A Complete Characterization of Projectivity for Statistical Relational Models

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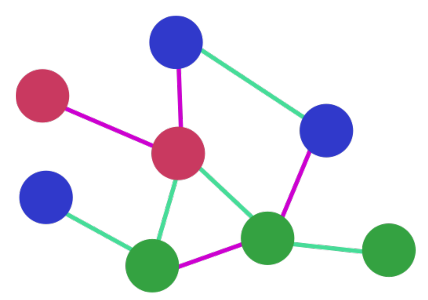
Statistical Relational Models

Possible Worlds

Heterogeneous relational structures that allow multiple types of relationships and nodes:

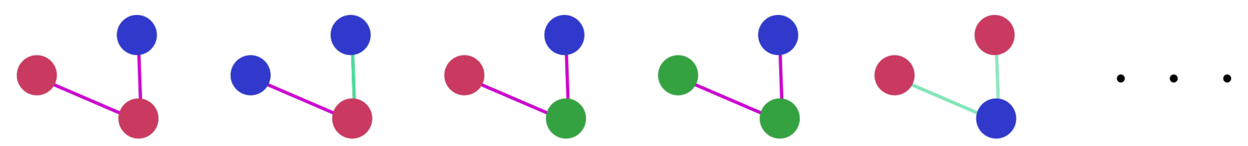


Generative probabilistic model

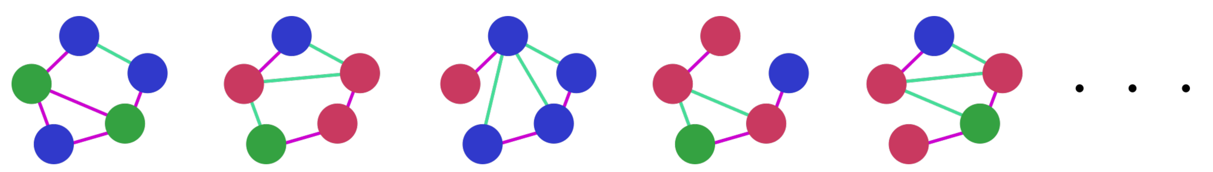
We are concerned with **families of distributions**:

$$P^{(n)} \quad (n \in \mathbb{N})$$

where $P^{(n)}$ is a distribution over n-worlds.



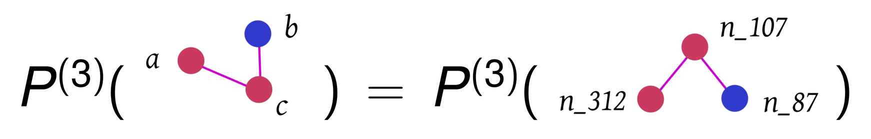
Size 3 worlds ➥Distribution $P^{(3)}$



Size 5 worlds ➥Distribution $P^{(5)}$

Exchangeability

We always assume that all distributions are **exchangeable**:

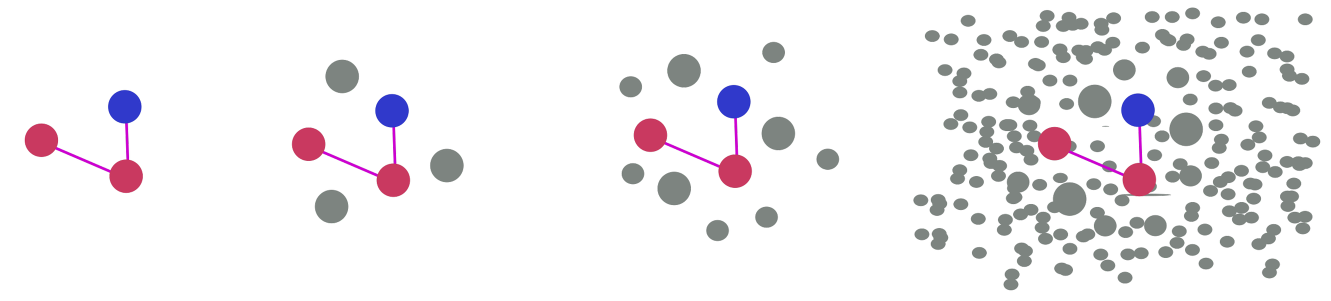


Why families of generative models?

- inherently **multi-relational**.
- **inductive:** learning from *training worlds* $\omega_1, \ldots, \omega_N$ (of different sizes); prediction for *query entities* embedded in a new world ω_q.
- **flexible inference:** computation of arbitrary conditional probabilities.

