Reformulating Portfolio Optimization for Bond Market Practitioners: A Linear Programming Approach

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1 Abstract

Since its introduction by Harry Markowitz in 1952, portfolio optimization theory has been primarily focused on equity portfolio construction. Fixed income portfolio management presents unique challenges due to differences in liquidity and characteristics of individual securities, requiring a shift away from the traditional approach. In particular, an intriguing yet little-studied problem frequently faced by bond traders is that of selecting which bonds to sell from an existing portfolio for cash generation. This paper introduces a Linear Programming (LP) formulation for this problem that seeks to offer a more practical solution to bond market participants compared to the standard mean-variance optimization method. The proposed LP approach incorporates several industry-standard risk constraints, which are often overlooked in academic portfolio optimizations but are crucial for a production system. The proposed formulation is implemented and tested on an anonymized real-world example, and the results are analyzed to demonstrate its potential utility.

2 Keywords

Portfolio Optimization, Linear Programming, Bond Portfolio Management, Fixed Income, Python

3 Introduction

Markowitz's portfolio optimization theory, introduced in 1952, was a revolutionary concept that remains at the core of optimization used across the investment industry to this day. While the idea has been enormously influential, its application in the bond market requires careful adaptation due to the fundamental differences between bonds and stocks. Differences in liquidity and security-specific characteristics demand a distinct approach for bond portfolio management.

In the fixed income domain, a more common and directly applicable theoretical problem compared to portfolio construction (choosing which bonds to buy) is the decision of which bonds to sell in order to generate cash. Moreover, bond portfolio managers and traders are oftentimes less concerned with the traditional portfolio optimization framework centered on portfolio expected return and variance, prioritizing portfolio duration management instead.

A crucial aspect that academic portfolio optimizations often overlook is the incorporation of risk limits such as limitations on allowed sector exposure and position sizing. For the optimization to be used in a production system, these limitations must be robustly modeled. To address these challenges, this paper proposes a linear programming formulation that explicitly incorporates several common risk limits.

The problem this paper aims to address can be formally stated as follows: "How can Linear

Programming approaches to standard portfolio optimization be adapted to develop a model that is more useful for bond market participants?". The hypothesis that guides this investigation is: "A model to select which bonds to sell (and in what amounts) can be developed by adapting LP approaches to standard portfolio optimization (specifically by changing the measure of risk to a measure of dispersion from target portfolio duration and expected return to required cash to generate)." The remainder of the paper will review related literature, detail the proposed mathematical formulation, present the associated programming implementation, and analyze the results of this implementation of the formulated optimization problem on an anonymized real-world example.

4 Literature Review

This section presents a review of the key concepts and methods in portfolio optimization, starting with the standard portfolio optimization framework and moving towards more sophisticated methods designed to address the limitations of the standard framework.

4.1 Portfolio Theory

The theory of portfolio selection was formally introduced by Harry Markowitz in his 1952 paper, "Portfolio Selection". This laid the foundation for modern portfolio theory, providing a mathematical formulation for the selection of a portfolio of assets.

The price of security j at time t is denoted as $P_j(t)$. The return of security j can be expressed as:

$$R_j(t) = \frac{P_j(t) - P_j(t-1)}{P_j(t-1)} = \frac{P_j(t)}{P_j(t-1)} - 1 \tag{Eq. 1}$$

The expected return of security j over T historical time periods is calculated as:

$$E\left(R_j\right) = \frac{1}{T}\sum_{t=1}^{T} R_j(t) \tag{Eq. 2}$$

A portfolio is a collection of securities, and the return of portfolio p is the weighted sum of the returns of the individual securities. If w_j represents the weight of security j in the portfolio (where $w_j = \frac{\text{Value}_j}{\text{Value}_{\text{port}}} = \frac{\text{Price}_j * \text{Amount}_j}{\text{Value}_{\text{port}}}$ and $\sum_{j=1}^{n} w_j = 1$), then the return of the portfolio R_p and its expected return $E\left(R_p\right)$ can be expressed as:

$$R_p = \sum_{j=1}^{n} w_j R_j \tag{Eq. 3}$$

$$E\left(R_p\right) = \sum_{j=1}^{n} w_j E\left(R_j\right) \tag{Eq. 4}$$

Note that this standard market value weighted average calculation of portfolio-level analytics appears repeatedly throughout this paper (for example in *Eq. 11.1* for calculating portfolio duration).

The variance of the portfolio return (or simply "portfolio variance"), which is often used as a measure of risk, is:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{Cov}\left(R_i, R_j\right) \tag{Eq. 5}$$

where the covariance of returns between securities i and j is calculated as $\mathrm{Cov}\left(R_i, R_j\right) = E[(R_i - E[R_i])(R_j - E[R_j])]$.

