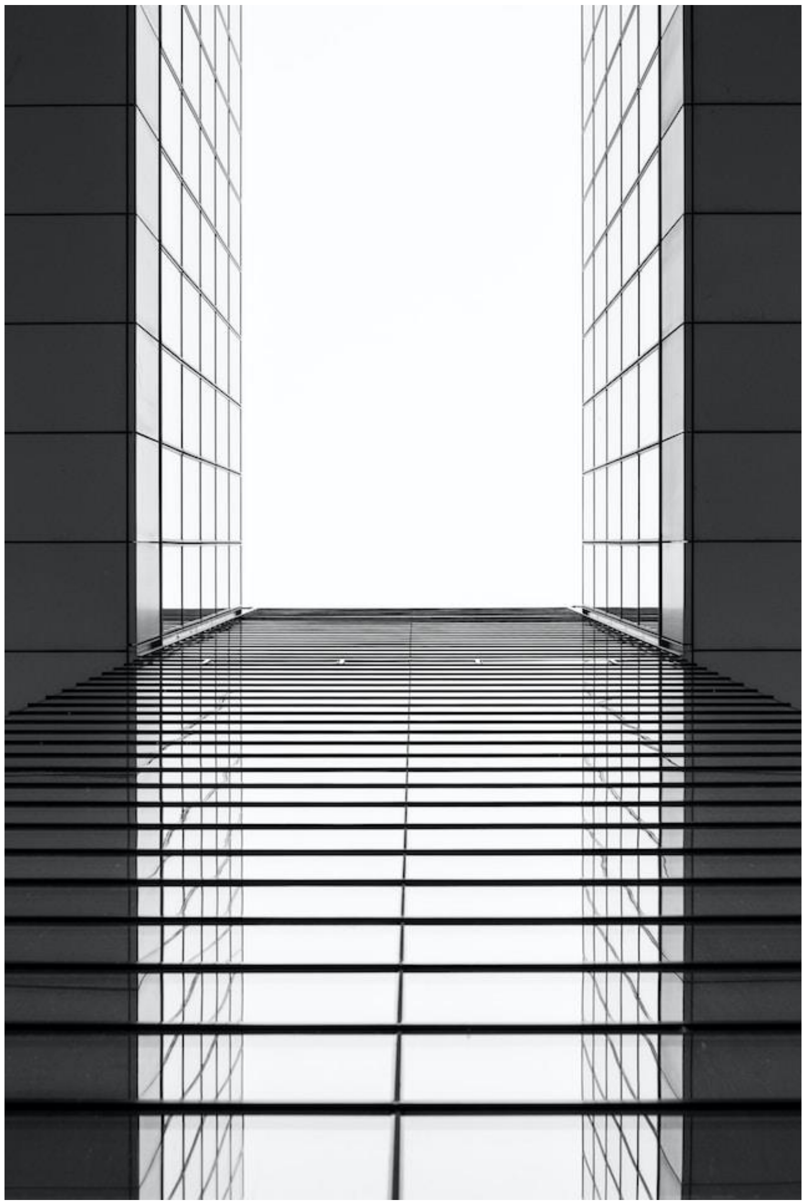
Shape Up and Solve It!

Learn Geometry Through Puzzles



by **Stephen Fratini**

Table of Contents

List of Figures

List of Puzzles

Preface

"Nothing will ever please me, no matter how excellent or beneficial, if I must retain the knowledge of it to myself. And if wisdom were given to me under the express condition that it must be kept hidden and not uttered, I should refuse it. No good thing is pleasant to possess, without friends to share it." – Seneca (Moral letters to Lucilius/Letter 6)

"Geometry, which before the origin of things was coeternal with the divine mind and is God himself." – Johannes Kepler

"Geometry is knowledge of the eternally existent." – Pythagoras

"I think the universe is pure geometry - basically, a beautiful shape twisting around and dancing over space-time." – Carl Sagan

This book features 100 engaging puzzles in the realm of "elementary" geometry. I use the term "elementary" because the puzzles don't delve into advanced geometric concepts, yet some of them present significant challenges. Solutions (or a reference to a solution) are provided for all the puzzles.

Section 2 offers a concise development of geometry that is grounded in the Birkhoff axioms. The book not only enumerates fundamental theorems in geometry (e.g., the Pythagorean theorem for triangles) but also proves many of the theorems. In cases where a proof of a theorem is not provided, a reference to a proof is given. Several puzzle solutions reference the theorems and concepts outlined in this section. Moreover, this section can serve as a handy refresher for those needing to revisit the basics of geometry. While the book isn't intended to replace a full-fledged geometry course textbook, it can function effectively as a supplementary resource.

The subsequent sections of the book contain various geometry puzzles along with their solutions. Some of the puzzles have been categorized into sections such as area puzzles and polygon puzzles. However, one could argue about the categorizations since many puzzles involve multiple elements such as areas, circles, polygons, and triangles.

There are also sections touching on Japanese temple geometry, probabilistic geometry, dissection puzzles, an overview of tessellations, analytic geometry puzzles and mass point geometry puzzles.

Caution: In this book, as well as in other books and articles on geometry, the statement of a puzzle takes precedence over the supporting figures (which may not be drawn exactly to scale, or which may only be an example of one aspect of the statement of the puzzle).

"Geometry is the science of correct reasoning on incorrect figures." – George Polya

Acknowledgements

The photo on the first page of this book is by Jonas Off on Unsplash.

The author would also like to thank the many geometers who post puzzles on YouTube and elsewhere. Many of the puzzles in this book are inspired by such postings. Some of my favorite folks on YouTube regarding geometry puzzles are

- Relish Math, https://www.youtube.com/@relishmath5632
- Math Booster, https://www.youtube.com/@MathBooster
- AMS Math, https://www.youtube.com/@AMSMath
- PreMath, https://www.youtube.com/@PreMath
- MindYourDecisions, https://www.youtube.com/@MindYourDecisions

Stephen Fratini
Sole Proprietor of The Art of Managing Things
Eatontown, New Jersey (USA)
Email: sfratini@artofmanagingthings.com
LinkedIn: www.linkedin.com/in/stephenfratini

Other books by the author:

- *The Art of Managing Things (2nd edition)*, self-published on Amazon, https://www.amazon.com/Art-Managing-Things-Stephen-Fratini-ebook/dp/B07N4H4YWH/, January 2019.
- *Mathematical Thinking: Exercises for the Mind (2nd Edition)*, self-published on Amazon, https://www.amazon.com/dp/B0CL34FRP1, October 2023.
- *Financial Mathematics with Python*, self-published on Amazon, https://www.amazon.com/gp/product/B08VKQR141, February 2021.
- *Math in Art, and Art in Math*, self-published on Amazon, https://www.amazon.com/dp/B091D1F8MB, March 2021.
- *Algebra through Discovery and Experimentation*, self-published on Amazon, https://www.amazon.com/dp/B09B5L9WL5, July 2021.
- *The Struggle Against Chaos*, self-published on Amazon, https://www.amazon.com/dp/B09BLPQ86Q, July 2021.
- *Mathematical Vignettes: Number theory, stochastic processes, game theory, cryptography, linear programming and more*, self-published on Amazon, https://www.amazon.com/Mathematical-Vignettes-stochastic-cryptography-programming-ebook/dp/B0BBP1PBJQ/, August 2022.
- *Learning Math through Puzzles: Number properties, counting, sequences and series, algebra, functions, and mathematical reasoning*, self-published on Amazon, https://www.amazon.com/dp/B0BZFRZP5B, March 2023.
- *Mathematical Vignettes: Volume II: Topics from combinatorial design, magic squares, finite geometry, abstract algebra, error correcting codes, geometric packing problems and much more, self-published on Amazon, https://www.amazon.com/dp/B0CM1CLSK8, October 2023.*

Electronic versions of my books are available (free of charge) at https://www.artofmanagingthings.com/home/my-books.

1 Introduction

"Geometry is the language in which God has written the universe." – Galileo Galilei

1.1 Purpose

Primarily, this is a book of geometry puzzles. A secondary purpose is to provide a brief overview of the axiomatic development of geometry using the approach taken by George David Birkhoff. Further, the puzzles tend to reinforce and test one's understanding of basic geometric concepts and theorems.

1.2 Intended Audience

There are two main audiences for this book, i.e., lovers of geometry puzzles and students studying geometry (typically in high school) who are looking for additional problems to test their knowledge of the subject matter.

1.3 Prerequisites

The prerequisites are an understanding of high school algebra and some familiarity with mathematical proofs and constructions.

1.4 Outline

The outline for the book is as follows:

- The front matter includes a preface, table of contents, table of figures and list of puzzles. The puzzle titles are mainly used as labels so that a list of puzzles could be automatically compiled (by Microsoft Word).
- Section 1 is this introduction.
- As noted in the preface, Section 2 provides an overview of basic geometry concepts and theorems. Solutions to many of the puzzles refer back to the concepts and theorems in this section.
- Sections 3, 4, 5 and 6 contain puzzles concerning triangles, circles, polygons and areas, respectively.
- Section 7 introduces what are known as Japanese temple geometry puzzles.
- Section 8 involves geometry puzzles whose solutions entail the use of probability.
- Section 9 contains dissection puzzles including tangram puzzles.
- Section 10 is an introduction to tessellations. There are no puzzles in this section.
- Section 11 has puzzles which are solved using analytic geometry, i.e., by placing the puzzle in a coordinate system and working with equations associated with the elements in the puzzle.
- Section 12 introduces mass point geometry, a technique from physics that can be used to solve some classes of geometry problems.

- A list of acronyms, references and an index of terms are provided at the end of the book.

1.5 Notation

The notation used in the puzzles is based on the notation stated in Section 2. However, we mention a few items here for emphasis:

- Right angles are indicated either by putting a small square at the angle in question, or by labeling the angle as $90°$.
- The notation $\triangle ABC$ means “triangle ABC”.
- As I started to draft this book, I decided to express the distance between two points (say A and B) as $d(AB)$. Many books just use AB and context to represent distance, e.g., $AB = 2$, but I use $d(AB) = 2$.
- Line segments are represented with a bar over top, e.g., the line segment between points A and B is represented by $\overline{AB}$. Some books will just use AB to represent the line segment between A and B, and also the line determined by points A and B. As one can see, there is a potential for confusion if one uses AB to denote both the line segment between points A and B, and the length of the line segment between points A and B.

2 Terms, Axioms and Basic Theorems

"Geometry is the language of God." – Albert Einstein

"Geometry is the only pure science." – Karl Frederick Gauss

2.1 Undefined Terms

Have you ever experienced the frustration of searching for a word's definition in a dictionary, only to find yourself going round in circles without getting a simple answer.? For example, consider the word "set". The following sequence of definitions comes from The Free Dictionary by Farlex:

- **Set**: A **collection** of distinct elements having specific common properties.
- **Collection**: A **group** of objects or works to be seen, studied, or kept together.
- **Group**: A **set**, together with a binary associative operation, such that the set is closed under the operation, the set contains an identity element for the operation, and each element of the set has an inverse element with respect to the operation.

This type of cycle is a general problem when one attempts to axiomatically define a branch of mathematics such as geometry. The problem is addressed by stipulating a set of undefined terms referred to as primitives. From the Wikipedia article "Foundations of geometry" [1]:

> **Primitives** (undefined terms) are the most basic ideas. Typically, they include objects and relationships. In geometry, the objects are things like points, lines and planes while a fundamental relationship is that of incidence – of one object meeting or joining with another. The terms themselves are undefined. *Famous mathematician David* Hilbert once remarked that instead of points, lines and planes one might just as well talk of tables, chairs and beer mugs. His point being that the primitive terms are just empty shells, place holders if you will, and have no intrinsic properties.

There are various formulations of axiomatic geometry, with the initial synthesis provided by Euclid in his Elements (published circa 300 B.C.). While Euclid's work stood the test of time for many centuries, eventually some issues were articulated regarding his work [1]:

- Lack of recognition of the concept of primitive terms, objects and notions that must be left undefined in the development of an axiomatic system.
- The use of superposition in some proofs without there being an axiomatic justification of this method. *(In the context of geometry, superposition refers to the idea that two geometric figures are identical in shape and size, and one can be exactly placed on top of the other by rigid motions such as translations, rotations, and reflections.)*
- Lack of a concept of continuity, which is needed to prove the existence of some points and lines that Euclid constructs.
- Lack of clarity on whether a straight line is infinite or boundaryless in the second postulate.
- Lack of the concept of betweenness used, among other things, for distinguishing between the inside and outside of various figures.

To address these issues, several enhancements to Euclid's formulation of the foundations of geometry were proposed. For example, Hilbert's axioms are a set of 20 postulates (axioms) proposed by David Hilbert in 1899 in his book *Grundlagen der Geometrie* as the foundation for a modern treatment of Euclidean geometry. A translation of Hilbert's work into English is available from the Gutenberg Project [3]. Other well-known modern formulations of Euclidean geometry were proposed by Alfred Tarski and by George Birkhoff. In what follows, we use the formulation offered by Birkhoff [4].

In his textbook "Basic Geometry" [4] (co-authored with Ralph Beatley), George David Birkhoff described the undefined terms in his formulation of geometry as follows:

> Any definitions that we need will be given as the need for them arises. Certain terms, namely, **number**, **order**, **equal**, **point**, **straight line**, **distance between two points**, and **angle between two lines**, we shall take as undefined terms. We shall also need those undefined terms commonly employed in every sort of logical reasoning, as, for example, is, are, not, and, or, but, if, then, all, every. The word "line" will ordinarily be understood to mean a "straight line". Part of our undefined notion of straight line and of plane is that each of these is a collection of points; also, that a straight line through any two points of a plane lies wholly within the plane. We shall also assume that a straight line divides a plane into two parts, though it is possible to prove this.

2.2 Axioms

In this section, we cover the 5 axioms (postulates) in Birkhoff's formulation of geometry, and some basic concepts that follow from these postulates.

Axiom 1 (Line Measure) The points on any line can be numbered so that the differences represent distances.

In Figure 1, we have points A, B and C on line ℓ. Each point is associated with a number. We can use the assigned numbers to compute distances. For example, the distance between points A and C is $3.5 - (-2) = 5.5$.

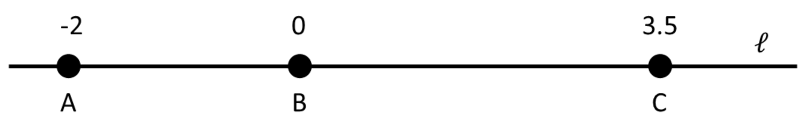


Figure 1. Line measure

Given three points A, B and C on a line, point B is said to reside **between** A and C if the numbers associated with A, B and C occur in order (could be ascending or descending order). In Figure 1, points B is between points A and C.

For points A and B on a line, the distance between the two points is represented as $d(AB)$.

The part of a line that contains all points between two given points (as well as the two points) is known as a **line segment**. In Figure 1, $\overline{AB}$ represents the line segment containing all points between A and B (including A and B). In some cases, the notation from a line segment, e.g., $\overline{AB}$, is used in lieu of the distance notation, i.e., $d(AB)$. It is a common abuse of notation used in geometry books, including this book 😊.

Given that points A, B and C are on a line. If $d(AB) = d(BC)$, the B is said to be the **midpoint** of the line segment $\overline{AC}$. In this case, B is said to **bisect** the line segment $\overline{AC}$.

Axiom 2 There is exactly one line through two given points.

Given Axiom 2, we can uniquely specify a line by two points on the line, e.g., example we could identify line ℓ in Figure 1 as line AB, line BC, or line AC.

If two lines have two or more points in common, they must be the same line; otherwise, Axiom 2 is contradicted. Thus, two lines have at most one point in common.

If two lines have one point in common, they are said to **intersect**.

A point P on a line divides the line into straight **half-lines (or rays)** each of which has P as an endpoint. Using Axiom 1, we can say that one half-line is made up of the end-point P and all points whose numbers are greater than the number assigned to P, and the other half-line is comprised of the end-point P and all points whose associated numbers are less than the number corresponding to P. Each half-line can be treated as a separate entity. A half-line is sometimes represented by its endpoint and some other point on the half-line. For example, the two half-lines in Figure 2 can be referred to as the half-lines OA and OB, with the understanding these entities extend indefinitely.

Two half-lines with the same endpoint form two **angles**, as shown in Figure 2. For the example, the smaller angle is represented as $\angle AOB$. This point O is known as the **vertex** of the angle. The half-lines OA and OB are referred to as the sides of the angle. When two half-lines form a straight line, the angle is referred to as a **straight angle**.

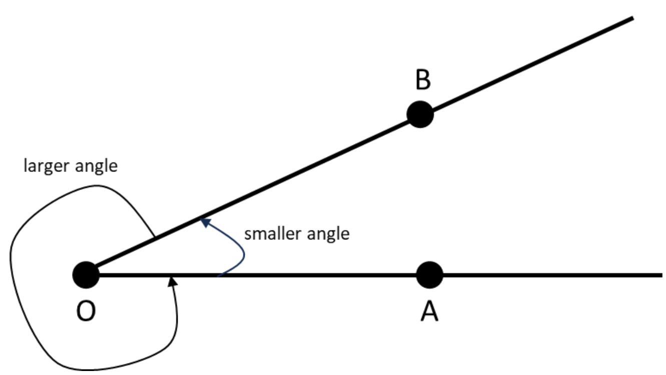


Figure 2. Example of the angles form by two half-lines

Axiom 3 (Angle Measure) All half-lines sharing the same endpoint can be numbered such that the differences represent angles.

The original statement of Axiom 3 in Birkhoff's book [4] (paraphrased above) did not specifically state how the numbering should be done, although the associated examples do imply a numbering scheme based on $360°$ (or 2π radians). The following version of Axiom 3, from the Wikipedia article "Birkhoff's axioms" [5], provides specific details on how to do the numbering of the half-lines (rays):

> The set of rays $\{\ell, m, n, \dots\}$ through any point O can be put into 1:1 correspondence with the real numbers $a(mod\ 2\pi)$ so that if A and B are points (not equal to O) of ℓ and m, respectively, the difference $(a_m - a_\ell)(mod\ 2\pi)$ of the numbers associated with the lines ℓ and m is $\angle\, AOB$. Furthermore, if the point B on m varies continuously in a line r not containing the vertex O, the number am varies continuously also.

In Figure 3, the angle between half-lines m and n is $\angle AOB = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

An alternate notation for an angle is to only use the vertex, e.g., $\angle O$ instead of $\angle AOB$ in Figure 3.

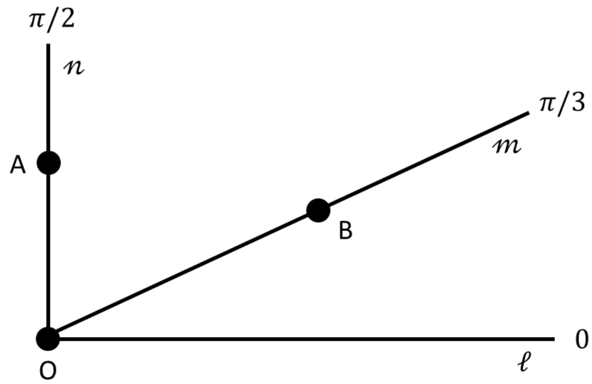


Figure 3. Angle measure example

Using the alternate statement of Axiom 3, we have the following definitions.

- An angle of measure $90°$ ($\frac{\pi}{2}$ radians) is known as a **right angle**.
- Angles of measure less than $90°$ are known as **acute angles**.
- Angles of measure more than $90°$ and less than $180°$ (π radians) are known **obtuse angles**.
- If two lines meet at a point O such that the angle between two of their half-lines has the measure $90°$, the lines are said to be **perpendicular**.

Axiom 4 All straight angles have the same measure.

Given half-lines ℓ, m and n with common vertex O such that the angle of m is less than the angle of n and greater than the angle of ℓ, then m is said to be **between** ℓ and n, e.g., see Figure 3.

A **polygon** is defined as a closed chain of connected line segments. The segments of a polygonal are called its **edges (or sides)**. Three polygons are shown in Figure 4. The interior angles of the polygon on the left are all less than $180°$. A polygon with this characteristic is known as a **convex polygon**. A polygon that is not convex is said to be **concave**. The middle polygon is concave since the interior angle $\angle EAB$ is greater than $180°$. The polygon on the right is concave and also self-intersecting.

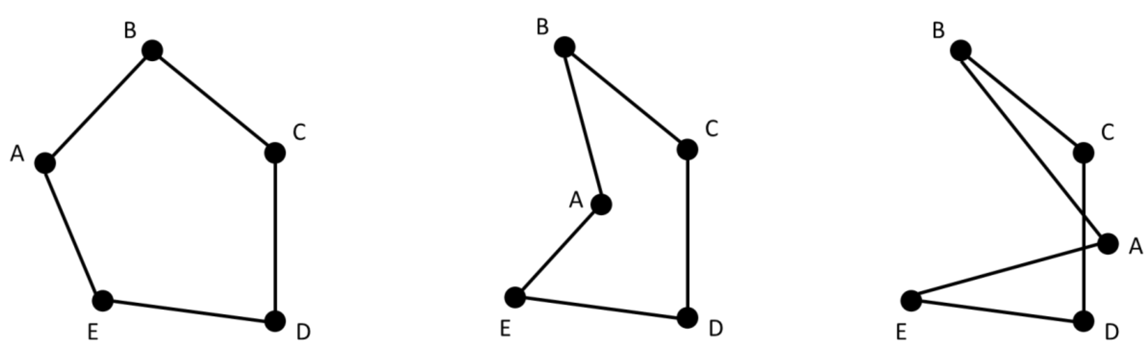


Figure 4. Example polygons

Two polygons are **similar** if all corresponding angles are equal and all corresponding side measures are proportional, i.e., by a multiple of a constant k. If $k = 1$, the polygons are congruent (i.e., equal).

A **chord of a polygon** $\mathcal{P}$ is a line segment connecting two non-adjacent vertices of $\mathcal{P}$.

. . .

Birkhoff's 5th axiom concerns similar triangles (three-sided polygons).

Axiom 5 Side-Angle-Side (SAS) Triangle Similarity Given two triangles $\triangle ABC$ and $\triangle DEF$ and some constant $k > 0$ such that $\overline{DE} = k\overline{AB}$, $\overline{DF} = k\overline{AC}$ and $\angle BAC = \angle EDF$, then $\overline{BC} = k\overline{EF}$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$, i.e., the triangles are similar.

[There is a slight abuse of notation here concerning the stated relationships between the measures of the corresponding sides, e.g., we should actually say $d(DE) = kd(AB)$.]

If triangles $\triangle ABC$ and $\triangle DEF$ are similar, we write $\triangle ABC \cong \triangle DEF$.

In Figure 5, assume we are given that $\angle BAC = \angle EDF$, $\overline{DE} = 2\overline{AB}$ and $\overline{DF} = 2\overline{AC}$. By Axiom 5.1, the two triangles are similar. In notation, we write $\triangle ABD \cong \triangle DEF$.

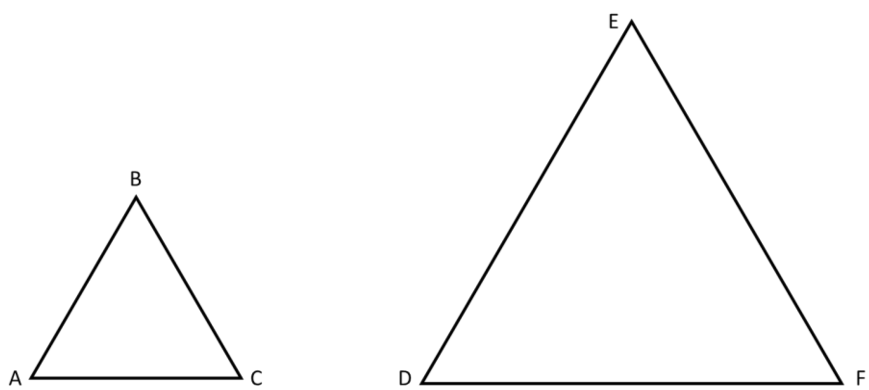


Figure 5. Similar triangles

In cases where the corresponding sides in two similar triangles are equal (i.e., the proportion constant $k = 1$), the triangles are said to be **congruent**. For congruent triangles, we use the equal sign $=$ as opposed to $\cong$.

2.3 Basic Theorems

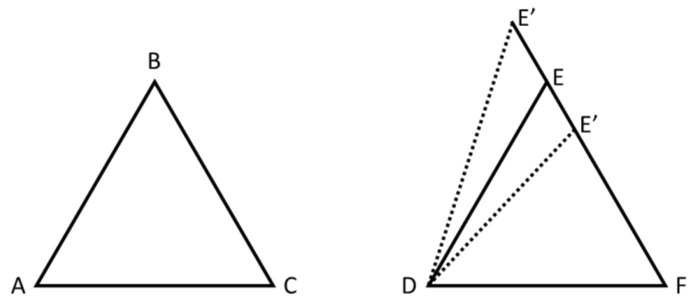
In this section, we discuss several fundamental theorems of geometry based on the Birkhoff axioms stated in the previous section.

Theorem 1 (Angle-Angle (AA) Triangle Similarity). If two triangles have two angles of equal measure, then the triangles are similar.

Proof: Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\angle A = \angle D$ and $\angle C = \angle F$.

Assume that $\overline{DF} = k\overline{AC}$. At this point, we don't know whether the other corresponding sides of the two triangles are in the same proportion k.

In the triangle $\triangle DEF$ and along the line containing side $\overline{EF}$, mark the point E' such that $\overline{E'F} = k\overline{BC}$. As shown in the figure below, E' could be on either side of E or be equal to E (which is what we intend to show).



We have $\angle C = \angle F, \overline{DF} = k\overline{AC}$ and $\overline{E'F} = k\overline{BC}$. Thus, by the SAS property (i.e., Axiom 5), $\triangle ABC \cong \triangle DE'F$.

The similarity of $\triangle ABC \cong \triangle DE'F$ implies that $\angle BAC = \angle E'DF$, but we were given that $\angle BAC = \angle EDF$ and so, $\angle E'DF = \angle EDF$. Thus, E' must be on the line ED as well as on the line EF. Since EF and ED can intersection in at most one point, it must be that $E = E'$ which implies $\overline{EF} = k\overline{BC}$.

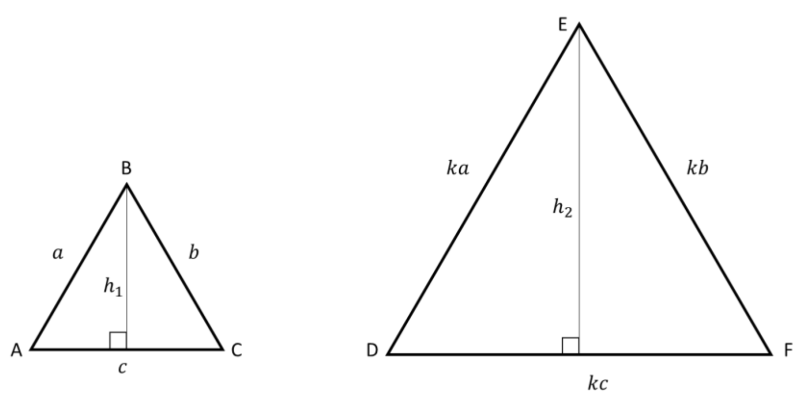
We have $\overline{DF} = k\overline{AC}, \overline{EF} = k\overline{BC}$ and $\angle C = \angle F$. Thus, by the SAS property, $\triangle ABC \cong \triangle DEF$. ■

Theorem 2 (Angle-Side-Angle (ASA) Triangle Congruence) If two triangles have two angles of equal measure with the sides between the two angles equal, then the triangles are congruent.

Proof: By Theorem 1, the triangles are similar. Since the triangles have a side of equal measure, the similarity proportion is 1, and thus, the triangles are congruent. ■

Theorem 3. If two triangles are similar with proportion k, then the heights of the two triangles are also in proportion k.

Proof: The figure below depicts the two similar triangles from the theorem statement.

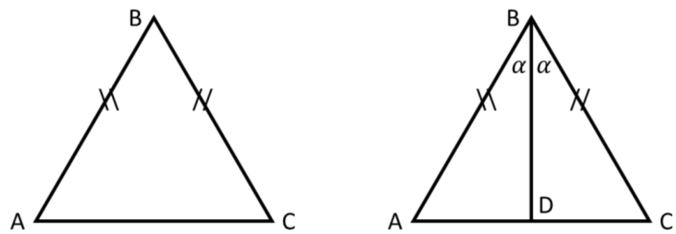


Using the Pythagorean theorem (Theorem 9) twice, we have that $h_1 = \sqrt{a^2 - \left(\frac{c}{2}\right)^2}$ and $h_2 = k\sqrt{a^2 - \left(\frac{c}{2}\right)^2}$. ■

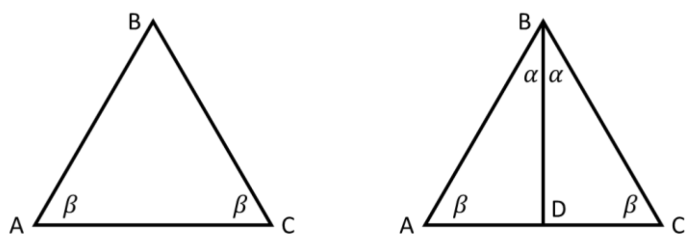
The next theorem concerns **isosceles triangles**, i.e., triangles that have two equal sides.

Theorem 4. If two sides of a triangle are equal, the angles opposite these sides are equal. Conversely, if two angles of a triangle are equal, the sides opposite these angles are equal.

Proof: In the left-side of the figure below, $\triangle ABC$ has two equal sides, i.e., $\overline{AB} = \overline{CB}$. We want to show that $\angle A = \angle C$. On the right-side of the figure, we draw a line segment $\overline{BD}$ that bisects $\angle B$. By the SAS principle, $\triangle ABD \cong \triangle CBD$ and thus, $\angle A = \angle C$.



Going in the other direction, assume $\triangle ABC$ has two equal angles, i.e., $\angle A = \angle B$ (as shown on the left of the figure below). Next, we again bisect $\angle B$ (as shown on the right of the figure below). By the AA triangle similarity principle, $\triangle ABD \cong \triangle CBD$. Further, the ratio of corresponding sides is 1 since the two triangles share a common side, i.e., $\overline{BD}$. Thus, $\triangle ABD = \triangle CBD$ (i.e., the two triangles are congruent), and so, $\overline{AB} = \overline{CB}$.

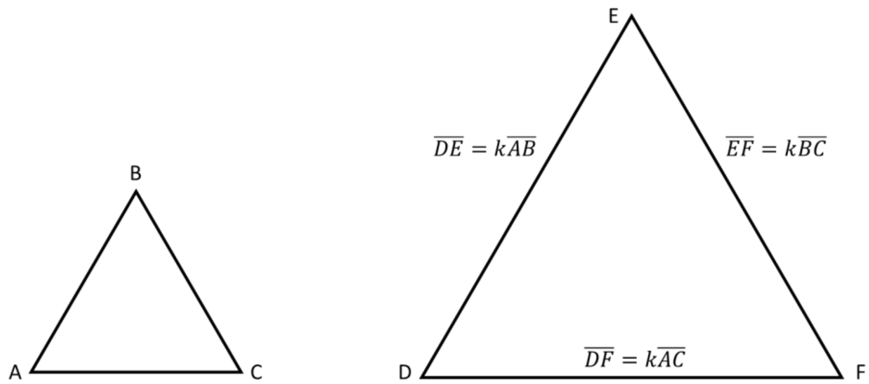


■

If all three sides of a triangle are of equal measure (known as an **equilateral triangle**), all the angles are also equal. This follows by applying Theorem 4 twice.

Similarly, we can apply Theorem 4 twice to a triangle with equal angles to prove that the triangle is equilateral.

Theorem 5 (Side-Side-Side (SSS) Triangle Similarity). Triangles $\triangle ABC$ and $\triangle DEF$ are similar if their corresponding sides are proportional (see the figure below).

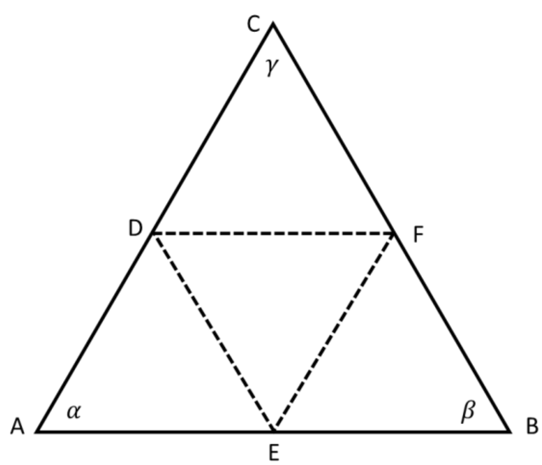


Proof: See the proof of Principle 8 in Birkhoff [4].

In some formulations of Euclidean geometry, the following theorem is assumed true as an axiom. In the Birkhoff formulation, we can prove this theorem using earlier results.

Theorem 6. The sum of the angles of a triangle is $180°$.

Proof: We want to show that the sum of the angles in $\triangle ABC$ is equal to $180°$. In the following figure, we have labeled the angles of $\triangle ABC$ as α, β and γ. Let D be the midpoint of $\overline{AC}$, E be the midpoint of $\overline{AB}$, and F be the midpoint of $\overline{BC}$. Form triangle $\triangle DEF$ from the three midpoints.



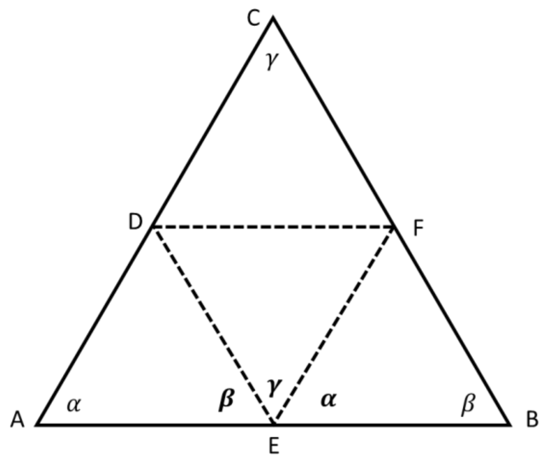
By definition, $\overline{AD} = \frac{1}{2}\overline{AC}$ and $\overline{AE} = \frac{1}{2}\overline{AB}$. Also, $\angle DAE = \angle CAB = \alpha$. By the SAS principle, we have $\triangle ABC \cong \triangle AED$ which implies $\angle DEA = \angle CBA = \beta$ and $\overline{DE} = \frac{1}{2}\overline{CB}$.

Similar to the argument above, we can show $\angle FEB = \angle CAB = \alpha$ and $\overline{EF} = \frac{1}{2}\overline{AC}$. Also, using a similar argument, we can show that $\overline{DF} = \frac{1}{2}\overline{AB}$.

We have shown that the sides of $\triangle DEF$ are in proportion $\frac{1}{2}$ to the corresponding sides of $\triangle ACB$.

By the SSS principle, $\triangle ACB \cong \triangle FED$ which implies $\angle ACB = \angle FED = \gamma$.

The updated information (concerning equal angles) is included in the figure below.



As one can see from the above figure, $\alpha + \beta + \gamma = \angle AEB = 180°$ (since AEB is a straight line) and thus, the angles of $\triangle ABC$ sum to $180°$. ■

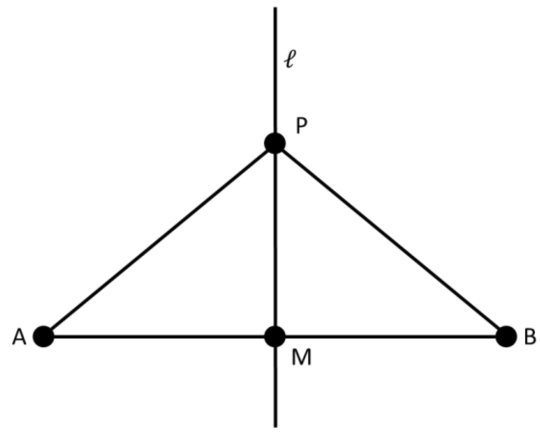
The following is a classification of triangles:

- A triangle all of whose angles are acute (i.e., less than $90°$) is called an **acute triangle**.
- A triangle one of whose angles is a right angle is called a **right triangle**.
- A triangle one of whose angles is obtuse (i.e., greater than $90°$) is called an **obtuse triangle**.

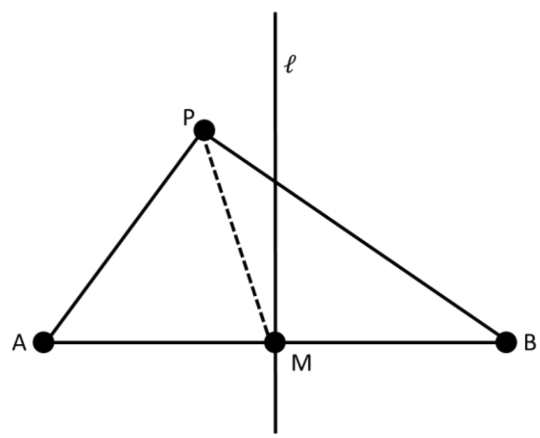
The following theorem may seem obvious but it still needs to be proven.

Theorem 7. All points equidistant from the endpoints of a line segment, and no others, lie on the perpendicular bisector of the line segment.

Proof: Consider the line segment AB and its perpendicular bisector (line ℓ) as shown in the figure below. Let P be a point on line ℓ. The triangles $\triangle AMP$ and $\triangle BMP$ are similar by the SAS principle since $\overline{AM} = \overline{BM}$, $\angle AMP = \angle BMP = 90°$ and they share edge $\overline{MP}$. Thus, $\overline{AP} = \overline{BP}$, i.e., P is equidistant from points A and B.



Next, we show that if a point P is equidistant from A and B then it must be on the perpendicular bisector of $\overline{AB}$. In the figure below, we have (by assumption) that $\overline{AP} = \overline{BP}$ which implies (by Theorem 4) that $\triangle APB$ is an isosceles triangle, and so, $\angle A = \angle B$. By the SAS principle, we have that $\triangle APM \cong \triangle BPM$ which implies $\angle PMA = \angle PMA$. However, $\angle PMA + \angle PMA = 180°$ and so, it must be that $\angle PMA = \angle PMA = 90°$. This forces $\overline{MP}$, and thus P, to be on line ℓ.



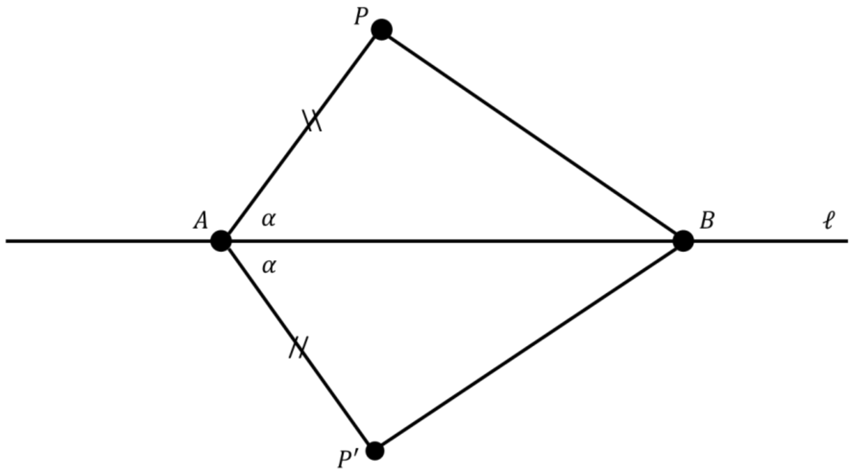
■

The next theorem relies on the result from the previous theorem.

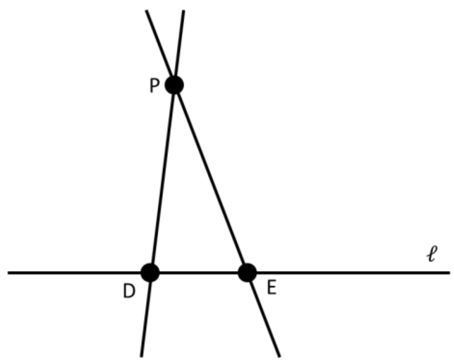
Theorem 8. Through a point P not on a line ℓ there is one and only one line through P and perpendicular line ℓ.

Proof: Consider a point P and line ℓ. Let A and B be distinct points on ℓ. Next, create the line segment $\overline{AP}$, and then reflect $\overline{AP}$ about ℓ to create line segment $\overline{AP'}$, as shown in the figure below. Create line segments $\overline{BP}$ and $\overline{BP}$'.

By the SAS triangle similarity principle, $\triangle APB \cong \triangle AP'B$ which implies $\overline{PB} = \overline{P'B}$. So, we have that P and P' are both equidistant from points A and B. By Theorem 7, P and P' determine a line that is the perpendicular bisector of $\overline{AB}$, and thus, also perpendicular to line ℓ. This establishes there is a at least one line containing point P and which is perpendicular to line ℓ.



Assume there are two lines containing P, with both being perpendicular to line ℓ, as shown in the following figure. In this case, $\angle PDE = \angle PED = 90°$. Since the angles of triangle $\triangle DPE$ must add to $180°$, we are forced to conclude that $\angle DPE = 0°$. This implies that line DP and EP are the same, and thus, we can only have one line containing a given point that is perpendicular to a given line.



■

The next theorem is quite famous and perhaps the only thing many folks remember from high school geometry. There are a large number of different proofs for this theorem.

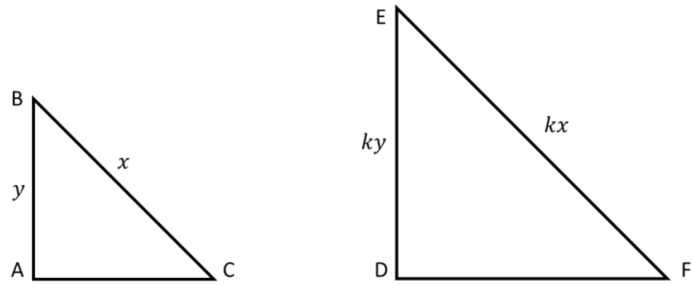
Theorem 9 (Pythagorean theorem). In any right triangle the square of the hypotenuse (i.e., the side opposite the right angle) is equal to the sum of the squares of the other two sides; and conversely.

Proof: Proof Wiki provides several proofs [6].

The following theorem is a similarity principle for right triangles. It depends on the Pythagorean theorem.

Theorem 10. If the hypotenuse and another side of two right triangles are in the same proportion k, the two triangles are similar. If $k = 1$, the two triangles are congruent.

Proof: Assume that we have two right triangles as shown in the following figure. The hypotenuses and one other side are in proportion k.



By the Pythagorean theorem, $\overline{AC}^2 = x^2 - y^2$ and $\overline{DF}^2 = k^2x^2 - k^2y^2 = k^2(x^2 - y^2)$. So, $\overline{DF} = k\sqrt{|x^2 - y^2|} = k\overline{AC}$. By the SSS triangle similarity principle, the two triangles are similar. ■

The following theorem appears obvious but the proof is in fact quite involved with several cases.

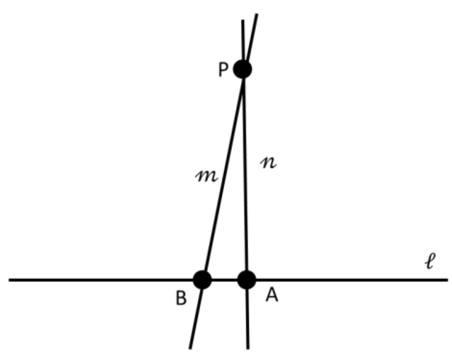
Theorem 11 (Triangle Inequality). The sum of two sides of a triangle is greater than the third side.

Proof: See Corollary 12c in Birkhoff [4].

The converse of the triangle inequality theorem is also true, i.e., if three real numbers are such that each is less than the sum of the other two, then there exists a triangle with these numbers as its side lengths and which has positive area (i.e., the it is not a degenerate triangle). For a proof of the converse, see the Wikipedia article “Triangle inequality” [7].

Theorem 12. The shortest distance between a point P and a line ℓ is via the line through P and perpendicular to ℓ.

Proof: By Theorem 8, there is a unique line (call it n) going through P and perpendicular to line ℓ. Take <u>any</u> other line m going through P and intersecting line ℓ. The situation is shown in the figure below.

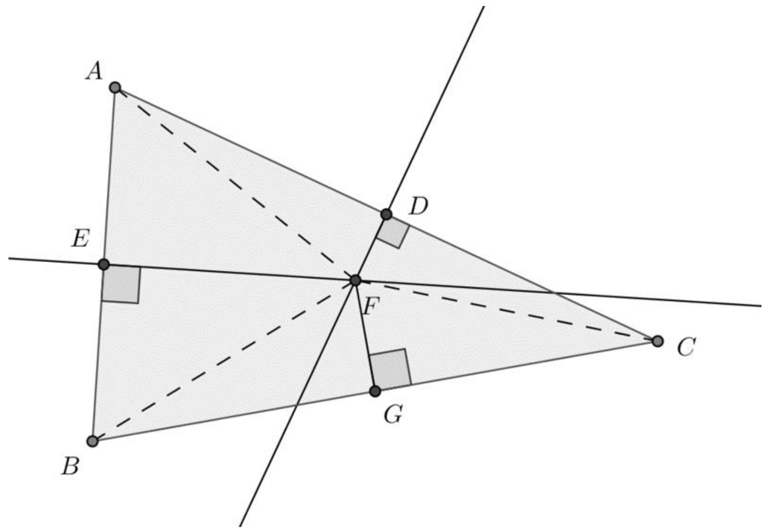


By the Pythagorean theorem, $\overline{AP}^2 + \overline{AB}^2 = \overline{BP}^2 \Rightarrow \overline{BP}^2 > \overline{AP}^2 \Rightarrow \overline{BP} > \overline{AP}$. ■

A **perpendicular bisector of a side of a triangle** is a line that is perpendicular to and bisects the side.

Theorem 13. The perpendicular bisectors of a triangle meet at point.

Proof: An example is provided in the figure below. See the Proof Wiki article "Perpendicular Bisectors of Triangle Meet at Point" [11] for a proof. This point is known as the **circumcenter of a triangle**.



2.4 Theorems about Parallel Lines

Lines in the same plane that do not meet are said to be **parallel**. (Note: It is possible for lines in 3-dimensional space to not meet and yet not be parallel.)

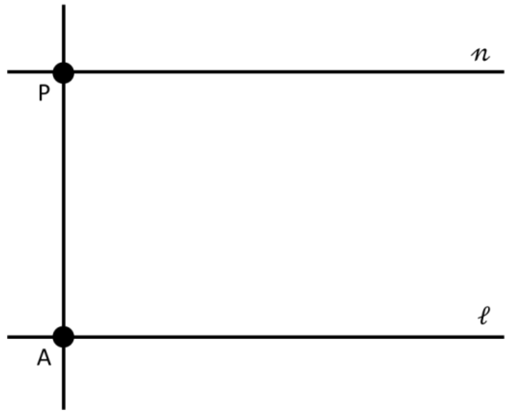
Theorem 14. For a point P not on a given line ℓ there is one and only one line through P which is parallel to ℓ. Further,

i. ***Given two parallel lines, if a third line intersects one of the two parallel lines, it must intersect the other.***

ii. ***Parallelism is transitive, i.e., if line a is parallel to line b, and line b is parallel to line c, then a is parallel to c.***

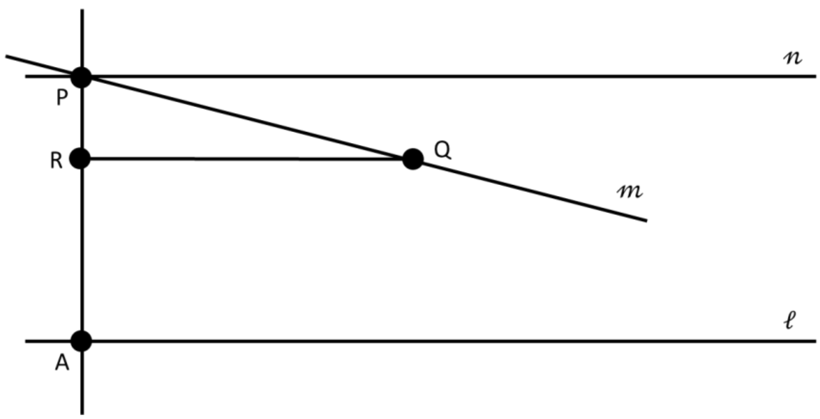
Proof: We first show there is one line through P and that is parallel to ℓ. By Theorem 8, there is one line containing P that is perpendicular to line ℓ. Let A be the point where the perpendicular line intersections ℓ. By Axioms 3 and 4, there is only one line through P that is perpendicular to line AP (call this line n). The situation is shown in the figure below.

If lines n and ℓ intersected (say at point B), then the angles of $\triangle ABP$ would be greater than 180° which contradicts Theorem 6. Thus n is parallel to l.



Next, we need to show that n is the only line through P that is parallel to ℓ. To prove this, we show that any line (other than n) going through P must intersect ℓ.

Assume m is a line through P that makes an angle less than 90° with line n. Pick a point Q on m and draw a perpendicular line from Q to point R on the line *AP*, as shown in the figure below.



Select a point S on line ℓ such that

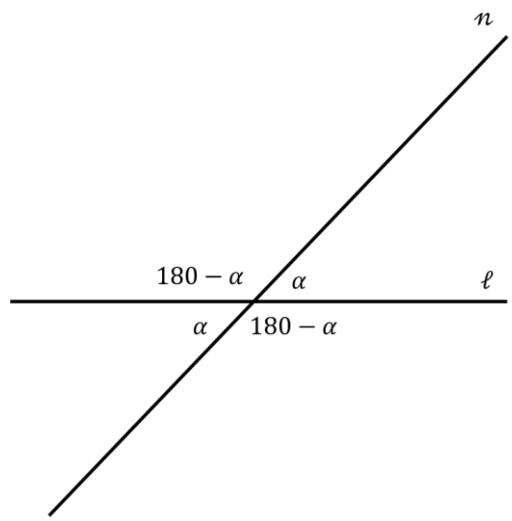
$$\frac{d(PR)}{d(RQ)} = \frac{d(PA)}{d(AS)}$$

If we let $k = \frac{d(RQ)}{d(AS)} = \frac{d(PR)}{d(PA)}$, then $d(RQ) = k \cdot d(AS)$ and $d(PR) = k \cdot d(PA)$. By the SAS triangle similarity principle, $\triangle PAS \cong \triangle PRQ$ which implies $\angle APS = \angle RPQ$. Thus, lines PS and m are the same line. So, we can conclude that line m intersects line ℓ (which is what we intended to prove).

If part i of the theorem were not true, then we would have two lines through a point on one line that is parallel to the other line which contradicts the main part of this theorem.

By a similar argument, part ii of the theorem is true. ■

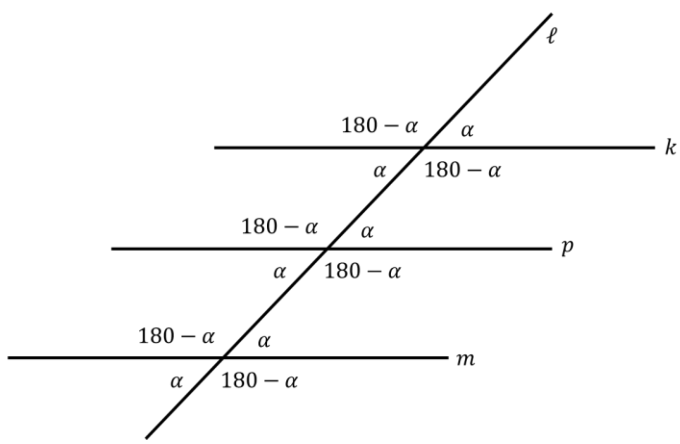
When two lines intersect each other, four angles are formed. Among these angles, there are two pairs of non-adjacent angles. These are called **opposite angles**, vertical angles or vertically opposite angles. By Axiom 3 (using the $360°$ numbering scheme) and Axiom 4, the angles at the point of intersection of two lines is as shown in the following figure.



A line that intersects several other lines is called a **transversal** of those lines.

Theorem 15. A transversal intersects each line in a set of parallel lines at the same angle.

Proof: See the Proof Wiki article "Parallelism implies Equal Corresponding Angles" [8]. An example of the theorem is shown in the figure below.



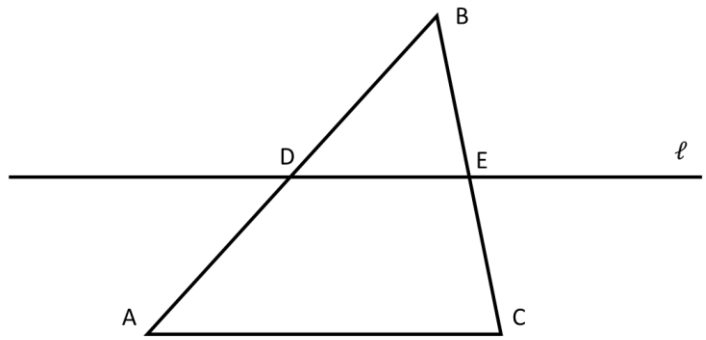
The converse of the previous theorem is also true (as recorded in the following theorem).

Theorem 16. Given a set of lines which are intersected by a transversal, if the corresponding angles are equal, then the lines are parallel.

Proof: See the Proof Wiki article "Equal Corresponding Angles implies Parallel Lines" [9]. ■

Theorem 17 (Triangle Proportionality Theorem or Intercept Theorem). If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Proof: The figure below illustrates the conditions of the theorem.



Triangles ABC and DBE share the angle at B. By Theorem 16, $\angle BDE = \angle BAC$ and $\angle BED = \angle BCA$. So, by the AA triangle similarity principle, triangles ABC and DBE are similar which implies

$$\frac{d(DB)}{d(AB)} = \frac{d(EB)}{d(CB)}$$

If we let $k = \frac{d(AB)}{d(CB)} = \frac{d(DB)}{d(EB)}$, then we have the following relationships

$$d(DB) = k \cdot d(EB)$$

$$d(AB) = k \cdot d(CB)$$

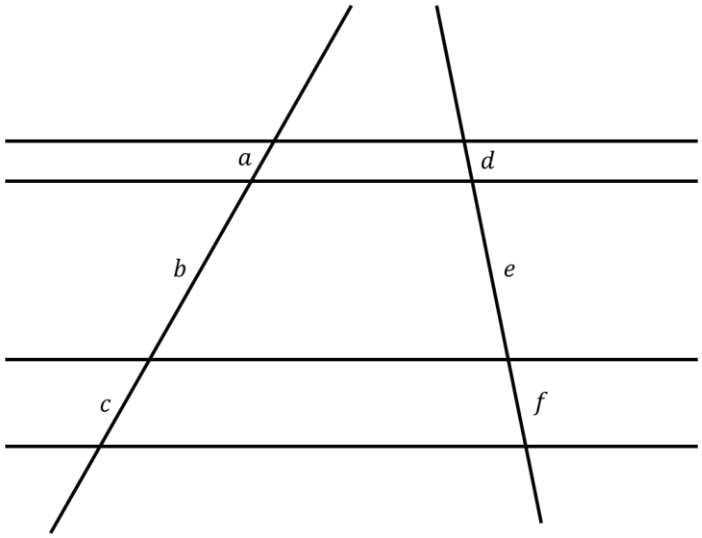
Subtracting the above two equations, we get

$$d(AD) = d(AB) - d(DB) = k\big(d(CB) - d(EB)\big) = k \cdot d(CE)$$

Thus, $\frac{d(AD)}{d(CE)} = k = \frac{d(DB)}{d(EB)}$ which is the result stated in the theorem. ■

The previous theorem can be extended to the case of several transversals intersecting a set of parallel lines. In the following figure, the letters represent distances along the transversal lines. The following equalities hold true:

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$



In general, we have the following theorem.

Theorem 18. If two or more parallel lines are intersected by two transversals, then the parallel lines divide the transversals proportionally.

Proof: See Theorem 16 in Birkhoff [4]. ■

2.5 Theorems about Circles

A **circle** is the set of all points in a plane that are equidistant from a given point which is known as the **center** of the circle. The distance from the center of a circle to the boundary (or **circumference**) of the circle is known as the **radius**.

A line segment joining two points on a circle is called a **chord**. An **arc** is any connected part of a circle. Figure 6 shows a circle with a chord (dashed line) and the smaller of the two arcs associated with the chord (thick curve). A chord that goes through the center of a circle is known as a **diameter**. A diameter of a circle is twice the measure of the radius.

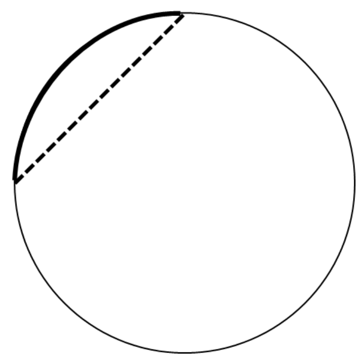
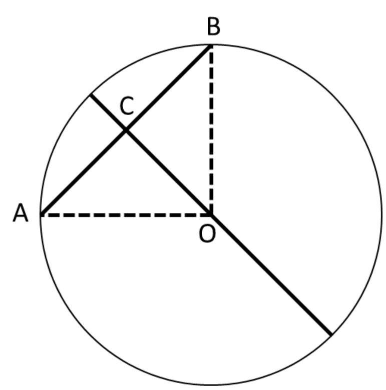


Figure 6. Circle, chord and arc

Theorem 19. A diameter (or radius) of a circle that is perpendicular to a chord (possibly another diameter) bisects the chord.

Proof: Draw radii from the center of the circle O to each of the endpoints of the chord (points A and B), as shown in the figure below.



Thus, $d(OA) = d(OB) = r$ where r is the radius of the circle. The condition of the theorem tells us the $\angle ACO = \angle BCO = 90°$ and we also known that triangles ACO and BCO share the side $\overline{CO}$. By Theorem 10, triangles ACO and BCO are congruent which implies $d(AC) = d(BC)$. ■

A **secant** is a line (as opposed to a segment) that intersects a circle in two points. A **tangent** is a line that intersects a circle in only one point. An example secant and tangent are shown in Figure 7.

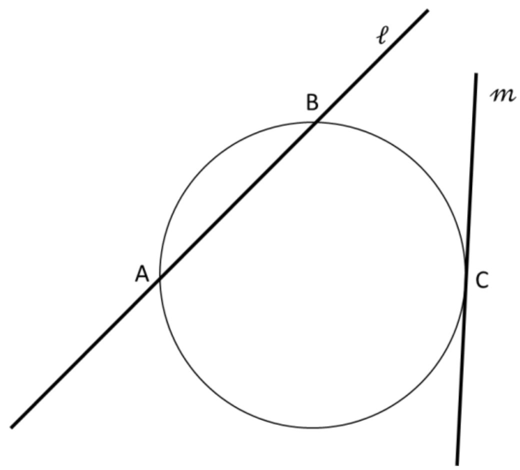
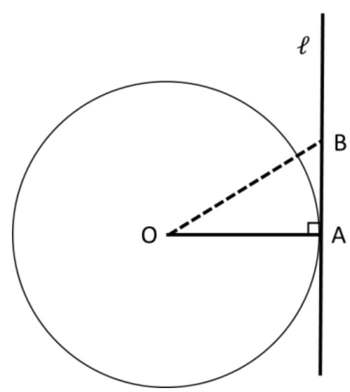


Figure 7. Secant and tangent lines to a circle

Theorem 20. If a line ℓ is perpendicular to a radius of a circle at a point A on the boundary of the circle, then line ℓ is tangent to the circle.

Proof: The situation described in the theorem is shown in the figure below. The little square near point A in the figure indicates a $90°$ angle.



Take any other point on line ℓ (point B in the figure). By the Pythagorean theorem, $d(OA)^2 + d(AB)^2 = d(OB)^2$ but $d(OA) = r$ (the radius of the circle). Thus, $d(OB) > r$ and therefore, B is exterior to the circle. So, A is the only point on ℓ that is also on the circle, i.e., ℓ is a tangent to the circle. ■

The following theorem is essentially the converse of the previous theorem.

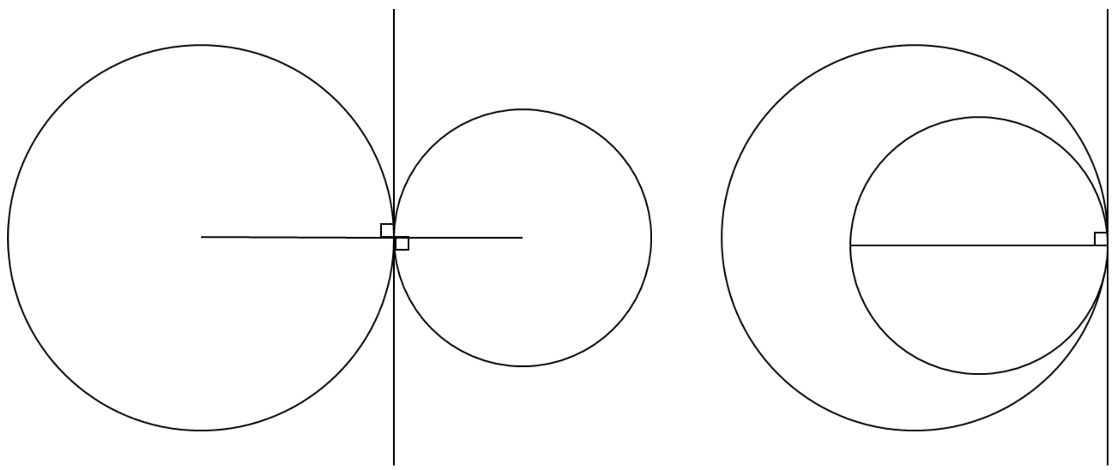
Theorem 21. A tangent to a circle is perpendicular to the radius drawn at the point of contact with the tangent.