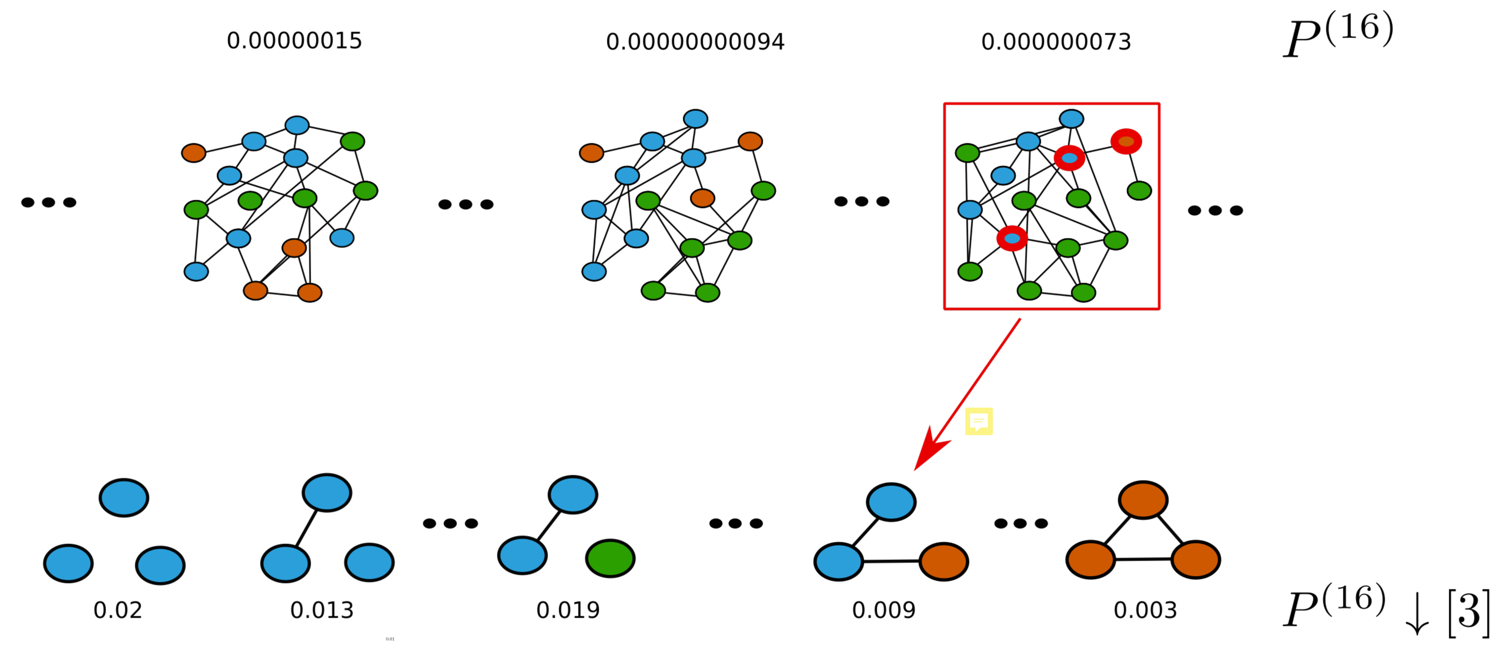
Projectivity and Extendability

The probability of a fixed substructure does not depend on the size of the domain it is embedded in:



$P^{(3)} = 0.035$ $P^{(6)} = 0.035$ $P^{(12)} = 0.035$ $P^{(3000)} = 0.035$

Marginalization: $P^{(n)} \rightsquigarrow P^{(n)} \downarrow [m]$ by Fenstad sampling:



Projectivity

The family $(P^{(n)})_n$ is **projective**, if for all $m < n$:

$$P^{(n)} \downarrow [m] = P^{(m)}.$$

Extendability

$P^{(m)}$ is **extendable** if for every n there exists $P^{(n)}$ such that

$$P^{(n)} \downarrow [m] = P^{(m)}$$

Why projective families?

- Exact size of domain may be unknown or changing over time.
- Inference complexity independent of domainsize (inference is *extremely lifted*!)
- Avoid *degeneracy:* extreme probabilities (0 or 1) for queries as $n \to \infty$

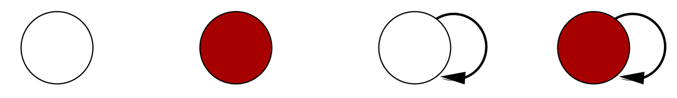
AHK Models

Adaptation of representation of infinite exchangeable arrays by Aldous (1981), Hoover (1979), Kallenberg (2006).

