overrightarrow{a-b}$ (not shown in the figure).

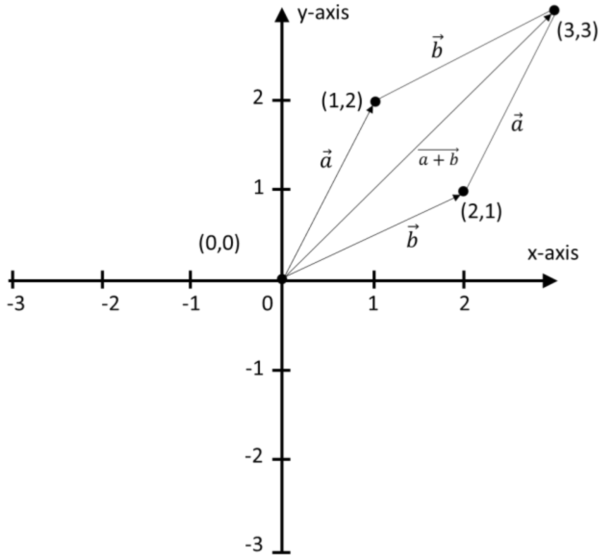

Figure 9. Vector addition – last step

Determine the following vector sums and show on a graph:

- $\vec{a}$ (with initial point at the origin, and (-3,1) as its termination point) added to $\vec{b}$ (with initial point at the origin, and (2,4) as its termination point)
- $\vec{a}$ (with initial point at (1,1), and (-2,2) as its termination point) added to $\vec{b}$ (with initial point at the origin, and (2,4) as its termination point)
- For the previous exercise, determine $\vec{a}-\vec{b}$. The solution for the previous exercise and this one can be found on GeoGebra at https://www.geogebra.org/calculator/terkdbk7. Note that $\vec{a}-\vec{b}$ is shown twice – once with the initial point at the origin, and a second time (via translation) to be in the parallelogram formed by vectors $\vec{a}$ and $\vec{b}$.

- $\vec{a}$ (with initial point at (-1,-1), and (-3,-2) as its termination point) added to $\vec{b}$ (with initial point at (1,-1), and (2,-3) as its termination point). The online graphing application GeoGebra can be used to assist with the drawing and addition of vectors, see https://www.geogebra.org/calculator/cd6gep9y for a solution to this exercise.

Further, it is possible to multiply a vector by a **scalar** (i.e., a constant). When multiplying a vector by a scalar, only the length changes (the slope remains the same).

- If the scalar is larger than 1, the resulting vector grows in length.
- If the scalar is between 0 and 1, the resulting vector shrinks.
- If the scalar is negative, the resulting vector goes in the opposite direction.

An example of each case is shown in Figure 10. (The figure was produced using the GeoGebra online graphing application, see https://www.geogebra.org/calculator/nwg9sptk.)

- $\vec{v} = 2\,\vec{u}$ where $\vec{u} = \overrightarrow{OB}$ and $\vec{v} = \overrightarrow{OA}$
- $\vec{w} = -1.5\,\vec{u}$
- $\vec{b} = .4\,\vec{a}$ where $\vec{a} = \overrightarrow{OC}$

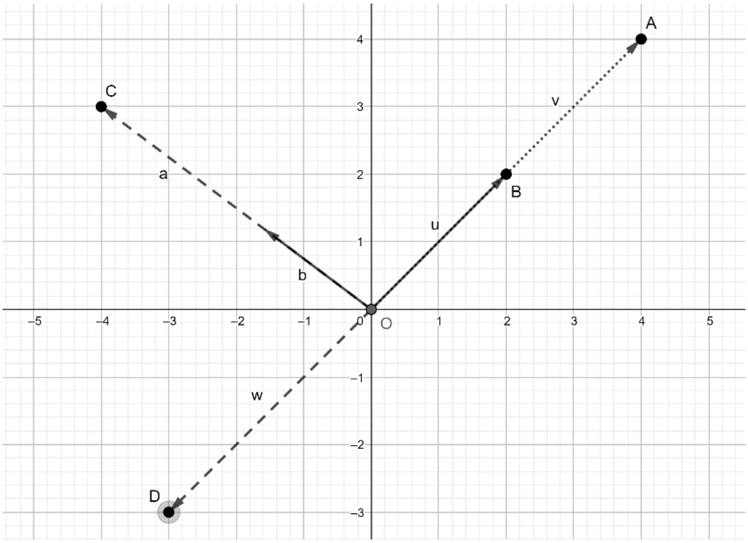

Figure 10. Vector scaling

7.2.4.2 Lines

A **line** (sometimes "straight" line to distinguish from various types of curved lines) is a set of continuous points with constant slope, extending infinitely in two directions. We've already seen an example in Section 7.2.3, i.e., $y = 5x + 12$. The equation for a line represents an infinite set of ordered pairs (x, y). For each x there is a unique y as determined by the equation for the line. In our example, if $x = 0$, then we can use the equation of the line to determine that $y = 12$. Similarly, if we are given a y value such as 17, we can use the equation to determine that x must be 1. The graph for $y = 5x + 12$ is shown in Figure 11 (technically, this is just part of the graph since the line extends indefinitely in both directions). The graph was generated using the GeoGebra graphing application at https://www.geogebra.org/calculator/hash7uyz. When drawing by hand, you only need to graph two points that lie on the line and then connect the points using a straightedge.

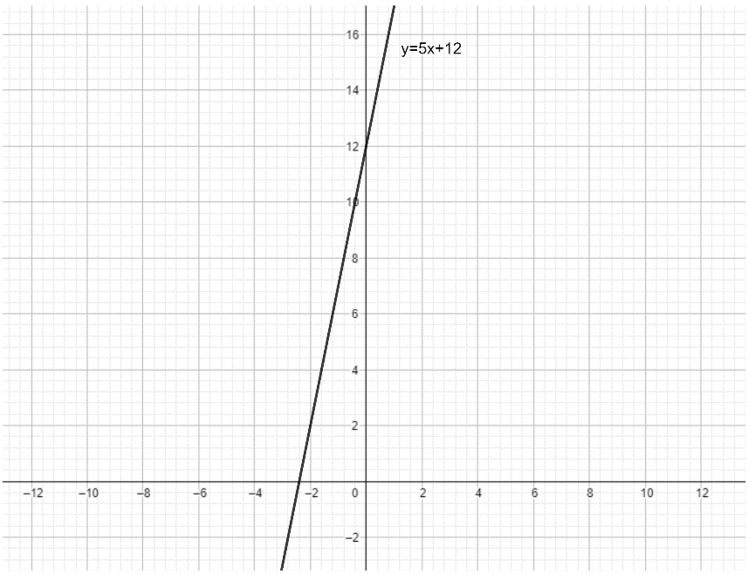

Figure 11. Graph of the line y=5x+12

In terms of definitions, the **slope** of a line is the rise over the run (change in y divide by a change in x). To determine the slope of a line, you need two points. If (x_1, y_1) and (x_2, y_2) are two points on a line then the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For our example, take the points $(0,12)$ and $(1,17)$, and then compute the slope as follows:

$$m = \frac{17 - 12}{1 - 0} = 5$$

The **y-intercept** of a line is where the line intersections the y-axis. For our example, the y-intercept is 5. To find the y-intercept of a line, simply set $x = 0$ and solve for y.

The equation for a line can be expressed in several formats:

- slope-intercept form: $y = mx + b$ where m is the slope of the line and b is the y-intercept
- point-slope form: $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) is a point on the line
- standard form: $Ax + By = C$.

All three forms are equivalent and can be derived from each other. For example, if we start with a line in the standard form, we can divide by B to get

$$\frac{A}{B}x + y = \frac{C}{B}$$

and solve for y to get

$$y = -\frac{A}{B}x + \frac{C}{B}$$

which is in the slope-intercept form, with slope $-\frac{A}{B}$ and y-intercept $\frac{C}{B}$.

To determine the equation of a line, you need either two points on the line, or one point on the line and the slope of the line.

Example 1: Find the equation of the line with slope -2 and lying on the point $(3,4)$.

Using the point-slope formula, we get

$$y - 4 = -2(x - 3)$$

which we can write in slope-intercept form

$$y = -2x + 10$$

Example 2: Find the equation of the line going through points $(1,2)$ and $(5,9)$.

The slope is $m = \frac{9-2}{5-1} = \frac{7}{4}$. Next, use $(1,2)$ in the point-slope formula to get

$$y - 2 = \frac{7}{4}(x - 1)$$

which we can also write in the slope-intercept form as

$$y = \frac{7}{4}x + \frac{1}{4}$$

Determine the equation for the line using $(5,9)$ in the slope-intercept formula. You should get the same answer after some simplification.

. . .

All lines perpendicular to a given line with slope m have slope $-\frac{1}{m}$ (for a proof see the article “A Simple Derivation of the Relationship of the Slope of Perpendicular Lines Using the Rotational Transformation Matrix” [11]).

As an example, consider the line (call it α) $y = 3x + 2$. We know that line α has slope 3. Find the line perpendicular $y = 3x + 2$ that goes through the point $(-1, -2)$. The perpendicular line (call it β) has slope $-\frac{1}{3}$. So, line β is given by

$$y + 2 = -\frac{1}{3}(x + 1)$$

which can be written as

$$y = -\frac{1}{3}x - \frac{7}{3}$$

The two lines are shown in Figure 12.

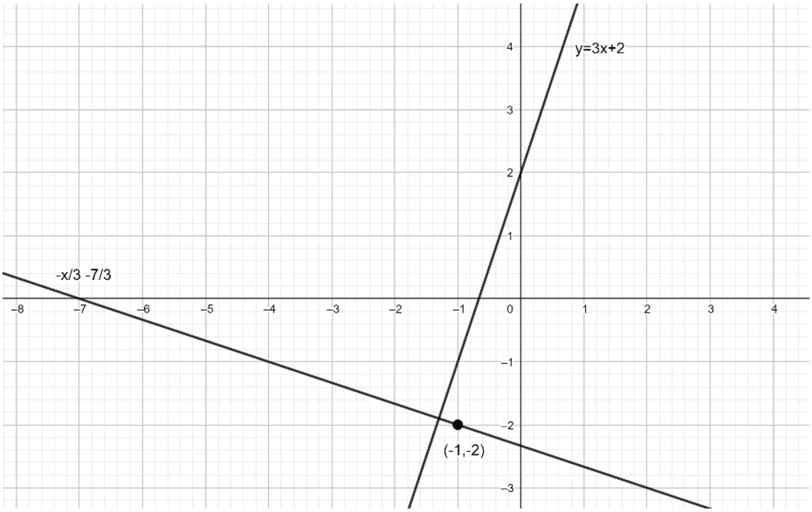

Figure 12. Perpendicular lines

. . .

Some exercises to try:

- Find the equation of the line with y-intercept -3 and slope -4. If you type “Find the equation of the line with y-intercept -3 and slope -4.” into Wolfram Alpha, it is smart enough to parse the sentence and correctly graph the line.
- Find the equation of the line with slope 2 and going through point $(-2,5)$. Wolfram Alpha can also parse the previous sentence and solve the problem correctly.

- Find the equation of the line going through points $(-1,-1)$ and $(3,3)$. Try putting the previous sentence into Symbolab.
- Find the equation of the line going through point $(7,4)$ and $(-1,8)$. Try putting the previous sentence into Wolfram Alpha.
- Rewrite the equation for the line $4x + 3y = 12$ in the slope-intercept and point-slope forms.
- Find the line through point $(3,2)$ and perpendicular to the line $y = -x$.

. . .

A line can also be described by vectors. For example, consider the equation $-6x + 3y = 12$. Rewrite the equation in slope-intercept form to get $y = 2x + 4$. All solutions to this equation can be written in the form $(x, 2x + 4)$ which can be written as

$$(x, 2x + 4) = (x, 2x) + (0,4) = x(1,2) + (0,4)$$

Thus, the line $y = 2x + 4$ can be written as the vector $\vec{u} = (0,4)$ plus multiples of the vector $\vec{v} = (1,2)$. Figure 13 shows $\vec{u}$ plus various multiples of $\vec{v}$ (dashed-lines), e.g.,

- $\vec{w} = \vec{u} + \vec{v}$
- $\vec{a} = \vec{u} + \frac{3}{2}\vec{v}$
- $\vec{b} = \vec{u} - \frac{3}{2}\vec{v}$

The terminal point for each of these vectors is on the line $y = 2x + 4$. The vector (1.2) effectively represents the slope of the line. Figure 13 was created using GeoGebra, see https://www.geogebra.org/calculator/andmtsy3.

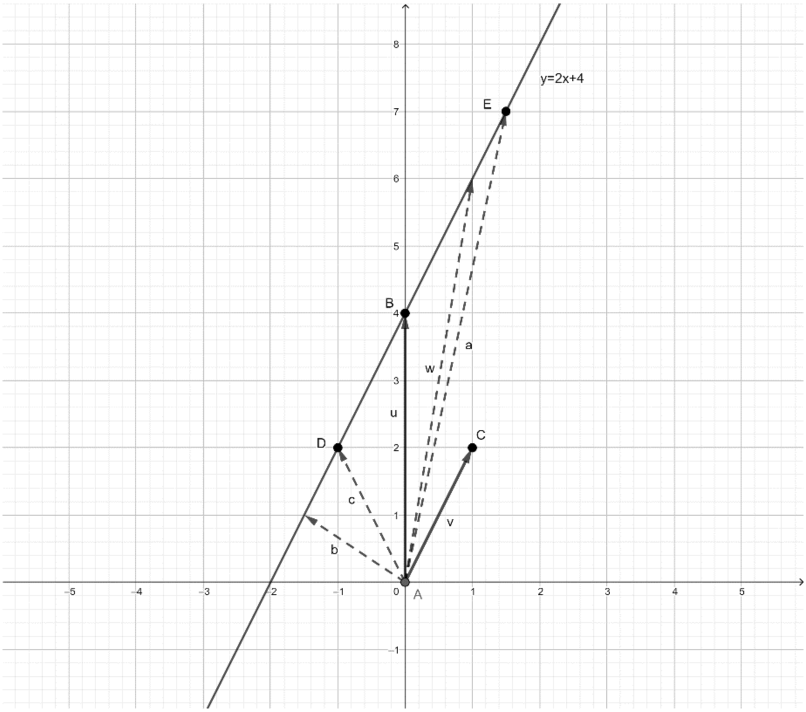

Figure 13. Line written in terms of vectors

The approach of using vectors to specify a line is not typical in two dimensional space. However, the vector approach extends very naturally to three and higher dimensions which is why this approach is mentioned here. For example,

$$(1,2,5) + t(-3,5,-7)$$

is the equation for a line in three dimensions. The line goes through the point $(1,2,5)$ and follows the direction of the vector $(-3,5,-7)$.

This is just as easy to do in higher dimensions. For example,

$$(4,5,-2,8,11) + t(1,2,3,-7,4)$$

is the equation for a line in five dimensional space that contains the point $(4,5,-2,8,11)$ and has direction $(1,2,3,-7,4)$.

. . .

In Section 7.2.3, we found the unique solution (4,1) to the system of equations

$$x - 3y = 1$$

$$2x + 4y = 12$$

The solution is shown graphically in Figure 14. The figure was generated using GeoGebra, see https://www.geogebra.org/m/nqaedvad.

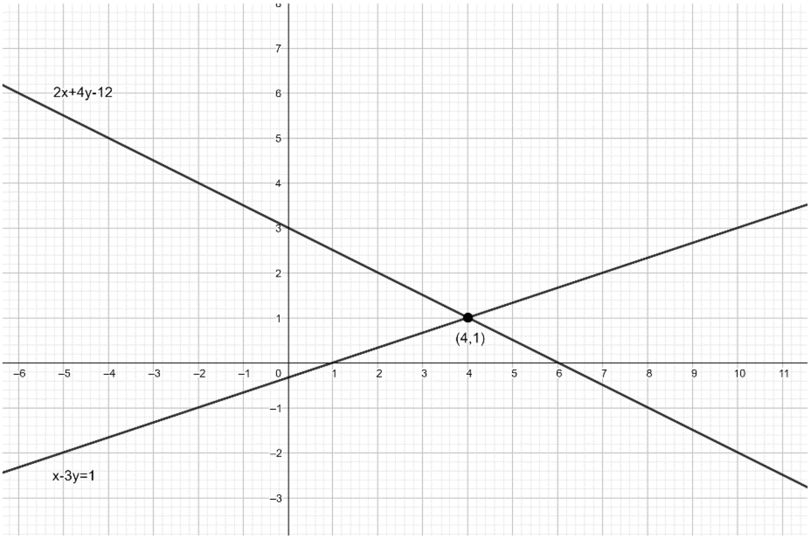

Figure 14. Intersecting lines

A linear system of equations with two variables but no solution is simply two parallel. For example, the equations

$$x + y = 3$$

$$x + y = 0$$

have no solution, and are parallel lines, as shown in Figure 15. The figure was generated using GeoGebra, see https://www.geogebra.org/m/fapp4zdx.

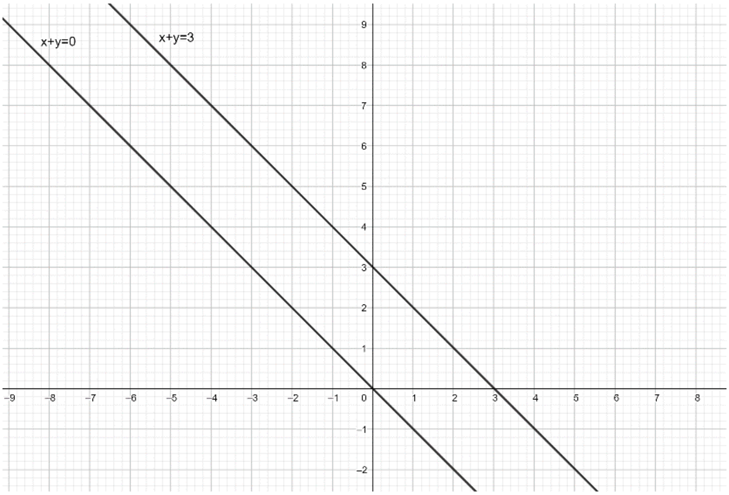

Figure 15. Parallel lines

7.2.4.3 Planes

In terms of points and lines, a plane is determined by either

- 3 non-collinear points (i.e., three points not on the same line)
- a line and a point not on the line
- two separate lines (could be parallel or could intersect in one point).

In three dimensions, the general equation for a plane has the form

$$Ax + By + Cz = D$$

where A, B, C and D are constants. For example, consider the equation

$$x + 2y - 3z = 7$$

A plot of the plane corresponding to the above equation is available on GeoGebra, see https://www.geogebra.org/calculator/tygkgmny. Further, GeoGebra (3D Calculator) lets you rotate figures such as the plane in our example.

Another way to view this equation is to solve for z in terms of x and y, i.e.,

$$z = \frac{1}{3}x + \frac{2}{3}y - \frac{7}{3}$$

and note that points satisfying the equation are of the form

$$(x, y, z) = \left(x, y, \frac{1}{3}x + \frac{2}{3}y - \frac{7}{3}\right) = x\left(1,0,\frac{1}{3}\right) + y\left(0,1,\frac{2}{3}\right) + (0,0,-\frac{7}{3})$$

This can be interpreted as all linear combinations of the vectors $(1,0,\frac{1}{3})$ and $(0,1,\frac{2}{3})$ added to the vector $(0,0,-\frac{7}{3})$. The two vectors determine an infinite number of planes, all with the same inclination. The point $(0,0,-\frac{7}{3})$ specifies one specific plane out of the infinite set. The GeoGebra 3D graphing calculator was used to graph the plane described here, see https://www.geogebra.org/calculator/rhyvuppy.

7.2.4.4 Graphs in Higher-dimensions

[**Author's Remark**: The ideas in this subsection are intended to give the student a glimpse of a more advanced topic known as linear algebra.]

As we saw in the examples just after Figure 13, the vector approach can be used to represent lines in higher dimensions. For example, the expression $\alpha\vec{u} + \vec{v}$ specifies a particular line in 9-dimensional space, where

- α is a scalar that ranges over the real numbers
- $\vec{u} = (3, -4, 8, 21, 4, 6, 8, -7, 11)$ is a vector in 9 dimensional space
- $\vec{v} = (-3, 2, 9, 4, -2, 6, 19, 8, 17)$ is a vector in 9 dimensional space.

The vector $\vec{u}$ can be viewed as the inclination (slope) of the line in 9 dimensional space. There are an infinite number of lines with the same inclination, but only one that goes through the terminal point of $\vec{v}$, i.e., the point $(-3,2,9,4,-2,6,19,8,17)$.

Since planes are essentially two-dimensional objects, two vectors and a point are sufficient to specify a unique plane (regardless of the dimension in which the plane is embedded). For example, the expression $\alpha\vec{u} + \beta\vec{v} + \vec{w}$ specifies a unique plane in 5-dimensional space, where

- α and β are scalars that range over the real numbers
- $\vec{u} = (4,-3,9,1,19)$
- $\vec{v} = (3,7,8,11,4)$
- $\vec{w} = (0,0,13,12,23)$

The vectors $\vec{u}$ and $\vec{v}$ determine the inclination of the plane within 5-dimensional space. The vector $\vec{w}$ determines the unique plane with the given inclination.

This pattern applies to higher dimensional objects. For example, we can specify a 3-dimensional space within any higher dimensional space using four vectors (three vectors to specify the inclination of the 3-dimensional space and one vector to select one 3-dimensional space out of the infinite number that have the same inclination). The vectors that specify the inclination are known as basis vectors, and the requirement is that basis vectors be independent, i.e., cannot write any one of the basis vectors as a linear combination of the others.

7.2.5 Word Problems

7.2.5.1 Currency Problems

Bert has 35 coins, consisting of dimes and quarters that total \$6.35. How many coins of each type does he have?

Answer:

Let x be the number dimes and y be the number of quarters.

The information about 30 coins converts to

$$x + y = 35$$

The information concerning the combined value of the coins can be expressed as

$$.1x + .25y = 6.35$$

which, if we multiply by 100, the above equation can be written as

$$10x + 25y = 635$$

Solving the first equation for y in terms of x gives us

$$y = 35 - x$$

Plug this into the second equation (as revised), we get

$$10x + 25(35 - x) = 635$$

$$10x + 875 - 25x = 635$$

$$-15x = 635 - 875 = -240$$

$$x = 16$$

Plugging this back into either of the given equations gives us that $y = 19$.

The above approach is known as "**solution via substitution**."

We can also use the Gauss-Jordan Elimination as follows

1	1	35
10	25	635

Subtract 10 times the first row from the second row to get

1	1	35
0	15	285

Divide the second row by 15

1	1	35
0	1	19

Subtract the second row from the first, and we have the final solution

1	0	16
0	1	19

. . .

Bugsy has 39 coins, consisting of nickels, dimes and quarters. The total value of the coins is $6.35. Further, we know that the number of dimes and quarters is 34. How many coins of each type does he have?

Let x be the number nickels, y be the number of dimes and z be the number of quarters. We are given the following

$$x + y + z = 39$$

$$y + z = 34$$

$$.05x + .1y + .25z = 6.35$$

Using the Gauss-Jordan Elimination, we start with the following augmented matrix

1	1	1	39
0	1	1	34
.05	.1	.25	6.35

Multiple the third row by 100 to get

1	1	1	39
0	1	1	34
5	10	25	635

Subtract 5 times the first row from the third row to get

1	1	1	39
0	1	1	34
0	5	20	440

Subtract 5 times the second row from the third row to get

1	1	1	39
0	1	1	34
0	0	15	270

Dividing the third row by 15 and we obtain the value for z

1	1	1	39
0	1	1	34
0	0	1	18

Subtract the third row from second row and from the first row

1	1	0	21
0	1	0	16
0	0	1	18

Finally, subtract the second row from the first

1	0	0	5
0	1	0	16
0	0	1	18

So, Bugsy has 5 nickels, 16 dimes and 18 quarters.

. . .

In the previous problem, remove the condition that the number of dimes and quarters is 34 and then determine all possible solutions.

Answer: In this case, we only have two equations, i.e.,

$$x + y + z = 39$$

$$.05x + .1y + .25z = 6.35$$

The augmented matrix is

1	1	1	39
.05	.1	.25	6.35

Multiply the second row by 100 to get

1	1	1	39
5	10	25	635

Subtract 5 times the first row from the second row to get

1	1	1	39
0	5	20	440

Divide the second row by 5 to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 39 \\ 0 & 1 & 4 & 88 \end{array}\right]$$

Subtract the second row from the first to get

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -49 \\ 0 & 1 & 4 & 88 \end{array}\right]$$

At this point, we can write both x and y in terms of z as follows:

$$x = 3z - 49$$

$$y = -4z + 88$$

All solutions to the problem are of the form

$$(x, y, z) = (3z - 49, -4z + 88, z)$$
$$= z(3, -4, 1) + (-49, 88, 0)$$

As noted earlier in this book, if there are fewer equations than variables, there are either an infinite number of solutions or no solutions to the system of equations. However, for the problem at hand, we have an additional condition, i.e., the solutions must be positive integers. As is shown below, this additional condition limits the number of solutions to a finite set.

Table 13 shows the possible solutions to this problem. For values of $z \leq 16$, x is negative and thus not a possible solution. For values of $z \geq 23$, y becomes negative and again, we do not have a valid solution. The only valid solutions occur for z (i.e., the number of quarters) equal to 17, 18, 19, 20, 21 or 22 (highlighted in bold in the table).

Table 13. Possible solutions to coin problem

z	x	y
15	-4	28
16	-1	24
17	**2**	**20**
18	**5**	**16**
19	**8**	**12**
20	**11**	**8**
21	**14**	**4**
22	**17**	**0**
23	20	-4
24	23	-8

. . .

You have \$1, \$5 and \$10 paper currency that add to a total of \$117. Further, you know that the number of \$1 and \$5 notes is 27, and the number of \$5 and \$10 notes is 15. Determine the number of \$1, \$5 and \$10 notes.

Hint: Check your answer at https://www.symbolab.com. Just enter the following command

$$\text{solve } x + 5y + 10z = 117; x + y = 27; y + z = 15$$

7.2.5.2 Investment Problem

Winnie invested various amounts of money at 5% and 7% (simple interest). She invested \$5000 more at 7% than at 5%. She earned \$950 over the course of a year. How much was invested at each rate?

Answer:

Let x be the amount invested at 5% and let y be the amount invested at 7%.

We are given that

$$x + 5000 = y$$

$$.05x + .07y = 950$$

Rearranging the first equation, and multiplying the second equation by 100, we get

$$x - y = -5000$$

$$5x + 7y = 95000$$

The augmented matrix is

1	−1	−5000
5	7	95000

Subtracting 5 times row 1 from row 2 gives

1	−1	−5000
0	12	120000

Divide row three by 12 to get

1	−1	−5000
0	1	10000

Add the second row to the first and we are done

1	0	5000
0	1	10000

So, Winnie invested \$5000 at 5% simple interest and \$1000 at 7% simple interest.

7.2.5.3 Number Digit Problems

The sum of the digits of a three digit number is 24. Three times the third digit (least significant digit) minus three times the first digit (most significant digit) is equal to $\frac{1}{3}$ the middle digit. The sum of the second and third digit minus the first digit is equal to 10. Find the number.

Answer:

Represent the number as xyz where x is the 100s, y is the 10s and z is the 1s. We are given

$$x + y + z = 24$$

$$3z - 3x = \frac{1}{3}y$$

$$y + z - x = 10$$

Rearranging the equations to get ready for creating of the augmented matrix, we have

$$x + y + z = 24$$

$$-3x - \frac{1}{3}y + 3z = 0$$

$$-x + y + z = 10$$

The augmented matrix is

1	1	1	24
-3	$-\frac{1}{3}$	3	0
-1	1	1	10

Using the Gauss-Jordan Elimination (steps left to the reader), we get the final answer

1	0	0	7
0	1	0	9
0	0	1	8

So, the number is 798.

. . .

Find the three digit number that meets the following conditions:

- the sum of the digits is 18
- the sum of the first two digits (100 and 10 positions) is equal to the third digit

- two times the sum of the first and third digits is equal to five times the middle digit plus 1

Hint: The clues in terms of equations are as follows

$$x + y + z = 18$$

$$x + y = z$$

$$2(x + z) = 5y + 1$$

Check you answer at https://www.symbolab.com by entering the following command

$$\text{solve } x + y + z = 18; x + y - z = 0; 2x - 5y + 2z = 1$$

7.2.5.4 Ages Problem

Abe, Beth and Cathy have an average age of 53. One-half of Cathy's age plus $\frac{1}{3}$ of Beth's age plus $\frac{1}{4}$ of Abe's age equals 65. Four years ago, Cathy was four times the age of Abe. How old are all three now?

Answer:

Let Abe's age be a, Beth's age be b and Cathy's age be c. [**Author's Remark**: We are not bound to use x, y and z as variables.]

In terms for equations, we are given

$$\frac{a + b + c}{3} = 53$$

$$\frac{1}{2}c + \frac{1}{3}b + \frac{1}{4}a = 65$$

$$c - 4 = 4(a - 4)$$

These equations can be rewritten as

$$a + b + c = 159$$

$$6c + 4b + 3a = 780$$

$$-4a + c + 12 = 0$$

The augmented matrix for this system of equations is

1	1	1	159
3	4	6	780
-4	0	1	-12

Using the Gauss-Jordan Elimination (steps left to the reader), we get the final solution

1	0	0	24
0	1	0	51
0	0	1	84

Abe is 24, Beth is 51 and Cathy is 84.

7.2.5.5 Travel Problem

The Red Barron flies a distance of 900 km in 6 hours, going directly into the wind (of unknown speed). On his return trip (with the wind at his back), it takes $4\frac{14}{19}$ (or $\frac{90}{19}$) hours to return to his starting point. The wind remains steady throughout the flight. Find the speed of the airplane and the wind speed.

Answer:

Let x be the speed of the airplane in still air, and y be the wind speed.

The given information is summarized in the following table

	Time	**Speed**	**Distance**
Outgoing trip	6	$x - y$	$6(x - y) = 900$
Return trip	$\frac{90}{19}$	$x + y$	$\frac{90}{19}(x + y) = 900$

The two equations for distance will be used to solve for x and y. Divide the first distance equation by 6, and multiple the second distance equation by $\frac{19}{90}$ to get

$$x - y = 150$$

$$x + y = 190$$

Adding the two equations together, we get $2x = 340$ or $x = 170$. Substituting back into either equation gives us $y = 20$.

7.2.5.6 Problems with Four or More Variables

Axel has 300 notes in the following denominations: \$1, \$2, \$5 and \$10. The sum of values of the notes is \$1120. The number of ones and fives is 20 more than the number of twos and tens. The number of ones, fives and tens is twice the number of twos. How many of each type of notes does Axel have?

Answer:

Let x be the number of ones, y be the number of twos, z be the number of fives and w be the number of tens. In terms of equations, we are given the following

$$x + y + z + w = 300$$

$$x + 2y + 5z + 10w = 1120$$

$$x + z = 20 + y + w$$

$$x + z + w = 2y$$

which can be rewritten as

$$x + y + z + w = 300$$

$$x + 2y + 5z + 10w = 1120$$

$$x - y + z - w = 20$$

$$x - 2y + z + w = 0$$

The augmented matrix is

1	1	1	1	300
1	2	5	10	1120
1	−1	1	−1	20
1	−2	1	1	0

Using the Gauss-Jordan Elimination or using an online application such as Symbolab, we get the following solution:

1	0	0	0	70
0	1	0	0	100
0	0	1	0	90
0	0	0	1	40

. . .

