True or False?

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Here is a big list of statements that may be true or false. For each one, prove that it is true or false. To prove that a statement is false, we just need to find a single counter-example to the statement. Many of the false statements are common errors we make while doing maths. One way to catch these mistakes is to check your work by looking for counter-examples. We often try to find counter-examples by starting with the simplest numbers we can think of, and if these don't work out to be counter-examples, we try other numbers. If a statement works out for multiple different examples with different characteristics, we begin to suspect that it might be true.

1. For all positive numbers x, $\sqrt{x} \leq x$.
2. For all positive numbers c, d, if $cd \leq d$, then $c \leq 0$.
3. For all real numbers a, b, c, if $ac \leq bc$, then $a \leq b$.
4. For all integers t, u, and v, $t^{u+v} = t^u + t^v$.
5. For all integers t, u, and v, $t^{uv} = t^u + t^v$.
6. For all integers t, u, and v, $t^{uv} = t^u \cdot t^v$.
7. For all positive numbers $\overline{\circ}$, $\log_{10}(1/\overline{\circ}) = -\log_{10}(\overline{\circ})$.
8. There exist distinct positive numbers p, q such that $p^q = q^p$.
9. For real numbers x and y, $(x+y)^2 = x^2 + y^2$.
10. For all positive numbers a and b, $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$.
11. For all positive numbers z and y, $\log_{10}(z) + \log_{10}(y) = \log_{10}(z+y)$.
12. If s and t are positive numbers such that $s > t$ then $\frac{s}{s+1} > \frac{t}{t+1}$.
13. For all integers a, b and c, $\frac{a+c}{b+c} = \frac{a}{b}$.
14. For all fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.
15. There exist fractions $\frac{a}{b}$ and $\frac{c}{d}$ such that $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.
16. If $f(x) = 2x$, then $f(x+1) = 2x+1$.
17. If $f(x+y) = f(x)+f(y)$ for all real numbers x and y, then $f(2x) = 2f(x)$ for all real numbers x.

18. If $f(x) = g(x)^2$, then $f(x^2) = g(x)^4$.

19. For any invertible function f, $f^{-1}(x) = f(x)^{-1}$.

20. For any function f, if $f(x) = f(y)$, then $x = y$.