Solution sheet 2nd Feb

1. (a) $\left\{\frac{\pi^n}{1+2^{2n}}\right\}$ converges, with limit 0.

This is because

$$\frac{\pi^n}{1+2^{2n}} = \frac{\pi^n}{1+4^n} \approx \frac{\pi^n}{4^n} = \left(\frac{\pi}{4}\right)^n,$$

and $\pi < 4$ so $\frac{\pi}{4} < 1$.

Formally we use the squeeze theorem.

Proof Let $\{a_n\}$ be defined by $a_n = 0$ for each n.

Let $\{b_n\}$ be defined by $b_n = \left(\frac{\pi}{4}\right)^n$ for each n.

Then for each n, $a_n \le \frac{\pi^n}{1+2^{2n}} \le b_n$.

Moreover, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{\pi}{4}\right)^n = 0$

Therefore by the squeeze theorem,

$$\lim_{n\to\infty} \frac{\pi^n}{1+2^{2n}} = 0.$$

(b) $\left\{\frac{n^2-n+7}{n+5}\right\}$ diverges.

Proof Divide each entry by n. We have

$$\frac{n^2-n+7}{n+5} = \frac{n-1+\frac{7}{n}}{1+\frac{5}{n}}.$$

Then $\lim_{n\to\infty} n-1+\frac{7}{n} = \infty$, but

$\lim_{n\to\infty} 1+\frac{5}{n} = 1.$

Therefore $\lim_{n\to\infty} \frac{n^2-n+7}{n+5} = \infty$ (so it diverges)

(c) $\left\{\dfrac{4^{n+1}+n^5 3^n}{2^{n+6}+2^{2n+1}}\right\}$ converges, with limit 2.

As in (b), we prove this by "getting rid of the ∞ in the denominator" by dividing each term by the largest thing occuring in the denominator.

proof

To make our work easier, rewrite each term to have the same power:

$$4^{n+1} = 4 \cdot 4^n$$
$$n^5 3^n = n^5 3^n$$
$$2^{n+6} = 2^6 \cdot 2^n$$
$$2^{2n+1} = 2 \cdot 2^{2n} = 2 \cdot 4^n.$$

Therefore

$$\frac{4^{n+1}+n^5 3^n}{2^{n+6}+2^{2n+1}} = \frac{4 \cdot 4^n + n^5 3^n}{2^6 \cdot 2^n + 2 \cdot 4^n}.$$

Divide each term by 4^n:

$$\frac{4 \cdot 4^n + n^5 3^n}{2^6 \cdot 2^n + 2 \cdot 4^n} = \frac{4 + n^5 \frac{3^n}{4^n}}{2^6 \frac{2^n}{4^n} + 2}.$$

We can now compute the limit, using the fact that

$$\lim_{n\to\infty} p^n = 0 \quad \text{if} \quad |p|<1.$$

We have:

$$\lim_{n\to\infty} \frac{4^{n+1} + n^5 3^n}{2^{n+6} + 2^{2n+1}} = \lim_{n\to\infty} \frac{4 + n^5 \frac{3^n}{4^n}}{2^6 \frac{2^n}{4^n} + 2}$$

$$= \frac{4 + \lim_{n\to\infty} n^5 \left(\frac{3}{4}\right)^n}{2^6 \lim_{n\to\infty} \left(\frac{1}{2}\right)^n + 2}$$

$$= \frac{4 + 0}{0 + 2}$$

$$= 2.$$

(d) $\left\{ \frac{(-1)^n \sin(n)}{n} \right\}$ converges, with limit 0.

We use the squeeze theorem,
because $-1 \leq (-1)^n \leq 1$, and
$-1 \leq \sin(n) \leq 1$,
so our hope is that we can eliminate these.

Proof. Define $a_n = -\frac{1}{n}$, and $b_n = \frac{1}{n}$.

Notice that for any n,

$$-1 \leq (-1)^n \sin(n) \leq 1,$$

So for any n,

$$a_n = -\frac{1}{n} \leq \frac{(-1)^n \sin(n)}{n} \leq \frac{1}{n} = b_n.$$

Because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,
By the squeeze theorem,

$$\lim_{n\to\infty} \frac{(-1)^n \sin(n)}{n} = 0.$$

$\boxed{\boxed{2}}$ We're given

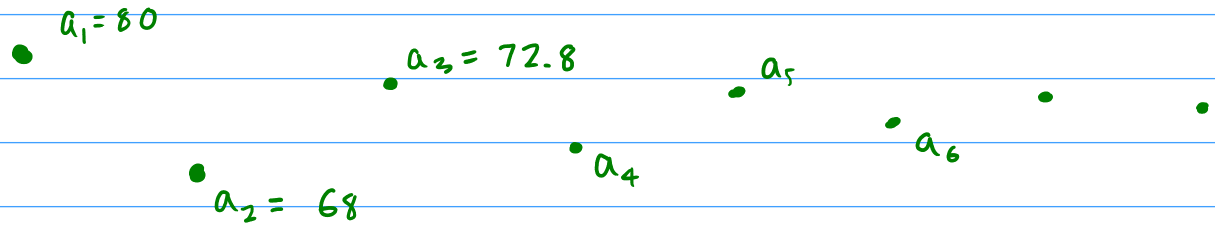
$$a_1 = 80$$
$$a_n = 100 - \frac{2}{5} a_{n-1}.$$

