12th JAN

1. Sin(x) approximation (optional)
2. Worksheet question 1
3. Worksheet question 2

1. Given an arbitrary function $f(x)$, there are many ways to approximate it with simpler functions. One such method is to approximate f by polynomials at a point $x = x_0$

The polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ is the nth order approximation of f at x_0 if its first n derivatives all agree with f at x_0.

* An "approximation" is "something similar to f". Here we use "has the same derivatives" to mean "similar". Other metrics can be used too!

Given a function f, one can show that if

$$p(x) = f(x_0) + f'(x_0)(x-x_0),$$

then $p(x)$ is the line tangent to f at x_0. In fact, p is the 1st order approximation to f, because

$$p(x_0) = f(x_0) \quad \text{and} \quad p'(x_0) = f'(x_0).$$

More generally,

$$p(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \dots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

is the nth order approximation to f.

What are the various approximations of $\sin(x)$, at $x=0$?

1st order approximation of $\sin(x)$ at $x=0$ is

$$p(x) = \sin(0) + \sin'(0)x$$
$$= x.$$

This is why "$\sin(x)$ is approximately x" for x near 0.

2nd order approximation? Also $p(x) = x$!
This means the "error" between $p(x) = x$ and $\sin(x)$ is at worst cubic. This is confirmed by calculating the 3rd order approximation.

$$p(x) = \sin(0)x + \sin'(x)x + \frac{\sin''(x)x^2}{2} + \frac{\sin'''(x)x^3}{6}$$
$$= x - \frac{x^3}{6}.$$

2. What is the limit

$$\lim_{x\to\infty} \frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{40}} ?$$

Idea behavior of a polynomial (as $x \to \infty$) is determined by the highest order term, so the limit should be

$$\lim_{x\to\infty} \frac{(2x)^{21}(3x)^{19}}{(2x)^{40}} = \lim_{x\to\infty} \frac{2^{21}3^{19}x^{40}}{2^{40}x^{40}} = \frac{3^{19}}{2^{19}}.$$

This is indeed the case! Start by expanding the fraction:

$$\frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{40}} = \frac{a_{40}x^{40} + a_{39}x^{39} + \cdots + a_1 x + a_0}{b_{40}x^{40} + b_{39}x^{39} + \cdots + b_1 x + b_0}.$$

* For now we don't know what a_i, b_i are, but we know they exist! Because numerator and denominator are polynomials.

$$= \frac{a_{40}x^{40}}{b_{40}x^{40} + \cdots + b_1 x + b_0} + \underbrace{\frac{a_{39}x^{39}}{b_{40}x^{40} + \cdots + b_0}}_{0} + \cdots + \underbrace{\frac{a_0}{b_{40}x^{40} + \cdots + b_0}}_{0}.$$

After expanding our expression, we find that the last 40 terms all go to zero as $x \to \infty$. This is because the numerator has a lower power of x than the denominator.

For the first term,

$$\frac{a_{40}x^{40}}{b_{40}x^{40} + b_{39}x^{39} + \cdots + b_1 x + b_0} = \frac{a_{40}}{b_{40} + \underbrace{\frac{b_{39}}{x}}_{0} + \cdots + \underbrace{\frac{b_1}{x^{39}}}_{0} + \underbrace{\frac{b_0}{x^{40}}}_{0}}$$

by dividing all terms by x^{40}.

Most of the new terms go to 0 again! This leaves

$$\lim_{x \to \infty} \frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{40}} = \frac{a_{40}}{b_{40}} = \frac{2^{21}3^{19}}{2^{40}}.$$

Q1 continued. <u>Parts 4 and 5</u>:

At $x = \frac{1}{2}$,
$$\frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{37}}$$
is <u>undefined</u> because $(2x-1)^{37} = 0$. On the other hand, $(2x-3)^{21}(3x-2)^{19}$ is a finite number.

This means $\lim\limits_{x \to \frac{1}{2}} \frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{37}}$ could be ∞, $-\infty$, or undefined.

We figure this out with signs. If x is slightly larger than $\frac{1}{2}$, then the fraction gives a <u>positive</u> value, so $\lim\limits_{x \to \frac{1}{2}^+} \frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{37}} = \infty$.

If x is slightly less than $\frac{1}{2}$, then the numerator is +ve but the denominator is -ve, so the whole expression is -ve. This gives
$$\lim_{x \to \frac{1}{2}^-} \frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{37}} = -\infty.$$
Overall $\lim\limits_{x \to \frac{1}{2}^-} \neq \lim\limits_{x \to \frac{1}{2}^+}$, so the limit of $\frac{(2x-3)^{21}(3x-2)^{19}}{(2x-1)^{37}}$ as $x \to \frac{1}{2}$ is <u>undefined</u>.

Q2. L'Hôpital's rule things.

* Can be applied exactly when both numerator and denominator are 0, or both are ∞.

e.g. for the 4th one:

$$\lim_{x\to 0} \frac{\log(\cos(2x))}{\log\cos(3x)} \xrightarrow{\text{L'Hôp}} \left(-\frac{2\sin(2x)}{\cos(2x)}\right) \Big/ \left(-\frac{3\sin(3x)}{\cos(3x)}\right)$$

$$= \frac{2\sin(2x)\cos(3x)}{3\sin(3x)\cos(2x)}$$

But
$$\left.\begin{aligned} \cos(3x) &\sim 1 \\ \cos(2x) &\sim 1 \\ \sin(2x) &\sim 2x \\ \sin(3x) &\sim 3x \end{aligned}\right\} \text{ for } x \sim 0$$

Therefore
$$\lim_{x\to 0} \frac{\log\cos(2x)}{\log\cos(3x)}$$

$$= \lim_{x\to 0} \frac{2\sin(2x)\cos(3x)}{3\sin(3x)\cos(2x)}$$

$$= \lim_{x\to 0} \frac{2\cdot 2x\cdot 1}{3\cdot 3x\cdot 1} = \frac{4}{9}.$$