Egyptian Fractions

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This activity was adapted from the NRICH Egyptian fractions activity, available here:
`https://nrich.maths.org/1173`.

In this exercise, we will be examining *Egyptian fractions*. The ancient Egyptians never wrote down fractions with numerators greater than 1. Instead, they liked to write fractions as sums of *unit fractions* with *distinct denominators*. For example, they would never write $\frac{2}{3}$. Instead, they would write $\frac{1}{2} + \frac{1}{6}$. These sums can potentially involve more than 2 fractions.

The goal of this exercise is to understand whether this is always possible, and if so, whether there are better and worse ways to do this.

To tackle a difficult problem like this, our first step is always to simplify the problem, try simple cases, and build intuition.

Exercise 1: Let us think first about the identities $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$, $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$, and $\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$. Verify them! Can you think of a general pattern these identities seem to follow?

Exercise 2: Using exercise 1, can you write $\frac{2}{5}$ as a sum of two distinct unit fractions? Suppose we have any fraction with numerator 2. Can we always write it as a unit fraction, or a sum of unit fractions? Use exercise 1, and consider the cases where the denominator is odd, and the denominator is even.

What if we have fractions with numerators greater than 2? The next 2 exercises will guide you through figuring out how we can write such fractions as a sum of unit fractions.

Exercise 3: Suppose we dropped the requirement that the denominators had to be distinct. What is a simple way in which we can write any fraction as a sum of unit fractions?

Exercise 4: Let us start with our representation from exercise 3 for the fraction $\frac{3}{7}$. Group some of the terms together, and use exercise 2 to convert this representation to one with distinct denominators. Try this with some other fractions!

Exercise 5: Now let us consider a different way to try to express a fraction as a sum of unit fractions. Start with a fraction, and at every stage, write down the largest possible unit fraction that is smaller than the fraction you're working on. Subtract off that fraction, and repeat.

For example, let's start with $\frac{11}{12}$: The largest possible unit fraction that is smaller than $\frac{11}{12}$ is $\frac{1}{2}$. Subtracting $\frac{1}{2}$, we see that $\frac{11}{12} - \frac{1}{2} = \frac{5}{12}$. The largest possible unit fraction that is smaller than $\frac{5}{12}$ is $\frac{1}{3}$. Subtracting $\frac{1}{3}$, we see that $\frac{5}{12} - \frac{1}{3} = \frac{1}{12}$. Hence, we know that $\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$.

Use the same process to find a representation for $\frac{3}{7}$.

Exercise 6: Which of the two representations you found for $\frac{3}{7}$ is "better"? How do you think we should define "better" in this context?

Exercise 7: Try out the process from exercise 5 with some other fractions, and compare them with the representations from exercise 4. Does one method always give a better representation? If not, are there situations where we should prefer one to the other?

Exercise 8: Can you convincingly demonstrate that the method of exercise 4 and the method of exercise 5 always work?

Exercise 9: Can you think of methods other than the ones in exercise 4 and exercise 5?