4.2 Mean-Variance Optimization

Markowitz (1952) introduced a formulation for portfolio optimization that seeks to simultaneously maximize return (in the form of portfolio expected return) and minimize risk (in the form of portfolio variance). This has come to be known as the Markowitz Mean-Variance Optimization. By introducing a scalar λ to represent the investor's level of risk aversion, this approach can be expressed by including both risk and return in the objective function (Vanderbei 1996):

$$\underset{\text{by choice of w's}}{\textbf{Maximize}}$$

$$\lambda\left(E\left(R_p\right) - \sigma_p^2\right) = \lambda\left(\sum_{j=1}^{n} w_j E(R_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \,\mathrm{Cov}\left(R_i, R_j\right)\right) \qquad \text{(Eq. 6)}$$

subject to:

$$\sum_{j=1}^{n} w_j = 1,$$

$$w_j \geq 0 \quad \text{for all } j$$

Rather than making an assumption about investor risk preferences, the standard formulation minimizes risk subject to a specified level of return. By specifying a range of returns, this formulation can be used to construct the efficient frontier (Markowitz 1959; DeFusco et al. 2004):

$$\underset{\text{by choice of w's}}{\textbf{Minimize}}$$

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \,\mathrm{Cov}\left(R_i, R_j\right) \qquad \text{(Eq. 7)}$$

subject to:

$$\sum_{j=1}^{n} w_j = 1,$$

$$w_j \geq 0 \quad \text{for all } j,$$

$$E\left(R_p\right) = \sum_{j=1}^{n} w_j E(R_j) = z$$

for all specified values of expected return z

in the window $r_{\min} \leq z \leq r_{\max}$

Both of the above approaches require quadratic programming approaches to solve due to the product $w_i w_j$ in the portfolio risk term (see *Eq. 5*). In practice, quadratic programming approaches can suffer from comutational complexity (or even infeasibility) issues for portfolios with a large number of holdings. Additionally, QP solvers need the covariance matrix of assets to be positive definite. This can become an issue in finance where the sample covariance matrix might be near singular (i.e., not full rank) due to high correlations among some assets (Lopez de Prado 2018).

4.3 LP Reformulation

Beginning with Young (1998), several authors have proposed solutions to address the issues of instability in the quadratic programming approach. Vanderbei (1996) proposed a reformulation that uses the absolute value to convert the quadratic optimization problem into a linear one in an approach known as the Mean Absolute Deviation (MAD) Markowitz formulation. In this approach, the portfolio variance is replaced with the Mean Absolute Deviation (MAD) as the measure of portfolio risk. Additionally, a “reward happiness parameter” μ is introduced, where a small μ indicates risk-averse behavior and a large μ indicates risk-seeking behavior.

Maximize
by choice of w’s

$$E\left(R_p\right) = \sum_{j=1}^{n} w_j E(R_j) \tag{Eq. 8}$$

subject to:

$$\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j} w_j \left(R_j(t) - E(R_j)\right)\right| \leq \mu,$$

$$\sum_{j=1}^{n} w_j = 1,$$

$$w_j \geq 0 \quad \text{for all } j$$

Note that, in contrast to *Eq. 7*, this formulation maximizes return subject to a specified level of risk. However, due to the absolute value, this formulation is still not linear.

To achieve a linear programming formulation, Vanderbei introduced an auxiliary variable y_t equal to the absolute value of the portfolio risk term. The y_t and $-y_t$ were then used as the threshold values for the portfolio risk constraint (an approach the author refers to as the “greedy substitution”), resulting in Vanderbei’s “MAD Markowitz” LP approach.

$$y_t = \left| \sum_j w_j \left(R_j(t) - E(R_j) \right) \right|$$

$$\underset{\text{by choice of w's}}{\textbf{Maximize}}$$

$$E\left(R_p\right) = \sum_{j=1}^{n} w_j E(R_j)$$

$$\textbf{subject to:}$$

$$-y_t \leq \sum_j w_j (R_j(t) - E(R_j)) \leq y_t \quad \text{for all } t \tag{Eq. 9}$$

$$\frac{1}{T} \sum_{t=1}^{T} y_t \leq \mu$$

$$\sum_j^j w_j = 1$$

$$x_j \geq 0 \text{ for all } j$$

$$y_t \geq 0 \text{ for all } t$$

The following section will present the methodology used in this paper, which builds upon these foundations to propose a novel approach to bond portfolio optimization focused on selecting sales.