Because $a_n = 100 - \frac{2}{5} a_{n-1}$ has that minus sign, it looks like a_n might be a decreasing sequence. Therefore we attempt to use the Monotone convergence theorem.

Proof.

DISCLAIMER I messed up!
I thought $\{a_n\}$ was decreasing but it's not, making the problem harder.

Computing the first few values of $\{a_n\}$, it looks something like this:



The odd indexed terms seem to decrease, and even seem to increase.

We will prove that $\{a_{2n-1}\}$ (the sequence of odd-indexed terms) is decreasing.

We have:
$$a_{2(n+1)-1}$$
$$= a_{2n+1}$$
$$= 100 - \frac{2}{5} a_{2n}$$
$$= 100 - \frac{2}{5}\left(100 - \frac{2}{5} a_{2n-1}\right)$$

$$= 60 + \frac{4}{25} a_{2n-1}.$$

Therefore writing $b_n = a_{2n-1}$,

$$\left.\begin{aligned} b_1 &= 80 \\ b_{n+1} &= 60 + \frac{4}{25} b_n. \end{aligned}\right\}$$ This is the sequence of odd-indexed terms.

To see that $\{b_n\}$ is <u>decreasing</u> we compute $\frac{b_{n+1}}{b_n}$ and hope it is <u>less than 1</u>.

But $\frac{b_{n+1}}{b_n} = \frac{60 + \frac{4}{25} b_n}{b_n} = \frac{60}{b_n} + \frac{4}{25}$.

For $\frac{60}{b_n} + \frac{4}{25}$ to be less than 1, we need b_n to be at least $\frac{500}{7}$. (Algebra).

<u>Lemma</u> For each n, $\frac{500}{7} \leq b_n$.

<u>proof</u>. $b_1 = 80 > \frac{500}{7}$.

Next, if $b_n > \frac{500}{7}$, then

$$\begin{aligned} b_{n+1} &= 60 + \frac{4}{25} b_n \\ &> 60 + \frac{4}{25}\left(\frac{500}{7}\right) \\ &= 60 + \frac{80}{7} \\ &= \frac{420}{7} + \frac{80}{7} = \frac{500}{7}. \end{aligned}$$

Therefore $\{b_n\}$ is <u>bounded below</u> by $\frac{500}{7}$.

Moreover, $\{b_n\}$ is then <u>decreasing</u> because

$$\frac{b_{n+1}}{b_n} = \frac{60 + \frac{4}{25} b_n}{b_n} = \frac{60}{b_n} + \frac{4}{25} < \frac{60}{(500/7)} + \frac{4}{25} = 1.$$

By the monotone convergence theorem,

$\{b_n\} = \{a_{2n-1}\}$ converges!

* Next, we must prove the <u>even terms</u> converge. That is, the sequence

$$\{c_n\} = \{a_{2n}\}$$

can be shown to be <u>bounded above</u> and <u>increasing</u> so it converges by MCT.

(You should do this! Similar proof to $\{b_n\}$ above!)

In summary, $\{b_n\} = \{a_{2n-1}\}$ and $\{c_n\} = \{a_{2n}\}$ are <u>convergent</u>.

We can then calculate their limits:

$$\begin{aligned} L = \lim_{n\to\infty} b_n &= \lim_{n\to\infty} b_{n+1} \\ &= \lim_{n\to\infty} \left(60 + \frac{4}{25} b_n\right) \\ &= 60 + \frac{4}{25} \lim_{n\to\infty} b_n \\ &= 60 + \frac{4}{25} L. \end{aligned}$$

Solving this gives

$$L = 60 + \frac{4}{25} L$$

$$\Rightarrow \frac{21}{25} L = 60$$

$$\Rightarrow L = \frac{60 \cdot 25}{21} = \frac{500}{7}.$$

Similarly, $\{c_n\}$ converges to $\frac{500}{7}$.

Since $\{b_n\}$ and $\{c_n\}$ both converge to $\frac{500}{7}$, so does $\{a_n\}$.

3

(a) For what values of p does $\{p^n \sin(n)\}$ converge? When it converges, what is the limit?