Proof: See Theorem 7.3.1 of Africk [10]. ■

It follows from Theorem 21 that there is only one tangent to a circle (with center O) at any given point P on a circle; otherwise, there would be at least two distinct perpendiculars to OP at P, which is not possible.

Theorem 22. If two circles are tangent, their centers and the point of tangency between the two circles are collinear.

Proof: There are two cases, i.e., the circles are exterior to each other (see the left of the figure below) or one circle is inside the other (see the right of the figure below). In either case, we draw radii from the center of each circle to the point of tangency. By Theorem 20, both radii are perpendicular to the tangent line, and thus, the line formed by the centers of the circles and the tangent point form a straight line.



■

An **inscribed angle** is the angle formed in the interior of a circle when two chords intersect on the circle. It can also be defined as the angle subtended at a point on the circle by two given points on the circle.

A **central angle** in a circle is an angle formed by two radii emanating from the center of the circle.

Figure 8 shows the three possible cases for an inscribed angle (solid lines) and the associated central angle (dashed lines). As stated in the following theorem, the measure of an inscribed angle is ½ the measure of its associated central angle. Note the subtlety in Case 2, i.e., the associated central angle is the larger external angle.

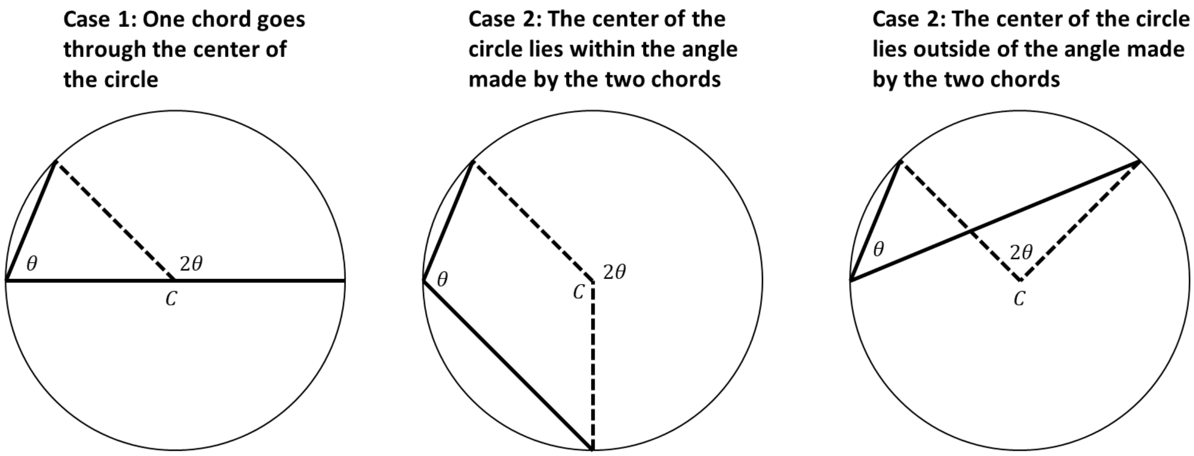


Figure 8. Inscribed angles

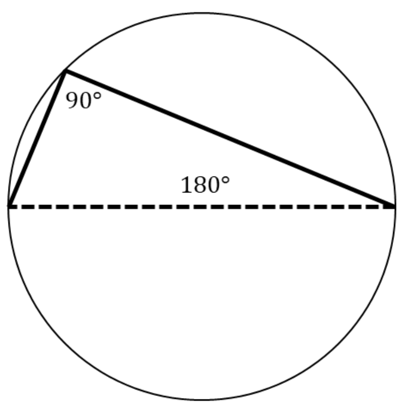
Theorem 23. The measure of an inscribed angle in a circle is half the measure of the associated central angle.

Proof: See the Wikipedia article "Inscribed angle" [12].

The **circumscribed circle or circumcircle** of a triangle is a circle that passes through all three vertices. Such a circle is unique since a circle is uniquely determined by three points (see Theorem 28).

Theorem 24. An angle "inscribed in a semicircle" is a right angle.

Proof: This follows from Theorem 23 for the case where the two endpoints of the chord fall on opposite sides of a diameter, as shown in the figure below. ■



Theorem 24 is an alternate (but equivalent) statement of Thales's theorem [13] which states that if A, B, and C are distinct points on a circle where the line segment AC is a diameter, the angle $\angle ABC$ is a right angle. The converse of Thales's theorem is also true, i.e.,

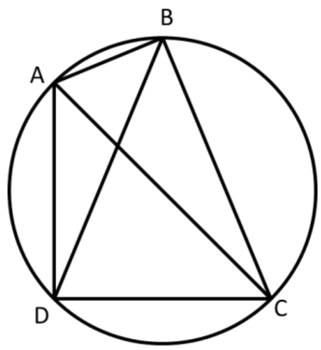
Theorem 25. The center of the circumcircle of a right triangle lies on its hypotenuse. (Equivalently, a right triangle's hypotenuse is a diameter of its circumcircle.)

Proof: See the "Converse" section of the Wikipedia article entitled "Thales's theorem" [13].

Theorem 26 (Inscribed Angle Theorem). Equal angles inscribed in a circle subtend equal arcs. Conversely, inscribed angles in a circle subtending the same arc are equal.

Proof: This follows from Theorem 23. ■

Theorem 26 also works for multiple angles. For example, in the figure below, $\angle ADC$ subtends the same arc as $\angle BAC$ and $\angle ACB$ combined. Thus, $\angle ADC = \angle BAC + \angle ACB$.



Theorem 27. Any given triangle can be circumscribed by a circle.

In other words, given any triangle, one can always find a circle such that the three vertices of the triangle lie on the circle.

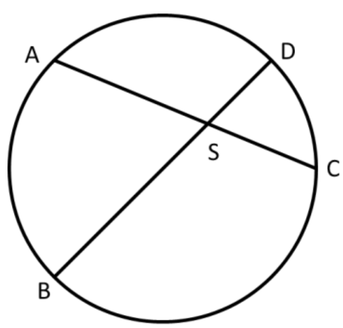
Proof: See the Proof Wiki article "Circumscribing Circle about Triangle" [14].

Theorem 28. Three non-collinear points describe a circle.

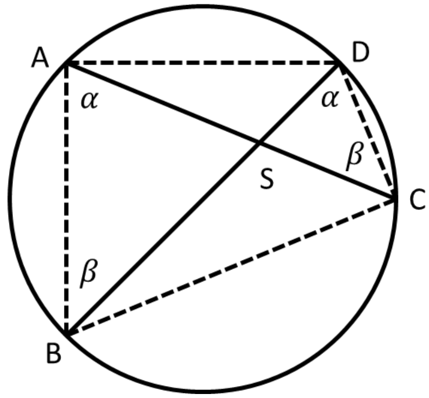
Proof: The three non-collinear points determine a triangle, and the result follows Theorem 27. ■

The intersecting chords theorem is a statement about the four line segments created by two intersecting chords within a circle. Specifically, it states that the products of the lengths of the line segments on each chord are equal. It is Proposition 35 of Book 3 of Euclid's Elements.

Theorem 29 (Intersecting Chords Theorem). If two chords of a circle ($\overline{AC}$ and $\overline{BD}$) intersect at a point S, then $d(AS) \cdot d(SC) = d(BS) \cdot d(SD)$, see the figure below.



Proof: Consider the figure below.



By the Inscribed Angle Theorem, $\angle ABS = \angle DCS = \beta$ since both are inscribed angles on chord AD, and $\angle BAS = \angle CDS = \alpha$ since both are inscribed angles on chord AD.

By the AA triangle similarity theorem, $\triangle ABS \cong \triangle DCS$. Thus, we have the following equality of ratios

$$\frac{d(AS)}{d(BS)} = \frac{d(SD)}{d(SC)}$$

which implies $d(AS) \cdot d(SC) = d(BS) \cdot d(SD)$. ■

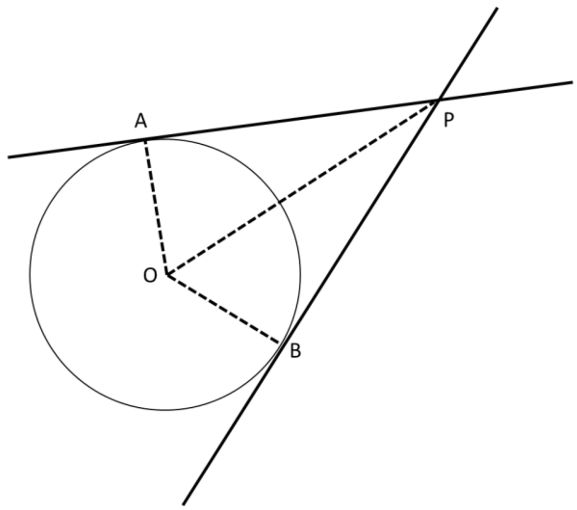
The converse of Intersecting Chords Theorem is also true, i.e.,

Theorem 30. If two line segments $\overline{AC}$ and $\overline{BD}$ intersect at a point S such that $(AS) \cdot d(SC) = d(BS) \cdot d(SD)$, then the points A, B, C, D lie on a common circle.

The following is sometimes referred to as "The Two Tangent Theorem."

Theorem 31. If two tangents to a circle intersect, they make equal angles with the line joining the point of intersection to the center, and their segments from the point of intersection to the points of tangency are equal.

The situation described in the theorem is depicted in the figure below. The theorem states that $\angle APO = \angle BPO$ and $d(AP) = d(BP)$.



Proof: By Theorem 21, $\angle OAP = 90° = \angle OBP$. Further, $\triangle\ OAP$ and $\triangle\ OBP$ share the same hypotenuse, and $d(OA) = d(OB)$ since they are both radii of the circle. By Theorem 10, we have that $\triangle\ OAP = \triangle\ OBP$ and it follows that $\angle APO = \angle BPO$ and $d(AP) = d(BP)$. ■

The **incircle of a triangle** is the circle tangent to each side of the triangle. See the example in Figure 9. The center of the incircle of a triangle is called the **incenter**.

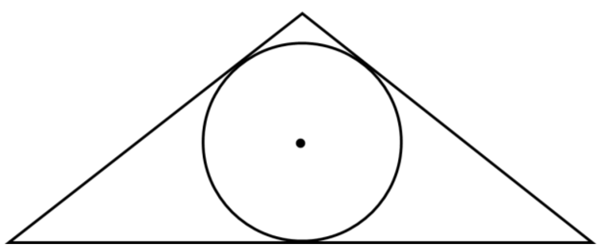
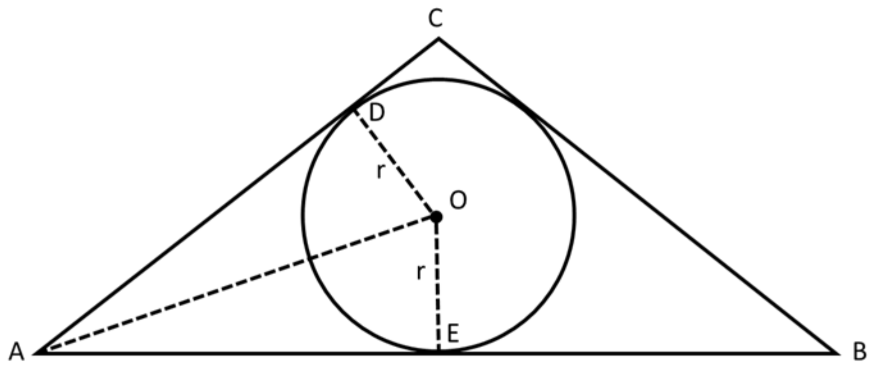


Figure 9. Incircle of a triangle

Theorem 32. The bisectors of the angles of a triangle meet at the incenter of the triangle.

Proof: In the figure below, O is the incenter and r is the radius of the incircle.



$\overline{OD}$ is perpendicular to $\overline{AC}$ (which we write in shorthand as $\overline{OD} \perp \overline{AC}$) and $\overline{OE} \perp \overline{AB}$. By Theorem 31, we have that $\angle DAO = \angle EAO$, i.e., the bisector of the angle at A intersects O. Similarly, the bisectors of $\angle C$ and $\angle B$ also intersect O. ■

So far, we have seen two different points that are in some sense the center of a triangle. Yet another such center-like point is the centroid. The **centroid of a triangle** is the intersection of the segments between one vertex and the midpoint of the opposite side (see Figure 10). The segments from each vertex to the midpoint of the opposite side of a triangle are known as **medians**.

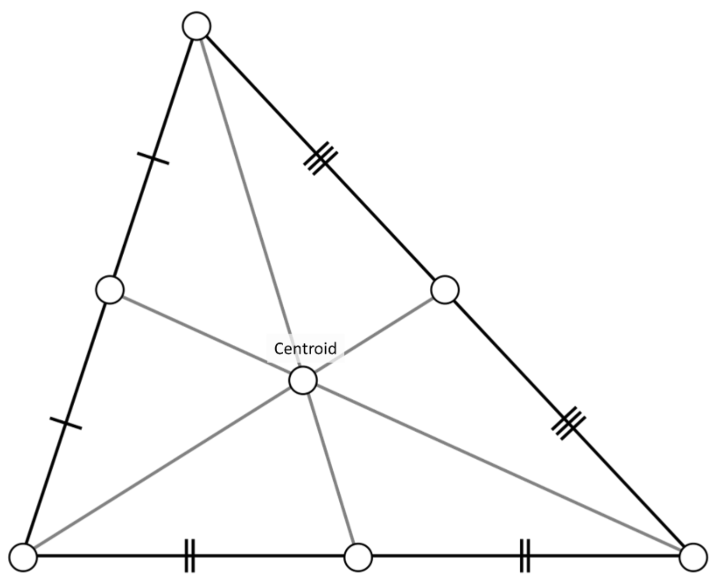
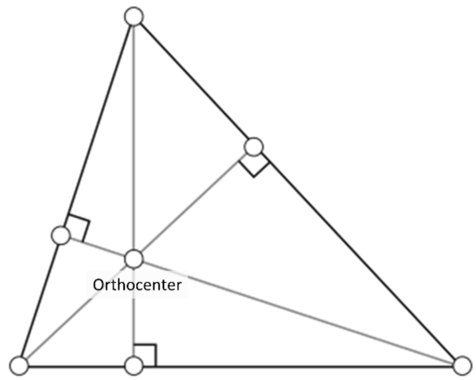


Figure 10. Centroid of a triangle

Theorem 33. The centroid of a triangle divides each median into segments in proportion 2 to 1, with the longer segment being between the vertex and the centroid.

Proof: See the Proof Wiki article “Position of Centroid of Triangle on Median” [15].

There are even more candidates for the center of a triangle [16], but we will discuss just one more, i.e., the orthocenter. The intersection of the altitudes (height from a given side) is defined to be the **orthocenter of a triangle**.



Some of the various centers of a triangle are related by what is called the Euler line. The **Euler line**, named after famous mathematician Leonhard Euler, is a line that passes through the orthocenter, circumcenter and centroid of a non-equilateral triangle. See the Wikipedia article “Euler line” [17] for further details and some examples.

In the case of an equilateral triangle, the Euler line collapses to a point (as further elucidated in the following theorem).

Theorem 34. The orthocenter, centroid and circumcenter of a triangle coincide if and only if the triangle is equilateral.

Proof: See the Proof Wiki article “Orthocenter, Centroid and Circumcenter Coincide iff Triangle is Equilateral” [18].

. . .

A **sector of a circle** is a region bounded by two radii of equal length with a common center and either of the two possible arcs, determined by this center and the endpoints of the radii. In the figure below, the sector subtends the smaller of two possible arcs.

A **segment of a circle** is a region bounded by a chord and one of the arcs connecting the chord's endpoints, see the example below.

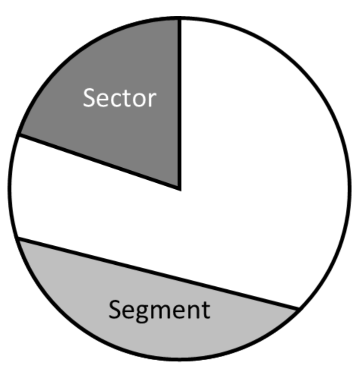


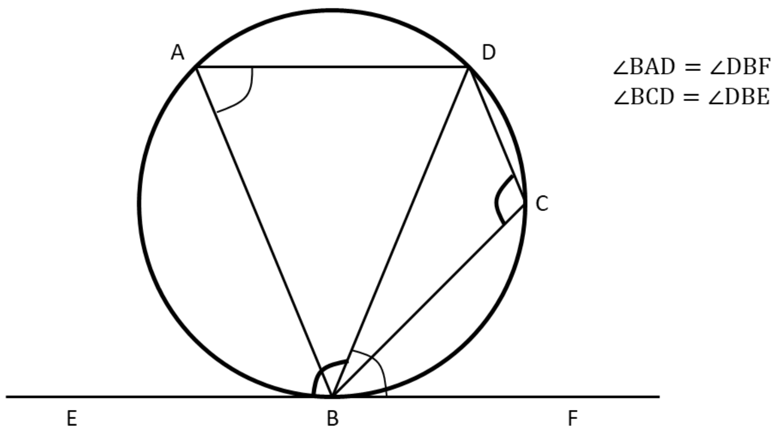
Figure 11. Example of a sector and a segment in a circle

. . .

Theorem 35 (Alternate Segment Theorem) If line EF is tangent to a circle (touching it at point B), BD is a chord of the circle, and A and C are points on the circle (on either side of chord BD), then $\angle BCD = \angle DBE$ and $\angle BAD = \angle DBF$.

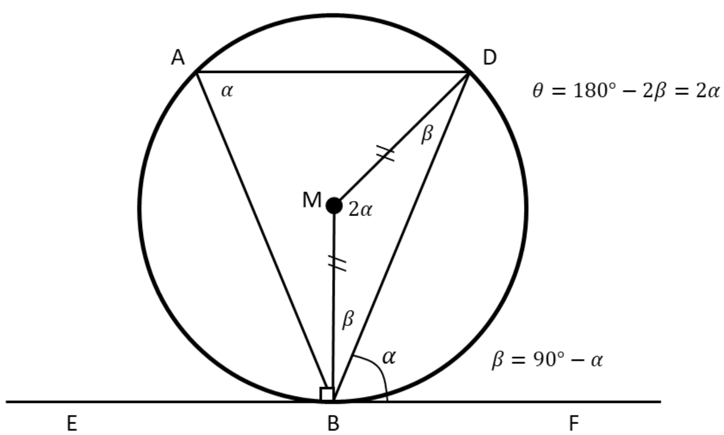
This is also known as the tangent-chord theorem.

The situation is shown in the figure below. The perhaps surprising fact is that point A can be anywhere on one side of chord BD, C can be anywhere on the other side of chord BD, and the theorem still holds true.



Solution: The proof is identical for points A and C. So, we just show the proof in the case of point A.

In the figure below, we have drawn a radius from the center of the circle to points B and D. Consequently, BMD is an isosceles triangle which implies $\angle MDB = \angle MBD = \beta$ where $\beta = 90° - \alpha$. In addition, $\angle DMB = 180° - 2\beta = 180° - 2(90° - \alpha) = 2\alpha$. By Theorem 23, $\angle BAD = \frac{1}{2}\angle BMD = \alpha$ which is what we wanted to prove.



. . .

The mathematical quantity π is defined to be the ratio of the circumference C of a divided by twice its radius r, i.e., $\pi = \frac{C}{2r}$. ■

Theorem 36. The area of a circle with radius r is given by the expression πr^2.

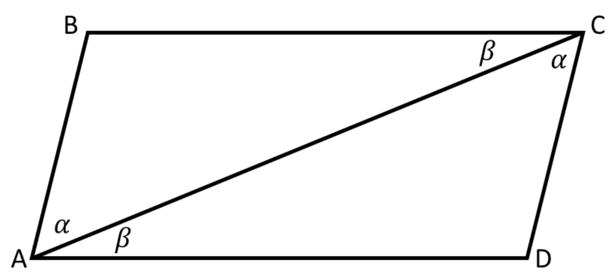
Proof: The article "Area of Circle" from Proof Wiki provides several proofs. Most of the proofs involve integral calculus. The proof by Kepler and the proof based on inscribed circles are not explicitly based on calculus but they do use the concept of a limit. ■

2.6 Theorems about Polygons

A **parallelogram** is a quadrilateral whose opposite sides are parallel to each other. **Squares** (quadrilaterals with all equal sides and $90°$ interior angles) and **rhomboids** (quadrilaterals with all equal sides such that opposite sides are parallel) are types of parallelograms. A **trapezoid** in North American English, or trapezium in British English, is a quadrilateral that has at least one pair of parallel sides.

Theorem 37. The opposite sides and angles of a parallelogram are equal to one another.

Proof: By Theorem 15, we have that $\angle BAC = \angle ACD = \alpha$ and $\angle CAD = \angle BCA = \beta$ (as shown in the figure below). This immediately gives us $\angle A = \alpha + \beta = \angle C$.



By the ASA triangle congruence principle, $\triangle ABC = \triangle CDA$ which implies $\angle B = \angle D, d(BC) = d(DA)$ and $d(AB) = d(DC)$. ■

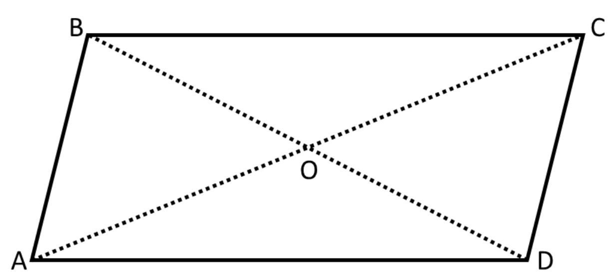
The converse of the previous theorem is true.

Theorem 38. If the opposite sides of a convex quadrilateral are equal, then the quadrilateral is a parallelogram.

Proof: See the Proof Wiki article "Opposite Sides Equal implies Parallelogram" [20]. ■

Theorem 39. The diagonals of parallelogram bisect each other.

Proof: A diagonal of a parallelogram is a line segment between opposite vertices, see dotted lines in the figure below. Our task is to prove that $d(BO) = d(DO)$ and $d(AO) = d(CO)$.



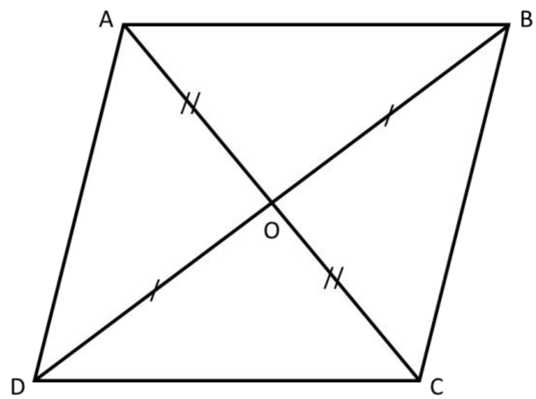
By Theorem 15, $\angle OAD = \angle OCB$ and $\angle ODA = \angle OBC$. By Theorem 37, $d(BC) = d(DA)$. By the ASA triangle congruence principle, $\triangle AOD = \triangle COB$ which implies $d(BO) = d(DO)$ and $d(AO) = d(CO)$. ■

A **rhombus** is a parallelogram whose sides are of equal length.

Theorem 40. The diagonals of a rhombus bisect each other at right angles.

Proof: From Theorem 39, we know that the diagonals of a parallelogram (and thus, also a rhombus) bisect each other.

In the generic rhombus in the figure below, we have the $\triangle ODA = \triangle OBA$ by the SSS triangle congruence principle. So, $\angle AOD = \angle AOB$ but $\angle AOD + \angle AOB = 180°$ since DB is a straight line. Thus, we conclude the $\angle AOD = \angle AOB = 90°$.



■

Theorem 41. The midpoints of the sides of a quadrilateral form a parallelogram.

The quadrilateral does not need to be convex (e.g., see Figure 12) but the sides of the quadrilateral cannot intersect.

Proof: See the Proof Wiki article "Midpoints of Sides of Quadrilateral form Parallelogram" [21]. ■

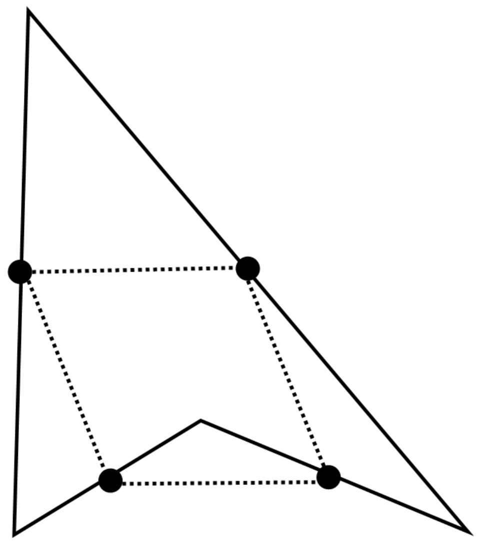


Figure 12. Parallelogram formed by the midpoints of a quadrilateral

The **internal angle of a vertex of a polygon** is the measure of the angle between the sides adjacent to that vertex, as measured inside the polygon. [22]

Contrary to intuition, the **external angle of a vertex of a polygon** is not the measure of the angle between the sides forming that vertex, as measured outside the polygon. An external angle is in fact an angle formed by one side of a polygon and a line produced from an adjacent side. [23]

The above definitions apply to convex and non-convex polygons, but not self-intersecting polygons.

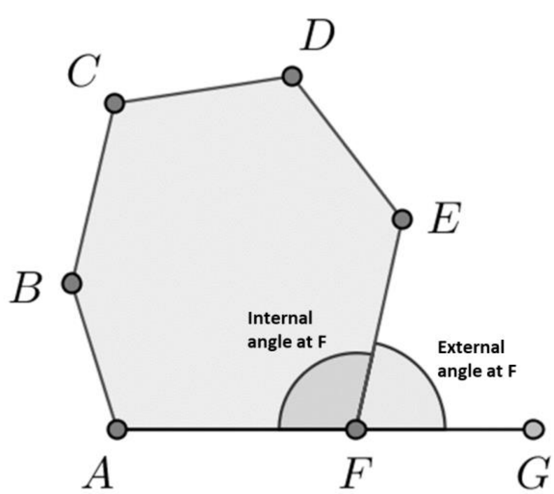


Figure 13. Internal and external angles at a vertex of a polygon.

The following two theorems concern the internal and external angles of simple polygons. A **simple polygon** is defined to be non-self-intersecting – it can be convex or non-convex.

Theorem 42. The sum of all internal angles of a polygon with n sides is given by the expression $(n-2)180°$ or $(n-2)\pi$ (in radians).

Proof: See the Proof Wiki article "Sum of Internal Angles of Polygon/Proof 1" [32]. ■

A regular polygon is a polygon which is both equilateral and equiangular.

Theorem 43. If is $\mathcal{P}$ a regular polygon with n sides, then each of the exterior angles of $\mathcal{P}$ is equal to $\frac{360°}{n}$ or $\frac{2\pi}{n}$ (in radians).

Proof: See the Proof Wiki article "Exterior Angle of Regular Polygon" [33]. ■

. . .

A **kite** is a quadrilateral with reflection symmetry across one of its diagonals. Equivalently, it is a quadrilateral whose four sides can be grouped into two pairs of adjacent equal-length sides. Kites can be either convex or concave, see Figure 14.

The source for the definition of kite and the associated figure is the Wikipedia article on kites [28]. The circles in the figure are not part of the kites. They were added in the Wikipedia article to emphasize a property of kites, i.e., for every convex kite there exists an inscribed circle that is tangent to all four sides. With a slight variation (as suggested in the figure), the property also holds for concave kites.

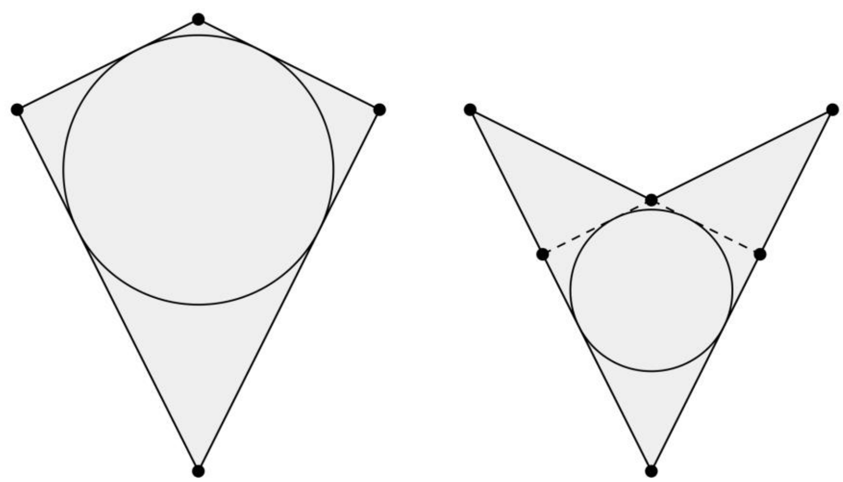


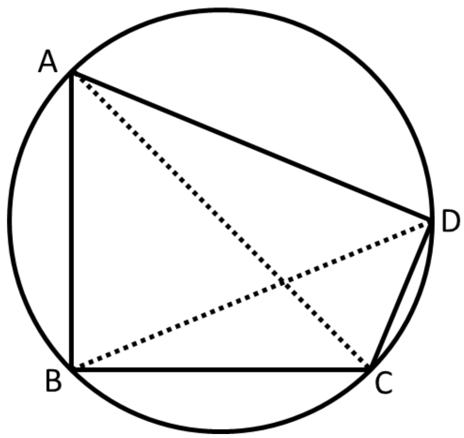
Figure 14. Convex and concave kites

Ptolemy's theorem [34] is a relation between the four sides and two diagonals of a **cyclic quadrilateral** (a quadrilateral whose vertices lie on a common circle).

Theorem 44 (Ptolemy's theorem). If a quadrilateral is cyclic then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

In terms of the figure below, Ptolemy's theorem states that

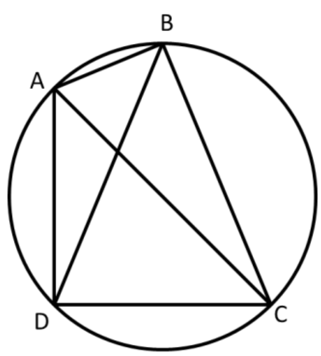
$$d(AC) \cdot d(BD) = d(AB) \cdot d(CD) + d(AD) \cdot d(BC)$$



Proof: The Wikipedia article "Ptolemy's theorem" [34] provides several proofs. ■

Theorem 45. The opposite angles of a cyclic quadrilateral sum to $\mathbf{180}°$***.***

Proof: In the figure below, we have that $\angle ADC = \angle BAC + \angle ACB$ by Theorem 26.



Add $\angle ABC$ to both sides of the equation that we derived above to get

$$\angle ADC + \angle ABC = \angle BAC + \angle ACB + \angle ABC = 180°$$

The last equality holds since $\angle BAC, \angle ACB, \angle ABC$ are the angles of a triangle.

Similarly, we can show that $\angle DAB + \angle DCB = 180°$. ■

The converse of Theorem 45 is also true.

Theorem 46. If the opposite angles of the convex quadrilateral are supplementary, then it is cyclic.

Proof: See the StackExchange article "Prove the opposite angles of a quadrilateral are supplementary implies it is cyclic" [35].

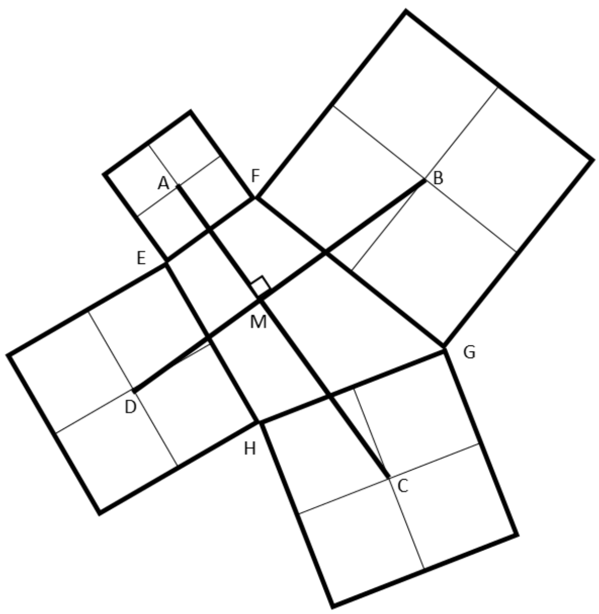
. . .

Theorem 47 (Van Aubel's theorem). Given any convex quadrilateral, if one draws a square on each side of the quadrilateral, then the two line segments between the centers of opposite squares are of equal lengths and perpendicular to each other.

Proof: See the Art of Problem Solving article on this topic [36]. ■

Van Aubel's theorem also holds for non-convex quadrilaterals and self-intersection quadrilaterals, but the construction of the associated squares needs to follow specific rules, as described in the Wikipedia article "Van Aubel's theorem" [37].

As an illustration of Van Aubel's theorem, consider the figure below. In the figure, four squares are draw along the sides of quadrilateral $EFGH$. If we connect the centers of squares on opposite sides of the quadrilateral, we get two perpendicular segments ($\overline{AC}$ and $\overline{BD}$) that are of equal length.



2.7 Area

In the Birkhoff development of geometry, two assumptions (axioms) are made concerning the concept of area.

Area Axiom 1. Every polygon has a number, called its area, such that

- equal polygons have equal areas, and
- the area of a polygon is equal to the sum of the areas of its constituent polygons.

The second item in Axiom 1 is quite powerful because all simple polygons can be decomposed into triangles (the areas of which we can compute using Theorem 50). For more background on polynomial triangulation, see the Wikipedia article "Polygon triangulation" [38].

Theorem 48. If $\mathcal{P}$ is a polygon with $n \geq 3$ sides, then there exists a triangulation of $\mathcal{P}$ that fulfills the following condition:

> ***If $\overline{AB}$ is a side of a triangle in the triangulation of $\mathcal{P}$, then $\overline{AB}$ is either a side of $\mathcal{P}$, or a chord of $\mathcal{P}$ that lies completely in the interior of $\mathcal{P}$.***

Further, all triangulations of $\mathcal{P}$ that fulfill this condition consist of exactly $n - 2$ triangles.

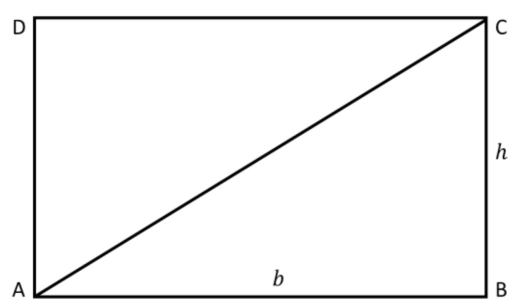
Proof: See the Proof Wiki article entitled "Polygon Triangulation Theorem" [39]. ■

Definition: The area of a square whose sides are one unit in length is defined to be a **unit of area**.

Area Axiom 2. The area of a rectangle is equal to the product of its length times its width.

Theorem 49. The area of a right triangle is equal to half the product of one side (the base) times the altitude (height) upon that side.

Proof: Starting with right triangle ABC (with right angle at B), append another right triangle ADC (with right angle at D) such that the two triangles form a rectangle, as shown in the figure below.



By Area Axiom 2, the area of the rectangle is bh. By the SAS triangle similarity principle, triangles ADC and ABC are similar with proportion $k = 1$, i.e., they are congruent. Thus, the area of triangle ABC (as well as that of ADC) is one half that of the rectangle, i.e., $\frac{1}{2}bh$. ■

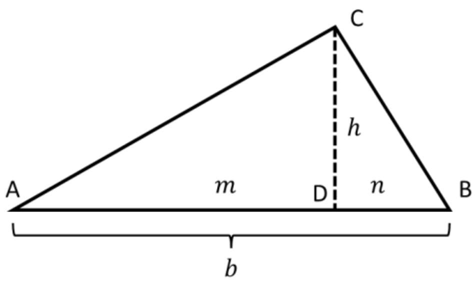
The area formula holds for any triangle (not just right triangles) as stated in the following theorem.

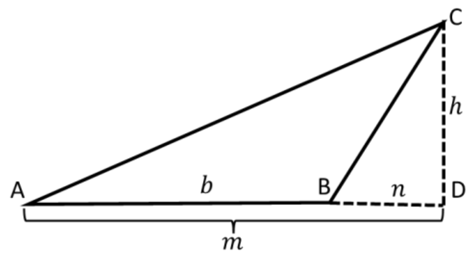
Theorem 50. The area of a triangle is equal to half the product of one side times the altitude upon that side.

Proof: The theorem is proven by reducing the problem to two right triangles within a non-right triangle. There are two cases, as shown in the figure below.

On the left, we drop a perpendicular segment from point C down to the base of the triangle. This leaves us with two right triangles, the sum of whose areas equal that of the triangle ABC.

On the right, we also drop a perpendicular down from C which intersects line AB at a point outside of triangle ABC. This slightly changes the calculation.





$$Area(\triangle ABC) = \frac{1}{2}mh + \frac{1}{2}nh = \frac{1}{2}bh$$

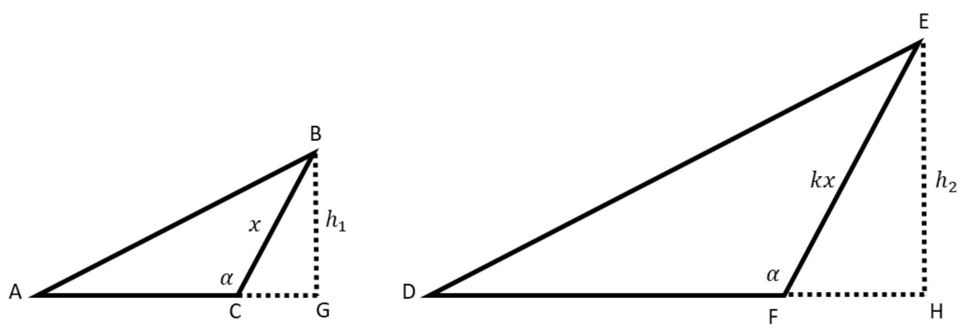
$$Area(\triangle ABC) = \frac{1}{2}mh - \frac{1}{2}nh = \frac{1}{2}bh$$

■

Notation: For many of the puzzles that follow, we represent the area of triangle ABC by the expression [ABC], and the area of the quadrilateral ABCD by the expression [ABCD].

Theorem 51. If the sides of two similar triangles are in proportion k, so are their heights.

Proof: Let ABC and DEF be two similar triangles whose sides are in proportion k, as shown in the figure below.



Let h_1 be the height of triangle ABC from base AC, and h_2 be the height of triangle DEF from base DF. Let G be the point where the altitude of ABC intersects the line AC, and H be the point where the altitude of DEF intersects line DF. By the AA triangle similarity principle, triangles CBG and FEH are similar triangles whose sides are in proportion k. Thus, $h_1 = kh_2$. ■

The following is a simple concept but easy to miss when solving a puzzle, e.g., see Puzzle 56.

Theorem 52. If two triangles have the same height, the ratio of their areas is equal to the ratio of their bases.

Proof: Let b_1 be the measure of the base of one triangle, and b_2 be the measure of the base of the other triangle. As stated in the theorem, both triangles have the same height (call it h). The ratio of the two areas is

$$\frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}$$

■

In a variation of the previous theorem, we have the following result.

Theorem 53. If two triangles are similar, the ratio of their areas is equal to the square of the ratio of any two corresponding sides.

Proof: Consider two similar triangles, $\triangle ABC$ and $\triangle DEF$, with corresponding sides AB, BC, CA, DE, EF and FD. Since the triangles are similar, we have that:

$$\frac{d(AB)}{d(DE)} = \frac{d(BC)}{d(EF)} = \frac{d(CA)}{d(FD)} = k$$

where k is the scale factor between the triangles.

Using the formula for the area of a triangle, and using BC and EF as the bases of the triangle, we have

$$[ABC] = \frac{1}{2} \cdot d(BC) \cdot h_{BC}$$

$$[DEF] = \frac{1}{2} \cdot d(EF) \cdot h_{DF}$$

where h_{BC} and h_{DF} are the heights of $\triangle ABC$ and $\triangle DEF$ with respect to based BC and EF, respectively.

Since the triangles are similar, we have by Theorem 51 that the ratio of the corresponding heights is also equal to the scale factor k, i.e.,

$$\frac{h_{BC}}{h_{DF}} = k$$

Substituting this expression into the formula for the area of triangle ABC, we have

$$[ABC] = \frac{1}{2} \cdot d(BC) \cdot kh_{DF}$$

Dividing these two equations for the areas of the triangles, we get

$$\frac{[ABC]}{[DEF]} = \frac{d(BC)}{d(EF)} \cdot k = k^2$$

This proves that the ratio of the areas of two similar triangles is equal to the square of the ratio of any two corresponding sides. ■

Theorem 54. If two triangles are similar and the ratio of their areas is x, then the ratio of their sides is $\sqrt{x}$.

Proof: If the ratio between the sides of the two triangles is k, then the ratio of their heights is also k, by Theorem 51. Thus, the ratio of the areas of the two triangles is

$$x = \frac{\frac{1}{2}(kb)(kh)}{\frac{1}{2}bh} = k^2$$

$$k = \sqrt{x}$$

In the above calculation, b and h are the base and height of one of the triangles. ■

Heron's (or Hero's) formula gives the area of a triangle in terms of the length of its sides. The formula is credited to Heron (or Hero) of Alexandria (known to have been active around 60 AD). Mathematical historian Thomas Heath suggested that Archimedes knew of the formula over two centuries earlier.

Theorem 55 (Heron's Formula). If a triangle has side lengths a, b, c then the area of the triangle is given by the expression

$$\sqrt{p(p-a)(p-b)(p-c)} \text{ where } p = \tfrac{1}{2}(a+b+c)$$

Proof: The Wikipedia article "Heron's formula" [40] provides several proofs. The Proof Wiki article "Heron's Formula" provides four different proofs of the formula [41]. ■

For example, take a 3, 4, 5 triangle (which we already know to be a right triangle by the converse of the Pythagorean). The area is $\frac{1}{2}(3)(4) = 6$. Let's check this via Heron's formula. We have that $p = \frac{1}{2}(3+4+5) = 6$ and $\sqrt{p(p-a)(p-b)(p-c)} = \sqrt{6(3)(2)(1)} = 6$. However, the true utility of Heron's formula is that it works for any type of triangle as long as you know the side lengths. ■

Brahmagupta's formula [42], named after the 7th century Indian mathematician, is used to find the area of any **cyclic quadrilateral** (i.e., one that can be inscribed in a circle) given the lengths of the sides. Heron's formula can be viewed as a special case of the Brahmagupta's formula for triangles (with $d = 0$ in the following formula, where two vertices of the quadrilateral collapse to form a triangle).

Theorem 56 (Brahmagupta's formula). If a cyclic quadrilateral has side lengths a, b, c, d then its area is given by the expression

$$\sqrt{(p-a)(p-b)(p-c)(p-d)} \text{ where } p = \tfrac{1}{2}(a+b+c+d)$$

Proof: The Wikipedia article "Brahmagupta's formula" [42] provides several proofs. ■

The area of a triangle circumscribed by a circle is related to the radius of the circle (known as the **circumradius**) in the following theorem. We use this theorem in the solution to Puzzle 23.

Theorem 57. If $\triangle ABC$ is a triangle with side lengths a, b and c opposite vertices A, B and C, respectively, then the area $\mathcal{A}$ of $\triangle ABC$ is given by the formula

$$\mathcal{A} = \frac{abc}{4R}$$

where R is the radius of the circle that circumscribes $\triangle ABC$.

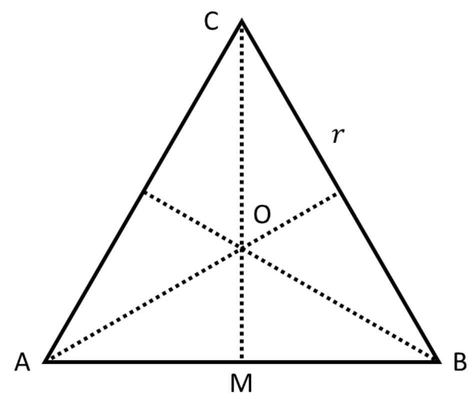
Proof: See the Proof Wiki article "Area of Triangle in Terms of Circumradius" [43].

3 Triangle Puzzles

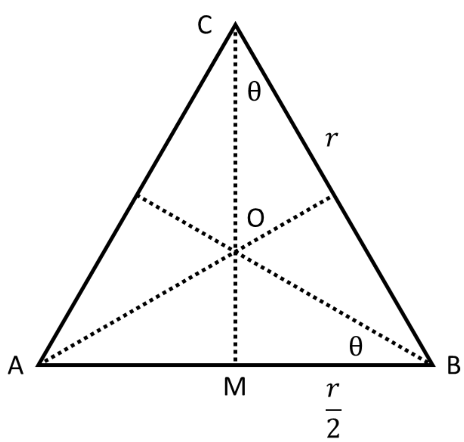
"Geometry is the foundation of all painting." – Albrecht Dürer

Puzzle 1. Distance from midpoint of equilateral triangle to midpoint of side

Prove that the distance from the midpoint of an equilateral triangle (i.e., the intersection of lines from each vertex to the midpoint of the opposite side) to any side of the triangle is half the distance between the midpoint and a vertex. In terms of the figure below, prove $d(OM) = \frac{1}{2}d(OB)$ where r is the length of a side of the triangle.



Solution: The solution is illustrated in the figure below.



By the Phythagorean theorem, $d(CM) = \frac{\sqrt{3}}{2}r$

By AA, $\triangle CMB \cong \triangle BMO$

Thus, $\frac{d(BO)}{d(CB)} = \frac{d(MB)}{d(CM)} \Rightarrow d(BO) = \frac{\sqrt{3}}{3}r$

By the Pythagorean theorem,

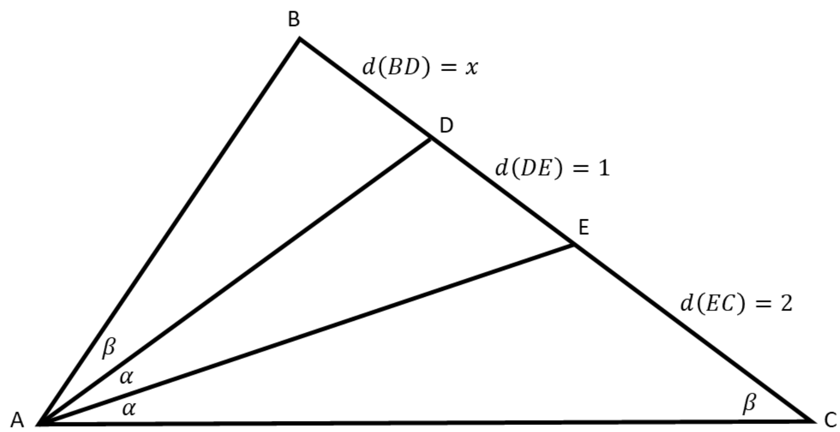
$$d(MO)^2 + d(MB)^2 = d(BO)^2$$

$$d(MO)^2 + \frac{r^2}{4} = \frac{r^2}{12}$$

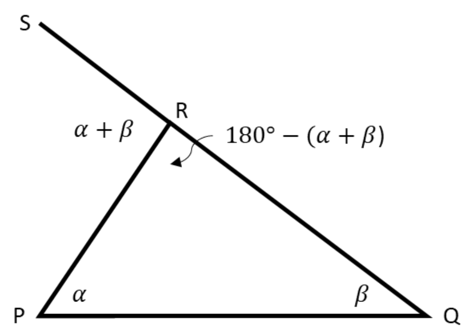
$$d(MO) = \frac{\sqrt{3}}{6}r = \frac{1}{2}d(OB)$$

Puzzle 2. Find the length of the unknown segment of a triangle

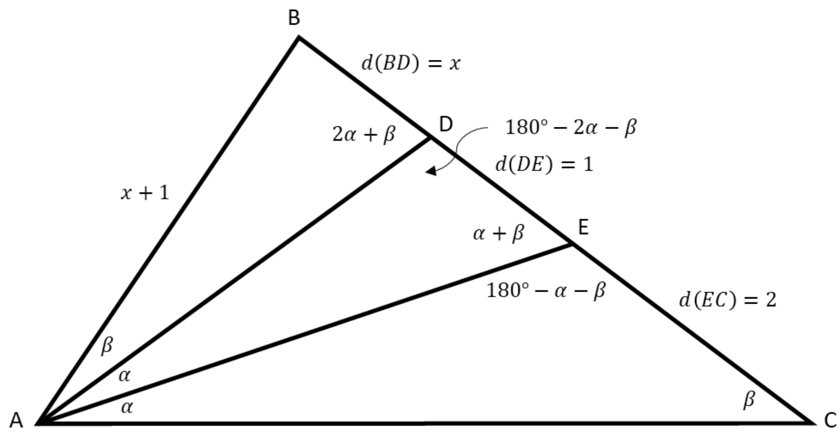
Given the various angle relationships and segment measures in the figure below, find the length of x.



Solution: We first need a preliminary result that we will use several times in the solution to the problem. The exterior angle at one vertex of a triangle is equal to the sum of the interior angles at the other two vertices, as shown in the figure below. This is because SRQ is a straight line and therefore, the angles $\angle SRP$ and $\angle QRP$ must add to $180°$.



Using our preliminary result, we have that $\angle DEA = \alpha + \beta$ and $\angle BDA = \alpha + 2\beta$. In turn, we see that triangle ABE is isosceles and thus, $d(AB) = d(BE) = d(BD) + d(DE) = x + 1$.



By the AA triangle similarity principle, $\triangle BAC \cong \triangle BDA$ which gives us the following ratio equivalence

$$\frac{d(BA)}{d(BC)} = \frac{d(BD)}{d(BA)}$$

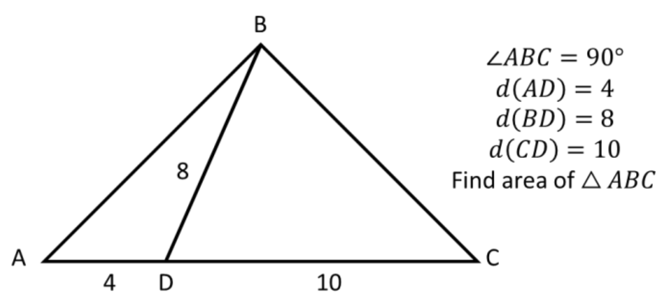
$$\frac{x+1}{x+3} = \frac{x}{x+1}$$

$$x^2 + 2x + 1 = x^2 + 3x$$

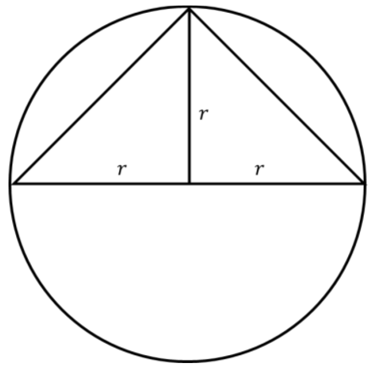
$$x = 1$$

Puzzle 3. Find the area of a triangle using Heron's formula

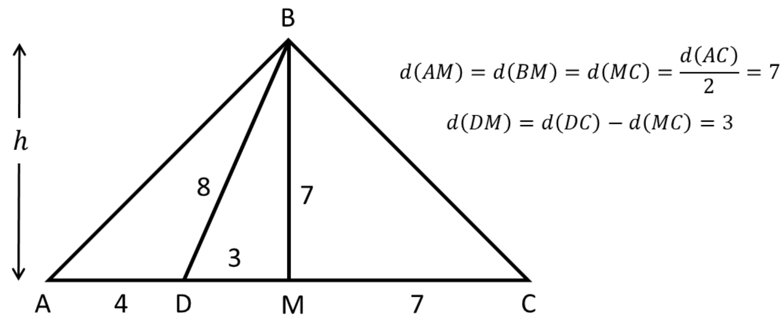
Find the area of a right triangle ABC given the information shown in the figure below.



Solution: In general, any triangle can be circumscribed by a circle (see Theorem 27). If we circumscribe a right triangle by a circle, then the line segment from the vertex corresponding to the right angle to the midpoint of the hypotenuse is a radius of the circle. See the figure below.



Applying the above concept to the problem at hand, let M be the midpoint of segment $\overline{AC}$. With this insight, we now know the lengths of all three sides of triangle BDM.



Next, we apply Heron's formula (Theorem 55) to triangle BDM. The value of p in the formula is $\frac{8+3+7}{2} = 9$, and so the area of the triangle is given by $\sqrt{p(p-a)(p-b)(p-c)} = \sqrt{9(1)(6)(2)} = 6\sqrt{3}$.

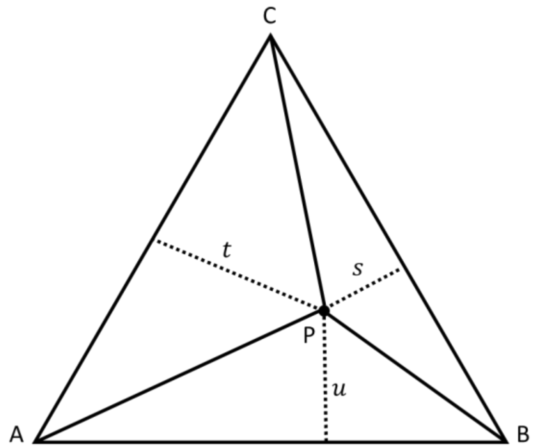
We can compute the area of triangle BDM a second way, using the height of triangle ABC as measure from the base AC, i.e., $[BDM] = 6\sqrt{3} = \frac{1}{2}h \cdot d(DM) = \frac{3}{2}h$ which implies $h = 4\sqrt{3}$. We now have sufficient information to calculate the area of triangle ABC, i.e., $\frac{1}{2}(4\sqrt{3})(14) = 28\sqrt{3}$.

Puzzle 4. Viviani's Theorem [48]

Prove that the sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude (height).

Solution: Let ABC be an equilateral triangle whose height is h and whose sides are of length a.

Let P be any point inside the triangle, and let u, s, t be the distances of P from the sides. Draw a line from P to each of A, B, and C, forming three triangles PAB, PBC, and PCA (as shown in the figure below).



The areas of triangles PAB, PBC and PCA are $\frac{1}{2}ua$, $\frac{1}{2}sa$ and $\frac{1}{2}ta$, respectively. The areas of the three triangles equals the area of the triangle ABC, and so,

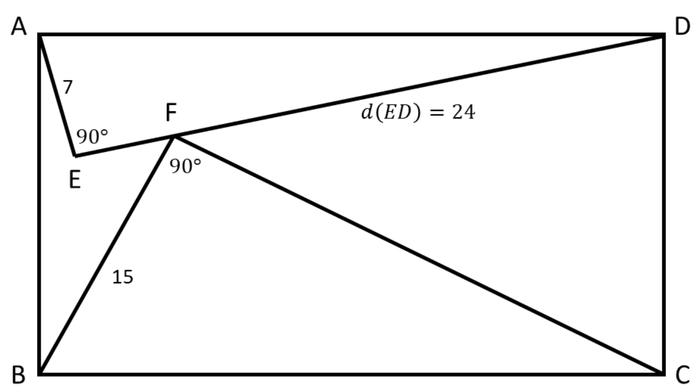
$$\frac{1}{2}ua + \frac{1}{2}sa + \frac{1}{2}ta = ah$$

$$h = u + s + t$$

The converse of this theorem also holds true, i.e., if the sum of the distances from an interior point of a triangle to the sides is independent of the location of the point, the triangle is equilateral [49].

Puzzle 5. Find the length of one side of a rectangle

For the configuration shown in the figure below, find the length of side AB in rectangle ABCD.



Source: YouTube video entitled "How To Solve The Rectangle Length Puzzle - Think INSIDE The Box!" [24].

Solution: Since we are given the length of two sides of triangle AED, we can use the Pythagorean theorem to find the length of the other side, i.e.,

$$7^2 + 24^2 = d(AD)^2$$

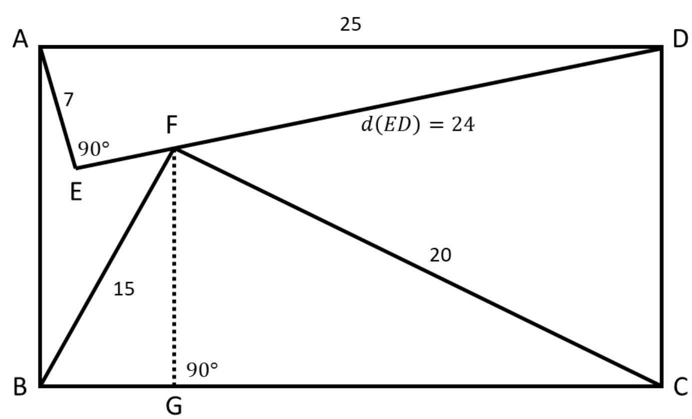
$$d(AD) = \sqrt{576 + 49} = \sqrt{625} = 25$$

Since ABCD is a rectangle, we also know that $d(BC) = 25$. So, we have sufficient information to solve for the other side of triangle BCF, i.e.,

$$15^2 + d(CF)^2 = 25^2$$

$$d(CF) = \sqrt{625 - 225} = \sqrt{400} = 20$$

Next, draw a perpendicular line segment from F to $\overline{BC}$, as shown in the figure below.



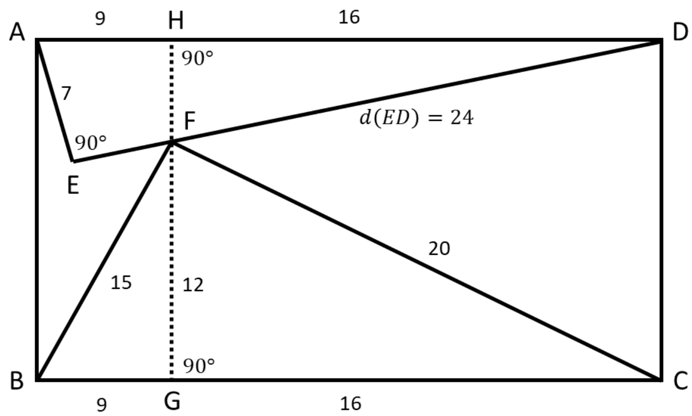
By the AA triangle similarity principle, $\triangle BFC \cong \triangle FGC$. Thus, we have the following ratio equalities:

$$\frac{d(GF)}{d(FC)} = \frac{d(BF)}{d(BC)}$$

$$\frac{d(GF)}{20} = \frac{15}{25} \Rightarrow d(GF) = 12$$

Using the Pythagorean theorem, we determine that $d(GC) = 16$, and so, $d(BG) = 25 - 16 = 9$.

Extend the line segment $\overline{FG}$ so that it touches $\overline{AD}$ at point H, and note that $d(AH) = d(BG) = 9$ and $d(HD) = d(GC) = 16$. See the updated quantities in the figure below.



By the AA triangle similarity principle, $\triangle AED \cong \triangle FHD$. Thus, we have the following ratio equalities:

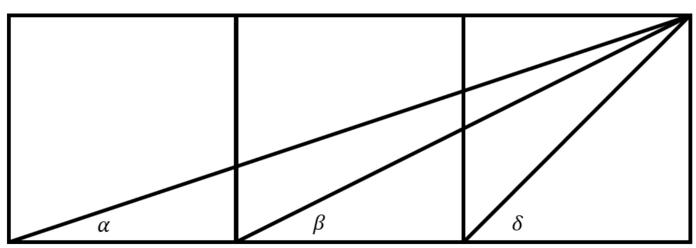
$$\frac{d(HF)}{d(HD)} = \frac{d(AE)}{d(ED)}$$

$$\frac{d(HF)}{16} = \frac{7}{24} \Rightarrow d(HF) = \frac{14}{3}$$

So, $d(AB) = d(GH) = d(GF) + d(FH) = 12 + \frac{14}{3} = \frac{50}{3}$.

Puzzle 6. Three squares and three angles, find the sum

Three identical squares (with side length 1) are lined-up in a row as shown in the following diagram. Find the sum of the three angles α, β and δ.

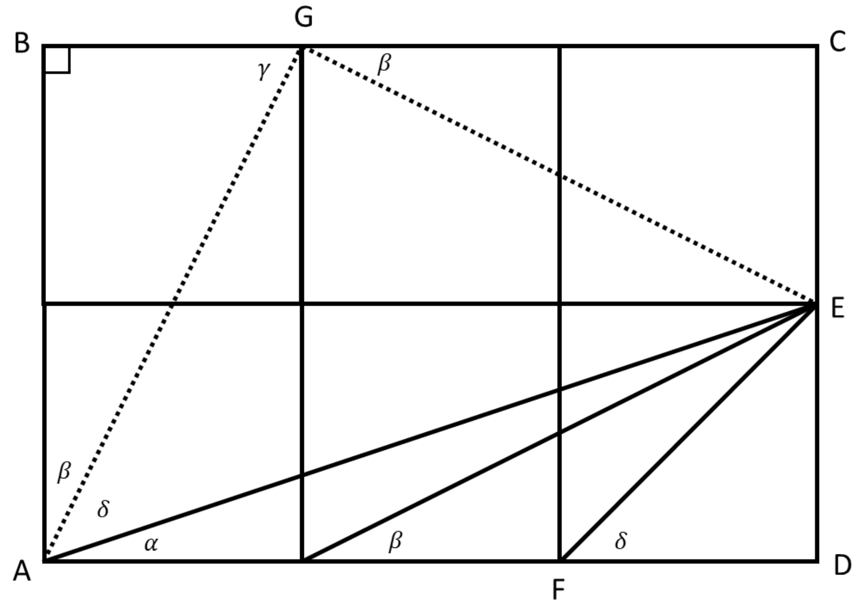


Solution: Draw lines segments $\overline{GE}$ and $\overline{AG}$, as shown in the figure below.