5 Methodology

The problem addressed by this paper is that of selecting which bonds to sell (and in what amounts) to generate a specified amount of cash from an existing portfolio. Similar to the above methods of portfolio selection, this problem has the dual objective of selecting sales to generate a specified amount of cash while simultaneously minimizing the portfolio’s deviation from several key strategy targets (the most important of these being portfolio duration). As such, the model is formulated to minimize the deviation from a target portfolio duration while satisfying several constraints related to risk limitations and strategy targets. The amount of cash generated from the selected sales is constrained to a range between the specified amount (which might, for example, be a client-requested withdrawal amount from the portfolio) and an acceptable post-withdrawal “cash cushion” value.

Though not motivated by the desire to eliminate a quadratic expression, the proposed formulation is, in essence, a combination of *Eq. 7* (with portfolio risk replaced by deviation from targeted duration and expected return by generated cash) and *Eq. 9* (with the “greedy substitution” appearing directly in the objective function rather than the primary constraint). The trading parameters, strategy/risk parameters, objective function, and constraints of the model are detailed below.

5.1 Variables and Parameters

5.1.1 Portfolio/Trading

Refer to Appendix A for a more detailed discussion of these variables.

- AUM * : AUM
- p_i : Price of bond i
- a_i : Amount of bond i initially held
- x_i : Amount of bond i sold
- d_i : Duration of bond i
- t_i : Tradeable increment of bond i (dictated by sector type)
- y_i : Number of tradable increments of bond i sold (**Decision Variable**)
- *min_cash_needed*: Minimum cash needed
- *max_cash_needed*: Maximum cash needed (acceptable "cash cushion")

*Variable is stated on a pre-withdrawal basis

5.1.2 Strategy/Risk

- *min_duration* † : Minimum portfolio duration
- *max_duration* † : Maximum portfolio duration
- *target_duration* † : Target portfolio duration
- $min\_percent_i$ † : Minimum position percentage of bond i (dictated by sector type)
- $max\_percent_i$ † : Maximum position percentage of bond i (dictated by sector type)
- *min_sector_allocation* † : Minimum allocation percentage (aggregated at sector type level)
- *max_sector_allocation* † : Maximum allocation percentage (aggregated at sector type level)
- *min_sector_duration* † : Minimum duration (aggregated at sector type level)
- *max_sector_duration* † : Maximum duration (aggregated at sector type level)
- S : Set of sector types held within the portfolio

† Variable has been adjusted to the pre-withdrawal equivalent for the purposes of the optimization.

5.2 Objective Function

The auxiliary variable *deviation* is introduced, representing the absolute deviation of the post-sale portfolio duration from the target duration:

$$deviation = \left| \frac{\sum_{i=1}^{n} d_i * (a_i - x_i) * p_i}{AUM} - target\_duration \right| \qquad \text{(Eq. 10.1)}$$

Then, to achieve LP formulation, Vanderbei's "greedy substitution" is performed:

$$-deviation \leq \left(\frac{\sum_{i=1}^{n} d_i * (a_i - x_i) * p_i}{AUM} - target\_duration \right) \leq +deviation \qquad \text{(Eq. 10.2)}$$

Which results in an objective function of, simply:

$$\underset{\text{by choice of y's}}{\textbf{Minimize}} \quad deviation \qquad \text{(Eq. 10.3)}$$

subject to the below constraints

5.3 Constraints

1. The total cash generated from the sales must be within the acceptable range:

$$min\_cash\_needed \leq \sum_{i=1}^{n} x_i * p_i \leq max\_cash\_needed \qquad \text{(Eq. 11.1)}$$

2. The portfolio duration after the sales (but before the withdrawal) must be within the target range:

$$min\_duration \leq \frac{\sum_{i=1}^{n} d_i * (a_i - x_i) * p_i}{AUM} \leq max\_duration \qquad \text{(Eq. 11.2)}$$

3. Each bond's position percentage allocation after sales must be within the minimum and maximum position percentage range:

$$min\_percent_i \leq \frac{(a_i - x_i) * p_i}{AUM} \leq max\_percent_i \quad \forall i \qquad \text{(Eq. 11.3)}$$

4. The total allocation percentage allocation by sector type must be within the minimum and maximum allocation percentage range:

$$min\_sector\_allocation \leq \frac{\sum_{i \in S} (a_i - x_i) * p_i}{AUM} \leq max\_sector\_allocation \quad \forall S \qquad \text{(Eq. 11.4)}$$

5. The portfolio duration by sector type after the sales must be within the minimum and maximum duration range:

$$min\_sector\_duration \leq \frac{\sum_{i \in S} d_i * (a_i - x_i) * p_i}{AUM} \leq max\_sector\_duration \quad \forall S \qquad \text{(Eq. 11.5)}$$

6. The number of tradable increments of each bond to be sold, y_i, must be a positive integer:

$$y_i \in \mathbb{Z}^+ \quad \forall i \qquad \text{(Eq. 11.6)}$$

Using these parameters, objective function, and constraints, standard LP optimization software libraries should be able to arrive at an optimal solution.

