① Absolute vs Conditional convergence

Defn Absolute convergence:
$\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Defn Conditional convergence:
$\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges, but not absolutely.

Thm Absolute convergence test.
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Thm Alternating series test.
Suppose $\sum_{n=1}^{\infty} a_n$ satisfies the following properties:

- Alternating (signs alternate $+, -, +, -, \ldots$)
- Absolutely decreasing ($|a_{n+1}| < |a_n|$ for all n)
- terms go to 0 ($\lim_{n\to\infty} a_n = 0$)

Then the series converges.

Which of the following converge conditionally? Which converge absolutely? Which diverge?

(a) $$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$$

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^3}$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$$

Solutions

(a) $\cos(\pi n) = -1, 1, -1, 1, \ldots$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Notice that:

- $\frac{(-1)^n}{n}$ is alternating
- $\frac{1}{n}$ is decreasing
- $\lim \frac{1}{n} = 0$.

Therefore by the alternating series test, it converges!

Next, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\cos(\pi n)}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges, so it doesn't converge absolutely.

Therefore (a) converges conditionally.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^3}$.

We first check absolute convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln(n)}{n^3} \right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$$

$$\leq \sum_{n=1}^{\infty} \frac{n}{n^3} \qquad \text{since } \ln(n) \leq n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \text{converges by } p\text{-test!}$$

By the comparison test, (b) converges absolutely.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$. This has no negative terms! This means

$$\sum_{n=1}^{\infty} \frac{1}{n\ln(n)} = \sum_{n=1}^{\infty} \left|\frac{1}{n \ln(n)}\right|.$$

i.e. <u>either</u> the series converges absolutely, <u>or</u> it diverges.

To see that it diverges, use the integral test:

$$f(x) = \frac{1}{x \ln(x)}$$

satisfies the premises of the integral test, and

$$\int_1^{\infty} \frac{1}{x \ln(x)}\, dx$$

diverges by the substitution $u = \ln(x)$.

(2) Re-indexing / tails

Let's go back to the alternating series test: it has two conditions which depend on every term in the series:

1. Absolutely decreasing
2. Alternating.

Sometimes achieving these requires reindexing, looking at tails, or something else similar.

E.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} + (-1)^n \frac{1}{n}$

Let's write the first few terms!

a_1	a_2	a_3	a_4	a_5	a_6	$\cdots$
$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{5}{24}$	$\frac{5}{16}$	$-\frac{27}{160}$	$\frac{35}{192}$	

Alternating ✓
terms go to zero ✓
Absolutely decreasing :(

e.g. $\frac{3}{4} > \frac{1}{2}$, $\frac{5}{16} > \frac{5}{24}$

We want to find N s.t. $|a_{n+1}| < |a_n|$ for all $n > N$.

$\iff$ solving the equation

$$\left|\frac{1}{2^{n+1}} + (-1)^{n+1}\frac{1}{n+1}\right| - \left|\frac{1}{2^n} + (-1)^n \frac{1}{n}\right| < 0.$$

It turns out that $N = 6$ works (I think)

$\therefore \displaystyle\sum_{n=6}^{\infty} \frac{1}{2^n} + (-1)^n\frac{1}{n}$ converges. By tail-convergence,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} + (-1)^n \frac{1}{n}$ converges.

Use re-indexing / tails / removing 0s to determine convergence / divergence of the following series:

(a) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-3.5)(n-6.5)}$$

(b) $$\sum_{n=1}^{\infty} \overset{\cos}{\cancel{\sin}}\left(\frac{\pi n}{2}\right) \frac{1+(-1)^n}{n}$$

(a) Consider $f(x) = \frac{1}{(x-3.5)(x-6.5)}$.
This has asymptotes at 3.5 and 6.5, but is <u>positive</u> and <u>decreasing</u> from 6.5 and onwards.

(You can use the 1st derivative test to show that it's decreasing)

This means $\displaystyle\sum_{n=7}^{\infty} \frac{(-1)^n}{(n-3.5)(n-6.5)}$ is

- alternating
- absolutely decreasing
- $\lim_{n\to\infty} a_n = 0$

So by the alternating series test, it converges.
Now by the convergence of the tail,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-3.5)(n-6.5)}$$

must <u>also</u> converge.