From triangle ABG, we see the $\beta + \gamma = 90°$. Since BG is a straight line, $\beta + \gamma + \angle AGE = 180°$ and since $\beta + \gamma = 90°$, we have that $\angle AGE = 90°$.

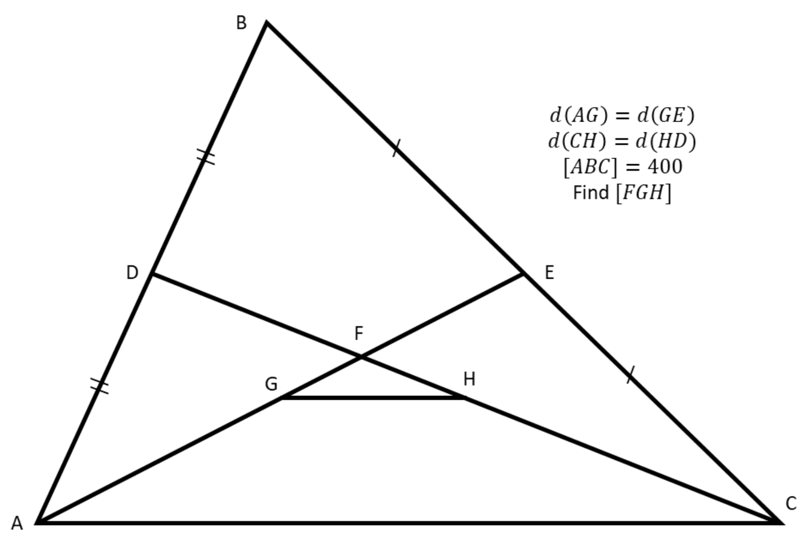
Since $\overline{GE}$ and $\overline{AG}$ are diagonals of rectangles of dimension 2×1, the two segments are of the same length, i.e., $\sqrt{5}$ units. Thus, $\triangle AGE$ is an isosceles triangle.

By the SAS triangle similarity principle, $\triangle AGE \cong \triangle EDF$, and thus, $\angle GAE = \delta$. In conclusion, $\angle BAD = 90° = \alpha + \beta + \delta$.



Puzzle 7. Find the area of the small triangle within a triangle

In the figure below, point D bisects segment $\overline{AB}$ and point E bisects segment $\overline{BC}$. Various segments are of the same length as indicated by the hash marks on the figure. Also, $d(AG) = d(GE), d(CH) = d(HD)$ and the area of triangle ABC is 400. Find the area of the small triangle *GFH*. The figure is not draw exactly to scale, and do not assume that $\overline{GH}$ is necessarily parallel to $\overline{AC}$.

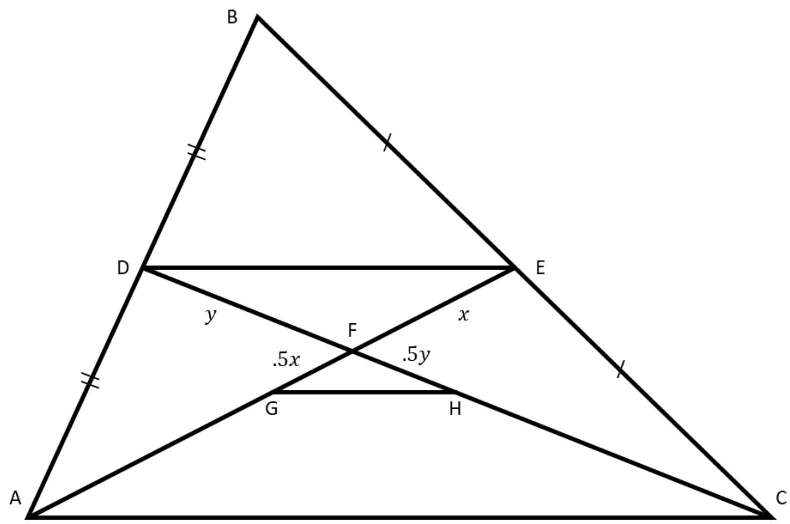


Solution: By definition, point F is the centroid of the triangle, and by Theorem 33, $d(AF) = 2d(FE)$ and $d(CF) = 2d(FD)$.

- If we let $d(FE) = x$, then $d(AF) = 2x$ and $d(AE) = 3x$. Further, $d(AG) = \frac{1}{2}d(AE) = \frac{3}{2}x$, and $d(GF) = d(AF) - d(AG) = 2x - \frac{3}{2}x = \frac{x}{2}$.
- If we let $d(FD) = y$, then $d(CF) = 2y$ and $d(CD) = 3y$. Further, $d(CH) = \frac{1}{2}d(HD) = \frac{3}{2}y$, and $d(HF) = d(CF) - d(CH) = 2y - \frac{3}{2}y = \frac{y}{2}$.

Draw a line segment from point D to E (see the figure below). By the SAS triangle similarity principle, triangle DEF is similar to triangle HGF with proportion constant $\frac{1}{2}$. By Theorem 53, we have

$$\frac{[DEF]}{[HGF]} = \left(\frac{d(DF)}{d(FH)}\right)^2 = \left(\frac{y}{.5y}\right)^2 = 4 \Rightarrow [HGF] = \frac{[DEF]}{4}$$



By Theorem 52 and the fact that $d(BE) = d(EC)$, we have that $[ABE] = [ACE]$ which implies that $[ABC] = 2 \cdot [ABE]$. However, we were given that $[ABC] = 400$ and so, $[ABE] = 200$.

Applying Theorem 52 to triangle ABE, we have that $[ADE] = [BED]$ which implies that $[ABE] = 2 \cdot [ADE]$, but $[ABE] = 200$, and so, $[ADE] = 100$.

Applying Theorem 52 to triangle ADE, we have

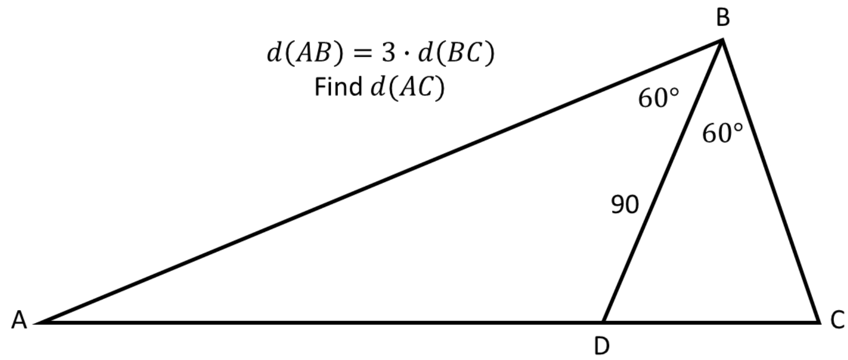
$$\frac{[ADF]}{[DEF]} = \frac{d(AF)}{d(FE)} = \frac{2x}{x} = 2 \Rightarrow [ADF] = 2 \cdot [DEF]$$

So, $[ADE] = 100 = [ADF] + [DEF] = 3 \cdot [DEF]$ which implies $[DEF] = \frac{100}{3}$.

Recall that we previously proved that $[HGF] = \frac{[DEF]}{4}$ and so, $[HGF] = \frac{1}{4} \cdot \frac{100}{3} = \frac{25}{3}$.

Puzzle 8. Deceptively difficult triangle puzzle

Given the configuration shown in the following figure, determine the length of line segment $\overline{AC}$.



Solution: To solve this puzzle, we will use the law of sines [44] and the law of cosines [45].

Let $d(BC) = x$ and $d(AB) = 3x$. In addition, let $\angle BDC = \alpha$ which means that $\angle BDA = 180° - \alpha$.

Using the law of sines on triangle DBC and then on ADB, gives us the following equations

$$\frac{d(DC)}{\sin(60°)} = \frac{x}{\sin(\alpha)} \Rightarrow d(DC) = \frac{x\sin(60°)}{\sin(\alpha)}$$

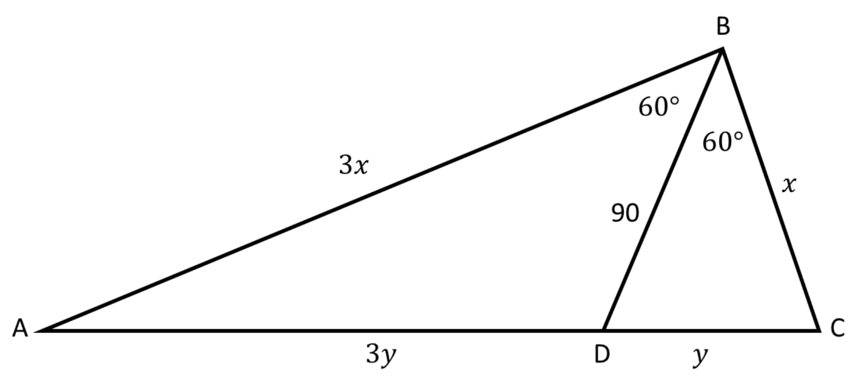
$$\frac{d(AD)}{\sin(60°)} = \frac{3x}{\sin(180° - \alpha)} = \frac{3x}{\sin(\alpha)} \Rightarrow d(AD) = \frac{3x\sin(60°)}{\sin(\alpha)}$$

We used the fact that $\sin(180° - \alpha) = \sin(\alpha)$ to simplify the above equation.

Using the above results, we have that

$$\frac{d(AD)}{d(DC)} = 3 \Rightarrow d(AD) = 3 \cdot d(DC)$$

If we let $d(DC) = y$, then $d(AD) = 3y$. Our updated figure is shown below



Applying the law of cosines to triangle BDC and recalling that $\cos(60°) = \frac{1}{2}$, we have

$$y^2 = x^2 + 90^2 - 2(x)(90)\cos(60°)$$

$$y^2 = x^2 - 90x + 90^2 \quad \text{(Equation 1)}$$

Next, apply the laws of cosines to triangle ABD to get

$$(3y)^2 = (3x)^2 + 90^2 - 2(3x)(90)\cos(60°)$$

$$9y^2 = 9x^2 - 270x + 90^2 \quad \text{(Equation 2)}$$

Multiply Equation 1 by 9 and then subtract Equation 2 to get

$$0 = (270 - 9 \cdot 90)x + (9 \cdot 90^2 - 90^2)$$

$$0 = -540x + 8 \cdot 90^2 \Rightarrow x = 120$$

Substitute the value of x into Equation 1 to get

$$y^2 = 120^2 - 90(120) + 90^2$$

$$y^2 = 11{,}700 \Rightarrow y = 30\sqrt{13}$$

So, $d(AC) = 4y = 120\sqrt{13}$.

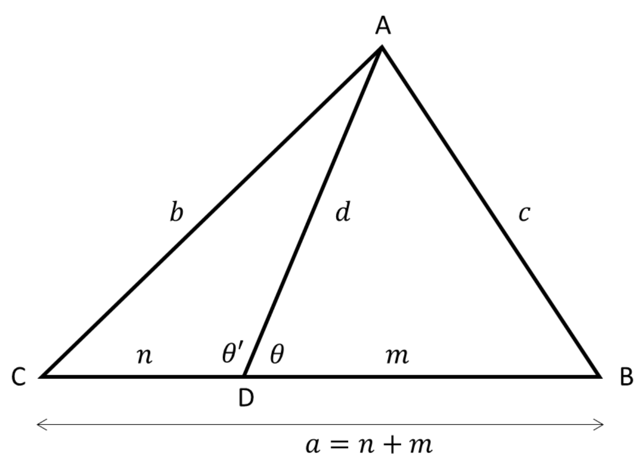
Puzzle 9. Prove Stewart's theorem

Stewart's theorem [46] states a relationship between the lengths of the sides and the length of a cevian in a triangle. (A **cevian** is a line segment which joins a vertex of a triangle to a point on the opposite side of the triangle). The theorem is named after the Scottish mathematician Matthew Stewart, who published the theorem in 1746.

With regard to the figure below, Stewart's theorem states the following:

$$b^2m + c^2n = man + d^2a$$

If $n = m$, we get what is called Apollonius's theorem [47].



Solution: In the figure, $\theta' = 180° - \theta$, and so $cos\ \theta' = -cos\ \theta$.

Applying the law of cosines [45] to triangle ADB regarding angle θ and to triangle ACD regarding angle θ', we get

$$c^2 = m^2 + d^2 - 2dm\cos\theta$$

$$b^2 = n^2 + d^2 - 2dn\cos\theta' = n^2 + d^2 + 2dn\cos\theta$$

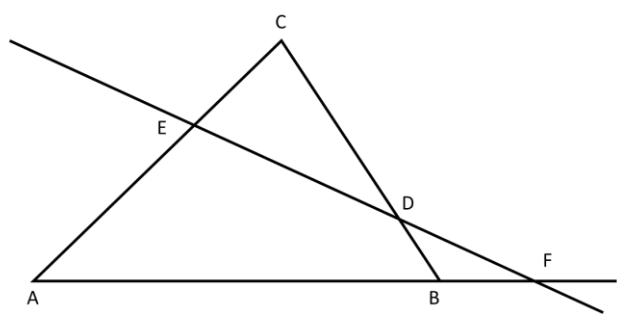
Multiplying the first equation by n and the second equation by m, and then adding, we get the equation

$$b^2m + c^2n = nm^2 + nd^2 + mn^2 + md^2$$
$$b^2m + c^2n = mn(m+n) + d^2(m+n)$$
$$b^2m + c^2n = man + d^2a$$

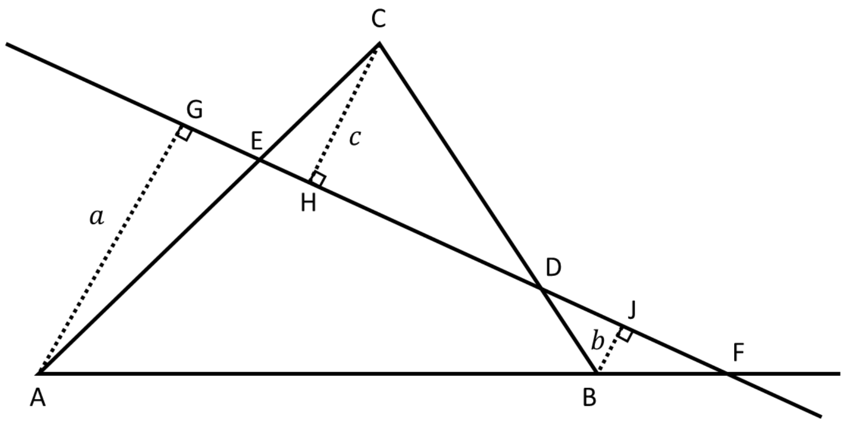
Puzzle 10. Prove Menelaus's theorem

Menelaus's theorem, named for Menelaus of Alexandria, is a proposition about a triangle and a transversal through two sides of the triangle. (A **transversal** is a line that passes through typically two (could be more) lines in the same plane at distinct points.) Consider a triangle ABC, and a transversal line that crosses $\overline{BC}, \overline{AC}, \overline{AB}$ at points D, E, F respectively, with D, E, F distinct from A, B, C (see the figure below). The theorem states that

$$\frac{d(AF)}{d(FB)} \times \frac{d(BD)}{d(DC)} \times \frac{d(CE)}{d(EA)} = 1$$



Solution: Draw perpendiculars line segments from points A, B, C to the line EDF and let their lengths be a, b, c, respectively (as shown in the figure below).



Using the AA principle of triangle similarity, we have the following results:

$$\triangle AGF \cong \triangle BJF \Rightarrow \frac{d(AF)}{d(FB)} = \frac{a}{b}$$

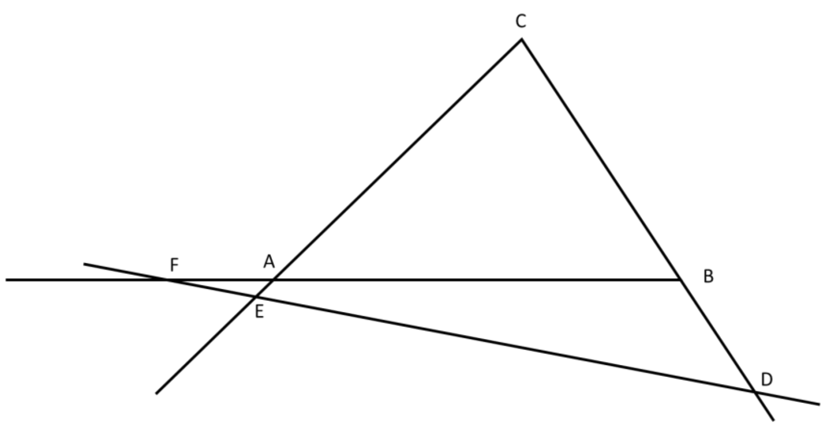
$$\triangle BJD \cong \triangle CHD \Rightarrow \frac{d(BD)}{d(DC)} = \frac{b}{c}$$

$$\triangle AGE \cong \triangle CHE \Rightarrow \frac{d(CE)}{d(EA)} = \frac{c}{a}$$

Thus, we have

$$\frac{d(AF)}{d(FB)} \times \frac{d(BD)}{d(DC)} \times \frac{d(CE)}{d(EA)} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{a} = 1$$

The theorem also holds true if the line DEF lies outside of the triangle ABC, e.g., the configuration shown in the figure below.



As noted in the Wikipedia article “Menelaus's theorem” [25], the theorem can be strengthened to a statement about signed lengths of segments, which provides some additional information about the relative order of collinear points. In this viewpoint, $d(AB)$ is taken to be positive or negative according to whether A is to the left or right of B in some fixed orientation of the line; for example, $\frac{d(AF)}{d(FB)}$ is defined as having positive value when F is between A and B and negative otherwise. The signed version of Menelaus's theorem states

$$\frac{d(AF)}{d(FB)} \times \frac{d(BD)}{d(DC)} \times \frac{d(CE)}{d(EA)} = -1$$

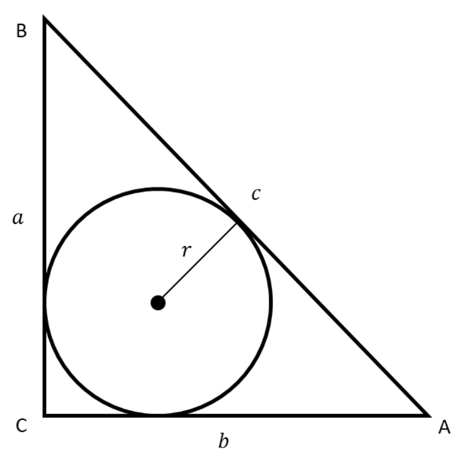
The converse of the strong version of the theorem is also true, i.e., if points D, E, F are chosen on lines BC, AC, AB respectively such that

$$\frac{d(AF)}{d(FB)} \times \frac{d(BD)}{d(DC)} \times \frac{d(CE)}{d(EA)} = -1$$

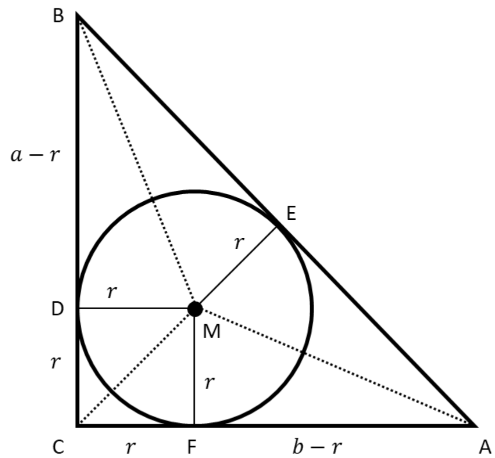
then the points D, E, F are collinear.

Puzzle 11. Inscribed circle in right triangle

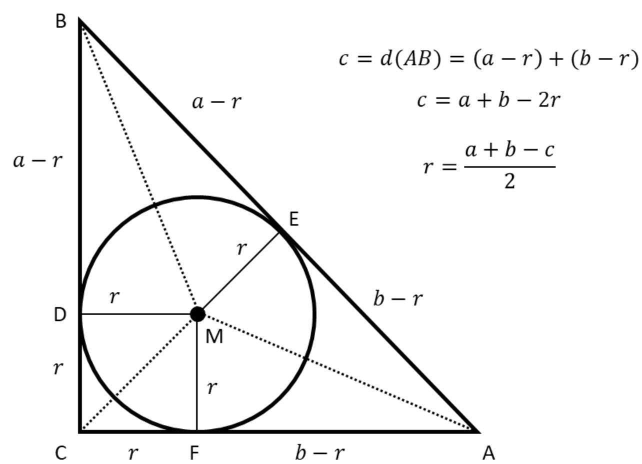
Given an inscribed circle of radius r within a right triangle, determine r in terms of the side lengths of the triangle. With respect to the figure below, the task is to find a formula for r in terms of a, b and c.



Solution: Draw radii from the center of the incircle to the point of tangency with each side, making an angle of 90° in each case (by Theorem 21).



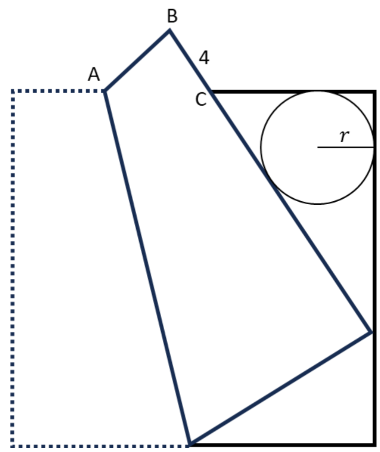
By Theorem 10, triangle BDM is congruent to triangle BEM, and so, $d(BE) = a - r$. Similarly, triangle AFM is congruent to triangle AEM, and so, $d(AE) = b - r$. From here, we can calculate r in terms of a, b and c (see the calculation on the right of the figure below).



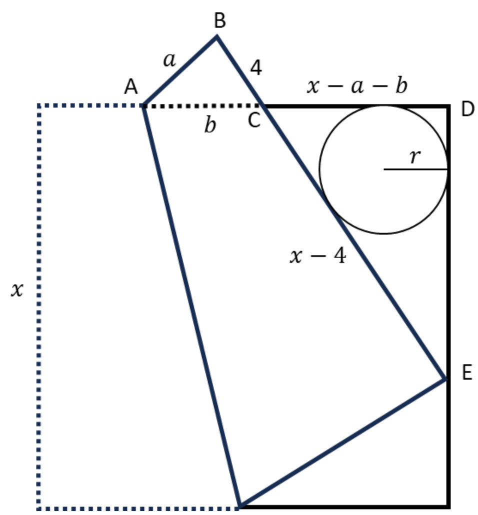
Puzzle 12. Folded square and a circle

A square piece of paper is folded as shown in the figure below, and a circle of radius r is inscribed in the triangle in the top right of the figure. We are given that $d(BC) = 4$. Determine the value of r.

Hint: Use the result from Puzzle 11.



Solution: Let x be the side length of the square, let $d(AB) = a$ and let $d(AC) = b$. These assignments imply that $d(CE) = x - 4$ and $d(CD) = x - a - b$.



By the AA triangle similarity principle, $\triangle ABC \cong \triangle CDE$. This give us the following ratio:

$$\frac{d(DE)}{a} = \frac{x-4}{b}$$

$$d(DE) \cdot b = a(x-4) \quad \text{(Equation 1)}$$

We also have

$$\frac{d(DE)}{a} = \frac{x - a - b}{4}$$

$$4 \cdot d(DE) = a(x - a - b) \quad \text{(Equation 2)}$$

Subtracting Equation 2 from Equation 1 yields

$$d(DE) \cdot (b - 4) = -4a + a^2 + ab = a^2 + a(b - 4) \quad \text{(Equation 3)}$$

Applying the Pythagorean theorem to triangle ABC, we get $a^2 = b^2 - 4^2$. Substituting this result into Equation 3 gives us

$$d(DE) \cdot (b - 4) = b^2 - 4^2 + a(b - 4)$$

$$d(DE) \cdot (b - 4) = (b - 4)(b + 4) + a(b - 4)$$

We can divide both sides of the above equation by $b - 4$ since $b - 4 \neq 0$ (if it did, $b = 4$ would violate the Pythagorean theorem regarding triangle ABC). Thus, we have

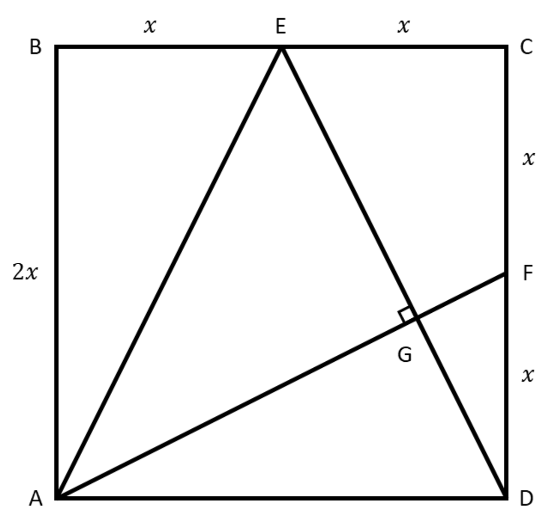
$$d(DE) = a + b + 4$$

Now we are in a position to apply the result from Puzzle 11 to triangle CDE, i.e.,

$$r = \frac{d(CD) + d(DE) - d(EC)}{2} = \frac{(x - a - b) + (a + b + 4) - (x - 4)}{2} = \frac{8}{2} = 4$$

Puzzle 13. Find the area of a right triangle in a square

The right triangle AEG is situated in square ABCD, as shown in the figure below. The square has side length $2x$. Find the area of triangle AEG in terms of x.



Solution: The area of the square is $(2x)^2 = 4x^2$. Triangles ABE, ECD and ADF each have area x^2 (using the area formula for triangles). If we subtract the areas of these three triangles from the area of the square and then add back the area of triangle GFD (which was subtracted twice), we get the area of triangle AEG.

Right triangles ECD and FGD are similar by the AA triangle similarity principle (note that they share the angle $\angle GDF$). The hypotenuse of triangle ECD is $\sqrt{5}x$ (using the Pythagorean theorem), and the hypotenuse of triangle GFD is x. Thus, the similarity proportion constant is $k = \sqrt{5}$ which implies

$$\frac{d(GF)}{d(FD)} = \frac{d(EC)}{d(ED)}$$

$$\frac{d(GF)}{x} = \frac{x}{\sqrt{5}x} \Rightarrow d(GF) = \frac{x}{\sqrt{5}}$$

Using the Pythagorean theorem on triangle GFD, we have that $d(GD)^2 = x^2 - \frac{x^2}{5} = \frac{4}{5}x^2$ which implies $d(GD) = \frac{2x}{\sqrt{5}}$.

So, the area of triangle GFD is

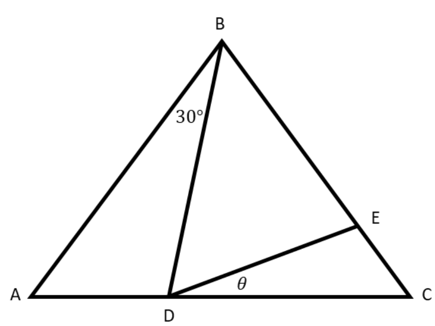
$$\frac{1}{2} \cdot d(GD) \cdot d(GF) = \frac{1}{2} \cdot \left(\frac{2x}{\sqrt{5}}\right)\left(\frac{x}{\sqrt{5}}\right) = \frac{x^2}{5}$$

Thus, the area of triangle AEG is

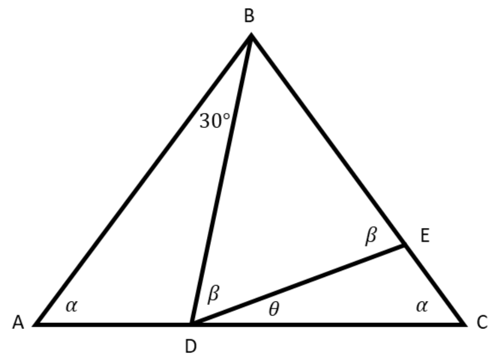
$$4x^2 - 3x^2 + \frac{x^2}{5} = \frac{6}{5}x^2$$

Puzzle 14. Angle relationship within a triangle – 1

In the figure below, we are given $\angle ABD = 30°, d(AB) = d(BC)$ and $d(DB) = d(BE)$. Find the measure of angle θ.



Solution: The condition $d(AB) = d(BC)$ implies that triangle ABC is isosceles, and so, $\angle BAC = \angle BCA$ (shown as α in the figure). The condition $d(DB) = d(BE)$ implies that triangle DBE is isosceles, and so, $\angle BDE = \angle BED$ (shown as β in the figure).



We have $\theta + \alpha + \angle DEC = 180°$ (since the sum of the angles in a triangle is $180°$), and $\beta + \angle DEC = 180°$ (since BE is a straight line), and therefore, $\beta = \theta + \alpha$. This is basically the exterior angle principle [26].

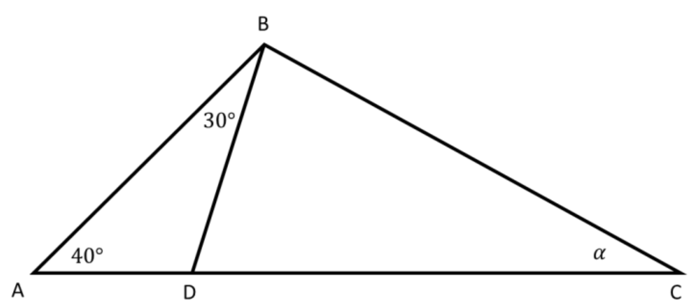
Applying the exterior angle principle at $\angle ADB$, we deduce that $\alpha + 30° = \beta + \theta$.

Substituting $\beta = \theta + \alpha$ into $\alpha + 30° = \beta + \alpha$ gives us

$$\alpha + 30° = (\theta + \alpha) + \theta$$
$$\theta = 15°$$

Puzzle 15. Angle relationships within a triangle – 2

Regarding the figure below, find the measure of angle α. We are also told that $d(AC) = d(AB) + d(DB)$.

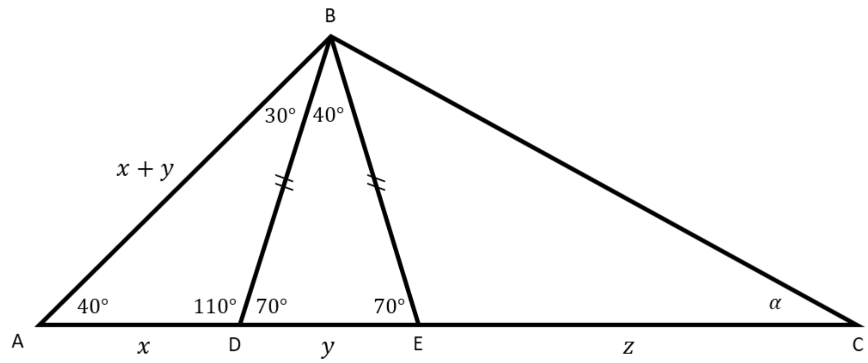


Solution: Since the angles of a triangle add to $180°$, $\angle ADB = 110°$. Since AD is a straight line, $\angle ADC$ must be 70°. See the figure below.

Next, draw a line segment from B down to line AC (call the point of intersection E) such that $d(BE) = d(BD)$ which implies that triangle DBE is isosceles and so, $\angle BED = 70°$.

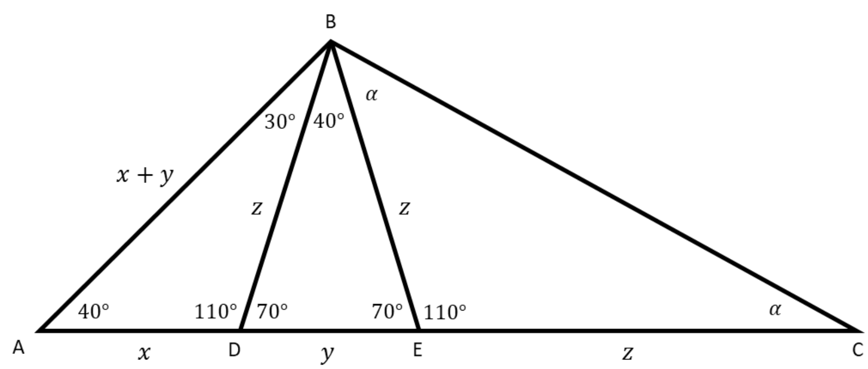
Make the following assignments: $d(AD) = x, d(DE) = y, d(EC) = z$.

Since $\angle ABE = 70° = \angle BED$, triangle ABE is isosceles which implies $d(AB) = d(AE) = x + y$.



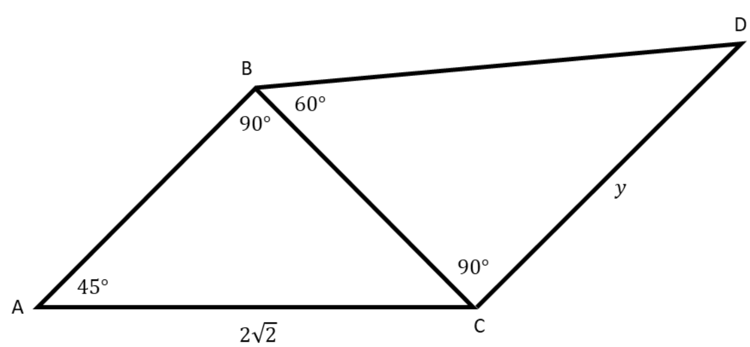
Making substitutions into the given condition $d(AC) = d(AB) + d(DB)$, we have $x + y + z = x + y + d(DB)$. So, $d(DB) = z$ and $d(BE) = z$.

Since $d(BE) = d(EC) = z$, BEC is an isosceles triangle which implies $\angle EBC = \alpha$. So, we have $2\alpha + 110° = 180°$ and we can conclude that $\alpha = 35°$.



Puzzle 16. Two adjacent triangles

In the configuration shown in the figure below, find the value of y.



Solution: Since the angles of a triangle add to $180°$, $\angle BCA = 45°$ and so, triangle ABC must be an isosceles triangle. Let $d(AB) = d(BC) = x$. Applying the Pythagorean theorem to triangle ABC, we have $2x^2 = \left(2\sqrt{2}\right)^2 = 8$ which implies $x = 2$.

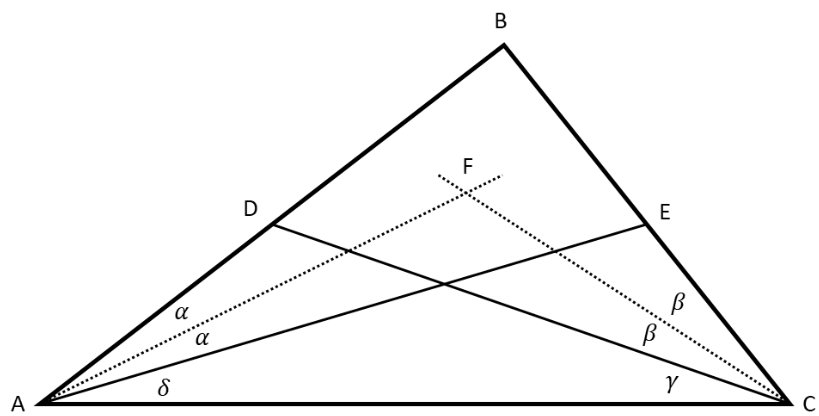
Turning our attention to triangle BCD, we have

$$\tan(\angle DBC) = \frac{y}{2}$$

$$\sqrt{3} = \tan 60° = \frac{y}{2} \Rightarrow y = 2\sqrt{3}$$

Puzzle 17. Bookkeeping triangle puzzle

In triangle ABC, segments $\overline{CD}$ and $\overline{AE}$ are as shown in the figure below. The line segment $\overline{AF}$ bisects $\angle DAE$ and the line segment $\overline{CF}$ bisects $\angle ECD$. Prove that $2\angle AFC = \angle ADC + \angle AEC$.



Solution: This problem appears formidable at first glance, but can be solved by writing down the various angle relationships in the figure, i.e.,

$$\angle ADC = 180° - (2\alpha + \delta + \gamma)$$

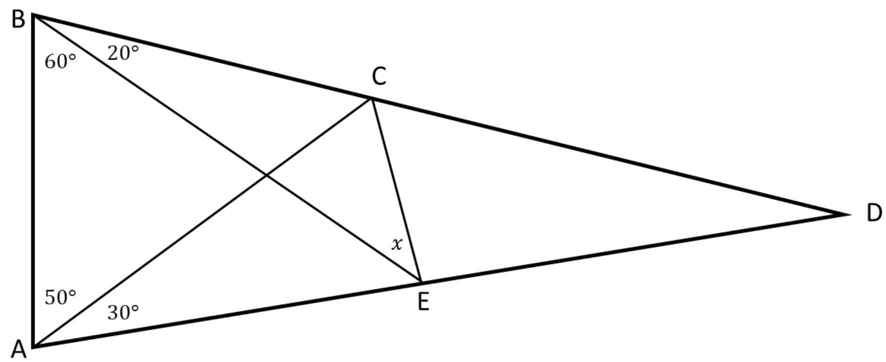
$$\angle AEC = 180° - (2\beta + \delta + \gamma)$$

Adding the above equations together, we get

$$\angle ADC + \angle AEC = 2\big(180° - (\ \alpha + \beta + \delta + \gamma)\big) = 2\angle AFC$$

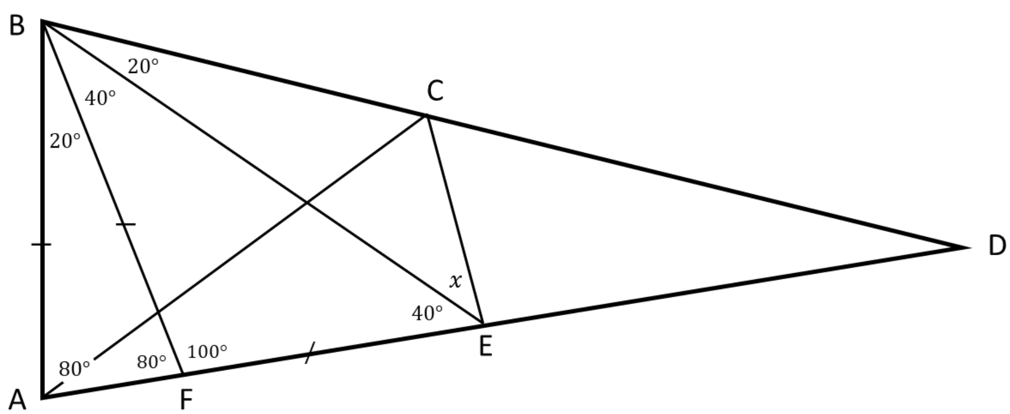
Puzzle 18. Langley's Adventitious Angles [27]

Find the measure of angle x in the figure below.

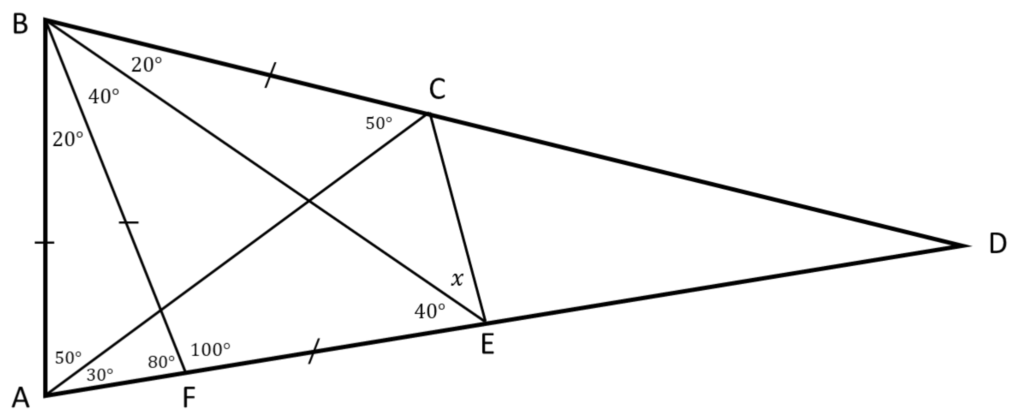


Solution: The approach taken here is to create a series of isosceles triangles, starting from the left of the figure and eventually including the angle x. To that end, draw the segment $\overline{BF}$ (shown in the figure below) such that $\angle ABF = 20°$ which in turn, implies $\angle BFA = 80°$. So, $\triangle ABF$ is an isosceles triangle and thus, $d(AB) = d(BF)$.

Since AD is a straight line, it must be that $\angle BFE = 100°$ which in turn, implies $\angle BEF = 40°$. Thus, $\triangle BEF$ is an isosceles triangle and $d(FE) = d(BF) = d(AB)$.



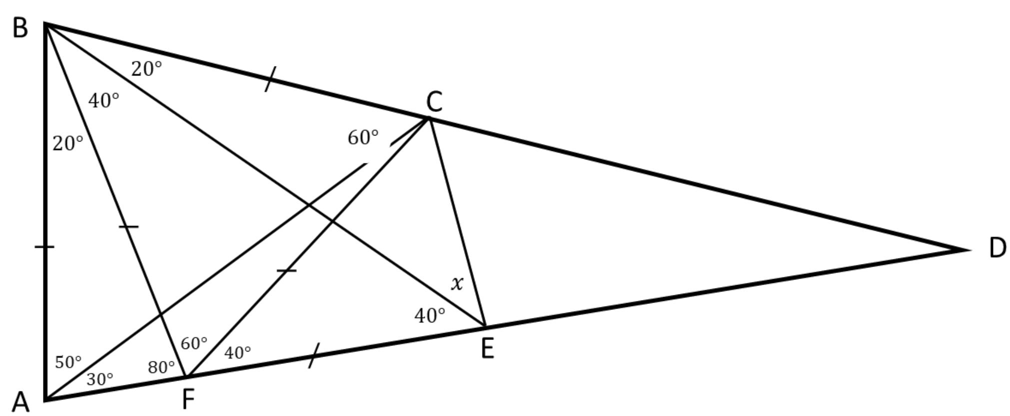
Next, consider triangle ABC in the figure below. Since the angles of a triangle add to $180°$, it must be that $\angle BCA = 50°$. Thus, ABC is an isosceles triangle, and $d(BC) = d(AB)$.



Since we have shown that $d(BF) = d(BC)$, it follows that BFC is an isosceles triangle. Since $\angle BFC = 60°$, it follows that $\angle BFC = \angle BCF = 60°$. Thus, $\triangle\, BCF$ is an equilateral triangle, and $d(FC) = d(BF) = d(BC)$. See the updated figure below.

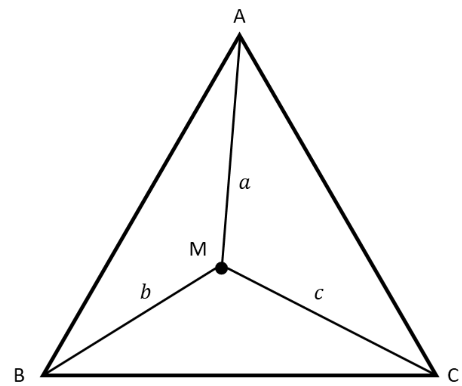
Finally, we see that FEC is an isosceles triangle. (Clearly, the figure is not drawn exactly.) So, $\angle FCE = \angle FEC = x + 40°$. Since the angles of a triangle sum to $180°$, we have (regarding triangle FEC)

$$2(x + 40°) + 40° = 180° \Rightarrow\ x = 30°$$



Puzzle 19. Find the angle

In the figure below, ABC is an equilateral triangle. The lengths of the three line segment interior to the triangle have the following constraint: $a^2 = b^2 + c^2$. Find the measure of $\angle BMC$.



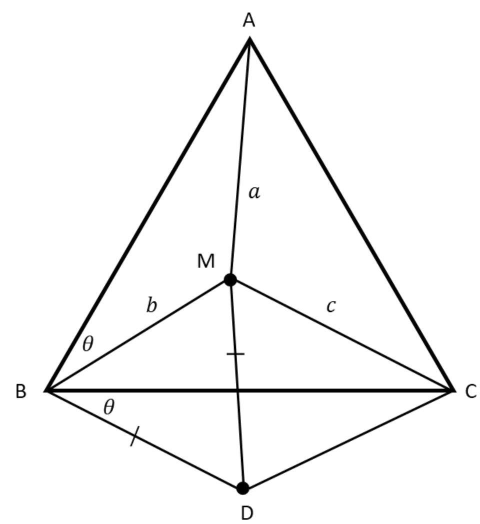
Solution: Select a point D (external to the triangle ABC) such that $d(BD) = d(MD) = d(BM) = b$. The resulting triangle BDM is equilateral.

Given that ABC and BDM are equilateral triangles, we have that $\angle ABC = \angle MBD = 60°$. Further,

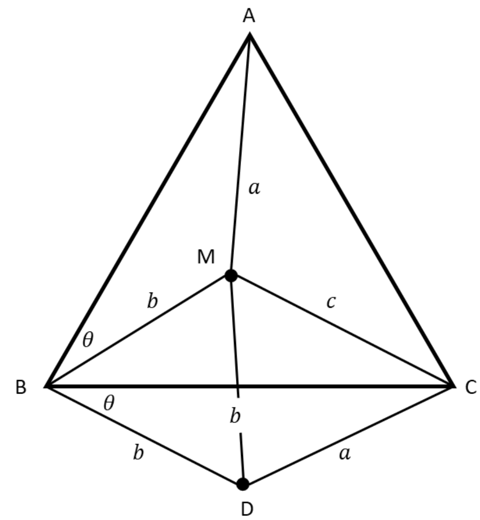
$$\angle ABC = \angle ABM + \angle MBC \Rightarrow \ \angle ABM = 60° - \angle MBC$$

$$\angle MBD = \angle CBD + \angle MBC \Rightarrow \angle CBD = 60° - \angle MBC$$

Thus, $\angle ABM = \angle CBD$ (shown as angle θ in the figure below).



Since $\angle ABM = \angle CBD, d(BM) = d(BD)$ and $d(AB) = d(BC)$, we have the $\triangle ABM = \triangle CBD$ by the SAS triangle congruence principle. Hence, $d(AM) = d(DC) = a$. The updated figure is shown below.



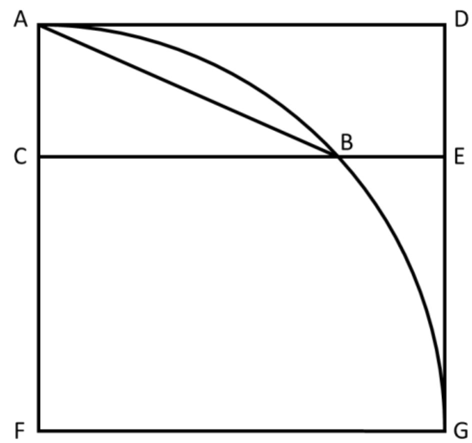
Since $\triangle BMD$ was constructed as an equilateral triangle, we known that $\angle BMD = 60°$. Since the side lengths of $\triangle DMC$ are a, b and c and we were given that $a^2 = b^2 + c^2$. From the converse of the Pythagorean theorem, we have that $\triangle DMC$ is a right triangle with a right angle at $\angle DMC$ (clearly, the figure is not drawn exactly). Thus, $\angle BMC = \angle BMD + \angle DMC = 60° + 90° = 150°$.

4 Circle Puzzles

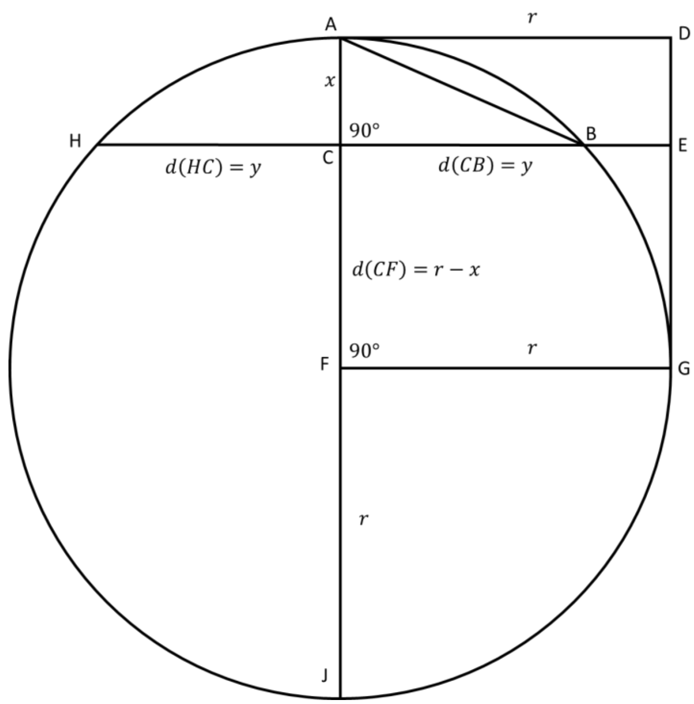
Our external physical reality is a mathematical structure. – Max Tegmark

Puzzle 20. Find the length of the chord

In the figure below, $AFGD$ is a square containing a quarter circle going from point A to point G. We are given that the rectangle $ACED$ has area 8. What is the length of chord $\overline{AB}$?



Solution: In the following figure, we have extended the quarter circle to a complete circle of radius r. Further, segment $\overline{CB}$ has been extended to make the chord $\overline{HB}$, and segment $\overline{AC}$ has been extended to make the chord $\overline{AJ}$. The length of several other segments within the figure are as indicated (all unknowns at this point).



In terms of our segment length labeling scheme, we can write $d(CJ) = 2r - x$.

Applying the Intersecting Chords Theorem (Theorem 29) to chords HB and AJ, we get the equation

$$x(2r - x) = y^2$$

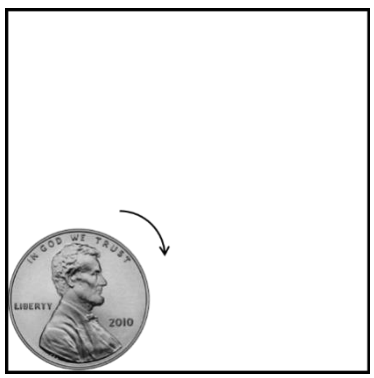
which implies $2rx = x^2 + y^2$

Applying the Pythagorean theorem to triangle ABC, we get $x^2 + y^2 = d(AB)^2$.

Combining the above two results gives us $2rx = d(AB)^2$ but rx is the area of the rectangle which we were told is 8. Thus, $d(AB) = \sqrt{16} = 4$.

Puzzle 21. Coin rotating around the interior of a square

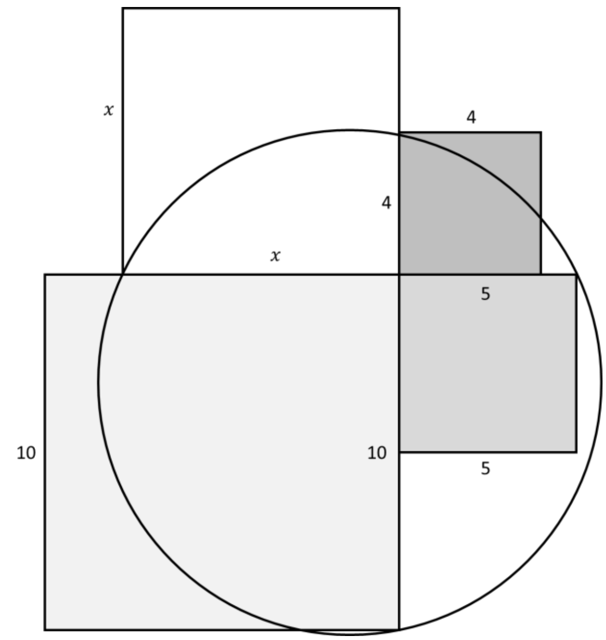
A coin rolls around the interior of a square and returns exactly to its starting position after one full traverse of the interior of the square. There is no slippage as the coin moves along the interior of the square. Find the side length of the square if the radius of the coin is 1.



Solution: At its starting position, the coin's base (tangent point with the bottom side of the square) is distance r from the left side of the square. The coin makes a $\frac{1}{4}$ rotation when going to the right side of the square. The distance covered by the rotation is $\frac{1}{4}(2\pi r) = \frac{\pi r}{2}$. So, the length of one side of the square is $r + \frac{\pi r}{2} + r = 2r + \frac{\pi r}{2} = 2 + \frac{\pi}{2}$ since we were given that $r = 1$.

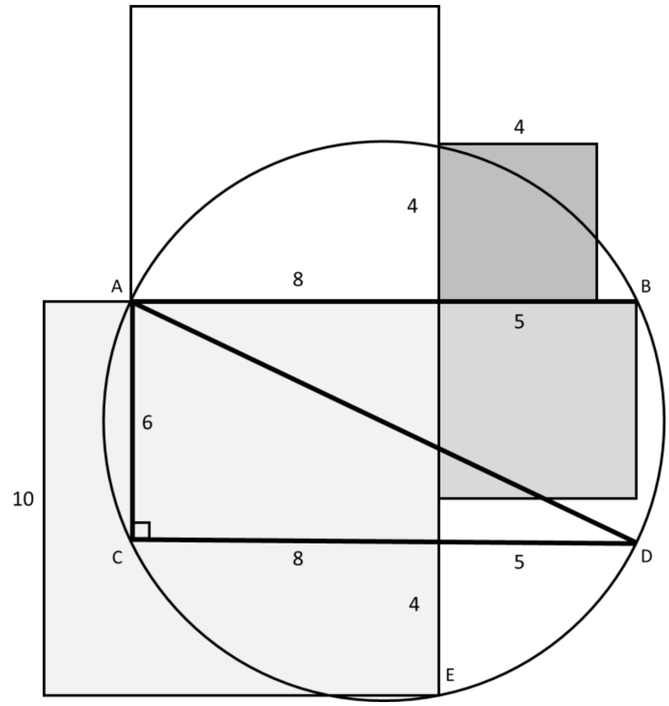
Puzzle 22. Four squares and a circle

We are given four squares with side lengths 10, 5, 4 and x. The squares are situated in relation to a circle as shown in the figure below. What is the area of the circle?



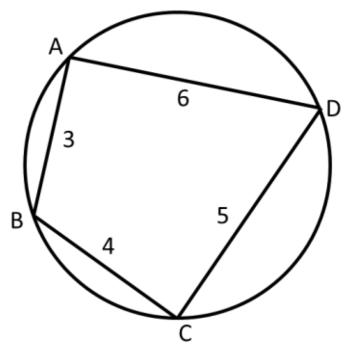
Solution: By the intersecting chord theorem (Theorem 29), we have $5x = 4(10)$ which implies $x = 8$. We can now compute $d(AB) = 8 + 5 = 13$.

Next, draw the chord CD such that it is parallel to chord AB and four units up from point E. Draw a line segment between points A and D to create triangle ACD. Since points A, C and D lie on the circle, and $\overline{AC}$ is perpendicular to $\overline{CD}$ (by construction), we have by Theorem 25 that chord $\overline{AD}$ is a diameter of the circle. By the Pythagorean theorem, $d(AD)^2 = d(AC)^2 + d(CD)^2 = 36 + 169 = 205$. So, the radius of the circle equals $\sqrt{205}/2$, and the area of the circle is $\frac{205}{4}\pi$.

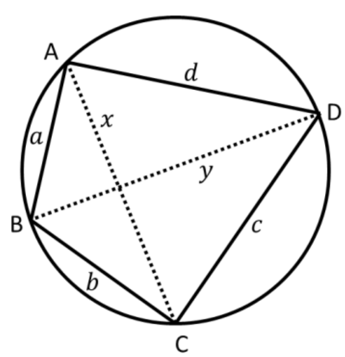


Puzzle 23. Determine the radius of the circle circumscribing the given quadrilateral

Find the radius of the circle in the figure below. The lengths of the various chords are as indicated in the figure.



Solution: We shall solve the problem generically, i.e., for any convex quadrilateral. To that end, we label the side lengths of the quadrilateral and its diagonals as shown in the figure below.



In what follows, we use the shorthand notation $[ABC]$ to denote the area of a triangle with vertices A, B and C, and $[ABCD]$ to denote the area of a convex quadrilateral with vertices A, B, C and D.

The circle that circumscribes quadrilateral $ABCD$ also circumscribes triangles ABC and ADC. Let the radius of this circle be R.

Applying Theorem 57 to triangles ABC and ACD, we get the following equations:

$$[ABC] = \frac{abx}{4R}, \quad [ACD] = \frac{cdx}{4R}$$

Adding the above two equations, we get

$$[ABCD] = [ABC] + [ACD] = \frac{abx}{4R} + \frac{cdx}{4R} = \frac{x}{4R}(ab + cd)$$

Similarly,

$$[ABCD] = [ABD] + [BCD] = \frac{y}{4R}(ad + bc)$$

Next, we multiple the two formulas that we have for $[ABCD]$ to get

$$[ABCD]^2 = \frac{xy}{16R^2}(ab + cd)(ad + bc)$$

Solving for R in the above equation gives us

$$R = \frac{\sqrt{xy(ab + cd)(ad + bc)}}{4[ABCD]}$$

By Ptolemy's theorem (Theorem 44), $xy = ac + bd$. By Brahmagupta's theorem (Theorem 56), $[ABCD] = \sqrt{(p - a)(p - b)(p - c)(p - d)}$ where $p = \frac{1}{2}(a + b + c + d)$. Substituting these two results into the above formula for R, we get

$$R = \frac{\sqrt{(ac + bd)(ab + cd)(ad + bc)}}{4\sqrt{(p - a)(p - b)(p - c)(p - d)}}$$

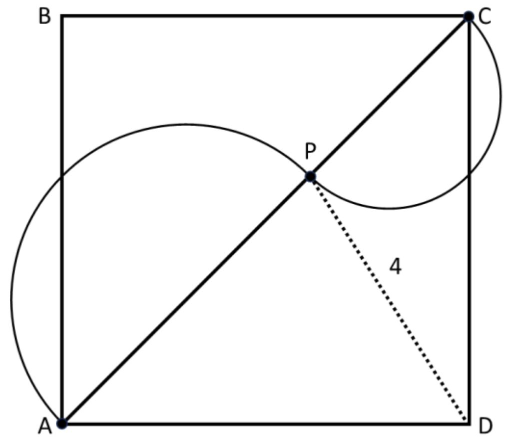
Thus, we have a general formula for R in terms of the side lengths of the circumscribed quadrilateral. As a final step, we need to substitute into the formula the given values for a, b, c and d, i.e.,

$$p = \frac{3 + 4 + 5 + 6}{2} = 9$$

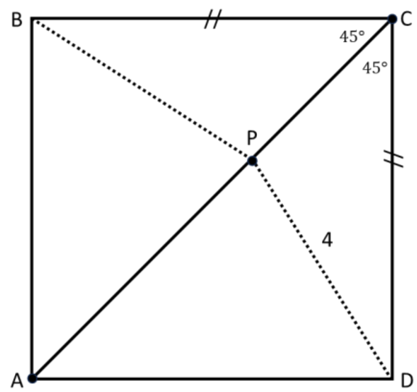
$$R = \frac{\sqrt{(15 + 24)(12 + 30)(18 + 20)}}{4\sqrt{6 \cdot 5 \cdot 4 \cdot 3}} = \frac{\sqrt{39 \cdot 42 \cdot 38}}{4\sqrt{6 \cdot 5 \cdot 4 \cdot 3}} = \frac{\sqrt{1729}}{4\sqrt{10}}$$

Puzzle 24. Find the area of two semicircles overlapping a square

In the figure below, two semicircles are drawn off a main diagonal of a square. The distance from vertex D to the point P (on the diagonal AC) is 4. Find the sum of the areas of the two semicircles.



Solution: By the SAS triangle congruence principle, triangles BCP and DCP are congruent (see the figure below). Thus, $d(BP) = 4$.



By the British flag theorem [51],

$$32 = 4^2 + 4^2 = d(BP)^2 + d(DP)^2 = d(AP)^2 + d(CP)^2$$

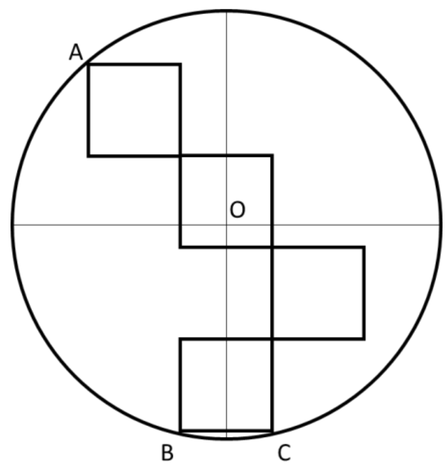
So, the area of the two semicircles is

$$\frac{1}{2}\left[\pi\left(\frac{d(AP)}{2}\right)^2 + \pi\left(\frac{d(CP)}{2}\right)^2\right] = \frac{\pi}{8}[d(AP)^2 + d(CP)^2] = 4\pi$$

We will encounter the British flag theorem again in Puzzle 28 and Puzzle 43.

Puzzle 25. Find the area of the circle containing four squares

Find the area of the circle shown in the figure below. Each of the four squares has area 4. The top square intersects the circle at point A, and the bottom square intersects the circle at points B and C. The points of tangency among the squares is as suggested in the figure. Point O is the center of the circle.



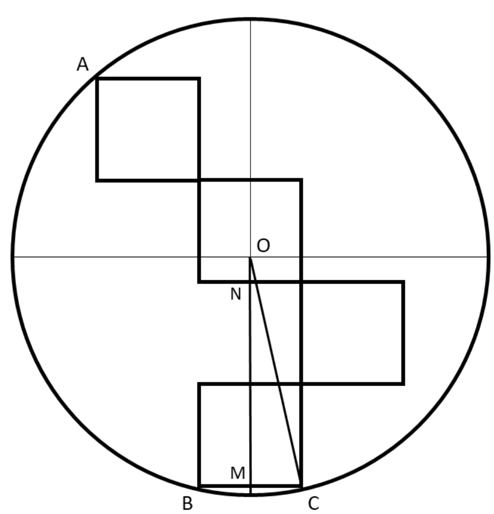
Solution: Draw a radius from point O that is perpendicular to chord BC. By Theorem 19, chord BC is bisected by the perpendicular radius. Label the midpoint of chord BC as M. Since each square is of area 4, each square has side length 2. Thus, $d(BM) = d(MC) = 1$.