6 Computational Experiments and Results

The proposed LP-based bond portfolio optimization methodology was implemented on a real-world anonymized bond portfolio dataset. The LP problem was defined using the Python library 'PuLP', a popular choice for LP and integer and mixed-integer LP formulations. The objective function and constraints were implemented in a nearly identical fashion to how they were expressed in the Methodology section above. The core programmatic optimization formulation is included in Appendix B.

A data-driven approach to sensitivity analysis was selected given the availability of historical production data. Specifically, the model was tested against five different cash raise values (one a real historical example from production and four theoretical values) for five different real-world anonymized portfolios. The portfolios and cash raise amounts were carefully chosen to test a wide yet reasonable range of the model parameters. During this sensitivity analysis, the formulation provided in the Methodology section, while theoretically sound, did reveal some limitations when applied to real-world data. In particular, the position sizing constraint (Eq 11.3) caused the model to produce an infeasible result for several portfolios at every tested cash raise amount. Upon further inspection, this was due to the portfolio entering the optimization with positions already sized below the minimum band of the constraint. Since the model is designed for selling (and thus is only able to decrease position sizing), it isn't able to resolve this constraint failure. As such, an adjustment was made in the code to apply conditional logic to relax this constraint when necessitated by portfolio's beginning positioning (see Appendix C).

After making this adjustment to the position sizing constraint, the solver successfully found an optimal solution to each test case and generated values for each holding's associated decision variable, y_i (indicating the number of tradable increments of each bond to sell) while adhering to all other constraints. Each result was additionally reviewed manually and deemed a real, "tradeable" solution. The produced solutions for the actual historical cash raises tested were extremely similar to–and in some cases superior to–the selections made manually by the trader (the author). The successful implementation of the model on real-world data and the favorable results achieved provide evidence in support of the hypothesis that LP approaches from standard portfolio optimization theory can be adapted in a way that is more relevant and useful for bond market practitioners.

7 Discussion and Conclusions

The LP-based bond portfolio optimization model presented in this paper addresses a significant problem in fixed income portfolio management, namely determining which bonds to sell in order to generate cash. The results of the computational experiment demonstrate that the proposed model can successfully find an optimal solution that minimizes the deviation from a target portfolio duration while meeting cash requirements. The inclusion of robust risk limits, such as constraints on sector exposure and position sizing, makes the model more practically useful for production systems subject to compliance review.

Though initial sensitivity analysis revealed a single limitation of the model (which was addressed), it would be beneficial to test the model on more diverse real-world data and in different market conditions to further validate its performance. Doing so will likely reveal similar issues that the current formulation will need to be adjusted to handle. While the model has several strengths, it also has limitations that could be addressed in future work. For instance, the model assumes that all bonds can be sold at their current prices, which may not be the case in a real trading environment. Future enhancements to the model could explore ensuring the most "tradeable" solution in all cases by incorporating a liquidity score at the individual bond level.

In conclusion, the proposed LP-based bond portfolio optimization model offers a novel and potentially more useful approach for fixed income portfolio management. The model's focus on portfolio duration and cash generation, rather than expected return and variance, aligns more closely with the priorities of fixed income managers. Further research and development could lead to even more powerful and practical tools for bond market practitioners.

8 References

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9 Appendix A

9.1 Detailed Presentation of Portfolio/Trading Variables and Parameters

- AUM * : AUM
- p_i : Price of bond i
  - “Dirty price” / 100 (for the purposes of the optimization)
- a_i : Amount of bond i initially held

$$a_i * p_i = \text{ initial value of bond } i$$

$$\frac{a_i * p_i}{AUM} = \text{starting } w_i$$

- x_i : Amount of bond i sold

$$a_i - x_i = \text{ amount remaining after sale}$$

$$x_i * p_i = \text{ sold value of bond } i$$

$$\frac{x_i * p_i}{AUM} = \text{sold } w_i$$

$$\frac{(a_i - x_i) * p_i}{AUM} = \text{ending } w_i$$

- d_i : Duration of bond i
- t_i : Tradeable increment of bond i (dictated by sector type)
- y_i : Number of tradable increments of bond i sold
  - **This is the actual decision variable**
  - This variable is necessary to implement as an Integer Programming formulation with sector-dependent increments, otherwise x would be used directly as the decision variable

$$x_i = t_i * y_i$$

- *min_cash_needed*: Minimum cash needed

$$min\_cash\_needed = \text{cash on hand - cash requested for withdrawal}$$

- *max_cash_needed*: Maximum cash needed (acceptable “cash cushion”)

$$max\_cash\_needed = \max(min\_cash\_needed{+}5000, (AUM{-}\text{cash requested for withdrawal}){*}.03)$$

*Variable is stated on a pre-withdrawal basis