(b) List the first few terms:

$$a_n = \cos\left(\frac{\pi n}{2}\right)\frac{1+(-1)^n}{n}$$

a_1	a_2	a_3	a_4	a_5	$a_6 \cdots$
0	-1	0	$\frac{1}{2}$	0	$-\frac{1}{3}$
	$\parallel$		$\parallel$		$\parallel$
	b_1		b_2		b_3

We'll define $b_m = (-1)^m \cdot \frac{1}{m}$. (where $2m = n$)

Then it should be the case that

$$\sum_{n=1}^{\infty} \cos\left(\frac{\pi n}{2}\right)\frac{1+(-1)^n}{n} = \sum_{m=1}^{\infty} b_m .$$

We next have to ~~prove~~ this.

To see this, first note that for odd n, $\cos\left(\frac{\pi n}{2}\right) = 0$, so $a_n = 0$.

For even n, write $n = 2m$. Then

$$\cos\left(\frac{\pi n}{2}\right)\frac{1+(-1)^n}{n} = \cos(\pi m)\frac{1+(-1)^{2m}}{2m}$$

$$= (-1)^m \frac{1+1}{2m}$$

$$= (-1)^m \frac{1}{m}.$$

This proves that

$$\sum_{n=1}^{\infty} a_n = \sum_{m=1}^{\infty} b_m .$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{m=1}^{\infty} b_m = \sum_{m=1}^{\infty} \frac{(-1)^m}{m}$ converges.

The latter converges by the alternating series test.

(3) Power series / Ratio and Root test.

Thm Ratio test

Consider $\sum_{n=1}^{\infty} a_n$. Define $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

Then:

$$\begin{cases} r<1, & \text{then } \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \\ r>1, & \text{then } \sum_{n=1}^{\infty} a_n \text{ diverges} \\ r=1, & \text{then the test provides no information.} \end{cases}$$

Thm Root test

Same as above, but using

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

When are they useful? Which one should I use?

They are useful when: a^n, $n!$ etc occur

not useful when: weird functions like $\sin(n)$, $\ln(n)$, or dominating term is n^p.

Typically ratio test is <u>much</u> easier to use than the root test.

However, sometimes the root test works even when the ratio test fails!

My bold claim: <u>in this class,</u> whenever root test works, I think the ratio test will work too!

Example:

$$\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{-(n+1)-(-1)^{n+1}}}{3^{-n-(-1)^n}}\right|$

If n is even, this is

$$\left|\frac{3^{-(n+1)+1}}{3^{-n-1}}\right| = \left|\frac{3^{-n}}{3^{-n-1}}\right| = 3.$$

If n is odd, this is

$$\left|\frac{3^{-(n+1)-1}}{3^{-n+1}}\right| = \left|\frac{3^{-n-2}}{3^{-n+1}}\right| = 3^{-3}$$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ doesn't exist!

(It keeps oscillating between 3 and 3^{-3}.)

Therefore we can't use the ratio test.

Now consider $\sqrt[n]{|a_n|}$.

This is

$$\sqrt[n]{3^{-n-(-1)^n}} = \sqrt[n]{\frac{1}{3^n}\cdot\frac{1}{3^{(-1)^n}}}$$

$$= \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt[n]{3^{(-1)^n}}}$$

Therefore $\lim_{n\to\infty}\sqrt[n]{|a_n|}$

$$= \frac{1}{\sqrt{3}}\lim_{n\to\infty}\frac{1}{\sqrt[n]{3^{(-1)^n}}}$$

$$= \frac{1}{\sqrt{3}} < 1.$$ Therefore the series converges by root test

Defn Power series

Any series that looks like $\sum a_n (x-b)^n$.

How to think about them: x is allowed to vary, you have a different series for each choice of x! It's like an infinite family of series that may converge or diverge at each different x.