Five numbers (call them a, b, c, d and e) add up to 80. The sum of a and b is equal to c. The sum of b and c is four more than d. The sum of c and d is three more than e. The sum of a, b and e is 12 more the c plus d. What are the values of the five numbers?

Answer:

In equation form, we are given the following information:

$$a + b + c + d + e = 80$$

$$a + b = c$$

$$b + c = d + 4$$

$$c + d = e + 3$$

$$a + b + e = c + d + 12$$

The equations can be rearranged as follows:

$$a + b + c + d + e = 80$$

$$a + b - c = 0$$

$$b + c - d = 4$$

$$c + d - e = 3$$

$$a + b - c - d + e = 12$$

Insert the following expression into Wolfram Alpha (at https://www.wolframalpha.com)

a+b+c+d+e=80; a+b-c=0; b+c-d=4; c+d-e=3; a+b-c-d+e=12

and the following solution is returned

$$a = 7, b = 8, c = 15, d = 19, e = 31$$

[**Author's Remark**: The original set of equations will also work with Wolfram Alpha. I also tried with Symbolab but the application does not currently support systems of linear equations with more than 4 variables.]

8 Polynomials of a Single Variable

8.1 Overview

A polynomial is an expression consisting of variables (also called indeterminates) and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponentiation of variables [12]. For example, the following are polynomials:

- $2x^4 - 7x^3 - x + \frac{4}{3}$
- $y + x + \frac{1}{3}xy^3 + zx + 7$
- $w^3 + wzx + 12.3z^3 + 9$

The following expressions are not polynomials:

- $7x^{-3} + 8y^{1/2}$ (not a polynomial since x is raised to a negative power, and y is raised to a non-integer power)
- $\frac{3x^3+4y}{11z+4z^2}$ (not a polynomial since the expression involves division)

In this section, we only consider polynomials of a single variable. The general form is

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are all constants (some of which may be zero).

For example, the following are polynomials of a single variable:

- $\frac{4}{3}x - 7$
- $8x^4 + 6x + 4$
- $2x^7 - \frac{3}{4}x^3 + 7x - 15$

8.2 Operations

8.2.1 Addition and Subtraction

Addition and subtraction of polynomials is a matter of reducing the problem to addition and subtraction of like terms, i.e., terms with the variable raised to the same power. This is analogous to arithmetic addition, as presented in Section 4.2, where we decomposed numbers into powers of 10 and then added like terms.

For example,

$$(8x^4 + 6x + 4) - (2x^7 - \frac{3}{4}x^3 + 7x - 15)$$

$$= -2x^7 + 8x^4 + \frac{3}{4}x^3 - x + 19$$

or we could add the two terms as follows:

$$(8x^4 + 6x + 4) + (2x^7 - \frac{3}{4}x^3 + 7x - 15)$$
$$= 2x^7 + 8x^4 - \frac{3}{4}x^3 + 13x - 11$$

In both cases, we were only able to combine the x terms and the constants since they appear in both expressions.

Try the following polynomial additions and subtractions:

- $\left(8x^3 + 9x^2 + \frac{9}{4}x + 6\right) - (4x^3 - 5x^2 + \frac{5}{4}x - 7)$
- $\left(\frac{2}{3}x^3 - 3x^2 + 8x - 3\right) + \left(\frac{1}{3}x^3 + 3x^2 + 9x + 7\right)$
- $\left(\frac{2}{3}x^3 - 3x^2 + 8x - 3\right) - \left(\frac{1}{3}x^3 + 3x^2 + 9x + 7\right)$
- $(8x^5 + 9x^4 - x + 6) + (4x^3 - 5x^2 + \frac{5}{4}x - 7)$

8.2.2 Multiplication

Multiplication is governed by the distributive law. In short, when two polynomials are multiplied together, every term in one polynomial is to be multiplied times every term in the other polynomial, and then the terms are to be added. This is best illustrated by an example.

$$(x^2 + 7x + 1)(3x^2 - 4x + 3)$$
$$= x^2(3x^2 - 4x + 3) + 7x(3x^2 - 4x + 3) + 1(3x^2 - 4x + 3)$$
$$= 3x^4 - 4x^3 + 3x^2 + 21x^3 - 28x^2 + 21x + 3x^2 - 4x + 3$$
$$= 3x^4 + 17x^3 - 22x^2 + 17x + 3$$

Equivalently, we could have expanded the problem as follows:

$$(x^2 + 7x + 1)(3x^2 - 4x + 3)$$
$$= (x^2 + 7x + 1)(3x^2) + (x^2 + 7x + 1)(-4x) + (x^2 + 7x + 1)(3)$$
$$= 3x^4 + 21x^3 + 3x^2 - 4x^3 - 28x^2 - 4x + 3x^2 + 21x + 3$$
$$= 3x^4 + 17x^3 - 22x^2 + 17x + 3$$

When doing the multiplication by hand, an effective way to keep track of the required multiplication is via a table, as shown below

	$\mathbf{3x^2}$	$\mathbf{-4x}$	$\mathbf{3}$
$\mathbf{x^2}$	$3x^4$	$-4x^3$	$3x^2$
$\mathbf{7x}$	$21x^3$	$-28x^2$	$21x$
$\mathbf{1}$	$3x^2$	$-4x$	3

After the table is populated, one needs to add all the terms and combine like terms to arrive at the final solution.

Try the following multiplication problems. You can check your answers with either Wolfram Alpha or Symbolab.

- $(x^2 - 3x + 1)(2x + 7)$
- $(x^3 - 2x + 4)(2x^2 + 6x)$
- $(x^4 + x^2 + 1)(x^3 + x + 1)$

8.2.3 Division

It is possible to divide one polynomial into another using a variation of the long-division technique we used for simple arithmetic. Powers of x replace the role played by powers of 10 in arithmetic.

The terms divisor, dividend and quotient (from Section 4.5) are used in the same manner in this section, but with a generalization to polynomials.

For example, Table 14 shows the division of $x + 1$ into $x^2 - 1$, i.e., the computation of $\frac{x^2-1}{x+1}$. The result is listed in the top row, i.e., $x - 1$. The answer checks out since $(x + 1)(x - 1) = x^2 - 1$. One word of caution, i.e., $\frac{x^2-1}{x+1} = x - 1$ except when $x = -1$ since when $x = -1$, the divisor is equal to 0 and the fraction is undefined in that case.

The steps are as follows:

- (setup) The numerator is placed under the division bar on the right. The denominator is placed on the left. We place zeros wherever there are missing powers of x.
- In the first step, we determine the highest power of x that we can multiple times $x + 1$ and not exceed x^2 (that would be x in this case).
- Multiple x times $x + 1$ and subtract from $x^2 - 1$.
- Repeat the process. This time we determine the highest power of x that can be multiplied times $x + 1$ and not exceed $-x - 1$ (that would be $-1x^0 = -1$ in this case).

Table 14. Long-division of polynomials – Example 1

			x	-1
x	1	x^2	0	-1
		x^2	x	0
			$-x$	-1
			$-x$	-1
			0	0

Let's try another example, i.e., $\frac{x^4-1}{x-1}$. The solution is shown in Table 15. The answer checks out since $x^4 - 1 = (x-1)(x^3 + x^2 + x + 1)$. Again, we have the restriction that the expression $\frac{x^4-1}{x-1}$ and by implication the quotient $(x^3 + x^2 + x + 1)$ are undefined when $x = 1$.

Table 15. Long-division of polynomials – Example 2

			x^3	x^2	x	1
x	-1	x^4	0	0	0	-1
		x^4	$-x^3$	0	0	0
			x^3	0	0	-1
			x^3	$-x^2$	0	0
				x^2	0	-1
				x^2	$-x$	0
					x	-1
					x	-1
					0	0

Next, we don an example where the coefficients are not just 1 and -1. Find the result when $21x^2 + 19x - 12$ is divided by $7x - 3$. The solution is shown in Table 16. So, $\frac{21x^2+19x-12}{7x-3} = 3x + 4$, except when $7x - 3 = 0$ or equivalently, $x = \frac{3}{7}$.

Table 16. Long-division of polynomials – Example 3

			$3x$	4
$7x$	-3	$21x^2$	$19x$	-12
		$21x^2$	$-9x$	0
			$28x$	-12
			$28x$	-12
			0	0

The three examples we have seen are the exceptions. It is more common to get a remainder when dividing one polynomial into another. For example, if we slightly modify the last example (change -12 to say 10), we get a remainder. The solution is shown in Table 17. The answer should be interpreted as follows:

$$\frac{21x^2 + 19x + 10}{7x - 3} = (3x + 4) + \frac{22}{7x - 3}$$

We can check our answer as follows:

$$(3x + 4) + \frac{22}{7x - 3} = \frac{(3x + 4)(7x - 3) + 22}{7x - 3}$$

$$= \frac{(21x^2 + 28x - 9x - 12) + 22}{7x - 3} = \frac{21x^2 + 19x + 10}{7x - 3}$$

Table 17. Long-division of polynomials with a remainder

			$3x$	4
$7x$	-3	$21x^2$	$19x$	10
		$21x^2$	$-9x$	0
			$28x$	10
			$28x$	-12
			0	22

Some exercises to try:

- $\frac{5x^2+32x-21}{5x-3}$
- $\frac{x^4-1}{x^3-x^2+x-1}$
- $\frac{x^4+3}{x+1}$ **Hint**: this one has a remainder

You can check your answers with Wolfram Alpha or Symbolab. To enter powers of x in Symbolab (and most other online applications of this sort), you need to use the ^ symbol. For example, enter x^2 as x^2.

8.3 Introduction to Functions

A **function** is a binary relationship between two sets that associates to each element of the first set exactly one element of the second set. For the discussions in this book, the sets are usually the real numbers or a subset of the real numbers. We have already seen functions but have not identified them as such. For example, the equation for a line (such as $y = \frac{2}{3}x + 7$) takes a real number value for x (e.g., 9) and maps it to another real number (e.g., 13). To emphasize the mapping, we write our example function for a line as

$$f(x) = \frac{2}{3}x + 7$$

and the particular mapping would be written as $f(9) = \frac{2}{3}(9) + 7 = 6 + 7 = 13$.

In the case of polynomials, we have also seen many instances in this section. For example, $f(x) = 7x^7 - 23x^3 + 5x - 15$ is a polynomial function of the single variable x which maps from the set of all real numbers to the set of all real numbers. The function $g(x) = x^2 + 1$ maps from all the real numbers to the set of positive real numbers greater than or equal to 1. This is true since the lowest output that can be generated is $g(0) = 1$. Also, keep in mind that the square of a negative number is a positive number.

8.4 Finding Roots

A root of a single-variable expression is a value of the variable that results in the expression being zero. Warning: this is a completely different usage of the term "root" from "root" in the sense of "square root" or "cube root."

For example, the function $f(x) = x^2 - 1$ has two roots, i.e., $x = 1$ and $x = -1$, since $f(1) = 0$ and $f(-1) = 0$.

In the case of polynomials, the number of real valued roots is at most the highest power of x. The highest power of x in a polynomial is known as the **degree of the polynomial**. As we saw, the second degree polynomial $x^2 - 1$ has two roots, and the second degree polynomial $x^2 + 1$ has no roots in the real numbers (it does have two complex number roots, i.e., $-i = -\sqrt{-1}$ and $i = \sqrt{-1}$ but complex numbers are beyond the scope of this book).

. . .

For first degree equations, it is easy to solve for the single root. For example,

$$3x - 8 = 0$$

$$3x = 8$$

$$x = 8/3$$

In general, if we have $ax + b = 0$ with a and b being real number constants, then the solution is

$$x = -b/a$$

For second degree equations, there is the **quadratic formula** that can be used to determine the roots (see the derivation below).

Take any second degree polynomial and set it equal to zero, i.e.,

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are real number constants}$$

Divide both sides of the equation by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Subtract $\frac{c}{a}$ from both sides of the equation

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add $\frac{b^2}{4a^2}$ to both sides of the equation (this is known as completing the square)

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Noting that the left-side of the equation is a perfect square, we have that

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Take the square root on both sides, and noting that the symbol $\pm$ means "plus or minus", we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides to get the two roots of the equation, we get the well-known quadratic formula

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

When the quantity within the radical (i.e., $b^2 - 4ac$) is negative, there are no real number roots but rather two complex roots.

For example, the two roots of $3x^2 + 7x - \frac{5}{4}$ are

$$x = -\frac{7}{6} \pm \frac{\sqrt{49 + 15}}{6} = -\frac{7}{6} \pm \frac{\sqrt{64}}{6} = -\frac{7}{6} \pm \frac{8}{6}$$

So, one root is $-\frac{7}{6} + \frac{8}{6} = \frac{1}{6}$ and the other is $-\frac{7}{6} - \frac{8}{6} = -\frac{15}{6} = -\frac{5}{2}$

Find the roots for the following polynomials:

- $\frac{3}{4}x - 4$
- $2x^2 - 9x + 4$
- $x^2 - 7x + 2$

You can check your answers in Wolfram Alpha or Symbolab. Just enter the expression as an equation equal to 0. You need to be very careful concerning the order of operations. For example, if you enter 3/4x – 4 into Symbolab, this get converted to $3/(4x - 4)$. To get the expression in the first exercise above, you need to enter (3/4)x – 4 in Symbolab. However, in Wolfram Alpha, 3/4x – 4 does get interpreted as $\frac{3}{4}x - 4$. So, the lesson here is that when using a symbolic computation application, make sure your input is interpreted as desired and if not add appropriate parenthesis to get the desired result.

. . .

For third and fourth degree polynomials, there are formulas that provide the value of the roots, see the articles from S.O.S. Mathematics [13][14]. These formulas involve multiple steps. The formula for solving the fourth degree (quartic) equation is incredibly complex, see https://planetmath.org/QuarticFormula). It has been proven that no closed form formula is possible for the general fifth and higher degree polynomials.

In some cases, higher degree polynomials can be simplified. There are an almost endless number of various tricks to simplify polynomial expressions, just a few are listed below.

Factoring out a power of x

Consider the expression $2x^3 - 9x^2 + 4x$. We can factor out an x to get $x(2x^2 - 9x + 4)$. Clearly, $x = 0$ is one root. The other two roots can be found by either applying the quadratic formula, or by noticing that

$$2x^2 - 9x + 4 = (2x - 1)(x - 4)$$

Thus, the other two roots are $\frac{1}{2}$ and 4.

Here's another example in the same category: $x^5 - 7x^4 + 2x^3$. In this case, we can factor out x^3 to get $x^3(x^2 - 7x + 2)$. From the x^3, we know that there is a triple root at $x = 0$. We can find the other two roots by applying the quadratic formula to $x^2 - 7x + 2$. This yields the roots $\frac{7+\sqrt{41}}{2}$ and $\frac{7-\sqrt{41}}{2}$.

Grouping of terms:

Consider the equation $x^3 - 7x^2 + 4x - 28$. If we factor x^2 from the first two terms and -4 from the second two terms, a pattern is revealed:

$$x^3 - 7x^2 - 4x - 28 = x^2(x - 7) - 4(x - 7) = 0$$

Now we can factor $(x-7)$ to get

$$(x-7)(x^2-4)=0$$

Thus, the three roots are $7, 2$ and -2.

Reduction to a lower-degree polynomial:

In some cases, it is possible to reduce a polynomial to a lower degree polynomial via a substitution. For example, take the polynomial x^4-8x^2+11. If we let $y=x^2$ and substitute into the original expression, we get $y^2-8y+11$. The modified expression can be solved using the quadratic formula, yielding

$$y=4-\sqrt{5} \text{ and } y=4+\sqrt{5}$$

But $y=x^2$ which can be expressed as $x=\pm\sqrt{y}$. Thus, the four roots to the original polynomial are

$$\sqrt{4-\sqrt{5}}, -\sqrt{4-\sqrt{5}}, \sqrt{4+\sqrt{5}}, -\sqrt{4+\sqrt{5}}$$

Note the $4-\sqrt{5}\cong 1.764>0$.

Factoring out $(x-r)$ when r is a root of the polynomial:

If r is a root of the polynomial $f(x)$ then we can factor $(x-r)$ out of $f(x)$ to get another polynomial, call it $g(x)$, such that $f(x)=(x-r)g(x)$.

For example, consider the polynomial x^3-x^2+x-1. By inspection, $x=1$ is seen to be a root. Next we use long-division to extract $(x-1)$ from x^3-x^2+x-1. As can be seen from the table below, $x^3-x^2+x-1=(x-1)(x^2+1)$.

			x^2	0	1
x	-1	x^3	$-x^2$	x	-1
		x^3	$-x^2$		
				x	-1
				x	-1
				0	0

Since x^2+1 has no real number roots, we can conclude that $x=1$ is the only root of the original equation.

Here's another example: $x^3+9x^2+26x+24$. By inspection (and some trial and error), we see that $x=-2$ is a root of the polynomial. Next, we divide $(x+2)$ into $x^3+9x^2+26x+24$ as follows:

			x^2	$7x$	12
x	2	x^3	$9x^2$	$26x$	24
		x^3	$2x^2$		
			$7x^2$	$26x$	24
			$7x^2$	$14x$	
				$12x$	24
				$12x$	24
				0	0

Thus, $x^3 + 9x^2 + 26x + 24 = (x + 2)(x^2 + 7x + 12)$. Further, we can see that $x^2 + 7x + 12 = (x + 3)(x + 4)$. So, $x = -3$ and $x = -4$ are the other two roots. Alternately, we could have applied the quadratic formula to $x^2 + 7x + 12$, with the same result.

Rational Root Theorem:

The rational root theorem states a constraint on rational roots of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all the coefficients are integers, and neither a_n nor a_0 equal 0. The term "rational root" includes possible roots that are fractions or integers.

The theorem states that each rational root $x = \frac{p}{q}$, written in lowest terms so that p and q are relatively prime (i.e., p and q don't have a common factor), satisfy the following conditions:

- p is an integer factor of the constant term a_0
- q is an integer factor of the leading coefficient a_n.

The theorem does not guarantee rational roots but only puts conditions on rational roots for a given polynomial if they exist.

For example, consider the polynomial $x^3 - 7x + 6$. The possible rational roots must have divisors of 6 in the numerator (i.e., $\pm 1, \pm 2, \pm 3, \pm 6$) and a divisor of 1 in the denominator (i.e., ± 1). So, the possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6$. Trying each of these possibilities in the given polynomial, we see that $1, 2$ and -3 are roots.

Just to emphasize that the theorem does not guarantee rational roots, take the polynomial $2x^3 + x - 1$. The possible rational roots must have divisors of 1 in the numerator and divisors of 2 in the denominator. Thus, the possible rational roots are $\pm 1, \pm \frac{1}{2}$ but none of these is a root. Also, Symbolab has a rational root capability. For the example at hand, try typing rational roots 2x^3+x-1

into the Symbolab command prompt.

Here's a more complex example: $3x^3 - 4x^2 - 17x + 6$. The possible rational roots can be expressed as follows:

$$\frac{\pm\{1,2,3,6\}}{\pm\{1,3\}}$$

Note that some of the combinations of numerators and denominators are repeats, e.g., 1/1, -1/-1, 3/3, -3/-3 all equal 1. After removing the duplicates, we have the following possible rational roots

$$\pm\left\{\frac{1}{3},\frac{2}{3},1,2,3,6\right\}$$

Plugging these into the given polynomial, we see that $-2, \frac{1}{3}$ and 3 are roots.

. . .

Find the roots of the following polynomials:

- $3x^2 - 14x - 24$
- $3x^2 - 11x + 4$
- $x^3 + 7x^2 + 12x$
- $x^4 + 8x^3 + 15x^2$
- $3x^3 + 15x^2 - 12x - 60$
- $x^5 - 5x^4 + 6x^3$
- $x^3 - 5x^2 + 7x - 3$ **Hint**: Use the rational root theorem
- $x^4 + 5x^3 + 5x^2 - 5x - 6$ **Hint**: Use the rational root theorem

You can check your answers at Wolfram Alpha or Symbolab.

8.5 Graphs

When graphing a polynomial, the vertical axis is typically referred to as the y-axis. This convention effectively adds a second variable. For example, Figure 16 depicts the graph of $y = 3x^3 + 15x^2 - 12x - 60$. This graph was generated using the online graphing application from Desmos at www.desmos.com.

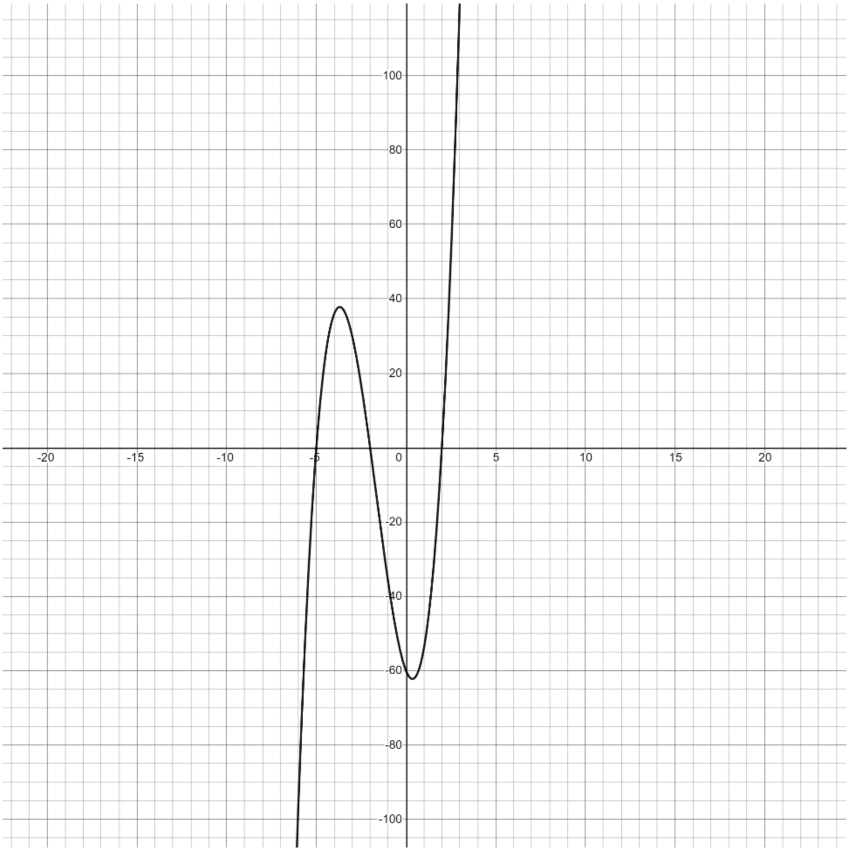

Figure 16. Graph of a cubic polynomial

Notice the graph in Figure 16 drops to a little over -60 at around $x = 0$. If we add 65 to the polynomial, the entire graph moves up by 65 and one would expect the modified polynomial to have only 1 real number root. This is, in fact, the case. To see this, type $3x^3 + 15x^2 - 12x + 5 = 0$ into the input box at www.symbolab.com.

Try graphing some of the polynomials from the previous section using Symbolab, Wolfram Alpha, Desmos and GeoGebra. Wolfram Alpha provides the real and complex roots (i.e., root in the form of a real number plus an imaginary number). Desmos and GeoGebra focus on graphing. Symbolab (like Wolfram Alpha) focuses on symbolic manipulation of equations as well as graphing.

8.6 Word Problems

8.6.1 Number Problems

The sum of a positive number and its square plus 7 is equal to 139.

Answer:

Let x be the number.

We are given $x^2 + x + 7 = 139$. Thus, we need to solve $x^2 + x - 132 = 0$. We could use the quadratic formula, or we could notice that

$$x^2 + x - 132 = (x + 12)(x - 11)$$

The two roots are 11 and -12, but the problem statement says "positive number". So, the answer to the problem is 11.

. . .

The product of two consecutive numbers is 342. What are the two numbers?

Hint: Let x be the smaller number. We are given that $x(x + 1) = 342$. Further, the negative root also provides a valid answer.

8.6.2 Age Problems

Aaron is 3 years older than his brother William. If the product of their ages is three more than five times the sum of their ages, how old are Aaron and William?

Answer:

If we let x be the age of Aaron, then William is $x - 3$.

We are given that

$$x(x - 3) = 5(x + x - 3) + 3$$

which is equivalent to

$$x^2 - 3x = 5(2x - 3) + 3$$

$$x^2 - 3x = 10x - 12$$

$$x^2 - 13x + 12 = 0$$

We can factor $x^2 - 13x + 12$ as $(x - 1)(x - 12) = 0$. Aaron is either 1 or 12, but William is three years younger and thus, Aaron must be 12 and not 1. William must be 9.

. . .

Three times Cupcake's age plus 7 is equal to Max's age. The product of their ages is 110. How old is each cat?

Hint: Let x be Cupcake's age, then Max is $3x + 7$. Further, we are given the $x(3x + 7) = 110$. Of the two solutions, we select the positive one.

. . .

Eric is 7 less than four times the age of Erica. The product of their ages is 47 more than the sum of their ages. How old are Eric and Erica?

Hint: Let x be Erica's age, then Eric is $4x - 7$. The other piece of information translates to

$$x(4x - 7) = (x + 4x - 7) + 47$$

which simplifies to

$$4x^2 - 12x - 40 = 0.$$

8.6.3 Geometry Problems

The length of a rectangular garden plot is 5 meters more than 2 times its width. The area of the plot is 150 square yards. Find the dimensions of the plot.

Answer:

Let x be the width of the plot. We are given that the length is $2x + 5$. In general, the area of a rectangle is its length times its width. So, we have

$$x(2x + 5) = 150$$

With some rearrangement, we get

$$2x^2 + 5x - 150 = 0$$

Applying the quadratic formula, yields the following possible solutions

$$x = -\frac{5}{4} \pm \frac{35}{4}$$

Of the two solutions, we select the positive one, i.e., $x = \frac{15}{2} = 7.5$ meters. The length is $2(7.5) + 5 = 20$.

. . .

A rectangular herb garden has a length that is 5 meters less than twice the width. There is a fence around the garden (4 meters away, as shown in Figure 17). The total area within the outer border is 255 square meters. What are the dimensions of the herb garden?

Answer:

If we let x be the width of the garden, then the length of the garden $2x - 5$. We need to add 8 meters to both the width and length of the garden to get the length and width of the outer rectangle (as shown in the figure). We are given that the area of the outer rectangle is 255. Thus, we have

$$(2x + 3)(x + 8) = 255$$

$$2x^2 + 19x - 231 = 0$$

Applying the quadratic formula, we get two solutions, i.e., $-\frac{33}{2}$ and 7. We choose the positive solution since the other solution is impossible for the problem at hand. Thus, the inner garden has dimensions 7 meters by 9 meters.

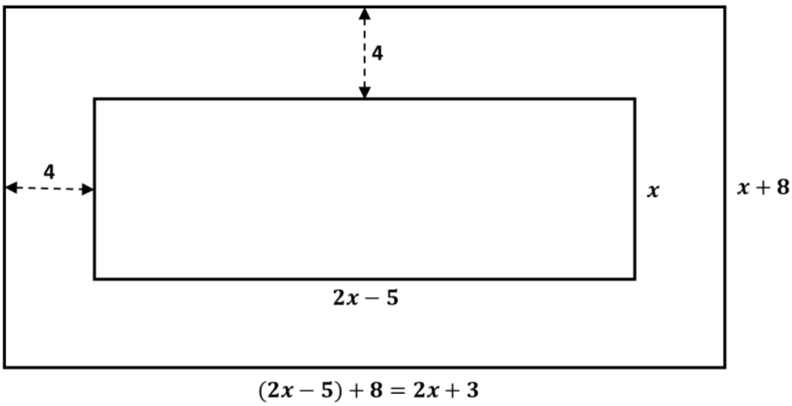

Figure 17. Diagram for herb garden problem

. . .

The length of a rectangle is 5 units more than its width. If the area of the rectangle is 126 square units, what are the dimensions of the rectangle?

Hint: If we let the width of the rectangle be x, then we are given that the length is $x + 5$. Further, the area is $x(x + 5) = 126$.

. . .

A telephone pole is supported by a wire anchored to the ground 5 feet from its base. The wire is 1 foot longer than the height that it reaches on the telephone pole (see Figure 18). Find the length of the wire.

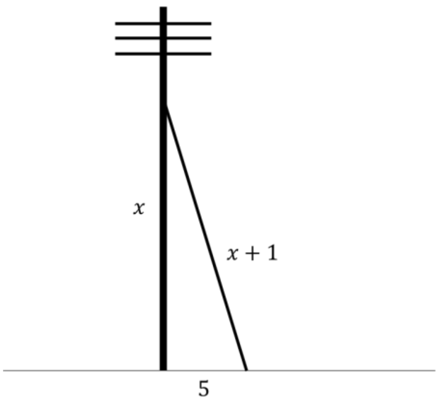

Figure 18. Telephone pole and support wire

Answer:

To solve this problem, we need to make use of the famous Pythagorean theorem which states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares on the other two sides of a right triangle (i.e., a triangle with one interior angle equal to 90 degrees). The Pythagorean theorem is illustrated in Figure 19.

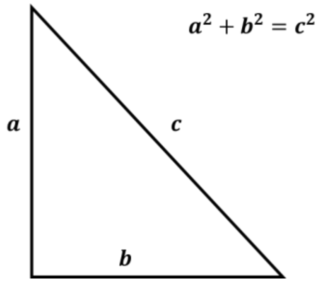

Figure 19. Pythagorean theorem

Applying the Pythagorean theorem to our problem, we have that

$$x^2 + 5^2 = (x + 1)^2$$

Expanding a bit, we get the following

$$x^2 + 25 = x^2 + 2x + 1$$

which reduces to

$$2x = 24 \text{ or } x = 12$$

Thus, the length of the wire is $x + 1 = 13$ feet.

. . .

The length of a rectangle is twice its width. If 3 meters are taken from its width, and 4 meters are taken from its length, the resulting rectangle has an area of 180 square meters. What are the dimensions of the original rectangle?

Answer:

Let x be the width of the original rectangle, which implies its length is $2x$. Further, we are given the following

$$(x - 3)(2x - 4) = 180$$

The above equation can be simplified as follows

$$2x^2 - 6x - 4x + 12 = 180$$

$$2x^2 - 10x - 168 = 0$$

$$x^2 - 5x - 84 = 0$$

This can be factored to get

$$(x+7)(x-12)=0$$

We select the positive root, i.e., $x=12$. Thus, the width of the original rectangle is 12 and its length is 24.

. . .

A room is $3x-6$ meters high, $x+2$ meters in length and $2x-3$ meters wide. If the volume of the room is 45 cubic meters, find the dimensions of the room.

Answer:

The volume of a rectangular solid is the product of its height, length and width. We are given

$$(3x-6)(x+2)(2x-3)=45$$

Expanding the above and collecting like terms, we get

$$6x^3-9x^2-24x-9=0$$

Dividing both sides of the above equation by 3, we get

$$2x^3-3x^2-8x-3=0$$

Using the rational root theorem, we have that the numerator of any possible rational roots are -3,-1,1 or 3 and the possible denominators are -2,-1,1 or 2. We can express this more succinctly as

$$\frac{\{\pm 1,\pm 3\}}{\{\pm 1,\pm 2\}}$$

Thus, the following are possible rational roots:

$$-3,-\frac{3}{2},-1,-\frac{1}{2},1,\frac{1}{2},\frac{3}{2},3$$

Trying these value for x in $2x^3-3x^2-8x-3$, we find that $-1,-\frac{1}{2}$ and 3 are roots. The two negative roots make no sense for the given problem and so we select the positive root $x=3$. Thus, the room's dimensions are 3 by 5 by 3, which checks out to be 45 cubic meters.