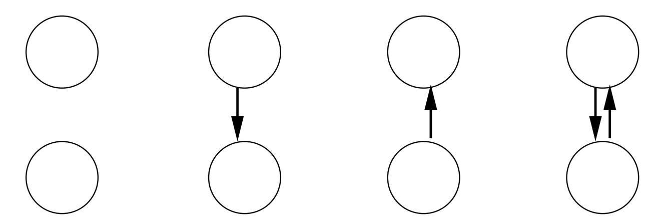
D variables

Describe a world by value assignment to the collection of variables that describe the *m-ary data* for induced sub-structures of size m ($m = 1, \ldots, arity(S)$):

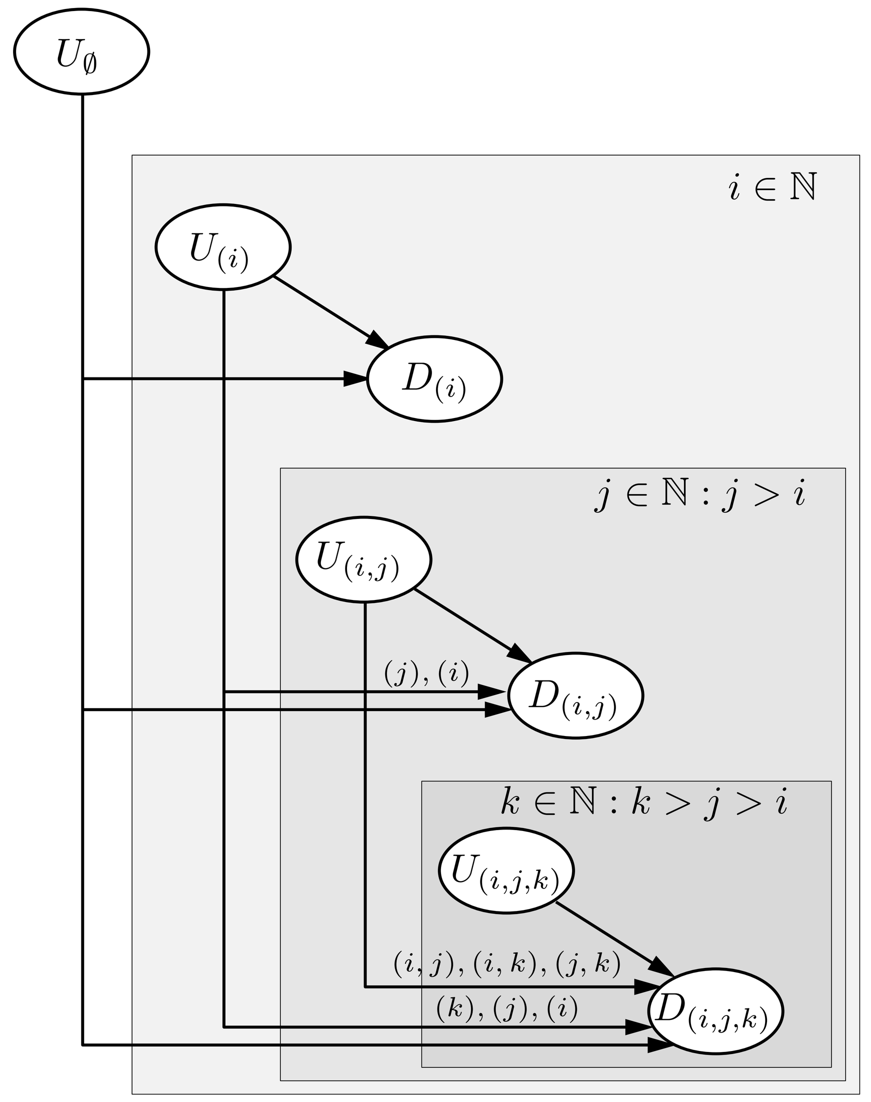
Example: $S = \{red/1, edge/2\}$.
State space for $D_{(i)}$ variables:



State space for $D_{(i,j)}$ variables:



AHK plate representation



Model components:

- D-variables
- U-variables: i.i.d., uniform on [0, 1]
- Conditional distribution of D's given U's defined by deterministic functions $f^{(m)}$ $m = 1, \ldots, a$.

Main Result and Discussion

Main Result

Equivalent for distribution $P^{(m)}$:

- $P^{(m)}$ extendable
- $P^{(m)}$ projective extendable
- $P^{(m)}$ has AHK representation

Equivalent for family $(P^{(n)})_n$:

- $(P^{(n)})_n$ projective
- $(P^{(n)})_n$ has AHK representation

Discussion

Done:

- Comprehensive characterization of projectivity and related properties
- Some understanding of implications for computational and statistical tractability

To do:

- Algorithmic solutions for learning and inference with AHK models

References

David J Aldous. Representations for partially exchangeable arrays of random variables. *Journal of Multivariate Analysis*, 1981.

D. N. Hoover. Relations on probability spaces and arrays of random variables. HPreprint, Institute for Advanced Study, Princeton, NJ, 2, 1979.

Olav Kallenberg. Probabilistic symmetries and invariance principles. Springer, 2006.