Draw another radius from O to point C. Let the length of the radius be r.

Let $d(ON) = x$. So, we can write $d(OM) = x + 4$ (since each square has side length 2).

Applying the Pythagorean theorem to triangle OMC, we have

$$(x + 4)^2 + 1^2 = r^2$$
$$r^2 = x^2 + 8x + 17$$



Next, construct the right triangle ADO (as shown in the figure below). We have that

$$d(OD) = 2 + d(OE) = 2 + 1 = 3$$

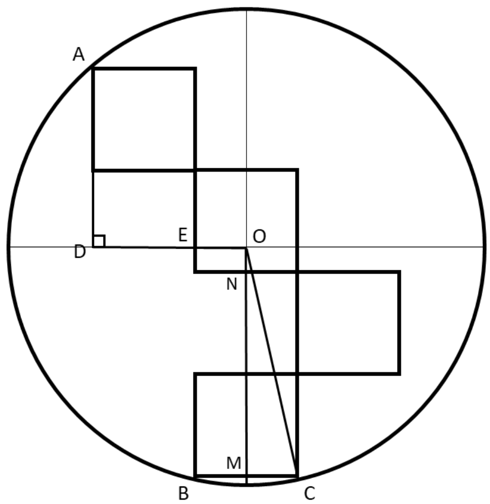
$$d(AD) = 2 + (2 - x) = 4 - x$$

$$d(OA) = r$$

Applying the Pythagorean theorem to triangle ADO, we have

$$3^2 + (4 - x)^2 = r^2$$

$$r^2 = x^2 - 8x + 25$$



Next, equate the two different expressions that we derived for r^2, i.e.,

$$x^2 + 8x + 17 = x^2 - 8x + 25$$

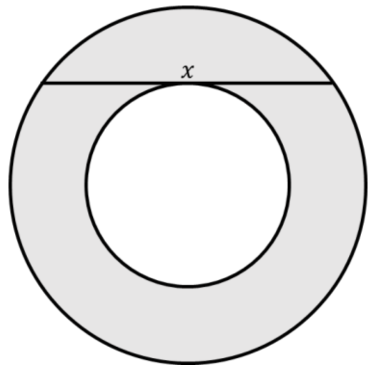
$$16x = 8 \Rightarrow x = \frac{1}{2}$$

Put the value for x back into either equation for r^2 and solve for r to get $r = \sqrt{\frac{85}{4}}$.

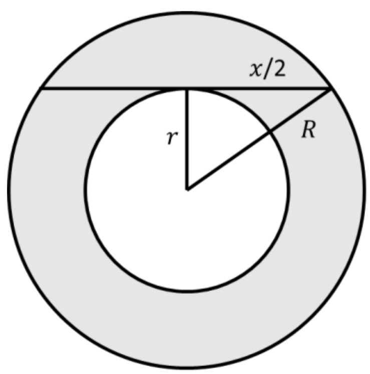
So, the area of the circle is $\frac{85\pi}{4}$.

Puzzle 26. A circle within a circle

Find the area of the shaded region between the two circles in the figure below, given that the length of the chord is x. Of course, the solution needs to be in terms of x.



Solution: Let r be the radius of the smaller circle and R be the radius of the larger circle. The area of the shaded region is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. So, the problem reduces to finding $R^2 - r^2$ in terms of x. To that end, draw a perpendicular line segment from the center of the two circles to the given chord. By Theorem 19, the chord is bisected by the perpendicular line segment. From here, it is just a matter of using the Pythagorean theorem (as shown in the figure below). So, the area of the shaded region is $\frac{\pi x^2}{4}$.



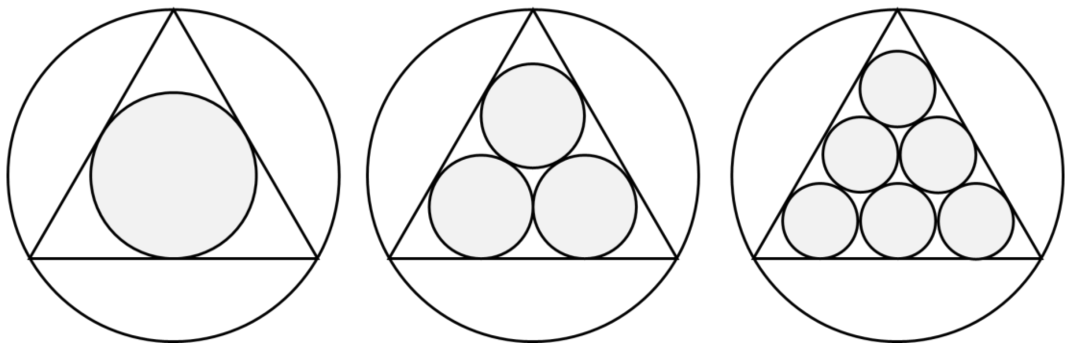
$$\left(\frac{x}{2}\right)^2 + r^2 = R^2$$
$$R^2 - r^2 = \frac{x^2}{4}$$

Puzzle 27. Find fraction of the configuration covered by the shaded area in the limiting case

The figure below shows the first three cases of circles packed into an equilateral triangle which is inscribed in a larger circle. The pattern continues indefinitely, with

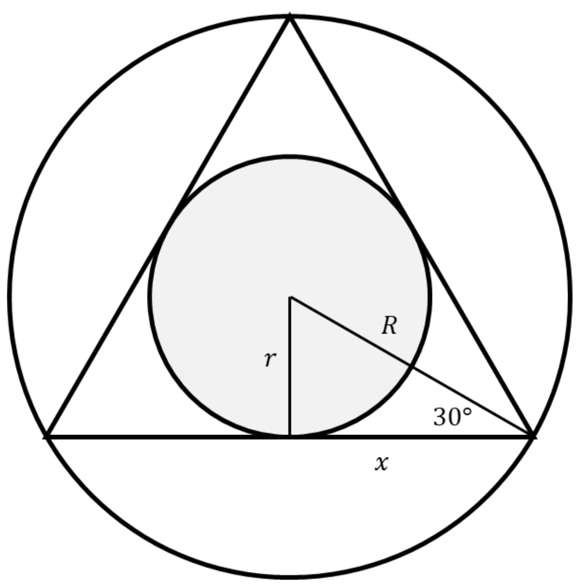
$$1, 3, 6, 10, \ldots, \frac{n(n+1)}{2}, \ldots$$

circles packed into the equilateral triangle. What fraction of the configuration (including the outer circle) is covered by the shaded circles as n approach infinity?



Solution: Let's start by computing the fraction of the shaded circle to the overall configuration in the case $n = 1$. Let the interior circle have radius r and the outer circle have radius R. We have that $r = \frac{R}{2}$ (see the first calculation in the figure below). For this case, the shaded fraction is

$$\frac{\pi\left(\frac{R}{2}\right)^2}{\pi R^2} = \frac{1}{4}$$



$$\frac{1}{2} = \sin 30° = \frac{r}{R}$$
$$r = \frac{R}{2}$$

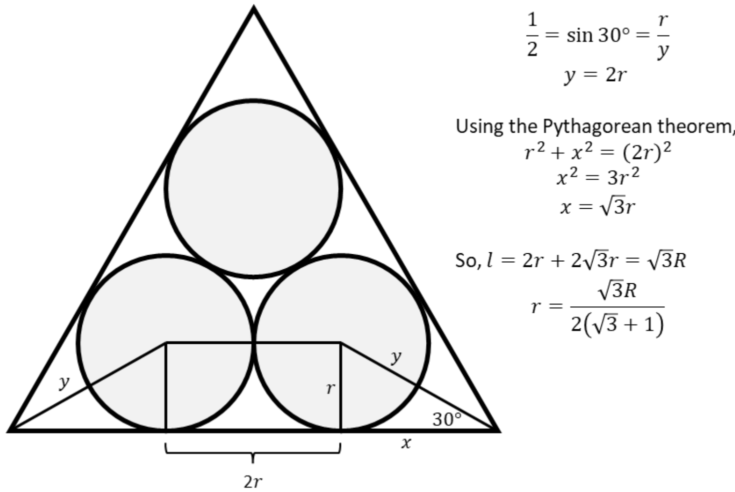
Using the Pythagorean theorem,

$$\left(\frac{R}{2}\right)^2 + x^2 = R^2$$
$$x^2 = \frac{3}{4}R^2$$
$$x = \frac{\sqrt{3}}{2}R$$

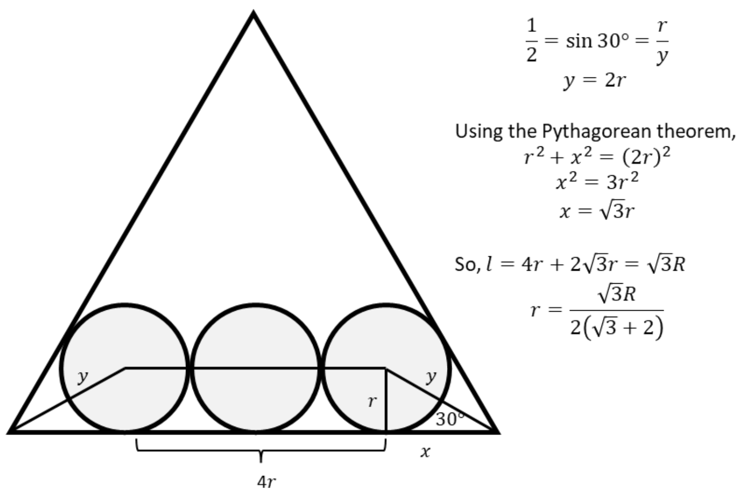
So, the length of the side of the triangle is

$$l = \sqrt{3}R$$

We next consider the case for three circles packed into the equilateral triangle. The calculations for r (i.e., the radius of the smaller circle) is shown in the figure below. The length of the side of the equilateral triangle remains the same in all cases, i.e., $l = \sqrt{3}R$.



Let's calculate r for one more case to establish the pattern, see the figure below.



For case n (where we have $\frac{n(n+1)}{2}$ circles packed in the triangle), the radius of each smaller circle is

$$r = \frac{\sqrt{3}R}{2(\sqrt{3}+n)}$$

The fraction of the shaded region for the n^{th} case is

$$\frac{\pi r^2}{\pi R^2} = \frac{\left(\frac{n(n+1)}{2}\right)\left(\frac{\sqrt{3}R}{2(\sqrt{3}+n)}\right)^2}{R^2} = \frac{n(n+1)}{2}\left(\frac{\sqrt{3}}{2(\sqrt{3}+n)}\right)^2$$

For the final step, we need to take the limit as n approaches infinity, i.e.,

$$\lim_{n\to\infty} \frac{n(n+1)}{2}\left(\frac{\sqrt{3}}{2(\sqrt{3}+n)}\right)^2 = \lim_{n\to\infty} \frac{3n^2+3n}{8n^2+16\sqrt{3}n+24)} = \frac{3}{8}$$

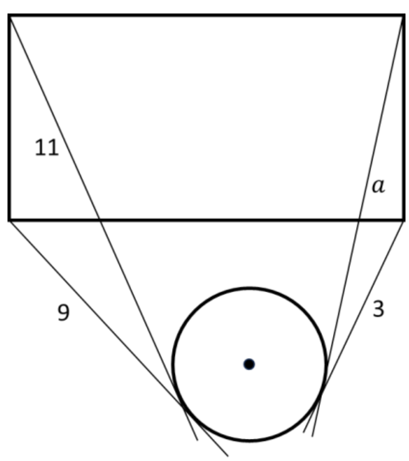
. . .

In general, circle packing within an equilateral triangle is a well-studied problem in discrete mathematics where the objective is to pack n unit circles (i.e., circles of radius 1) into the smallest possible equilateral triangle. Optimal solutions are known for $n < 13$ and for any triangular number of circles, and conjectures are available for $n < 28$. The triangular numbers are $1,3,6,10, \dots, \frac{n(n+1)}{2}, \dots$ (basically, what we studied in Puzzle 27).

A conjecture by Paul Erdős and Norman Oler states that "if n is a triangular number, then the optimal packings of $n-1$ and of n circles have the same side length for the encompassing equilateral triangle". In other words, the conjecture states that an optimal packing for $n-1$ circles can be found by removing any single circle from the optimal hexagonal packing of n circles, see the discussion and diagrams in the Wikipedia article "Circle packing in an equilateral triangle" [29]. This conjecture is now known to be true for $n \leq 15$.

Puzzle 28. Tangents to a circle from four corners of a rectangle

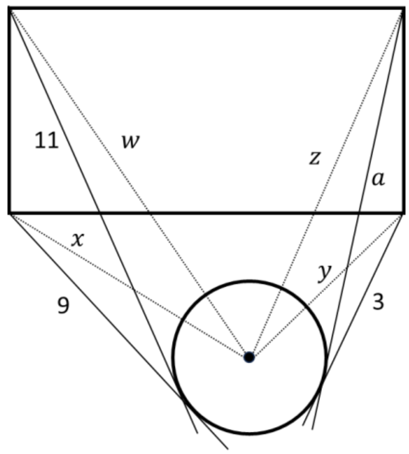
In the figure below, tangent lines are drawn from the four corners of a rectangle to a circle. The distance between each corner of the rectangle to the point of tangency is given for three of the four vertices. Find the length of a.



Hint: The solution makes use of the British flag theorem which we discuss in Puzzle 43.

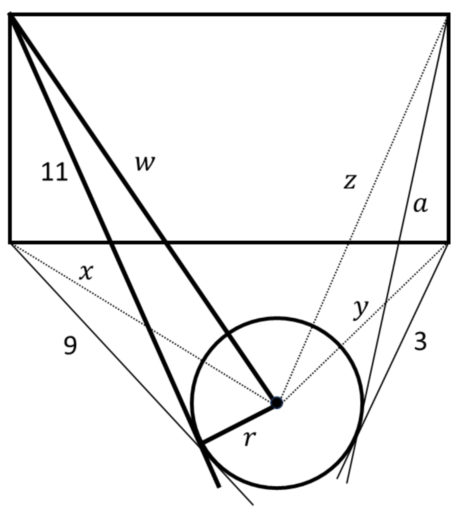
Solution: Draw a line segment from each vertex of the rectangle to the center of the circle (see the figure below). By the British flag theorem (which works for a point either within or outside of a given rectangle), we have

$$w^2 + y^2 = x^2 + z^2$$



Next, draw a radius from the center of the circle to the line of length 11 at the point of tangency to the circle (see the figure below). By Theorem 21, the radius, so drawn, is perpendicular to the line. Thus, we can use the Pythagorean theorem on the triangle drawn with the thick lines in the figure below. This gives us

$$11^2 + r^2 = w^2$$



We do the same procedure for the other three tangent lines to the circle and obtain the following equations:

$$9^2 + r^2 = x^2$$

$$3^2 + r^2 = y^2$$

$$a^2 + r^2 = z^2$$

Using the above equations, we have

$$w^2 + y^2 = 121 + 9 + 2r^2$$

$$x^2 + z^2 = 81 + a^2 + 2r^2$$

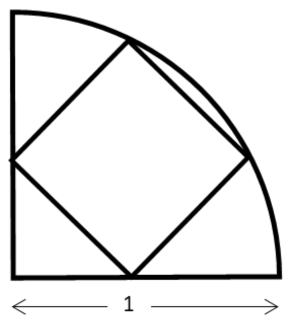
But we know from our earlier use of the British flag theorem that $w^2 + y^2 = x^2 + z^2$ and so,

$$121 + 9 + 2r^2 = 81 + a^2 + 2r^2$$

$$a^2 = 49 \Rightarrow a = 7$$

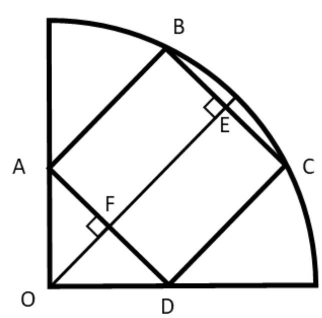
Puzzle 29. A square inscribed in a quarter-circle

A square is inscribed in a quarter-circle of radius 1 (as shown in the figure below). Find the area of the square.

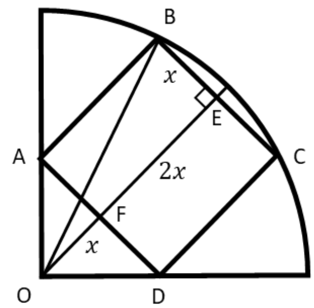


Solution: Draw a line segment from the center of the circle that is perpendicular to the chord $\overline{BC}$ (see the figure below). By Theorem 19, the line segment $\overline{OE}$ bisects the chord $\overline{BC}$. Since ABCD is a square, $\overline{OE}$ also bisects the line segment $\overline{AD}$.

By the SAS triangle congruence principle, $\triangle AOF \cong \triangle DOF$ which implies that $\angle AOF = \angle DOF = 45°$ and in turn, it must be that $\angle OAF = \angle ODF = 45°$. So, triangles AOF and DOF are isosceles triangles which means $d(AF) = d(OF) = d(DF) = d(OD) = x$. Since $ABCD$ is a square, we also have $d(BE) = d(EC) = x$ and $d(FE) = 2x$.

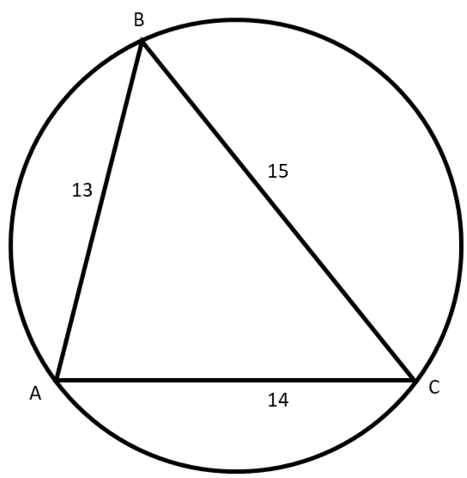


Consider the right triangle OBE in the figure below. By the Pythagorean theorem, $(3x)^2 + x^2 = 1$ which implies $x^2 = \frac{1}{10}$. So, the area of the square is $(2x)^2 = 4x^2 = \frac{4}{10} = \frac{2}{5}$.

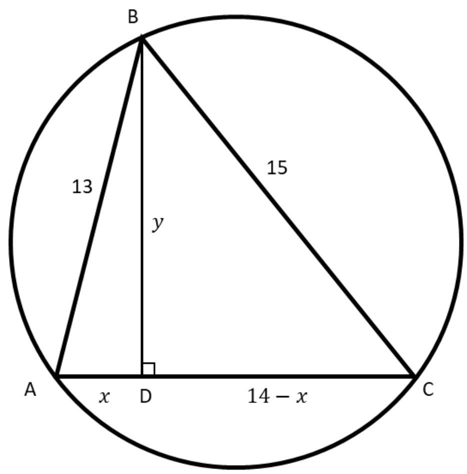


Puzzle 30. Find the radius of the circle with inscribed triangle

In the figure below, triangle ABC is inscribed within the circle. The lengths of the sides of the triangle are as shown in the figure. Find the radius of the circle.



Solution: Draw a perpendicular line from point B to line AC (as shown in the figure below). Let $d(AD) = x$ which implies $d(DC) = 14 - x$. Let $d(BD) = y$.



Applying the Pythagorean theorem to triangle ABD, we have

$$x^2 + y^2 = 169 \Rightarrow y^2 = 169 - x^2$$

Applying the Pythagorean theorem to triangle CDB, we have

$$(14 - x)^2 + y^2 = 225$$

$$y^2 = 29 + 28x - x^2$$

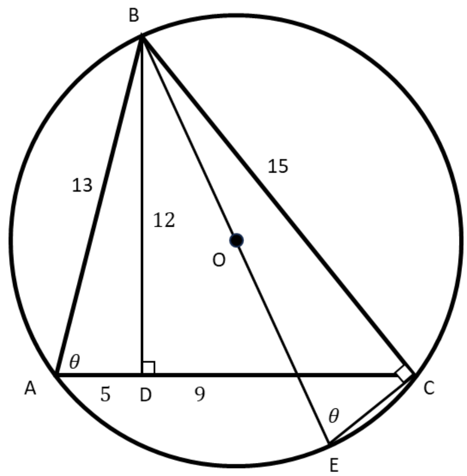
Equating the two expressions that we have for y^2, we get

$$169 - x^2 = 29 + 28x - x^2$$

$$28x = 140 \Rightarrow x = 5$$

Substitute $x = 4$ into the equation $y^2 = 169 - x^2$ to get $y = 12$.

Next, draw a diameter from B through the center of the circle O to point E. By Theorem 24, $\angle BCE = 90°$ since it subtends an angle of $180°$. By Theorem 26, $\angle BAC = \angle BEC$ since both angles subtend the arc between B and C. See the updated information in the figure below.



By the AA triangle similarity principle, $\triangle BAD \cong \triangle BEC$ which implies

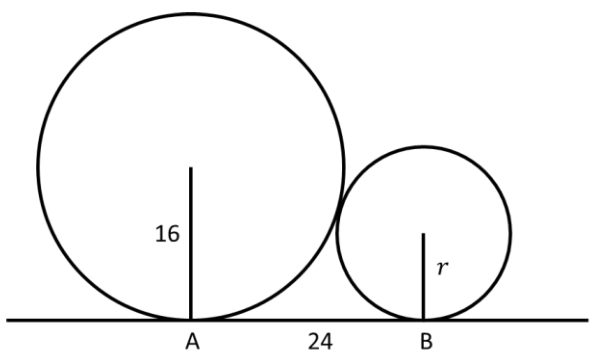
$$\frac{d(BE)}{d(BC)} = \frac{d(AB)}{d(BD)}$$

$$\frac{d(BE)}{15} = \frac{13}{12} \Rightarrow d(BE) = \frac{65}{4}$$

The radius of the circle is half $d(BE)$, i.e., $\frac{65}{8}$.

Puzzle 31. Two circles sitting on a line

Two circles are tangent to a line at points A and B, respectively. The radius of the larger circle is 16, and the distance between points A and B is 24. Find the radius of the smaller circle.



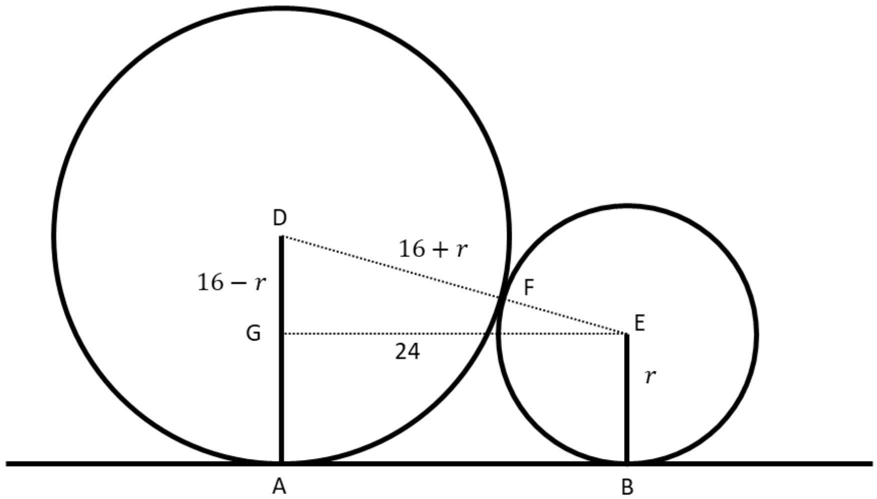
Solution: Draw a line segment from point E that is perpendicular to $\overline{AD}$, and parallel to $\overline{AB}$. The line segment $\overline{EG}$ will have the same length as $\overline{AB}$, i.e., 24.

Draw a radius from D to the point of tangency F with the smaller circle, and draw a radius from E to the F. Both radii will be perpendicular to the tangent line between the two circles, and thus, DE is a straight line (see Theorem 22). Further, $d(DE) = 16 + r$.

We also have that $d(GD) = 16 - r$.

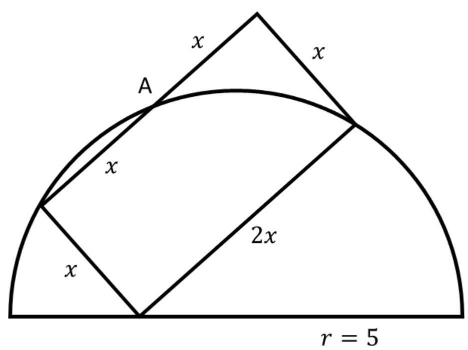
By the Pythagorean theorem, we have

$$(16 - r)^2 + 24^2 = (16 + r)^2$$
$$256 - 32r + r^2 + 24^2 = 256 + 32r + r^2$$
$$64r = 24^2 \Rightarrow r = 9$$



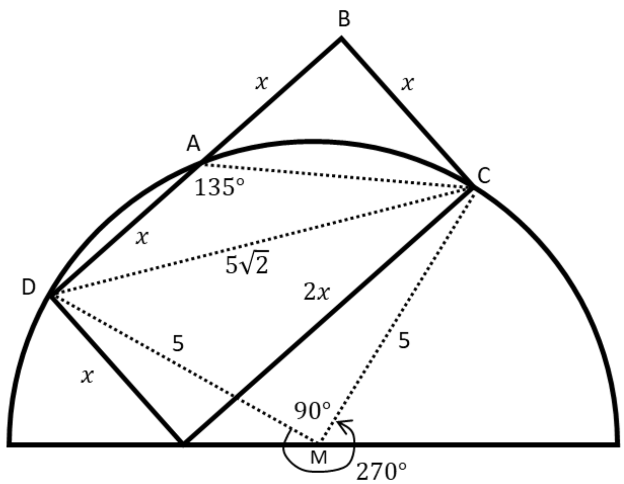
Puzzle 32. Rotated rectangle in a semicircle

A rectangle of dimensions x by $2x$ is situated in relation to a semicircle of radius 5 as shown in the figure below. The intersection point A exactly divides the associated edge of the rectangle into two segments. Find the area of the rectangle.



Solution: As a first step, we find the distance between D and C in the figure below.

Since ABC is an isosceles triangle and the angle at B is $90°$, $\angle BAC = \angle BCA = 45°$ which implies that $\angle DAC = 135°$. By Theorem 23, the central angle associated with $\angle DAC$ is $270°$ (this is measured the "long way" around point M) which that implies that $\angle DMC = 90°$ (measuring the "short way" around point M). We can now apply the Pythagorean theorem to triangle DMC to determine that $d(DC) = 5\sqrt{2}$.



Applying the Pythagorean theorem to triangle DBC, gives us

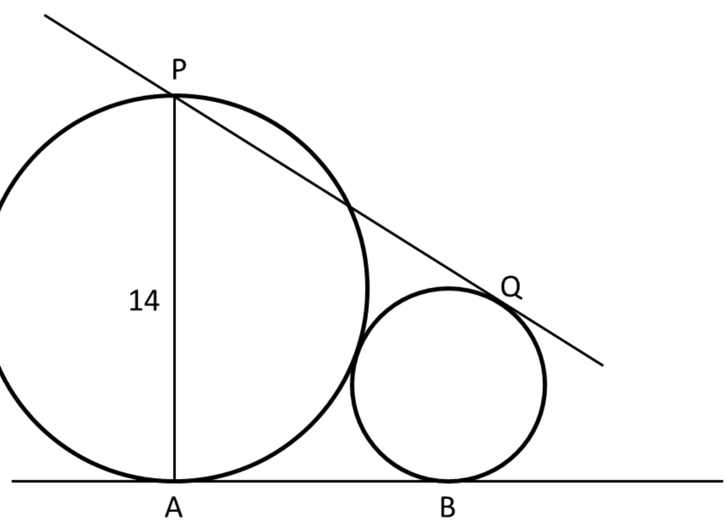
$$(2x)^2 + x^2 = \left(5\sqrt{2}\right)^2$$

$$5x^2 = 50 \Rightarrow x = \sqrt{10}$$

So, the area of the rectangle is $\sqrt{10} \cdot (2\sqrt{10}) = 20$.

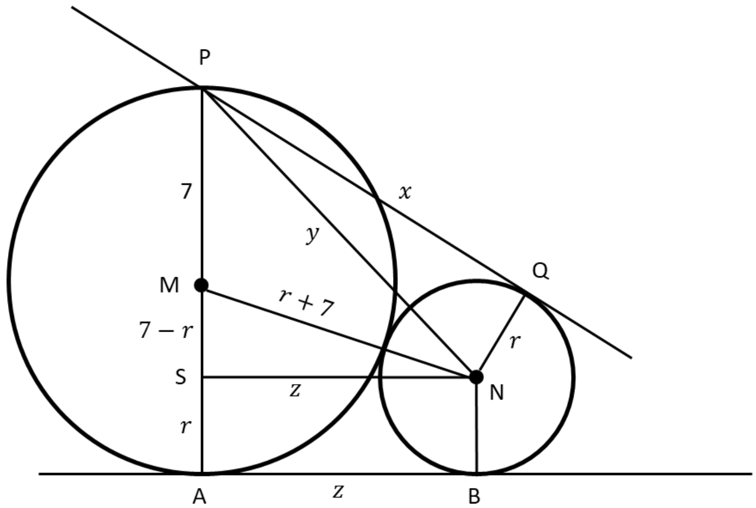
Puzzle 33. Big circle – small circle with common roof

Two tangent circles are as shown in the figure below. The diameter of the circle on the left is 14. Line AB is tangent to both circles. Line PQ is tangent to the circle on the right. Find the distance between P and Q.



Discussion: If the radius of the circle on the right approaches 0 (which is not excluded by the statement of the puzzle), then B coincides with A, and $d(PQ) = 14$. If the radius of the circle on the right is 7 (i.e., same size as the other circle), then again, we get that $d(PQ) = 14$. It appears the answer may be 14 but we need to prove this.

Solution: Regarding the figure below, let $d(PG) = x$, $d(PN) = y$ and $d(AB) = z$.



Applying the Pythagorean theorem to triangle QNP, we have

$$r^2 + x^2 = y^2 \text{ (Equation 1)}$$

Draw line SN parallel to line AB and note that $d(SN) = d(AB) = z$. This implies $d(AS) = r$ and $d(PS) = 14 - r$

Applying the Pythagorean theorem to triangle PSN, we have

$$(14 - r)^2 + z^2 = y^2$$

Substituting Equation 1 into the above equation gives us

$$(14 - r)^2 + z^2 = r^2 + x^2 \text{ (Equation 2)}$$

Applying the Pythagorean theorem to triangle MSN,

$$(7 - r)^2 + z^2 = (r + 7)^2$$

$$49 - 14r + r^2 + z^2 = r^2 + 14r + 49$$

$$z^2 = 28r \text{ (Equation 3)}$$

Substitute Equation 3 into Equation 2 to get

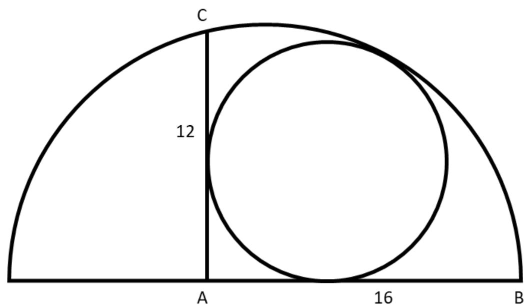
$$(14 - r)^2 + 28r = r^2 + x^2$$

$$14^2 - 28r + r^2 - 28r = r^2 + x^2 \Rightarrow x = 14$$

[**Author's remark**: The above proof assumes the circle on the right is smaller than the one on the left. The reader may want to try solving the problem in the case where the circle on the right is larger than the one on the left.]

Puzzle 34. Circle inside a semicircle

In the figure below, a circle of radius r is positioned inside a semicircle of radius R, with tangencies as suggested in the figure. We are given that $d(AC) = 12$ and $d(AB) = 16$. Find r.

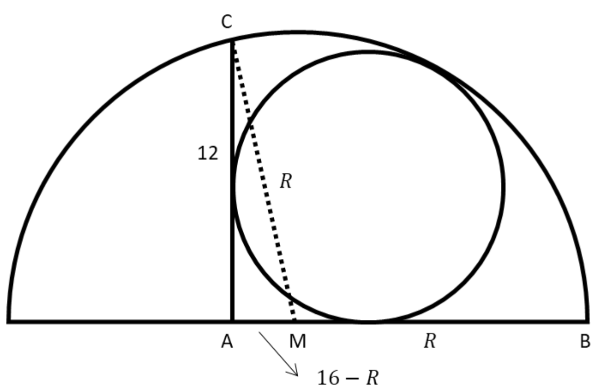


Solution: Draw a line from C to the center of the semicircle M, and note that $d(AM) = 16 - R$. Applying the Pythagorean theorem to triangle ACM, we have

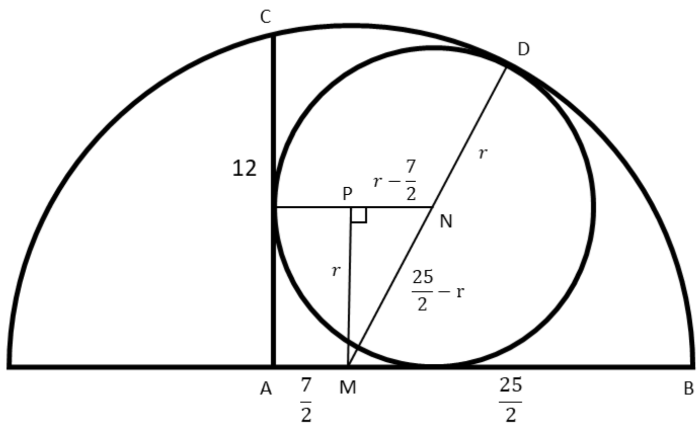
$$12^2 + (16 - R)^2 = R^2$$

$$144 + 256 - 32R + R^2 = R^2$$

$$32R = 400 \Rightarrow R = \frac{25}{2}$$



Next, draw a radius from M to D, and a radius from N to D. By Theorem 22, both radii are perpendicular to the tangent point D, and thus, MND is a straight line. Create the triangle MPN as shown in the figure below.



Applying the Pythagorean theorem to triangle MPN, we have

$$r^2+\left(r-\frac{7}{2}\right)^2=\left(\frac{25}{2}-r\right)^2$$

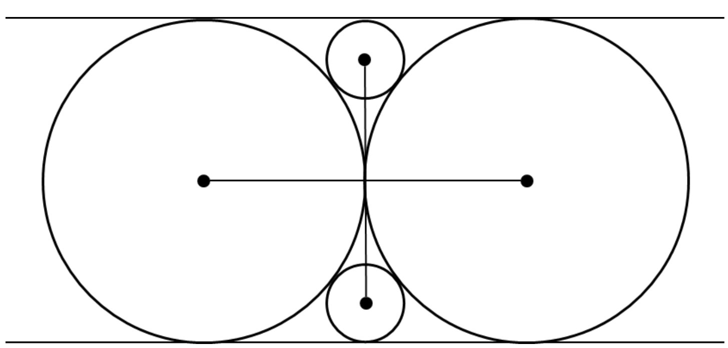
$$r^2+r^2-7r+\frac{49}{4}=\frac{625}{4}-25r+r^2$$

$$r^2+18r-144=0$$

$$(r+24)(r-6)=0 \Rightarrow \boldsymbol{r=6}$$

Puzzle 35. Two big circles and two little circles

Two identical big circles and two identical smaller circles are configured as shown in the figure below (tangencies are as suggested in the figure). If the distance between the centers of the two smaller circles is 96, what is the distance between the centers of the two larger circles?



Solution: By Theorem 22, C, M and E lie on a straight line. By construction, the tangent line between the two larger circles goes through the center of the two smaller circles, as shown in the figure below. Let x be the radius of each of the larger circles, and y be the radius of each of the smaller circles. We seek $d(CE)=2x$.

Consider the line segment $\overline{AB}$. We have that $d(AB)=2x=96+2y \Rightarrow y=x-48$.

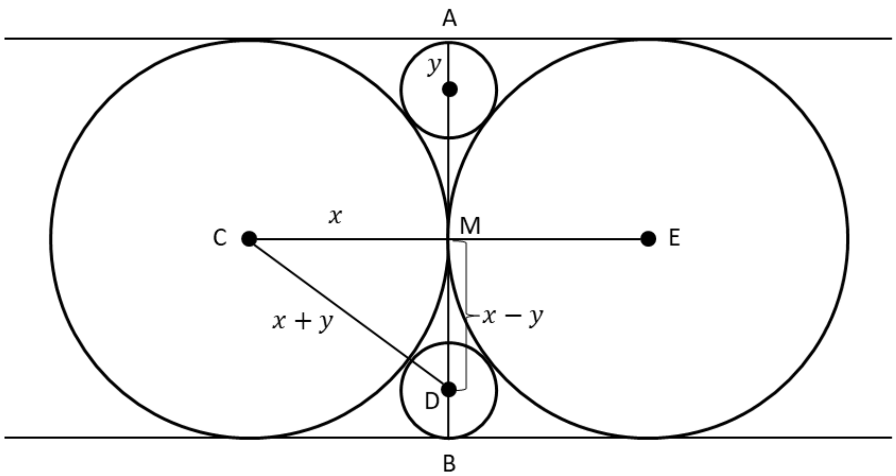
Applying the Pythagorean theorem to triangle CDM, we get

$$x^2+(x-y)^2=(x+y)^2$$

Substitute $y = x - 48$ into the above equation to get

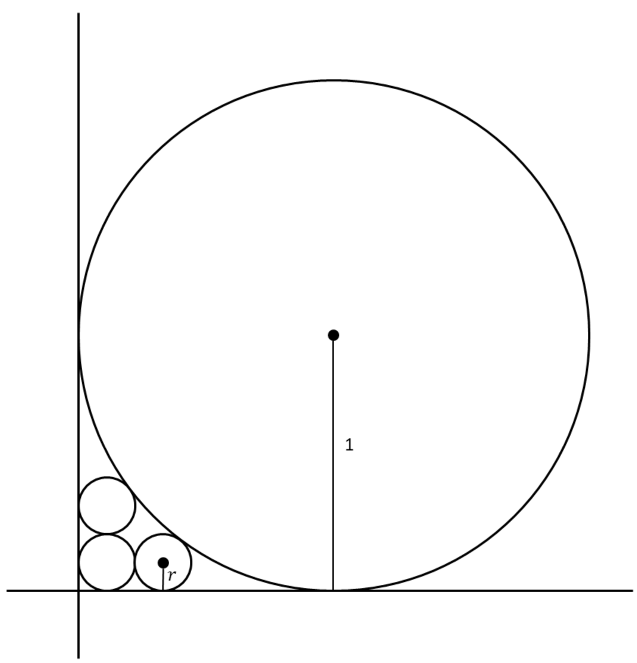
$$x^2 + (x - (x - 48))^2 = (x + (x - 48))^2$$
$$x^2 + 48^2 = (2x - 48)^2 = 4x^2 - 192x + 48^2$$
$$3x^2 - 192x = x(3x - 192) = 0 \Rightarrow x = 64$$

Thus, $d(CE) = 2x = 128$.



Puzzle 36. Three small circles bounded on the right by a big circle

In the figure below, three small circles of radius r are bounded by tangent lines on the left and bottom, and by a larger circle of radius 1 on the right. What is the value of r?



Solution: Draw the line segment $\overline{AC}$ parallel to the line at the bottom of the circles (as shown in the figure below). Since A is a distance $3r$ from the vertical line on the left, $d(AC) = 1 - 3r$. We also have that $d(BC) = 1 - r$ and $d(AB) = 1 + r$. Applying the Pythagorean theorem to triangle ABC gives us the answer, i.e.,

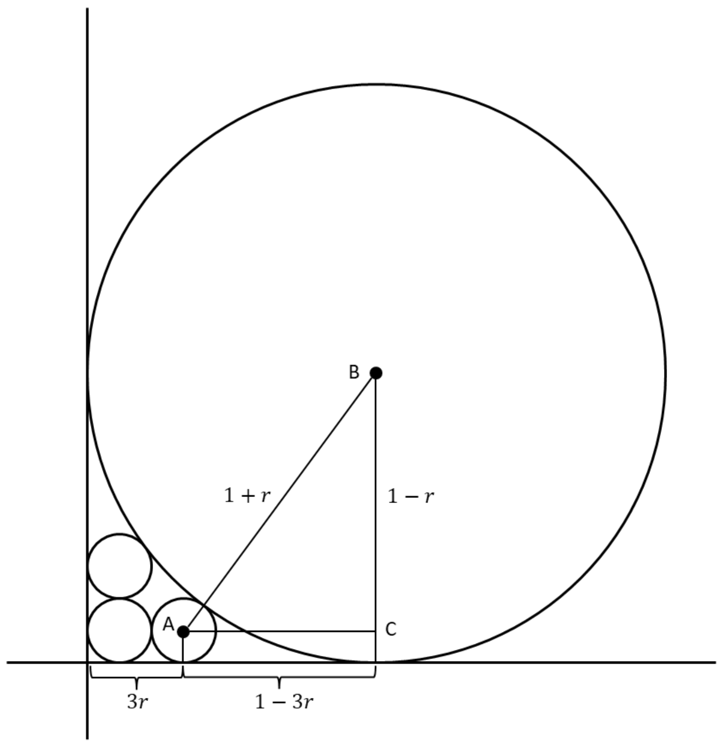
$$(1 - r)^2 + (1 - 3r)^2 = (1 + r)^2$$

$$1 - 2r + r^2 + 1 - 6r + 9r^2 = 1 + 2r + r^2$$

$$9r^2 - 10r + 1 = 0$$

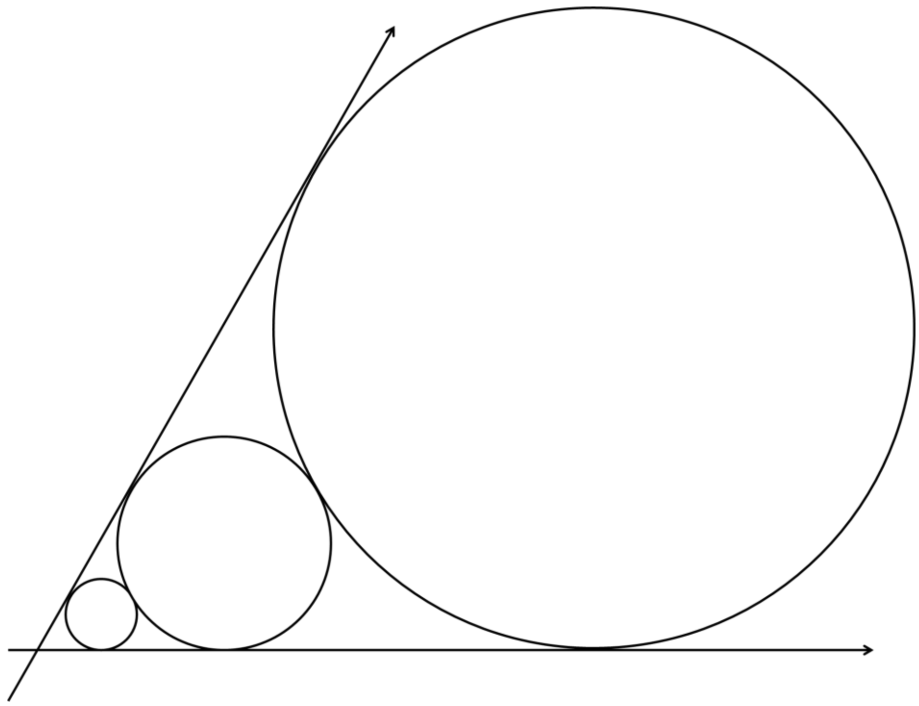
$$(9r - 9)\left(r - \frac{1}{9}\right) = 0 \Rightarrow r = \frac{1}{9}$$

(Note that the other solution $r = 1$ is not possible since $r < 1$ by the construction of the puzzle.)

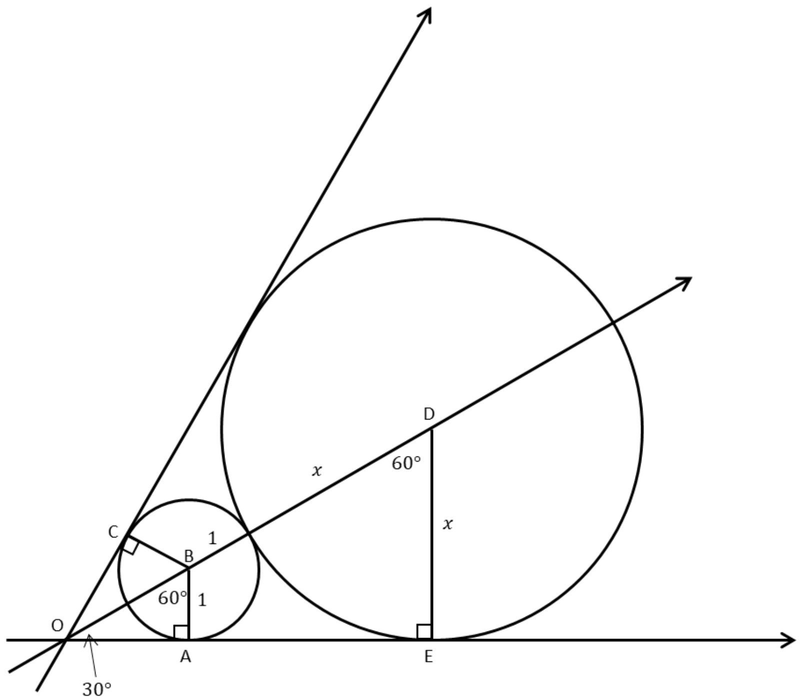


Puzzle 37. Cascade of every larger circles

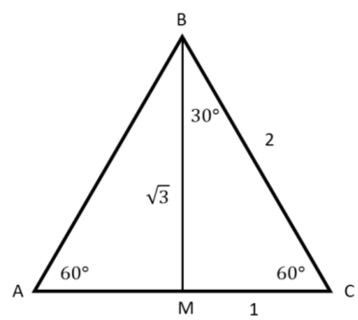
In the figure below, a string of tangent circles are placed between two lines that intersect at $60°$. The radius of the smallest circle is 1. If we place n circles (of increasing size) in this configuration, what is the radius of the n^{th} circle?



Solution: By repeated application of Theorem 22, we can deduce that the centers of the circles and their points of tangency lie on a straight line (line BD in the figure below). At this point, we don't know that O is on line BD.



In the above figure, we've drawn perpendicular radii from B to line OA, and from B to line OC. By Theorem 10, triangles OCB and OAB are congruent. Thus, $\angle COB = \angle AOB = 30°$. This implies that OAB (as well as OCA) are 30-60-90 triangles. Recall from basic trigonometry that the sides of a 30-60-90 are in the ratio $2:1:\sqrt{3}$ (see the figure below).



So, we have that $d(OB) = 2$ (also $d(OA) = d(OC) = \sqrt{3}$ but we don't need this for the puzzle solution). Using the same logic, we can show that triangle ODE is a 30-60-90 triangle. Thus, lines OB and OD are both at $30°$ to line OA, and we can now conclude that O is on line BD.

Next, let $d(DE) = x$. Adding up the various segments, we see the $d(OD) = 2 + 1 + x = 3 + x$. Since ODE is a 30-60-90 triangle, we have that

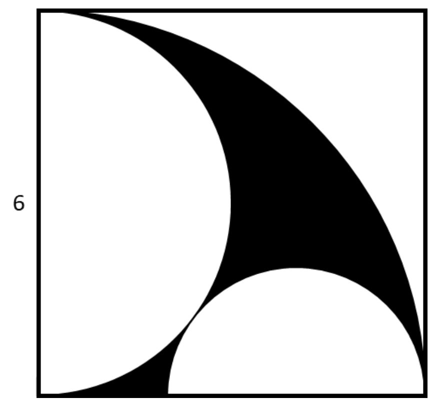
$$d(OD) = 2d(DE)$$

$$3 + x = 2x \Rightarrow x = 3$$

We can repeat the same logic to show that the radius of the third circle in the sequence is 3 times the radius of the second, i.e., 9. Continuing the process, we conclude that the radius of the n^{th} circle is 3^{n-1}.

Puzzle 38. The Ulu Knife puzzle

In the figure below, we have square of side length 6 that encloses a quarter-circle (black) of radius 1.5 which is overlapped by a white semicircle (left) of radius 3 and a smaller semicircle (bottom right) of radius r. Find the area of the exposed black region (which looks like an Ulu knife blade, at least to me).

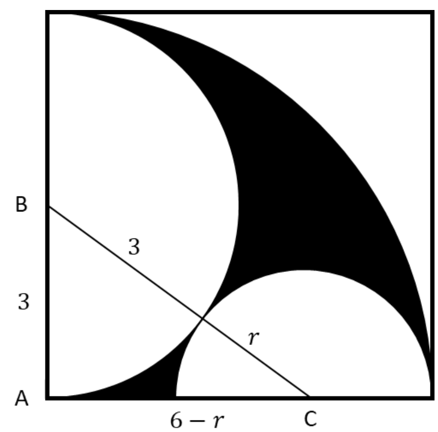


Solution: The segment between the centers of the semicircles (i.e. points B and C in the following figure) also contains the point of tangency between the two semicircles (by Theorem 22). Applying the Pythagorean theorem to right triangle ABC, we have

$$(3+r)^2 = 3^2 + (6-r)^2$$

$$9 + 6r + r^2 = 9 + 36 - 12r + r^2$$

$$18r = 36 \Rightarrow r = 2$$

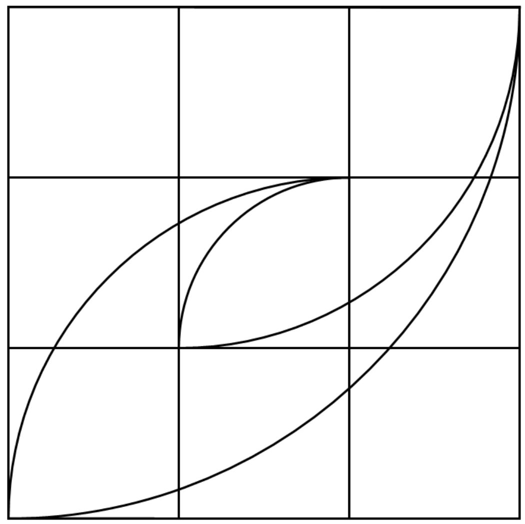


The area of the shaded region equals the area of the quarter-circle minus the combined areas of the two semicircles, i.e.,

$$\frac{1}{4}\pi(6)^2 - \frac{1}{2}\pi(3)^2 - \frac{1}{2}\pi(2)^2 = \frac{5\pi}{2}$$

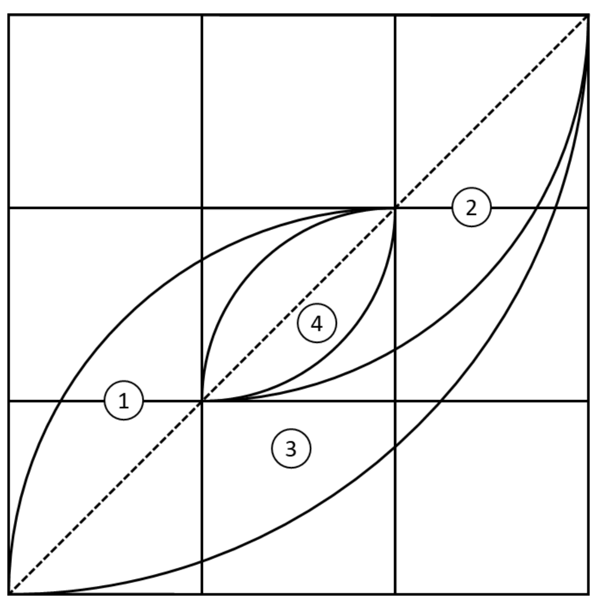
Puzzle 39. Lobster claw puzzle

The figure below consists of nine 4×4 squares in the background, and what appears to be the outline of a lobster claw. The lobster claw is made of quarter circles of various sizes (whose radii can be determined by the positioning of each quarter circle relative to the grid in the background). Find the area of the lobster claw.

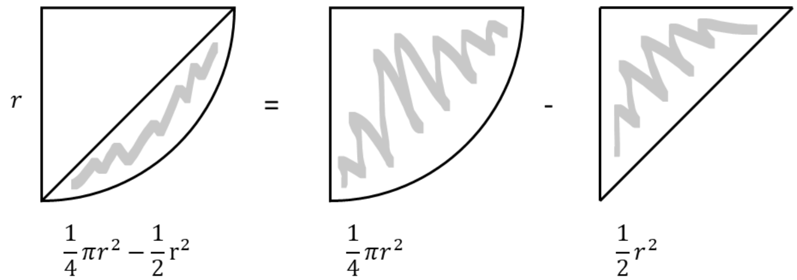


Solution: The solution presented here is based on Solution #2 in the YouTube video "How To Solve The Area Of A Claw" [30].

Draw a diagonal from bottom left to top right, as shown in the figure below. Next take region 1 and move it into the position labeled as region 2 (noting that the two regions are congruent). Regions 2 and 3 are equivalent to the area of the original claw but are easier to compute.



The approach is to compute the area of region 2,3 and 4 as a unit, and then subtract the area of region 4, with the result being the area of region 2 and 3. Regions 2,3 and 4 combined form a secant of a quarter-circle of radius 12. Region 4 is a secant of a quarter-circle of radius 4. So, the problem reduces to finding the formula for the secant of a quarter-circle (which we illustrate in the figure below).



Using the above formula, the area of regions 2,3 and 4 is

$$\frac{1}{4}\pi \cdot 12^2 - \frac{1}{2} \cdot 12^2$$

and the area of region 4 is

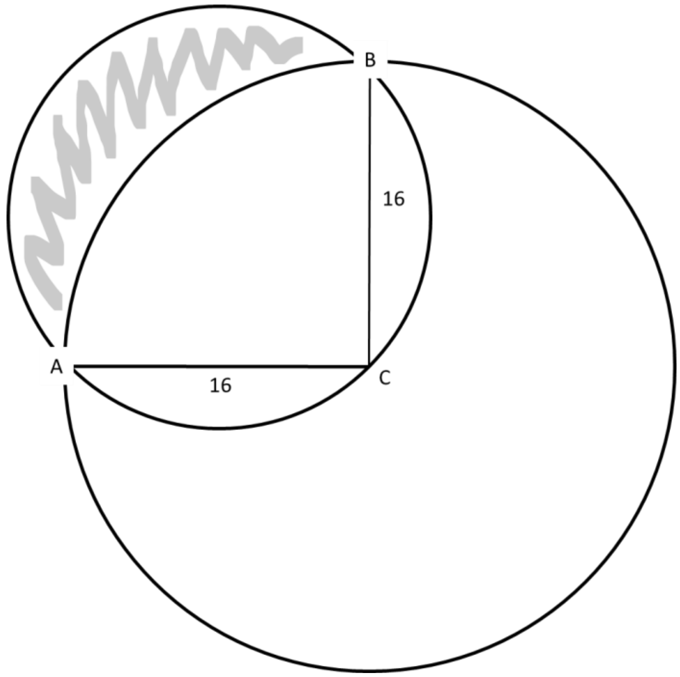
$$\frac{1}{4}\pi \cdot 4^2 - \frac{1}{2} \cdot 4^2$$

Subtracting the area of region 4 from the combined areas of regions 2,3 and 4 gives us the final result, i.e., $32\pi - 64$.

[**Author's Remark**: The figures in this document were drawn using Microsoft PowerPoint which has several drawbacks concerning the construction of complex geometric figures. One of the major drawbacks is that it is very hard to fill regions in a complex diagram which is why I resorted to using the squiggly pattern to indicate a region in several puzzles, including this puzzle and the next.]

Puzzle 40. Overlapping circles

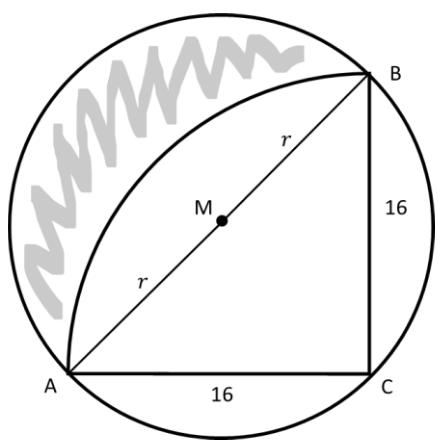
In the figure below, a larger circle of radius 16 overlaps a smaller circle. Point C is the center of the larger circle. The two circles intersect at points A and B. Find the area in the smaller circle but outside the larger circle (the region with the squiggly line in the figure).



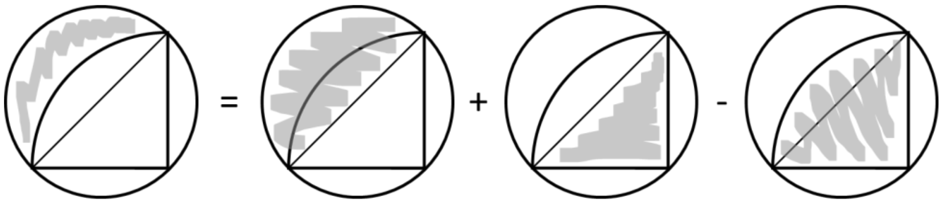
Solution: In the figure below, we focus on the smaller circle and its intersection with the larger circle. Applying the Pythagorean theorem to triangle ABC, we have

$$(2r)^2 = 16^2 + 16^2 = 512$$

$$4r^2 = 512 \Rightarrow r = 8\sqrt{2}$$



To find the area of the shaded region we add the area of the upper-left semicircle to the area of the triangle and then subtract the area of the quarter circle, see the figure below. In general, if the radius of the quarter circle is R then the area of the shaded region in question is $\frac{R^2}{2}$.



$$\frac{1}{2}\pi\left(8\sqrt{2}\right)^2 + \frac{1}{2}(16)(16) - \frac{1}{4}\pi 16^2$$

$$= 64\pi + 128 - 64\pi = 128$$

5 Polygon Puzzles

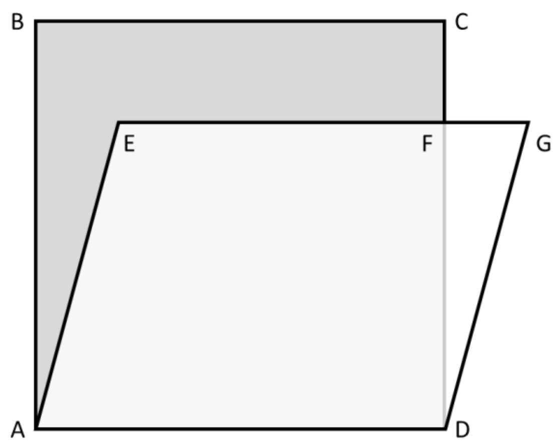
Where there is matter, there is geometry. – Johannes Kepler

Puzzle 41. Prove that the diagonals of a convex kite are perpendicular

Solution: Recall that there are two cases for kites, i.e., convex and concave (see Figure 14). The stated property does not hold for concave kites. A proof of the result can be found in the Proof Wiki article "Diagonals of Kite are Perpendicular" [31].

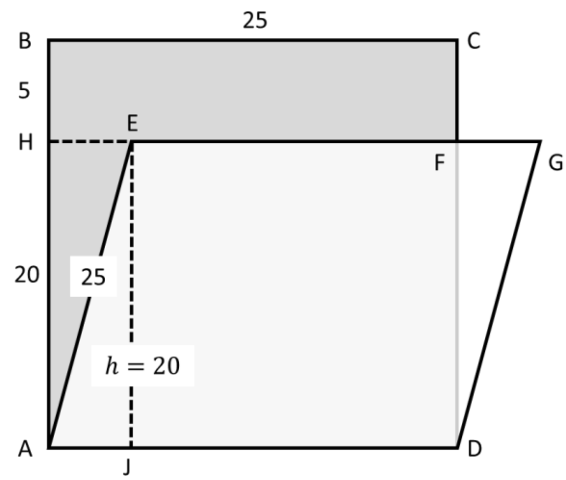
Puzzle 42. A square and a rhombus walk into a bar

A square and a rhombus walk into a bar. The rhombus has too much to drink and falls on top of the square. How much of the square is not covered by the rhombus, i.e., find the area of the shaded region (the polygon $ABCFE$)? You are given that the area of the square $ABCD$ is 625, and the area of the rhombus $AEGD$ is 500.



Solution: Since the area of the square is 625, the length of each side is 25. Since the rhombus shares side $\overline{AD}$ with the square, we can say the length of the base of the rhombus is 25. Further, since the sides of a rhombus are of equal length, we know each side is of length 25.

The area of a rhombus is given by bh where b is the length of the base and h is its height. So, we have $500 = 25h$ which implies $h = 20$. From this, we can deduce that $d(AH) = 20$ and $d(BH) = 5$. The various quantities are shown in the figure below (not drawn exactly to scale).

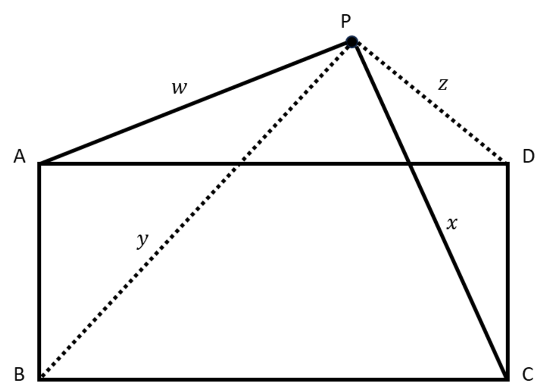


Applying the Pythagorean theorem to triangle AHE, we find that $d(HE) = 15$, and so the triangle has area $(.5)(15)(20) = 150$. The rectangle $BHFC$ has area 125. Thus, the area of the shaded region is 275.