8.6.4 Problems Involving Long-Division

If the area of a rectangle is given by $x^3-2x^2-6x+12$ and its length given by $x-2$, with the restriction that $x>2$. Find the width of the rectangle.

Answer:

Since $Area\ =\ Length\ \cdot Width$, we have that

$$Width=\frac{Area}{Length}=\frac{x^3-2x^2-6x+12}{x-2}$$

Use long-division, as shown in the table below:

			x^2	0	-6
x	-2	x^3	$-2x^2$	$-6x$	12
		x^3	$-2x^2$		
			0	$-6x$	12
			0	$-6x$	12
				0	0

So, the width is given by $x^2 - 6$.

. . .

If the area of a rectangle is given by $10x^3 + 51x^2 + 62x + 21$ and the width is given by $10x^2 + 41x + 21$, find an expression for the length. What are the restrictions on x? ***Hint***: Division by zero is not allowed, and the length and width of the rectangle must be positive numbers.

. . .

Find a and b if the polynomial $x^5 - ax + b$ is divisible by $x^2 - 4$.

Answer:

Using long-division, as shown below, we must have $a = 16$ and $b = 0$ for $x^2 - 4$ to exactly divide $x^5 - ax + b$

					x^3	0	$4x$	
x^2	0	-4	x^5	0	0	0	$-ax$	b
			x^5	0	$-4x^3$			
					$4x^3$	0	$-ax$	$-b$
					$4x^3$	0	$-16x$	0
							$(16 - a)x$	$-b$

9 Polynomials with Multiple Variables

9.1 Overview

This section covers polynomials with multiple variables. A **multi-variable polynomial** is an expression consisting of 2 or more variables and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponentiation of variables. Some examples

- $5x^2 + y^2$
- $x^2 - 3xy + y^2 + 7$
- $3zy^2 + wxyz - z^2w + 3$

Each group of multiplied variables and constants (separated by a plus or minus sign) in a polynomial is called a **term**. In the expression $3zy^2 + wxyz - z^2w + 3$, there are four terms, i.e., $3zy^2, wxyz, -z^2w$ and 3. The degree of a term is the sum of the exponents of the variables. For example, the **degree of the term** $wxyz$ is four and the degree of the term $3zy^2$ is three. The **degree of a polynomial** is defined to be the degree of the term with highest degree. For example, the degree of $3zy^2 + wxyz - z^2w + 3$ is four.

In what follows in this section, we focus on a particular type of multi-variable polynomial known as a conic section.

9.2 Conic Sections

9.2.1 Overview

A **conic section** is a multi-variable polynomial of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where a, b, c, d, e and f are constants, and x and y are variables.

The term “conic section” comes from the fact that the various curves defined by the above formula can be obtained as the intersection of the surface of a one or two cones with a plane. The three types of conic sections are the hyperbola (on the right of Figure 20), the parabola (on the left of Figure 20), and the ellipse (middle-top diagram in Figure 20); the circle (middle-bottom diagram in Figure 20) is a special case of the ellipse. However, the typical definitions of these curves are typically not based on the intersection of a plane with a cone, as will be explained further in the following subsection.

Figure 20. Conic Sections

9.2.2 Circles

A **circle** is the set of all points equidistant from a given point.

In order to derive a formula for a circle, we first need to determine a formula for the distance between two points. Figure 21 shows two points, i.e., (x_0, y_0) and (x_1, y_1). The horizontal distance between the two points is $x_1 - x_0$ and the vertical distance between the two points is $y_1 - y_0$. The straight-line **distance between two points** (labeled as d in the figure) is what we seek. From the Pythagorean theorem, we have that $d^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$. Solving for d, we have $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$.

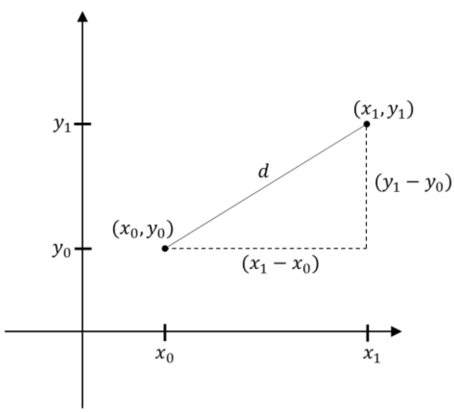

Figure 21. Distance between two points

To determine the equation for the circle with center (x_0, y_0) and radius r, we will use the formula for the distance squared. All points (x, y) of distance r from the point (x_0, y_0) satisfy the following equation:

$$r^2 = (x - x_0)^2 + (y - y_0)^2$$

In other words, the above expression is the equation for a circle with center (x_0, y_0) and radius r.

For example, the circle with center $(2,3)$ with radius 5 is given by the following equation and depicted in Figure 22.

$$25 = (x - 2)^2 + (y - 3)^2$$

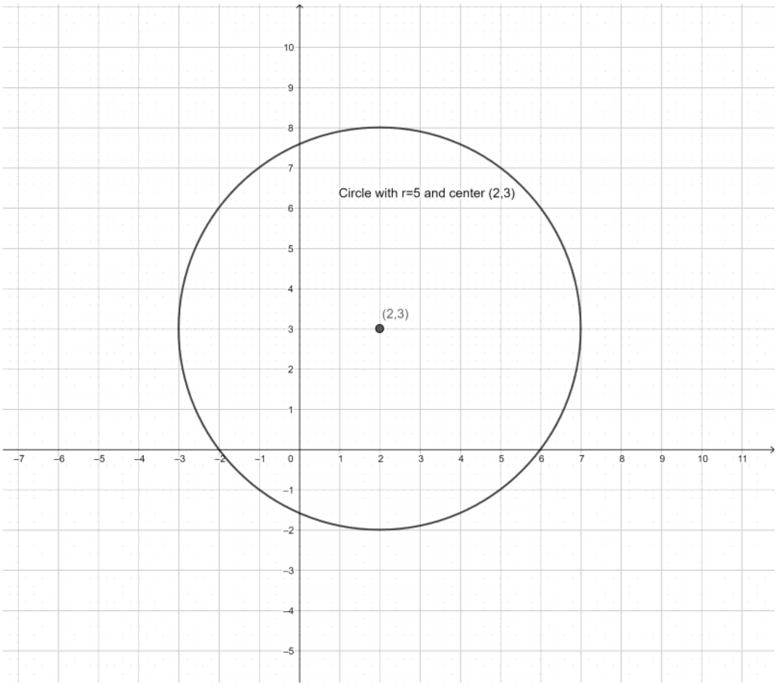

Figure 22. Graph of circle

9.2.3 Parabolas

A **parabola** is the set of points that are equidistant from a given point (known as the **focus**) and given line (known as the **directrix**).

The general equation for the parabola with focus $F(f_1, f_2)$ and directrix $ax + by + c = 0$ is given by

$$\frac{(ax + by + c)^2}{a^2 + b^2} = (x - f_1)^2 + (y - f_2)^2$$

As a first example, consider the parabola with focus $(0, \frac{1}{4})$ and directrix $y = -1/4$. We have that

$$a = 0, b = 1, c = \frac{1}{4}, f_1 = 0, f_2 = 1/4$$

Plugging into the general equation, we get

$$(y + \frac{1}{4})^2 = x^2 + (y - \frac{1}{4})^2$$

Expanding both sides of the equation, yields

$$y^2 + \frac{1}{2}y + \frac{1}{2} = x^2 + y^2 - \frac{1}{2}y + \frac{1}{2}$$

Cancelling like terms on both side, we arrive at

$$y = x^2$$

The parabola $y = x^2$, along with its focus and directrix, is shown in Figure 23. The **axis of symmetry** divides a parabola into symmetrical halves (the line $x = 0$ in the example). The intersection of the axis of symmetry with a parabola is called the **vertex** of the parabola (the point $(0,0)$ in the example). The distance between the vertex and the focus, measured along the axis of symmetry, is the **focal length** ($\frac{1}{4}$ in the example).

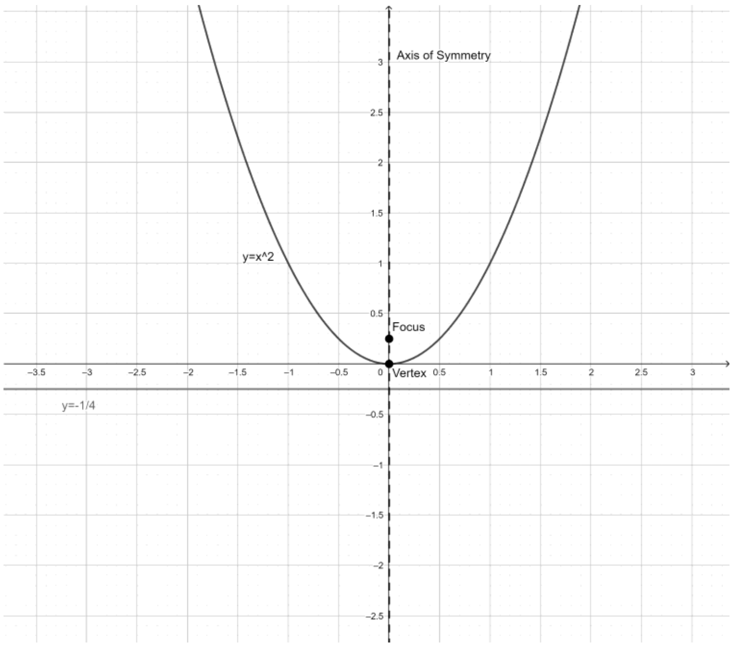

Figure 23. Simple parabola

In general, if $a = 0$, the directrix is a horizontal line, i.e., $y = -\frac{c}{b}$. In this case, the equation for a parabola reduces to the form $y = Ax^2 + Bx + C$, where A, B and C are constants. So, if we are presented with an equation of the above form, we know it is an equation for a parabola with a horizontal line as its directrix.

Similar, if $b = 0$, the directrix is a vertical line, i.e., $x = -\frac{c}{a}$. An Equation of the form $x = Dy^2 + Ey + F$, with D, E and F being constants, is a parabola with a vertical line as its directrix.

. . .

As a more complex example, consider the parabola with focus $(1,1)$ and directrix given by $x + y + 1 = 0$. Whenever the directrix is not a vertical or horizontal line, the parabola will be rotated with respect to the axes of the graph.

Using the general formula for a parabola, with $a = b = c = 1$, and $f_1 = f_2 = 1$, we get the equation

$$\frac{(x + y + 1)^2}{2} = (x - 1)^2 + (y - 1)^2$$

After some work (steps omitted), the above equation reduces to

$$x^2 - 2xy + y^2 - 6x - 6y = -3$$

The graph of the parabola, along with its focus and directrix, is shown in Figure 24.

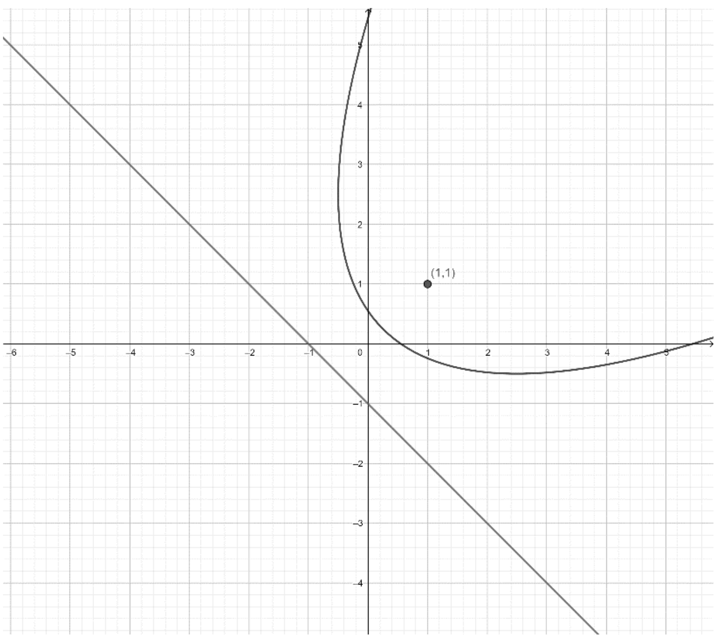

Figure 24. Rotated parabola

Some exercises to try:

- Determine the equation for the parabola with focus $(\frac{1}{4}, 0)$ and directrix $x = -\frac{1}{4}$ and then graph the parabola
- Determine the equation for the parabola with focus $(0, \frac{5}{4})$ and directrix $y = \frac{3}{4}$ and then graph the parabola
- Determine the equation for the parabola with focus $(1, \frac{9}{4})$ and directrix $y = \frac{3}{4}$ and then graph the parabola

You can use GeoGebra to check your answers. Just enter the point and equation, and then use the Parabola(Point, Equation) function.

. . .

As we have seen, the general equation for a parabola allows for rotated parabolas with respect to the coordinate axes. There is also something called the **standard form of a parabola** which requires (assumes) either a horizontal or vertical line for the directrix. This assumption results in a simpler equation for parabolas.

Consider a parabola with vertex (h, k), focus $(h, k + p)$ and directrix $y = k - p$, where h, k and p are constants. Associating this information with the general formula for a parabola, we have

$$a = 0, b = 1, c = p - k, f_1 = h \text{ and } f_2 = k + p$$

Substituting into the general formula

$$(y + p - k)^2 = (x - h)^2 + (y - k - p)^2$$

$$y^2 + 2(p - k)y + (p - k)^2 = (x - h)^2 + y^2 - 2(p + k)y + (p + k)^2$$

Subtracting y^2 from both sides of the equation and collecting like terms, we have

$$(2p - 2k + 2p + 2k)y + (p - k)^2 - (p + k)^2 = (x - h)^2$$

$$4py - 4kp = (x - h)^2$$

$$\boldsymbol{4p(y - k) = (x - h)^2}$$

The last equation in the above sequence is the standard form for a parabola with a horizontal directrix. In this format, the focal length of the parabola is p.

If a parabola has the vertical line $x = h - p$ as its directrix, vertex (h, k) and focus $(h + p, k)$ then the parabola has the following standard form

$$\boldsymbol{4p(x - h) = (y - k)^2}$$

For example, if we are given that a parabola has directrix $y = 7$ and focus $(3,13)$ then we know the parabola satisfies the standard form. Next, we solve for h, k and p. We have

$$k - p = 7$$

$$k + p = 13$$

$$h = 3$$

Adding the first two equations together and solving for k, we get $k = 10$. Substituting $k = 10$ back into either of the first two equations, we get $p = 3$. Now, we have sufficient information to use the standard formula, i.e.,

$$12(y - 10) = (x - 3)^2$$

The standard forms are also useful in working in the opposite direction, i.e., given the equation for a parabola, find the directrix and focus.

For example, find the directrix and focus of the parabola given by the equation

$$y = x^2 - 6x + 2$$

To solve this problem, we need to use the completing the square technique that we saw in Section 8.4. The first step is to take $\frac{1}{2}$ the coefficient of the x term, and add and subtract this number from the right-side of the equation, i.e.,

$$y = x^2 - 6x + 9 - 9 + 2$$

This can be written as

$$y = (x - 3)^2 - 7$$

which can be arranged into the standard form for a parabola, i.e.,

$$y + 7 = (x - 3)^2$$

Comparing the above with the standard form, we see that $p = \frac{1}{4}$, $k = -7$ and $h = 3$. So, the directrix is $y = -7 - \frac{1}{4} = -\frac{29}{4}$ and the focus is $(3, -\frac{27}{4})$. The graph of this parabola is shown in Figure 25.

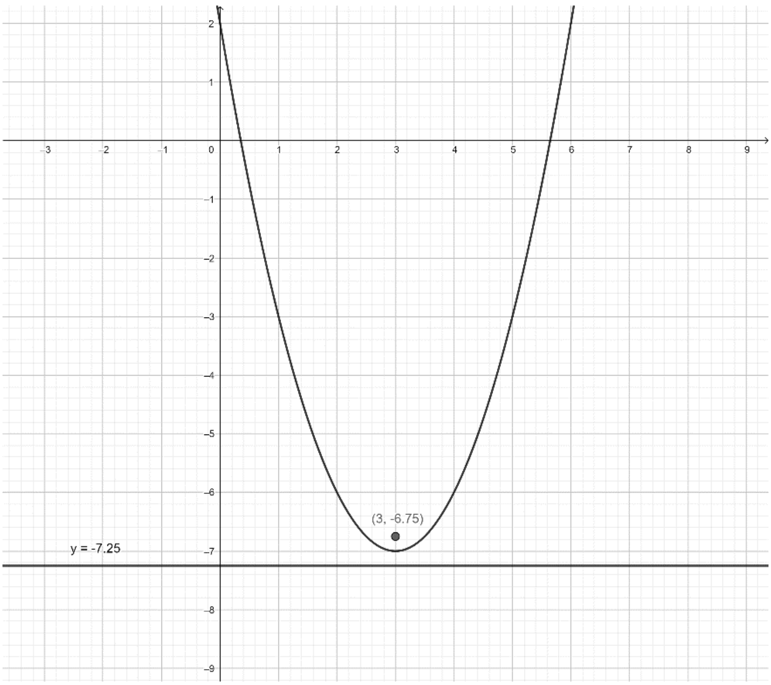

Figure 25. Graph of parabola in standard form

A few problems to try:

- Find the directrix and focus of the parabola given by $y = x^2 + 4x + 8$
- Find the directrix and focus of the parabola given by $4y = x^2 + 4x + 8$
- Find the directrix and focus of the parabola given by $x = y^2 + 4y + 8$.

. . .

Yet another approach for describing parabolas is to start with the simple parabolas $y = x^2$ and $x = y^2$, and then do various transformations to obtain other parabolas.

The parabolas in Figure 26 were all obtained by taking multiples of $y = x^2$ (shown as the dashed curve in the top-half of the figure). The parabolas above the x-axis (starting from the widest) are $\frac{1}{5}x^2, \frac{1}{2}x^2, x^2, 2x^2$ and $5x^2$. The parabolas below the x-axis (starting from the widest) are $-\frac{1}{5}x^2, -\frac{1}{2}x^2, -x^2, -2x^2$ and $-5x^2$. As one can see, multiplying by a fraction less than 1 widens the parabola and multiplying by a number greater than 1 makes the parabola narrower.

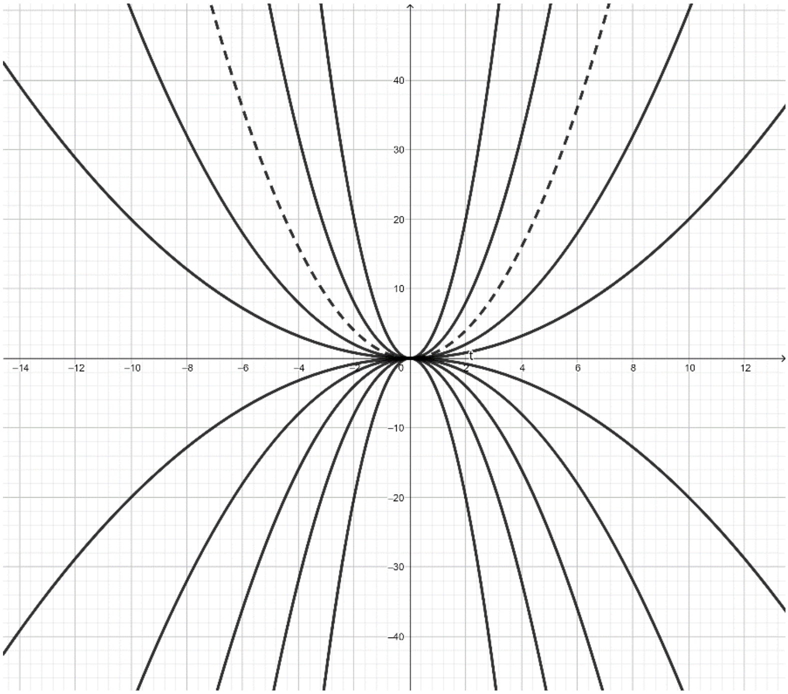

Figure 26. Parabola variations – 1

Similarly, we can start with $x = -y^2$ (dashed curve in Figure 27) and define many other parabolas by taking multiples of the original curve (as shown in Figure 27).

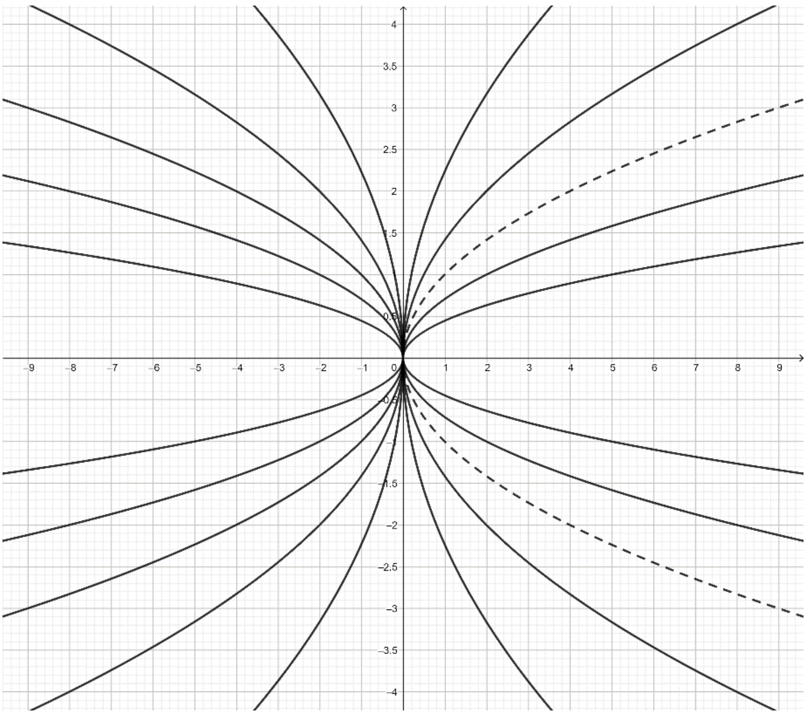

Figure 27. Parabola variations – 2

In addition to multiplying the expression of a parabola by a constant to make the parabola wider or narrower, we can also move (translate) the parabola to another location. Several examples of parabola translations are shown in Figure 28

- $(y-2)=(x-3)^2$ is a translation of $y=x^2$ three units to the right and two units up
- $(y+2)=-\frac{1}{3}(x+1)^2$ is a translation of $y=-\frac{1}{3}x^2$ one unit to the right and two units down
- $(x-3)=3(y+2)^2$ is a translation of $x=3y^2$ three units to the right and two units down.

In general, $(y-a)=c(x-b)^2$ is a translation of the parabola $y=cx^2$ by a units along the x-axis and b units along the y-axis. Similarly, $(x-a)=c(y-b)^2$ is a translation of the parabola $x=cy^2$ by a units along the x-axis and b units along the y-axis.

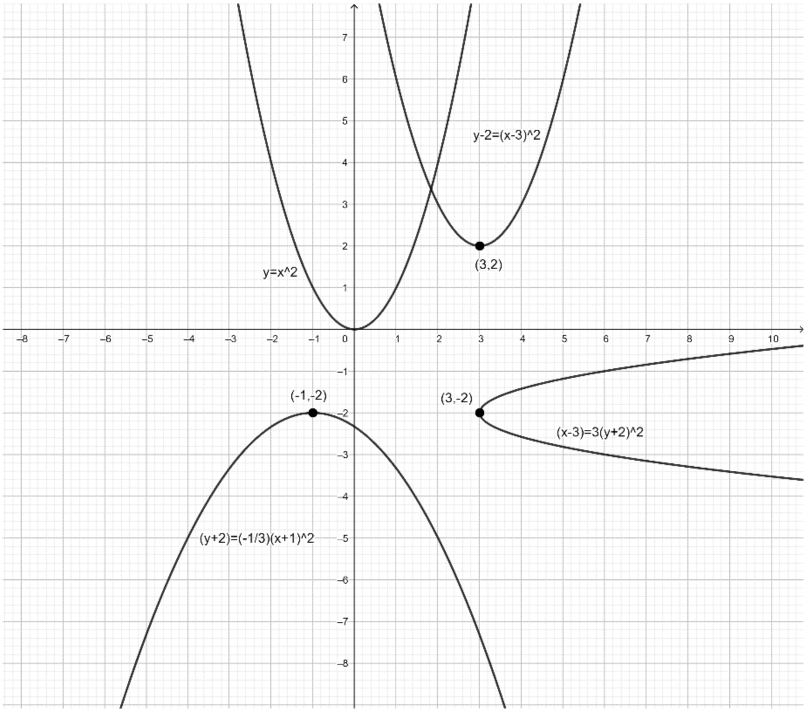

Figure 28. Parabola translations

It is also possible to do rotations of $y = x^2$ and $x = y^2$ but this involves trigonometry and matrix multiplication (neither of which has been covered thus far in this book).

9.2.4 Ellipses

An **ellipse** is a curve defined by two focal points, such that for all points on the ellipse, the sum of the two distances to the focal points is a constant. When the two foci coincide, the ellipse degenerates into a circle.

Figure 29 illustrates some basic concepts related to the ellipse:

- The foci are F_1 and F_2. The center is midway between the two foci. In the example, the center in $(0,0)$.
- The ellipse has a minor and major axis, each cutting the ellipse in half. The minor axis cuts along the shorter width of the ellipse and the major axis cuts along the longer width of the ellipse.
- The semi-minor axis (half the minor axis) is of length b.
- The semi-major axis (half the major axis) is of length a.

- For every point (x, y) on the ellipse, the sum of the distances from the foci is the same, i.e., $r_1 + r_2 = 2a$ for all points on the ellipse.
- The ellipse has two vertices along the major axis, shown as V_1 and V_2 in the figure.

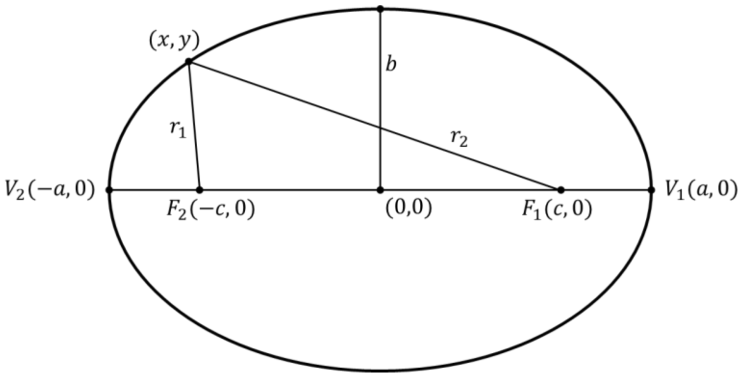

Figure 29. Ellipse terminology

The equation for the general ellipse, with center at the point (0,0) and as labeled in Figure 29, is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The derivation of the above formula only uses basic algebra but is a bit tedious, see the Wolfram MathWorld article for the details [15].

When $r_1 = r_2$, then both r_1 and r_2 must be equal to a. This results in the situation shown in Figure 30. By the Pythagorean theorem, we have $b^2 + c^2 = a^2$. Thus, to determine the equation of an ellipse with center at the origin, we need to know the values of at least two of the three variables a, b and c. This formula assumes that the foci are along a horizontal line, and that $a \geq b$. If the ellipse is oriented along a vertical line, with $a \leq b$, then we would have $a^2 + c^2 = b^2$.

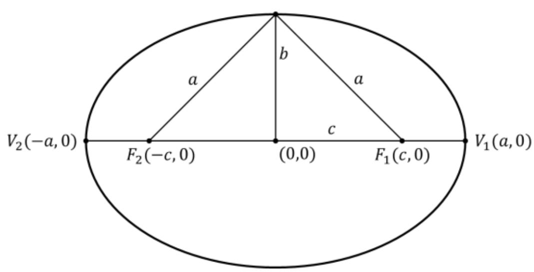

Figure 30. Relationship among the variables for an ellipse

The **eccentricity** of an ellipse is a measure of its elongation (or flatness). It is a number from 0 (the limiting case of a circle) to 1 (the limiting case of infinite elongation, no longer an ellipse but a parabola). The eccentricity (represented by the parameter e) can be compute from the values of a and b as follows:

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}, \text{ if } a \geq b$$

$$e = \frac{c}{b} = \sqrt{1 - \frac{a^2}{b^2}}, \text{ if } a \leq b$$

. . .

As a first example, consider the equation

$$\frac{x^2}{49} + \frac{y^2}{16} = 1$$

This is an ellipse with center at the origin, $a = 7$ and $b = 4$. We can compute c as follows:

$$c = \sqrt{a^2 - b^2} = \sqrt{49 - 16} = \sqrt{33} \cong 5.75$$

Thus, the foci are $F_1\left(-\sqrt{33}, 0\right)$ and $F_2(\sqrt{33}, 0)$. The eccentricity is $e = \frac{c}{a} \cong .82$.

The graph is shown in Figure 31.

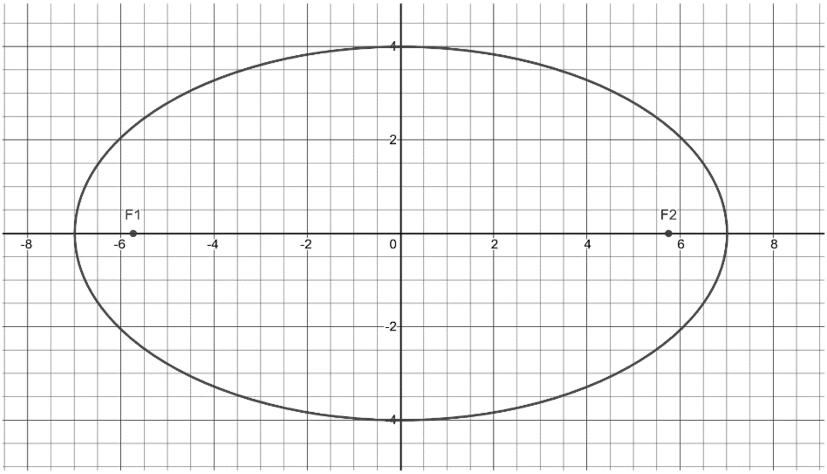

Figure 31. Ellipse example

When a is much larger than b, the result is a very flat ellipse. For example, Figure 32 depicts the graph of $\frac{x^2}{225} + \frac{y^2}{4} = 1$. The eccentricity is approximately $.991$.

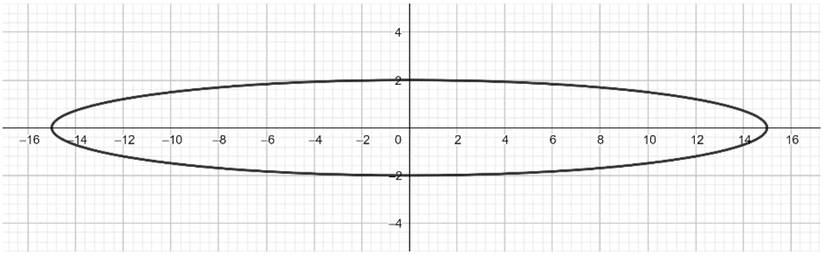

Figure 32. Very flat ellipse with high eccentricity

On the other hand, if b is much larger than a, the result is a very narrow ellipse. For example, Figure 33 depicts the graph of $\frac{x^2}{4}+\frac{y^2}{225}=1$. The eccentricity is approximately $.991$.

