19th Jan

1. General questions
2. (Limit) comparison test summary
3. Worksheet questions

1. What do we need to know for the quiz?
   Will update this ASAP when I hear back

   Where do we submit the problem set?
   On Gradescope, this should be available!

   Where are the "answers to polls"? The link seems to be broken!
   Will update ASAP

2. Limit comparison test:
   Let f and g be continuous +ve functions.
   Then:

Case 1: $\frac{f(x)}{g(x)} \to$ non-zero number

$$\int_a^\infty f(x)\,dx \text{ converges} \iff \int_a^\infty g(x)\,dx \text{ converges}$$

Case 2: $\frac{f(x)}{g(x)} \to 0$

$$\int_a^\infty f(x)\,dx \text{ converges} \Leftarrow \int_a^\infty g(x)\,dx \text{ converges}$$

Case 3: $\frac{f(x)}{g(x)} \longrightarrow \infty$

$$\int_a^\infty f(x)\,dx \text{ converges} \Rightarrow \int_a^\infty g(x)\,dx \text{ converges}$$

Compare this to the "Comparison test":
If f, g are +ve continuous functions and
$f(x) \le g(x)$ <u>for all x</u>, then

$$\int_a^\infty f(x)\,dx \text{ converges} \Leftarrow \int_a^\infty g(x)\,dx \text{ converges.}$$

* Can you see why the limit comparison test implies the comparison test?

So how do we <u>use</u> these tests?
How do we use the limit comparison test?

<u>Step 1</u> Guess whether or not the integral converges or diverges. (Look at the "overall" power of x in the function).

e.g. $$\int_5^\infty \frac{x^2+4}{\sqrt{\cos(x)+x^6}}\,dx$$

should <u>diverge</u>, because

$$\frac{x^2+4}{\sqrt{\cos(x)+x^6}} \approx \frac{x^2}{\sqrt{x^6}} = \frac{x^2}{x^3} = \frac{1}{x}$$

(fails the p-test).

<u>Step 2</u> If you think $\int f(x)$ <u>converges</u>, choose $g(x)$ to be some function such that $\int g(x)$ <u>converges</u>.

Likewise, if you think $\int f(x)$ diverges, choose $g(x)$ s.t. $\int g(x)$ diverges.

Next, we want $g(x)$ to satisfy

$\frac{f(x)}{g(x)} \to k, 0$ (if we think $\int f$ converges)

$\to k, \infty$ (if we think $\int g$ diverges)

Typically, choosing

$$g(x) = \frac{1}{x^p}, \quad \text{where } p \text{ is}$$

the "overall power of x" in $f(x)$ as mentioned above should work.

Final remark on the limit comparison test: what if the integral is

$$\int_0^\infty f(x)\,dx \;?$$

If we're integrating from 0, then

$$\int_0^\infty \frac{1}{x^p}\,dx \quad \text{diverges} \quad \text{whatever } p \text{ is!}$$

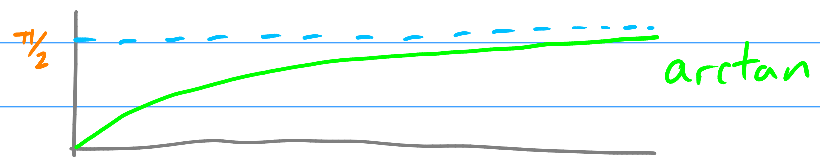
In these instances, we can usually write

$$\underbrace{\int_0^1 f(x)\,dx}_{\substack{\text{Converges if} \\ f \text{ is continuous on} \\ \text{this domain}}} + \underbrace{\int_1^\infty f(x)\,dx}_{\substack{\text{Apply limit} \\ \text{comparison} \\ \text{here.}}} = \int_0^\infty f(x)\,dx,$$

3. Actual question example!

Does $\displaystyle\int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx$ converge?

Step 1. $\arctan(x)$ is positive, and bounded above by $\pi/2$.



Therefore, in the limit,

$$\frac{\arctan(x)}{1+x^2} \approx \frac{\pi/2}{1+x^2} \approx \frac{1}{x^2}.$$

By the p-test, the integral *should* converge.

Step 2. We choose $g(x) = \frac{1}{x^2}$ because the power cancels that from above, and $\int_1^\infty \frac{1}{x^2}$ converges.

Step 3. Checking the premise of the limit comparison test:

$$\lim_{x\to\infty} \left(\frac{\arctan(x)}{1+x^2}\right) \Big/ \left(\frac{1}{x^2}\right) = \pi/2.$$

Hence $\displaystyle\int_1^\infty \frac{\arctan(x)}{1+x^2}\,dx$ converges!!

Finally, write

$$\int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx = \int_0^1 \frac{\arctan(x)}{1+x^2}\,dx + \int_1^\infty \frac{\arctan(x)}{1+x^2}\,dx.$$

Converges because the integrand is continuous on $[0,1]$

Converges by the limit comp. test as shown above.

Therefore $\int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx$ converges!