1. Taylor series and remainders

Defn The Taylor series of $f(x)$ centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Thm Let $I = (a-r, a+r)$ be an interval on which the Taylor series converges. Then for $x \in I$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Examples: Find the Taylor series of $\sin(x)$ about $a = \frac{\pi}{2}$?

Step 1: Find derivatives:

	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f''''(x)$
$=$	$\sin(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$

Step 2: Evaluate at $a = \frac{\pi}{2}$:

	$f(a)$	$f'(a)$	$f''(a)$	$f^{(3)}(a)$	$f^{(4)}(a)$
$=$	1	0	-1	0	1

Step 3: Write first few terms of series:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

$$= 1 + \frac{(-1)(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \frac{(-1)(x-\frac{\pi}{2})^6}{6!} + \dots$$

Step 4: Find pattern, write in Σ notation:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$$

Defn The degree N Taylor polynomial of $f(x)$ at a is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N .$$

Defn The <u>remainder</u> is

$$R_N(x) = f(x) - P_N(x)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Intuitively, $P_N(x)$ is an "approximation" of $f(x)$ that we can compute, and $R_N(x)$ is the error in our approximation.

Thm Taylor's remainder theorem.

Suppose I is an interval containing "b" and "a".

We want to estimate how bad the error

$$R_N(b) = f(b) - P_N(b)$$

$$= \sum_{n=N+1}^{\infty} \frac{f^{(N)}(a)}{n!}(x-a)^n$$

is, given our approximation of $f(b)$ with $P_N(b)$.

Suppose that $|f^{(N+1)}(x)| \leq M$ for all x in I.

Then $$|R_N(b)| \leq \frac{M}{(N+1)!}|b-a|^{N+1} .$$

Example

"Let $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

For what N is $P_N(2)$ within 10^{-4} of $f(2)$?"

Strategy:

1. Choose an interval I containing "a" and "2". (The smaller the easier.)

2. Find some number M (in terms of N) so that for $x \in I$,
$$|f^{(N+1)}(x)| \leq M.$$

3. Now we know that
$$|R_N(2)| \leq \frac{M}{(N+1)!}|2-a|^{N+1}$$
by Taylor's theorem. Therefore we want to find N so that
$$\frac{M}{(N+1)!}|2-a|^{N+1} \leq 10^{-4}.$$
This "N" is the answer.

Example: We want to estimate e^2 using Taylor polynomials of e^x.

How many terms do we need to ensure that the error is at most 10^{-3}?

Step 1: Choose an interval I.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{we want to estimate} \quad e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

This series is centered at $a=0$, we want to approximate at $b=2$.
$\therefore$ Use the interval $I=(-1,3)$ which contains both 0 and 2.

Step 2: Find M.

We need a number M which is <u>at least</u> $|f^{(N+1)}(x)|$, for all $x \in I$.

In our case, $f^{(N+1)}(x) = e^x$, and the largest value this can attain in $(-1,3)$ is e^3.

Therefore we can use $M = e^3$.
(We can also use 30 which is bigger than e^3.)

$$* \quad |f^{(N+1)}(x)| \leq 30 \quad \text{for all } x \in I.$$

Step 3: Use the remainder theorem.

We now have that

$$|R_N(2)| \leq \frac{M}{(N+1)!}|b-a|^{N+1} = \frac{30}{(N+1)!}2^{N+1}$$

We now solve for $\frac{30}{(N+1)!}2^{N+1} < 10^{-3}$.

This gives (e.g. using a calculator)

$$N = 11.$$

Example: We want to estimate e^{-1} using Taylor polynomials. How many terms do we need to ensure error $< 10^{-3}$?

Writing the series, we have

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

This is alternating!

Recall: If $\sum_{n=0}^{\infty} a_n$ is an alternating, absolutely decreasing series whose terms converge to 0, then

$$|R_N| \leq |a_{N+1}|$$

where $R_N = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N} a_n$.

By the alternating remainder theorem, we simply need to find N such that

$$|a_{N+1}| = \frac{1}{(N+1)!} \leq 10^{-3}.$$

$N = 7$ works.

As a general rule: when faced with a remainder problem.

1. See if the alternating test applies.
2. If not, use the Taylor remainder thm.

2. Finding Taylor series.

Thm Uniqueness of Taylor series.

If $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(x-a)^n$, then

this is the Taylor series of $f(x)$.

i.e. $a_n = f^{(n)}(a)$.

Strategy:

- We know Taylor series for common functions: $\sin(x)$, $\cos(x)$, $e(x)$, $\arctan(x)$, $\ln(1+x)$, $\frac{1}{1-x}$.

  Try to write the given function in terms of "common" functions.

  - might require trig identities, integration / differentiation,

- Once this has been done, write the common functions as Taylor series, and add / multiply etc as needed.

Strategy 2:

If this fails, write out terms using the definition of Taylor series.

Examples:

1. Find Taylor series of the following functions

(a) $\sin(x) + \cos(x)$ centered at $a = 0$

(b) $e^x \sin(x)$ centered at $a = 0$

(c) $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ centered at $a = 0$

(d) $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ centered at $a = \frac{\pi}{4}$

2. Write the following as "function values"

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(2n+1)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!}$

3. Find series converging to:

(a) $\arctan\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)$

(b) $\cos^2(3)$

1. (a) $\sin(x) + \cos(x)$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{x^n}{n!}$$

(b) $e^x \sin(x)$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)$$

where $a_n = (0, 1, 0, -1, 0, 1, 0, -1, \ldots)$

$$= \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \sum_{i=0}^{n} \frac{a_i}{i!\,(n-i)!}.$$

I don't think there is a nice formula for this! Sorry :(

(c) $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin(x)$ by the double angle formula

$$\therefore \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(d) $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2}\sin(x)$.

Now we need a Taylor series for $\sin(x)$ about $\frac{\pi}{4}$.

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

The derivatives of $\sin(x)$ are

$$\sin(x), \quad \cos(x), \quad -\sin(x), \quad -\cos(x), \quad \sin(x), \ldots$$

Therefore

$$\sin(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) - \frac{1}{\sqrt{2}\,2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{\sqrt{2}\,3!}\left(x-\frac{\pi}{4}\right)^3 + \ldots$$

$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{2} \rfloor}\frac{\left(x-\frac{\pi}{4}\right)^n}{n!}$$

Thus $$\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}(-1)^{\lfloor \frac{n}{2} \rfloor}\frac{\left(x-\frac{\pi}{4}\right)^n}{n!}$$

2. (a) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(1/4)^n}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{2\cdot(1/2)^{2n+1}}{(2n+1)!}$$

$$= 2\sin(1/2)$$

(b) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{1^{n+1}}{n+1} = \ln(1+1) = \ln(2)$$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2}$

3 (a) $\arctan\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1}$

$\ln\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^n}{n}$

$\therefore \arctan\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\left(\frac{1}{2}\right)^{2n+1}}{2n+1} + \frac{\left(\frac{1}{2}\right)^n}{n} \right)$

(can try to simplify more if desired!)

(b) $\cos^2(3) = \cdots ?$

Trig identities!

- $\cos(2x) = \cos^2(x) - \sin^2(x)$
- $\cos^2(x) + \sin^2(x) = 1$

$\therefore 2\cos^2(x) = 1 + \cos(2x)$

It follows that

$\cos^2(3) = \frac{1 + \cos(6)}{2}$

$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{6^{2n}}{(2n)!}.$