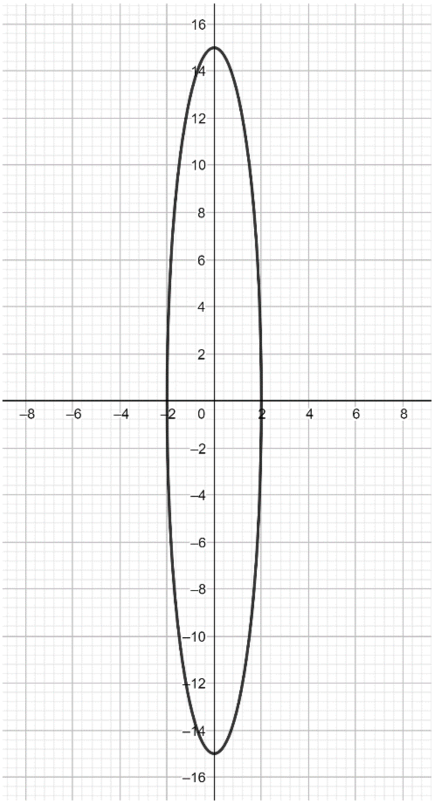

Figure 33. Very narrow ellipse with high eccentricity

In a manner similar to circle and parabolas, we can translate an ellipse up or down, and left or right. The equation

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

translates the center of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from the point $(0,0)$ to the point (x_0, y_0).

Figure 34 depicts several translations of the ellipse $\frac{x^2}{49}+\frac{y^2}{16}=1$

- The equation for the ellipse with center $(7,3)$ is $\frac{(x-7)^2}{49}+\frac{(y-3)^2}{16}=1$.
- The equation for the ellipse with center $(-9,6)$ is $\frac{(x+9)^2}{49}+\frac{(y-6)^2}{16}=1$.
- The equation for the ellipse with center $(-2,-5)$ is $\frac{(x+2)^2}{49}+\frac{(y+5)^2}{16}=1$.

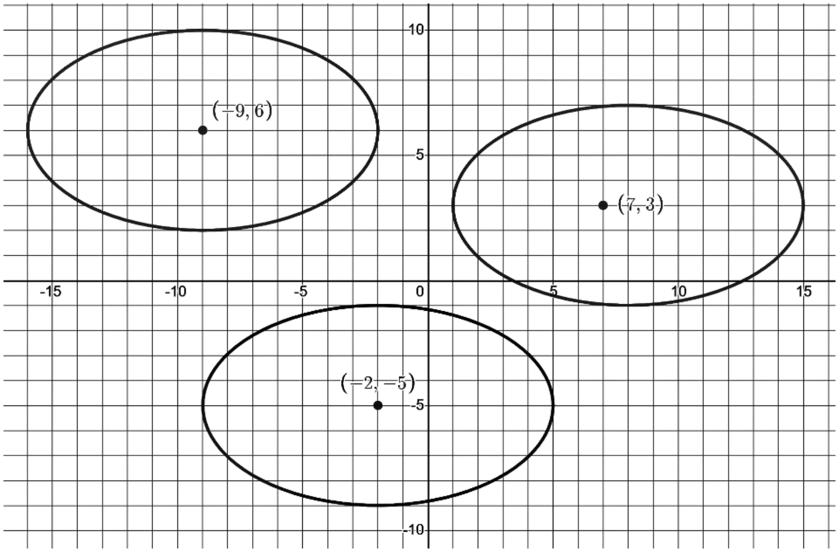

Figure 34. Ellipse translations

In addition to translations, we can also elongate (or shrink) an ellipse in either the vertical or horizontal direction.

Elongation in the horizontal direction is done by multiplying the x term by a positive number less than 1. Elongation in the vertical direction is done by multiplying the y term by a positive number less than 1.

Figure 35 shows examples of elongating the ellipse $\frac{x^2}{49}+\frac{y^2}{16}=1$ (labeled as eq1 in the figure) in both the vertical and horizontal. The equation for the ellipse that is elongated in the horizontal direction is

$$\frac{1}{4}\cdot\frac{x^2}{49}+\frac{y^2}{16}=1 \text{ (labeled as eq2 in the figure)}$$

and the equation for the ellipse that is elongated in the vertical direction is

$$\frac{x^2}{49}+\frac{1}{8}\cdot\frac{y^2}{16}=1 \text{ (labeled as eq 3 in the figure)}$$

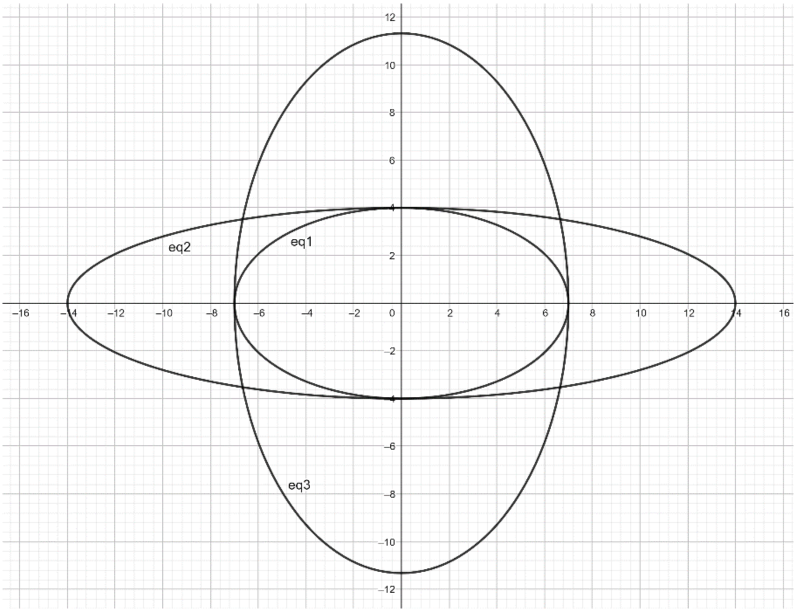

Figure 35. Ellipse elongation

The eccentricities of the three ellipses are as follows:

- original ellipse (eq1 in the figure): $e \cong .821$
- ellipse eq2 in the figure: $e \cong .958$
- ellipse eq3 in the figure: $e \cong .786$

Shrinking in the horizontal direction is done by multiplying the x term by a positive number greater than 1. Shrinking in the vertical direction is done by multiplying the y term by a positive number greater than 1.

Figure 36 shows examples of shrinking the ellipse $\frac{x^2}{49}+\frac{y^2}{16}=1$ (labeled as eq1) in both the vertical and horizontal directions.

The equation for the ellipse that is shrunk in the vertical direction (labeled as eq2) is

$$4 \cdot \frac{x^2}{49} + \frac{y^2}{16} = 1$$

The equation for the ellipse that is elongated in the horizontal direction (labeled as eq3) is

$$\frac{x^2}{49} + 8 \cdot \frac{y^2}{16} = 1$$

The eccentricities of the three ellipses are as follows:

- original ellipse (eq1 in the figure): $e \cong .821$
- ellipse eq2 in the figure: $e \cong .484$
- ellipse eq3 in the figure: $e \cong .979$

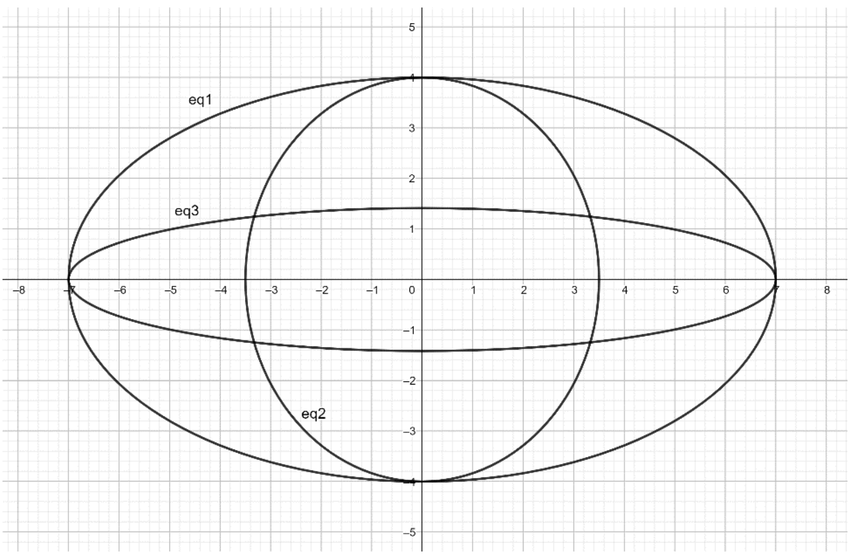

Figure 36. Ellipse flattening

Use GeoGebra to graph the following variations of $\frac{x^2}{49} + \frac{y^2}{16} = 1$ and also compute the eccentricity in each case

- $\frac{x^2}{49} + \frac{32y^2}{16} = 1$
- $\frac{98x^2}{49} + \frac{y^2}{16} = 1$
- $\frac{1}{53} \cdot \frac{x^2}{49} + \frac{y^2}{16} = 1$

- $\frac{x^2}{49} + \frac{1}{53} \cdot \frac{y^2}{16} = 1$

GeoGebra has an eccentricity function which takes that name of a conic as its argument and then returns the eccentricity.

. . .

The equation

$$\frac{1}{7} \cdot \frac{(x-5)^2}{49} + 5 \cdot \frac{(y+7)^2}{16} = 1$$

is a translation, elongation in the horizontal direction and shrinking in the vertical direction of the ellipse $\frac{x^2}{49} + \frac{y^2}{16} = 1$. A drawing of the original graph and the modified graph is available at https://www.desmos.com/calculator/qojdoybk90.

Try making some variations of your own at www.desmos.com.

. . .

Ellipses can be rotated. In such cases, there is an xy term in the equation. For example,

$$220x^2 - 96xy + 192y^2 + 220x - 48y = 2441$$

is the equation for a rotated ellipse with foci at A(-2,2) and B(1,2), and $a = 4$ (i.e., the length of the semi-major axis). The graph shown in Figure 37 was generated using the GeoGebra graphing tool (see https://www.geogebra.org/calculator/wjfkvn2v).

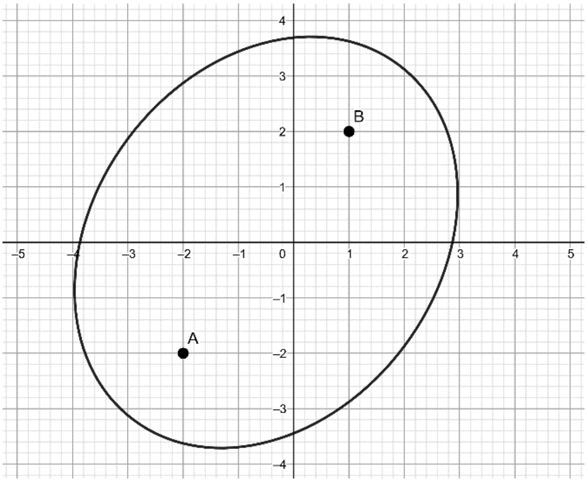

Figure 37. Rotated ellipse

9.2.5 Hyperbolas

Similar to an ellipse, a **hyperbola** is defined by two foci F_1 and F_2. However, in the case of a hyperbola, each point P on the hyperbola satisfies the following constraint:

$$|d(P, F_1) - d(P, F_2)| = 2a$$

such that

- a is a constant
- $d(P, F_1)$ is the distance between P and F_1. The distance $d(P, F_1)$ is as defined in the text associated with Figure 21. $d(P, F_2)$ is defined similarly.
- The vertical bars mean **absolute value**. The absolute value of a number changes a negative number to a positive number and leaves positive numbers as-is. For example, $|-4| = 4$ and $|4| = 4$.

The equation for a hyperbola is very similar to that of an ellipse, i.e.,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The graph for this hyperbola is shown in Figure 38 (see the two curves drawn with heavy solid lines):

- The hyperbola is made of two curves which approach two straight lines as the curves extend to infinity. These lines are known as the **asymptotes of the hyperbola**. For the given equation, the asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ (shown as dashed lines in the figure).
- For this form of a hyperbola, the center is at the origin, i.e., the point $(0,0)$. It is possible to translate the center as we did for ellipses.
- The vertices of the hyperbola are $V_1(a, 0)$ and $V_2(-a, 0)$.
- The constant b in the formula for the hyperbola determines the slope of the asymptotes lines for a given value of a.
- The foci of the hyperbola are at $F_1(c, 0)$ and $F_2(-c, 0)$. While not obvious, it can be shown that $a^2 + b^2 = c^2$ for a, b and c as defined here.

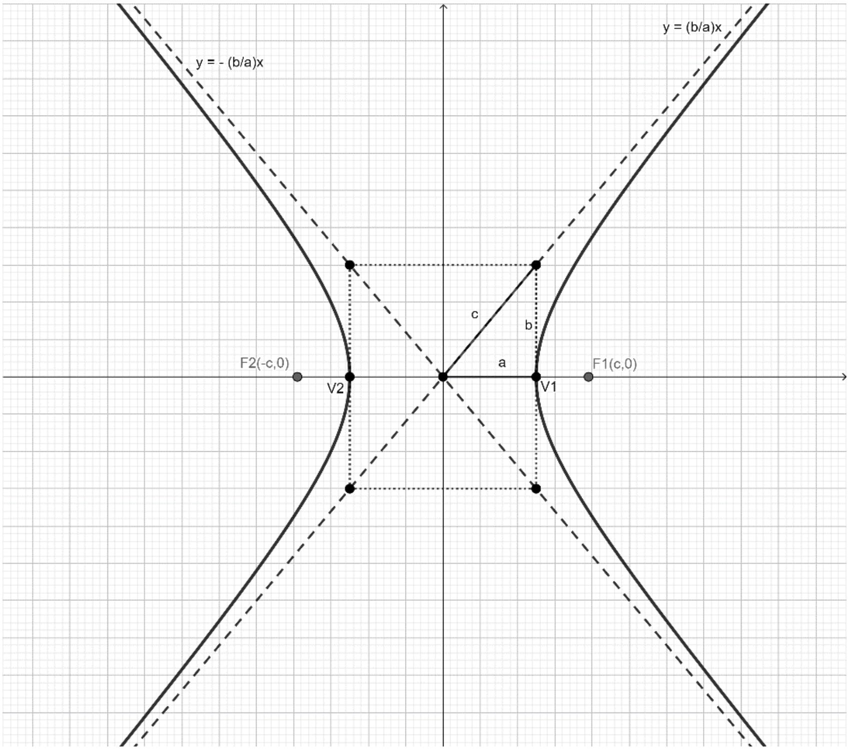

Figure 38. General characteristics of a hyperbola

Switching the negative sign to the y term, i.e.,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

results in the same hyperbola but rotated 90 degrees. The two curves are referred to as **conjugate hyperbolas**. For example, Figure 39 depicts the hyperbola

$\frac{x^2}{12^2} - \frac{y^2}{5^2} = 1$ (labeled as eq1 and shown as the solid curve)

and its conjugate

$\frac{y^2}{5^2} - \frac{x^2}{12^2} = 1$ (labeled as eq2 and shown as the dashed-line curve)

For both hyperbolas, we have $c = \sqrt{144 + 25} = \sqrt{169} = 13$.

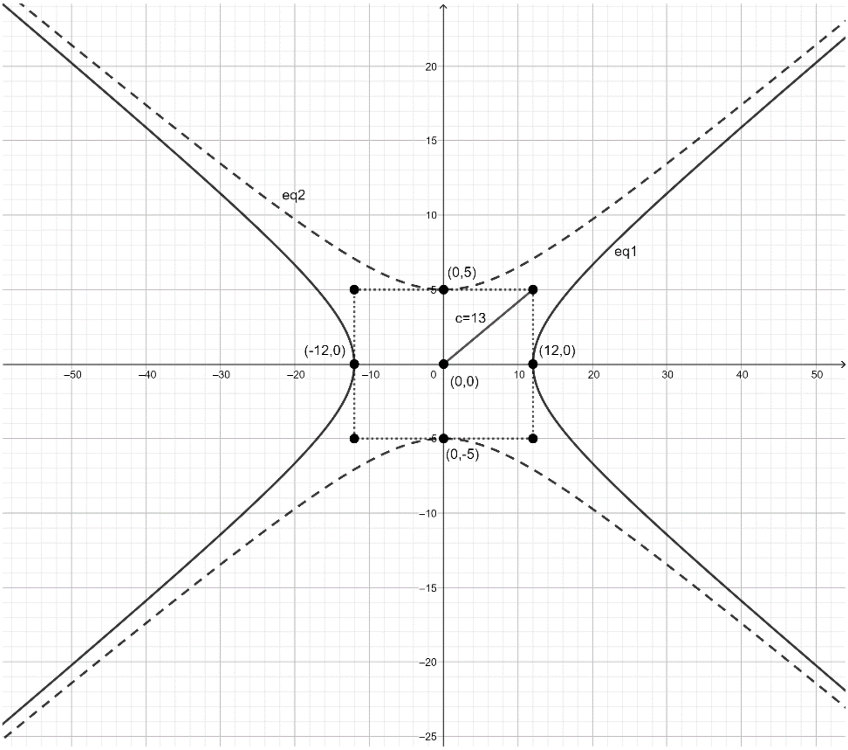

Figure 39. Conjugate hyperbolas

The center of a hyperbola can be moved to any point (x_0, y_0). The equation

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

has the same graph as $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ but with its center moved to the point (x_0, y_0).

For example, Figure 40 depicts the hyperbola given by the equation

$$\frac{(x-13)^2}{12^2}-\frac{(y+19)^2}{5^2}=1$$

This hyperbola has the same shape as $\frac{x^2}{12^2}-\frac{y^2}{5^2}=1$ but translated 13 units to the right and 19 units down. The center of the translated hyperbola is $(13, -19)$. The asymptote lines are

$$(y+19)=\frac{5}{12}(x-13)$$

and

$$(y+19)=-\frac{5}{12}(x-13)$$

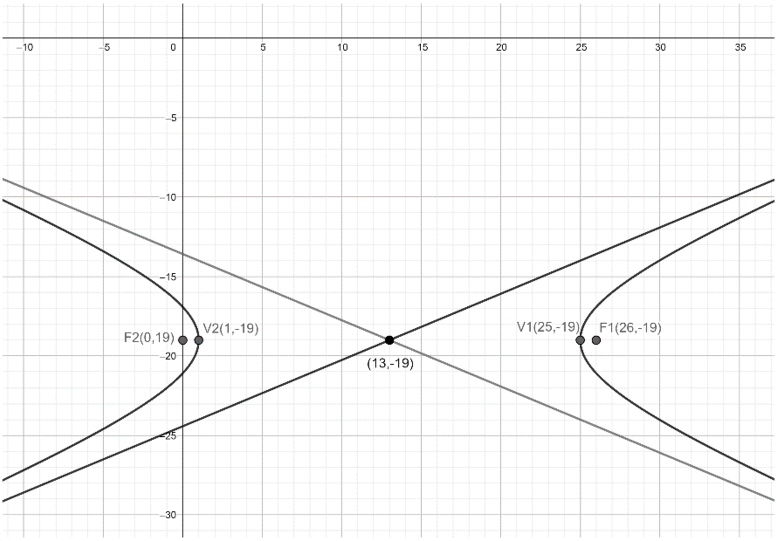

Figure 40. Hyperbola translation

As with the ellipse and parabola, it is possible to have a rotated hyperbola. For example, plot the graph for $-28x^2 - 64xy + 20y^2 - 56x - 64y = -71$ using GeoGebra (www.geogebra.org) or Desmos (www.desmos.com).

9.2.6 Identification of Conics

For most of the examples that we have seen thus far in this section, we have either been given the equation for a conic in standard form or have been given key characteristics such as foci or vertices. In some cases, one is presented with an equation in reduced form, e.g., $-28x^2 - 64xy + 20y^2 - 56x - 64y + 71 = 0$. Is there a way to determine that type of conic is represented by a given second degree equation of two variables when presented in such a form?

The answer to this question is "yes." Consider the general equation for a conic

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

We have the following facts:

- If $b^2 - 4ac$ is less than zero and if a conic exists, it will be either a circle or an ellipse.
    - if $a = c$ and $b = 0$, the equation represents a circle, which is a special case of an ellipse.
- If $b^2 - 4ac$ equals zero and if a conic exists, it will be a parabola.
- If $b^2 - 4ac$ is greater than zero and if a conic exists, it will be a hyperbola.

The “if a conic exists” condition refers to cases where the general second degree equation with two variables is a degenerate conic (discussed further in Section 9.3).

For the example stated above, we have $a = -28, b = -32$ and $c = 20$. Thus, $b^2 - 4ac = (-32)^2 - 4(-28)(20) = 1024 + 2240 = 3264 > 0$ which implies from the above rules that we have a hyperbola in this case. The graph is shown in Figure 41. The foci are at points $(1,1)$ and $(-3, -1)$. The asymptotes are shown as dashed lines.

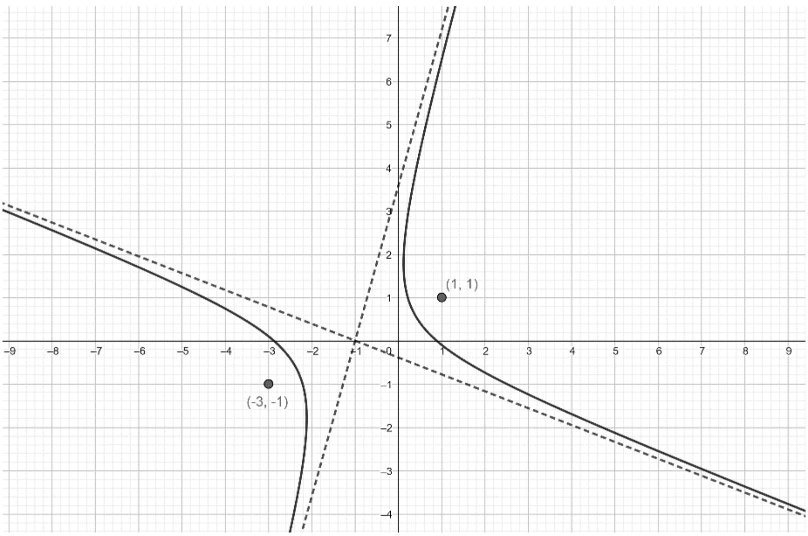

Figure 41. Graph of rotated hyperbola

Let’s try another example, i.e., $49x^2 + 70xy + 25y^2 + 10x - 20y - 25 = 0$. Doing the test, we have $b^2 - 4ac = 70^2 - 4(49)(25) = 4900 - 4900 = 0$ and so, the equation is that of a parabola. The graph shown in Figure 42.

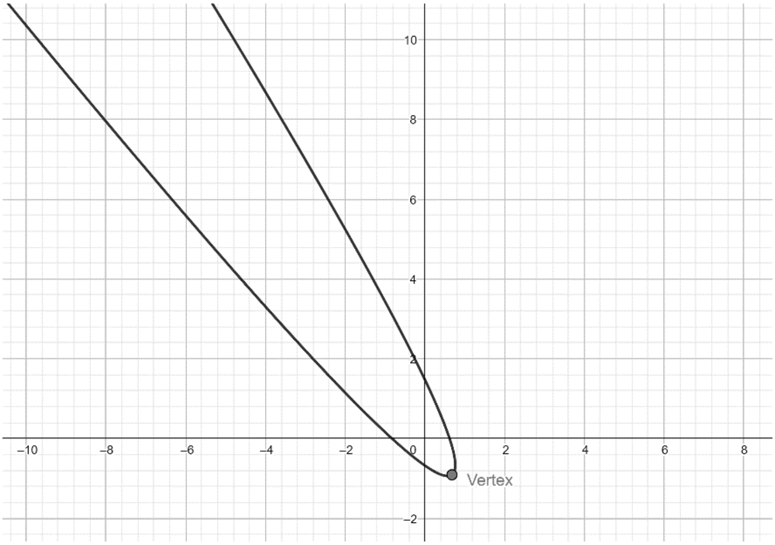

Figure 42. Graph of rotated parabola

Determine the type of conic described by the following equations, determine the vertices and foci, and graph each curve. Make use of the Focus and Vertex functions in GeoGebra.

- $x^2 - 3xy - 7y^2 - 6x + 5y - 11 = 0$
- $4x^2 - 12xy + 9y^2 - 2x + 7y + 5 = 0$
- $x^2 - 8x + y^2 + 14y + 40 = 0$
- $2x^2 - 12xy + 11y^2 - 3x + 9y + 13 = 0$
- $7x^2 - 12xy + 9y^2 - 5x + 12y - 11 = 0$

9.3 Degenerate Conics

There are several cases where the equation for a conic degenerates to a straight line, a pair of intersecting straight lines, a single point, or no points at all in the set of real numbers.

- If, in the general equation for a conic, $a = b = c = 0$, then we are left with the straight line $dx + ey + f = 0$.
- The equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 0$ is only satisfied by the single point (x_0, y_0).
- The equation $\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 0$ is satisfied by the two intersecting lines $y - y_0 = \frac{b}{a}(x - x_0)$ and $y - y_0 = -\frac{b}{a}(x - x_0)$.

- The equations $x^2 + 2 = 0$ and $3x^2 + y^2 + 4 = 0$ have no solutions in the real numbers. Note that the left-side of each equation is always greater than zero and thus equality cannot be achieved.

The Wikipedia article degenerate conics [16] provides an advanced discussion on this topic.

9.4 Solving

9.4.1 Intersection of a Conic and a Line

A line can intersect an ellipse or circle in zero, one or two points. All three cases are shown in Figure 43. The equation for the circle in the figure is

$$(x-3)^2 + (y-2)^2 = 9$$

In the case of the line $y = x$, there are two points of intersection. We can see this by substituting $y = x$ into the equation for the circle

$$(x-3)^2 + (x-2)^2 = 9$$

$$x^2 - 6x + 9 + x^2 - 4x + 4 = 9$$

$$2x^2 - 10x + 4 = 0$$

$$x^2 - 5x + 2 = 0$$

Using the quadratic formula, we get the two solutions, i.e.,

$$x = \frac{5+\sqrt{17}}{2}, x = \frac{5-\sqrt{17}}{2}$$

The line $y = -x - 2$ does not intersect the circle at all.

The line $y = 5$ intersects the circle only at the point $(3,5)$.

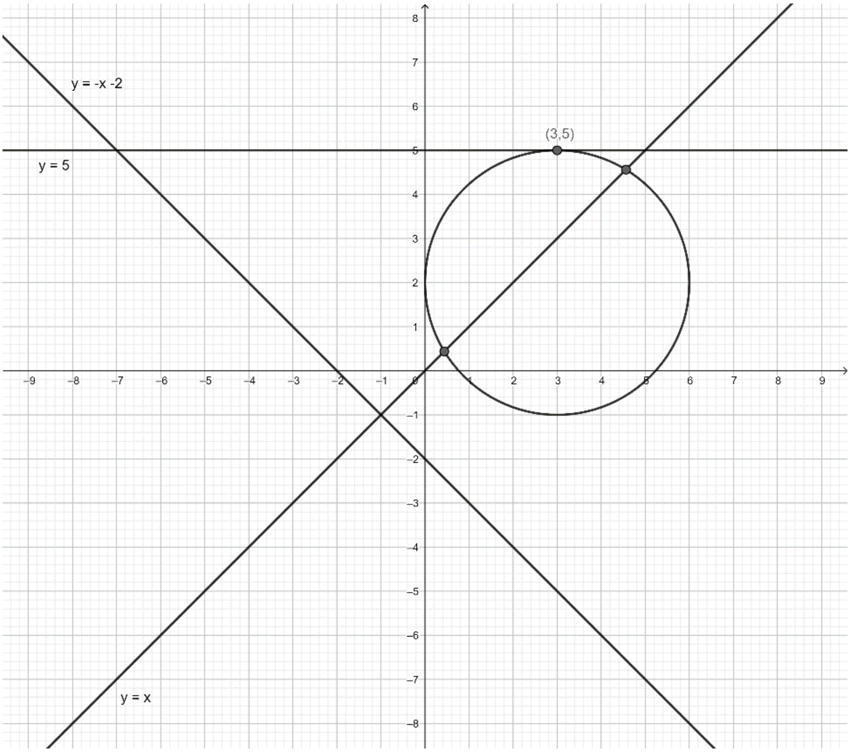

Figure 43. Lines intersecting a circle

A line can intersect a parabola in zero, one or two points. Figure 44 shows the line $y = x$ intersecting the parabola $y = \frac{1}{4}(x + 1)^2$ at the point (1,1), and the line $y = 4$ intersecting the same parabola at points $(-5,4)$ and $(3,4)$. The $y = x - 2$ does not intersect the parabola at all.

Concerning the intersection point with $y = x$, we substitute $y = x$ into the equation for the parabola to get

$$x = \frac{1}{4}(x + 1)^2$$

which expands to

$$x = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$$

$$\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} = 0$$

$$x^2 - 2x + 1 = 0$$

which can be factored to get

$$(x-1)^2 = 0$$

So, the intersection is at $x = 1$ and $y = \frac{1}{4}(1+1)^2 = 1$, i.e., at the point $(1,1)$.

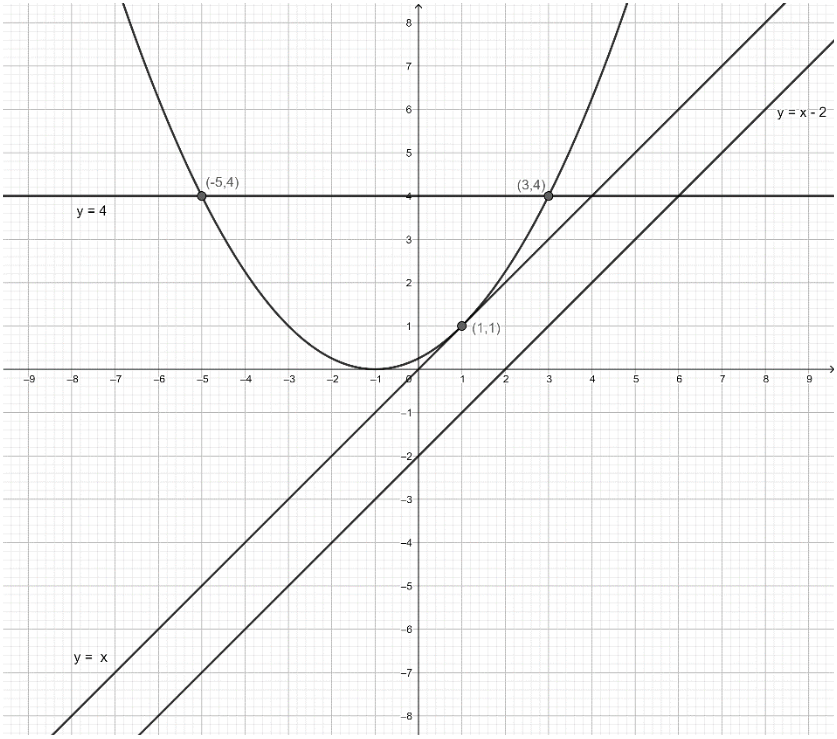

Figure 44. Lines intersecting a parabola

A line can intersect a hyperbola in zero, one or two points. Figure 45 shows two different ways that a line can intersect a hyperbola in two points. The equation for the hyperbola in the figure is

$$\frac{x^2}{121} - \frac{y^2}{81} = 1$$

The line $y = \frac{x}{3} + 7$ intersects each half of the hyperbola in one point. The line $x = 30$ intersects the right-half of the hyperbola in two points.

