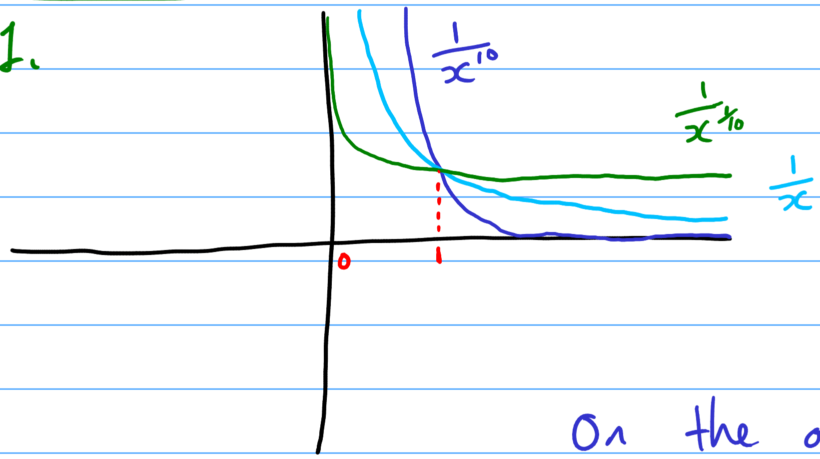
26th Jan sections.

1. • Intuition behind p-test
   • When do we need it / how do we apply it?

2. The second integral from the worksheet

3. The third integral from the worksheet

1.



Look at the $\frac{1}{x^{10}}$ curve first. The are between 0 and 1. It looks infinite! On the other hand, the area to the right of 1 seems to go to 0. Therefore

$$\int_0^1 \frac{1}{x^{10}}\,dx \text{ diverges, and } \int_1^\infty \frac{1}{x^{10}}\,dx \text{ converges.}$$

Next, look at the $\frac{1}{x^{\frac{1}{10}}}$ curve. This time, the area from 0 to 1 seems to be finite, and 1 to ∞ diverges! That is,

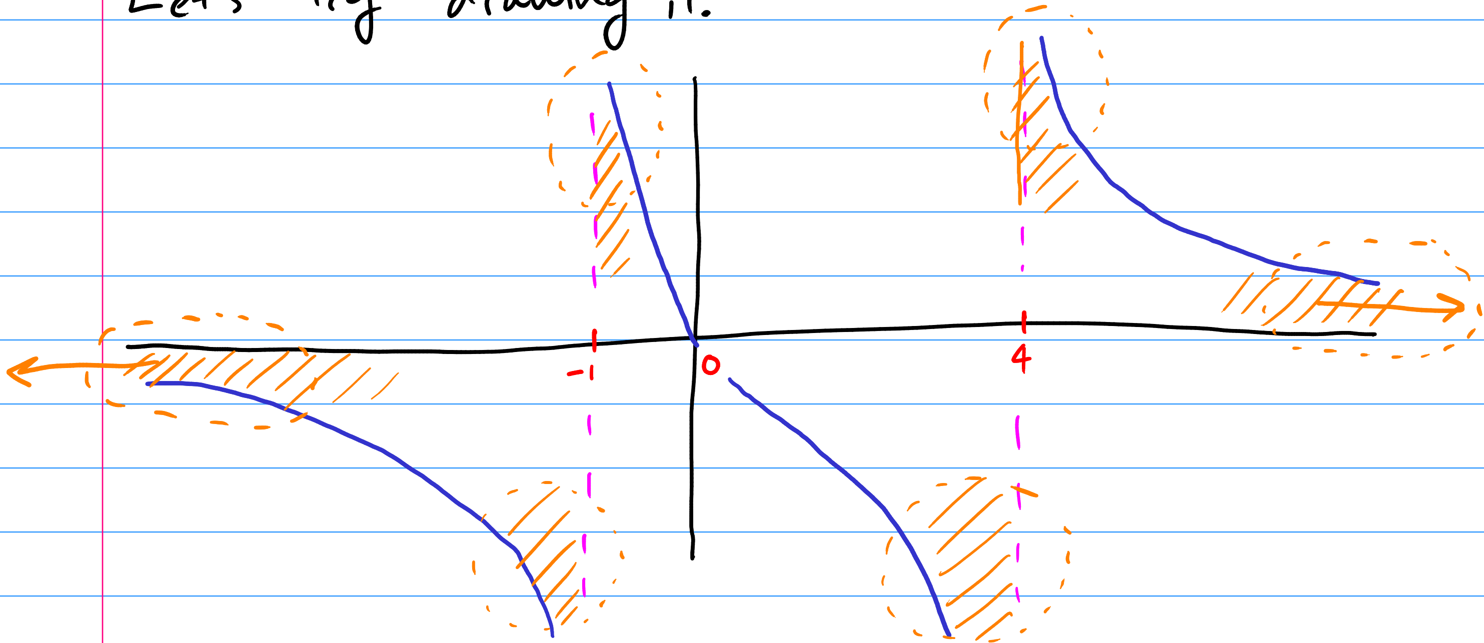
$$\int_0^1 \frac{1}{x^{\frac{1}{10}}}\,dx \text{ converges, and } \int_1^\infty \frac{1}{x^{\frac{1}{10}}}\,dx \text{ diverges.}$$