A remark on exponents: 0^0.

$$\sum_{n=0}^{\infty} x^n = x^0 + x^1 + x^2 + x^3 + \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots$$

Why is $x^0 = 1$? If $x = 0$, it's 0^0, is this really defined? Usually, no. But for power series, we declare that $x^0 = 1$ for any x.

What are the intervals of convergence of the following series?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^{5n+1}}{3n^2\, 2^{n-3}}$

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n$

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}\, n}{x^n\,(n+1)}\right|$$

$$= \lim_{n\to\infty} |x| \frac{n}{n+1}$$

$$= |x|$$

Therefore $\sum a_n$ converges when $|x|<1$ and diverges when $|x|>1$.

What if $|x|=1$?

When $x=1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges by alternating series test.

When $x=-1$, $\sum_{n=1}^{\infty} \frac{1}{n}$, diverges by p-test.

Therefore the interval of convergence is $(-1, 1]$.

(b) $\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$

Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-a)^{n+1}\, n!}{(x-a)^n\,(n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x-a}{n+1}\right| = 0.$$

The limit is 0, independent of x or a. This means the interval of convergence is $(-\infty, \infty)$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^{5n+1}}{3n^2\, 2^{n-3}}$

Root test: (Ratio test works too! I just thought I should give a root test example too)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(x+3)^{5n+1}}{3n^2\, 2^{n-3}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{(x+3)}\ \sqrt[n]{(x+3)^{5n}}}{\sqrt[n]{2^{-3}}\ \sqrt[n]{3n^2}\ \sqrt[n]{2^n}}$$

$$= \frac{1 \cdot (x+3)^5}{1 \cdot 1 \cdot 2}$$

Rmk $\displaystyle\lim_{n\to\infty} \sqrt[n]{x+3} = \lim_{n\to\infty} \sqrt[n]{2^{-3}} = 1$, because $x+3$ and 2^{-3} are constant.

Why is $\displaystyle\lim_{n\to\infty} \sqrt[n]{3n^2} = 1$? This is harder!

Write $\sqrt[n]{3n^2} = (3n^2)^{\frac{1}{n}} = e^{\ln((3n^2)^{\frac{1}{n}})}$

$$= e^{\ln(\sqrt{3}\cdot n)\cdot \frac{2}{n}}$$

But $\displaystyle\lim_{n\to\infty} 2\cdot\frac{\ln(\sqrt{3}n)}{n} = 0$,

so $\lim e^{\frac{\ln(\sqrt{3}n)}{n}2} = 1$.

This means we have

$$r = \left|\frac{(x+3)^5}{2}\right|$$

Thus $r<1$ when

$$\left|\frac{(x+3)^5}{2}\right| < 1, \quad \text{i.e. } |x+3| < \sqrt[5]{2},$$

so $-\sqrt[5]{2}-3 < x < \sqrt[5]{2}-3$.

$r>1$ when $x < -\sqrt[5]{2}-3$ or $x > \sqrt[5]{2}-3$.

What happens when $x = -\sqrt[5]{2}-3$, or $x = \sqrt[5]{2}-3$?

(i.e. when $r=1$, so the root test is inconclusive).

If $x = \sqrt[5]{2}-3$, then

$$\sum_{n=0}^{\infty} \frac{(x+3)^{5n+1}}{3n^2\, 2^{n-3}} = \sum_{n=0}^{\infty} \frac{(\sqrt[5]{2})^{5n+1}}{3n^2\, 2^{n-3}}$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt[5]{2}) \cdot 2^n}{3n^2 \cdot 2^{-3} \cdot 2^n}$$

$$= \sum_{n=0}^{\infty} \left(\sqrt[5]{2} \cdot 2^3\right) \frac{1}{3n^2}.$$

Converges by limit comparison or direct comparison against $\frac{1}{n^2}$.

What if $x = -\sqrt[5]{2}-3$?

Then $\displaystyle\sum_{n=0}^{\infty} \frac{(-\sqrt[5]{2})^{5n+1}}{3n^2\, 2^{n-3}}$.

Converges, by similar working to above!

Therefore the interval of convergence is $[-\sqrt[5]{2}-3, \sqrt[5]{2}-3]$.