To find the intersection of $y = \frac{x}{3} + 7$ with $\frac{x^2}{121} - \frac{y^2}{81} = 1$, you need to substitute $\frac{x}{3} + 7$ for y in the equation of the hyperbola and then solve a very messy quadratic equation. Try typing x^2/121 - y^2/81 = 1; y=x/3 + 7

into the "Enter a problem" area at https://www.symbolab.com to see a summary of the steps and the final solution. Alternatively, enter the equation for the hyperbola and the line into GeoGebra and then use the intersection function. The intersection points A and B are as follows

$$A = \left(\frac{33\left(77 - 9\sqrt{1049}\right)}{608}, \frac{11\left(77 - 9\sqrt{1049}\right)}{608} + 7\right)$$

$$B = \left(\frac{33\left(77 + 9\sqrt{1049}\right)}{608}, \frac{11\left(77 + 9\sqrt{1049}\right)}{608} + 7\right)$$

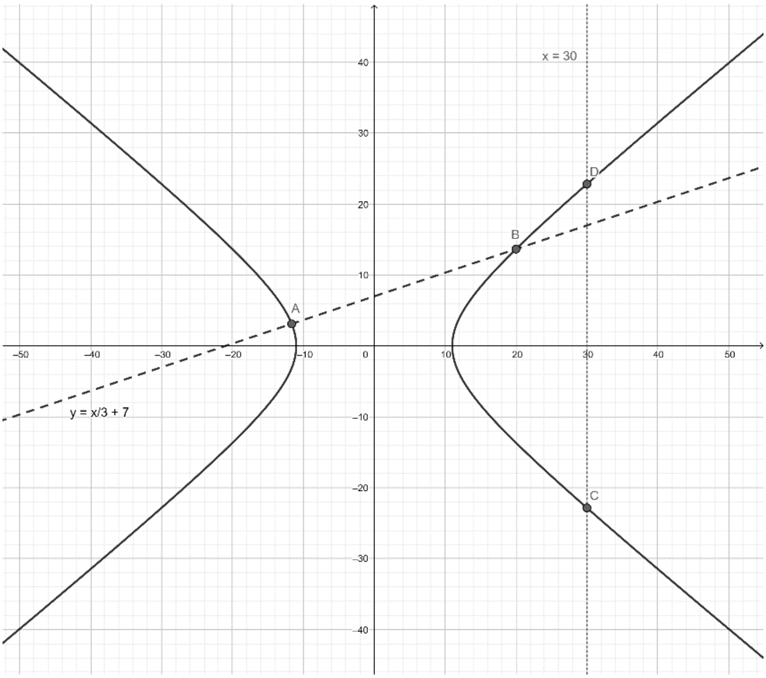

Figure 45. Lines intersecting a hyperbola

9.4.2 Intersection of Two Conics

As a byproduct of Bézout's theorem [17], we have that any two conics intersect in four points (some of which may be repeated and some of which may be complex numbers).

9.4.2.1 Intersecting Parabolas

Figure 46 shows the 3 possible cases for intersecting parabolas. The heavy-line parabola ($y = x^2$) is intersected once by the dotted-line parabola ($y = -x^2$), and twice by the dashed-line parabola ($y = -\frac{1}{2}x^2 + 2$). Also, dotted-line and dashed-line parabolas don't intersect at all. To find the exact intersection points of $y = x^2$ and $y = -\frac{1}{2}x^2 + 2$, we substitute the first equation into the second to get

$$x^2 = -\frac{1}{2}x^2 + 2$$

$$\frac{3}{2}x^2 = 2$$

$$x^2 = \frac{4}{3}$$

So, the two roots are $\pm\frac{2}{\sqrt{3}}$ and the points of intersection are $(\frac{2}{\sqrt{3}}, \frac{4}{3})$ and $(-\frac{2}{\sqrt{3}}, \frac{4}{3})$.

The figure was drawn using GeoGebra, see https://www.geogebra.org/calculator/awp4nxbp.

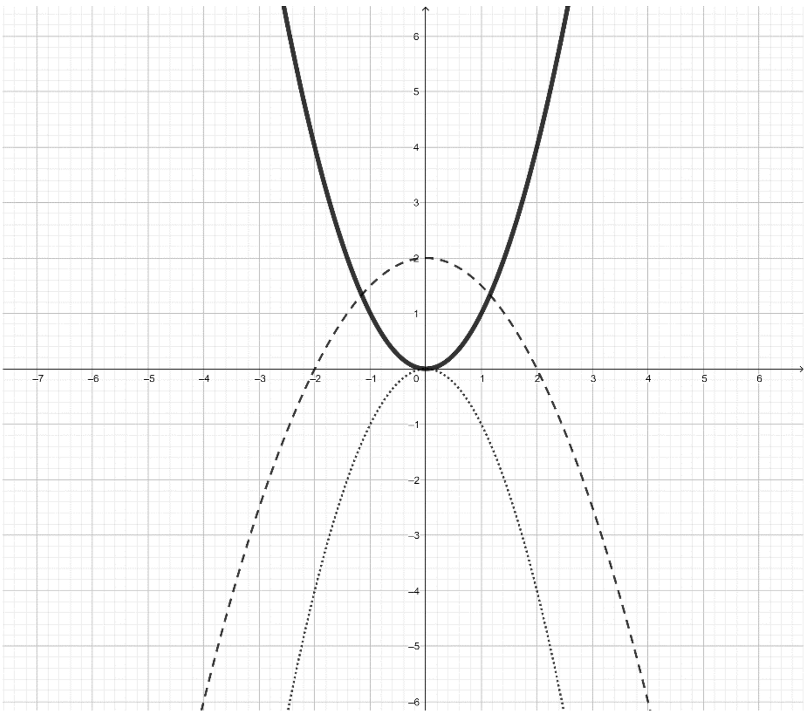

Figure 46. Intersecting parabolas

9.4.2.2 Intersecting Ellipses

Figure 47 shows two ellipses intersecting in four points. The ellipses are given by the equations:

$$2x^2 + 3y^2 = 5 \text{ (shown with the dashed line)}$$

$3x^2 + 2y^2 = 5$ (shown with the solid line)

From the first equation, we have $y^2 = \frac{5}{3} - \frac{2}{3}x^2$ and substituting this into the second equation, we get

$$3x^2 + \frac{10}{3} - \frac{4}{3}x^2 = 5$$

which simplifies to

$$\frac{5}{3}x^2 = \frac{5}{3}$$

So, $x = 1$ or -1. Using either of the two equations for the ellipses, $x = 1$ gives us that $y = 1$ or -1. This gives of two of the points of intersection, i.e., $(1,1)$ and $(1,-1)$. For $x = -1$, we get two additional points of intersection, i.e., $(-1,1)$ and $(-1,-1)$.

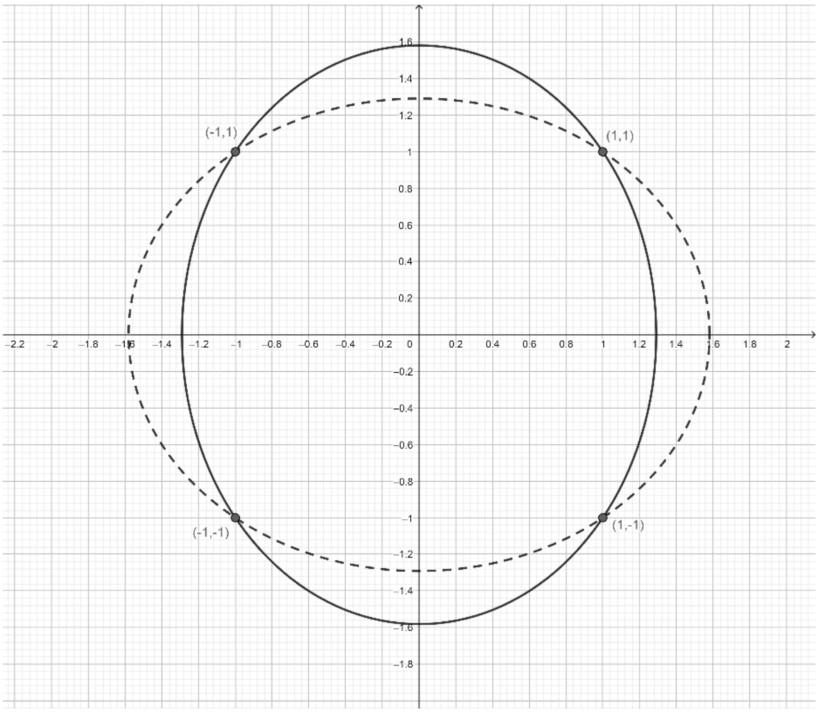

Figure 47. Ellipses intersecting in four points

Figure 48 shows two examples of ellipses intersecting at one point, and one example of ellipses intersecting at two points. In particular,

- The thin-line ellipse (given by $\frac{x^2}{9}+\frac{y^2}{16}=1$) intersects the dashed-line ellipse (given by $\frac{(x-6)^2}{9}+\frac{y^2}{16}=1$) and the dotted-line ellipse (given by $\frac{x^2}{9}+\frac{(y+2)^2}{16}=\frac{1}{4}$) each in one point.
- The thin-line ellipse (given by $\frac{x^2}{9}+\frac{y^2}{16}=1$) intersects the thick-line ellipse (given by $\frac{x^2}{9}+\frac{(y-4)^2}{16}=\frac{1}{4}$) in two points.

The figure was drawn using GeoGebra, see https://www.geogebra.org/calculator/b8ptkg9r. Try modifying the equations via translations and scaling of either the x or y term.

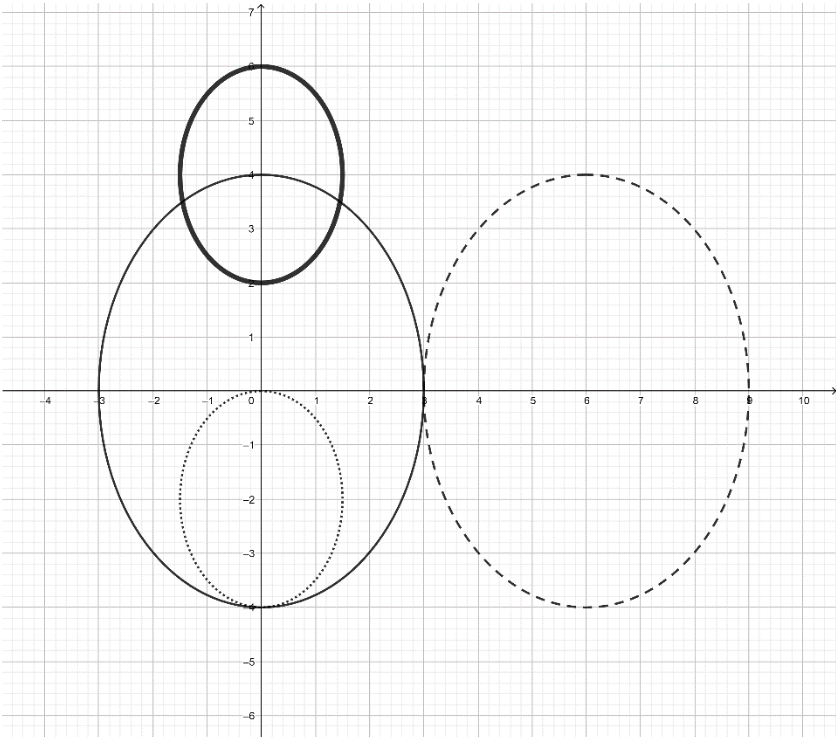

Figure 48. Intersecting ellipses

9.4.2.3 Intersecting Hyperbolas

Figure 48 depicts an example of two hyperbolas that intersect in exactly two points. The equations for the hyperbolas are

$x^2-3x+2-y^2-2y-1=3$ (shown as a dashed-line curve)

$y^2+5y+4-x^2+3x-2=5$ (shown as a solid-line curve)

If we add the two equations, we can eliminate the x terms as well as the y^2 term. This results in the equation

$$3y + 3 = 8$$

Thus, $y = 5/3$. Substituting back into the first equation, we get

$$x^2 - 3x + 2 - \frac{25}{9} - \frac{10}{3} - 1 = 3$$

which simplifies to

$$x^2 - 3x - \frac{73}{9} = 0$$

Using the quadratic formula, we get that $x = \frac{3}{2} \pm \frac{\sqrt{373}}{6}$.

Thus, the two points of intersection are $(\frac{3}{2} + \frac{\sqrt{373}}{6}, \frac{5}{3})$ and $(\frac{3}{2} - \frac{\sqrt{373}}{6}, \frac{5}{3})$.

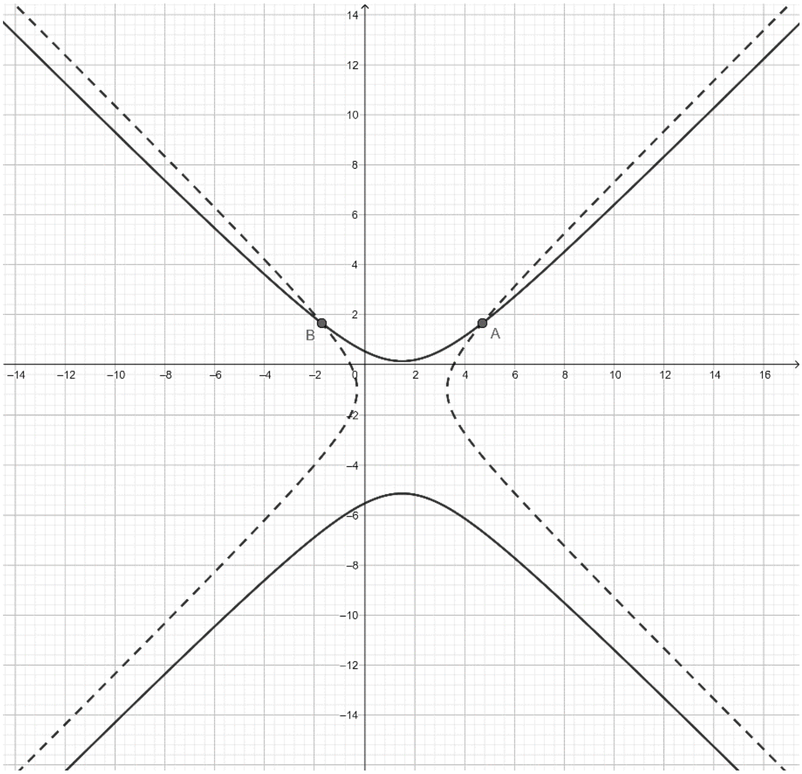

Figure 48. Hyperbolas intersecting in two points

9.4.2.4 Parabola-ellipse Intersections

Let’s work through some parabola-ellipse intersection examples, with an emphasis on how the examples are created.

In Figure 49, we start with a simple parabola, i.e., $y = x^2$ (labeled as eq1 in the figure).

The next step is to find an ellipse that intersects the parabola in two points. An ellipse with center at the origin should work, and we selected $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (labeled as eq2 in the figure). We can move eq2 up 2 units to get exactly three points of intersection with the parabola. This could be done by translating the origin of the ellipse given by eq2 two units up, i.e., $\frac{x^2}{3} + \frac{(y-2)^2}{4} = 1$ (not shown in the figure). For visual effect, we took a slightly different approach in the figure, i.e., replaced the 1 on the right-side by a 5 to make the ellipse large, and then multiplied the y term to make the ellipse narrower. This results in the ellipse $\frac{x^3}{3} + \frac{5(y-2)^2}{4} = 5$ (labeled as eq3 in the figure). Using GeoGebra, graph $\frac{x^3}{3} + \frac{(y-2)^2}{4} = 5$ and $\frac{x^2}{3} + \frac{5(y-2)^2}{4} = 1$ to see the individual effect of each change to eq2. Try experimenting with some other modifications of eq2.

Finally, we expand, flatten and translate upwards eq1 to get eq4, i.e.,

$$\frac{x^2}{3} + \frac{11(y-7)^2}{4} = 11$$

which intersects the parabola in four points.

- The $y - 7$ term translates eq1 up 7 units, but the translated ellipse no longer intersects the parabola but rather is “inside” the parabola.
- Replacing 1 by 11 on the right-side of the equation makes the ellipse larger so that we now have 4 points of intersection with the ellipse. However, this adjustment causes overlap (which is technically fine, but visually unpleasing).
- The 11 times $(y - 7)^2$ makes the ellipse flatter, avoiding the overlap while still preserving 4 intersection points.

To determine the intersection points, we substitute $y = x^2$. For example, to determine the four intersection points between eq1 (the parabola) and eq4, we substitute $y = x^2$ into eq4 to get

$$\frac{x^2}{3} + \frac{11(x^2-7)^2}{4} = 11$$

Multiple both sides of the equation by 12

$$4x^2 + 33(x^2 - 7)^2 = 132$$

Expanding the second term on the left, we get

$$4x^2 + 33(x^4 - 14x^2 + 49) = 132$$
$$33x^4 - 458x^2 + 1617 = 132$$
$$33x^4 - 458x^2 + 1485 = 0$$

If we let $u = x^2$, the above equation can be reduced to the following quadratic equation:

$$33u^2 - 458u + 1485 = 0$$

Applying the quadratic formula, we get that

$$u = \frac{229 \pm 2\sqrt{859}}{33}$$

Since $u = x^2$, we have that $x = \pm u$. For each value of u, we get two values for x and thus four roots to the quartic (4th degree) equation in x:

$$x = \sqrt{\frac{229 + 2\sqrt{859}}{33}}, \sqrt{\frac{229 - 2\sqrt{859}}{33}}, -\sqrt{\frac{229 + 2\sqrt{859}}{33}}, -\sqrt{\frac{229 - 2\sqrt{859}}{33}}$$

Try using Wolfram Alpha, to determine the intersection points of eq1 and eq4.

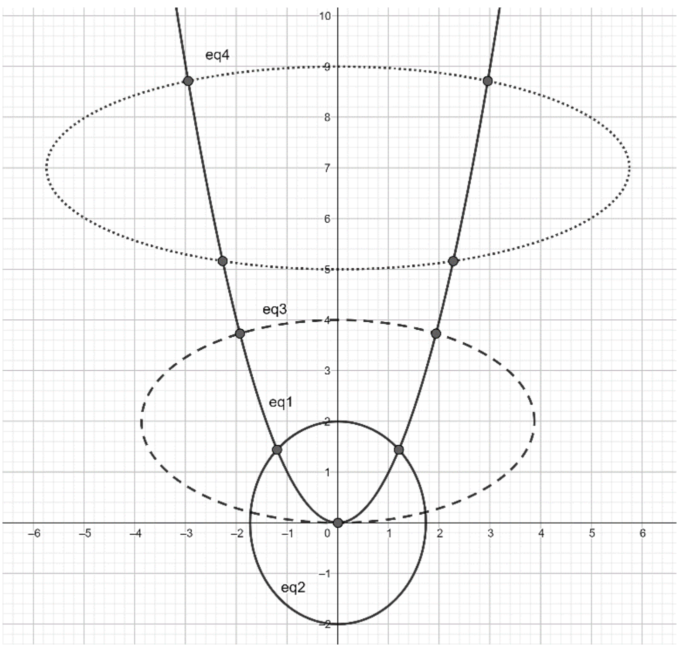

Figure 49. Intersect of a parabola with ellipses

Figure 49 was generated using GeoGebra, see https://www.geogebra.org/calculator/zcxgmrza.

9.4.2.5 Ellipse-hyperbola Intersections

Analysis of the intersection possibilities between an ellipse and a hyperbola are left as exercises for the reader with a lot of hints to help with the analysis:

- Using an online graphing tools such as GeoGebra or Desmos, start with a simple hyperbola, e.g., $\frac{x^2}{16} - \frac{y^2}{25} = 1$.
- Place an ellipse on the same graph, e.g., $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Since the ellipse does not intersect the hyperbola, we have covered one case, i.e., the case where an ellipse and hyperbola have no intersection points.
- The next step is to expand the ellipse so that it intersects the hyperbola in four points. This can be done by changing the right-hand side of the equation to a large number
- . Let's try 16, i.e., $\frac{x^2}{4} + \frac{y^2}{3} = 16$. This works! Also, note that the ellipse expanded by a factor of $\sqrt{16} = 4$. So, if we change the right-side to 4 that should double the size of the original ellipse and give us exactly 2 intersection points with the ellipse.
- To get exactly 3 intersection points, we can take $\frac{x^2}{4} + \frac{y^2}{3} = 16$ and translate it 4 units to the right by replacing x with $x - 4$, i.e., use the equation $\frac{(x-4)^2}{4} + \frac{y^2}{3} = 16$.
- To find the two intersection points of $\frac{x^2}{16} - \frac{y^2}{25} = 1$ and $\frac{x^2}{4} + \frac{y^2}{3} = 4$, solve the for equation for x^2, substitute into the second equation and then solve for y (you should get $y = 0$). Substitute $y = 0$ into either of the two equations to get that $x = 1$ or -1.
- To find the four intersection points of $\frac{x^2}{16} - \frac{y^2}{25} = 1$ and $\frac{x^2}{4} + \frac{y^2}{3} = 16$, subtract 4 times from the first equation from the second (to eliminate the x^2 terms) and solve for y (there will be two values). Substitute each value of y back into either of the original equations (there will be two values of x for each value of y). Check your answer with Wolfram Alpha or Symbolab.

9.4.2.6 Hyperbola-parabola Intersections

The final case concerns intersections of a hyperbola and a parabola. The possibilities are 0,1,2,3 and 4 points of intersection. An example of the 1,2,3 and 4 intersection cases is shown in Figure 50 and also available at https://www.geogebra.org/calculator/rkmycast.

The equations in the figure are as follows:

- eq1: $\frac{x^2}{9} - \frac{y^2}{4} = 1$ (hyperbola)
- eq2: $x = -y^2$ (parabola with 2 points of intersection with hyperbola)

- eq3: $x - 3 = -\frac{y^2}{2}$ (parabola with 3 points of intersection with hyperbola)
- eq4: $x - 6 = -\frac{y^2}{3}$ (parabola with 4 points of intersection with hyperbola)
- eq5: $x + 3 = -2y^2$ (parabola with 1 point of intersection with hyperbola)

Use Symbolab to find the points of intersection between eq1 and eq4. You need to separate the equations by a semicolon, and use the ^ symbol to represent exponents, e.g., write x^2 as x^2.

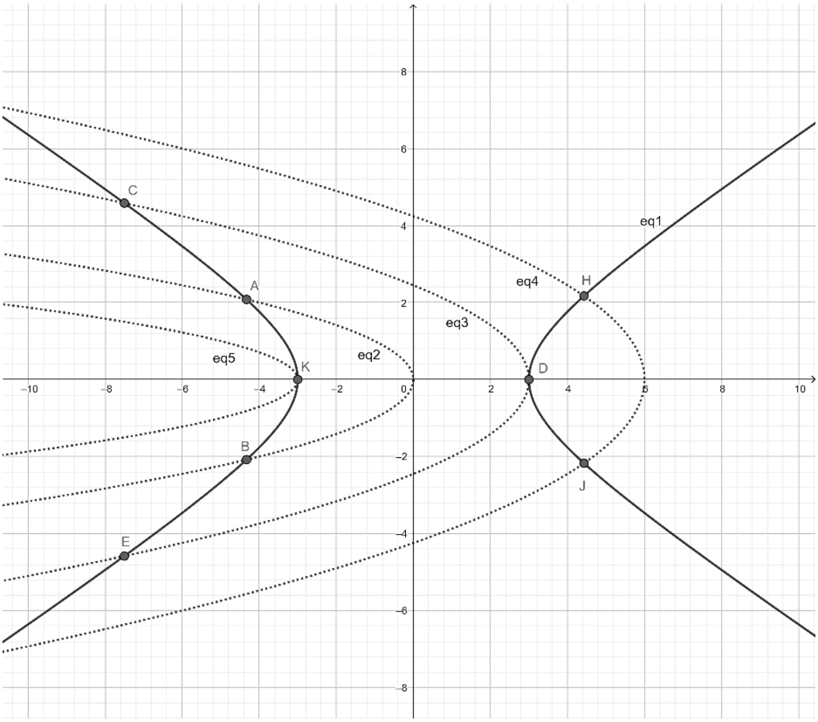

Figure 50. Intersection of parabolas with a hyperbola

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Most of the examples that we have seen in this section were intentionally designed to allow for relatively easy determination of the intersection points. In general, the determination of the intersection points for two conic sections is a difficult problem. For a discussion of the various approaches, see “Algorithm: Intersection of two conics” [18].

9.5 Word Problems

An arched entry to a park has a parabolic shape, with height 25 feet and a base of width 30 feet. Find the equation which models this shape, where the x-axis represents the ground. Also, find the focus and directrix of the parabola.

Answer:

Orient the parabola so that the axis of symmetry is the y-axis, the parabola opens downward, and the vertex is at $(0,25)$. Given the width of 30 feet, the parabola intersection the x-axis at the points $(-15,0)$ and $(15,0)$, as shown in Figure 51.

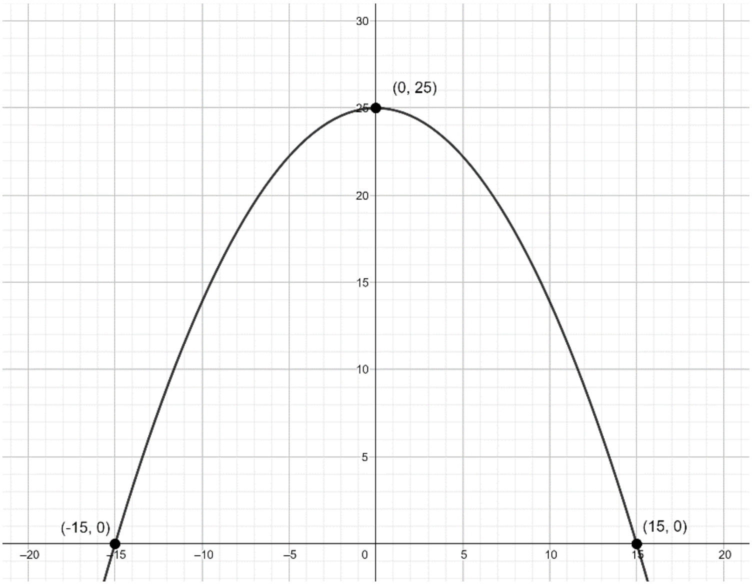

Figure 51. Arched entry to park

Given the assumed orientation and input information, we have that the parabola is of the standard form $4p(y-k) = (x-h)^2$. We know the vertex is $(0,25)$ and so, $k = 25$ and $h = 0$. Thus, the equation for the arch is $4p(y-25) = x^2$.

Substituting the point $(15,0)$ into the equation, we get

$$4p(0-25) = 15^2 = 225$$

$$p = -\frac{9}{4}$$

Thus, the equation for the arch is $-9(y-25) = x^2$, the directrix is $y = k - p = 25 + \frac{9}{4} = \frac{109}{4}$ and the focus is $(h, k+p) = \left(0, 25 - \frac{9}{4}\right) = (0, \frac{91}{4})$.

. . .

Try the same problem as the previous one but in this scenario, the arch is 10 meters high and $2\sqrt{10}$ meters wide at the base.

. . .

A radio telescope has a parabolic dish with a diameter of 100 meters. The received RF signals are reflected to the focal point of the parabola. If the focal length is 50 meters, find the depth of the dish.

Answer:

Situate the parabola so that its vertex is at $(0,0)$ and the axis of symmetry is the y-axis. So, $h = k = 0$ in the standard formula for a parabola. We are given that the focal length is 50 meters, i.e., $p = 50$. Thus, we have the equation for the parabolic dish:

$$200y = x^2$$

We are looking for the height of the parabola (depth of the dish as described in the problem statement) when $x = 50$. Plugging $x = 50$ into the equation of the parabola, we get that $y = \frac{2500}{200} = 12.5$ meters.

A diagram of the radio telescope dish is shown in Figure 52. The dish is the part of the parabola between points A and B. The parabola itself extends indefinitely.

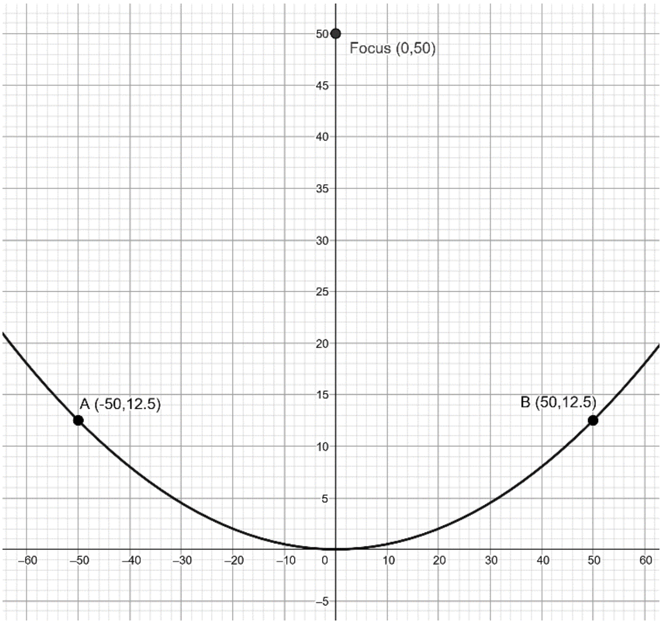

Figure 52. Radio telescope diagram

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Consider a half-ellipse tunnel that allows for one-way car and truck traffic. The tunnel is 10 meters wide and 6 meters high. Can a 3-meter-wide and 5-meter-high truck pass through the tunnel without hitting the sides of the tunnel?

Answer:

If we place the ellipse so that the center is at $(0,0)$, we have the ellipse with equation

$$\frac{x^2}{5^2}+\frac{y^2}{6^2}=1$$

The scenario is shown in Figure 53. The tunnel is represented by the top-half of the ellipse. A cross-section of the truck is shown as the shaded rectangle, and as one can see, the truck does fit.

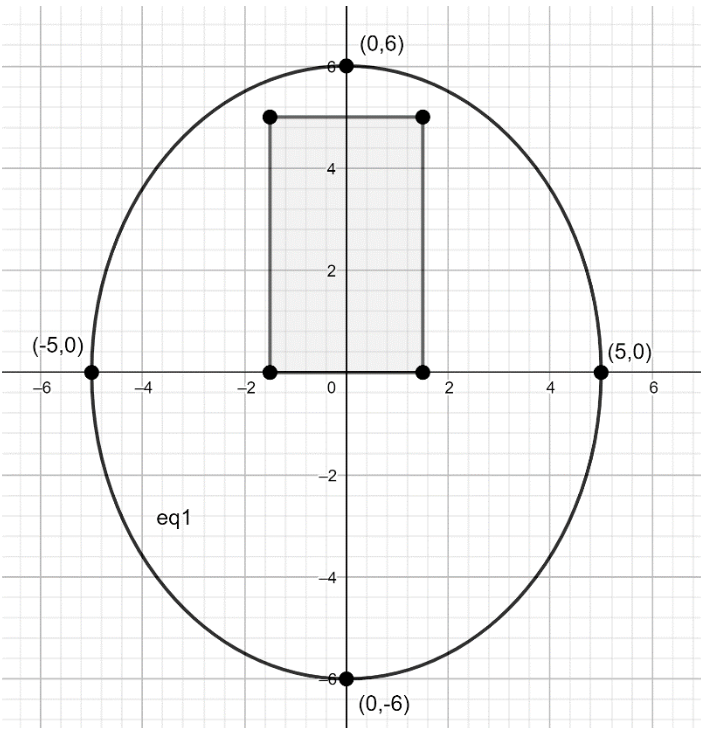

Figure 53. Truck in elliptical tunnel

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In the previous problem, what is the maximum height of the truck that can still fit in the tunnel? Assume the width is held constant at 6 meters.

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An elliptical-shaped garden is 50 meters at its major axis and 30 meters at its minor axis. The owner of the garden wants to put a fountain at each focus of the ellipse. Find an equation for the ellipse, and determine the location of the foci.