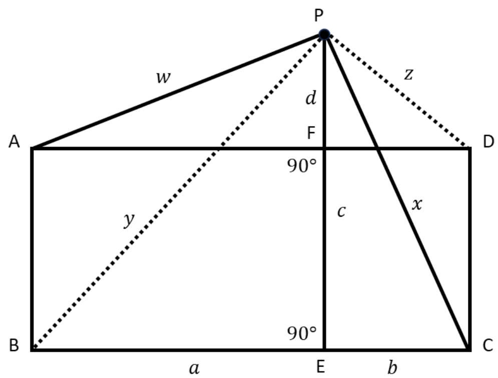
Puzzle 43. British Flag Puzzle

For any point P, exterior to a rectangle, show that the sum squares of the distances to pairs of opposite vertices of the rectangle are equal. In the figure below, you are to prove

$$w^2 + x^2 = y^2 + z^2$$



Solution: The proof is illustrated in the following figure.



Let $d(BE) = a, d(EC) = b, d(EP) = c$ and $d(PF) = d$.
Since ABCD is a rectangle, $d(AF) = a$ and $d(FD) = b$.
Using the Pythagorean theorem four times, we get

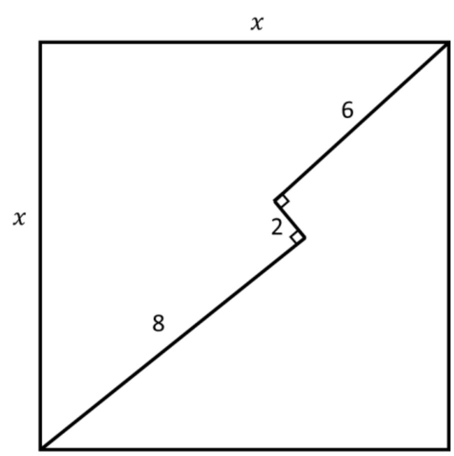
$$y^2 = a^2 + c^2$$
$$z^2 = d^2 + b^2$$
$$w^2 = a^2 + d^2$$
$$x^2 = c^2 + b^2$$

Thus, $w^2 + x^2 = y^2 + z^2$.

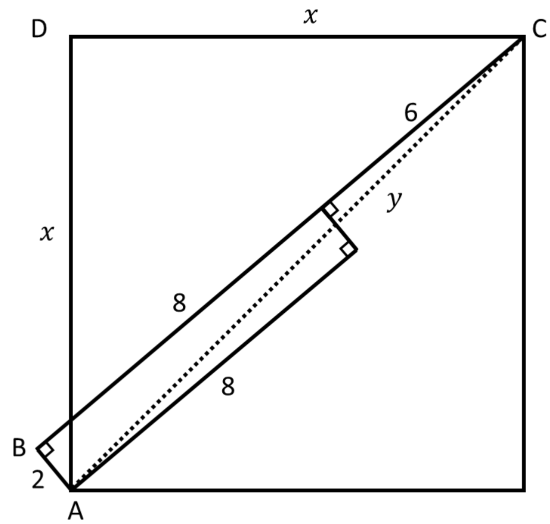
The result is also true if P is inside the rectangle, or if P is outside of the plane containing the rectangle in 3-dimensional space. The Wikipedia article "British flag theorem" [51] explains why the puzzle is named as such (not very convincing but the name has stuck).

Puzzle 44. Find the side length of a square given some internal information

For the square in the following diagram, find x. The segments within the square meet at right angles.



Solution: Construct the 2×8 rectangle shown in the figure below. Next, notice that ABC is a right triangle and we know the length of two of the sides. The necessary calculations are shown in the figure.



Applying the Pythagorean theorem to BAC

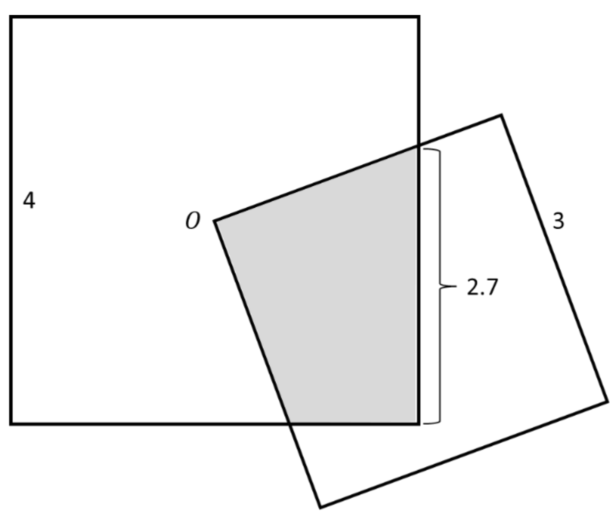
$$2^2 + 14^2 = y^2$$
$$y = 10\sqrt{2}$$

Applying the Pythagorean theorem to ADC

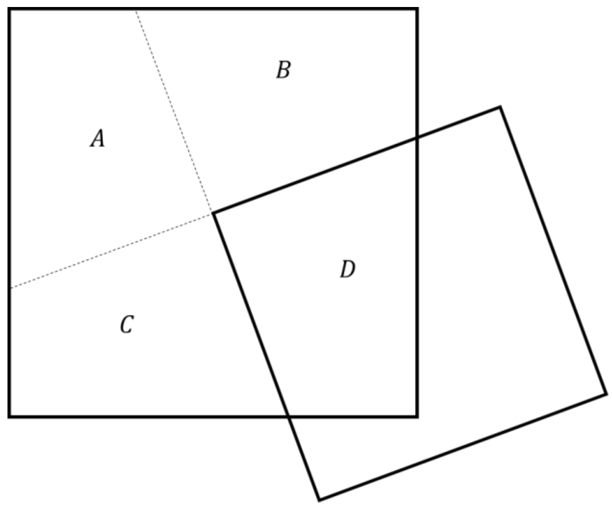
$$x^2 + x^2 = \left(10\sqrt{2}\right)^2 = 200$$
$$x^2 = 100 \Rightarrow x = 10$$

Puzzle 45. Two overlapping squares

Two squares overlap as shown in the figure below. One square has sides of length 4 and the other if length 3. The upper left vertex of the small square coincides with the center of the larger square (point O in the figure). Find the area of the shaded region.

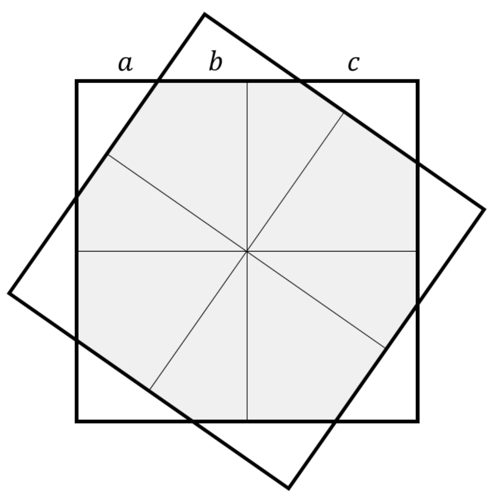


Solution: The information concerning the length 2.7 is not needed to solve the puzzle, and can easily throw one off (as it did with me). If we extend the two sides of the smaller square as shown in the figure below, we see that the larger square is divided into four regions of equal area. Thus, each region is of area $\frac{16}{4} = 4$. In addition, no matter how we rotate the smaller square about the center of the larger square, the overlapping area still results in $\frac{1}{4}$ of the area of the larger square.

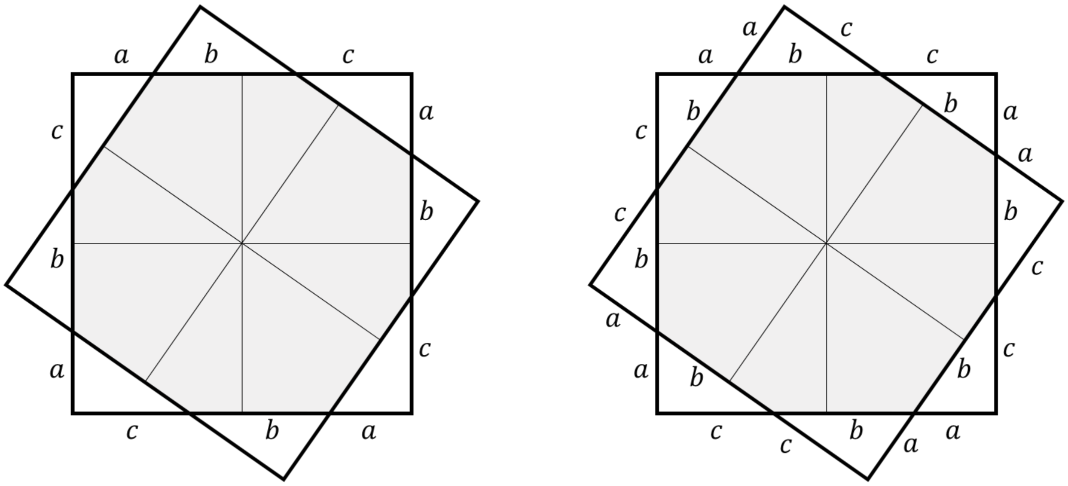


Puzzle 46. One square overlapping and rotated over another square

One square is placed over another square as shown in the figure below. The two squares are of the same size, and have the same center (i.e., one square is rotated over the other). The top edge of one square is divided into segments of lengths a, b and c by the placement of the other square. What proportion of the total area of the resulting star-shaped figure is shaded?



Solution: Since the squares are identical with a common center, the rotation of one square over the other makes the same cuts on all four edges (see the left side of the figure below). By repeated use of the AA triangle similarity principle, we see that all the triangles exterior to the shaded region are similar (see the right side of the figure). By the Pythagorean theorem, we have the $a^2 + c^2 = b^2$.



Let x be the side length of each square. We have that $x - b = a + c$. By squaring both side of this equation, we get

$$(x-b)^2 = (a^2 + c^2) + 2ac$$

$$x^2 - 2xb + b^2 = b^2 + 2ac$$

$$x^2 - 2xb = 2ac$$

The area of the star-shaped figure formed by the two squares is

$$x^2 + 4\left(\frac{ac}{2}\right) = x^2 + 2ac = x^2 + (x^2 - 2xb) = 2x^2 - 2xb$$

The area of the shaded region is

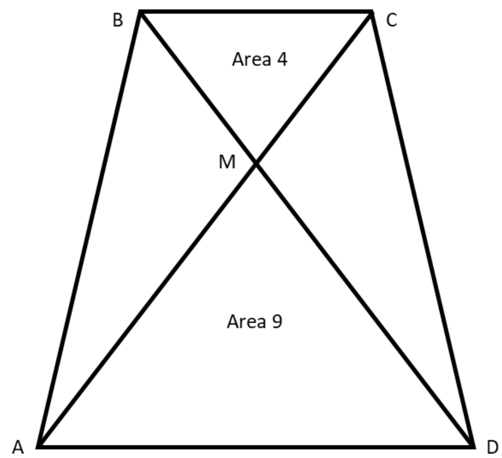
$$x^2 - 4\left(\frac{ac}{2}\right) = x^2 - 2ac = x^2 - (x^2 - 2xb) = 2xb$$

So, the proportion of the shaded region to the overall start-shaped figure is

$$\frac{2xb}{2x^2 - 2xb} = \frac{b}{x-b} = \frac{b}{a+c}$$

Puzzle 47. Find the area of a trapezoid given the area of two triangles within

Find the area of the trapezoid in the figure below, given that the area of triangle BMC is 4 and the area of triangle AMD is 9.



Solution: By Theorem 15, $\angle CBM = \angle ADM$ and $\angle BCM = \angle MAD$. Thus, by the AA triangle similarity principle, $\triangle BMC \cong \triangle DMA$.

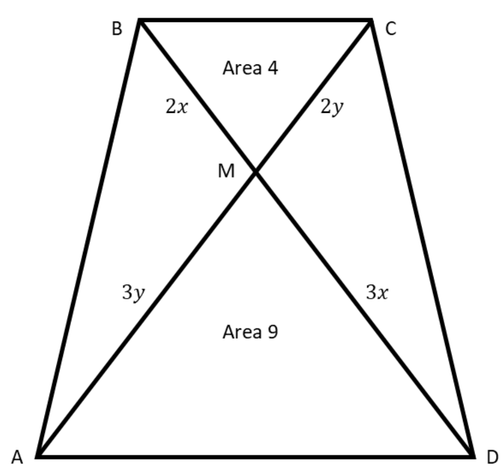
By Theorem 53,

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9} = \frac{[BMC]}{[DMA]} = \frac{d(BC)^2}{d(AD)^2} = \frac{d(BM)^2}{d(MD)^2} = \frac{d(MC)^2}{d(AM)^2}$$

which implies

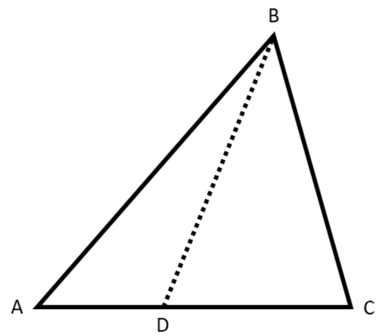
$$\frac{d(BC)}{d(AD)} = \frac{d(BM)}{d(MD)} = \frac{d(MC)}{d(AM)} = \frac{2}{3}$$

If we let $d(BM) = 2x$ and $d(MC) = 2y$, then $d(AM) = 3y$ and $d(MD) = 3x$. See the updated figure below.



Before proceeding, we need a preliminary result. Take any triangle ABC and draw a line segment from B to $\overline{AC}$, as shown in the following figure. Triangles ADB and CDB are both of height h. Thus, we have that

$$\frac{[ABD]}{[CBD]} = \frac{\frac{1}{2}d(AD) \cdot h}{\frac{1}{2}d(DC) \cdot h} = \frac{d(AD)}{d(DC)}$$



Using our preliminary result, we have

$$\frac{[ABM]}{4} = \frac{[ABM]}{[CBM]} = \frac{d(AM)}{d(CM)} = \frac{3y}{2y} = \frac{3}{2}$$

$$[ABM] = 6$$

Using our preliminary result again, we have

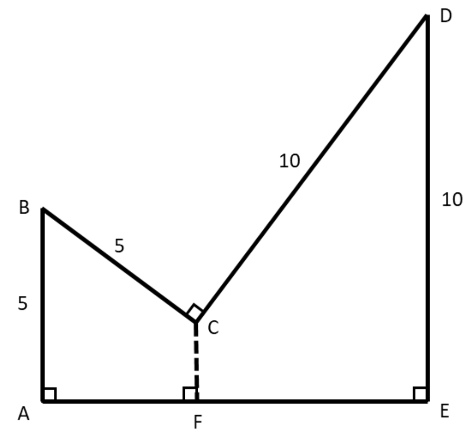
$$\frac{[DCM]}{4} = \frac{[DCM]}{[CBM]} = \frac{d(DM)}{d(BM)} = \frac{3x}{2x} = \frac{3}{2}$$

$$[DCM] = 6$$

So, the area of the trapezoid is $4 + 6 + 6 + 9 = 25$.

Puzzle 48. 5-gon distance problem

In the 5-gon shown in the figure below, find the distance between points C and F. In addition to the right angles and distances shown in the figure, we are given that $d(AE) = 10$.



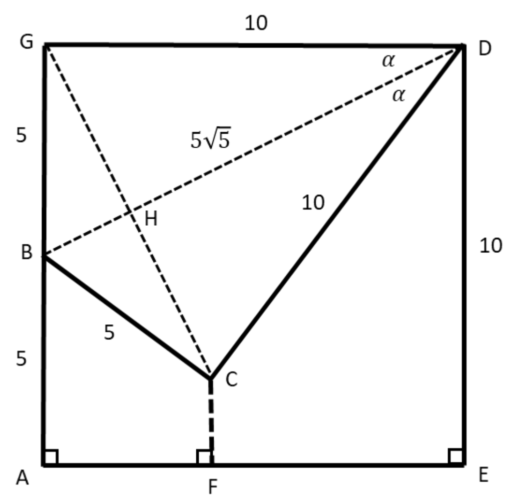
Solution: Draw a line segment from point B to D, thereby creating triangle BCD. Reflect triangle BCD about line BD to create congruent triangle BGD.

By the Pythagorean theorem, $d(BD) = 5\sqrt{5}$.

Since $\angle BGH = \angle BCD = 90°$ and triangles BGH and BGD share the angle $\angle GBH$, we have (by the AA triangle similarity principle) that $\triangle BGH \cong \triangle BGD$. Thus,

$$\frac{d(GH)}{d(BG)} = \frac{d(GD)}{d(BD)} \Rightarrow \frac{d(GH)}{5} = \frac{10}{5\sqrt{5}} \Rightarrow d(GH) = \frac{10}{\sqrt{5}}$$

Since (by construction) triangles BCD and BGD are congruent, we also have that $d(HC) = \frac{10}{\sqrt{5}}$.



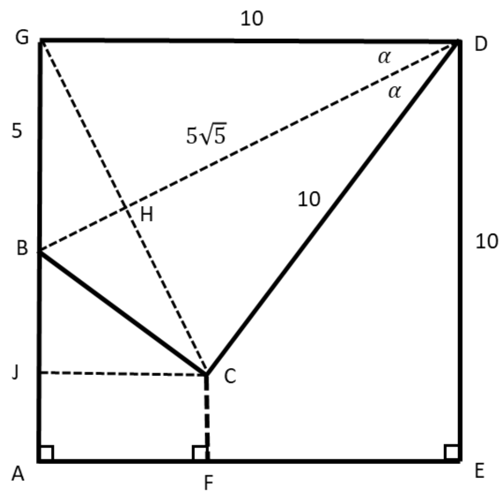
Next, draw a line segment from C to line AG that is parallel to line AE (see the line segment $\overline{JC}$ in the figure below). By the AA triangle similarity principle, $\triangle BGH \cong \triangle JGC$. Since we already know that $\triangle BGH \cong \triangle BGD$, we have by the transitivity property of triangle similarity that

$$\triangle JGC \cong \triangle BGH \cong \triangle BGD$$

Thus,

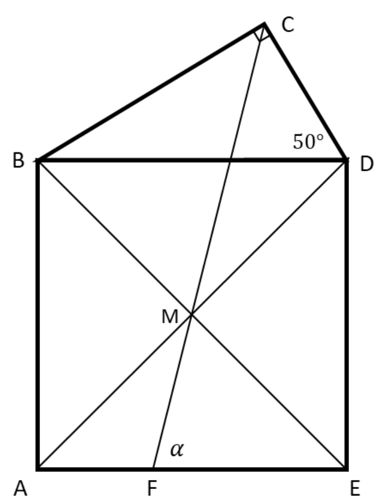
$$\frac{d(GJ)}{d(GC)} = \frac{d(GD)}{d(BD)} \Rightarrow \frac{d(GJ)}{\frac{20}{\sqrt{5}}} = \frac{10}{5\sqrt{5}} \Rightarrow d(GJ) = 8$$

Since $d(GA) = d(GJ) + d(JA)$, we have $10 = 8 + d(JA)$. Thus $d(CF) = d(JA) = 2$.



Puzzle 49. Square with a right triangle on top

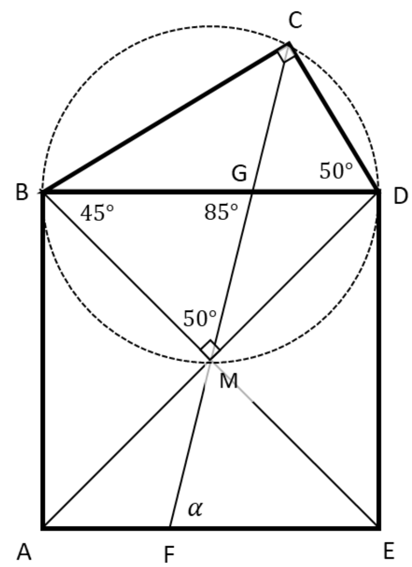
In the figure below, we have a right triangle on top of a rectangle. The line CF passes through the center of the rectangle (i.e., the intersection of the diagonals) and $\angle CDB = 50°$. What is the measure of angle α?



Solution: The diagonals of a square intersect at $90°$, and so, $\angle BMD = 90°$. Since the interior angles of a quadrilateral (in this case BCDM) add to $360°$, and $\angle BCD + \angle BMD = 90° + 90° = 180°$, it must be that $\angle CBM + \angle CDM = 180°$. By Theorem 46, BCDM is a cyclic quadrilateral and thus, it can be circumscribed by a circle (as shown in the figure below).

Since angle CDB and BMC subtend the same arc on the circle, they are of equal measure, i.e., $\angle BMC = 50°$.

Since the diagonals of a square bisect each other, we have that $d(BM) = d(DM)$ and thus, BDM is an isosceles triangle. Since $\angle BMD = 90°$, it must be that $\angle MBD = \angle BDM = 45°$. Since the angles of a triangle equal $180°$, it must be that $\angle BGM = 85°$. By Theorem 15, $\alpha = 85°$.

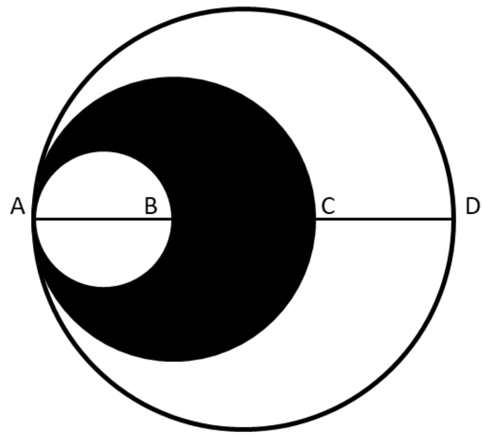


6 Area Puzzles

Equations are just the boring part of mathematics. I attempt to see things in terms of geometry.
– Stephen Hawking

Puzzle 50. Shady circle in the middle

Three circles are tangent at point A as shown in the figure below. We are given the $d(AB) = d(BC) = d(CD)$. What fraction of the configuration is shaded?

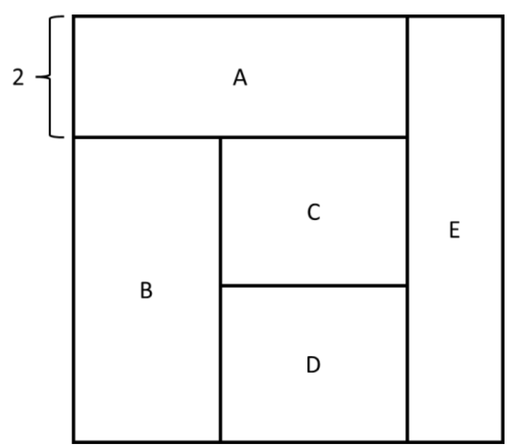


Solution: If we let r be the radius of the smallest circle, then the radius of the middle circle is $2r$ and the radius of the largest circle is $3r$. From smallest to largest, the respective areas of the circles are $\pi r^2, 4\pi r^2, 9\pi r^2$.

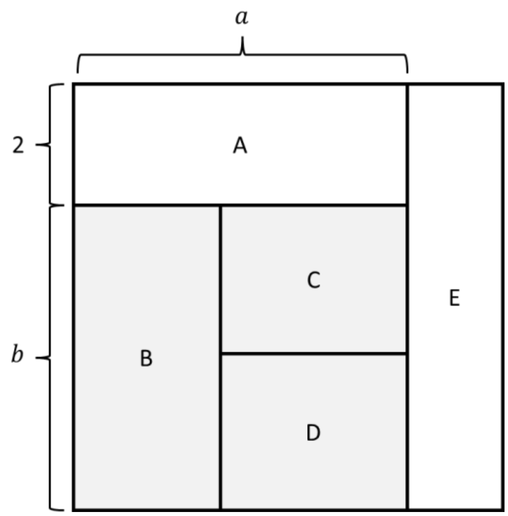
So, the area of the shaded area is $4\pi r^2 - \pi r^2 = 3\pi r^2$ and the shaded fraction is $\frac{3\pi r^2}{9\pi r^2} = \frac{1}{3}$.

Puzzle 51. Find the area of the square

Find the area of the large square in the figure below if all the smaller regions (A, B, C, D and E) are of the same area. Further, one side of rectangle A is 2 units in measure.

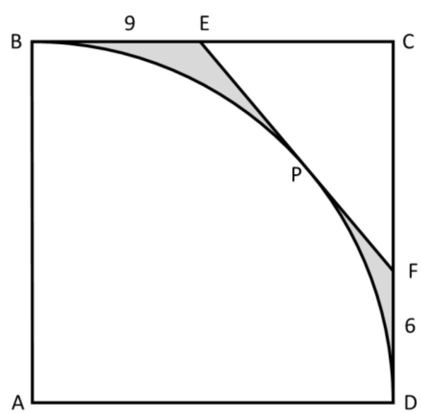


Solution: Label with variables the side of rectangles A and B as shown in the figure below. With this labeling, the area of rectangle A (and all other rectangles by the condition of the puzzle) is $2a$. The area of the shaded region is ab which must equal $3(2a) = 6a$. So, $ab = 6a \Rightarrow b = 8$. Thus, each side of the square is $b + 2 = 8$ and the area of the square is 64.

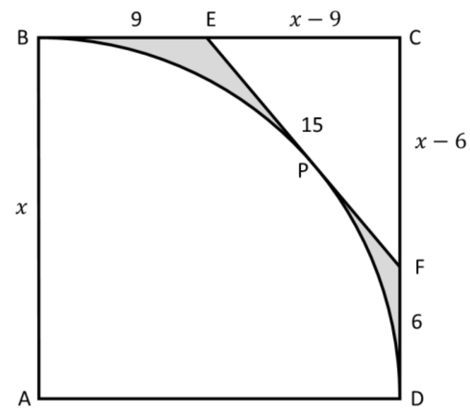


Puzzle 52. Find the area of the unusual shape

Find the area of the shaded area in the figure below (not necessarily drawn exactly to scale). The polygon ABCD is a square which contains a quarter circle and the right triangle CEF. We are given $d(BE) = 0$ and $d(DF) = 6$.



Solution: By Theorem 31, $d(EP) = d(BE) = 9$ and $d(PF) = d(DF) = 6$. So, the hypotenuse of triangle CEF is of length 15. In the figure below, let the length of one side of the square be x.



Applying the Pythagorean theorem to the triangle ECF, we get the equation

$$(x-9)^2 + (x-6)^2 = 15^2$$
$$x^2 - 18x + 81 + x^2 - 12x + 36 = 225$$
$$2x^2 - 30x - 108 = 0$$
$$x^2 - 15x - 54 = 0$$
$$(x-18)(x+3) = 0$$

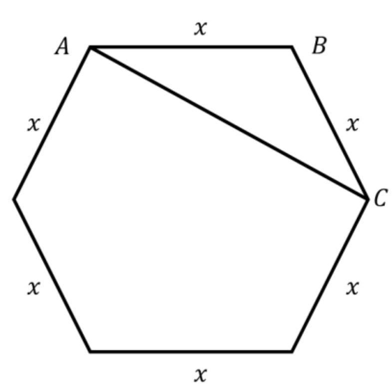
We take the positive of the two solutions, i.e., $x = 18$.

So, the square has area $18^2 = 324$, the area of the triangle is $\frac{1}{2} \cdot 9 \cdot 12 = 54$ and the area of the quarter circle is $\frac{1}{4}\pi 18^2 = 81\pi$. Thus, the area of the shared region is

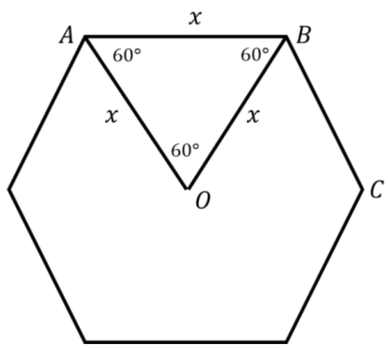
$$324 - 54 - 81\pi = 270 - 81\pi$$

Puzzle 53. Find the area of the triangle in a regular hexagon

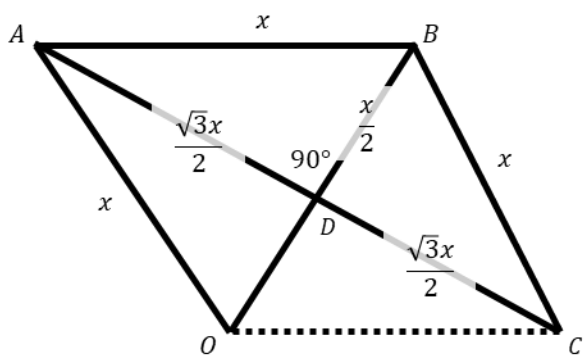
Find the area of the triangle shown in the upper right of the regular hexagon.



Solution: By Theorem 42, the sum of the interior angles of a hexagon is $(6-2)180° = 720°$. In a regular hexagon, all the interior angles are equal and so, each interior angle is of measure $\frac{720°}{6} = 120°$. Form a triangle with one vertex at the center of the hexagon (i.e., at the intersection of the line segments between opposite vertices) and the other two vertices coinciding with two adjacent vertices of the hexagon, as shown in the figure below. All the angles of this triangle of $60°$ and thus, it is an equilateral triangle.

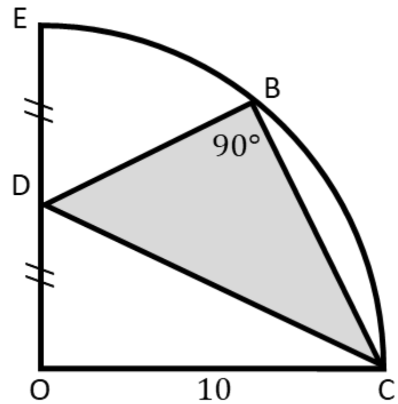


By Theorem 40, line segment $\overline{AC}$ bisects and is perpendicular to line segment $\overline{OB}$, since OABC is a rhombus. From here, we use the Pythagorean theorem triangle ADB to determine that $d(AD) = \frac{\sqrt{3}}{2}x$ and so, $d(AC) = 2d(AD) = \sqrt{3}x$ (which is the measure of the base of the triangle in question). So, the area of the triangle is $\frac{1}{2}\left(\sqrt{3}x\right)\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{4}x^2$.

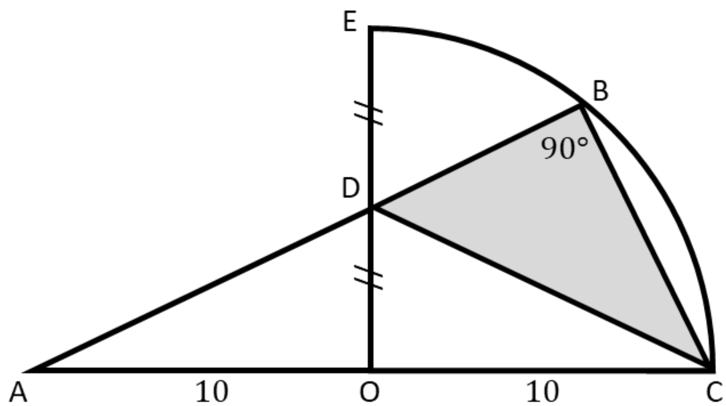


Puzzle 54. Find the area of the triangle in the quarter circle

Find the area of the triangle in the figure below. The measure of $\overline{OC}$ is 10, and $d(OD) = d(DE) = 5$. The shape around the triangle is a quarter circle.



Solution: Reflect the triangle ODC about the line OE to get triangle ODA, as shown in the figure below. Using the Pythagorean theorem on triangle ADO, we have that $d(AD) = 5\sqrt{5}$.



By the AA triangle similarity principle, triangles CBA and ADO are similar. Based on this similarity, we have the following two relationships:

$$\frac{d(CB)}{d(AC)} = \frac{d(DO)}{d(DA)} \Rightarrow \frac{d(CB)}{20} = \frac{5}{5\sqrt{5}} \Rightarrow d(CB) = \frac{20}{\sqrt{5}} = \frac{20}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = 4\sqrt{5}$$

$$\frac{d(BA)}{d(AC)} = \frac{d(AO)}{d(DA)} \Rightarrow \frac{d(BA)}{20} = \frac{10}{5\sqrt{5}} \Rightarrow d(BA) = 8\sqrt{5}$$

Further, $d(BD) = d(BA) - d(AD) = 3\sqrt{5}$.

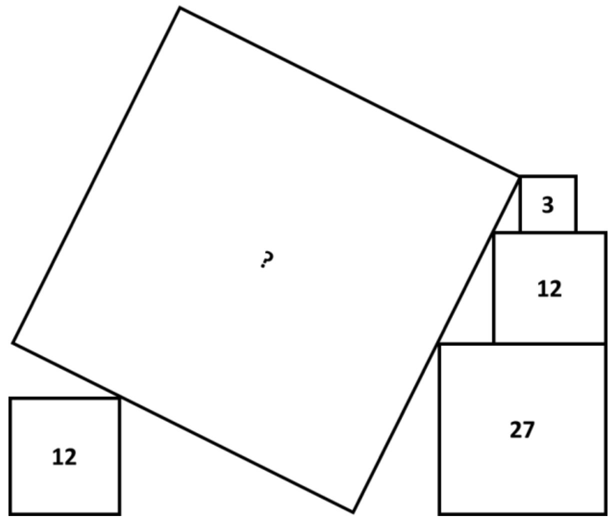
Using the formula for the area of a triangle, we have that the area of triangle BCD is

$$\frac{1}{2}(4\sqrt{5})(3\sqrt{5}) = 30$$

[**Author's Remark**: The solution to this problem entails a common technique (trick) where it helps to add structure to the given configuration to solve the puzzle. I personally find this type of puzzle to be very difficult.]

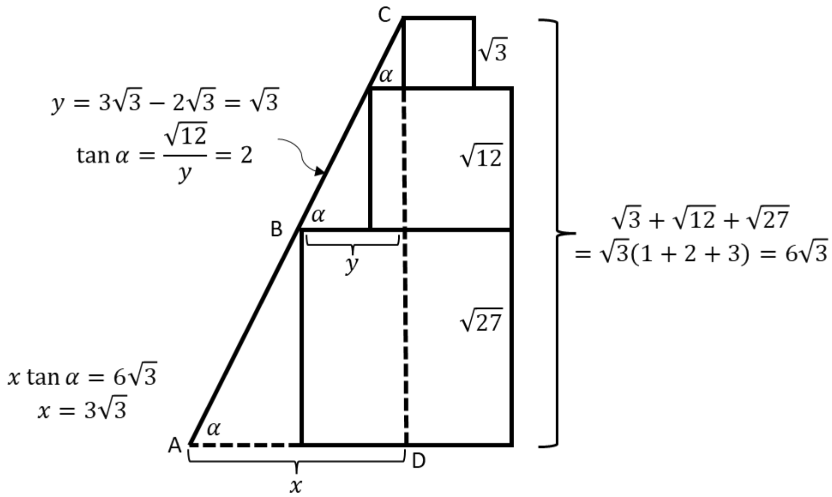
Puzzle 55. Find the area of the toppled square

Find the area of the large square in the following figure. All the items in the figure are squares. The numbers represent areas. The squares of areas 3, 12 and 27 line-up with the larger square exactly as shown.



Solution: It turns out that the square of area 12 on the lower left is not necessary to solve the problem, although it does give the appearance of supporting the large square. To solve the puzzle, we focus on the right side of the large square and the three smaller squares to its right (as shown in the figure below).

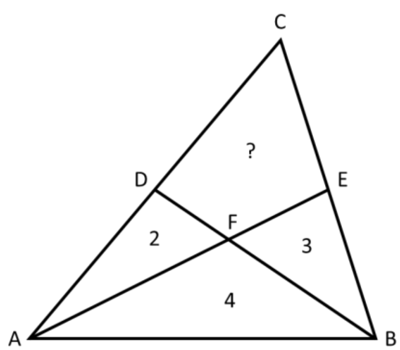
The right side of the large square is part of a transversal to the parallel lines going through the base of each of the smaller squares. By Theorem 15, the three angles labeled as α in the figure are of the same measure. First, we calculated the tangent of α using the middle triangle in the figure. From here, we can determine the measure of the base of the triangle ADC (shown as x in the figure). Further, we are essentially given the height of triangle ADC. Finally, we use the Pythagorean theorem to compute $d(AC)^2$ which is the square of the side of the large triangle, i.e., its area.



$$d(AC)^2 = d(AD)^2 + d(DC)^2$$
$$= x^2 + \left(6\sqrt{3}\right)^2 = 108 + 27 = 135$$

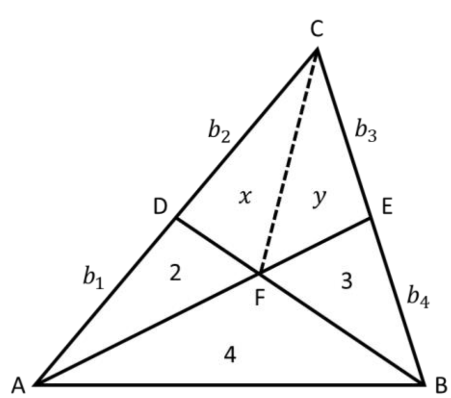
Puzzle 56. Find the area of the unknown region in a triangle

The triangle below is divided into 4 regions by two line segments, i.e., $\overline{AE}$ and $\overline{BD}$. The areas of three of the regions are shown in the figure. Find the area of the quadrilateral $CDFE$.



Solution: We will make repeated use of Theorem 52 to solve the problem.

Divide the unknown region into regions of area x and y.



Comparing triangles CDB and ADB (noting that both have the same height), and applying Theorem 52, we have

$$\frac{x+y+3}{2+4} = \frac{b_2}{b_1} \Rightarrow \frac{x+y+3}{6} = \frac{b_2}{b_1}$$

Comparing triangles DCF and DAF in a similar manner to the above, we get

$$\frac{x}{2} = \frac{b_2}{b_1}$$

The previous two results give us the following (which we label as Equation 1)

$$\frac{x}{2} = \frac{x+y+3}{6}$$

Next, compare the ratio of the areas of triangles ACE and ABE to get

$$\frac{x+y+2}{7} = \frac{b_3}{b_4}$$

Comparing the ratio of the areas of triangles FCE and FBE, we get

$$\frac{y}{3} = \frac{b_3}{b_4}$$

Combing the previous two results gives us the following (which we label as Equation 2)

$$\frac{x+y+2}{7} = \frac{y}{3}$$

Simplifying Equations 1 and 2, we get the following system of two equations in two unknowns

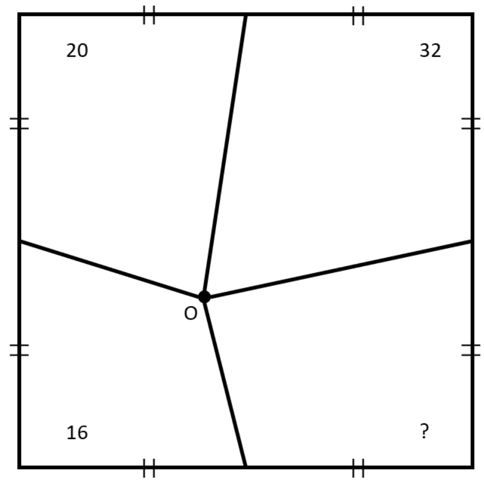
$$y = 2x - 3$$

$$3x + 6 = 4y$$

Substituting the equation for y in the first equation into the second equation and solving for x, we get $x = \frac{18}{5}$. Plugging the value for x into the first equation yields $y = \frac{21}{5}$. So, the area of the previously unknown region is $\frac{18}{5} + \frac{21}{5} = \frac{39}{5}$.

Puzzle 57. Find the area of the unknown region in a square

Find the area of the unknown region (bottom right) of the square shown below. The four interior line segments go from point O to the midpoint of each side of the square. The areas of the other three regions are shown in the figure (not drawn to scale).



Solution: Label the diagram as shown in the figure below. Uppercase letters are points. The lower case letters are areas (as yet unknown). Triangles that have the same base and height measure also have the same area. For example, triangle ABO and BCO have the same base and height measure, and thus, we label their areas with the same variable, i.e., a. Similar arguments hold for the other areas labeled in the diagram. On the other hand, we were given the areas of three regions each of which are comprised of two triangles. So, we have

$$a + d = 16$$

$$a + b = 20$$

$$b + c = 32$$

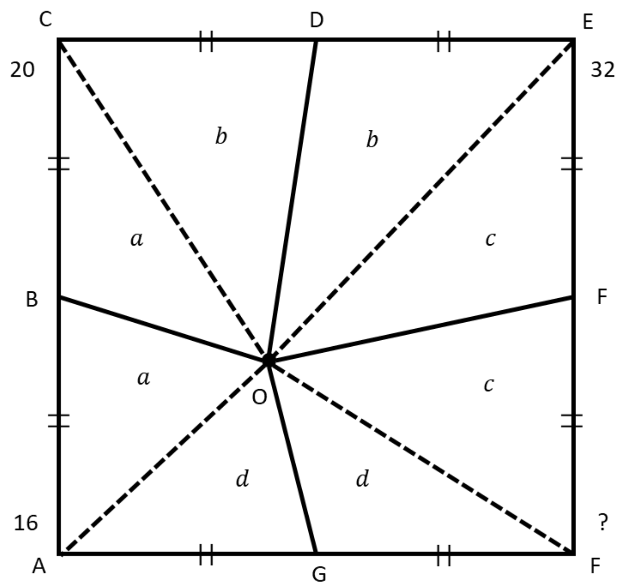
By the associative law of real numbers, we have the following

$$(a + b) + (d + c) = (a + d) + (b + c)$$

Substituting in the various values for $a + d$, $a + b$ and $b + c$ into the above equation yields

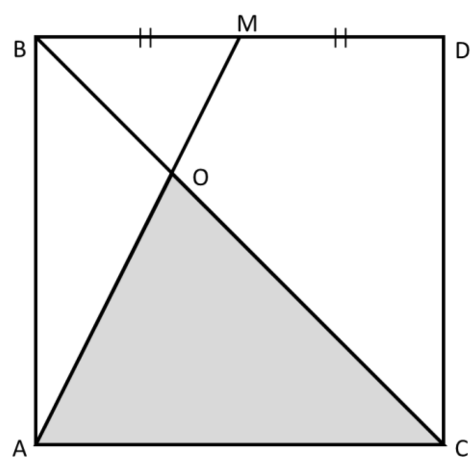
$$20 + (d + c) = 16 + 32$$

Thus, the unknown area $(d + c)$ is 28.



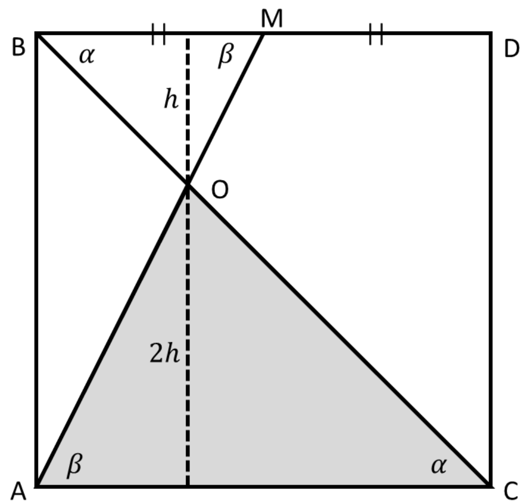
Puzzle 58. What fraction of the square is occupied by the triangle?

In the figure below, find the fraction of the area of the square that is occupied by the shaded triangle.



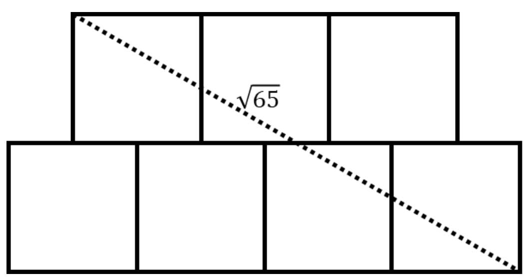
Solution: Since the line through BD is parallel to the line through AC, we can use Theorem 15 to establish that $\angle OBM = \angle OCA$ (angle α in the figure below) and $\angle OAC = \angle BMO$ (angle β in the figure below). By the AA triangle similarity principle, triangles BOM and AOC are similar with proportion constant $k = 2$. Since $\triangle BOM \cong \triangle AOC$ are in proportion 2, their heights are also in proportion 2 (by Theorem 3).

So, the height of the entire square is $3h$ and the area of the square is $9h^2$. The area of the shaded triangle is $\frac{1}{2}(3h)(2h) = 3h^2$. Thus, the fraction of the area of the square covered by the shaded triangle is $\frac{3h^2}{9h^2} = \frac{1}{3}$.



Puzzle 59. Find the area of the squares

Find the area of the 7 identical squares shown in the figure below. The top blocks are shifted exactly ½ of a side length over the bottom blocks.



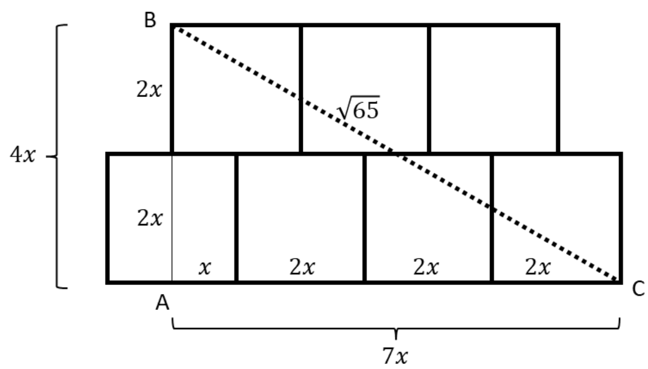
Solution: Let the side length of a square be $2x$. The triangle ABC, as labeled in the figure below, is a right triangle. Using the Pythagorean theorem, we have

$$(4x)^2 + (7x)^2 = \left(\sqrt{65}\right)^2$$

$$16x^2 + 49x^2 = 65$$

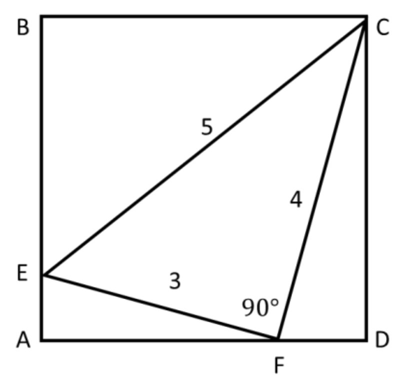
$$65x^2 = 65 \Rightarrow x = 1$$

So, each square has area 4 and the entire configuration has area 28.



Puzzle 60. Find the area of the square with embedded 3-4-5 triangle

Find the area of the square shown in the figure below. The right triangle EFC intersects the square at points E, F and C.



Solution: We first note the following

- Since the angles of a triangle sum to $180°$, $\angle AEF + 90° + \angle EFA = 180°$. (Equation 1)
- Since straight lines have angle $180°$, $\angle EFA + 90° + \angle CFD = 180°$. (Equation 2)

From Equation 1, $\angle EFA = 90° - \angle AEF$. Plugging this into Equation 2, we get

$$90° - \angle AEF + 90° + \angle CFD = 180°$$

$$\angle AEF = \angle CFD$$

Since two of the three angles of triangles AEF and DFC are equal, we have by the AA triangle similarity principle that $\triangle AEF \cong \triangle DFC$. The ratio of corresponding side measure are equal for similar triangles, and so,

$$\frac{3}{4} = \frac{d(EF)}{d(CF)} = \frac{d(AF)}{d(CD)}$$

Thus, $d(CD) = \frac{4}{3} d(AF)$. Let $d(AF) = x$.

Being two sides of a square, we have that $d(AD) = d(CD) = \frac{4}{3}x$, and so,

$$d(FD) = d(AD) - d(AF) = \frac{4}{3}x - x = \frac{x}{3}$$

Applying the Pythagorean theorem to triangle FDC, we have

$$\left(\frac{4x}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 4^2$$

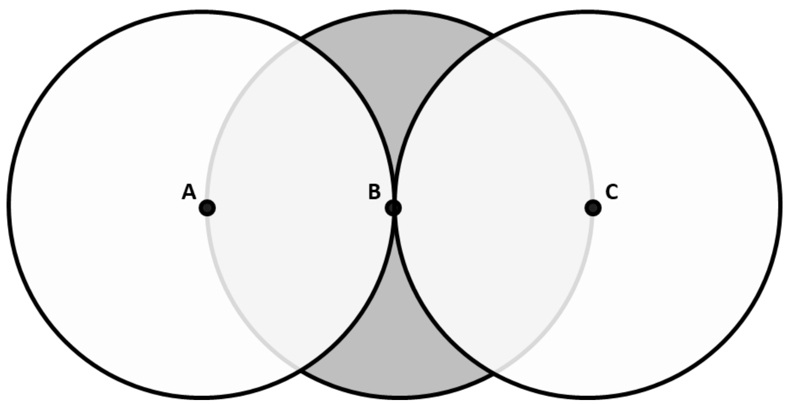
$$16x^2 + x^2 = 12^2$$

$$x = \frac{12}{\sqrt{17}}$$

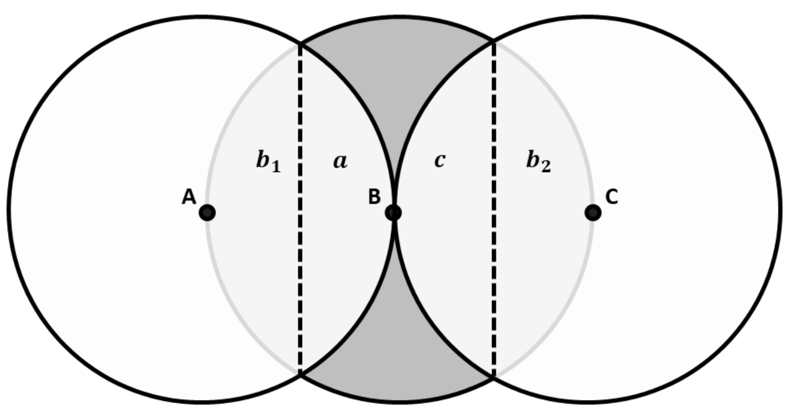
Hence, the length of a side of the square is $d(AD) = \frac{4}{3} \cdot \frac{12}{\sqrt{17}} = \frac{16}{\sqrt{17}}$ and the area of the square is $\frac{256}{17}$.

Puzzle 61. Overlapping circles area problem

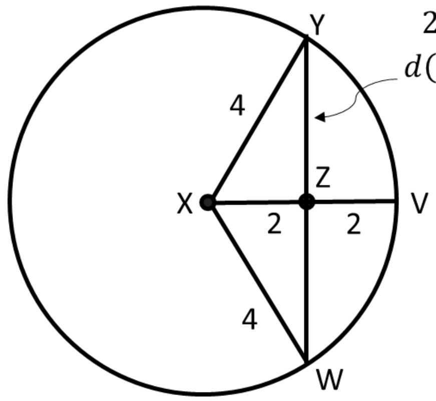
Find the area of the exposed shaded area of the center circle in the figure. Each of the circles have the same radius. We are given that $d(AB) = d(BC) = 4$.



Solution: In the figure below, we have divided each of the light gray regions in half, yielding 4 segments. Segments b_1 and b_2 pertain to the middle circle, segment a pertains to the circle on the left and segment c pertains to the circle on the right. The segments are identical in area. Our approach is to determine the area of one segment, multiply by 4 and then subtract from the area of the middle circle to get the area of the dark shaded region.



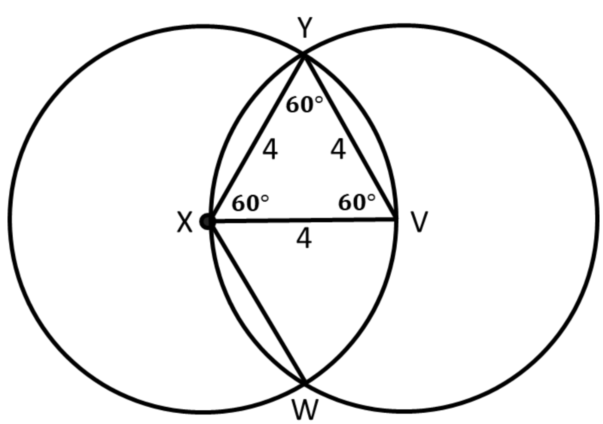
To compute the area of one of the segments, we compute the area of the triangle shown in the figure below (i.e., triangle XYW), compute the area of the sector spanned by $\angle YXW$, and then subtract the area of the triangle from the area of the sector (which gives us the area of one segment).



$$2^2 + d(ZY)^2 = 4^2$$
$$d(ZY) = \sqrt{12} = 2\sqrt{3}$$

$$[XYZ] = \frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$$
$$[XYW] = 2[XYZ] = 4\sqrt{3}$$

To compute the area of the sector spanned by $\angle YXW$, we need to compute the angle. The figure below highlights two of the three circles to emphasize that $d(VY)$ is 4 since it is the radius of the circle on the right (see the figure below). So, triangle XYV is an equilateral triangle, and all of its angles are $60°$. Thus, the measure of $\angle YXW$ is $120°$ ($\frac{1}{3}$ of the area of the circle), and the area of the sector is $\frac{1}{3}(16\pi) = \frac{16}{3}\pi$.

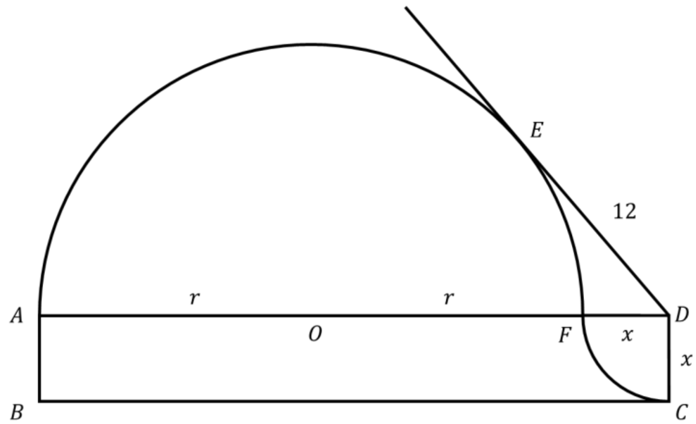


Collecting the results, we have that the area of the dark shaded region is the area of the middle circle minus 4 times the area of a segment, i.e.,

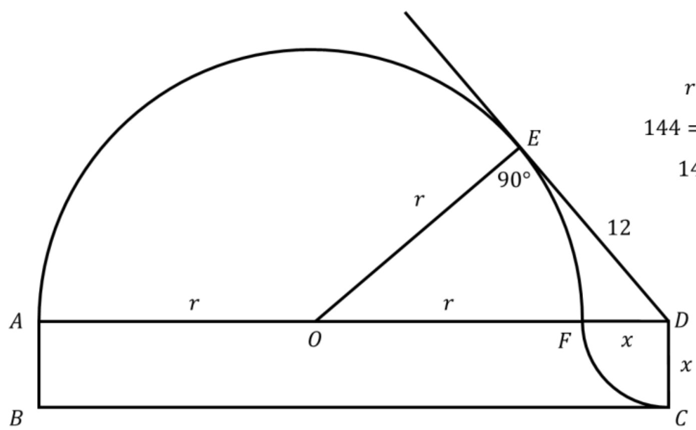
$$16\pi - 4\left(\frac{16\pi}{3} - 4\sqrt{3}\right) = 16\pi - \frac{64}{3}\pi + 16\sqrt{3}$$
$$= \mathbf{16}\left(\sqrt{\mathbf{3}} - \frac{\boldsymbol{\pi}}{\mathbf{3}}\right)$$

Puzzle 62. Find the area of the rectangle

In the figure below, find the area of rectangle $ABCD$. Above the rectangle is a semicircle of radius r and center at point O. In the right corner of the rectangle is a quarter circle of radius x. Line DE is tangent to the semicircle. The length of segment $\overline{DE}$ is 12.



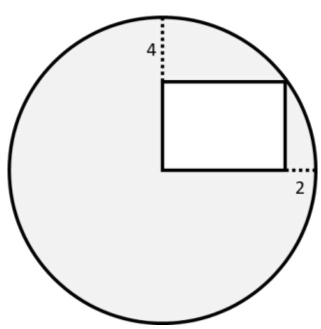
Solution: Draw a radius from point O to point E. By Theorem 21, segment $\overline{OE}$ is perpendicular to line ED. Applying the Pythagorean theorem to triangle OED, we find that the area of the rectangle is 144. The calculation is shown in the following figure.



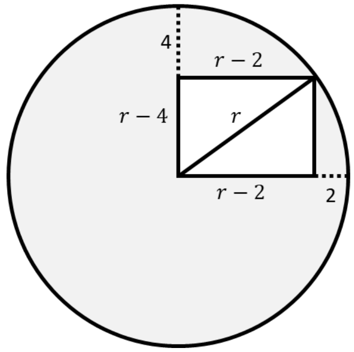
$$r^2 + 12^2 = (r + x)^2$$
$$144 = 2rx + x^2 = (2r + x)x$$
$$144 = d(AD) \cdot d(DC)$$

Puzzle 63. Find the area of the circle minus the rectangle

Find the area of the shaded region in the figure below. The lengths of the two dotted lines are given.



Solution:



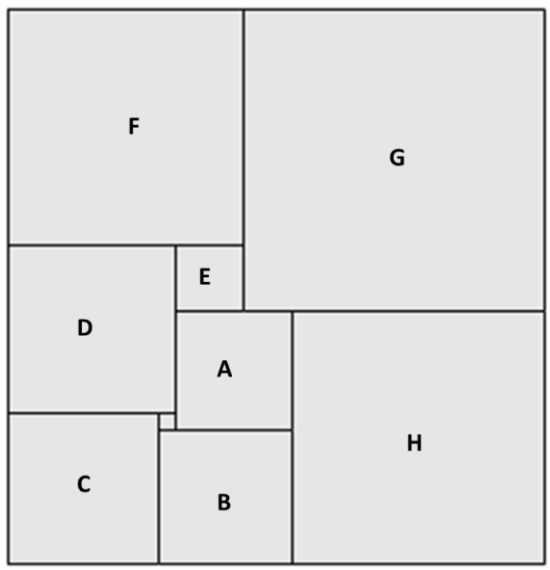
$$(r-4)^2 + (r-2)^2 = r^2$$
$$r^2 - 12r + 20 = 0$$
$$(r-2)(r-10) = 0$$

Select $r = 10$ and reject $r = 2$

Area of shaded region= $100\pi - 48$

Puzzle 64. Find the area of the rectangle comprised of 9 perfect squares

Find the area of the large rectangle in the figure below. The regions A through H are all squares as is the very small square (unlabeled) which has side length 4 (and thus, area 16).



Solution: Let a be the side length of square A, b be the side length of square B, and so on.

We will solve the problem by examining the relationships among the side lengths of the squares.

$$b = a + 4$$
$$c = b + 4 = (a + 4) + 4 = a + 8$$
$$d = c + 4 = (a + 8) + 4 = a + 12$$

Next, we see that

$$d + 4 = a + e$$
$$(a + 12) + 4 = a + e$$
$$e = 16$$

Continuing with the analysis of relationships among the square, we have

$$f = d + e = (a + 12) + 16 = a + 28$$

$$g = f + e = (a + 28) + 16 = a + 44$$

$$h = a + b = 2a + 4$$

At this point, we only need to find a and then substitute back into the other equations. There are several relationships that yield the value of a, e.g.,

$$a + h = e + g$$

$$a + (2a + 4) = 16 + (a + 44)$$

$$a = 28$$

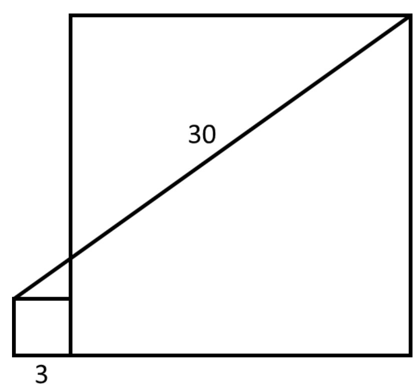
So, we have

$$a = 28, b = 32, c = 36, d = 40, e = 16, f = 56, g = 72, h = 60$$

Thus, the dimension of the rectangle is 132×128, and the area is $16{,}896$.

Puzzle 65. Big square – little square

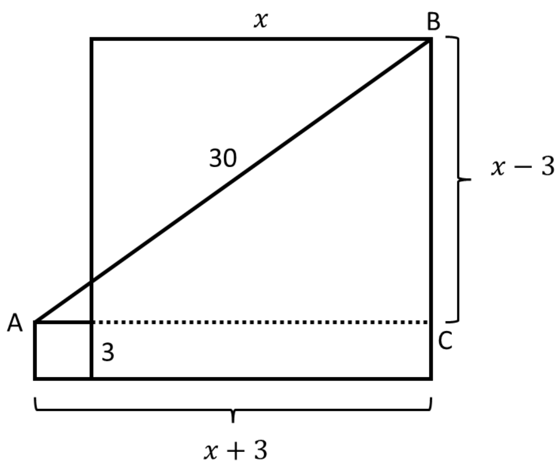
In the figure below, each side of the smaller square is of length 3. The distance between the upper left vertex of the smaller square and the upper right vertex of the larger square is 30. Find the area of the larger square.



Solution: Let x be the side length of the square. Applying the Pythagorean theorem to triangle ABC, we have

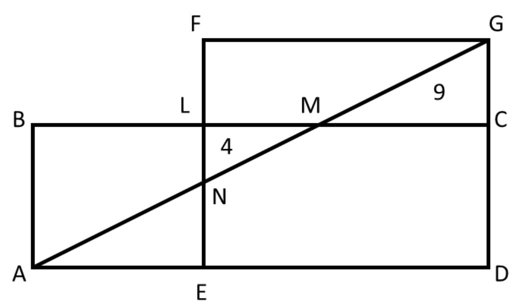
$$(x-3)^2 + (x+3)^2 = 30^2$$
$$2x^2 + 18 = 900$$
$$x^2 = 441$$

Thus, the area of the larger square is 441.