First consider $\{p^n\}$. This has limits

$$\lim_{n\to\infty} p^n = \begin{cases} 0 & -1 < p < 1 \\ 1 & p = 1 \\ \infty & p > 1 \\ \text{divergent} & p < -1. \end{cases}$$

[We use the squeeze theorem to prove that $\{p^n \sin(n)\}$ converges when $-1 < p < 1$.]

Because $-1 \leq \sin(n) \leq 1$, we have

$$-p^n \leq p^n \sin(n) \leq p^n.$$

For $-1 < p < 1$, $\lim -p^n = \lim p^n = 0$.

Therefore by the squeeze theorem,

$$\lim p^n \sin(n) = 0.$$

[On the other hand, for $p \leq -1$ and $p \geq 1$, $p^n \sin(n)$ diverges because $\sin(n)$ oscillates, and p^n doesn't go to zero.]

More formally, for any N, there exists $m_1, m_2 > N$ such that $\sin(m_1) > \frac{1}{2}$, $\sin(m_2) < -\frac{1}{2}$, with m_1 and m_2 <u>even</u>. But then

$$p^{m_1} \sin(m_1) \geq \sin(m_1) > \frac{1}{2}$$

$$p^{m_2} \sin(m_2) \leq \sin(m_2) < -\frac{1}{2}$$

Therefore $p^n \sin(n)$ doesn't converge for $p \le -1$ or $p \ge 1$.

(b) Same question, but for $\left\{\frac{p^n}{n!}\right\}$.

For simplicity, we first assume p is positive.

Then is $\frac{p^n}{n!}$ increasing? decreasing? neither?

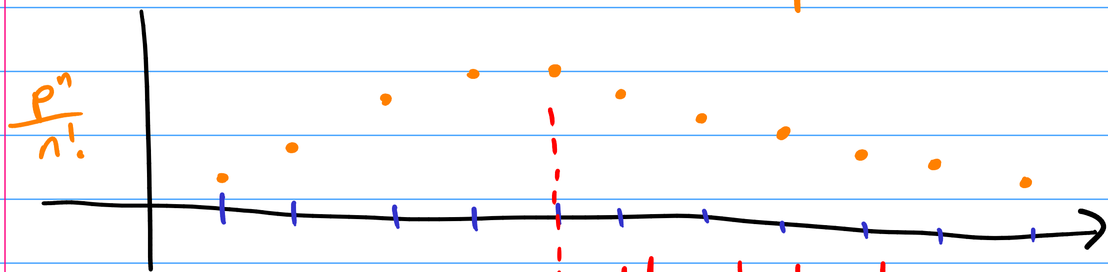
The numerator is

$$p \times p \times p \times \cdots \times p$$

denominator

$$1 \times 2 \times 3 \times \cdots \times n$$

Therefore eventually the denominator wins, i.e. when $n > p$. In a picture,



should start decreasing when the index is at least p.

Remark $\{a_n\}_{n=1}^{\infty}$ converges iff $\{a_n\}_{n=N}^{\infty}$ converges, from our table we made in the discussion.

Therefore, we will show that

$\left\{\frac{p^n}{n!}\right\}_{n=N}^{\infty}$ converges, where N is some number at least p.

1. $\left\{\frac{p^n}{n!}\right\}_{n=N}^{\infty}$ is decreasing.

To see this, write

$$a_n = \frac{p^n}{n!}$$

$$a_{n+1} = \frac{p^{n+1}}{(n+1)!}.$$

Then $\frac{a_{n+1}}{a_n} = \frac{p^{n+1}}{(n+1)!} \cdot \frac{n!}{p^n} = \frac{p}{n+1}.$

But $n+1 \geq N > p$, so $\frac{p}{n+1} < 1.$

② $\{a_n\}$ is bounded below (by 0).

This is because all terms are positive!

By the monotone convergence theorem,

$\lim_{n\to\infty} \frac{p^n}{n!}$ exists!

Since we know the limit exists, we can now compute it as follows:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$= \lim_{n\to\infty} \frac{p^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{p}{n+1} \cdot \frac{p^n}{n!}$$

$$= \lim_{n\to\infty} \frac{p}{n+1} \cdot \lim_{n\to\infty} \frac{p^n}{n!}$$

$$= 0 \cdot L.$$

But if $L = 0 \cdot L = 0$, then the limit is 0.

For $p < 0$, $\left\{\frac{p^n}{n!}\right\}$ converges as well!

This can be proved using the squeeze theorem:

$$-\left|\frac{p^n}{n!}\right| \leq \frac{p^n}{n!} \leq \left|\frac{p^n}{n!}\right|.$$

But the left and right limits go to 0 because of the first part of this proof.

In summary,

$$\lim_{n\to\infty} \frac{p^n}{n!} = 0$$ for all values of $p \in \mathbb{R}$.