Answer:

Situate the ellipse so that its center is at the origin. We are given that $a = 25$ and $b = 15$. So, the equation for the ellipse is

$$\frac{x^2}{25^2} + \frac{y^2}{15^2} = 1$$

The graph is shown in Figure 54.

To determine the foci, we need to compute $c = \sqrt{a^2 - b^2} = \sqrt{625 - 225} = \sqrt{400} = 20$. So, the foci are at $(-20,0)$ and $(20,0)$.

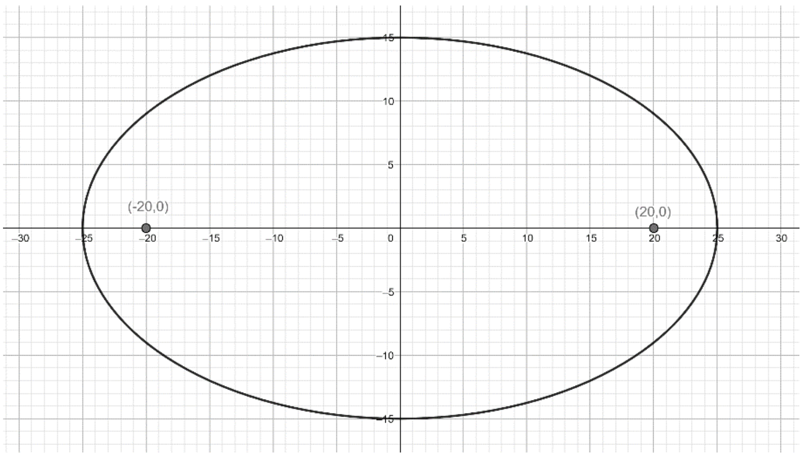

Figure 54. Elliptical garden with fountains at the foci

. . .

An underwater tunnel is to have a cross-section in the form of a half ellipse. The tunnel is planned to support 4 lanes of traffic (2 lines in each direction). Each lane is 3 meters wide. The tunnel is to be 16 meters wide at the base. There is a requirement that the tunnel support vehicles with a maximum height of 3.3 meters. To be on the safe side, the engineers decide to design for a 3.5 meters maximum, with the knowledge that a 3.3 meter restriction will be listed on the road leading to the tunnel. What should be the height of the tunnel?

Answer:

We are given that the tunnel's base is 16 meters. This translates to $a = 8$ in the formula for an ellipse. Situate the ellipse so that its center is $(0,0)$, as shown in Figure 55. As noted in the problem statement, the engineers are assuming a maximum vehicle size (i.e., 3.5 meters) larger than the requirement displayed on the sign leading to the tunnel (i.e., 3.3 meters). The maximum vehicle height comes into play for vehicles in the right lane as shown in the figure. This gives us two points on the ellipse, i.e., $(-6,\frac{7}{2})$ and $(6,\frac{7}{2})$.

We need to determine b in the equation

$$\frac{x^2}{64}+\frac{y^2}{b^2}=1$$

Plugging $(6,\frac{7}{2})$ into the above equation, we get

$$\frac{36}{64}+\frac{49}{4b^2}=1$$

Solving for b^2, we get

$$b^2=\frac{49}{4}\cdot\frac{64}{28}=28$$

which implies that $b=\sqrt{28}\cong 5.2915$. Thus, the height of the tunnel should be about 5.2915 meters.

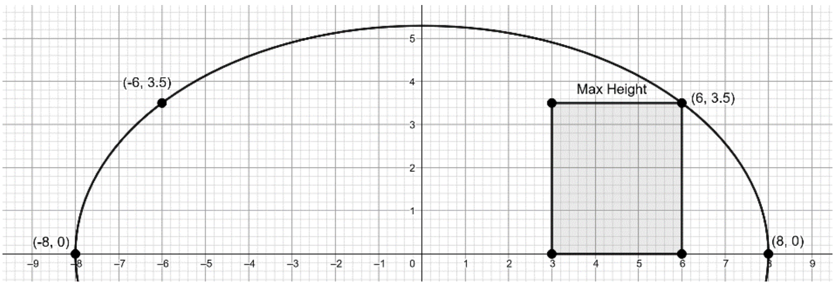

Figure 55. Elliptical underwater tunnel

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A vase artist wants to order hyperbola-shaped porcelain vases from a manufacturer. The artist will hand paint each of the vases. The vases are to be 12 inches tall, 4 inches at the narrowest (in the middle) and 5 inches at the top and base. Convert the requirements into an equation for an ellipse.

Answer:

If we place the hyperbola so that the center is at $(0,0)$, we have the situation shown in Figure 56. The requirements for the vase tell us that $a=2$. We need to solve for b to fully determine the equation for the hyperbola. We can use one of the 4 points (at the top and bottom of the vase) to solve for b.

We have

$$\frac{x^2}{4}-\frac{y^2}{b^2}=1$$

Plugging $(\frac{5}{2},6)$ into the above equation, we get

$$\frac{1}{4}\cdot\frac{25}{4}-\frac{36}{b^2}=1$$

$$\frac{36}{b^2}=\frac{25}{16}-1=\frac{9}{16}$$

$$b^2=36\left(\frac{16}{9}\right)=64$$

So, $b=8$ and the equation for the hyperbola is

$$\frac{x^2}{4}-\frac{y^2}{64}=1$$

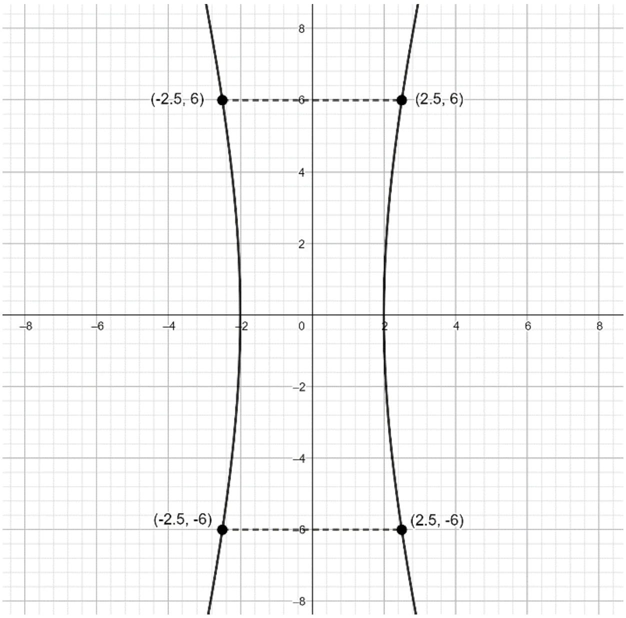


Figure 56. Hyperbolic vase

10 Functions

10.1 Definition

We briefly introduced functions in Section 8.3. Recall that a **function** maps an input to a **single** output. Some examples are

- $f(x) = \frac{2}{3}x - 7$ (a straight line)
- $g(x) = 5x^2 + 4$ (a parabola)
- $h(x) = 2x^3 - 7x^2 + 4x + 3$ (cubic polynomial)

The key word in the definition is “single”. The equation $x^2 + y^2 = 1$ is not a function since each value of x is mapped to two values of y. For example, if $x = 0$, then y is either -1 or 1.

. . .

It is possible to apply functions in series. For example (using the functions defined above), we can first apply function f to x and then apply function g to the result. This is written as $g(f(x))$ and is computed as follows:

$$g\big(f(x)\big) = g\left(\frac{2}{3}x - 7\right) = 5(\frac{2}{3}x - 7)^2 + 4 = \frac{20}{9}x^2 - \frac{140}{3}x + 249$$

The application of a function to the result of another function is known as **composition of functions**.

If $f(x) = x + 5$ and $g(x) = x^2$, what is $g(f(x))$ and $f(g(x))$?

. . .

If $g\big(f(x)\big) = x$, then g is said to be the **inverse function** of f, and is written as f^{-1}. In general, if f^{-1} is the inverse of f, then $f\big(f^{-1}(x)\big) = f^{-1}(f(x)) = x$

> **Caution**: this notation is standard but nevertheless confusing since f^{-1} is not $\frac{1}{f(x)}$. If you need to write $f(x)$ to the power a, it is safer (in terms of understanding to your intended audience) to write $[f(x)]^a$ rather than $f^a(x)$.

To find the inverse of the function $y = f(x)$, we solve for x in terms of y, and then exchange x and y. For example, take the function $y = f(x) = 3x$. Solving for x in terms of y, we get $x = y/3$. Next, we exchange x and y to get the inverse, i.e., $y = f^{-1}(x) = x/3$. We can check our answer as follows:

$$f\big(f^{-1}(x)\big) = f\left(\frac{x}{3}\right) = 3\left(\frac{x}{3}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(3x) = \frac{3x}{3} = x$$

As a more complex example, let's find the inverse function of $y = f(x) = 5x^2 + 4$. First, we solve for x in terms of y as follows:

$$y = 5x^2 + 4$$

$$5x^2 = y - 4$$

$$x^2 = \frac{y}{5} - \frac{4}{5}$$

$$x = \pm\sqrt{\frac{y}{5} - \frac{4}{5}}$$

To obtain the inverse of $f(x)$, we switch the variables in the above to get

$$y = f^{-1}(x) = \pm\sqrt{\frac{x}{5} - \frac{4}{5}}$$

$f(x)$ and the two parts of its inverse are shown in Figure 57. In the figure, eq1 is $y = \sqrt{\frac{x}{5} - \frac{4}{5}}$ and eq2 is $y = -\sqrt{\frac{x}{5} - \frac{4}{5}}$ (dashed-line curve). For this example and in general, the inverse of a function is the reflection of the original function about the line $y = x$ (dotted line in the figure).

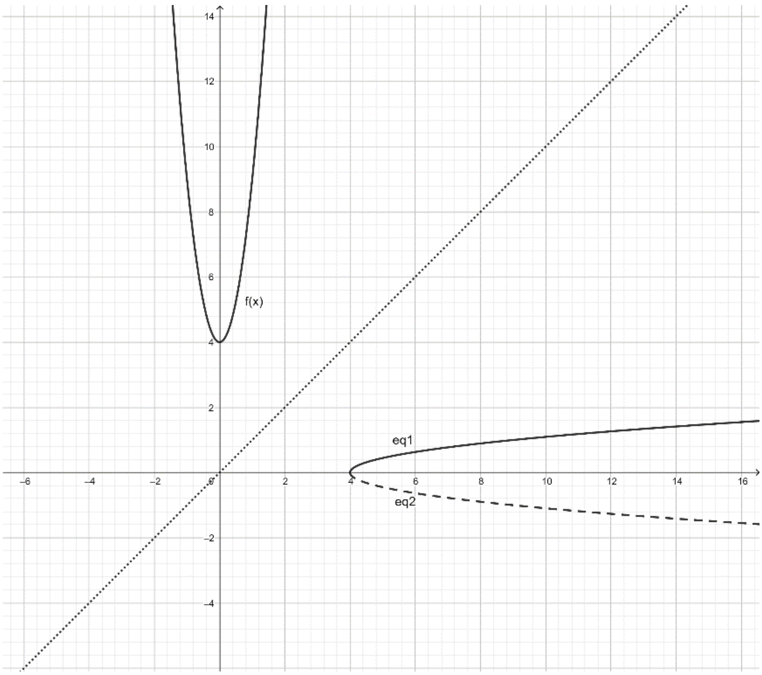


Figure 57. Inverse function example

We can verify that f^{-1} is the inverse of f as follows:

$$f\big(f^{-1}(x)\big) = 5\left(\pm\sqrt{\frac{x}{5}-\frac{4}{5}}\right)^2 + 4 = 5\left(\frac{x}{5}-\frac{4}{5}\right) + 4 = x - 4 + 4 = x$$

In the above, it does not matter if we use the positive or negative part of the inverse. The result is the same in both cases.

Going in the other direction, we need to be careful as to what part of the inverse to use. If $x \geq 0$, we must use the positive part of the inverse:

$$f^{-1}(f(x)) = \sqrt{\frac{5x^2+4}{5}-\frac{4}{5}} = \sqrt{x^2} = |x|$$

If $x < 0$, we must use the negative part of the inverse:

$$f^{-1}(f(x)) = -\sqrt{\frac{5x^2+4}{5}-\frac{4}{5}} = -\sqrt{x^2} = -|x|$$

Note that $|x|$ is the absolute value of x and is always a positive number. The absolute value function is covered in more detail in the next subsection.

. . .

The conic sections ellipse and hyperbola (with a few exceptions) are not functions. the reason is that for a single value of x, one gets two values of y. Figure 58 shows the circle with equation

$$x^2 + y^2 = 4$$

and the hyperbola with equation

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

Neither equation is a function. For example, regarding the circle, $x = 1$ maps to $\sqrt{3}$ and $-\sqrt{3}$; thus, violating the definition of a function. Regarding the hyperbola, $x = 5$ maps to 3.75 and -3.75. Again, violating the definition of a function.

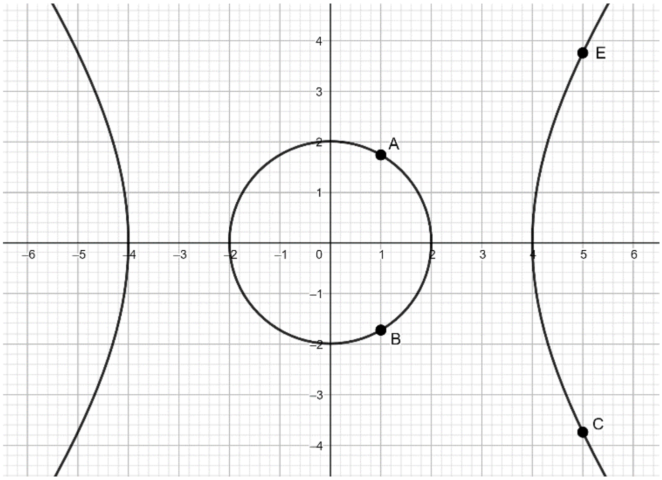


Figure 58. Example of circle and hyperbola not being functions

A parabola with a directrix other than a horizontal line is also not a function. For example, see Figure 24.

It is possible, however, to use part of an ellipse or hyperbola as a function. For example, take the hyperbola from the example above and solve for y to get

$$y = \pm 5\sqrt{\frac{x^2}{16} - 1}$$

If we select the positive sign, we get the top half of the hyperbola, i.e., $y = f(x) = 5\sqrt{\frac{x^2}{16} - 1}$. The negative sign yields the bottom half of the hyperbola. In both cases, we have a function.

. . .

In the remainder of this section, we introduce some of the more common functions that appear throughout mathematics and science.

10.2 Absolute Value Function

The absolute value function takes a real number (positive or negative) and returns a positive real number of the same magnitude. The absolute value function is defined as

$$f(x) = \sqrt{x^2}$$

The input number is first squared (yielding a positive number in all cases) and then we take the square to get the original magnitude. For example,

$$f(-5) = \sqrt{(-5)^2} = \sqrt{25} = 5$$

Alternately, the absolute can be defined as follows

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The graph is quite simple and shown in Figure 59.

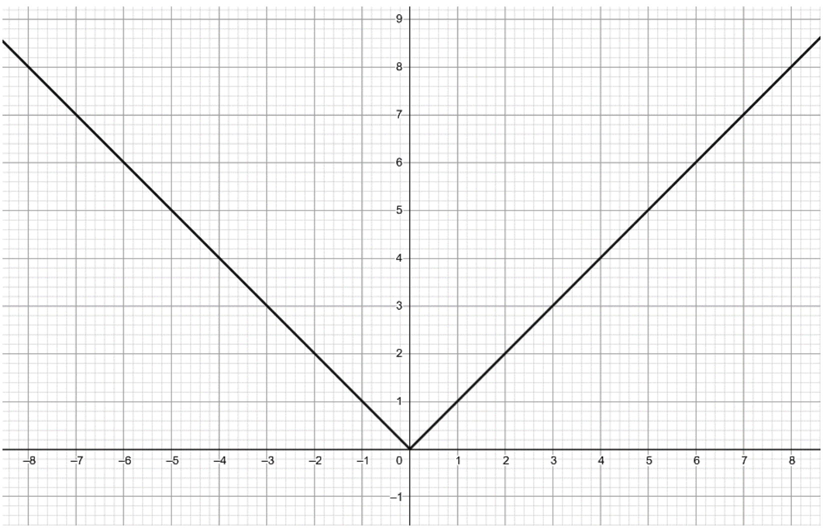


Figure 59. Absolute value function

10.3 Exponential and Logarithmic Functions

10.3.1 Concepts

An exponential function is simply a positive number raised to the power x. The general format for an exponential function is $h(x) = ab^x$ where a is a constant and b is a positive real number.

For example, $f(x) = 2^x$ is an exponential function whose graph is shown in Figure 60, along with 2 example points. The graph approaches zero as x takes on larger negative values, e.g., $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ is almost zero.

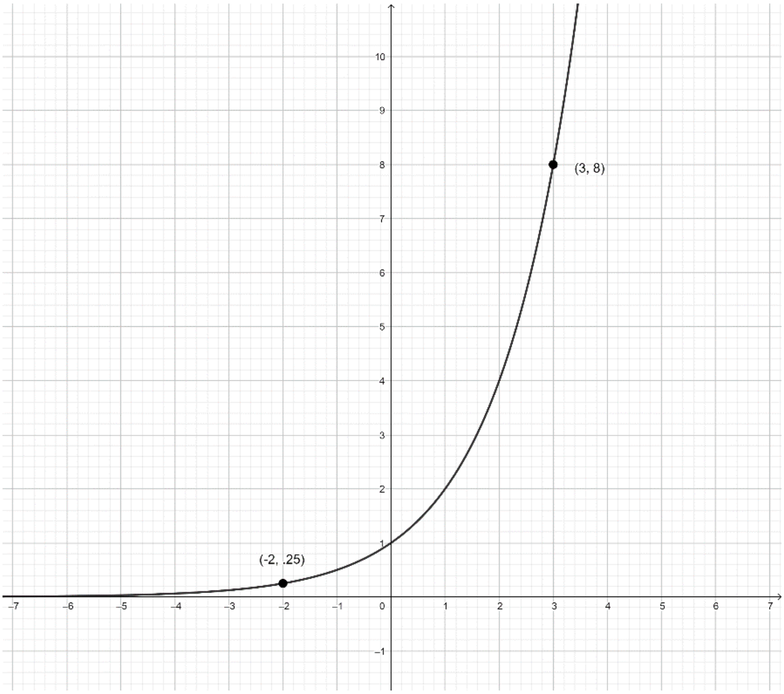


Figure 60. Exponential function example

Try using GeoGebra or Desmos to graph $g(x) = (\frac{1}{3})^x$ and $k(x) = \frac{1}{8}(2^x)$.

Also, think about why exponential functions are not defined for $x < 0$. ***Hint***: Try to graph $f(x) = (-2)^x$.

The rules below apply concerning the combination of exponents. In the following rules, a and b are constants, $a > 0$, and x and y are variables.

- $a^x \cdot a^y = a^{x+y}$ (product rule for exponents)
- $\frac{a^x}{a^y} = a^{x-y}$ (quotient rule for exponents)
- $(a^x)^y = a^{xy}$ (power rule for exponents)
- $(ab)^x = a^x b^x$

For example,

- $2^x \cdot 2^y = 2^{x+y}$
- $(3^x \cdot 3^y)/3^z = 3^{x+y}/3^z = 3^{x+y-z}$
- $(5^x \cdot 5^y)^z = (5^{x+y})^z = 5^{z(x+y)}$
- $(10 \cdot 11)^x = 10^x \cdot 11^x$

Simplify the following expressions:

- $2^x \cdot 2^y \cdot 2^z$
- $7^z/(7^x \cdot 7^y)$
- $((5^x)^y)^z$ **Hint**: do the inner operation first and then raise to the z power.

. . .

The inverse of the function a^x (where a is a real number greater than 0) is called $\log_a x$. (“Log” is the shorthand term “logarithm”.)

By definition,

$$\log_a(a^x) = x, \text{ and } a^{\log_a x} = x$$

Further, if $\log_a x = y$ then $a^y = x$, and conversely. This is under the conditions $x > 0, a > 0$ and $a \neq 1$. The constant a is called the **base of the logarithm**. The problem with $a = 1$ is that $1^y = 1 = x$ is not a function.

Some examples

- The equation $2^3 = 8$ is equivalent to $\log_2 8 = 3$.
- The equation $\log_3 81 = 4$ is equivalent to $3^4 = 81$.

The graph of $y = 3^x$ and its inverse $y = \log_3 x$ are shown in Figure 61. As with all pairs of inverse functions, one is the reflection of the other with respect to the line $y = x$.

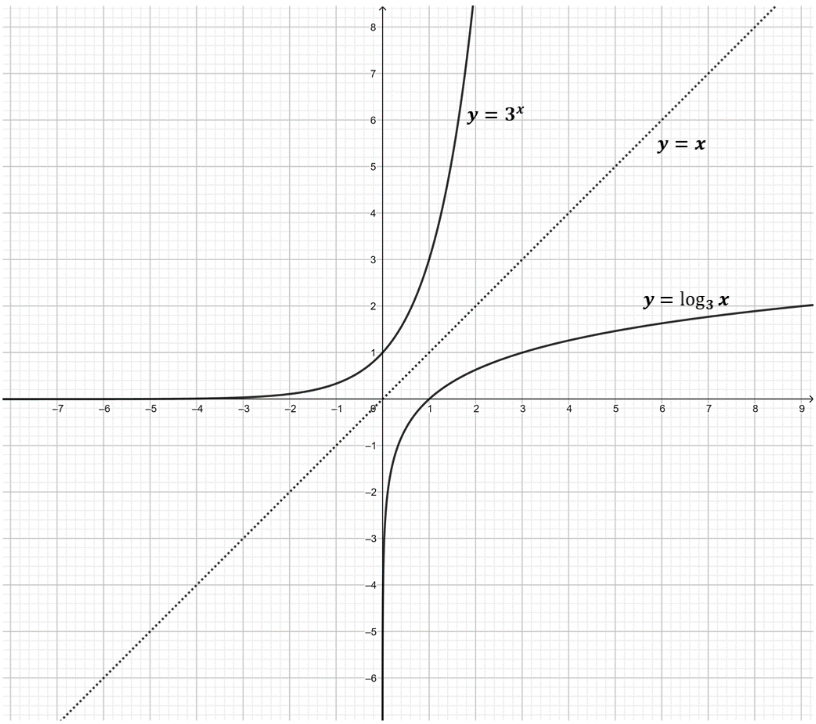


Figure 61. Exponential and Log function graphs

For a fractional base, the log function approaches infinity as x approaches 0 and approaches negative infinite as x approaches infinite. Figure 62 depicts the graph of $y = \log_{\frac{2}{3}} x$ (i.e., log base $\frac{2}{3}$) and its inverse $y = \left(\frac{2}{3}\right)^x$.

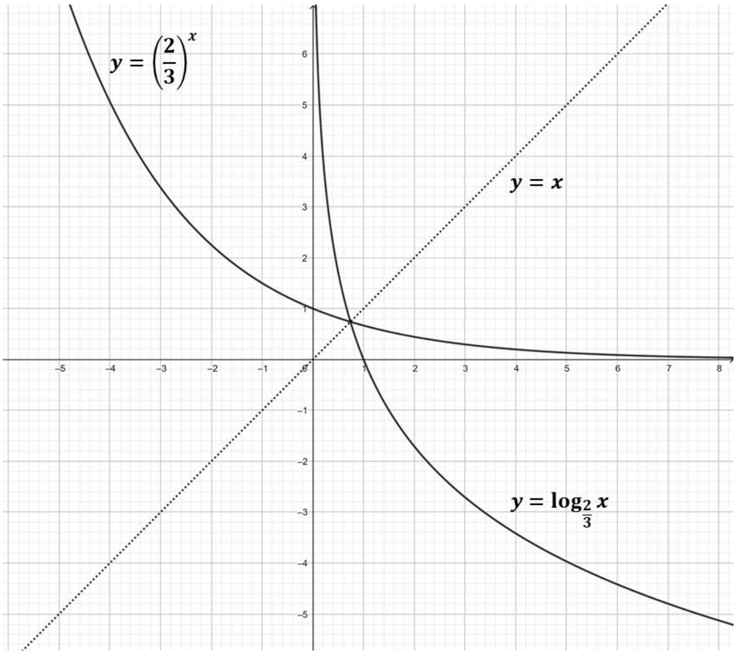


Figure 62. Exponential and Log with fractional base

The rules and identities for combining logs are as follows:

- $\log_a(xy) = \log_a x + \log_a y$ (product rule for logs)
- $\log_a(\frac{x}{y}) = \log_a x - \log_a y$ (quotient rule for logs)
- $\log_a(x^b) = b \log_a x$ (power rule for logs)
- $\log_a(1) = \log_a(x^0) = 0 \log_a(x) = 0$
- $\log_a(a) = 1$ since $a^1 = a$

Some examples

- $\log_3(3^{10}) = 10 \log_3(3) = 10$
- $\log_5\left(\frac{1}{125}\right) = \log_5\left(\frac{1}{5^3}\right) = \log_5(5^{-3}) = -3 \log_5(5) = -3$
- $\log_3(\sqrt{81}) = \log_3\left(81^{\frac{1}{2}}\right) = \frac{1}{2}\log_3(81) = \frac{1}{2}\log_3(3^4) = \frac{1}{2} \cdot 4 \log_3(3) = 2$
- Alternately, we could have done the above exercise as follows: $\log_3(\sqrt{81}) = \log_3(9) = \log_3(3^2) = 2 \log_3(3) = 2.$
- $\log_a((x+4)(x+7)) = \log_a(x+4) + \log_a(x+7)$ for any $a > 0$

The above examples were intentionally constructed so that we could arrive at a solution by hand. Usually that is not the case, and one needs to use a calculator to compute the log of a number.

A few exercises to try

- Evaluate $\log_{10}(1000)$
- Evaluate $\log_7(1/49)$
- Evaluate $\log_2(512)$

...

In some cases, it is required to convert from one log base to another. If you have $\log_a x$ and want to determine x with respect to the logarithm with base b, use the following formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

For example,

$$\log_3 6 = \frac{\log_{10} 6}{\log_{10} 3} \cong \frac{0.77815125}{.4771212547} \cong 1.63092975$$

Since some calculators only do logarithms base 10, the above type of calculation can come in handy.

10.3.2 Word Problems and Examples

Julius invests $1000 in an interest bearing bank account. Interest is paid on a monthly basis and reinvested (thus also earning interest). The monthly interest rate is .5% or .005. After 3 months, how much does Julius have in his account?

Answer:

The following table tracks the value of Julius' investment

Time	**Value of Investment**
0	1000
1 month	$1000 + (.005)(1000) = 1005$
2 months	$1005 + (.005)(1005) = 1005 + 5.025 = 1010.25$
3 months	$1010.25 + (.005)(1010.25) = 1010.25 + 5.05125$ $= 1015.30125$

The reader may be thinking that this does not look like an exponential or logarithm problem, but what if we changed the problem to say P dollars invested over n compounding periods at an interest rate of i period?

For the more general problem, the exponent function reveals itself. Table 18 shows the derivation of the compound interest formula. A pattern emerges in the first three step and then we generalize to n steps.

Table 18. Derivation of Compound Interest Formula

Initial principal	P
Principal after first compounding interval	$P + iP = P(1 + i)$
Principal after second compounding interval	$P(1 + i) + \mathrm{P}(1 + i)(i)$ $= P(1 + i)[1 + i] = P(1 + i)^2$
...	...
Principal after n compounding intervals	$\boldsymbol{P(1 + i)^n}$

Now we can answer more difficult problems that are hard to compute manually (i.e., step by step). For example, how much money will Julius have in 10 years given the same starting principal and interest rate per period?

We have $P = 1000$, $i = .005$ and $n = 10(12) = 120$. Using a spreadsheet for the calculation, we determine that Julius will have about \$1819.40 in ten years.

. . .

Continuing with the previous problem, let's say that Julius wants to reach \$2000 in savings. How long will that take, with the same assumptions concerning the interest per compounding period and initial principal?

Answer:

We want to know the value of n for which $1000(1.005)^n = 2000$.

First divide both sides of the equation by 1000 to get $(1.005)^n = 2$.

Next, take the log base 10 on both sides of the equation. (There is nothing special about the log base here. We could have used any base and would still get the same answer).

$$\log_{10}(1.005)^n = \log_{10}(2)$$

Using the power rule for logs, we get

$$n \log_{10}(1.005) = \log_{10}(2)$$

$$n = \frac{\log_{10}(2)}{\log_{10}(1.005)} \cong 138.97$$

So, if Julius saves his initial amount for 139 periods, he will have more than \$2000.

We can check by plugging $n = 139$, $P = 1000$ and $i = .005$ into the compound interest formula:

$$1000(1 + .005)^{139} \cong 2000.24$$

In general, if you want to have Q dollars, given initial principal P and interest rate i per compounding period, and want to know how long it will take, you need to solve the following equation for n

$$P(1 + i)^n = Q$$

$$(1 + i)^n = \frac{Q}{P}$$

taking the log on both sides

$$\log_{10}(1 + i)^n = \log_{10}\left(\frac{Q}{P}\right)$$

using the power rule for logs

$$n \log_{10}(1 + i) = \log_{10}\left(\frac{Q}{P}\right)$$

$$n = \frac{\log_{10}(1 + i)}{\log_{10}\left(\frac{Q}{P}\right)}$$

. . .

In yet another variation of the above problems, let's say that Julius wanted to know how much to invest in the present so that in 20 years he would have \$10,000, given an interest rate of .005 per month (with compounding).

Answer:

We are given $Q = 10{,}000,\ n = 120$, and $i = .005$. We need to solve for P in the equation

$$10000 = P(1.005)^{120}$$

Divide both side of the equation by $(1.005)^n$ to get

$$P = (10000)(1.005)^{-120} \cong 5496.33$$

. . .

What if Julius' investment is compounded more frequently than once per month? If compounding is done m times per month, then the interest rate for each compounding period is $\frac{i}{m}$ where i is the interest rate per month. In this case, the formula for the value of the investment in n months is

$$P\left(1 + \frac{i}{m}\right)^{mn}$$

As m get larger (approaches infinity), the value of $\left(1+\frac{1}{m}\right)^m$ approaches what is known as Euler's number e. The value of e is about 2.71828. Euler's number appears extensively in many areas of mathematics. Further, it can also be shown, that as m approaches infinity, $\left(1+\frac{x}{m}\right)^m$ approach e^x.

For the problem at hand, as m approaches infinity, the value of the investment is given by

$$Pe^{in}$$

This is known as continuous compounding and it is a special case of what is called exponential growth.

Under continuous compound, Julius's \$1000 investment (with a monthly interest rate of .005) will be worth the following amount in 10 years (120 months):

$$1000e^{(.005)(120)} = 1822.12$$

which is slightly more than the $\$1819.40$ that we calculated earlier in the case of monthly compounding.

As we shall see in some of the following problems, exponential growth is used in fields other than finance.

For additional coverage of basic financial mathematics, see the book "Financial Mathematics with Python: A Concise Guide" [19].

. . .

A virologist is studying a newly-created virus. At time $t = 0$ hours, she puts one thousand virus instances into a suitable growth environment. Six hours later, she measures (actually estimates) 45000 virus instances. Assuming exponential growth, what is the growth constant k for the bacteria?

Answer:

The assumption of an exponential growth, translates into the following equation

$$Q = Pe^{kt}$$

where P is the starting quantity (in this case, number of virus instances), t is times, Q is the quantity at time t, and k is the growth factor (analogous to the interest rate in continuous compounding).