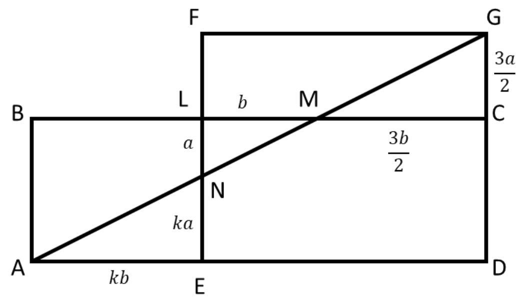
Puzzle 66. Two overlapping rectangles with same area

In the figure below, ABCD and EFGD are two rectangles with the same area. The area of triangle LMN is 4, and the area of triangle CGM is 9. Find the area of either rectangle.



Solution: Since $\angle NLM$ and $\angle MCG$ are right angles, and $\angle LMN = \angle GMC$, we have that $\triangle\, LMN \cong \triangle\, MGC$ by the AA triangle similarity principle. By Theorem 54, the ratio of the sides of the triangles LMN and MCG is the square root of the ratios of their areas, i.e., $\sqrt{\frac{9}{4}} = \frac{3}{2}$. If we let $d(LM) = a$ and $d(LN) = b$, then $dCG) = \frac{3}{2}a$ and $d(MC) = \frac{3b}{2}$, as indicated in the diagram below.

In addition, triangle AEN is similar to triangle MLN. Assume the ratio of the sides is k. So, $d(AE) = kb$ and $d(EN) = ka$, as shown in the diagram below.



We were given that $[ABCD] = [EFGD]$, where the square brackets refer to area. Further, we have

$$[ABCD] = [ABLE] + [ELCD] = [FLCG] + [ELCD] = [EFGD]$$

which implies $[ABLE] = [FLCG]$.

We also have

$$[FLCG] = \left(b + \frac{3b}{2}\right)\left(\frac{3a}{2}\right) = \frac{15ab}{4}$$

$$[ABLE] = (kb)(a + ka)$$

which implies

$$(kb)(a + ka) = \frac{15ab}{4}$$

$$4abk(1 + k) = 15ab$$

$$4k(k + 1) = 15$$

$$4k^2 + 4k - 15 = 0$$

Using the quadratic formula on the above equation and taking the positive solution, we have that $k = \frac{3}{2}$.

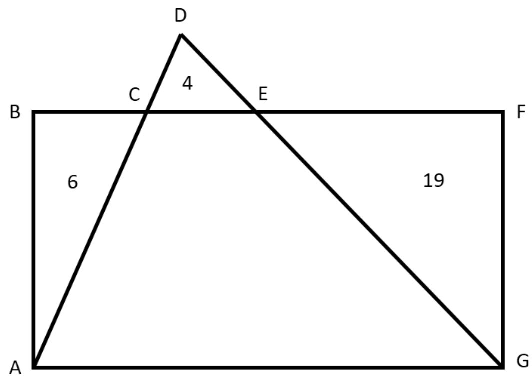
Since we were given that the area of triangle LMN is 4, we have that $\frac{1}{2}ab = 4$ or $ab = 8$.

We now have sufficient information to compute the area of rectangle $EFGD$, i.e.,

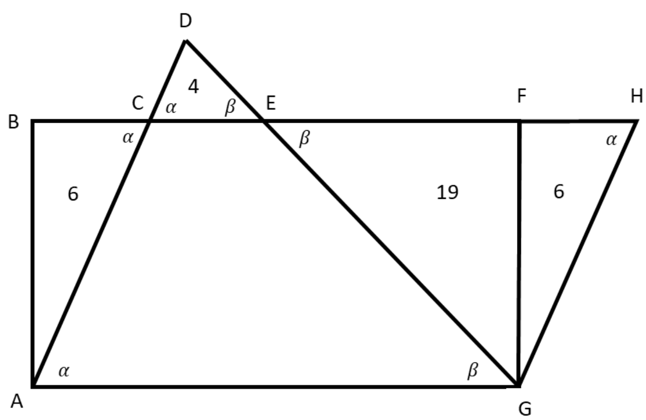
$$[EFGD] = \left(a + ka + \frac{3a}{2}\right)\left(b + \frac{3b}{2}\right) = (4a)\left(\frac{5b}{2}\right) = 10ab = 80$$

Puzzle 67. Find the area of the quadrilateral

In the configuration below, $ABFG$ is a rectangle, $[ABC] = 6, [CDE] = 4$ and $[EFG] = 19$. Find the area of the quadrilateral $ACEG$.



Solution: Draw a line through point G that is parallel line AC, and extend line segment $\overline{BF}$ to intersect the newly created line (as shown in the figure below). By construction, $ACHG$ is a parallelogram.



By the AA triangle similarity principle, triangles CDE and HGE are similar. By Theorem 54,

$$\frac{4}{19+6} = \frac{[CDE]}{[HGE]} = \frac{d(CE)^2}{d(EH)^2}$$

$$\frac{2}{5} = \frac{d(CE)}{d(EH)} \Rightarrow d(EH) = \frac{5}{2}d(CE)$$

Since $ACHG$ is a parallelogram, we have that

$$d(AG) = d(CH) = d(CE) + d(EH) = \frac{7}{2}d(CE)$$

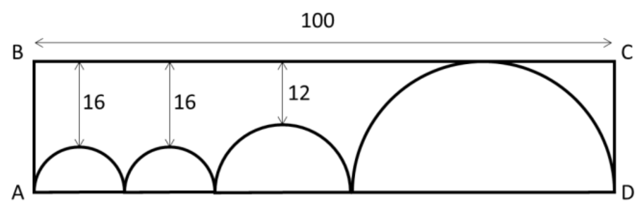
By the AA triangle similarity principle, triangles CDE and ADG are similar. By Theorem 54,

$$\frac{[ACEG]+4}{4}=\frac{[ADG]}{[CDE]}=\frac{d(AG)^2}{d(CE)^2}=\frac{\left(\frac{7}{2}d(CE)\right)^2}{d(CE)^2}=\frac{49}{4}$$

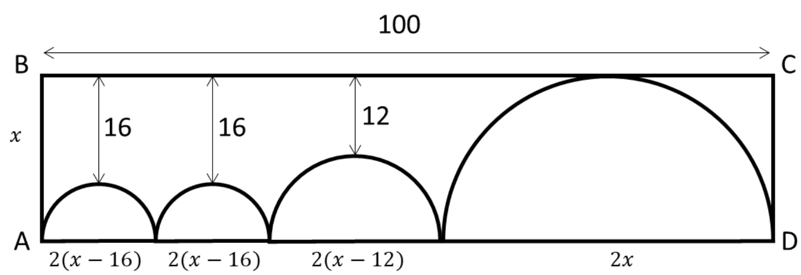
$$[ACEG]+4=49 \Rightarrow\ [ACEG]=45$$

Puzzle 68. Rectangle and 4 semicircles within

Find the area of the rectangle in the figure below. The length of the rectangle is 100 units. The distances from the top of each semicircle to the top of the rectangle are shown in the figure. The semicircle on the right is tangent to the top of the rectangle. Find the area of the rectangle, with the usual warning that the figure is not drawn exactly to scale.



Solution: Let x be the height of the rectangle. The diameter of each semicircle can be represented in terms of x, as shown in the figure below.



We have that

$$2(x-16)+2(x-16)+2(x-12)+2x=100$$

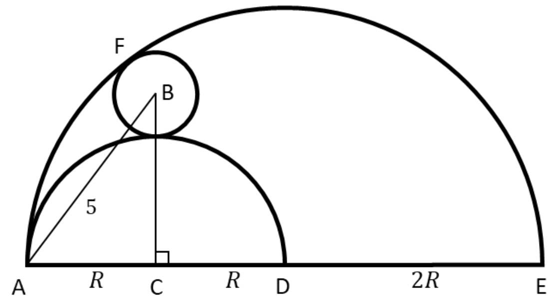
$$8x-88=100$$

$$8x=188 \Rightarrow x=\frac{188}{8}=\frac{47}{2}$$

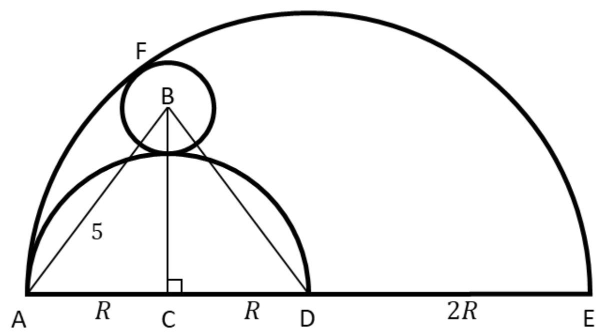
So, the area of the rectangle is 2350.

Puzzle 69. Find the area of the triangle inside a semicircle

In the figure below, we have a semicircle of radius $2R$ containing a semicircle of radius R. The center of the larger semicircle is at point D and the center of the smaller semicircle is at point C. There is a small circle (with center B and radius r) directly above the smaller semicircle, i.e., point B is directly above point C, and line BC is perpendicular to line AE. Further, we are given that $d(AB) = 5$. Find the area of triangle ABC.



Solution: Draw a line segment from point B to point D. By the SAS triangle congruence principle, triangle ABC is congruent to triangle DBC and thus, $d(BD) = d(AB) = 5$. Next, apply the Pythagorean theorem to triangle BCD (see the calculations on the right of the figure below).



$d(BD) = 2R - r = 5$

$d(BC) = R + r$

By the Pythagorean theorem,

$d(BC)^2 + d(CD)^2 = d(BD)^2$

$(R + r)^2 + R^2 = (2R - r)^2$

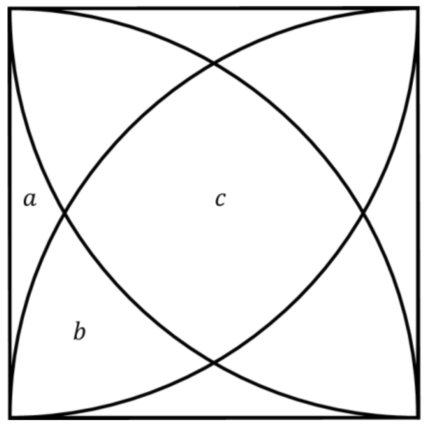
$6rR = 2R^2 \Rightarrow R = 3r$

Substitute $R = 3r$ into the equation $2R - r = 5$ to get $6r - r = 5$ which implies $r = 1$ and $R = 3$.

We can now compute the area of the triangle, i.e., $[ABC] = \frac{1}{2}R(R + r) = \frac{1}{2}(3)(4) = 6$.

Puzzle 70. Four rotated quarter-circles in a square

In the figure below, we have four quarter-circles within a square of side length x. Find formulas for the areas a, b and c in terms of x.



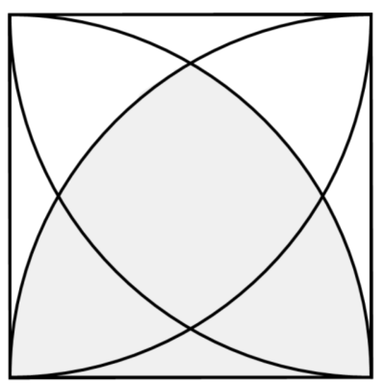
Solution: Our approach will be to determine 3 equations with the variables a, b and c, and then solve for each variable in terms of x. To that end, we use the area of the square to get our first equation, i.e.,

$$4a + 4b + c = x^2 \quad \text{(Equation 1)}$$

We can get an second equation by computing the area of any one of the four quarter-circles, i.e.,

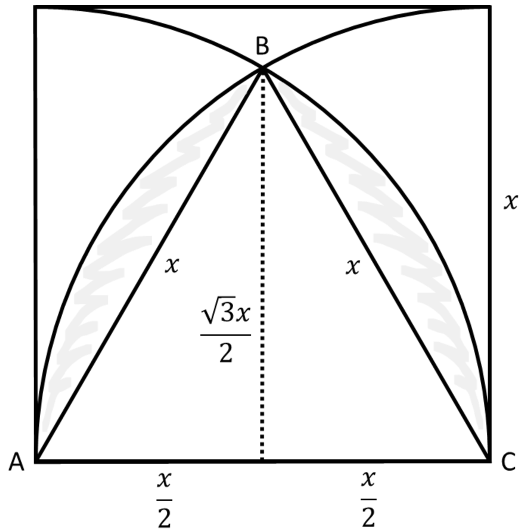
$$2a + 3b + c = \frac{\pi x^2}{4} \quad \text{(Equation 2)}$$

To get the third equation, we compute the gray area in the figure below.



To compute the gray area, we first compute the area of the equilateral triangle ABC shown in the figure below. Note that the diameter of each quarter-circle is x.

$$\frac{1}{2}\left(\frac{x}{2}\right)\left(\frac{\sqrt{3}x}{2}\right) = \frac{\sqrt{3}x^2}{4}$$



Next, we compute the sector of the circle between arc BC, i.e.,

$$\frac{60°}{360°}(\pi x^2) = \frac{\pi x^2}{6}$$

If we subtract the area of the triangle ABC from the area of the sector, we get the area of section on the right (indicated with the gray scribble), i.e.,

$$\frac{\pi x^2}{6} - \frac{\sqrt{3}x^2}{4}$$

The area of the gray area in question is the area of triangle ABC plus the area of the left and right sections (gray scribble), i.e.,

$$\frac{\sqrt{3}x^2}{4} + 2\left(\frac{\pi x^2}{6} - \frac{\sqrt{3}x^2}{4}\right) = \frac{\pi x^2}{3} - \frac{\sqrt{3}x^2}{4}$$

But the area of the gray area is also equal to $a + 2b + c$ and so, we have our third equation

$$a + 2b + c = \frac{\pi x^2}{3} - \frac{\sqrt{3}x^2}{4} \quad \text{(Equation 3)}$$

Subtract Equation 2 from Equation 1 to get

$$2a + b = x^2 - \frac{\pi x^2}{4} = x^2\left(1 - \frac{\pi}{4}\right) \quad \text{(Equation 4)}$$

Next, Equation 4 from Equation 2 to get

$$2b + c = x^2\left(\frac{\pi}{2} - 1\right) \quad \text{(Equation 5)}$$

Substitute the value for $2b + c$ from Equation 5 into Equation 3 to get

$$a + x^2\left(\frac{\pi}{2} - 1\right) = \frac{\pi x^2}{3} - \frac{\sqrt{3}x^2}{4}$$

$$a = x^2\left(1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

Now that we have an expression for a in terms of x, we can substitute the expression for a into Equation 4 to get an express for b in terms of x:

$$b = x^2\left(\frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1\right)$$

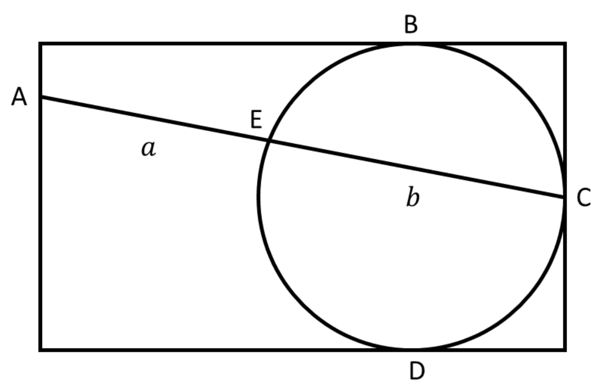
Finally, we substitute the expression for b into Equation 5 to get an expression for c in terms of x:

$$c = x^2\left(1 + \frac{\pi}{3} - \sqrt{3}\right)$$

Several other methods of solution are possible for this puzzle, see the article " Area bounded by arcs of quarter circles" [54].

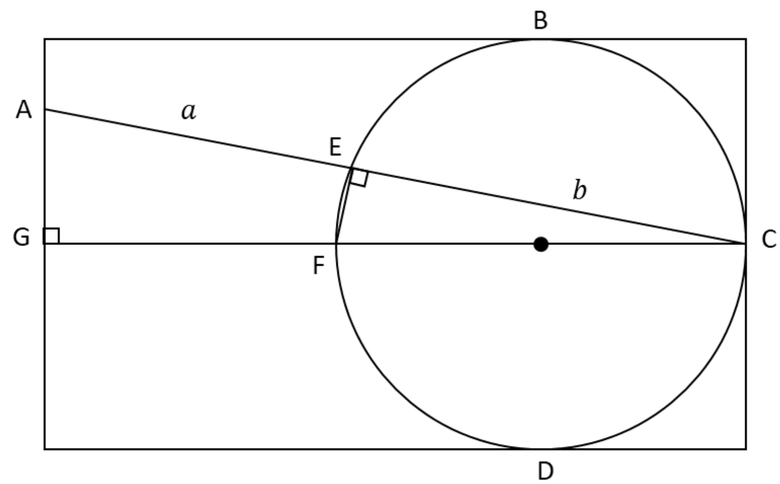
Puzzle 71. Rectangle with a circle and line segment crossing both

Find the area of the rectangle shown in the figure below in terms of a and b where $d(AE) = a$ and $d(EC) = b$. The circle is tangent to the rectangle at points B, C and D.



Solution: There is a hidden piece of information in the puzzle statement, i.e., the situation of the circle in the rectangle (with tangent points at B and D) implies the height of the rectangle is equal to the diameter of the circle. If we let r be the radius of the circle, then the rectangle has height $2r$. Let l be the length of the rectangle. So, the area of the rectangle is $2rl$ which we need to represent in terms of a and b.

Draw the line segment $\overline{GC}$ going through the center of the circle and parallel to the base of the rectangle as shown in the figure below.



By Theorem 24, FEC is a right triangle. By the AA triangle similarity principle, triangles AGC and FEC are similar since they are both right triangle and share a common angle at point C. Thus, we have the following ratio equation

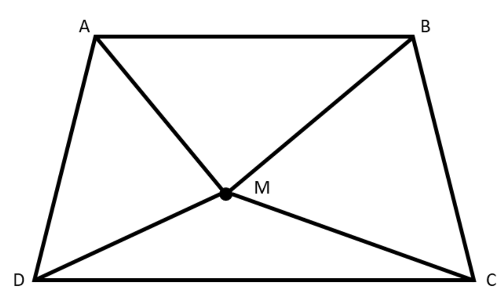
$$\frac{d(FC)}{d(EC)} = \frac{d(AC)}{d(GC)}$$

$$\frac{2r}{b} = \frac{a+b}{l}$$

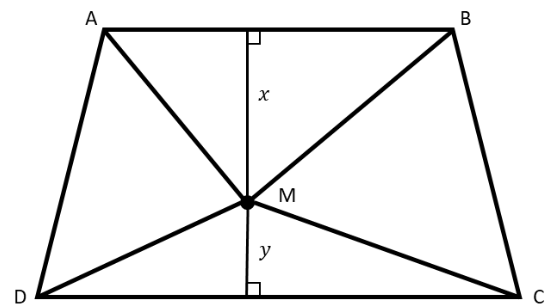
So, the area of the rectangle is $2rl = b(a+b)$.

Puzzle 72. Two triangles in a rhombus

In the rhombus shown in the figure below, $[AMB] = 10, [DMC] = 8$ and $\frac{d(AB)}{d(CD)} = \frac{4}{5}$. Find the **sum** of the areas of triangles AMD and BMC. The figure is not drawn to scale.



Solution: Draw perpendicular line segments from M to line AB, and from M to line DC. Let the lengths of the line segments be x and y, respectively (as shown in the figure below).



We have that

$$\frac{5}{4} = \frac{10}{8} = \frac{[AMB]}{[DMC]} = \frac{\frac{x}{2} \cdot d(AB)}{\frac{y}{2} \cdot d(CD)} = \frac{4}{5}\left(\frac{x}{y}\right)$$

$$\frac{x}{y} = \frac{25}{16}$$

Next, we parameterize the $x, y, d(AB)$ and $d(CD)$, i.e., make the following assignments:

$$x = 25t$$

$$y = 16t$$

$$d(AB) = 4s$$

$$d(CD) = 5s$$

Using the above assignments, we can write

$$10 = [AMB] = \frac{1}{2}d(AB) \cdot x = \frac{1}{2}(4s) \cdot x = 2sx = 2s(25t) = 50st$$

$$st = \frac{1}{5}$$

The area of the entire rhombus is given by

$$[ABCD] = \frac{1}{2}\big(d(AB) + d(CD)\big)(x + y)$$

$$= \frac{1}{2}(4s + 5s)(25t + 16t) = \frac{1}{2} \cdot 9s \cdot 41t = \frac{369}{2}st = \frac{369}{2} \cdot \frac{1}{5} = \frac{369}{10}$$

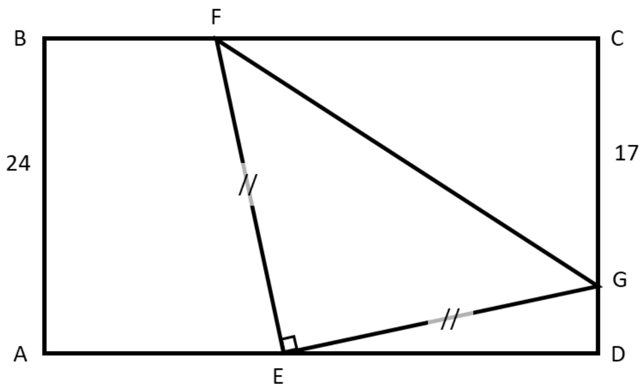
So, the area that we seek is

$$[AMD] + [BMC] = [ABCD] - [AMB] - [DMC]$$

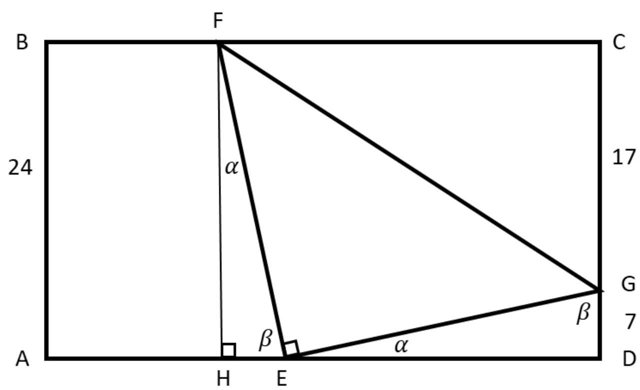
$$= \frac{369}{10} - 10 - 8 = \frac{369 - 100 - 80}{10} = \frac{189}{10} = 18.9$$

Puzzle 73. Right isosceles triangle in a rectangle

In the rectangle, shown in the figure below, we have an isosceles right triangle EFG. We are given $d(AB) = 24, d(CG) = 17$ and $d(EF) = d(EG)$. Find the area of the triangle. The figure is not drawn exactly to scale.



Solution: Draw a line segment from F that is perpendicular to line AD (intersecting the line at point H), as shown in the figure below. Since ABDC is a rectangle, we can conclude that $d(GD)$ must be 7.



Label $\angle GED$ as α and $\angle EGD$ as β. Since EGD is a right triangle, we have that $\alpha + \beta = 90°$ which implies that $\angle HEF$ must be equal to β and in turn, $\angle HFE$ must be equal to α.

By the AA triangle similarity principle, $\triangle EHF \cong \triangle GDE$ and since $d(EF) = d(EG)$, the triangles must be congruent. This implies that $d(HE) = 7$. Now, applying the Pythagorean theorem to triangle HEF, we get

$$24^2 + 7^2 = d(EF)^2$$

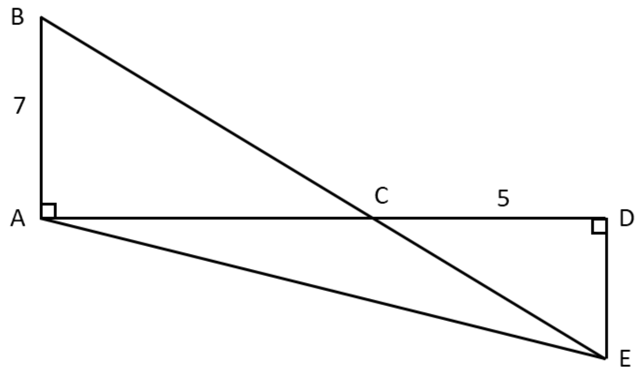
$$d(EF) = 25$$

So, the area of triangle EFG is $\frac{1}{2}(25)(25) = \frac{625}{2}$.

Strangely enough, we don't have enough information to determine the base of the rectangle or its area. The problem would have the same answer if we extended $d(AH) = d(BF)$ as far to the left as we wish. Since $d(HE) = 7$ and $d(ED) = 24$, we can say the $d(AD) = d(BC) \geq 31$.

Puzzle 74. Three adjacent triangles

In the figure below, $\angle BAC = \angle CDE = 90°$, ACD is a straight line, $d(AB) = 7$ and $d(CD) = 5$. What is the area of triangle ACE?



Solution: By the AA triangle similarity principle, $\triangle ABC \cong \triangle DEC$ and thus, we have the following equality of ratios

$$\frac{d(AB)}{d(AC)} = \frac{d(DE)}{d(CD)}$$

$$\frac{7}{d(AC)} = \frac{d(DE)}{5} \Rightarrow d(AC) \cdot d(DE) = 35$$

In terms of areas, we have

$$[ACE] = [ADE] - [DEC]$$

$$= \frac{1}{2}(d(AC) + 5) \cdot d(DE) - \frac{1}{2}\big(5 \cdot d(DE)\big)$$

$$= \frac{1}{2}d(AC) \cdot d(DE) + \frac{5}{2}d(DE) - \frac{5}{2}d(DE) = \frac{35}{2}$$

7 Japanese Temple Geometry Puzzles

If you accept the idea that both space itself, and all the stuff in space, have no properties at all except mathematical properties, then the idea that everything is mathematical starts to sound a little bit less insane. – Max Tegmark

Sangaku are Japanese geometrical problems or theorems on wooden tablets which were placed as offerings at Shinto shrines and Buddhist temples during the Edo period by members of all social classes. The paper "Japanese temple geometry" [55] provides the following background on the topic:

> Between the 17th and 19th centuries, the Japanese government closed its borders to the outside world in an attempt to become more powerful. Foreign books were banned, people could not travel, and foreigners were not allowed to enter the country. One result of this isolation was the flourishing of sangaku – wooden tablets inscribed with intricately decorated geometry problems that were hung beneath the eaves of Shinto shrines and Buddhist temples all over Japan. The hanging of tablets in Japanese shrines is a centuries-old custom, but earlier tablets generally depicted animals. It is believed that the sangaku arose in the second half of the 17th century, with the oldest surviving sangaku tablet dating from 1683. About 900 tablets have survived, as well as several collections of sangaku problems in early 19th century hand-written books or books produced from wooden blocks. Although their exact purpose is unclear, it would appear that they are simply a joyful expression of the beauty of geometric forms, designed perhaps to please the gods.

Fukagawa and Pedoe authored an entire book on the topic. An updated version of this book is available, see "Sacred Mathematics: Japanese Temple Geometry" [56]. In this book, the following tribute is paid to the creators of the sangaku puzzles:

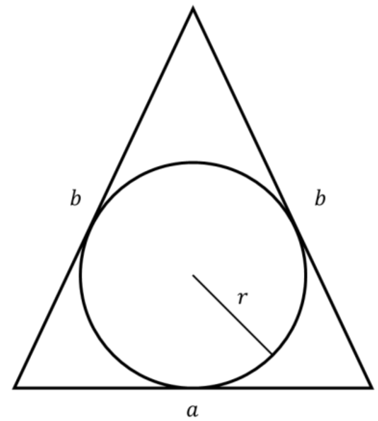
> One cannot accuse the Japanese mathematicians of lacking ingenuity. Seki developed a theory of determinants before Leibnitz, and other Japanese geometers proved a handful of famous theorems prior to their Western counterparts, or at least independently of them. They include the Descartes circle theorem, the Malfatti problem, the Casey theorem, the Soddy hexlet, and a few others.

In what follows, we will state and solve a small sampling of the sangaku puzzles.

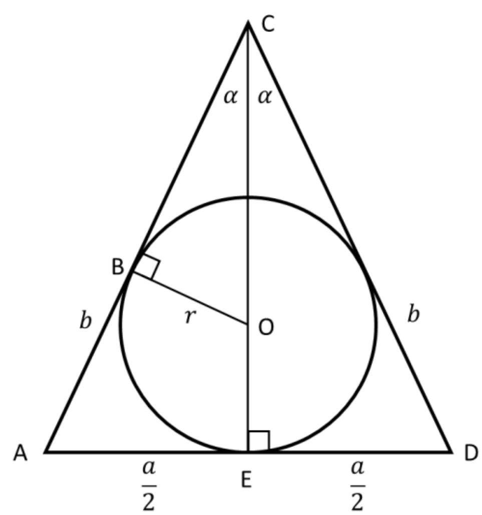
Puzzle 75. Find the radius of a circle inscribed in an isosceles triangle

Find the radius of a circle in terms of the side lengths of the isosceles triangle in the diagram below.

Source for puzzle: Problem 5 from Chapter 4 of "Sacred Mathematics: Japanese Temple Geometry" [56]



Solution: Draw a perpendicular line segment from the center of the circle to one side of the triangle, and bisect the angle at point C (as shown in the figure below). Let $d(CE) = h$. By Theorem 10, $\triangle AEC \cong \triangle DEC$ and thus, $\angle ACO = \angle DCO$ and $d(AE) = d(DE) = \frac{a}{2}$.



By the AA triangle similarity principle, $\triangle OBC \cong \triangle DEC$ and so,

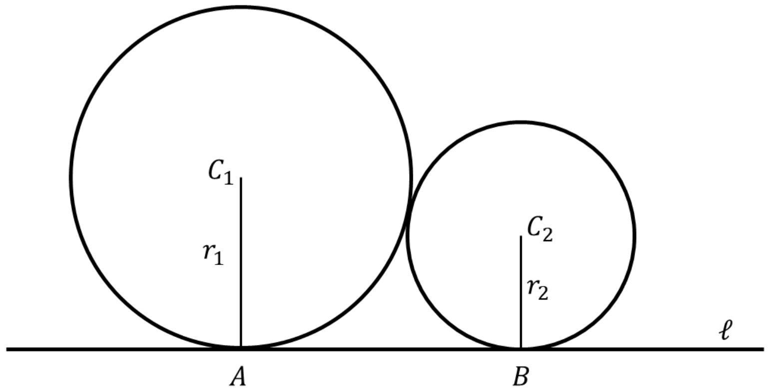
$$\frac{d(BO)}{d(CO)} = \frac{d(DE)}{d(CD)} \Rightarrow \frac{r}{h-r} = \frac{a/2}{b} \Rightarrow 2br = ah - ar \Rightarrow r(2b+a) = ah \Rightarrow r = \frac{ah}{2b+a}$$

By the Pythagorean theorem, $h = \sqrt{b^2 - \frac{a^2}{4}}$ and so, we can now write r in terms of a and b, i.e.,

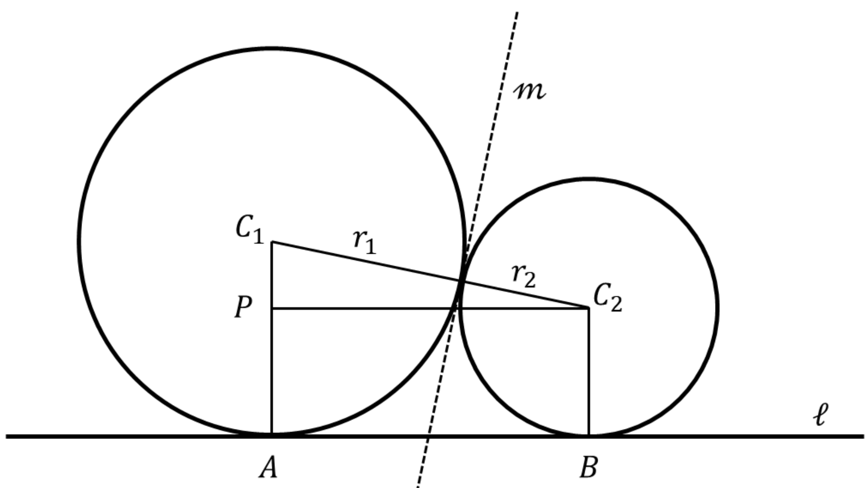
$$r = \frac{a\sqrt{b^2 - \frac{a^2}{4}}}{2b+a}$$

Puzzle 76. Relationship between two tangent circles

In the figure below, the two circle are tangent to each other, and also tangent to line ℓ at points A and B, respectively. The circle on the left has center C_1 and radius r_1, and the circle on the right has center C_2 and radius r_2. Prove that $d(AB)^2 = 4r_1r_2$.



Solution: In the figure below, point P is selected such that $\overline{PC_2}$ is parallel to $\overline{AB}$. By Theorem 22, C_1, C_2 and the point of tangency between the two circles lie on a straight line which is perpendicular to line m.



By the Pythagorean theorem, we have

$$d(PC_1)^2 + d(PC)^2 = d(C_1C_2)^2$$

$$(r_1 - r_2)^2 + d(AB)^2 = (r_1 + r_2)^2$$

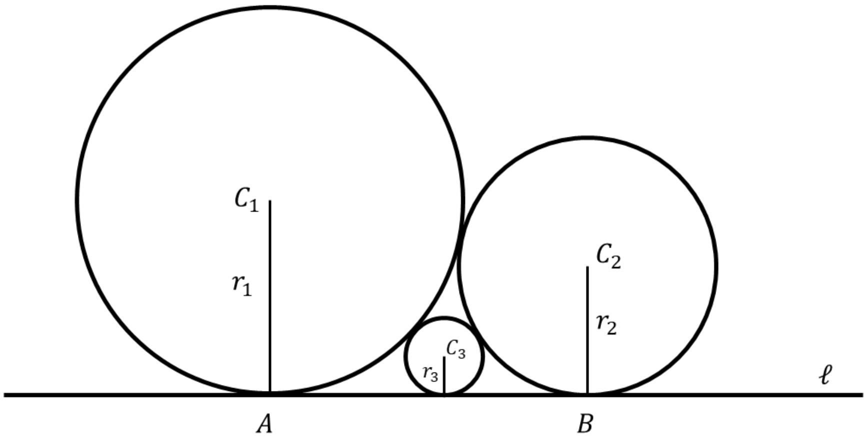
$$r_1^2 - 2r_1r_2 + r_2^2 + d(AB)^2 = r_1^2 + 2r_1r_2 + r_2^2$$

$$d(AB)^2 = 4r_1r_2$$

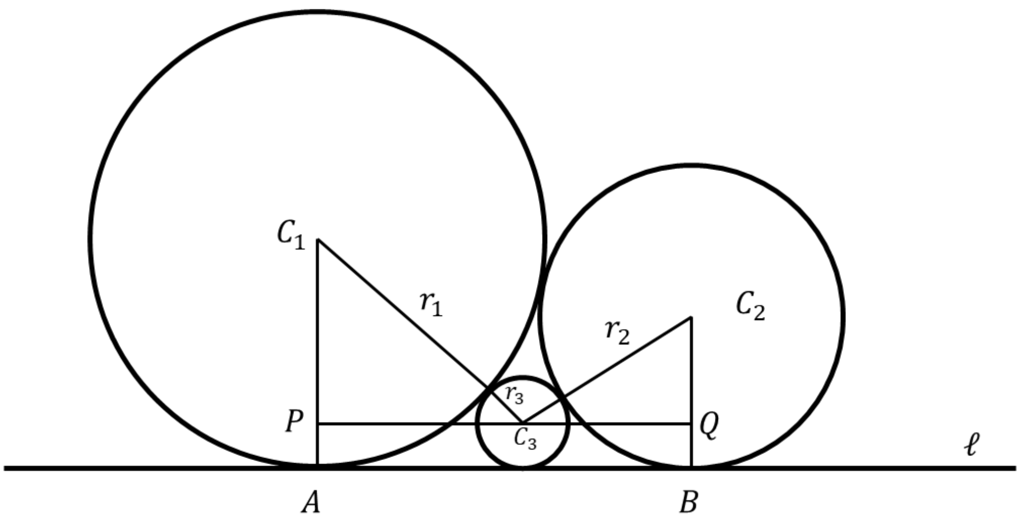
Puzzle 77. Relationship among three tangent circles

Add another circle to the configuration in Puzzle 76 such that the new circle is tangent to and in between the other two circles. The new circle is also tangle to line ℓ. Prove that

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$



Solution: In the figure below, points P and Q are selected such that $\overline{PQ}$ is parallel to $\overline{AB}$. From Puzzle 76, we know that $d(AB)^2 = 4r_1r_2$ which implies $d(AB) = \sqrt{4r_1r_2}$.



By construction, we have that $d(PC_3) + d(C_3Q) = d(AB) = \sqrt{4r_1r_2}$. Applying the Pythagorean theorem to each of the terms on the left, we have

$$\sqrt{(r_1 + r_3)^2 - (r_1 - r_3)^2} + \sqrt{(r_2 + r_3)^2 - (r_2 - r_3)^2} = \sqrt{4r_1r_2}$$
$$\sqrt{4r_1r_3} + \sqrt{4r_2r_3} = \sqrt{4r_1r_2}$$

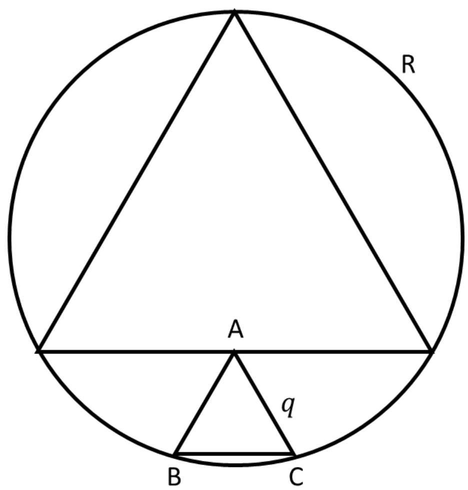
Dividing the above by $\sqrt{4r_1r_2r_3}$, gives us the desired result

$$\frac{1}{\sqrt{r_2}}+\frac{1}{\sqrt{r_1}}=\frac{1}{\sqrt{r_3}}$$

Puzzle 78. Two triangles in a circle

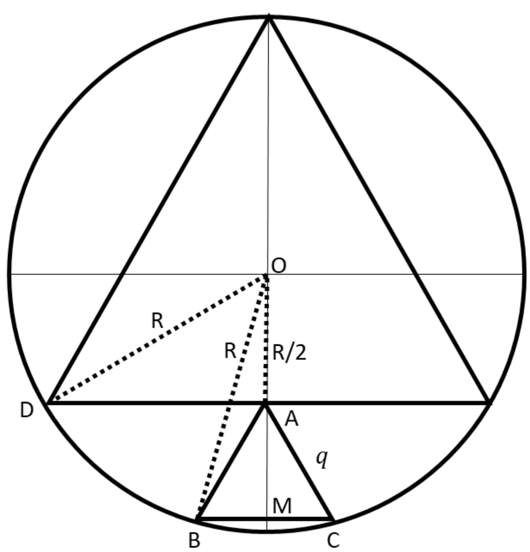
In the figure below, there are two equilateral triangles in a circle of radius R. The side length of the smaller triangle is q. The larger triangle touches the circle in exactly three points, and the small triangle touches the circle in exactly two points. Point A is the midpoint of one side of the larger triangle and the point of intersection with the smaller triangle. Find q in terms of R.

Source for puzzle: Problem 6 from Chapter 4 of "Sacred Mathematics: Japanese Temple Geometry" [56]



Solution: In the figure below, O is the center of the larger circle. Let M be the midpoint of $\overline{BC}$.

By the result in Puzzle 1, $d(OA)=\frac{1}{2}d(OD)=\frac{R}{2}$.



Applying the Pythagorean theorem to triangle AMC, we get that $d(AM) = \frac{\sqrt{3}}{2}q$.

Applying the Pythagorean theorem to triangle OBM, gives us

$$d(BM)^2 + d(OM)^2 = d(OB)^2$$

$$\left(\frac{q}{2}\right)^2 + \left(\frac{R}{2} + \frac{\sqrt{3}}{2}q\right)^2 = R^2$$

Using the quadratic formula to solve for q and selecting the positive value solution, we get

$$q = \frac{\sqrt{15} - \sqrt{3}}{4}R$$

Puzzle 79. Find the relationship among the circles

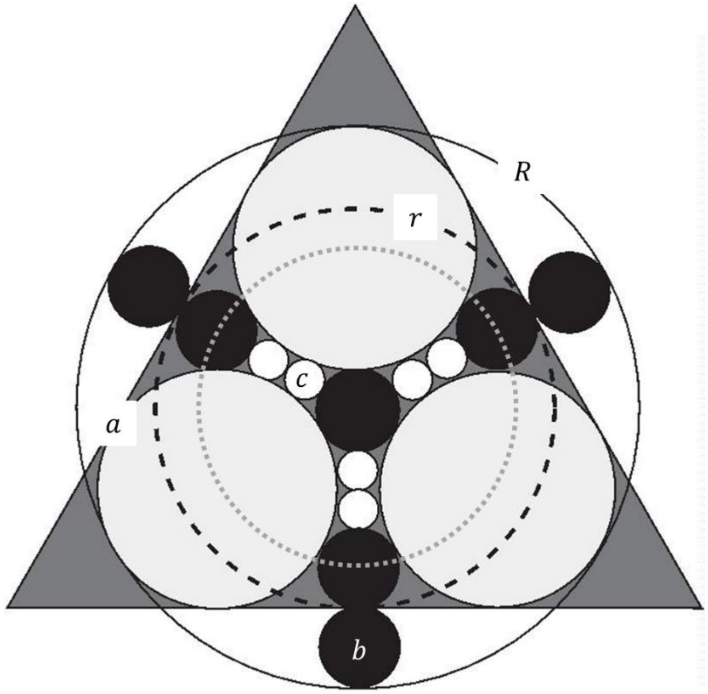
In the figure shown in the figure below, we have the following

- one equilateral triangle
- three light gray circles of radius a
- seven black circles of radius b (four of which are inside the triangle)
- six white circles of radius c which touch each other as shown
- a large outer circle of radius R
- a dashed line circle of radius r

Find b, a and r in terms of c.

Source for puzzle and figure: Problem 10 from Chapter 4 of “Sacred Mathematics: Japanese Temple Geometry” [56]

[**Author’s Remark**: My initial thought was that there is insufficient information to solve the puzzle. So, I have added the gray, dotted-line circle. This circle goes through the centers of the three light gray circles, and through the center of three of the black circles. I later figured out this additional information was not needed, and will explain in the solution.]



Solution: From the stated information and structure of the figure, we have the following equations:

$$r = 3b + 4c \quad \text{(Equation 1)}$$

$$R = b + 2a \quad \text{(Equation 2)}$$

$$R = 5b + 4c \quad \text{(Equation 3)}$$

$$a + b = 2b + 4c \quad \text{(Equation 4)}$$

Equation 4 follows from the condition concerning the gray, dotted-line circle.

From Equations 2 and 3, we have

$$b + 2a = 5b + 4c$$

$$2a = 4b + 4c$$

$$a = 2b + 2c$$

Plug the above expression for a into Equation 4 to get

$$(2b + 2c) + b = 2b + 4c$$

$$b = 2c$$

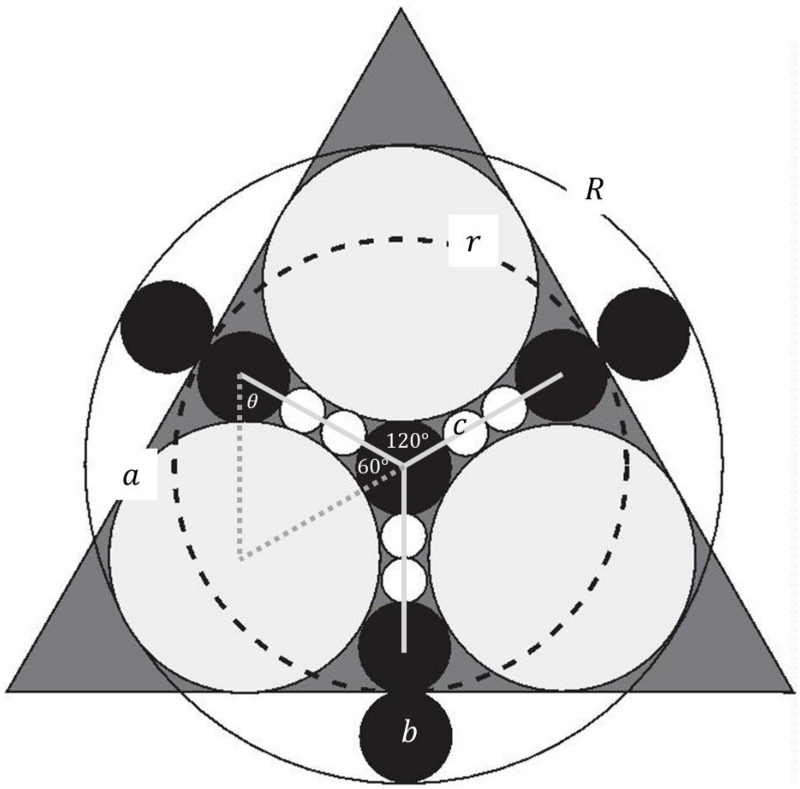
Putting $b = 2c$ into the equation $a = 2b + 2c$ yields $a = 6c$.

Making substitutions into Equation 2 and 1, we get $R = 14c$ and $r = 10c$, respectively.

. . .

If we remove the condition related to the gray, dotted-line circle, we can still determine Equation 4. In the figure below, the three light gray line segments (emanating from the center of the black circle) make three angles of $120°$ (this comes from the symmetric construction of the puzzle). Next, we draw the two dotted line segments from one of the gray circles to two adjacent black circles.

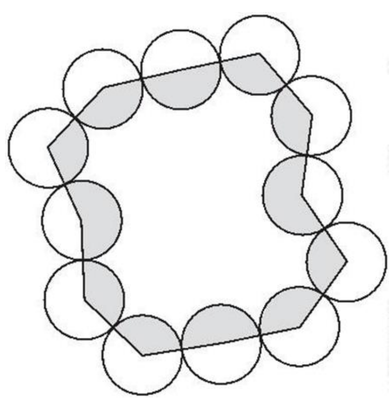
These line segments are of the same length, and thus, we have an isosceles triangle. One of the dotted line segments bisects the $120°$ angle to make an angle of $60°$. Since we have an isosceles triangle, it must be that θ also equals $60°$. Further, since the angles of a triangle add to $180°$, the third angle is also $60°$ and thus, we have an equilateral triangle. Equating the segment lengths of a dotted line segment of the triangle with the light gray segment, we get Equation 4.



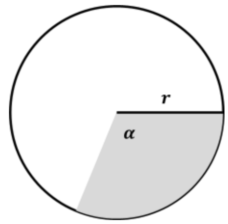
Puzzle 80. Find the shaded area of circles arranged in a polygon

In the figure below, the centers of a ring of tangent circles (each with radius r) are connected by line segments to form a non-self-intersecting polygon. Let S_1 be the sum of the areas of the circles inside of the polygon (shaded regions in the figure below), and S_2 the sum of the areas of the circles outside of the polygon. Find $S_2 - S_1$.

Source for puzzle and figure: Problem 4 from Chapter 4 of "Sacred Mathematics: Japanese Temple Geometry" [56].



Solution: First, note that the area of a sector in a circle of angle α is $\frac{\alpha}{2\pi}(\pi r^2) = \frac{\alpha}{2}r^2$ where r is the radius of the circle (see the figure below).



For the problem at hand, assume that we have n circles with sectors (measured inside the polygon) that span angles of measure $\alpha_1, \alpha_2, \dots, \alpha_n$. The sum of the shaded regions is

$$S_1 = \sum_{i=1}^{n} \frac{\alpha_i}{2} r^2 = \frac{r^2}{2} \sum_{i=1}^{n} \alpha_i = \frac{r^2}{2}(n-2)\pi$$

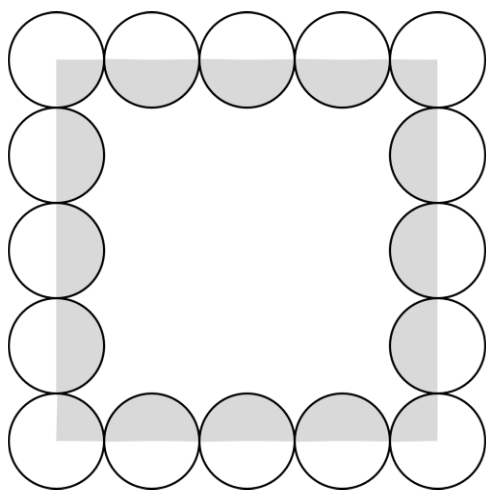
The last equality in the above expression is a result of Theorem 42.

The area of the circles lying outside of the polygon is

$$S_2 = n\pi r^2 - S_1$$

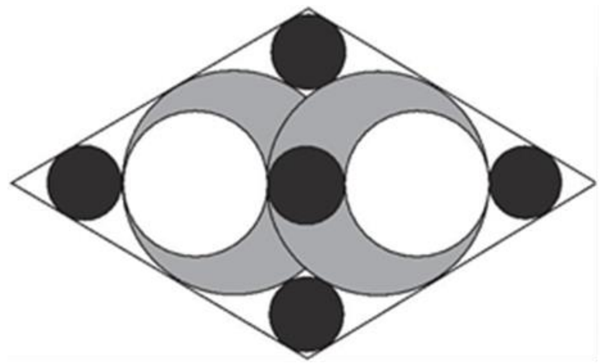
So, $S_2 - S_1 = n\pi r^2 - 2S_1 = n\pi r^2 - (n-2)\pi r^2 = 2\pi r^2$.

Intuitively, this result makes sense. For example, consider the special case in the figure below where the polygon is a square. The inside and outside areas cancel each other except for the four circles at the corners of the square. The outside area of the corner circles is $4\left(\frac{3}{4}\right)\pi r^2$ and the inside area of the corner circles is $4\left(\frac{1}{4}\right)\pi r^2$. So, the difference is $2\pi r^2$.

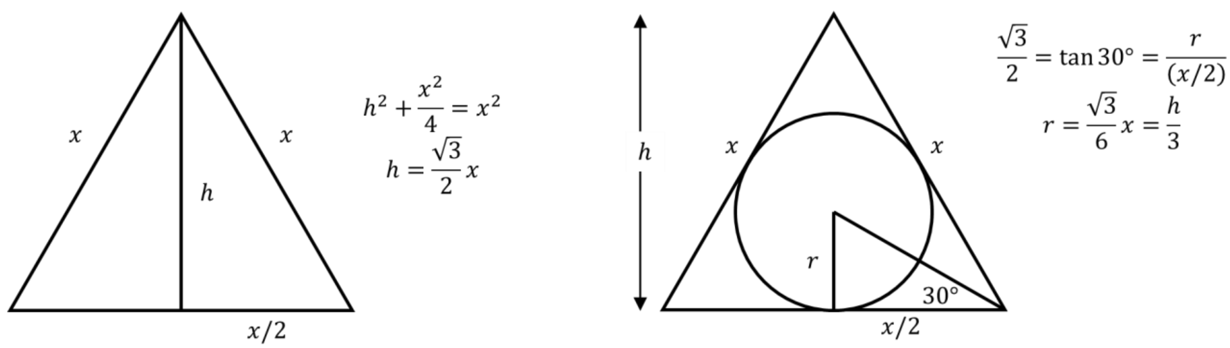


Puzzle 81. Pretty circle pattern in a rhombus

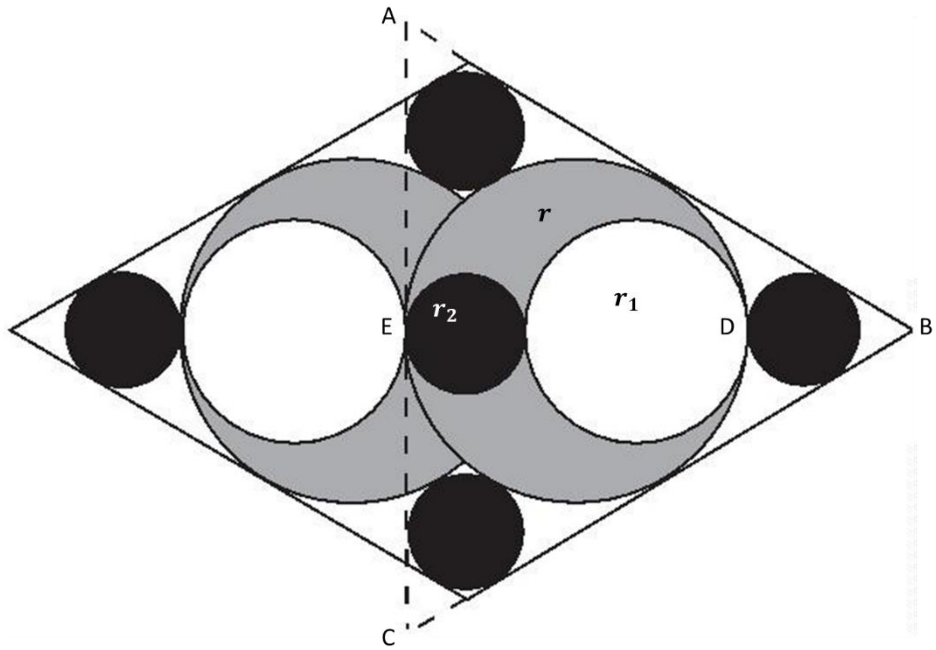
The outer shape in the figure below is a rhombus. Within the rhombus, there are two gray circles of radius r, two white circles of radius r_1, and five black circles of radius r_2. Prove that $r_2 = \frac{r_1}{2}$.



Solution: Before getting to the substance of the solution, we need two preliminary results. On the left side of the figure below, we prove that an equilateral triangle of side length x has height $h = \frac{\sqrt{3}}{2}x$. On the right side of the figure, we prove that the $r = \frac{h}{3}$ where r is the radius of an incircle within an equilateral triangle of height h. The angle of $30°$ follows from Theorem 32.



Superimpose the triangle ABC as shown in the figure below. By the symmetry of the diagram, we see that $\triangle ABC$ is an equilateral triangle. In $\triangle ABC$, let x represent the length of a side and h be the height.



Using our preliminary results, we have

$$r = \frac{h}{3} = \frac{\sqrt{3}}{6}x$$

Next, compute the measure of segment $\overline{DB}$, i.e.,

$$d(DB) = h - 2r = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{3}x = \frac{\sqrt{3}}{6}x = r$$

By our second preliminary result, $r_2 = \frac{d(DB)}{3}$ and using the above result, we get that $r_2 = \frac{1}{3}r \Rightarrow$

$r = 3r_2$.

From the diagram, we see that $d(ED) = 2r = 2r_1 + 2r_2$ which implies $r = r_1 + r_2$.

Thus, $3r_2 = r_1 + r_2$ which implies $r_2 = \frac{r_1}{2}$.

8 Probabilistic Geometry Puzzles

It is a part of probability that many improbable things will happen. – Aristotle

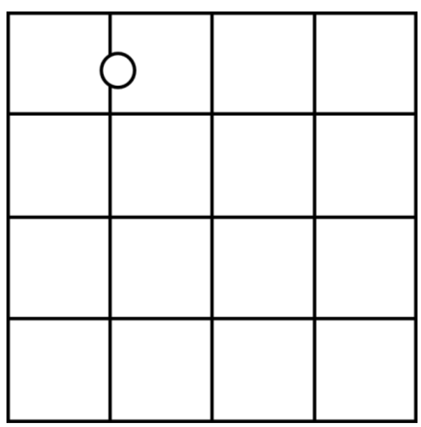
As the name suggests "probabilistic geometry" (or "geometric probability) entails probability as applied to geometry. From the Wikipedia article "Geometric probability" [57]

> Problems of the following type, and their solution techniques, were first studied in the 18th century, and the general topic became known as **geometric probability**.
>
> - (Buffon's needle) What is the chance that a needle dropped randomly onto a floor marked with equally spaced parallel lines will cross one of the lines?
> - What is the mean length of a random chord of a unit circle? (cf. Bertrand's paradox).
> - What is the chance that three random points in the plane form an acute (rather than obtuse) triangle?
> - What is the mean area of the polygonal regions formed when randomly oriented lines are spread over the plane?

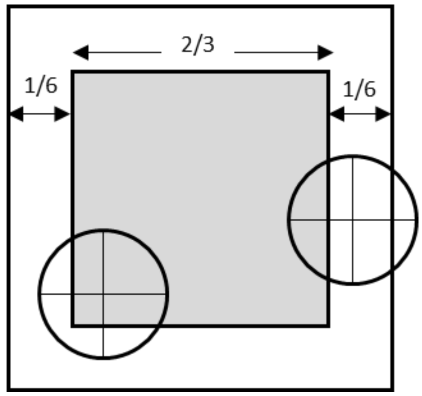
Puzzle 82. Tossing a coin onto a checked board

A coin is randomly tossed onto a board as shown in the figure below. The board consists of 16 squares of dimension 1 cm by 1 cm. The coin is a circle of radius 1/6 cm. What is the probability that the coin will land completely within one of the squares (without overlapping any edge)?

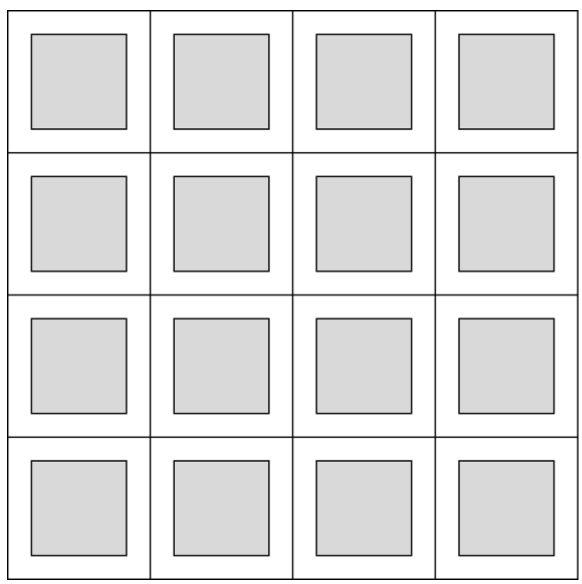
Also, note that the toss does not count unless the center of the coin lands within the boundary of the board.



Solution: The key to this puzzle is to realize that the location of the center of the circle determines whether or not the coin falls completely within one of the squares. Since the coin is tossed randomly, we can assume that all locations for the landing point of the center of the coin are equally probable. To solve the puzzle, we only need to consider one of the squares, as shown in the figure below. If the center of the coin lands within the gray region, then the circle will be entirely within the particular square. The gray region has area $2/3 \cdot 2/3 = 4/9$ and the area of the surrounding square is 1.



In the context of the entire 16×16 board. The following figure shows the areas (in gray) where the coin's center can land without the coin overlapping any of the edges of the board. The area of the entire board is $4 \times 4 = 16$ and the area of the gray areas is $16 \cdot \frac{4}{9} = \frac{64}{9}$. Thus, the probability of the coin's center landing on a gray area (and thus not touching an edge) is $\frac{\left(\frac{64}{9}\right)}{16} = \frac{4}{9}$.



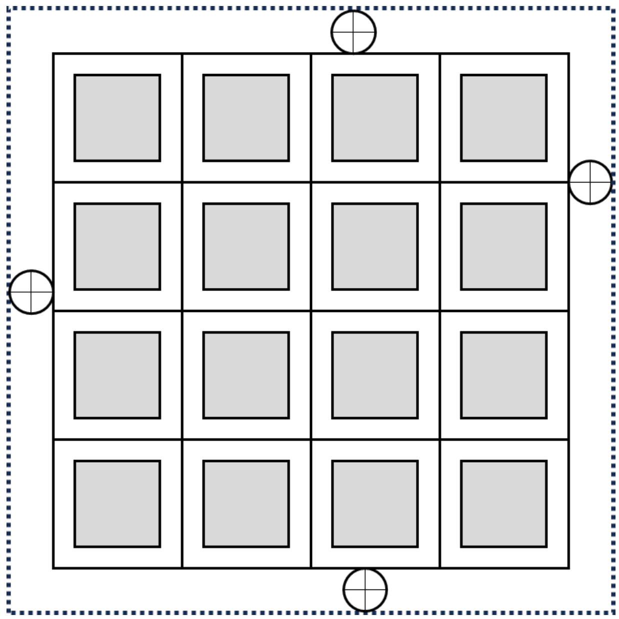
Puzzle 83. Variation on tossing a coin onto a checked board

Compute the probability that the coin will land completely within a square if we modify the definition of a valid coin toss to include tosses where any part of the coin touches the board, i.e., the coin does not need to land completely within the board, and a legal tosses counts even if only part of the coin overlaps the board.

Solution: The modified condition concerning a legal toss of the coin effectively extends the boundary of the board by the diameter of the coin, i.e., 1/3 cm. The figure below shows four coins at the boundary of what constitutes a legal toss. The dotted-line square has side length $4 + \frac{2}{3} = \frac{14}{3}$.

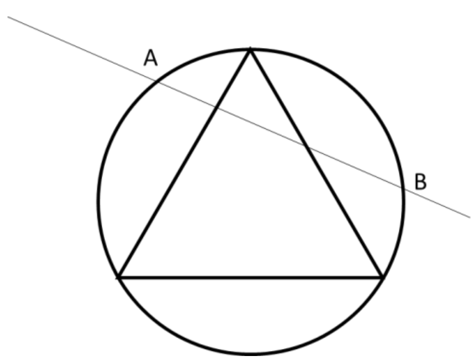
So, the probability of the coin landing in the gray area is now

$$\frac{\left(\frac{64}{9}\right)}{\left(\frac{14}{3}\right)^2} = \frac{64}{9} \cdot \frac{9}{196} = \frac{16}{49}$$



Puzzle 84. Bertrand's Paradox [58]

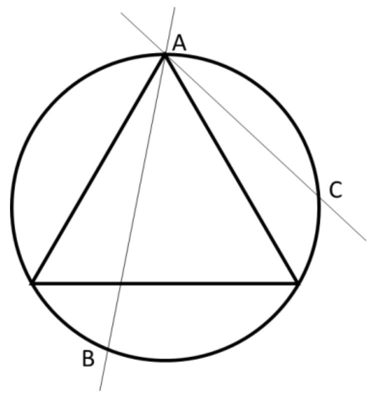
A straight line is randomly superimposed on top of a circle that circumscribes an equilateral triangle. What is the probability that the chord thus formed (e.g., chord AB in the figure below) has length greater than the side of the triangle?



