Definition: The *dot product* of two n-vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix}$ is defined to be $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$.

Example: $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 2 \cdot (-1) + (-3) \cdot 1 + 1 \cdot 2 = -3$

Conjecture: Two vectors v and w are perpendicular if and only if $v \cdot w = 0$.

Lemma:

- $v \cdot w = w \cdot v$
- $v \cdot v = ||v||^2$
- $v \cdot (cu) = c(v \cdot u)$
- $v \cdot (u + w) = v \cdot u + v \cdot w$

Use the lemma to **prove** the following:

Proposition: $||v - w||^2 = ||v||^2 + ||w||^2 - 2v \cdot w$

Use the proposition and the Cosine law to **prove**:

Theorem: $v \cdot w = ||v||||w|| \cos(\theta)$ where θ is the angle between the two vectors v and w.

Prove the following as an immediate consequence:

Corollary: Two non-zero vectors v and w are perpendicular if and only if $v \cdot w = 0$.

Remark: This approach depends on assuming that the length of a vector v is $||v||$.