We have that $P = 1000$, $Q = 45000$ and $t = 6$. Plugging into the exponential growth equation, we get

$$45000 = 1000e^{6k}$$

which implies

$$45 = e^{6k}$$

Taking the log base e on both sides (this is known as the natural log and represented as $\ln x$):

$$\ln 45 = \ln e^{6k} = 6k\,(\ln e) = 6k$$

$$k = \frac{\ln 45}{6} \cong .6344$$

. . .

A known type of coronavirus, given a favorable environment, doubles in population every 6.5 hours. Given that there were approximately 1000 virus instances to start with, how many virus instances will there be in 2 days?

Answer:

We are given that the number of virus instances doubles in 6.5 hours (regardless of the starting point) which can be translated into the equation

$$2P = Pe^{6.5k}$$

Dividing both sides of the above equation by P and then taking the natural log on both sides, gives us

$$6.5k = \ln 2$$

and so $k = \frac{\ln 2}{6.5}$.

For the problem at hand, we have $P = 1000$ and $t = 48$ hours.

The number of virus instances in 48 hours should be

$$Q = 1000e^{48\left(\frac{\ln 2}{6.5}\right)} \cong 167{,}106$$

. . .

Assuming continuous compounding, how long will it take an investment to double at a nominal 5% interest?

Answer:

To model the problem, we use the continuous compounding formula with $i = .05$ and solve for n in the following equation

$$2P = Pe^{.05n}$$

$$2 = e^{.05n}$$

$$\ln 2 = .05n$$

$$n \cong 13.86 \text{ years.}$$

. . .

What nominal yearly interest rate is needed to double your money in exactly 10 years, assuming continuous compounding?

Hint: This is similar to the previous problem, but you need to solve for i rather than n.

. . .

The value of a new car depreciates as time passes. Assume the value of the car decreases according to an exponential decay model. Further, we are given that the value of the car is $20,000 at the end of 5 years, with the rate of depreciation being 9% per year. What was the value of the car when it was new?

Answer:

We have the current value of the car $Q = 20{,}000$, $n = 5$ and $k = -.09$. Putting these values into the exponential growth (decay in this case) model, we have

$$20000 = Pe^{(-.09)(5)} = Pe^{-.45} \cong .6376P$$

$$P \cong \$31{,}366.24$$

. . .

A **decibel** (db) is a measure of the loudness or power of a sound or signal. Decibels are defined in terms of the following logarithmic formula:

$$db = 10\log_{10}(\frac{P}{P_0})$$

where P_0 is a reference power level (sound level that can barely be perceived) and P is the measured power-level of a given sound.

For example, if a sound is 30 times the reference power level, i.e., $P = 30P_0$. This converts to

$$10\log_{10}(\tfrac{30P_0}{P_0}) = 10\log_{10}(30) \cong 14.77 \text{ decibels}$$

A jet engine generates 140 db of noise at take-off. What is the power level of the jet engine compared to the base level?

Using the decibel formula, we have

$$140 = 10\log_{10}(\frac{xP_0}{P_0})$$

which simplifies to

$$14 = \log_{10}(x)$$

Converting the above to exponents, we have that

$$x = 10^{14}$$

For additional examples of noise levels, see that chart at Noise Help [20].

. . .

The Richter scale for measuring and classifying earthquakes is defined in an analogous manner to decibels, using logs base 10. From the Wikipedia article on the Richter scale [21]:

> The Richter magnitude of an earthquake is determined from the logarithm of the amplitude of waves recorded by seismographs (adjustments are included to compensate for the variation in the distance between the various seismographs and the epicenter of the earthquake). The original formula is:
>
> $$M_L = \log_{10}\left(\frac{A}{A_0(\delta)}\right)$$
>
> where A is the maximum excursion of the Wood–Anderson seismograph, the empirical function $A_0(\delta)$ depends only on the epicentral distance of the station, δ. In practice, readings from all observing stations are averaged after adjustment with station-specific corrections to obtain the M_L value.

In simpler terms, A is a reading taken by a seismograph and A_0 is a base level for comparison (similar in concept to P_0 in the definition of "decibel").

Because log base 10 is used in the scale, an earthquake with a reading of 7.0 has 10 times the amplitude of a 6.0 earthquake and 100 times the amplitude of a 5.0 earthquake.

10.4 Trigonometric Functions

10.4.1 Concepts

10.4.1.1 Definitions

Definition of a **triangle** from Wikipedia: A triangle is a polygon with three edges and three vertices. It is one of the basic shapes in geometry.

Some common definitions of **trigonometry**:

- from *The American Heritage® Dictionary of the English Language, 5th Edition*: Trigonometry: *noun* The branch of mathematics that deals with the relationships between the sides and the angles of triangles and the calculations based on them, particularly the trigonometric functions.
- from *Wikipedia*: Trigonometry (from Greek trigōnon, "triangle" and metron, "measure") is a branch of mathematics that studies relationships between side lengths and angles of triangles.

The three angles of a triangle always add to 180 degrees. (In what follows, we will use the shorthand notation for degrees, i.e., $180°$ in lieu of "180 degrees.")

If one of the angles in a triangle is $90°$, the triangle is known as a **right triangle**. The triangle on the left of Figure 63 is an example of a right triangle. The side opposite the right angle in a right triangle is known as the **hypotenuse**.

If all the sides are equal in length, then all the angles are $60°$, and conversely. Such a triangle is known as an **equilateral triangle**. The triangle in the middle of Figure 63 is an example of an equilateral triangle.

If two of the sides of a triangle equal, then so are two of the sides, and conversely. Such a triangle is known as an **isosceles triangle**. The triangle on the right of Figure 63 is an example of an isosceles triangle.

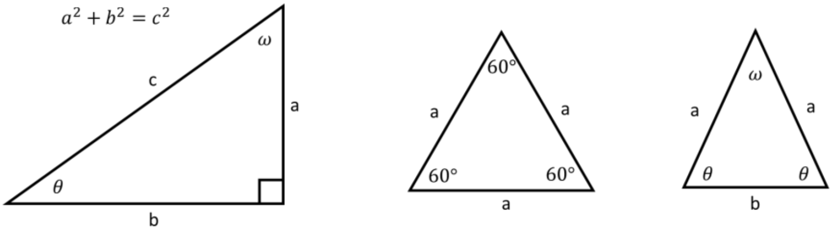


Figure 63. Several types of triangles

If all the angles of a triangle are less than $90°$, the triangle is said to be **acute** (e.g., the triangles in the middle and right of Figure 63 are acute).

If one of the angles in a triangle is more than $90°$, the triangle is said to be **obtuse**. See the example of an obtuse triangle in Figure 64.

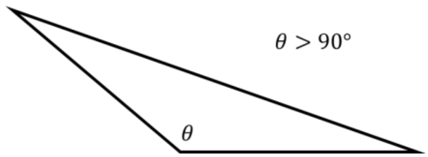


Figure 64. Obtuse triangle

Two triangles are said to be **congruent** if they are exactly the same size and shape, i.e., all pairs of corresponding angles are equal, and all pairs of corresponding sides are of the same length. Two triangles are **similar** if they have the same shape (i.e., the same angles) but not necessarily the same size. Congruent triangles are similar but not necessarily the other way around.

Each of the following statements provide necessary and sufficient conditions for a pair of triangles to be congruent:

- Side-Side-Side (SSS): If the three side (lengths) of one triangle are equal to the three side (lengths) of another triangle, the two triangles are congruent.
- Side-Angle-Side (SAS): If the two sides and the included angle of one triangle (i.e., between the two sides) are equal to the two sides and the included angle of another triangle, the two triangles are congruent.
- Angle-Side-Angle (ASA): If the two angles and the included side (i.e., between the two angles) of one triangle are equal to the two angles and the included side of another triangle, the two triangles are congruent.
- Angle-Angle-Side (AAS): If two angles and the non-included side of one triangle are congruent to two angles and the non-included side of another triangle, the two triangles are congruent.
- Hypotenuse-Leg congruence: If the hypotenuse and a side (other than the hypotenuse) of one right triangle are equal to the hypotenuse and a side (other than the hypotenuse) of another right triangle, the two triangles are congruent.

Not all combinations of three elements (angles or sides) guarantee congruent triangles. For example, angle-angle-angle only ensures similar but not congruent.

In what follows, we give a brief introduction to trigonometry, with a focus on right triangles.

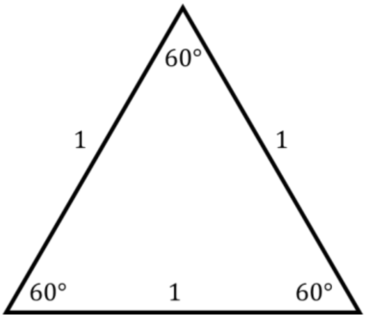
10.4.1.2 Some Preliminary Results

The trigonometric functions (to be defined in the next section) depend on the ratios of side lengths in a triangle. For the most part, it is not easy to manually determine the ratios of sides in a triangle (given all three angles) but there are a few exceptions (some of which we cover below).

Figure 65 depicts an equilateral triangle on the left. On the right of the figure, the triangle is divided into two equal triangles, with angles 30-60-90. The base (bottom) of each 30-60-90 triangle in the figure is of length $\frac{1}{2}$. The common side (call it x) can be calculate using the Pythagorean theorem:

$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

which implies $x^2 = \frac{3}{4}$ or $x = \frac{\sqrt{3}}{2}$. If we choose a as the side length of the equilateral triangle, then the sides of the 30-60-90 triangles would be $a, \frac{a}{2}$ and $\frac{\sqrt{3}}{2}a$. The ratios between pairs of sides are the same, regardless of the value of a which is why we took the easy path and choose $a = 1$.



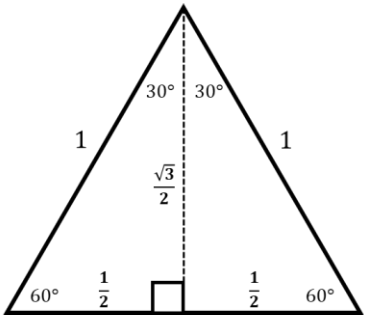


Figure 65. 30-60-90 Triangle

Figure 66 depicts a 45-90-45 triangle. Using the Pythagorean theorem, the hypotenuse is determined to be $\sqrt{2}$.

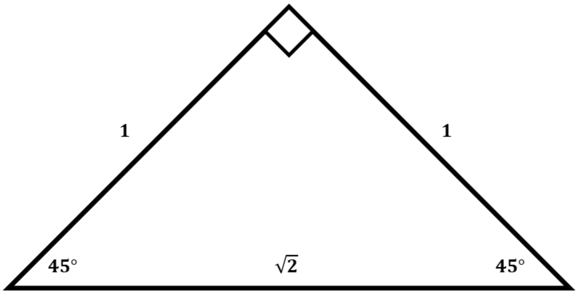


Figure 66. 45-90-45 Triangle

10.4.1.3 Trigonometric Functions

In this subsection, we cover some of the basic trigonometric functions. These functions are based on the ratio of side lengths for a given angle within a right triangle.

Figure 67 depicts the terminology used to describe a right triangle. The sides are labeled relative to the angle θ. If we labeled the sides relative to the angle ω, the roles of adjacent and opposite would be reversed. In either case, the hypotenuse remains the same (it is the side opposite the right angle).

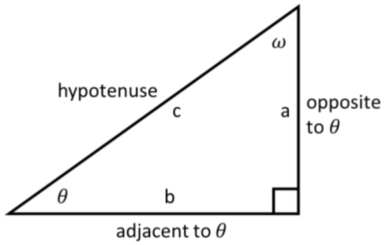


Figure 67. Terminology for a right triangle

The following trigonometric functions are defined for right triangles:

- The sine function of θ is defined as $\sin\theta = \frac{a}{c}$ (just to emphasis the point about the roles of the side being relative to the angle, we note that $\sin\omega = \frac{b}{c}$)
- The cosine function of θ is defined as $\cos\theta = \frac{b}{c}$
- The tangent function of θ is defined as $\tan\theta = \frac{a}{b}$. Alternately, we can define $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

As θ approaches 0, the opposite side gets smaller and $\sin\theta$ approaches 0. We have that $\sin 0° = 0$. As θ approaches 90 degrees, the length of the opposite side approaches the length of the hypotenuse, and $\sin\theta$ approaches 1 and so $\sin 90° = 1$. In between $0°$ and $90°$, $0 < \sin\theta < 1$. Based on the analysis in the previous section, we have that $\sin 30° = \frac{\sqrt{3}}{2}$, $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\sin 60° = \frac{1}{2}$.

Using similar arguments, $\cos 0° = 1$ and $\cos 90° = 0$. In between $0°$ and $90°$, $0 < \cos\theta < 1$. Based on the analysis in the previous section, we have that $\cos 30° = \frac{1}{2}$, $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\cos 60° = \frac{\sqrt{3}}{2}$.

The tangent function equals 0 when $\theta = 0$ and increases to 1 when $\theta = 45°$. For $\theta > 45°$, the tangent continues to increase and approaches infinite as θ approaches $90°$ since the denominator (the length of the adjacent side) approaches 0 as θ approaches 90°. Based on the analysis in the previous section, we have that $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, $\tan 45° = 1$ and $\tan 60° = \sqrt{3}$.

The above analysis is summarized in Table 19. A longer list of values can be found in the Wikipedia article entitled "List of trigonometric identities" [22].

Table 19. Summary of some trigonometric function values

	$0°$	$30°$	$45°$	$60°$	$90°$
sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	infinite

The trigonometric functions can also be defined in terms of the unit circle, as shown in Figure 68. Since the hypotenuse is of length 1, the $\sin\theta$ equals the length of the opposite side from θ and $\cos\theta$ equals the length of the adjacent side to θ. This also gives us the identity $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

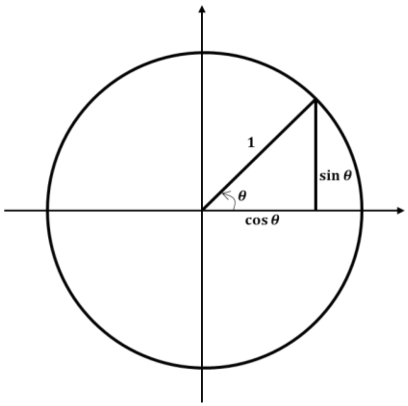


Figure 68. Defining trigonometric functions based on the unit circle

The advantage of this approach is that it allows us to define trigonometric functions for values of θ greater than $90°$. The diagram on the left of Figure 69 shows how the trigonometric functions are defined for values of θ between $90°$ and $180°$. For example, take $\theta = 135°$. We first draw the triangle as shown in the figure and then use the complement of θ (i.e., $45°$) to compute $\cos\theta$, i.e., $\cos\theta = -\cos 45° = -\frac{\sqrt{2}}{2}$. The same approaches would be used to compute $\cos\theta$, $\sin\theta$ or $\tan\theta$ of any angle θ between $90°$ and $180°$.

The diagram in the middle of Figure 69 shows how to define the trigonometric functions for values of θ between $180°$ and $270°$. Here both the opposite and adjacent sides have negative values. For example, take $\theta = 210°$. The angle in the triangle in the middle of the figure would then be $30°$. So, we have (for example) $\cos\theta = -\cos 30° = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -\sin 30° = -\frac{1}{2}$.

The diagram on the right of Figure 69 shows how to define the trigonometric functions for values of θ between $270°$ and $360°$. In this case, the adjacent side is positive and the opposite side is negative.

If we continue beyond $360°$, the pattern repeats indefinitely. We could also go in the clockwise direction for negative angles, which again allows for repeating pattern beyond $-360°$.

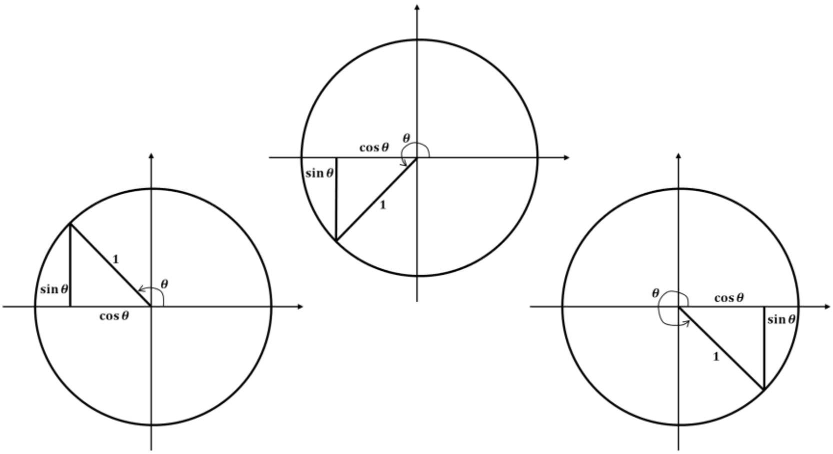


Figure 69. Trig functions in Quadrants II, III and IV

For a more detailed presentation of this particular topic, the following YouTube video entitled “Redefining the Trig Functions on the Unit Circle” [23] is recommended.

10.4.2 Radians

When discussing angles in calculus and higher mathematics, it is more common to talk about angles in terms of something called **radians** rather than degrees. Radians are defined in terms of the unit circle discussed in the previous section. From geometry, we know that the distance around a circle (the circumference) is equal to $2\pi r$ where r is the radius of the circle. For the unit circle, the circumference is 2π. A radian specifies the location of the end of a radius on a unit circle in terms of the distance along the circumference starting from the point $(0,0)$. For example, if the end of the radius line is at point $(0,1)$, then it is a distance $\pi/2$ from $(0,0)$ if you travel along the circumference of the unit circle. This is equivalent to $90°$.

Figure 70 shows some example equivalences between degrees and radians.

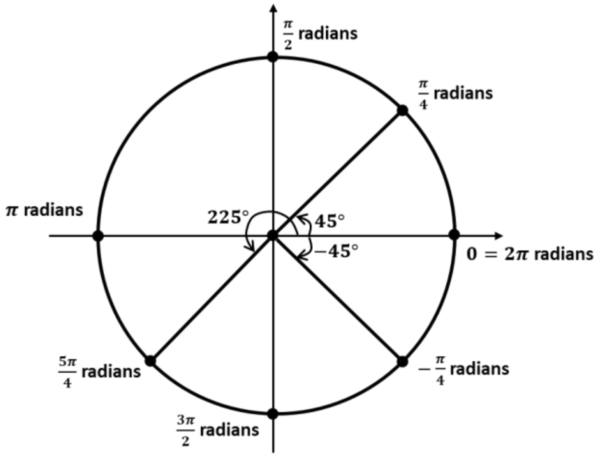


Figure 70. Radians and degrees

What are the equivalents in radians for the following angles in degrees: $120°, 150°, 240°, 300°$ and $-120°$?

The conversion formula for degrees to radians is

$$\text{angle in radians} = \text{angle in degrees} \cdot \frac{\pi}{180}$$

Going in the other direction, the conversion formula

$$\text{angle in degrees} = \text{angle in radians} \cdot \frac{180}{\pi}$$

10.4.3 Graphs

In the previous section, we computed a few values for the sine, cosine and tangent function. However, these functions are defined for all real values, with the exception of the tangent which has points of discontinuity.

Figure 71 shows the graphs of the sine (solid line) and cosine (dashed line). The horizontal axis represents the various angles (arguments to the sine and cosine functions).

The graphs extend indefinitely in both the positive and negative directions, with the same repeating pattern. Both functions repeat every 2π radians. So, $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$. For this reason, the sine and cosine are classified as **periodic functions**, with period 2π.

The sine and cosine functions are copies of each other but translated by $\pi/2$ radians. So, $\sin(x + \frac{\pi}{2}) = \cos x$.

We also have the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Examine the graphs of sine and cosine to convince yourself that these identities are true.

The points shown on the sine function are $A\left(\frac{\pi}{2}, 1\right)$, $B(\pi, 0)$, $C(\frac{3\pi}{2}, -1)$ and $D(2\pi, 0)$. Points $E(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ and $F(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2})$ are two of the infinite number of intersection points between the sine and cosine functions.

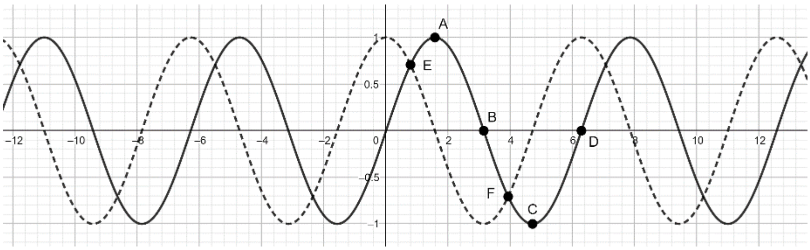


Figure 71. Graphs of sine and cosine functions

Figure 72 shows the tangent function (solid curved lines). Since the denominator of the tangent ratio is zero at ... $-\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, ... the function approaches plus or minus infinity at these angles. The dashed lines are approached by the tangent function as it goes off to infinity at the aforementioned angles.

As can be inferred from the graph, $\tan(x + \pi) = \tan x$ and thus, the tangent function is periodic with period π.

We also have that $\tan(-x) = \tan x$.

The point A on the graph is $(\frac{\pi}{4}, 1)$ and the point B is $(-\frac{\pi}{4}, -1)$.

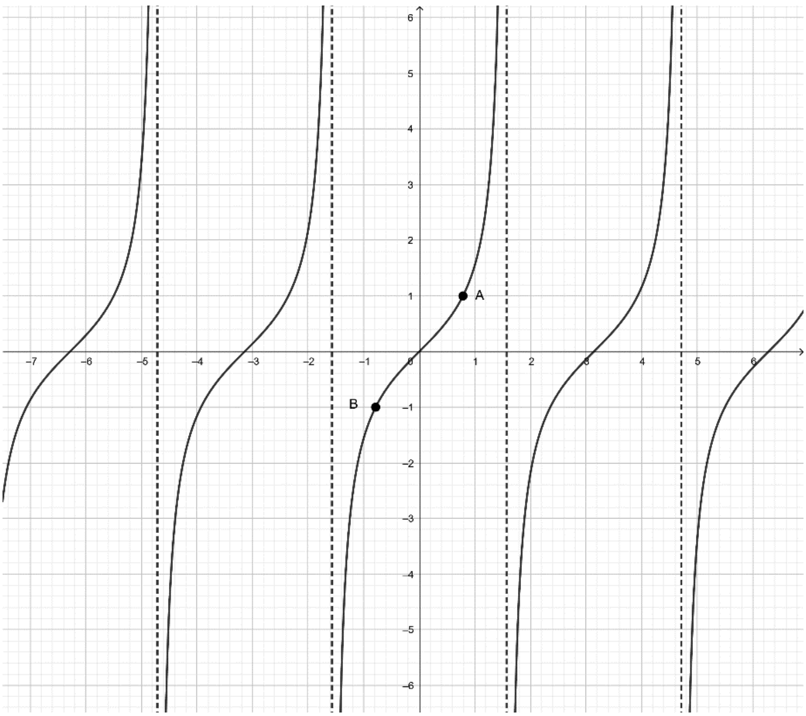


Figure 72. Tangent function

10.4.4 Exercises

10.4.4.1 Concepts and Identities

1. Given an example of two non-congruent triangles with the same three angles.

2. Given an example of two non-congruent triangles that have the same sequence of side, side and angle. (The answer is at the end of the exercises.)

3. What does one radian equal in terms of degrees?

4. The trig identity $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin\beta$ allows one to compute $\sin(\alpha+\beta)$ if the sine and cosine of α and β are known. Use this identity to determine $\sin(\frac{7\pi}{12})$. ***Hint***: Note that $\frac{7\pi}{12} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$ and we know the sine and cosine for $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

5. There is also a trig identity for the sine of the difference of two angles, i.e., $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin\beta$. Use this identity to compute the sine of $\frac{\pi}{12}$.

10.4.4.2 Graphing

For the following exercise that involve graphing make use of GeoGebra, Desmos or your own favorite graphing tool.

1. Graph the functions $\sin x$, $\sin(\frac{x}{3})$ and $3\sin x$ on the same diagram. In general, multiplying the argument of a function by a positive fraction (less than 1) squishes (contracts) the function, and multiplying by a positive number greater than 1 stretches the function. The amplitude (height) remains unchanged.
2. Modify the arguments of the cosine and tangent functions in a similar manner to Exercise 1.
3. The amplitude of a function can be increased by multiplying the function with a number greater than one, and decreased by multiplying the function with a positive fraction (less than one). For example, graph $\cos x$, $5 \cos x$ and $\frac{1}{2}\cos x$.
4. It is possible to modify the amplitude and stretch (or contract) the graph of a function, e.g., graph $5\cos(3x)$ and compare to $\cos x$.
5. Trigonometric functions can be translated. For example, try the graph $(y + 3) = \sin(x - 2)$. The function $\sin x$ is moved down three units and two units to the right.
6. What transformations does $y + 4 = 2\sin(3(x - 2))$ do to $y = \sin x$?
7. Graph $\frac{100}{x}\sin x$, $\frac{x^2 \cos x}{100}$ and $\frac{x^3 \sin x}{1000}$. If you are using Symbolab, you may need to zoom out to see the full effect.
8. Describe the intersection points of $\tan x$ and $\sin x$. ***Hint***: graph the two functions on the same diagram.
9. Describe the intersection points of $y = \sin x$ and $y = \sin(\frac{x}{2})$.
10. Graph the chirp function, i.e., $y = \sin(x^2)$.
11. Graph $y = (\log_{10} x)\sin(7x)$ and $y = \frac{\sin x}{\log_{10} x}$.
12. A few wild trig functions to graph: $\sin(\frac{10}{x})$, $x\sin(\frac{10}{x})$, $x^2\sin(\frac{10}{x})$ and $x^2 \sin\left(\frac{10}{x^2}\right)$. Try changing the 10 in the numerator to some other number.
13. The hyperbolic sine function is defined to be $\sinh x = \frac{(e^x - e^{-x})}{2}$. Graph $\sinh x$ and $\frac{1}{\sinh x}$.

Answer to Exercise 2 in Section 10.4.4.1

In Figure 73, triangle ABD and ABE have two sides of the same length and a common angle (i.e., α) but they are not congruent. Note that segments AD and AE are both radii of the same circle and thus of the same length.

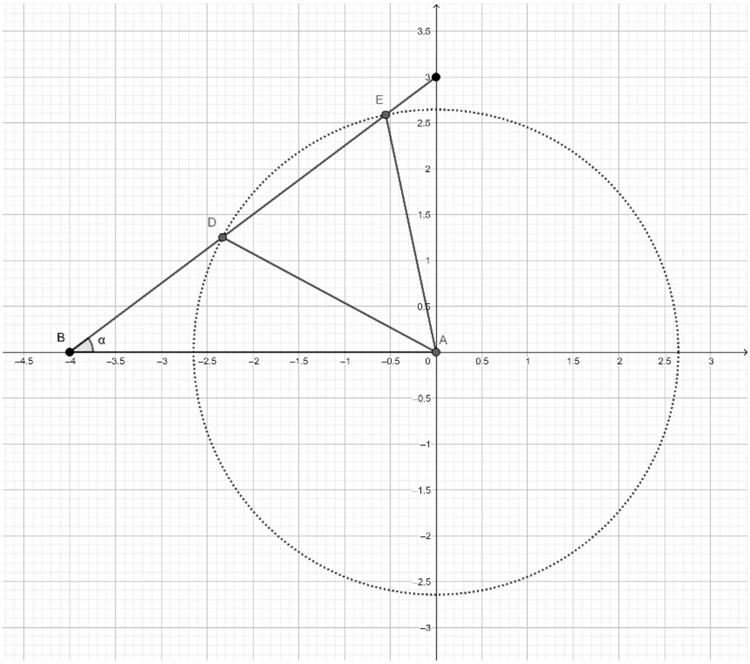


Figure 73. Side-Side-Angle counter-example

11 Radicals

11.1 Overview

In Section 4.8, we discussed square roots, cube roots and other roots of real numbers. Collectively, the various roots are known as **radicals** and in fact, $\sqrt{}$ is called the radical sign. Radicals can be applied to algebraic expressions as well as real numbers. For example, $\sqrt{x^2 - 3x - 4}$ is a valid expression. In general, we can have $\sqrt[n]{f(x)}$ which reads "the n^{th} root of the function $f(x)$". This is equivalent to the expression $f(x)^{1/n}$. Written either way, this refers to the expression that when multiplied times itself n times results in $f(x)$. For example, $\sqrt[3]{x^6} = x^2$ since x^2 multiplied times itself 3 times equals x^6. The number n is referred to as the **degree of the radical**.

Going back to our example, i.e., $\sqrt{x^2 - 3x - 4}$, and noting that $x^2 - 3x - 4 = (x + 1)(x - 4)$, we can see that the function inside the radical is negative (and thus undefined) when $-1 < x < 4$. This is even more apparent from the graph of $y = x^2 - 3x - 4$ (dotted line curve in Figure 74). The solid line curve in the figure is $y = \sqrt{x^2 - 3x - 4}$ which is undefined for $-1 < x < 4$.

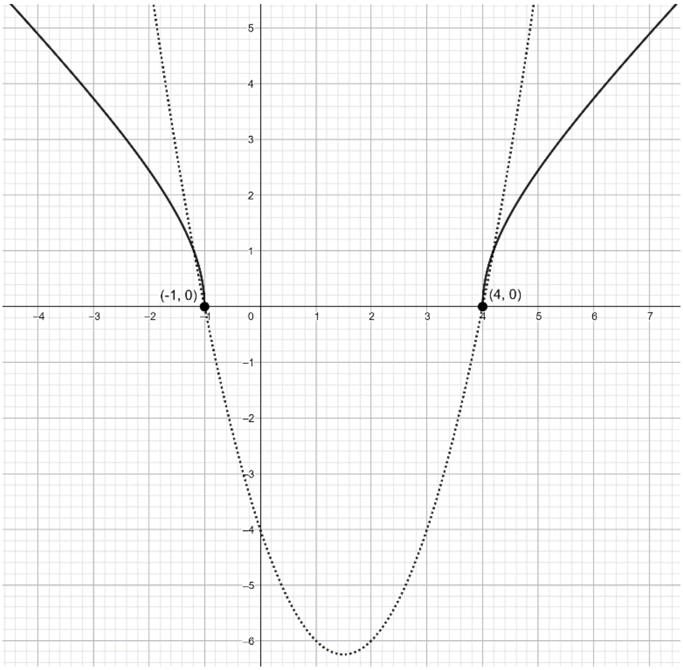


Figure 74. Function and its square root

In general, $\sqrt[n]{f(x)}$ is undefined when n is an even integer and $f(x) < 0$, but is defined when n is odd, regardless of whether $f(x)$ is positive or negative. For example, $g(x) = \sqrt{x-1}$ is undefined for $x < 1$. To see this, just pick any number less than 1 (say -2) and observe that $g(-2) = \sqrt{-3}$ is undefined since there is no real number that when multiplied times itself equals -3. On the other hand, $h(x) = \sqrt[3]{x-1}$ is defined for all values of x. For example, $h(0) = \sqrt[3]{-1} = -1$. Graphs of the two functions are shown in Figure 75. The solid line curve is $g(x) = \sqrt{x-1}$ and the dotted line curve is $h(x) = \sqrt[3]{x-1}$.