Finally, with $\frac{1}{x}$, they <u>both</u> diverge. In summary,

$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges iff } p < 1$$

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges iff } p > 1.$$

1. Let's say we want to integrate $\frac{x}{(x-4)(x+1)}$.
Firstly, is it integrable?
i.e. is the area under the curve finite?
Let's try drawing it.



It has vertical asymptotes at $x=4$ and $x=-1$.
There are **6 places** where the area might get infinite! (as indicated)

We must break up the integral so that each piece has only 1 bad point (potential infinity), e.g.

$$\int_{-\infty}^{\infty} \rightsquigarrow \underbrace{\int_{-\infty}^{-2} + \int_{-2}^{-1} + \int_{-1}^{0} + \int_{0}^{4} + \int_{4}^{5} + \int_{5}^{\infty}}_{\text{Notice there are 6 integrals, for the 6 "possible infinities".}}$$

Use the comparison/p-test on each piece.

- If any of them diverge, the original integral diverges.
- If all of them converge, the integral converges.

2. The second integral from the sheet!

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{5-x^2}}\,dx.$$

First observation: $\sqrt{5-x^2}$ $\boxed{\text{isn't defined}}$ for $x^2>5$. In that instance, we have the sqrt of a negative number! Therefore

" $\frac{1}{\sqrt{5-x^2}}$ " doesn't make sense from $-\infty$ to ∞.

The largest subset of $(-\infty,\infty)$ on which $\frac{1}{\sqrt{5-x^2}}$ is defined is $(-\sqrt{5},\sqrt{5})$. Therefore we consider $\int_{-\sqrt{5}}^{\sqrt{5}} \frac{1}{\sqrt{5-x^2}}\,dx.$

We have two bad points, $-\sqrt{5}$ and $\sqrt{5}$. Therefore we break the integral up:

$$\int_{-\sqrt{5}}^{\sqrt{5}} \frac{1}{\sqrt{5-x^2}}\,dx = \int_{-\sqrt{5}}^{0} \frac{1}{\sqrt{5-x^2}}\,dx + \underbrace{\int_{0}^{\sqrt{5}} \frac{1}{\sqrt{5-x^2}}\,dx}.$$

$$\int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}-x}\,\sqrt{\sqrt{5}+x}}\,dx$$

$$= \int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}+x}}\cdot\frac{1}{\sqrt{\sqrt{5}-x}}\,dx$$

$$\leq \int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}-x}}\,dx \quad ☆$$

* in this step, we use that for x between 0 and $\sqrt{5}$, the largest value of $\frac{1}{\sqrt{\sqrt{5}+x}}$ is $\frac{1}{\sqrt{\sqrt{5}}}$, and this is less than 1.

Therefore
$$\frac{1}{\sqrt{\sqrt{5}+x}}\frac{1}{\sqrt{\sqrt{5}-x}} \leq \frac{1}{\sqrt{\sqrt{5}-x}}.$$
(we're about to use the comparison test!)