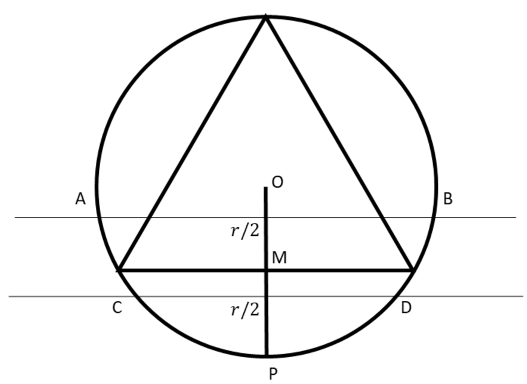
Solutions: As we shall see, there are several solutions with different results (thus, the paradox).

In the first approach, we rotate the circle and triangle such that one end of the chord intersects a vertex of the triangle. If the other end of the chord intersects the side of the triangle opposite the particular vertex, then the chord is larger than the side length of the triangle. For example, chord

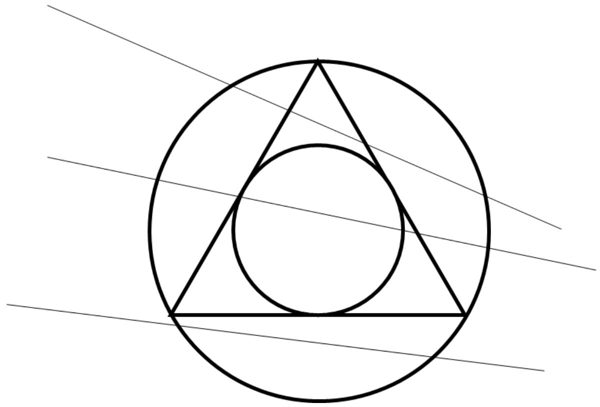
AB in the figure is longer than the side length of the triangle, and chord AC is shorter. The other end of the chord from the vertex intersect the opposite side of the triangle $\frac{1}{3}$ of the time, and that is the probability that the randomly selected chord is longer than the side of the triangle. It is critical to understand that rotating the circle and in the manner described do **not** change the length of the chord.



In the second measuring approach, we construct a radius (segment $\overline{OP}$ in the figure below) from the center of the circle such that it is perpendicular to one side of the triangle. We rotate the circle, triangle and radius $\overline{OP}$ such that the line to be measured is perpendicular to $\overline{OP}$. If the line intersects $\overline{OP}$ between points O and M, the associated chord is longer than the side of the triangle. If the line intersects $\overline{OP}$ between points M and P, the associated chord is shorter than the side of the triangle. It follows from Theorem 33 and Theorem 34 that $d(OM) = d(MP) = \frac{r}{2}$ where r is the radius of the circle. Thus, the probability that the line is longer than the side of the triangle is $\frac{1}{2}$.



In the third approach, we draw a circle within the triangle such that the circle is tangent to the three sides of the triangle (i.e., an incircle of the triangle). As a consequence of Theorem 33 and Theorem 34, the radius of the incircle is $\frac{1}{2}$ the radius of the outer circle. If the center of the chord associated with a line falls within the incircle, then the chord is longer than the side of the triangle. The area of the incircle is $\frac{1}{4}$ that of the outer circle, and so, the probably that the chord is longer than the side length of the triangle is $\frac{1}{4}$.



The Wikipedia article on this topic [58] summarizes one possible explanation for this paradox as follows:

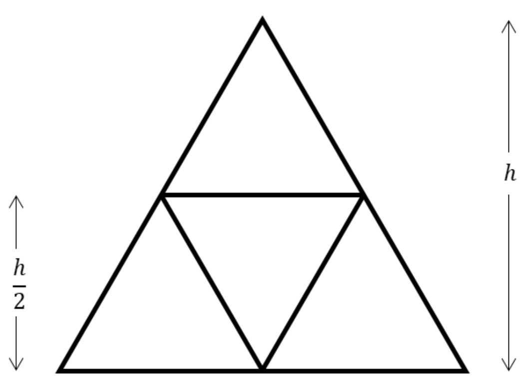
> The problem's classical solution (presented, for example, in Bertrand's own work) hinges on the method by which a chord is chosen "at random". The argument is that if the method of random selection is specified, the problem will have a well-defined solution (determined by the principle of indifference [59]). The three solutions presented by Bertrand correspond to different selection methods, and in the absence of further information there is no reason to prefer one over another; accordingly, the problem as stated has no unique solution.

Author's remark: The above explanation does not ring true for me. Look at it this way, we randomly drop a line over a circle. At this point the length of the chord formed by the intersection of the line with the circle is fixed. We then create three different measuring mechanisms which we superimpose over the circle. Each mechanism yields a different probability. So, the issue is with the measurement approach and not how the chord is selected. For another explanation, see the Khan Academy video "Bertrand's Paradox" [60].

Puzzle 85. Relationship between interior point and distance to sides of equilateral triangle

Given a randomly placed point within an equilateral triangle, what is the probability that the perpendicular (i.e., shortest) distance from that point to one side of the triangle is greater than the sum of the distances to the other two sides?

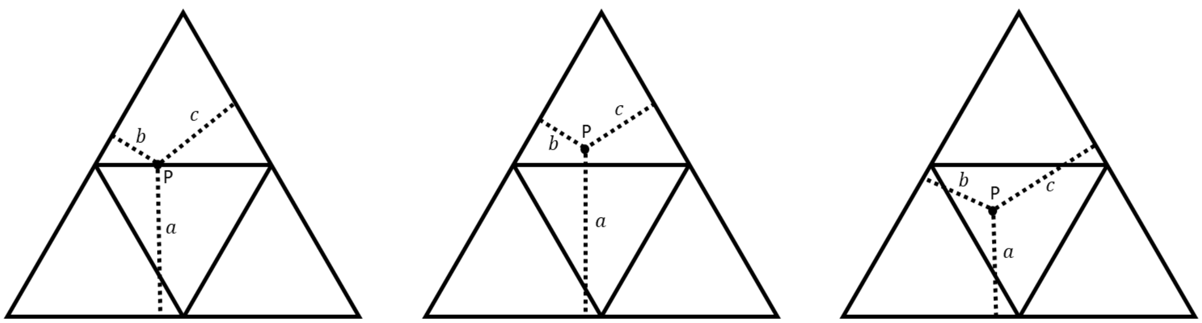
Solution: Divide the equilateral into four equal triangles by connecting the midpoints of each side, as shown below.



As shown in the figure below, there are three cases for the location of the point P, i.e., it falls on the center triangle, if falls within the center triangle, or it falls within one of the three triangles associated with a vertex of the encompassing triangle. From Viviani's theorem (see Puzzle 4), $a + b + c = h$.

- For the case where P is on the perimeter of the center triangle (left of figure below), $a = \frac{h}{2}$ and thus, $a = b + c$. This is true if P falls on any of the three sides of the center triangle.
- If P falls outside the center triangle (middle of figure), then $a > \frac{h}{2}$, and thus, $a > b + c$.
- If P falls within the center triangle (right of figure), then neither a, b nor c are greater than $\frac{h}{2}$, and thus, neither a, b nor c has length greater than the sum of the other two.

All four of the interior triangles are congruent and thus, have the same area, i.e., $\frac{1}{4}$ the area of the larger triangle. So, the probability that P falls within the center triangle is $\frac{1}{4}$ and that is the answer to the puzzle.



Puzzle 86. The Broken Stick Puzzle

Two points are randomly selected on a stick and the stick is then broken at those two points, thereby making three smaller sticks. What is the probability that the three smaller sticks can be arranged in a triangle?

Solution: We have already solved this puzzle but in a different guise. In Puzzle 85, let the unbroken stick be represented by the height of the large equilateral triangle, and the three segments be represented by the distance from point P to each of the sides of the triangle. The broken pieces of the stick can only be arranged in a triangle when none of the stick segments has length greater than the sum of the other two, which happens with probability $\frac{1}{4}$.

The broken stick puzzle is a special case of a more general puzzle which goes as follows:

> Consider a stick of fixed length and a positive integer $k \geq 3$. Pick $k - 1$ distinct interior points on the stick, independently and at random, and cut the stick at these $k - 1$ points. What is the probability that the resulting k pieces form a k-gon?

The general problem is discussed and solved in the paper "Broken bricks and the pick-up sticks problem" [61]. The probability that the resulting k pieces form a k-gon is $1 - \frac{k}{2^{k-1}}$.

9 Dissection Puzzles

Divide each difficulty into as many parts as is feasible and necessary to resolve it. – Rene Descartes

Mathematical dissection puzzles entail the decomposition of a geometric shape, often a polygon, into smaller pieces which can be rearranged to form another shape. The goal is to find a sequence of cuts (typically, but not always, straight line cuts) that dissect the original shape into pieces that can be reassembled to create a different shape (or shapes).

Such puzzles often require creative thinking, spatial reasoning, and a good understanding of geometric principles. The challenge lies in finding the right set of cuts and arrangements to achieve the desired result. The pieces obtained through the dissection are typically rearranged without any overlaps or gaps to form the target shape.

There are various types of mathematical dissection puzzles, each with its own set of rules and challenges. Some puzzles involve dissecting a shape into pieces that can be rearranged to form a different shape with the same area or perimeter. Other puzzles may focus on achieving specific mathematical ratios or relationships between the pieces.

The Wallace-Bolyai-Gerwien theorem [62] states that a polygon can be formed from another by cutting it into a finite number of pieces and recomposing these by translations and rotations if and only if the two polygons have the same area. Figure 15 shows the decomposition of a square and reassembly into a triangle of equal area (the white space is added to highlight the cuts).

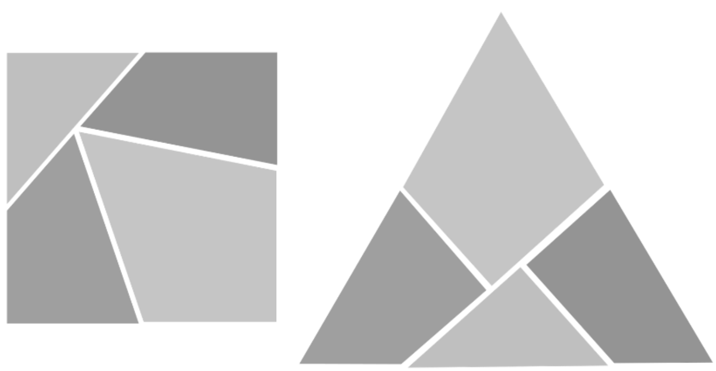


Figure 15. Dissecting a square and recomposing into a triangle

One classic example of a mathematical dissection puzzle is the **tangram** [63], which originated in China and consists of seven flat pieces that can be combined to form a square. The pieces include two large triangles, one medium triangle, two small triangles, a square, and a parallelogram. The seven tangram pieces as arranged in a square are shown in Figure 16.

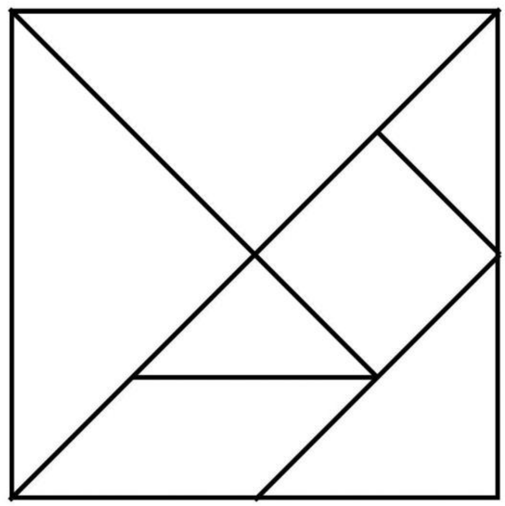


Figure 16. Tangram pieces arranged in a square

The tangram shapes can be arranged in an endless collection of configurations. A few simple examples are shown in Figure 17.

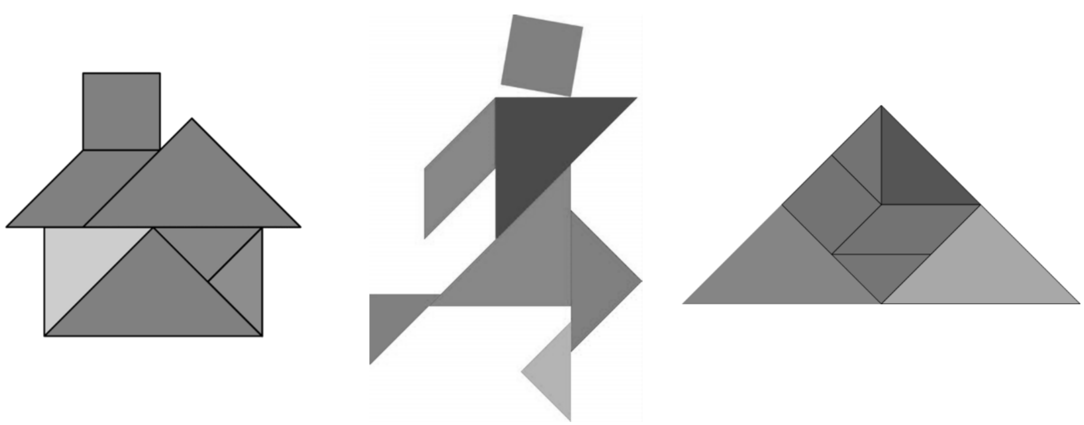


Figure 17. Tangram examples

Tangram puzzles are often presented with a given final shape but without showing the demarcation lines of the individual piece. The puzzle solver is asked to reproduce the shape using the tangram pieces. For example, the left side of the figure below shows how a tangram puzzle might be presented, and the right side of the figure depicts the solution.

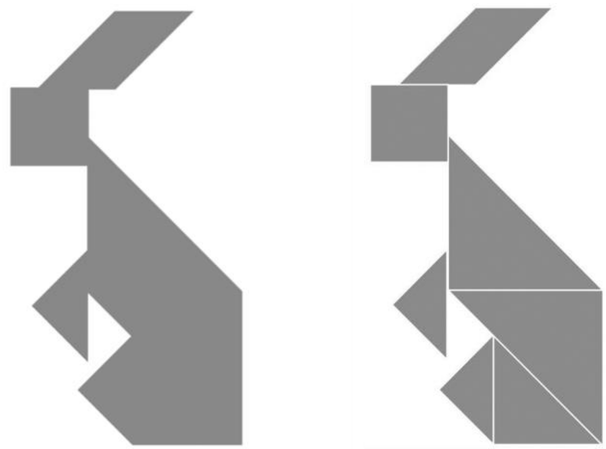


Figure 18. Tangram rabbit

. . .

Famous puzzle master Sam Loyd [64] created a type of puzzle similar to tangram. Like tangram, the pieces are cut out of a square. On the left of Figure 19, line segments are drawn from each vertex of the square to the midpoint of the opposite side. On the right of the figure, some of the line segments are truncated to form the pieces of Loyd's dissection of the square.

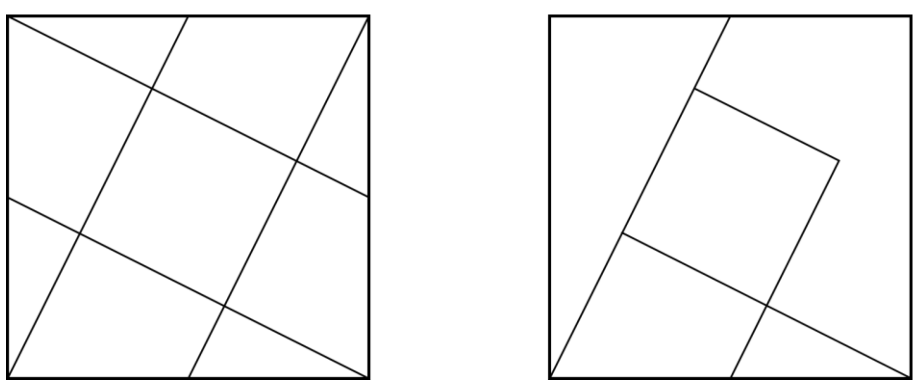
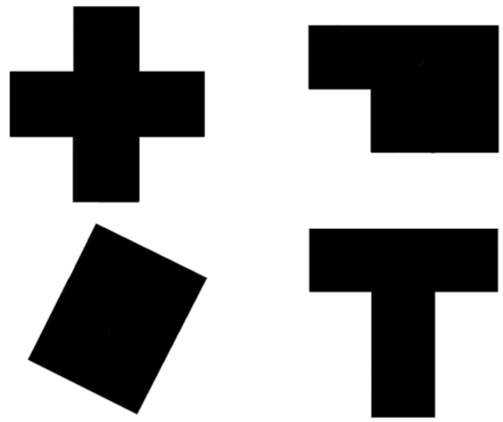


Figure 19. Sam Loyd's square dissection

Note: The solutions for the puzzles in this section are provided at the end of the section.

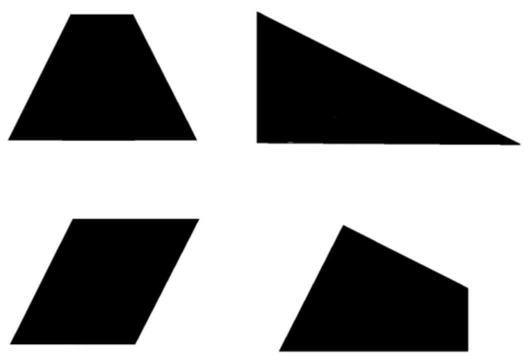
Puzzle 87. Sam Loyd square dissection and reassembly problems

Using the pieces from the Sam Loyd decomposition of a square create each of the shapes in the figure below. To be clear, this is four separate problems.



Puzzle 88. Another Sam Loyd square dissection and reassembly set of problems

The instructions from Puzzle 87 also apply here.



. . .

Another famous dissection puzzle comes from puzzle master Henry Dudeney. The problem is to dissect a square into four pieces and then reassemble the pieces into an equilateral triangle. A high-level illustration of the solution is shown in Figure 20.

Source of the figure is the Wikipedia article "Hinged dissection" [65]

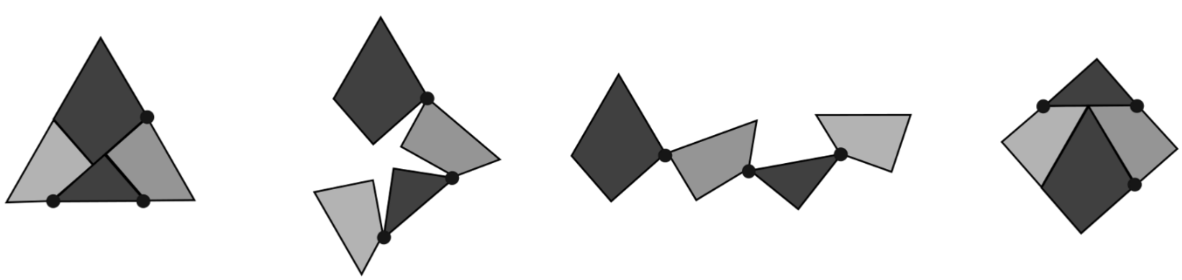


Figure 20. Dudeney's hinged dissect of a square into an equilateral triangle

The figure is “hinged” at three points, and it is possible to go back and forth between the equilateral triangle and square by moving the pieces about the hinges. Such a dissection is known as a “hinged dissection”. As noted in “Hinged dissection” [65]

> The Wallace–Bolyai–Gerwien theorem, first proven in 1807, states that any two equal-area polygons must have a common dissection. However, the question of whether two such polygons must also share a hinged dissection remained open until 2007, when Erik Demaine et al. proved that there must always exist such a hinged dissection, and provided a constructive algorithm to produce them. This proof holds even under the assumption that the pieces may not overlap while swinging, and can be generalized to any pair of three-dimensional figures which have a common dissection (see Hilbert's third problem). In three dimensions, however, the pieces are not guaranteed to swing without overlap

For a detailed explanation concerning the location of the cuts and hinge points, see the YouTube video “Equilateral Triangle To Square Dissection and Problem Set” [66].

. . .

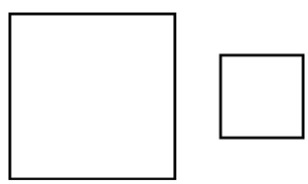
Puzzle 89. Carpenter’s dissection puzzle

A carpenter has a 25×36 piece of wood. However, she needs a 30×30 piece of wood to finish a project. How can the piece of wood be cut in two and glued together to form the required 30×30 piece of wood?

. . .

Puzzle 90. Cut two squares into 5 pieces and reassemble into a larger square

Given the two squares shown in the figure below, divide the squares into 5 pieces (using straight line cuts) and reassemble into a larger square. While not critical to the problem, you can assume the larger of the two squares has twice the side length of the smaller square.



. . .

Puzzle 91. Make one larger triangle out of three smaller ones

Given three congruent equilateral triangles, dissect each and to exactly form a larger equilateral triangle.

. . .

The Banach-Tarski paradox is an inscrutable construction that defies common sense. It states that:

> Given a solid ball in three-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Indeed, the reassembly process involves only moving the pieces around and rotating them without changing their shape. However, the pieces themselves are not "solids" in the usual sense, but infinite scatterings of points. The reconstruction can work with as few as five pieces. [67]

This seemingly impossible feat is achieved by utilizing a complex and counterintuitive technique involving the decomposition of the ball into sets of different sizes and complexities, including uncountable infinite sets. These sets are then rearranged and combined in a specific way, ultimately resulting in two identical copies of the original ball.

Even more confounding is the pea and the Sun paradox, i.e.,

> Given any two "reasonable" solid objects (such as a small ball and a huge ball), the cut pieces of either one can be reassembled into the other. This is often stated informally as "a pea can be chopped up and reassembled into the Sun" and called the "pea and the Sun paradox".

Here are some key points concerning such paradoxes:

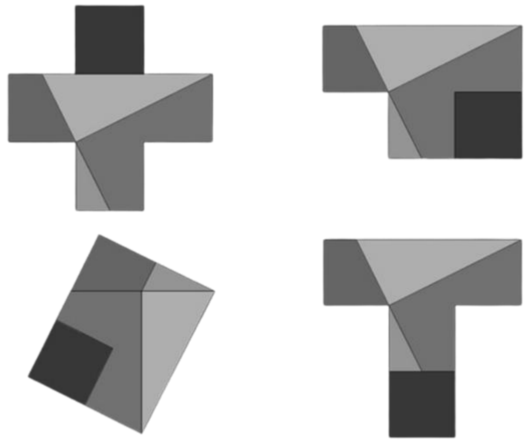
- It doesn't violate the laws of physics. The paradox operates within the realm of pure mathematics and doesn't directly address the physical world.
- It relies on specific mathematical definitions. The concepts of set, volume, and decomposition used in the theorem are complex.
- It has limitations. The paradox only applies to specific types of sets and transformations, and it doesn't imply that we can actually duplicate or transform physical objects.
- It remains a subject of debate. While the paradox is mathematically sound, its implications and interpretations continue to be discussed and debated by mathematicians and philosophers.

The Banach-Tarski paradox serves as a fascinating example of how seemingly impossible things can be true within the realm of mathematics. It challenges our intuition and pushes the boundaries of our understanding of space, volume, and infinity.

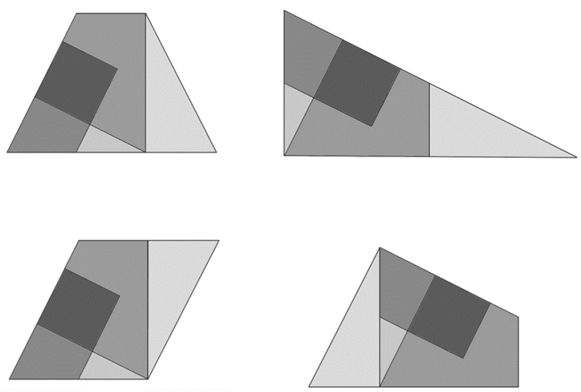
Here are some resources for further exploration:

- Wikipedia article entitled “Banach–Tarski paradox” [67]
- YouTube video entitled “The Banach-Tarski Paradox” [68]
- Quanta Magazine article entitled "Banach-Tarski and the Paradox of Infinite Cloning” [68]

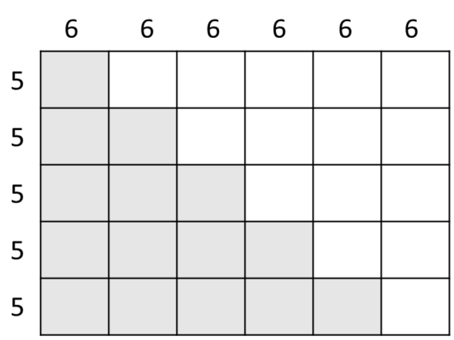
Solution to Puzzle 87:



Solution to Puzzle 88:

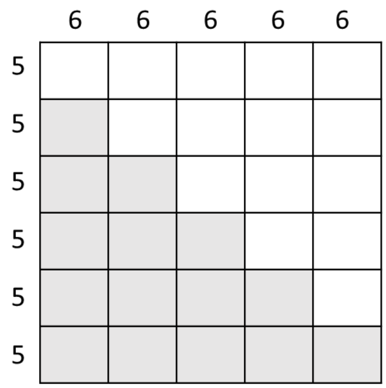


Solution to Puzzle 89: We first check to make sure that we have enough material to do the job, i.e., $25 \times 36 = 30 \times 30 = 900$. The required cut is along the boundary between the shaded and white squares in the figure below.



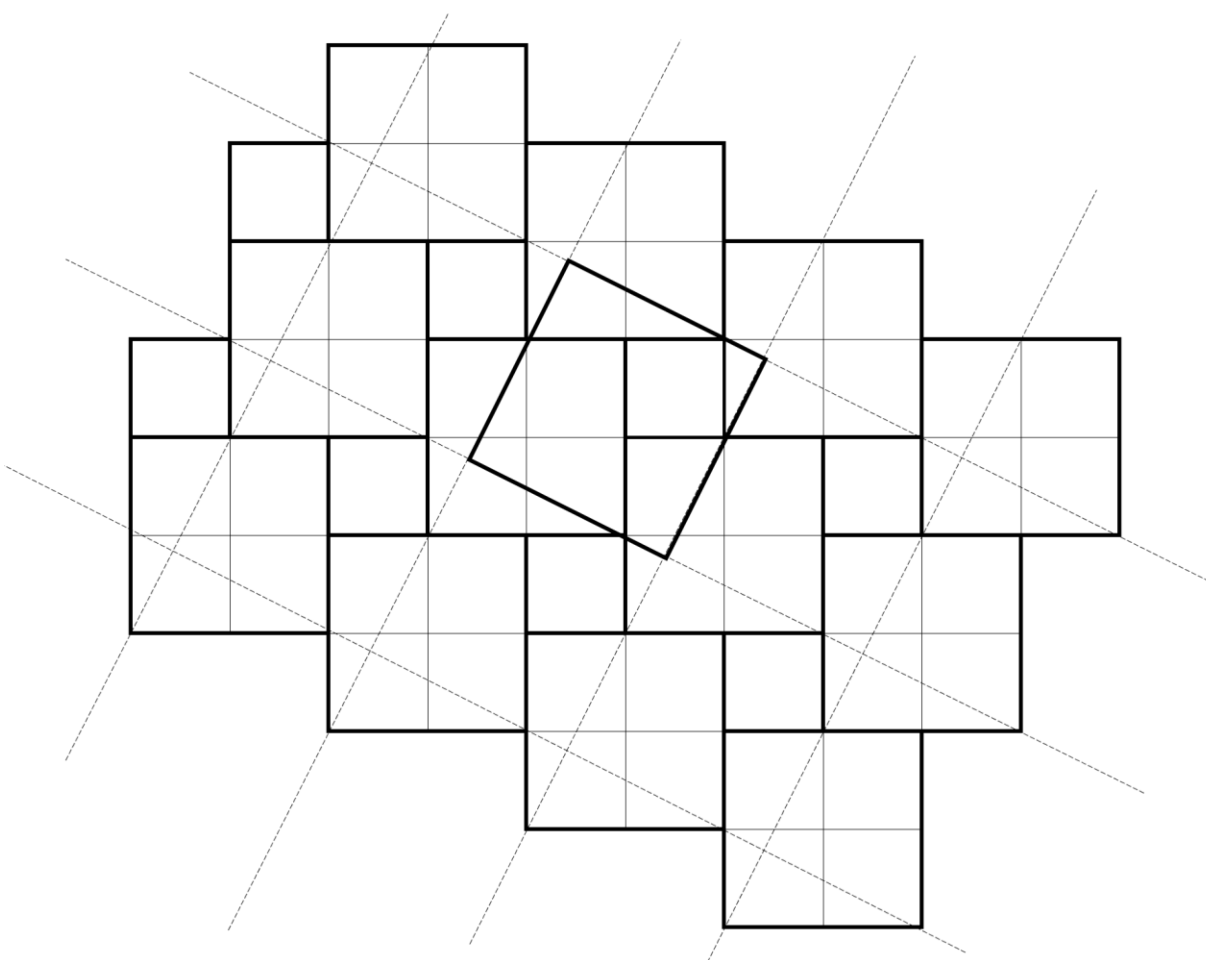
25×36

The required reassembly is shown in the figure below.



30×30

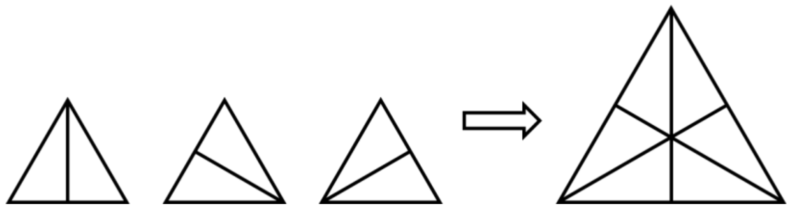
Solution to Puzzle 90: The puzzle is solved by first tiling the plane with a regular pattern using the two squares given in the puzzle statement (such a tiling is formally known as a **tessellation**). One then superimposes a grid that creates larger squares that effectively tile the plane. One of the larger squares (consisting of five pieces that comprise the two given squares) is highlighted in the center of the figure below.



This is a very old puzzle. In his book “Dissections: Plane & Fancy” [70], Frederickson states that the puzzle dates back to Thābit ibn Qurra (836-901 AD).

Solution to Puzzle 91:

The solution is depicted in the figure below. The problem goes back to Plato's dialog “Timaeus.”



10 Tessellations

Our lives are unique stones in the mosaic of human existence - priceless and irreplaceable.
— Henri Nouwen

10.1 Overview

[**Author's remark**: Unlike other sections in this book that present puzzles to the reader, this section offers no puzzles but rather provides an overview of tessellations, a type of puzzle.]

A **tessellation** or tiling is a covering of a surface using one or more geometric shapes, called tiles, with no overlaps and no gaps. We used a tessellation in the solution of Puzzle 90.

Without qualification, the concept typically refers to flat (planar) shapes, but the concept can be extended to higher dimensions. In terms of language derivations, the Wikipedia article on tessellations [71] states the following:

> In Latin, tessella is a small cubical piece of clay, stone or glass used to make mosaics. The word "tessella" means "small square" (from tessera, square, which in turn is from the Greek word τέσσερα for four). It corresponds to the everyday term "tiling", which refers to applications of tessellations, often made of glazed clay.

In what follows, the terms "tessellation" and "tiling" are used interchangeably. Further, keep in mind that, by definition, a tiling of the plane is infinite and as such, only part of each tiling is shown in the figures that follow.

10.2 Classifications

10.2.1 Regular Tessellations

There are many categories of tessellations. The simplest category is that of tessellations formed by regular polygons, i.e., polygons that are equiangular (all angles are equal in measure) and equilateral (all sides have the same length). These are known as **regular tessellations**. There are only three possible tilings of the plane with a single type of regular polygon – only equilateral triangles, squares or regular hexagons will work (for a proof of this fact, see Regular Tessellation article in Wolfram MathWorld [72]). Figure 21 shows a portion of each type of tiling. The black and white colorings are purely for visual effect.

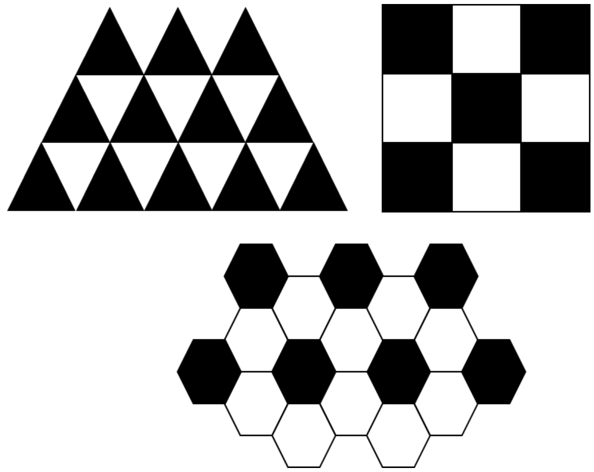


Figure 21. Examples of regular tessellations in the plane

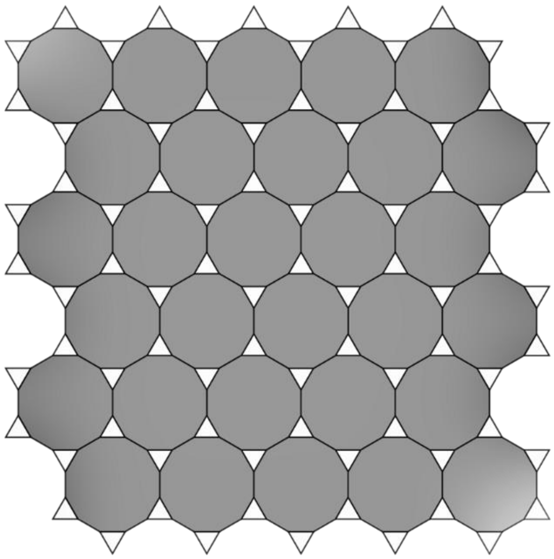
When classifying various types of tessellations, the concept of vertex configuration is very useful. A **vertex configuration** is a sequence of numbers that represents the number of sides on each face surrounding a given vertex. For example, the notation a.b.c.d describes a vertex that has 4 faces surrounding it, with a, b, c and d sides, respectively. In Figure 21, all the vertices in the triangle-based tiling have vertex configuration 3.3.3.3.3.3, all the vertices in the square-based tiling have vertex configuration 4.4.4.4 and all the vertices in the hexagon-based tiling have vertex configuration 6.6.6.

10.2.2 Semi-regular Tessellations

Figure 22, Figure 23, Figure 24 and Figure 25 depict what are called **semi-regular tessellations** (also known as Archimedean or uniform tilings). Semi-regular tessellations are comprised of two or more regular polygons such that all the vertices have the same vertex configuration (shown above each tiling in the figures below). The colorings are for visual effect only. Some of the components of the figures appear to be circles but they are not. For example, the gray components in the tessellation on the left of Figure 22 are 12-gons.

Figure credits for all the semi-regular tessellations go to Tom Ruen, see the Wikipedia article "Euclidean tilings by convex regular polygons" [75].

3.12.12



3.4.6.4

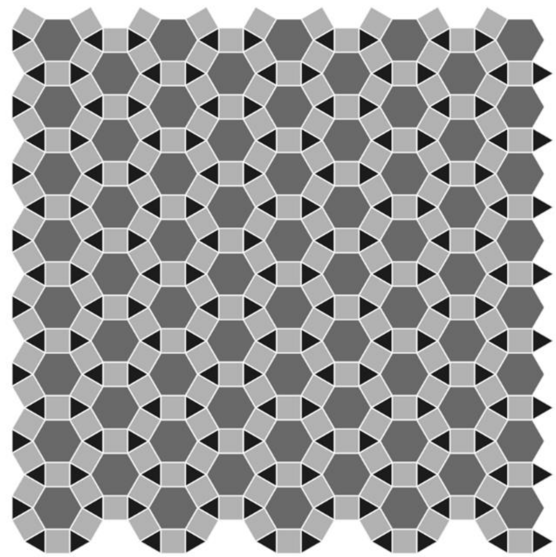
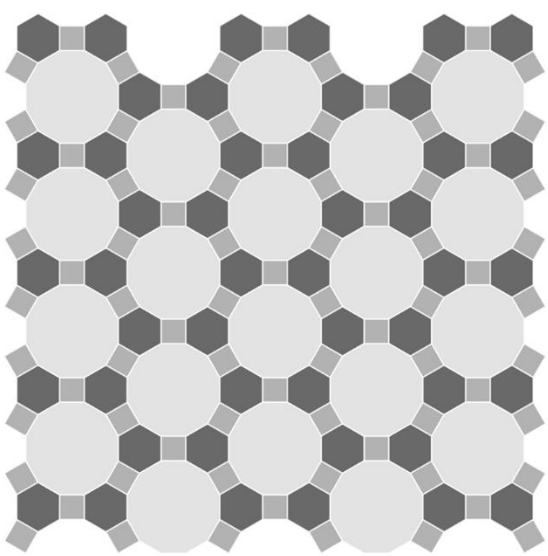


Figure 22. Semi-regular tessellations - Part 1

Surprisingly, there are only 8 possible semi-regular tessellations. A proof of this fact can be found in Section 2.2 of the paper by Swanson [76]. The general idea of the proof is that the interior angles of the regular polygons surrounding a vertex must add to 360 degrees (2π radians) and this only allows for a limited number of possibilities.

4.6.12



3.6.3.6

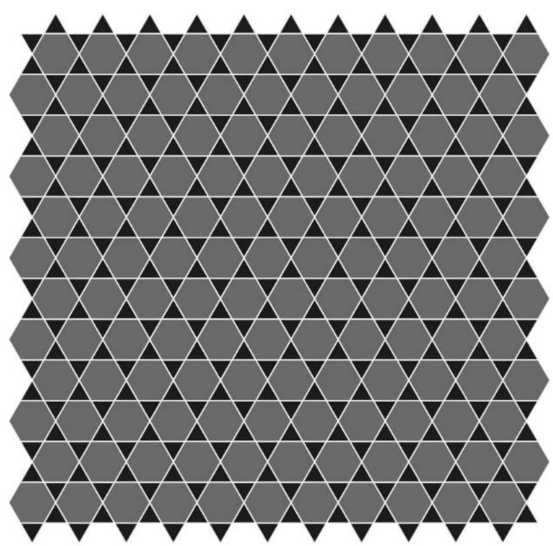


Figure 23. Semi-regular tessellations - Part 2

The famous astronomer Johannes Kepler (1571-1630) is generally credited with the first systematic explanation of semi-regular tessellations. He wrote about regular and semi-regular tessellations in his book *Harmonices Mundi* (1619). Unfortunately, Kepler's work on semi-regular tessellations was neglected (perhaps "forgotten" is more accurate), resulting in many mathematicians needlessly replicating his work, and in some cases, with incorrect results.

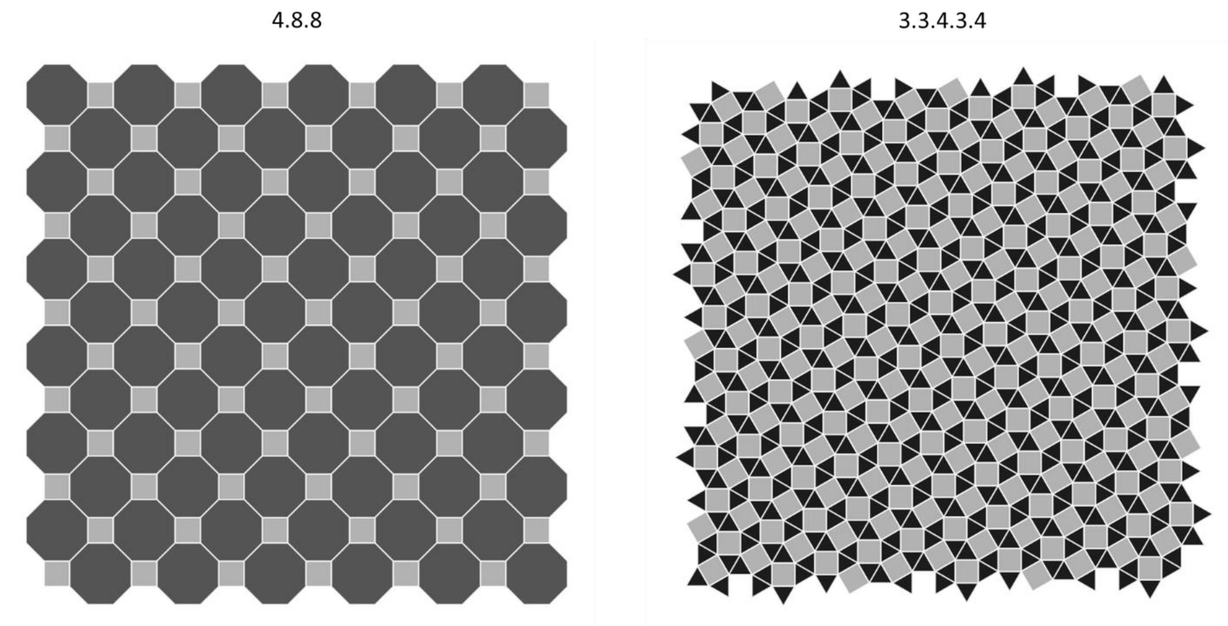


Figure 24. Semi-regular tessellations - Part 3

An extensive list and associated drawings of many tessellations (including the semi-regular tessellations) can be found in the Wikipedia article entitled *List of tessellations* [73].

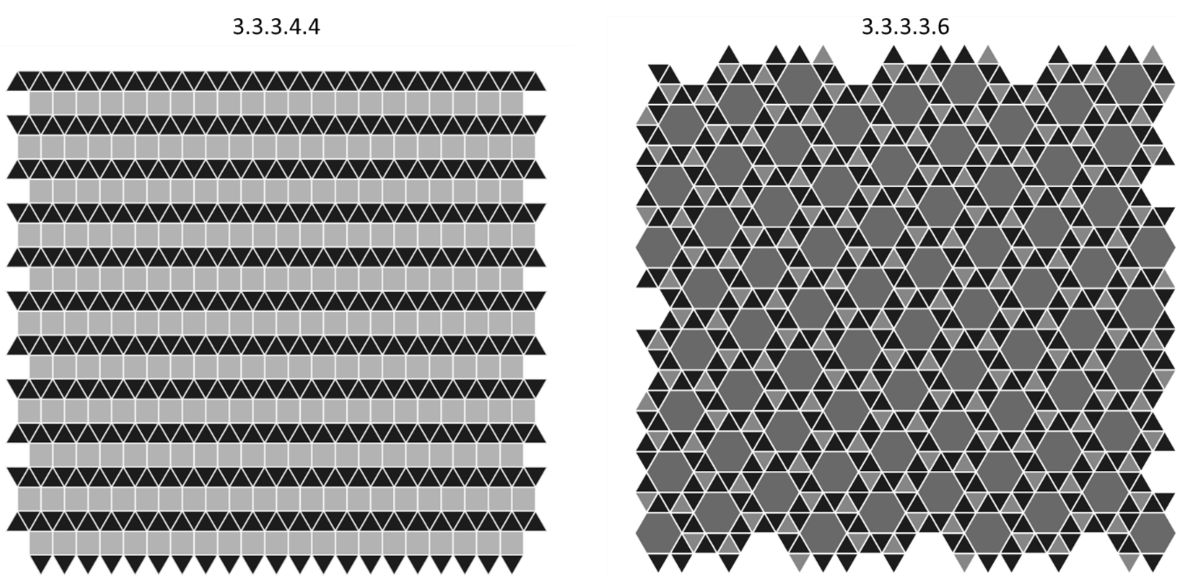


Figure 25. Semi-regular tessellations – Part 4

10.2.3 K-uniform Tessellations

A **k-uniform tessellation** is a tiling of the plane using regular polygons with k different vertex patterns. We've already seen the set of 1-uniform tessellations, i.e., the three regular tilings, and the eight semi-regular tessellations.

Although a bit dated, the paper entitled *Tilings by Regular Polygons* [74] gives a good overview of k-uniform tessellations. The Wikipedia article entitled *Euclidean tilings by convex regular polygons* [75] provides an up-to-date status on k-uniform tessellations and includes many beautifully illustrated examples.

Figure 26 shows an example of a 4-uniform tessellation. The small white circles, at the top and center of the figure, highlight the four types of vertices. The tessellation is described by the following notation:

$$[3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 6^3]$$

The description for each vertex is separated by a semi-colon. The vertex notation is the same as that used for semi-regular tessellation with one further compression, i.e., repeats such as 3.3.3.3 are written as 3^4. For example, 6^3 represents the pattern for the vertices surrounded by 3 hexagons in Figure 26. [**Author's remark**: Yes, I could have used the compressed notation for semi-regular tessellations but thought it was better to use the expanded form when first introducing the notation.]

Credits for Figure 26 go to Tom Ruen, https://commons.wikimedia.org/wiki/File:4-uniform_6.svg. The original figure has been changed to grayscale and the 4 small white circles have been added.

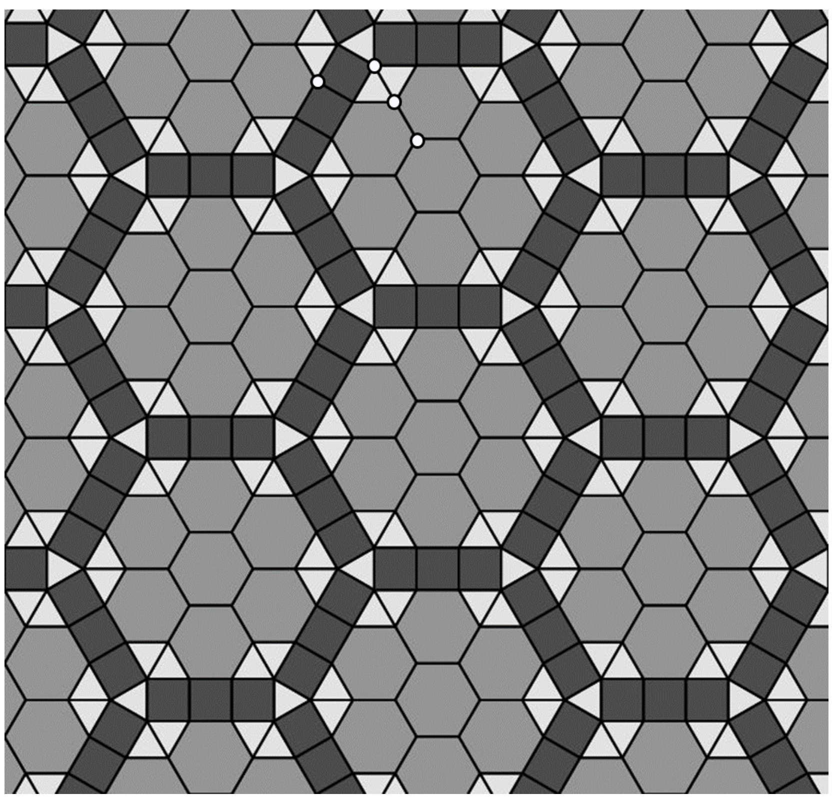


Figure 26. Example of a 4-uniform tessellation

Figure 27 illustrates a complication in the classification scheme regarding k-uniform tessellations, i.e., several vertices of the same type can be situated differently within a tessellation. The tessellation in Figure 27 is considered to be a 3-uniform tessellation even though there are only two different vertices types, i.e., $3.4.6.4$ and $3.4^2.6$. However, the vertices of type 3.4.6.4 are situated within the tessellation in two different ways (see, for example, the two vertices labeled as 1 and 2 in the figure). Another way to view the situation is to imagine a second identical tiling. There is no combination of translations, rotations or reflections that allows one to place the second tiling on top of the first tiling with Vertex #1 placed directly over Vertex #2, while completely aligning the patterns.

An example of a $3.4^2.6$ vertex is covered by the circle with horizontal stripes.

In terms of notation, the tessellation in Figure 27 is represented as $[(3.4.6.4)2;\ 3.4^2.6]$. The "2" after $(3.4.6.4)$ indicates that the particular vertex type is positioned in two different ways within the tessellation.

Credits for Figure 27 go to Tom Ruen, see https://commons.wikimedia.org/wiki/File:3-uniform_26.svg. The figure has been changed to grayscale, with the addition of three circles to highlight examples of the two types of vertices.

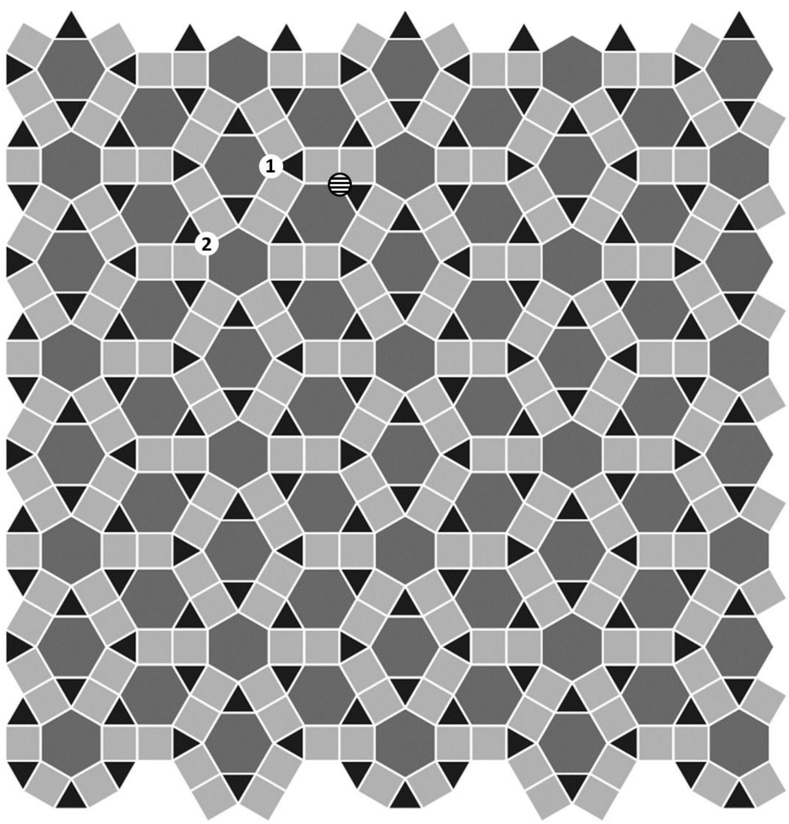


Figure 27. Example of a 3-uniform, 2 vertex type tessellation

Figure 28 depicts a 5-uniform tessellation with only 2 vertex types, i.e., one vertex is of type 4.6.12 and another vertex is of type 3.4.6.4. The vertex of type 3.4.6.4 is situated within the tessellation in four different ways (see the four vertices highlighted with white circles in Figure 28). Notice that each of the white vertices is situated differently with respect to the 12-gon. The vertex notation for this tessellation is [(3.4.6.4)4; 4.6.12].

Credits for Figure 28 go to Tom Ruen, see https://commons.wikimedia.org/wiki/File:5-uniform_36.svg. The original figure has been changed to grayscale and four small circles have been added.

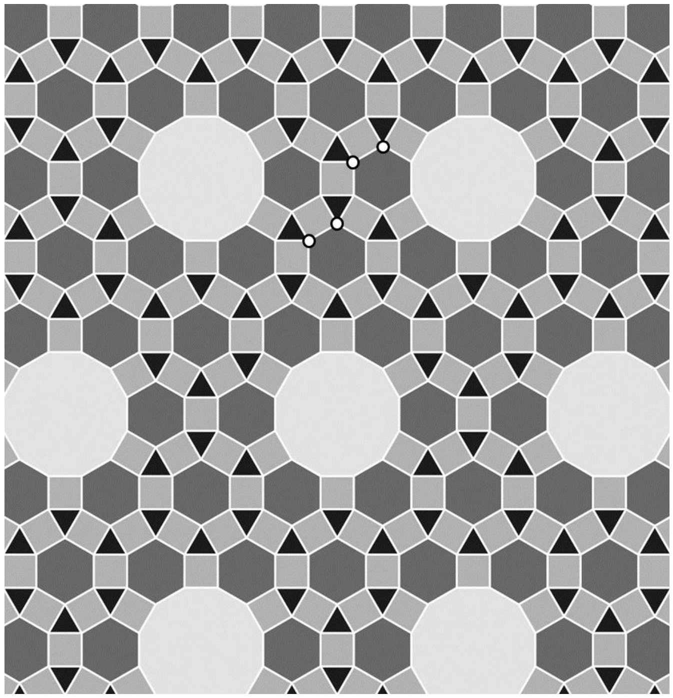


Figure 28. Example of a 5-uniform tessellation with only 2 vertex types

10.2.4 Non-periodic Tessellations

All the tessellations that we have seen so far are periodic in the sense that one can make a copy of the tessellation, translate the copy to another position in the plane, and then drop the copy on top of the original and get an exact match. If this is not possible, then the tessellation is classified as **non-periodic** (also known as aperiodic). Non-periodic tilings are said to lack translation symmetry. The definitions of periodic and non-periodic apply to any type of tiling (not just those involving polygons).

Figure 29 is a very simple but uninteresting example of a non-periodic tiling. The two triangles (which repeat nowhere else) disrupt the periodicity of the tiling. To be clear, the pattern of rectangles extend in all directions.

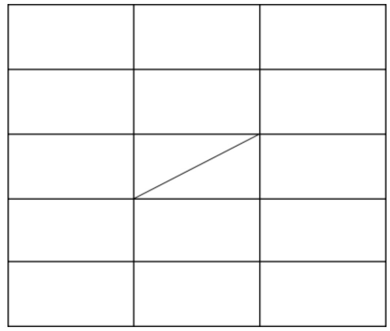


Figure 29. Simple non-periodic tiling

A complex example of an aperiodic tessellation is the rhomb tiling in Figure 30. The source for Figure 30 and the details for constructing similar figures can be found in the online article by Tjipke Hibma [77], see the figure entitled "Quarter-rhomb tiling for n=11." Unlike the figures for periodic tilings, the rhomb tiling figure itself does not provide sufficient information to continue the pattern. The figure looks very ordered, and one might think (without detailed observation) that the rhomb tiling is periodic, but it is not.

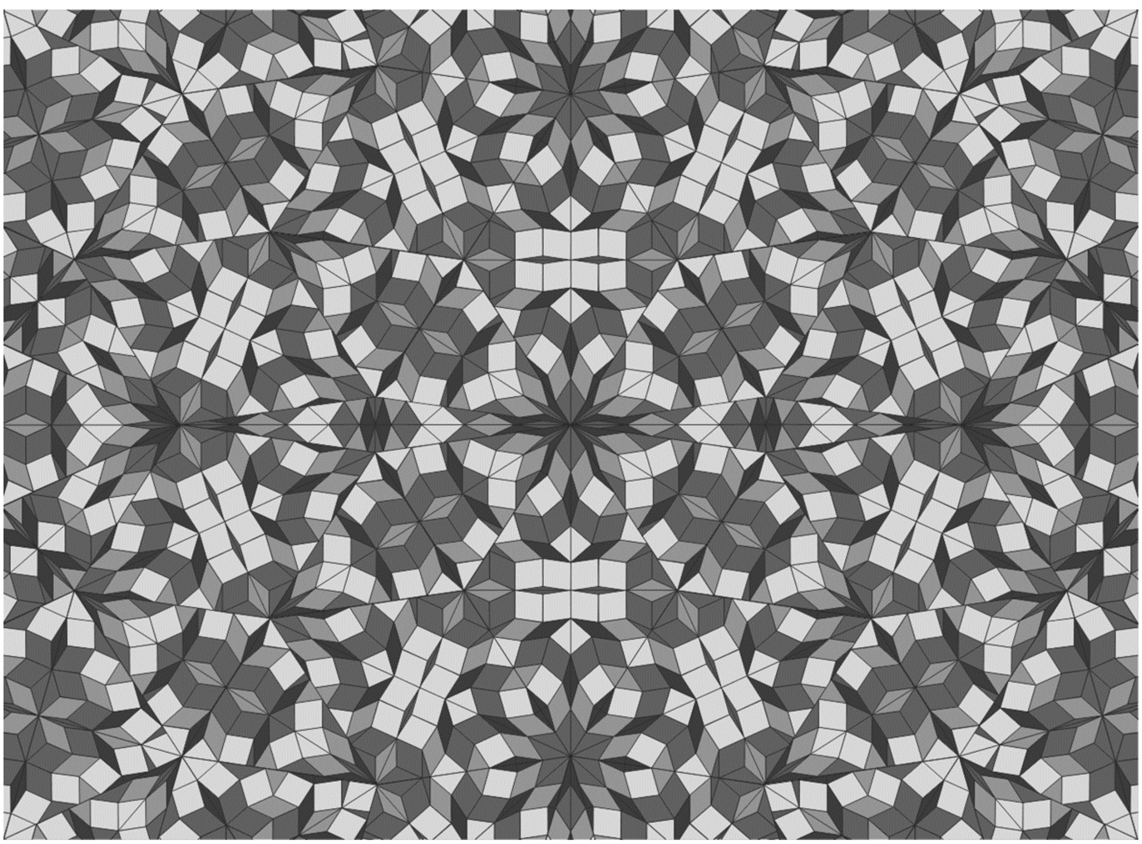


Figure 30. Example of a rhomb tiling

Perhaps the most famous non-periodic tiling is the eponymous Penrose tiling [78]. While the Penrose tiling lacks translational symmetry, it does have reflection symmetry and five-fold rotational symmetry. An example Penrose tiling is shown in Figure 31. (Note that "P3" in the title of the figure refers to one of three types of Penrose tilings, and yes, the other two types are called P1 and P2.)

The following YouTube videos cover various aspects of periodic and non-periodic tilings.

- A two-part series on fundamentals of periodic (https://youtu.be/yKyZlK1Ch0M) and aperiodic (https://youtu.be/hraRYYPjtCk) tilings
- Penrose tiles: *Helsinki Maths Mystery* at https://youtu.be/yxlEojkVJ0c and the video entitled *The Penrose Tessellation* at http://penrose.dmf.unicatt.it/html5_penrosenext.html.en
- Roger Penrose - Forbidden crystal symmetry in mathematics and architecture: https://youtu.be/th3YMEamzmw (this is a lecture from Roger Penrose).

Also, see the online book entitled "Math and the Art of MC Escher" [79] which has a section on aperiodic tessellations as well as several other sections on different aspects of tessellations.

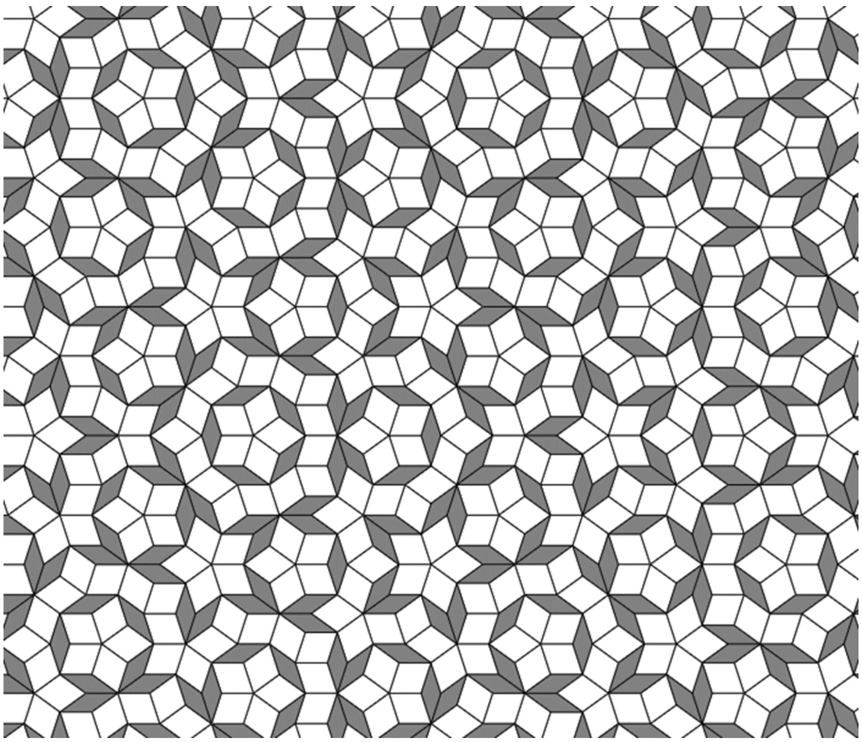


Figure 31. Example of a Penrose P3 tiling

Until recently, it was not known whether there existed an aperiodic tiling that consisted of a single tile. However, in early 2023, a retired print technician named David Smith discovered a single type of tile (known as the Hat tile, given its shape) that can cover the entire plane without a repeating pattern, see Figure 32. The tile is sometimes referred to as the "einstein" tile, not in reference to Albert Einstein, but in reference to the German meaning of the word, i.e., "one stone" or "one tile".

For further details on the hat tile and its discovery, see the article "Mathematicians have finally discovered an elusive 'einstein' tile" [80].