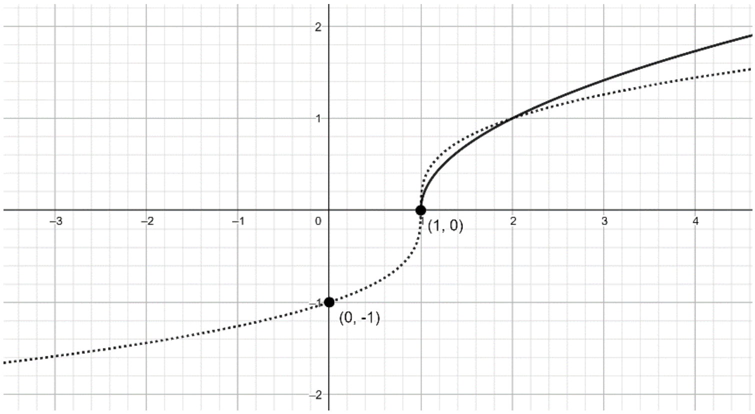


Figure 75. Square and cube root of $x - 1$

11.2 Operations

11.2.1 Addition and Subtraction

Radicals with exactly the same "contents" can be added or subtracted, for example

- $3\sqrt{x^2+2x+1} + 7\sqrt{x^2+2x+1} = 10\sqrt{x^2+2x+1}$
- $11\sqrt[5]{7x+9} - 4\sqrt[5]{7x+9} = 7\sqrt[5]{7x+9}$

However, $\sqrt{x^2+3x+1} + \sqrt{3x^3+5x} \neq \sqrt{3x^3+x^2+7x+1}$. To see this, evaluate both sides of inequality for $x = 1$. On the left, we have $\sqrt{5} + \sqrt{8} \cong 5.065$ and on the right, we have $\sqrt{13} \cong 3.61$.

11.2.2 Multiplication and Division

Two radicals of the same degree can be multiplied, i.e., $\sqrt[n]{f(x)} \cdot \sqrt[n]{g(x)} = \sqrt[n]{f(x) \cdot g(x)}$. For example,

$$\sqrt{x-1} \cdot \sqrt{x+4} = \sqrt{(x-1)(x+4)} = \sqrt{x^2+3x-4}$$

Two radicals of the same degree can also be divided, i.e., $\frac{\sqrt[n]{f(x)}}{\sqrt[n]{g(x)}} = \sqrt[n]{\frac{f(x)}{g(x)}}$. For example,

$$\frac{\sqrt{x^2+3x-4}}{\sqrt{x-1}} = \sqrt{\frac{x^2+3x-4}{x-1}} = \sqrt{\frac{(x-1)(x+4)}{x-1}} = \sqrt{x+4}$$

We need to be careful here regarding the possible values for x. Even though we cancelled out $(x-1)$, the restriction $x > 1$ still holds in the final solution. Further, we must have that $x + 4 \geq 0$ or equivalently, $x \geq -4$. Combining the two restriction and noting that the former obviates the later, we have that the final solution must satisfy $x > 1$.

11.2.3 Conjugates

The **conjugate** of an expression of the form $a + b\sqrt{c}$ is $a - b\sqrt{c}$. When such an expression is multiplied by its conjugate, the radical is removed. For example, the conjugate of $2 + 3\sqrt{5}$ is $2 - 3\sqrt{5}$ and the product of the two numbers is

$$(2+3\sqrt{5})(2-3\sqrt{5}\,) = 4 - 6\sqrt{5} + 6\sqrt{5} - 9(\sqrt{5})^2 = 4 - 9(5) = -41$$

Conjugates can sometimes be helpful in simplifying expressions. For example, consider the expression

$$\frac{(1+\sqrt{3})}{2+\sqrt{3}}$$

Multiple the expression by $1 = \frac{(2-\sqrt{3})}{(2-\sqrt{3})}$ and we get

$$\frac{(1+\sqrt{3})}{2+\sqrt{3}} \cdot \frac{(2-\sqrt{3})}{(2-\sqrt{3})} = \frac{2+2\sqrt{3}-\sqrt{3}-3}{4+2\sqrt{3}-2\sqrt{3}-3} = \sqrt{3}-1$$

which is simpler than (but equal to) the original expression.

In other cases, it is just a matter of style not to have a radical in the denominate regardless of whether the altered expression is any simpler than the original.

For example, consider the expression $\frac{2+\sqrt{5}}{2-\sqrt{7}}$. If we multiply by $1 = \frac{2+\sqrt{7}}{2+\sqrt{7}}$, we get

$$\frac{2+\sqrt{5}}{2-\sqrt{7}} = -\frac{1}{3}(2+\sqrt{5})(\,2+\sqrt{7})$$

While it is true the final expression on the right does not have a radical in the denominator, it is arguable whether it is any simpler than the expression on the left.

11.3 Graphs

Graphing equations with a single radical can be made easier by eliminating the radical which is done by raising the radical to the power of its degree. For example, take the equation $y = \sqrt{x-1}$. If we square both sides of the equation, we get

$$y^2 = x - 1$$

This is the equation for a parabola, facing to the right and with vertex $(1,0)$. However, from the initial equation, we see that $y > 0$, and so, we are restricted to the top-half of the parabola. In Figure 76, the solid line curve is $y = \sqrt{x-1}$. The bottom-half of the parabola is shown as the dotted line curve.

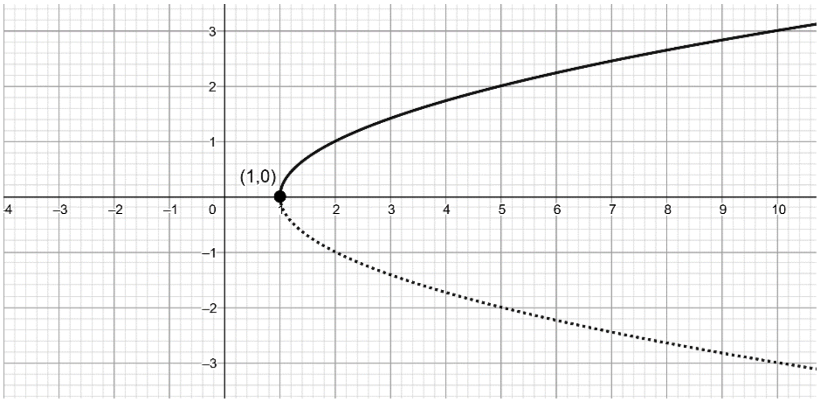


Figure 76. Radicalized parabola

As a second example, consider the equation $y = \sqrt{x^2 + 3x - 4}$. Noting that the degree of the radical is 2, we square both sides of the equation to get

$$y^2 = x^2 + 3x - 4$$

Completing the square on the right-side of the equation, we have

$$y^2 = \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} - \frac{25}{4}$$

$$y^2 = (x + \frac{3}{2})^2 - \frac{9}{4} - 4$$

$$y^2 = (x + \frac{3}{2})^2 - \frac{25}{4}$$

$$(x + \frac{3}{2})^2 - y^2 = \frac{25}{4}$$

which is the equation for a hyperbola, with center at $(-\frac{3}{2}, 0)$. But there is a problem, i.e., in the original equation, it is implied that $y \geq 0$. Thus, the graph of the original equation is just the top half of the hyperbola. The graph of $y = \sqrt{x^2 + 3x - 4}$ is shown in Figure 77 (just the top-half of the hyperbola, shown as a solid line curve). The bottom-half of the hyperbola is shown as a dotted line curve and is not part of $y = \sqrt{x^2 + 3x - 4}$.

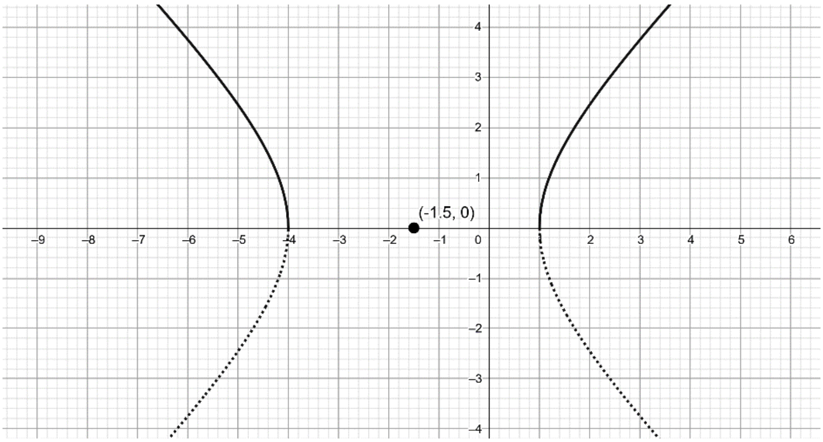


Figure 77. Graph of a radicalized hyperbola

Next, we put the two previous graphs together, and then graph the quotient $\frac{\sqrt{x^2+3x-4}}{\sqrt{x-1}}$ which, as we saw in the previous section, reduces to $\sqrt{x+4}$ with the restriction that $x > 1$. Figure 78 show the numerator of the quotient $\sqrt{x^2+3x-4}$ (dashed line curve), the denominator of the quotient $\sqrt{x-1}$ (dotted line curve) and the final solution $\sqrt{x+4}$ (solid line).

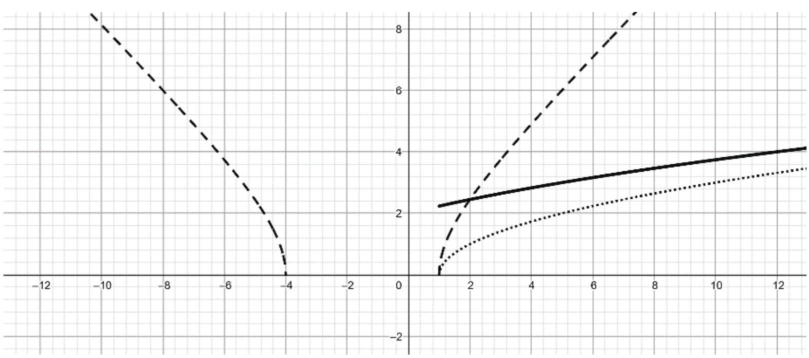


Figure 78. Graph of quotient of radicals

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Graph and identify the following equations using an analysis similar to that used in the previous examples:

1. $y = \sqrt{5-3x}$ ***Hint***: This is the top-half of a parabola that points to the left.

2. $y = -4\sqrt{1-\frac{x^2}{81}}$ ***Hint***: This is the bottom-half of an ellipse.

3. $y = \sqrt[3]{x-2} + 3$ ***Hint***: This is a cubic equation of the variable y.

11.4 Solving Equations with Radicals

We've already seen a few examples of removing radicals from equations with just one radical. As another example, take the exercise from the previous section, i.e., $y = \sqrt[3]{x-2} + 3$. To reduce this equation into something easier to graph, we subtract 3 from both sides of the equation and then cube both sides to get

$$(y-3)^3 = (x-2)$$

which is the cubic $y^3 = 2$ moved two units up and three units the right (see Figure 79). Since the radical is of an odd degree, there is no restriction on the values that x can take.

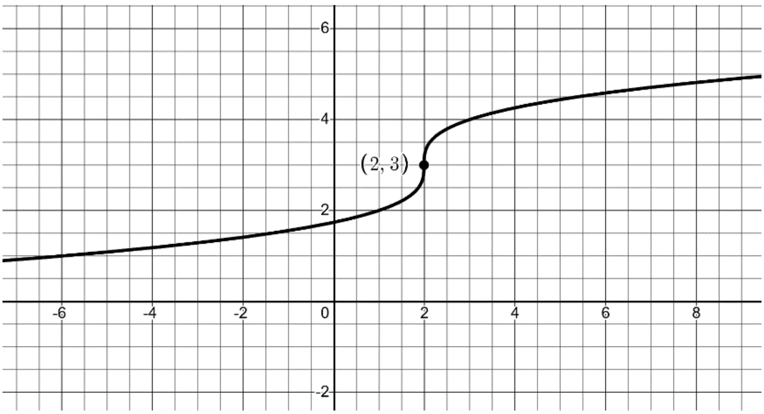

Figure 79. Graph of a cubic radical

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While it takes more work, it is possible to solve equations with two radicals. For example, solve the equation $\sqrt{3x+19} - \sqrt{5x-1} = 2$ for values of x. We first move one of the radicals to the other side of the equation to get

$$\sqrt{3x+19} = \sqrt{5x-1} + 2$$

Squaring both sides of the equation yields

$$3x + 19 = (5x-1) + 4\sqrt{5x-1} + 4$$

which simplifies to

$$2\sqrt{5x-1} = -x + 8$$

Square both sides of the equation to eliminate the remaining radical:

$$4(5x-1) = x^2 - 16x + 64$$

which simplifies to

$$x^2 - 36x + 68 = 0$$

We could use the quadratic formula on the above equation, or factor the equation as follows:

$$(x - 2)(x - 34) = 0$$

Thus, $x = 2$ or 34. We need to verify if either solution is valid by substituting the value back into the original equation.

Check for $x = 34$: $\sqrt{3(34) + 19} - \sqrt{5(34) - 1} = \sqrt{121} - \sqrt{169} = 11 - 13 = -2$ which should equal 2 but does not. So, $x = 34$ is not a valid solution. This is known as an **extraneous solution**.

Check for $x = 2$: $\sqrt{3(2) + 19} - \sqrt{5(2) - 1} = \sqrt{25} - \sqrt{9} = 5 - 3 = 2$. So, $x = 2$ checks out and is the only valid solution to the original problem.

For some rationale as to why extraneous solutions can occur, see the article " How do extraneous solutions arise from radical equations?" [24].

. . .

Some exercises to try:

1. Solve $\sqrt{x - 9} + \sqrt{x} = 9$ for x.

2. Solve $\sqrt{3x - 5} + \sqrt{x - 1} = 2$ for x. ***Hint***: When solve the equation, one finds $x = 2$ or 10. Checking the solution when $x = 10$, leads to a contradiction, i.e., $\sqrt{3(10) - 5} + \sqrt{10 - 1} = \sqrt{25} + \sqrt{9} = 5 + 3 = 8 \neq 2$. So, the only solution is $x = 2$. The other answer $x = 10$ is an extraneous solution. The lesson here is that when solving equations with radicals, you must check the solutions back in the original equation to make sure they are valid.

3. Solve the equation $\sqrt{2x + 7} + 4 = x$ and determine the solution. ***Hint***: There is an extraneous solution.

12 Inequalities

12.1 Definitions and Concepts

We've already seen some usage of inequalities in the preceding section of this book. Inequalities include the following concepts:

- greater than, written as $>$, e.g., $2 > 1$
- greater than or equal, written as $\geq$, e.g., $3 \geq 3$
- less than, written as $<$, e.g., $4 < 5$
- less than or equal, written as $\leq$, e.g., $7 \leq 7$
- not equal, written as $\neq$, e.g., $2 \neq 3$

The following properties are helpful in dealing with inequalities. The statements are written in terms of $<$ and $>$ which can be replaced by the corresponding $\leq$ and $\geq$, unless stated otherwise. It is assumed that a, b and c are real numbers.

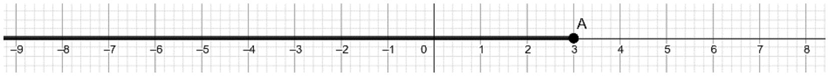
- Converse: $a < b$ and $b > a$ are equivalent statements
- Transitivity: if $a > b$ and $b > c$, then $a > c$
- Addition and subtraction: if $a > b$, then $a + c > b + c$ and $a - c > b - c$. This rule is helpful in simplifying inequalities.
- Multiplication and division:
  - If $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. A similar result can be stated if $a < b$. The general idea is that multiplication or division by a positive number does not change the direction of an inequality.
  - If $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$. A similar result can be stated if $a < b$. The general idea is that multiplication or division by a negative number reverses the direction of an inequality.

The following rules can be derived from the above rules and are commonly used when dealing with inequalities:

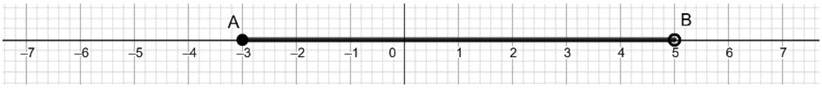
- If $a > b$, then $-a < -b$.
- If $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

12.2 Single Variable – First Degree – One Dimension

Inequalities can involve variables. The simplest case entails inequalities with a single variable. For example, $x \leq 3$ indicates all real numbers less than or equal to 3. In the diagram below, the inequality is shown as the ray starting at point A and extending to negative infinity.



Inequalities can be combined to create a bounded set of values. For example, the inequalities $x < 5$ and $x \geq -3$ combine to define the interval $-3 \leq x < 5$. The inequality is shown as the segment AB in the diagram below. The open circle at B indicates that 5 is not included in the interval.

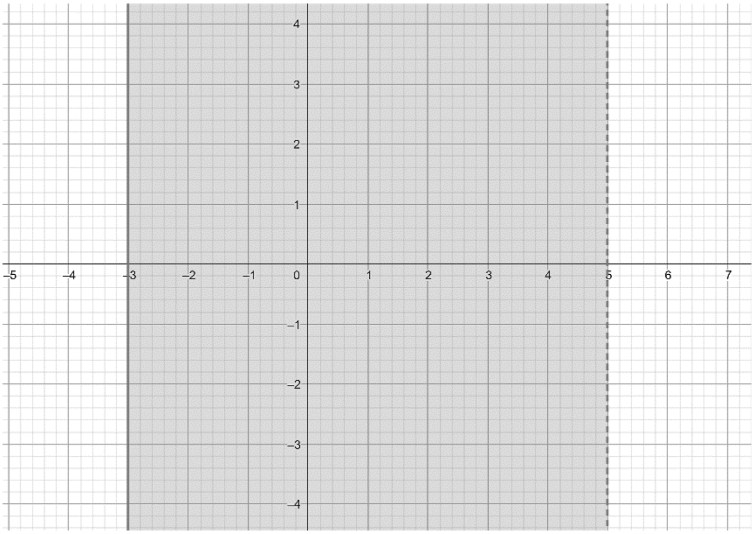


In some cases, a collection of inequalities may have no points in common. For example, the intersection of $x > 7$ and $x < 3$ is the empty set.

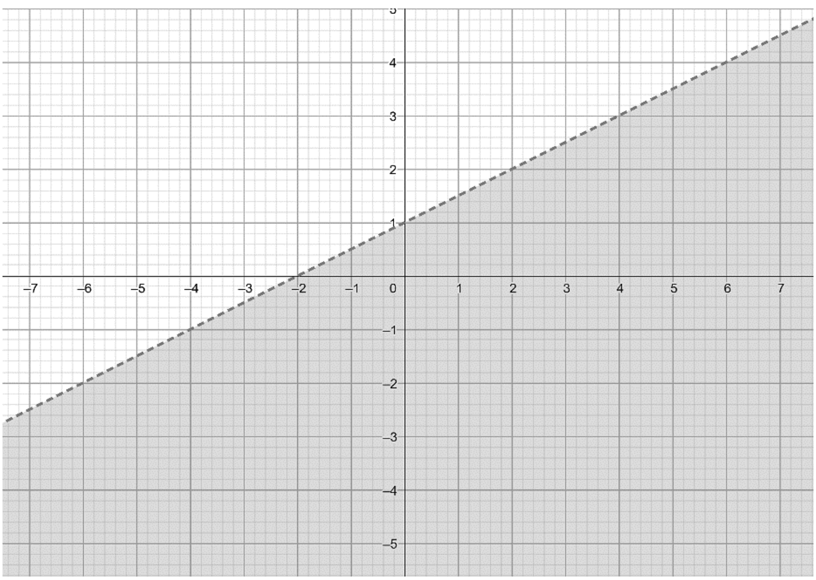
12.3 Two Variables – First Degree – Two Dimensions

12.3.1 Single Inequality

In two dimensions, the inequalities from the previous section look different. For example, $-3 \leq x < 5$ is an infinite strip. The reason is that we have a second dimension (represented by the variable y) which is unconstrained in this case. The graph of $-3 \leq x < 5$ is shown in the figure below. The gray area extends indefinitely upwards and downwards. The line $x = 5$ is not included (as indicated by the dashed line).



Consider the inequality $y < \frac{1}{2}x + 1$. This represents all the points below (but not including) the line $y = \frac{1}{2}x + 1$, as shown in the figure below. The inequality $y \geq \frac{1}{2}x + 1$ is the line $y = \frac{1}{2}x + 1$ and all the points above the line.



12.3.2 Multiple Inequalities

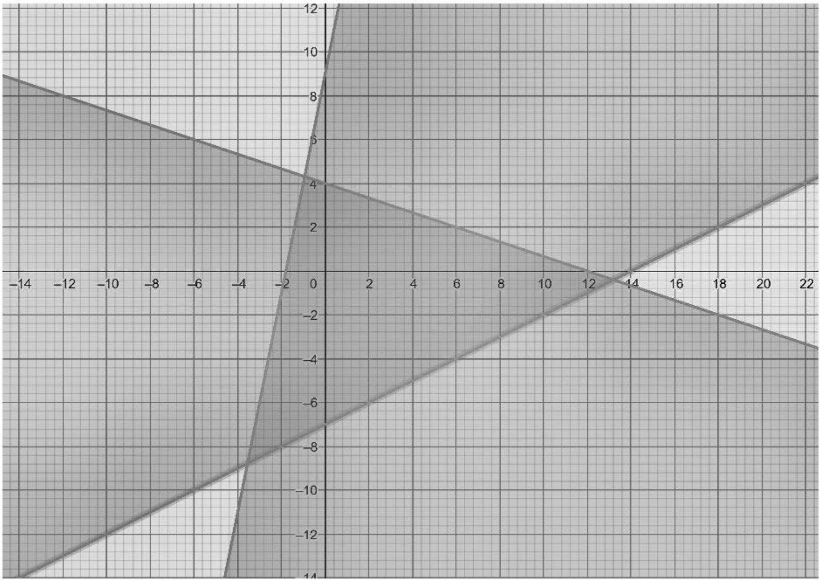
Several inequalities of two variables can be used to define a bounded region in the plane. For example, the inequalities

$$y \leq -\frac{1}{3}x + 4$$

$$y \geq \frac{1}{2}x - 7$$

$$y \leq 5x + 9$$

define the darker region in the center of the following figure. The individual lines can be identified in the graph by their y-intercepts, i.e., 4 for the first line in the list above, -7 for the second line and 9 for the third line.



If the above collection of inequalities were graphed in three dimensions, the region of intersection would be a triangular prism extending indefinitely in the positive and negative directions in the third dimension.

Use GeoGebra or Desmos to graph the following inequalities and find the intersection of the regions:

- $y \geq x - 5$
- $y \leq \frac{x}{4} + 3$
- $y \geq -x - 7$

An answer is provided at https://www.geogebra.org/calculator/xqcbc9bb.

12.4 Single Variable – Higher Degrees – One Dimension

Higher degree inequalities with a single variable can give rise to several simpler inequalities. For example, take the inequality $x^2 - 3x \geq 10$.

This can be written as

$$x^2 - 3x - 10 \geq 0$$

and then factor as

$$(x - 5)(x + 2) \geq 0$$

For the above to be true, both terms must be greater than zero or less than zero.

In the first case, we have $x - 5 > 0$ and $x + 2 > 0$ which implies $x > 5$ and $x > -2$. So, $x > 5$ in this case.

In the second case, we have $x - 5 < 0$ and $x + 2 < 0$ which implies $x < 5$ and $x < -2$. So, $x < -2$ in this case.

Thus, for the $(x - 5)(x + 2) \geq 0$ to hold true either $x > 5$ or $x < -2$.

An alternate way to solve the solve problem is to find the roots of $f(x) = x^2 - 3x - 10$ and then test whether the function is positive or negative on the intervals defined by the roots. Given the factorization above, we know that the roots of $f(x)$ are -2 and 5. The roots bound three intervals, i.e., $(-\infty, -2)$, $(-2, 5)$ and $(5, \infty)$. On each of these intervals, $f(x)$ is either positive all the time or negative all the time. If the function switched between positive and negative (or vice versa) on one of these intervals, then another root would exist but we know that is not the case. We do the tests as follows:

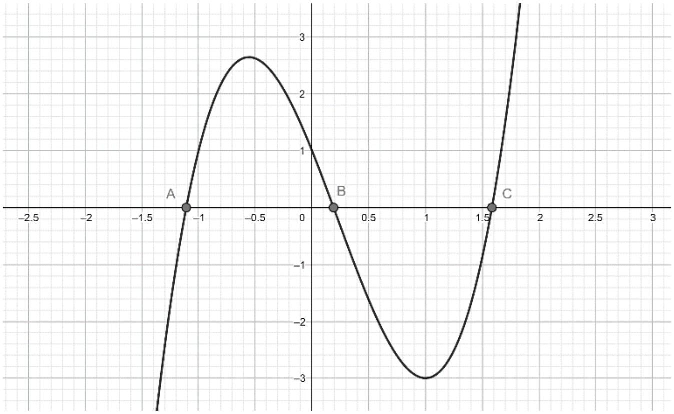
- $f(-3) = 8$, so $f(x)$ is positive for x in the interval $(-\infty, -2)$
- $f(0) = -10$, so $f(x)$ is negative for x in the interval $(-2, 5)$
- $f(6) = 8$, so $f(x)$ is positive for x in the interval $(5, \infty)$.

Thus, $x^2 - 3x - 10 > 0$ holds true for x in the interval $(-\infty, -2)$ or $(5, \infty)$.

. . .

For what values of x is the function $g(x) = 3x^3 - 2x^2 - 5x + 1$ greater than zero?

To solve this problem, we use an online graphing tool (in this case GeoGebra). The part of $g(x)$ where its roots occur is shown in the figure below. There is also a capability in GeoGebra that returns the roots of a given function (see https://www.geogebra.org/calculator/habfpbyp). In this case, the roots are approximately -1.11, $.19$ and 1.59. As can be seen from the graph, $g(x) > 0$ when $-1.11 < x < .19$ or $x > 1.59$.

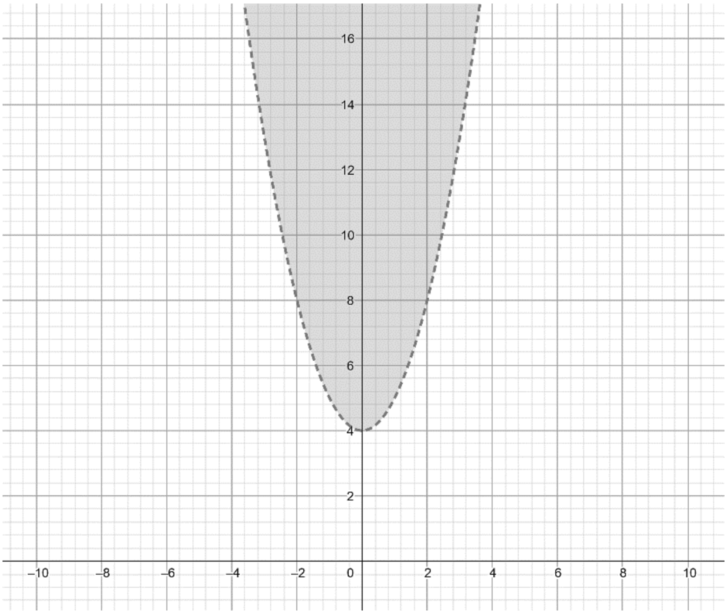


12.5 Two Variables – Higher Degrees – Two Dimensions

Describe the region such that $\frac{x^2}{100} + \frac{y^2}{25} \leq 1$. From our work with conic sections, we know that $\frac{x^2}{100} + \frac{y^2}{25} = 1$ is an ellipse. If we take a test point in the interior of the ellipse, e.g., $(0,0)$, we have $\frac{0^2}{100} + \frac{0^2}{25} = 0 \leq 1$ which is true. So, we conclude that the region defined by the inequality $\frac{x^2}{100} + \frac{y^2}{25} \leq 1$ is the ellipse $\frac{x^2}{100} + \frac{y^2}{25} = 1$ and its interior.

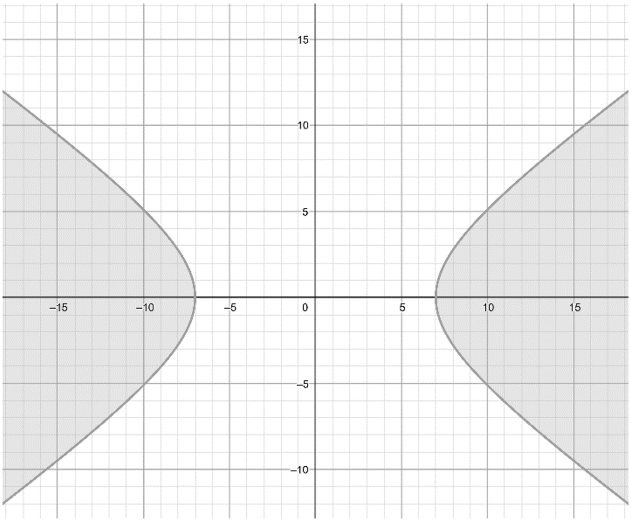
. . .

Describe the region bounded by the inequality $y > x^2 + 4$. We know that $y = x^2 + 4$ is the equation for a parabola. If we try the test point $(0,0)$, the inequality is false, i.e., $0 > 0^2 + 4 = 4$ is false. So, the bounded region must be above the parabola, as shown in the figure below.



. . .

Here's a slightly more complicated problem, with three regions to check. Find the region(s) bounded by the inequality $\frac{x^2}{49} - \frac{y^2}{25} \geq 1$. We know that $\frac{x^2}{49} - \frac{y^2}{25} = 1$ is a hyperbola and as such, divides the plane into three regions. If we try a test point in each region, e.g., $(-10,0)$, $(0,0)$ and $(10,0)$, we see that the inequality holds true for $(-10,0)$ and $(10,0)$. Thus, the regions bounded by the inequality are the two gray areas shown in the figure below.



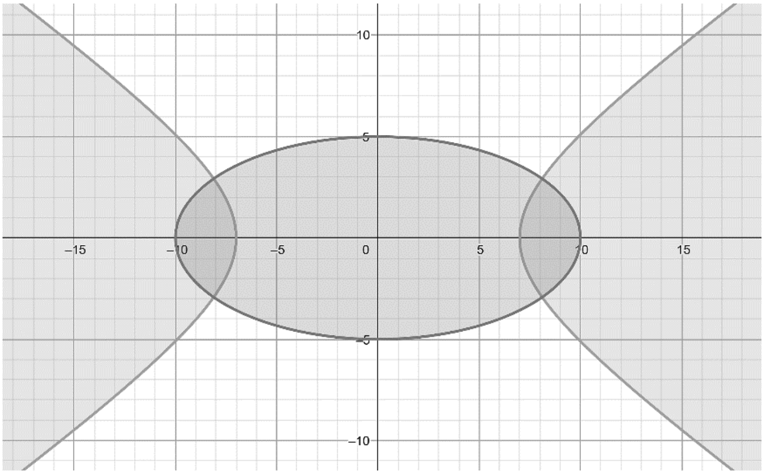
. . .

As a final problem, find the region that satisfies the inequalities:

$$\frac{x^2}{100} + \frac{y^2}{25} \leq 1$$

$$\frac{x^2}{49} - \frac{y^2}{25} \geq 1$$

We've already seen both inequalities in the previous problems. It is just a matter of taking the intersection of the two regions (see the two darker gray regions in the figure below).



. . .

A few exercises to try:

- Find the region defined by the inequality $x > y^2$.
- Find the region defined by the inequality $\frac{x^2}{9} + \frac{y^2}{16} > 1$.
- Find the region bounded by $x \geq y^2$ and $\frac{x^2}{9} + \frac{y^2}{16} \leq 1$.

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Acronyms

ASA – Angle Side Angle

AAS – Angle Angle Side

db – decibel

FOIL – First Outer Inner Last

GCD – Greatest Common Divisor

LCM – Least Common Multiple

RF – Radio Frequency

SAS – Side Angle Side

SSS – Side Side Side

Index of Terms