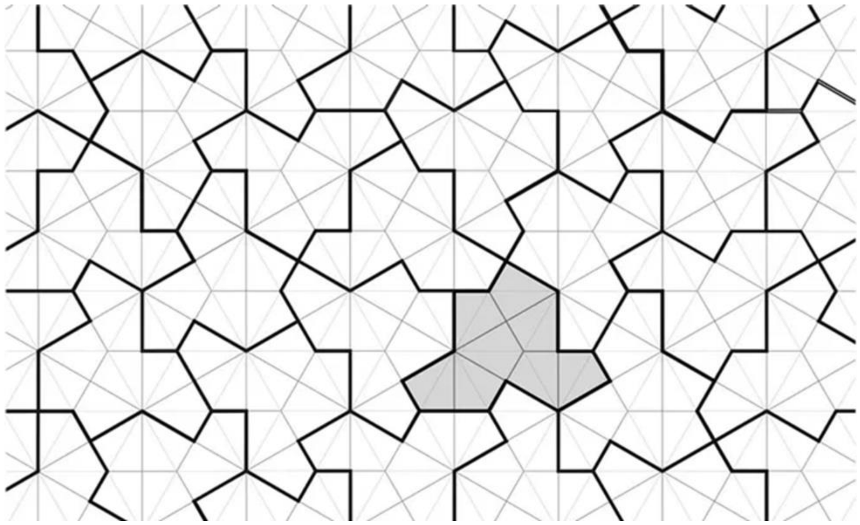


Figure 32. Hat tile

Smith's discovery was made using a software program called the PolyForm Puzzle Solver. He was experimenting with different shapes when he came across the hat-shaped tile that would eventually become his discovery. He noticed that the tile could be used to cover an infinite plane without a repeating pattern, and he sent his discovery to Craig Kaplan, a computer scientist at the University of Waterloo in Canada.

Kaplan and his colleagues investigated Smith's discovery and found that it was indeed a new type of aperiodic tile. They also found that there are infinitely many different tiles of this type, see the YouTube video entitled "See an 'einstein' tile morph into different shapes" [81].

The discovery of the hat tile is a significant breakthrough in mathematics. Aperiodic tilings have been studied for many years, but they were not thought to exist with only one tile. The discovery of the hat tile shows that there are many more possibilities for aperiodic tiles than previously thought.

10.3 Open Problems concerning Tessellations

The following are some open (and very difficult) problems concerning tessellations:

- Heesch's Tiling Problem [82]: The Heesch number of a shape is the maximum number of layers of copies of the same shape that can surround it with no overlaps and no gaps. Heesch's problem is the problem of determining the set of numbers that can be Heesch numbers.
- Regular Tessellations in Higher Dimensions: Regular tessellations of the plane are well-understood (e.g., triangles, squares, hexagons). However, extending the concept of regular tessellations to higher dimensions, such as in three-dimensional space and beyond, poses interesting challenges. For further details on this topic, see "Chapter V - Regular Tessellations in Higher Dimensions" in the book "Groups and Geometry" [83].
- Aperiodic Tessellations: The study of aperiodic tessellations, which do not repeat regularly, has led to fascinating discoveries like Penrose and hat tiles. Exploring new classes of aperiodic tessellations or finding a general understanding of their properties is still an active area of research.

- Algorithms for Tessellation Design [84]: Developing efficient algorithms for generating tessellations with specific properties or constraints is an ongoing challenge. This is particularly relevant in computer graphics, where tessellations are used for modeling and rendering.
- Tessellations on Curved Surfaces: Extending tessellations beyond flat surfaces to curved surfaces or non-Euclidean geometries introduces new challenges. Understanding the rules and properties of tessellations on such surfaces is an open problem. See the example in Figure 33.

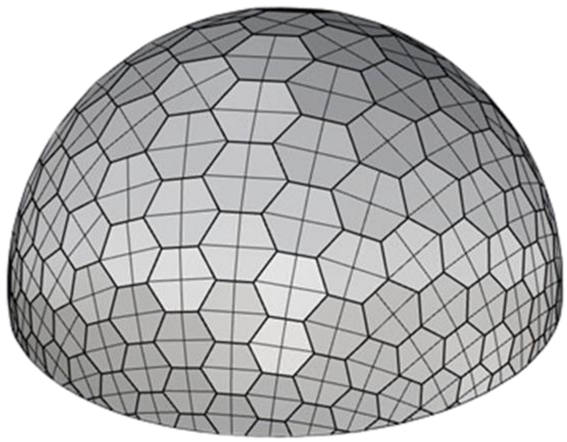


Figure 33. Hexagon tiling of sphere surface

The source for Figure 33 is Daniel Rocco, see https://discourse.mcneel.com/t/hexagon-tessellation-freeform-surface-with-constant-angles/122170.

11 Analytic Geometry Puzzles

Why is there anything at all rather than nothing whatsoever? – Gottfried Leibniz

Analytic geometry, also known as coordinate geometry or Cartesian geometry, is a branch of mathematics that studies geometry using a coordinate system. This means that geometric shapes are represented by equations and inequalities, and geometric problems are solved using algebraic methods.

Analytic geometry was developed in the 17th century by René Descartes and Pierre de Fermat. It is a fundamental tool in many areas of mathematics, including calculus, linear algebra, and differential geometry. It is also used in many other fields, such as physics, engineering, and computer graphics. According to the Wikipedia article "Analytic geometry", there were several much earlier concepts that could be considered as analytic geometry [85]:

> The Greek mathematician Menaechmus solved problems and proved theorems by using a method that had a strong resemblance to the use of coordinates and it has sometimes been maintained that he had introduced analytic geometry.
>
> Apollonius of Perga, in "On Determinate Section", dealt with problems in a manner that may be called an analytic geometry of one dimension; with the question of finding points on a line that were in a ratio to the others. Apollonius in the Conics further developed a method that is so similar to analytic geometry that his work is sometimes thought to have anticipated the work of Descartes by some 1800 years. His application of reference lines, a diameter and a tangent is essentially no different from our modern use of a coordinate frame, where the distances measured along the diameter from the point of tangency are the abscissas, and the segments parallel to the tangent and intercepted between the axis and the curve are the ordinates. He further developed relations between the abscissas and the corresponding ordinates that are equivalent to rhetorical equations (expressed in words) of curves. However, although Apollonius came close to developing analytic geometry, he did not manage to do so since he did not take into account negative magnitudes and in every case the coordinate system was superimposed upon a given curve a posteriori instead of a priori. That is, equations were determined by curves, but curves were not determined by equations. Coordinates, variables, and equations were subsidiary notions applied to a specific geometric situation.
>
> The 11th-century Persian mathematician Omar Khayyam saw a strong relationship between geometry and algebra and was moving in the right direction when he helped close the gap between numerical and geometric algebra with his geometric solution of the general cubic equations, but the decisive step came later with Descartes. Omar Khayyam is credited with identifying the foundations of algebraic geometry, and his book "Treatise on Demonstrations of Problems of Algebra" (1070), which laid down the principles of analytic geometry, is part of the body of Persian mathematics that was eventually transmitted to Europe. Because of his thoroughgoing geometrical approach to algebraic equations, Khayyam can be considered a precursor to Descartes in the invention of analytic geometry.

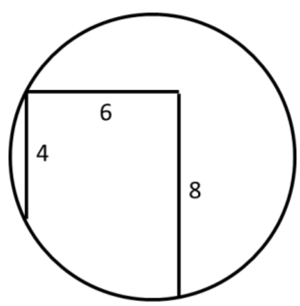
Here are some of the key concepts in analytic geometry:

- Coordinate systems: A coordinate system is a way of assigning numbers to points in space. The most common coordinate system is the Cartesian coordinate system, which uses two perpendicular axes to assign an ordered pair of numbers (x, y) to each point in the plane.
- Equations of geometric shapes: The equation of a geometric shape is an algebraic equation that describes all the points that belong to the shape. For example, the equation of a circle with center (h, k) and radius r is given by the equation $(x - h)^2 + (y - k)^2 = r^2$.
- Distance formula: The distance formula can be used to calculate the distance between two points (x_1, y_1) and $x_2, y_2)$. The formula is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Slope formula: The slope formula can be used to calculate the slope of a line given points (x_1, y_1) and $x_2, y_2)$ on the line. The formula is: $\frac{y_2 - y_1}{x_2 - x_1}$.
- Intercepts: The x-intercept of a line is the point where the line crosses the x-axis. The y-intercept of a line is the point where the line crosses the y-axis.

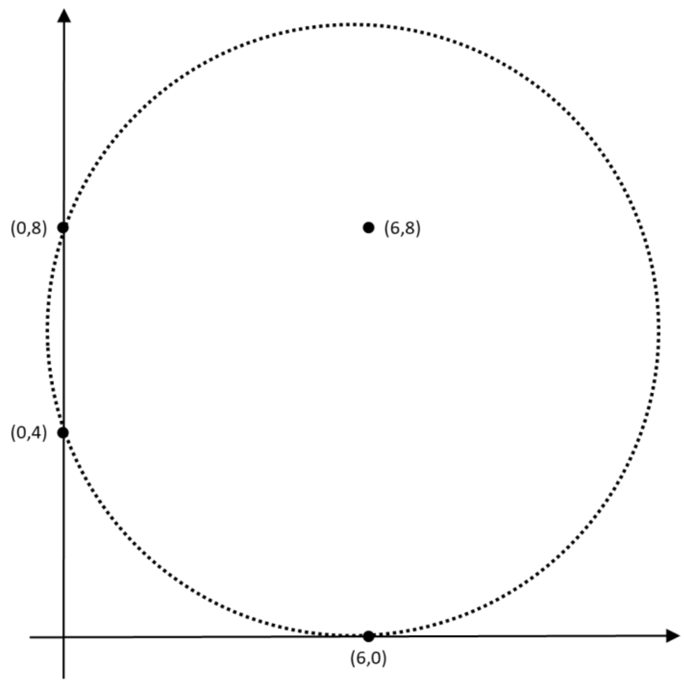
Analytic geometry is a powerful tool that can be used to solve a wide variety of geometric problems. It is an essential part of the mathematical toolkit for anyone who wants to study mathematics, physics, engineering, or any other field that relies on geometry. A few examples are given in the following puzzles.

Puzzle 92. Determine the area of the circle given unusual information

Determine the area of the circle shown in the figure below. The segments meet at right angles.



Solution: This problem is more easily solved using analytic geometric. To that end, place the endpoints of the four given segments in the coordinate system shown below. The problem is made yet easier if we put the segment of length 4 along the y-axis, and the lower endpoint of the segment of length 8 on the x-axis.



From Theorem 28, we know that 3 non-collinear points determine a circle. Further, we know that the equation of a circle with center (a, b) and radius r is given by $(x-a)^2 + (y-b)^2 = r^2$.

Next, substitute the three points on the circle into the equation for a circle. This gives us three equations with 3 unknowns.

$$a^2 + (8-b)^2 - r^2 = 0 \quad \text{(Equation 1)}$$

$$a^2 + (4-b)^2 - r^2 = 0 \quad \text{(Equation 2)}$$

$$(6-a)^2 + b^2 - r^2 = 0 \quad \text{(Equation 3)}$$

Subtracting Equation 2 from Equation 1, we get

$$(8-b)^2 - (4-b)^2 = 0$$

$$64 - 16b + b^2 - 16 + 8b - b^2 = 0$$

$$b = 6$$

Substitute $b = 6$ into Equations 2 and 3 to get

$$a^2 + 4 - r^2 = 0 \quad \text{(Equation 4)}$$

$$(6 - a)^2 + 36 - r^2 = 0 \quad \text{(Equation 5)}$$

Subtract Equation 4 from Equation 5, and solve for a.

$$a^2 - 12a + 36 + 36 - r^2 - a^2 - 4 + r^2 = 0$$

$$-12a + 68 = 0$$

$$a = \frac{17}{3}$$

Plugging the values for a and b into Equation 2 gives us

$$\left(\frac{17}{3}\right)^2 + (4 - 6)^2 = r^2$$

$$r^2 = \frac{325}{9}$$

Thus, the area of the circle is $325\pi/9$ and the equation for the circle is

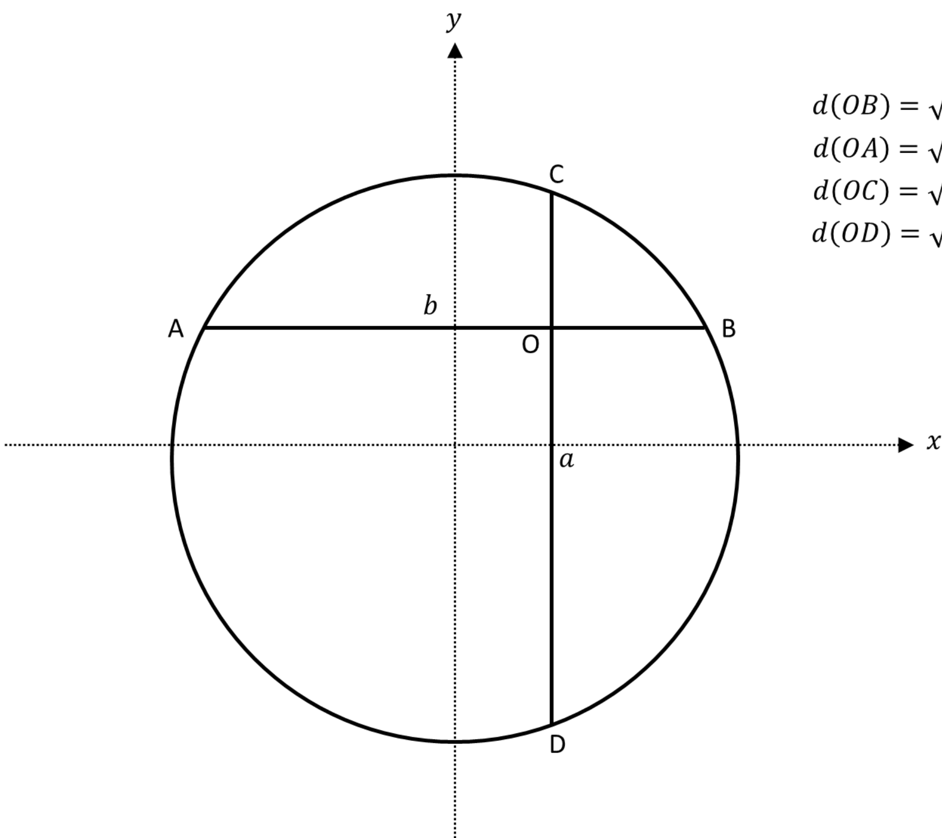
$$\left(x - \frac{17}{3}\right)^2 + (y - 6)^2 = \frac{325}{9}$$

The problem can be solved using only geometric principles, but the solution is much harder, see the YouTube video entitled "Very difficult for most students" [52].

Puzzle 93. Summation formula concerning intersecting perpendicular chords

Given two perpendicular chords $\overline{AB}$ and $\overline{CD}$ of a circle with radius r such that the chords intersect at point O, prove that $d(AO)^2 + d(BO)^2 + d(CO)^2 + d(DO)^2 = 4r^2$.

Solution: This is another example of where analytic geometry provides an easier solution than using purely geometric principles. Without loss of generality, assume the circle has center at point $(0,0)$ and the coordinates for the intersection point of the two chords is (a, b). The following figure shows the circle as placed in a coordinate system. We first determine the coordinates from points A, B, C and D. For example, the y coordinate for point B is b, and we get the x coordinate by solving for x in the equation $x^2 + b^2 = r^2$, i.e., $x = \sqrt{r^2 - b^2}$. Similarly, we get the other coordinates, i.e., $A = \left(-\sqrt{r^2 - b^2}, b\right), C = \left(a, \sqrt{r^2 - a^2}\right)$, and $D = (a, -\sqrt{r^2 - a^2})$. From here, it is simple to compute the distance from each of the four points to the point O (see the right side of the figure below).



$$d(OB) = \sqrt{r^2 - b^2} - a$$
$$d(OA) = \sqrt{r^2 - b^2} + a$$
$$d(OC) = \sqrt{r^2 - a^2} - b$$
$$d(OD) = \sqrt{r^2 - a^2} + b$$

From here, it is just a matter of computing the stated distance in the formula.

$$d(OA)^2 + d(OB)^2 + d(OC)^2 + d(OD)^2$$

$$= \left(\sqrt{r^2 - b^2} + a\right)^2 + \left(\sqrt{r^2 - b^2} - a\right)^2 + \left(\sqrt{r^2 - a^2} - b\right)^2 + \left(\sqrt{r^2 - a^2} + b\right)^2$$

$$= \left[(r^2 - b^2) + 2a\sqrt{r^2 - b^2} + a^2\right] + \left[(r^2 - b^2) - 2a\sqrt{r^2 - b^2} + a^2\right]$$

$$+ \left[(r^2 - a^2) - 2b\sqrt{r^2 - a^2} + b^2\right] + \left[(r^2 - a^2) + 2b\sqrt{r^2 - a^2} + b^2\right]$$

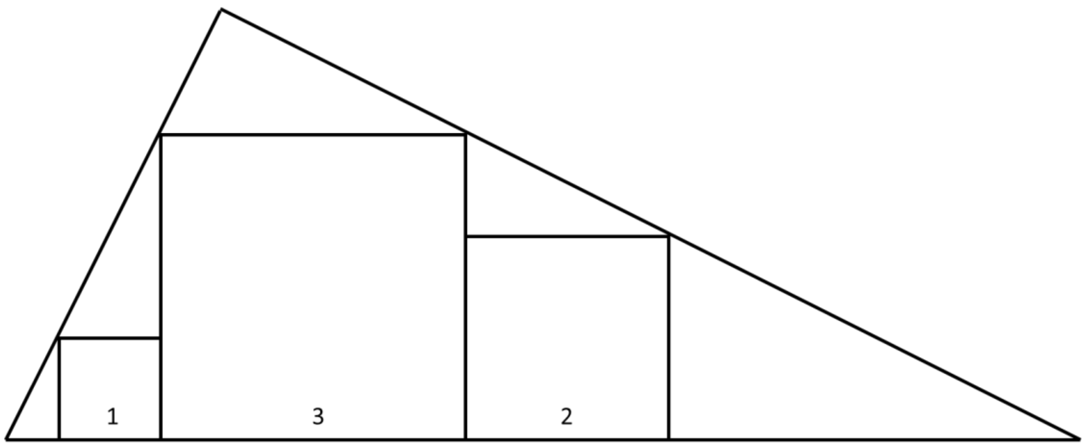
$$= [(r^2 - b^2) + a^2] + [(r^2 - b^2) + a^2] + [(r^2 - a^2) + b^2] + [(r^2 - a^2) + b^2]$$

$$= 4r^2$$

For a solution using purely geometric concepts, see the article at Socratic Q&A [53].

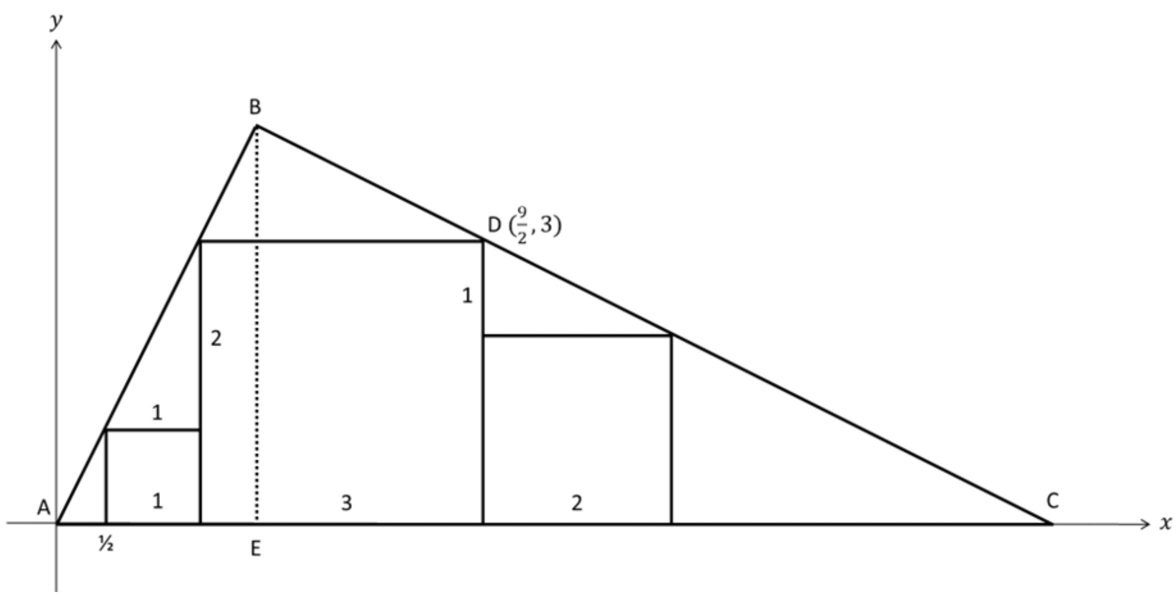
Puzzle 94. Find the area of the triangle contain three squares

Find the area of the triangle shown in the figure below. The three squares have side lengths 1,3 and 2, respectively. Tangencies among the squares and the triangle are as suggested in the figure.



Solution:

The first step is to place the configuration within a coordinate system. We choose the vertex A of the triangle to be at the origin (as shown in the figure below).



The line AB has slope 2 and goes through the point $(0,0)$. Using the slope-intercept formula, the line AB is represented by $y = 2x$. When $y = 1$ on line AB, $x = \frac{1}{2}$.

The line BC has slope $-\frac{1}{2}$ and goes through the point D with coordinates $(\frac{9}{2}, 3)$. So, line BC is represented by $y - 3 = -\frac{1}{2}(x - \frac{9}{2})$ or $y = -\frac{x}{2} + \frac{21}{4}$.

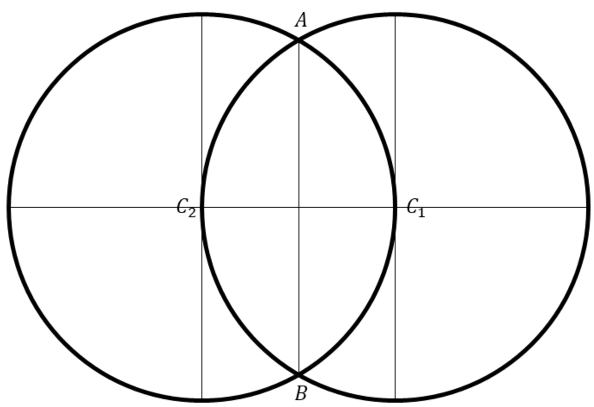
Next, we solve for the intersection of lines AB and BC, i.e.,

$$2x = -\frac{x}{2} + \frac{21}{4} \Rightarrow 8x = -2x + 21 \Rightarrow x = \frac{21}{10}$$

Plugging x into either of the equations involving y, we get $y = \frac{42}{10}$ (which is the height of the triangle ABC). To compute $d(AC)$, we need to find where line BC intersects the x-axis. Putting $y = 0$ into $y = -\frac{x}{2} + \frac{21}{4}$, we get $x = \frac{21}{2}$ (which is the length of the base of triangle ABC). Thus, triangle ABC has area $\frac{1}{2}\left(\frac{21}{2}\right)\left(\frac{42}{10}\right) = \frac{441}{20} = 22.05$.

Puzzle 95. Find the distance between the point of intersection of two circles

In the configuration shown in the figure below, find the distance between points A and B. The radius of each circle is 1. Each circle goes through the center of the other.



Solution: Place the configuration of circles in a coordinate system, as shown in the following figure. The equations for the circles are

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = 1 \quad \text{(Equation 1)}$$

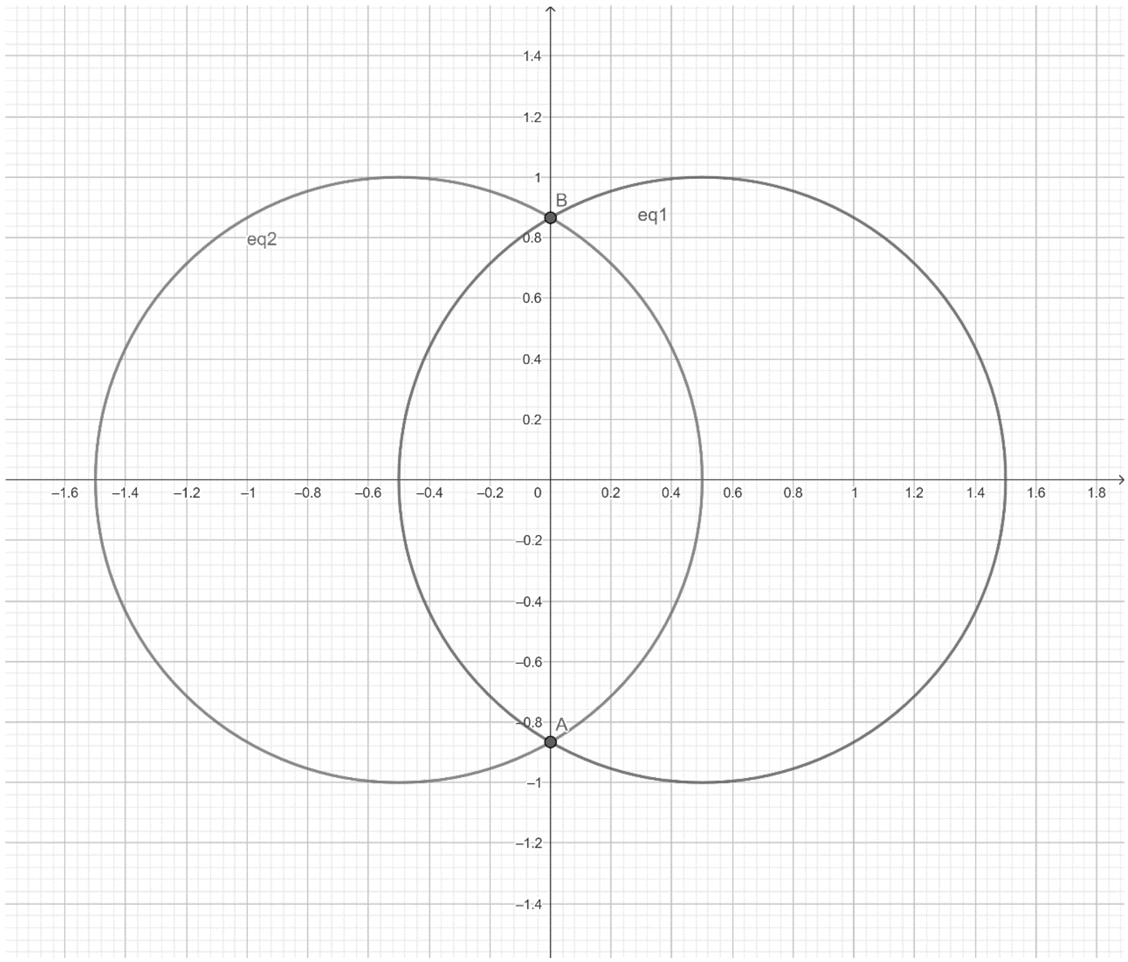
$$\left(x + \tfrac{1}{2}\right)^2 + y^2 = 1 \quad \text{(Equation 2)}$$

Subtract Equation 1 from Equation 2 to get

$$\left(x + \frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2 = 0$$

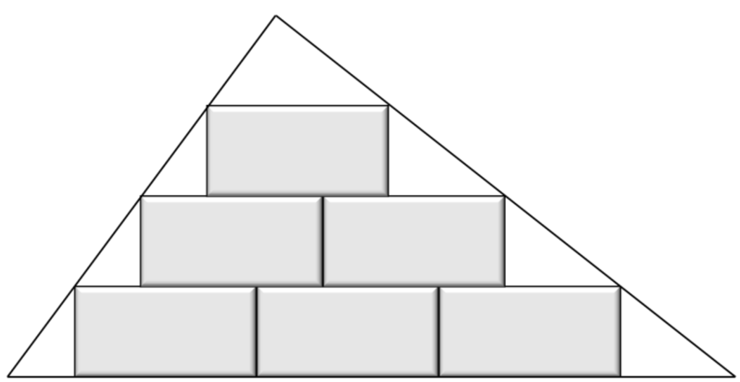
$$\left(x^2 + x + \frac{1}{4}\right) - \left(x^2 - x + \frac{1}{4}\right) = 0 \Rightarrow x = 0$$

Substitute $x = 0$ into Equation 1 to get $y = \pm\frac{\sqrt{3}}{2}$. So, the distance between the two points of intersection is $\sqrt{3}$.

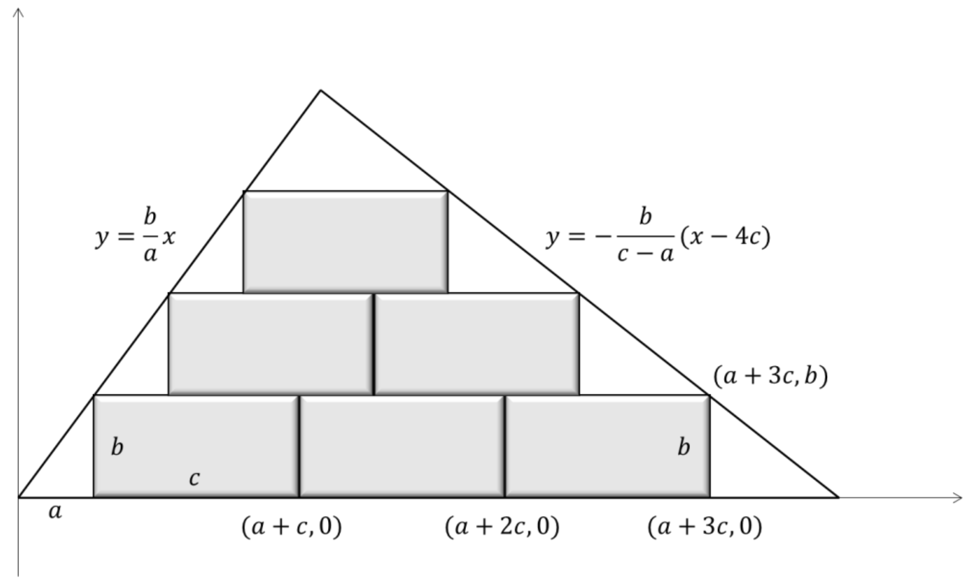


Puzzle 96. Bricks in a triangle

The figure below depicts a cross-sectional view of identical bricks enclosed within a triangle. The tangencies are as suggested in the figure. What fraction of the figure is shaded, i.e., covered by bricks?



Solution: Place the configuration in a coordinate system as shown in the figure below. Let the dimensions of each brick be $b \times c$, and let a be the offset of the bottom left brick from the bottom left vertex of the triangle.



The equation for the left edge of the triangle (call this line ℓ) is $y = \frac{b}{a}x$ since the associated line goes through point $(0,0)$ and has slope $\frac{b}{a}$.

The line associated with the right edge of the triangle (call this line m) has slope $-\frac{b}{c-a}$ and goes through the point $(a + 3c, b)$. Using the point-slope form, the equation for this line is

$$y - b = -\frac{b}{c-a}\big(x - (a + 3c)\big)$$

$$y = -\frac{bx}{c-a} + \frac{b(a+3c)}{c-a} + b$$

$$y = -\frac{bx}{c-a} + \frac{4bc}{c-a}$$

Next, we solve for the x-coordinate of the intersection of the two lines noted above.

$$\frac{b}{a}x = -\frac{bx}{c-a} + \frac{4bc}{c-a}$$

$$x\left(\frac{b}{a} + \frac{b}{c-a}\right) = \frac{4bc}{c-a}$$

$$x\left(\frac{bc - ab + ab}{a(c-a)}\right) = \frac{4bc}{c-a}$$

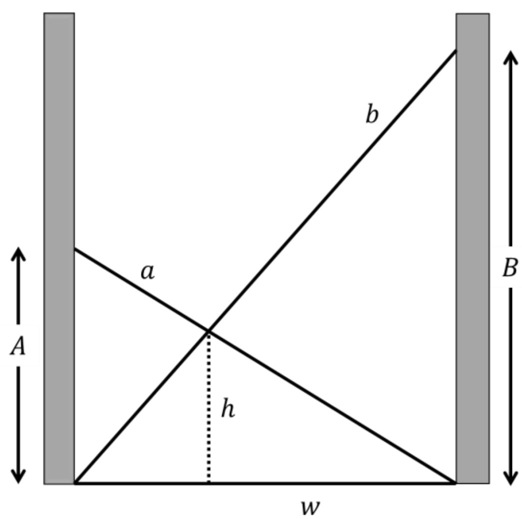
$$x = \frac{a(c-a)}{bc} \cdot \frac{4bc}{c-a} = 4a$$

The associated value for y is $4b$. So, the point of intersection of the left and right segments of the triangle is $(4a, 4b)$. Further, when $y = 0$ in line m, we have that $x = 4c$ and thus, the base of triangle ABC is of length $4c$. So, the area of the triangle is $\frac{1}{2}(4c)(4b) = 8cb$. Since we have 6 bricks, their total area is $6cb$. Thus, the ratio of the shaded region (i.e., the bricks) to the entire area of the triangle is $\frac{6cb}{8cb} = \frac{3}{4}$.

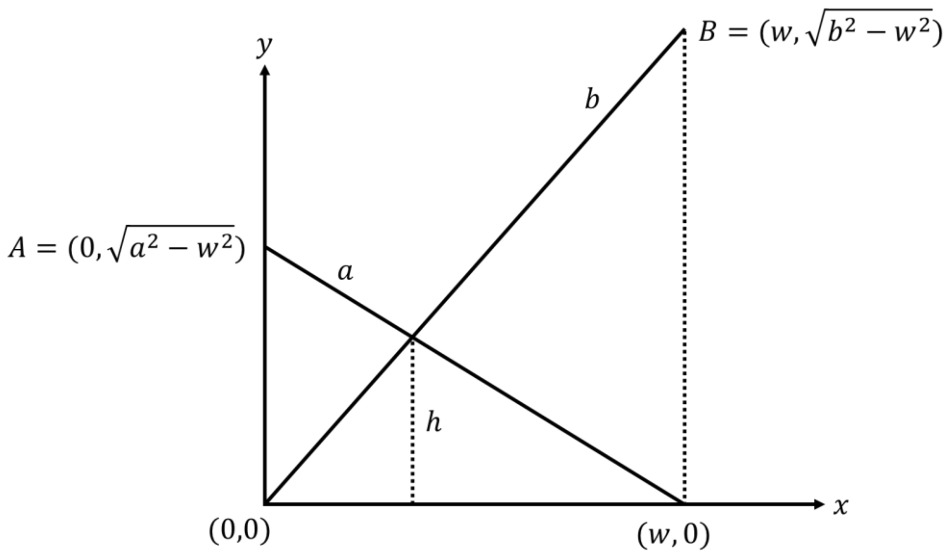
Puzzle 97. Crossed ladder puzzle

Two ladders of lengths a and b are positioned in an alley as shown in the figure below. The width of the alley is w. Given a, b and w, find the height h where the two ladders cross.

This is a version of a more complex puzzle where *a, b* and h are given, and you are asked to determine w, see the Wikipedia article "Crossed ladders problem" [50].



Solution: Place the configuration onto an xy-coordinated system, as shown below. The heights A and B are calculated using the Pythagorean theorem.



Using the information on the graph, we can determine the equations for the lines representing each ladder, i.e.,

$$y = \frac{\sqrt{b^2 - w^2}}{w} x$$

$$y = -\frac{\sqrt{a^2 - w^2}}{w} x + \sqrt{a^2 - w^2}$$

The two ladders cross at the intersection point of the above two equations. Setting the two expressions for y equal and then solving for x, we get

$$x = \frac{w\sqrt{a^2 - w^2}}{\sqrt{b^2 - w^2} + \sqrt{a^2 - w^2}}$$

and then plugging x back into either equation for y, we get the height where the ladders cross

$$y = \frac{\sqrt{b^2 - w^2}\,\sqrt{a^2 - w^2}}{\sqrt{b^2 - w^2} + \sqrt{a^2 - w^2}}$$

12 Mass Point Geometry Puzzles

Everything we call real is made of things that cannot be regarded as real. – Niels Bohr

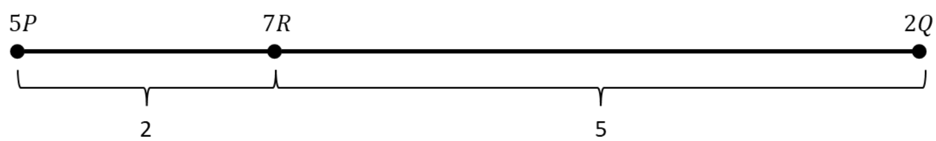
Mass point geometry is a problem-solving technique that makes use of concepts from mechanics such as center of mass and balancing about a fulcrum point. All problems that can be solved using mass point geometry can also be solved using either similar triangles, vectors, or area ratios. Mass point geometry was developed in the 1960s by New York high school students [86]. However, the concept dates back to as early as 1827 concerning the ideas of August Ferdinand Möbius in his theory of homogeneous coordinates.

The following basic definitions are used to develop mass point geometry:

- A **mass point** is a pair (m, P), also written as mP, where m is a mass and P is a point in a plane.
- Two points mP and nQ coincide if and only if $m = n$ and $P = Q$.
- The sum of two mass points mP and nQ has mass $m + n$ and point R where R is the point on line PQ such that $\frac{d(PR)}{d(RQ)} = \frac{n}{m}$ or $m \cdot d(PR) = n \cdot d(RQ)$. In other words, R is the fulcrum point that balances the points P and Q.

See the example in the figure below. In the example, point R should be viewed as the fulcrum point between the masses at points P and Q.

$$\frac{d(PR)}{d(RQ)} = \frac{2}{5}$$



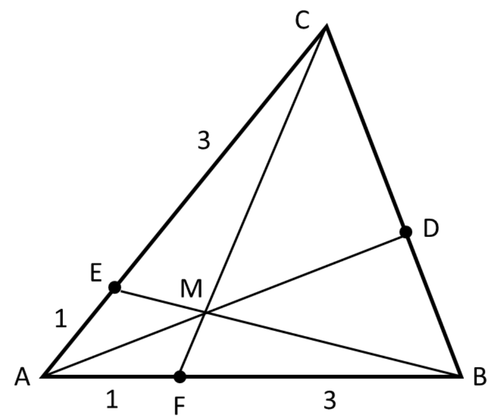
Given a mass point mP and a positive real scalar k, we define scalar multiplication as $k(m, P) = (km, P)$.

The following properties hold true for mass points:

- (Closure) Addition produces a unique sum, i.e., there is a unique fulcrum point that balances the two masses (points).
- (Commutativity) $nP + mQ = mQ + nP$
- (Associativity) $nP + (mQ + kR) = (nP + mQ) + kR = nP + mQ + kR$
- (Distributivity) $k(nP + mQ) = knP + kmQ$
- (Subtraction) If $n > m$ then $nP = mQ + xX$ can be solved for the unknown mass point xX. For example, let's say that we knew $5P$ and the fulcrum point $7R$ in the previous example but did not know the details of the point R. Using subtraction, we can determine the mass of point R, i.e., $7 - 5 = 2$ and its distance from R, i.e., 5.

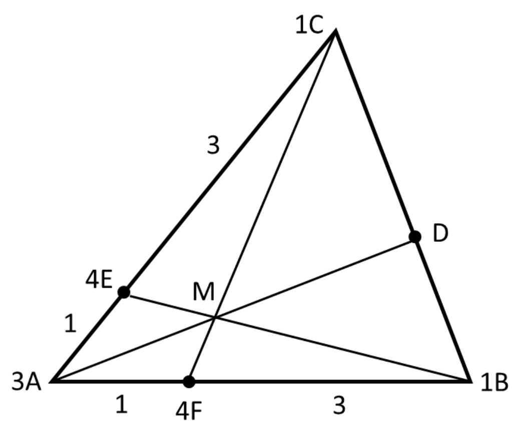
Puzzle 98. Mass point triangle problem #1

In triangle ABC, point E is on line segment $\overline{AC}$ such that $d(CE) = 3d(AE)$, point F is on line segment $\overline{AB}$ such that $d(BF) = 3d(AF)$. If $\overline{BE}$ and $\overline{CF}$ intersect at point M, and lines AM and BC intersect at point D, determine $\frac{d(MB)}{d(ME)}$ and $\frac{d(MD)}{d(MA)}$. The given data is shown in the figure below.

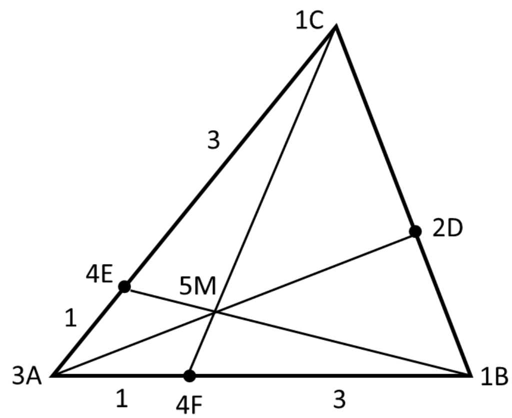


Source: Problem One from Wikipedia article "Mass point geometry" [86]

Solution: First, we assign masses to point A, E and C so that point E balances the masses at A and C. We could have used different sets of masses, e.g., 3.3, 4.4 and 1.1, but it is typical to use the smallest integer values that meet the conditions. We do similar for point A, F and B.



Next, we add the point masses $4F$ and $1C$ to get the point mass $5M$. We use subtraction to determine the mass at D, i.e., $5M - 3A = 2D$. See the updated figure below.

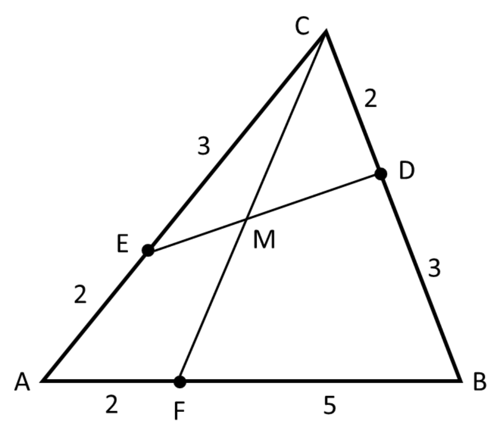


Now that we have assigned masses to all the points, we can use the definition of "point mass" to determine the requested ratios, i.e.,

$$\frac{d(MB)}{d(ME)} = \frac{4}{1} = 4, \quad \frac{d(MD)}{d(MA)} = \frac{3}{2}$$

Puzzle 99. Mass point triangle problem #2

In triangle ABC, points D, E and F are on line segments $\overline{BC}, \overline{CA}$ and $\overline{AB}$, respectively, such that $d(AE) = d(AF) = d(CD) = 2$, $d(BD) = d(CE) = 3$ and $d(BF) = 5$. If $\overline{DE}$ and $\overline{CF}$ intersect at point M, compute $\frac{d(MD)}{d(ME)}$ and $\frac{d(MC)}{d(MF)}$. The given data is shown in the figure below.



Source: Problem Two from Wikipedia article "Mass point geometry" [86]

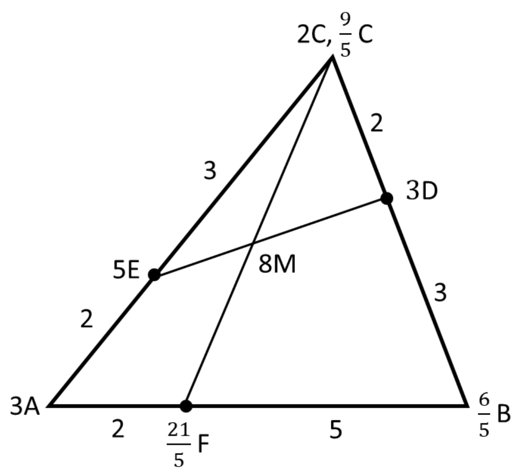
Solution: We start by assigning a mass of 3 to point A and 2 to point C. This implies the fulcrum at point E must have mass 5.

Next, we need to determine the mass at F and at B. We already known that $d(AF) = 2$ and $d(FB) = 5$ and we have set of mass to A.   Since $2d(AF) = 6$, we set the mass at point B to $\frac{6}{5}$ so that $\frac{6}{5}d(FB) = 6$. This implies that the mass at F is $\frac{21}{5}$ since $3A + \frac{6}{5}B = \frac{21}{5}F$.

Given the distances from B to D and from D to C, we will get a contradiction unless we split the mass at C. So, with regard to the line BC. If we set the mass at C to $\frac{9}{5}$, then $\frac{9}{5}d(CD) = \frac{6}{5}d(BD)$ which implies the mass at D must be $\frac{15}{5} = 3$.

Adding the combined mass at C $\left(\frac{19}{5}\right)$ to the mass at F gives us a mass of $\frac{40}{5} = 8$ at point M.

The various masses are shown in the diagram below. This mass distribution is not unique (e.g., see the solution for Problem Two in "Mass point geometry" [86]).

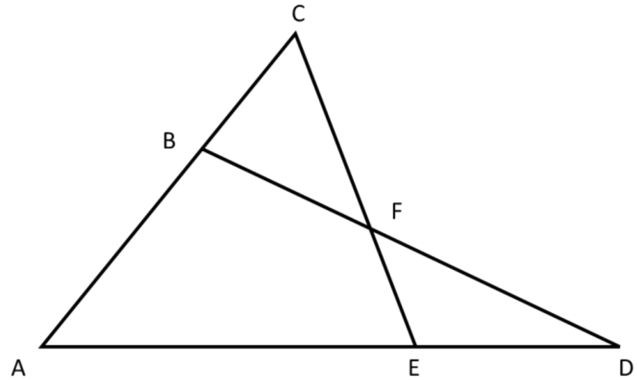


Using the definition of "point mass" to determine the requested ratios, we get

$$\frac{d(MD)}{d(ME)} = \frac{5}{3}, \quad \frac{d(MC)}{d(MF)} = \frac{\left(\frac{21}{5}\right)}{\left(\frac{19}{5}\right)} = \frac{12}{19}$$

Puzzle 100. Mass point triangle problem #3

In the figure below, $\frac{d(AB)}{d(BC)} = 2$ and $\frac{d(AE)}{d(ED)} = \frac{7}{3}$. Determine $\frac{d(BF)}{d(FD)}$ and $\frac{d(CF)}{d(FE)}$.



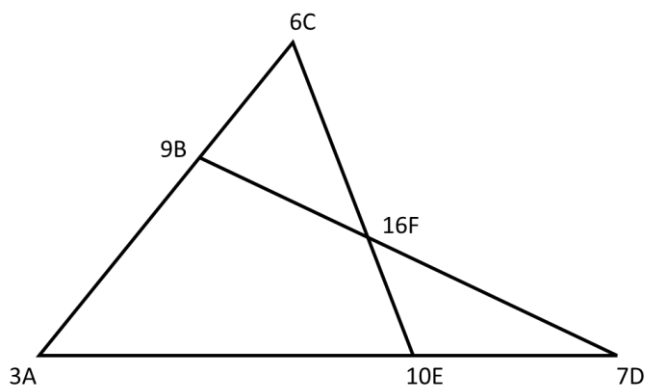
Solution: To preserve the 2:1 ratio between $\overline{AB}$ and $\overline{BC}$, we assign a mass of 3 to point A and 6 to point C. Mass point addition gives us a mass of 9 at point B.

If we let the mass at D be x, then the mass at E is $x + 3$. So, $\frac{d(AE)}{d(ED)} = \frac{7}{3} = \frac{x}{3}$ which implies $x = 7$, and in turn, the mass at E is 10.

Mass addition gives us a mass of 16 at point F.

Using the definition of point mass, we have

$$\frac{d(BF)}{d(FD)} = \frac{7}{9}, \qquad \frac{d(CF)}{d(FE)} = \frac{10}{6} = \frac{5}{3}$$



Acronyms

AA – Angle-Angle

ASA – Angle-Side-Angle

SAS – Side-Angle-Side

SSS – Side-Side-Side

References

[1] *Foundations of geometry*, Wikipedia, https://en.wikipedia.org/wiki/Foundations_of_geometry, accessed on 17 October 2023.

[2] *Euclid's Elements*, Wikipedia, https://en.wikipedia.org/wiki/Euclid's_Elements, accessed on 17 October 2023.

[3] Hilbert, D., *The Foundations of Geometry*, Project Gutenberg, https://dev.gutenberg.org/ebooks/17384.

[4] Birkhoff, G.D., Beatley, R., *Basic Geometry (3rd Edition)**, American Mathematical Society (AMS), 2000. https://archive.org/details/basicgeometry0000birk

[5] *Birkhoff's axioms*, *Wikipedia*, https://en.wikipedia.org/wiki/Birkhoff's_axioms, accessed on 19 October 2023.

[6] *Pythagoras's Theorem*, Proof Wiki, https://proofwiki.org/wiki/Pythagoras%27s_Theorem, accessed on 22 October 2023.

[7] *Triangle inequality*, Wikipedia, https://en.wikipedia.org/wiki/Triangle_inequality#Converse, accessed on 23 October 2023.

[8] *Parallelism implies Equal Corresponding Angles*, Proof Wiki, https://proofwiki.org/wiki/Parallelism_implies_Equal_Corresponding_Angles, accessed on 25 October 2023.

[9] *Equal Corresponding Angles implies Parallel Lines*, Proof Wiki, https://proofwiki.org/wiki/Equal_Corresponding_Angles_implies_Parallel_Lines, accessed on 25 October 2023.

[10] Africk, H., *Elementary College Geometry*, LibreTexts™, https://math.libretexts.org/Bookshelves/Geometry/Elementary_College_Geometry_(Africk).

[11] *Perpendicular Bisectors of Triangle Meet at Point*, Proof Wiki, https://proofwiki.org/wiki/Perpendicular_Bisectors_of_Triangle_Meet_at_Point, accessed on 30 October 30, 2023.

[12] *Inscribed angle*, Wikipedia, https://en.wikipedia.org/wiki/Inscribed_angle, accessed on 27 October 2023.

[13] *Thales's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Thales%27s_theorem, accessed on 22 November 2023.

[14] *Circumscribing Circle about Triangle*, Proof Wiki, https://proofwiki.org/wiki/Circumscribing_Circle_about_Triangle, accessed on 14 November 2023.

[15] *Position of Centroid of Triangle on Median*, Proof Wiki, https://proofwiki.org/wiki/Position_of_Centroid_of_Triangle_on_Median, accessed on 19 November 2023.

[16] *Triangle center*, Wikipedia, https://en.wikipedia.org/wiki/Triangle_center, accessed on 19 November 2023.

[17] *Euler line*, Wikipedia, https://en.wikipedia.org/wiki/Euler_line, accessed on 19 November 2023.

[18] *Orthocenter, Centroid and Circumcenter Coincide iff Triangle is Equilateral*, Proof Wiki, https://proofwiki.org/wiki/Orthocenter,_Centroid_and_Circumcenter_Coincide_iff_Triangle_is_Equilateral, accessed on 19 November 2023.

[19] *Area of Circle*, Proof Wiki, https://proofwiki.org/wiki/Area_of_Circle, accessed on 5 November 2023.

[20] *Opposite Sides Equal implies Parallelogram*, Proof Wiki, https://proofwiki.org/wiki/Opposite_Sides_Equal_implies_Parallelogram, accessed on 29 October 2023.

[21] Midpoints of Sides of Quadrilateral form Parallelogram, Proof Wiki, https://proofwiki.org/wiki/Midpoints_of_Sides_of_Quadrilateral_form_Parallelogram, accessed on 29 October 2023.

[22] *Polygon/Internal Angle*, Proof Wiki, https://proofwiki.org/wiki/Definition:Polygon/Internal_Angle, accessed on 29 October 2023.

[23] *Polygon/External Angle*, Proof Wiki, https://proofwiki.org/wiki/Definition:Polygon/External_Angle, accessed on 29 October 2023.

[24] *How To Solve The Rectangle Length Puzzle - Think INSIDE The Box!*, Mind Your Decisions YouTube video, https://youtu.be/GIOAG0SO0VQ?si=9rwItcUZ1IKO-dnq, accessed on 21 November 2023.

[25] *Menelaus's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Menelaus%27s_theorem, accessed on 7 December 2023.

[26] *Exterior angle theorem*, Wikipedia, https://en.wikipedia.org/wiki/Exterior_angle_theorem, accessed on 29 December 2023.

[27] *Langley's Adventitious Angles*, Wikipedia, https://en.wikipedia.org/wiki/Langley's_Adventitious_Angles, accessed on 22 December 2023.

[28] *Kite (geometry)*, Wikipedia, https://en.wikipedia.org/wiki/Kite_(geometry), accessed on 9 November 2023.

[29] *Circle packing in an equilateral triangle*, Wikipedia, https://en.wikipedia.org/wiki/Circle_packing_in_an_equilateral_triangle, accessed on 6 December 2023.

[30] *How To Solve The Area Of A "Claw"*, YouTube video from Mind Your Decisions, https://youtu.be/B_yNI3gAHNk?si=P-_VD3MoFNskqAgI, accessed on 22 December 2023.

[31] *Diagonals of Kite are Perpendicular*, Proof Wiki, https://proofwiki.org/wiki/Diagonals_of_Kite_are_Perpendicular, accessed on 9 November 2023.

[32] *Sum of Internal Angles of Polygon/Proof 1*, Proof Wiki, https://proofwiki.org/wiki/Sum_of_Internal_Angles_of_Polygon/Proof_1, accessed on 29 October 2023.

[33] *Exterior Angle of Regular Polygon*, Proof Wiki, https://proofwiki.org/wiki/Exterior_Angle_of_Regular_Polygon, accessed on 29 October 2023.

[34] *Ptolemy's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Ptolemy's_theorem, accessed on 25 November 2023.

[35] *Prove the opposite angles of a quadrilateral are supplementary implies it is cyclic*, StackExchange Mathematics, https://math.stackexchange.com/questions/114783/prove-the-opposite-angles-of-a-quadrilateral-are-supplementary-implies-it-is-cyc, accessed on 17 December 2023.

[36] *Van Aubel's Theorem*, Art of Problem Solving (AoPS) Online, https://artofproblemsolving.com/wiki/index.php/Van_Aubel%27s_Theorem, accessed on 4 December 2023.

[37] *Van Aubel's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Van_Aubel's_theorem, accessed on 4 December 2023.

[38] *Polygon triangulation*, Wikipedia, https://en.wikipedia.org/wiki/Polygon_triangulation, accessed on 27 December 2023.

[39] *Polygon Triangulation Theorem*, Proof Wiki, https://proofwiki.org/wiki/Polygon_Triangulation_Theorem, accessed on 27 December 2023.

[40] *Heron's formula*, Wikipedia, https://en.wikipedia.org/wiki/Heron's_formula, accessed on 13 November 2023.

[41] *Heron's Formula*, Proof Wiki, https://proofwiki.org/wiki/Heron%27s_Formula, accessed on 13 November 2023.

[42] *Brahmagupta's formula*, Wikipedia, https://en.wikipedia.org/wiki/Brahmagupta's_formula, accessed on 25 November 2023.

[43] *Area of Triangle in Terms of Circumradius*, Proof Wiki, https://proofwiki.org/wiki/Area_of_Triangle_in_Terms_of_Circumradius, accessed on 25 November 2023.

[44] *Law of sines*, Wikipedia, https://en.wikipedia.org/wiki/Law_of_sines, accessed on 2 December 2023.

[45] *Law of cosines*, Wikipedia, https://en.wikipedia.org/wiki/Law_of_cosines, accessed on 2 December 2023.

[46] *Stewart's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Stewart%27s_theorem, accessed on 12 December 2023.

[47] *Apollonius's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Apollonius's_theorem, accessed on 12 December 2023.

[48] *Viviani's theorem*, Wikipedia, https://en.wikipedia.org/wiki/Viviani's_theorem, accessed on 28 December 2023.

[49] Chen, Zhibo, and Tian Liang. “The Converse of Viviani’s Theorem.” The College Mathematics Journal 37, no. 5 (2006): 390–91. https://doi.org/10.2307/27646392.

[50] *Crossed ladders problem*, Wikipedia, https://en.wikipedia.org/wiki/Crossed_ladders_problem, accessed on 13 November 2023.

[51] *British flag theorem*, Wikipedia, https://en.wikipedia.org/wiki/British_flag_theorem, accessed on 15 November 2023.

[52] *Very difficult for most students*, YouTube video by Math Booster, https://youtu.be/N2JrwKUXZPY?si=5Bhh4yFOATFpTuRN, accessed on 17 November 2023.

[53] *If two chords AB and CD of a circle perpendicular to each other at point O …*, Socratic Q&A: Geometry, https://socratic.org/questions/if-two-chords-ab-and-cd-of-a-circle-perpendicular-to-each-other-at-point-o-then-, accessed on 18 November 2023.

[54] *Area bounded by arcs of quarter circles*, MATHalino, https://mathalino.com/reviewer/plane-geometry/02-area-common-arcs-quarter-circles, accessed on 3 December 2023.

[55] Vincent, J., Vincent, C., *Japanese Temple Geometry*, Australian Senior Mathematics Journal, 2004. https://files.eric.ed.gov/fulltext/EJ720042.pdf

[56] Fukagawa Hidetoshi, Tony Rothman, *Sacred Mathematics: Japanese Temple Geometry**, Princeton University Press, 2008. https://archive.org/details/fukakgawa-hidetoshi-sacred-mathematics-japanese-temple-geometry

[57] *Geometric probability*, Wikipedia, https://en.wikipedia.org/wiki/Geometric_probability, accessed on 18 November 2023.

[58] *Bertrand paradox (probability)*, Wikipedia, https://en.wikipedia.org/wiki/Bertrand_paradox_(probability), accessed on 18 November 2023.

[59] *Principle of indifference*, Wikipedia, https://en.wikipedia.org/wiki/Principle_of_indifference, accessed on 19 November 2023.

[60] *Bertrand’s Paradox*, Kahn Academy, https://www.khanacademy.org/partner-content/wi-phi/wiphi-metaphysics-epistemology/probability-philosophy/v/bertrands-paradox-probability-wireless-philosophy, accessed on 19 November 2023.

[61] T. Kyle Petersen, Bridget Eileen Tenner, *Broken bricks and the pick-up sticks problem*, arXiv:1906.10237 [math.CO], https://arxiv.org/abs/1906.10237.

[62] *Wallace–Bolyai–Gerwien theorem*, Wikipedia, https://en.wikipedia.org/wiki/Wallace%E2%80%93Bolyai%E2%80%93Gerwien_theorem, accessed on 23 November 2023.

[63] *Tangram*, Wikipedia, https://en.wikipedia.org/wiki/Tangram, accessed on 23 November 2023.

[64] *Sam Loyd*, Wikipedia, https://en.wikipedia.org/wiki/Sam_Loyd, accessed on 27 November 2023.

[65] *Hinged dissection*, Wikipedia, https://en.wikipedia.org/wiki/Hinged_dissection, accessed on 28 November 2023.

[66] Niles, R., *Equilateral Triangle To Square Dissection and Problem Set*, YouTube video, https://youtu.be/SLz8IRuwt94?si=SzMVdJoI87PxL9JO, accessed on 28 November 2023.

[67] *Banach–Tarski paradox*, Wikipedia, https://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox, accessed on 6 December 2023.

[68] *The Banach-Tarski Paradox*, YouTube video by Vsauce, https://youtu.be/s86-Z-CbaHA?si=pUww2g-YpBAwG-ZR, accessed on 6 December 2023.

[69] Max G. Levy, *Banach-Tarski and the Paradox of Infinite Cloning*, Quanta Magazine, https://www.quantamagazine.org/how-a-mathematical-paradox-allows-infinite-cloning-20210826/.

[70] Frederickson, G.N., *Dissections: Plane & Fancy**, Cambridge University Press, 1997.

[71] *Tessellation*, Wikipedia, https://en.wikipedia.org/wiki/Tessellation, accessed on 5 December 2023.

[72] *Regular Tessellations*, Wolfram MathWorld, https://mathworld.wolfram.com/RegularTessellation.html, accessed 5 December 2023.

[73] *List of tessellations*, Wikipedia, https://en.wikipedia.org/wiki/List_of_tessellations, accessed on 5 December 2023.

[74] Grunbaum, B., Shephard, G.C., *Tilings by Regular Polygons*, Mathematics Magazine, Vol. 50, No. 5, Page 227-247, November 1977.

[75] *Euclidean tilings by convex regular polygons*, Wikipedia, https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons, accessed on 5 December 2023.

[76] Swanson, I., *Quilting semi-regular tessellations*, available at http://people.reed.edu/~iswanson/semireg.pdf. In addition, this paper is the first three chapters of the book ”Crafting by Concepts”, edited by Sarah-Marie Belcastro and Carolyn Yackel, published by A K Peters.

[77] Hibma, T., Aperiodic Rhomb Tiling, https://www.aperiodictiling.org/wpaperiodictiling/index.php/fractional-rhomb-tilings/quarter-rhombs/, accessed on 5 December 2023.

[78] Penrose, R., The role of aesthetics in pure and applied mathematical research, Bulletin of the Institute of Mathematics and Its Applications, 10: 266ff, 1974.

[79] Math and the Art of M. C. Escher, online wiki book with multiple editors, https://mathstat.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher, accessed on 5 December 2023.

[80] Conover, E., *Mathematicians have finally discovered an elusive “einstein” tile*, Science News, https://www.sciencenews.org/article/mathematicians-discovered-einstein-tile, accessed on 5 December 2023.

[81] *See an ‘einstein’ tile morph into different shapes*, YouTube video from Science News, https://youtu.be/ugnvucpcfPA?si=BnXXJ5zwszXvEb90, accessed on 5 December 2023.

[82] *Heesch's problem*, Wikipedia, https://en.wikipedia.org/wiki/Heesch's_problem, accessed on 5 December 2023.

[83] Lyndon, R., *Groups and Geometry**, Cambridge University Press, 1985.

[84] *Tessellation (computer graphics)*, Wikipedia, https://en.wikipedia.org/wiki/Tessellation_(computer_graphics), accessed on 5 December 2023.

[85] *Analytic geometry*, Wikipedia, https://en.wikipedia.org/wiki/Analytic_geometry, accessed on 10 December 2023.

[86] Rike, T., *Mass Point Geometry, unpublished paper, https://mathcircle.berkeley.edu/sites/default/files/archivedocs/2007_2008/lectures/0708lecturespdf/MassPointsBMC07.pdf, accessed on 10 December 2023.*

* Indicates the book or article is available for borrowing from the Internet Achieve at https://archive.org/. In some cases, only an earlier edition of a book will be available.

Index of Terms