Next, by the p-test,
$$\int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}-x}}\,dx \quad \text{converges.}$$
This is because changing variables to
$$t = \sqrt{5}-x$$
gives
$$\int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}-x}}\,dx = \int_0^{\sqrt{5}} \frac{1}{\sqrt{t}}\,dt = \int_0^{\sqrt{5}} \frac{1}{t^{\frac{1}{2}}}\,dt.$$
Therefore by the p-test $\int_0^{\sqrt{5}} \frac{1}{\sqrt{\sqrt{5}-x}}\,dx$ converges, and now by the direct comparison test
$$\int_0^{\sqrt{5}} \frac{1}{\sqrt{5-x^2}}\,dx \quad \text{converges.}$$

Now if $\int_{-\sqrt{5}}^{0} \frac{1}{\sqrt{5-x^2}}\,dx$ converges, you're done!

3. The last integral.

$$\int_0^{2\pi} \frac{\ln(x)}{\cos(x)}\,dx$$

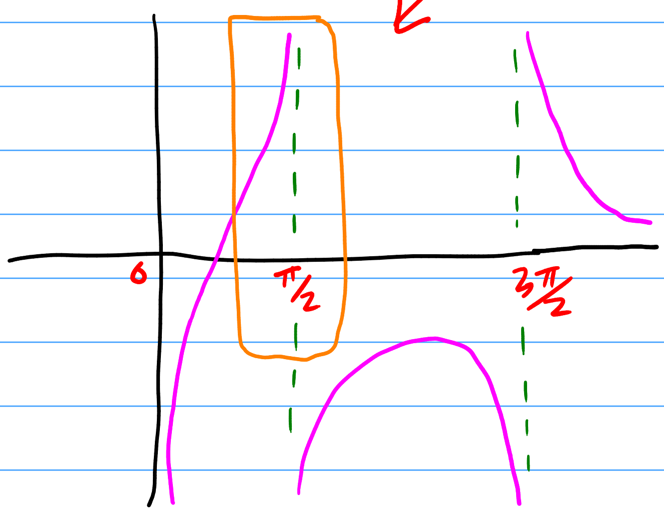
First, where are the problem points?

$\ln(x) \to -\infty$ as $x \to 0$

$\cos(x) = 0$ at $x = \frac{\pi}{2}$

$\cos(x) = 0$ at $x = \frac{3\pi}{3}$.

Graph should look like



Let's focus on the orange region.

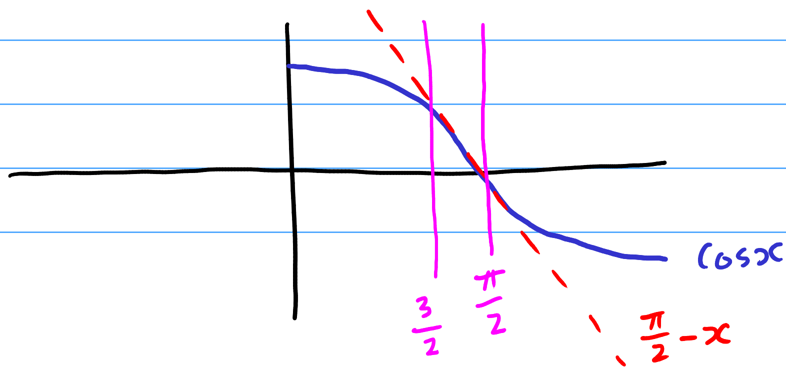
This is, for example,

$$\int_{\frac{3}{2}}^{\pi/2} \frac{\ln(x)}{\cos(x)}\,dx$$

Between $\frac{3}{2}$ and $\frac{\pi}{2}$, $\ln(x)$ is a positive continuous function! We can bound it above or below by constants, i.e.

$$\boxed{\int_{\frac{3}{2}}^{\pi/2} \frac{\ln(x)}{\cos(x)}\,dx \text{ converges} \iff \int_{\frac{3}{2}}^{\pi/2} \frac{K}{\cos(x)}\,dx \text{ converges.}}$$

Next we want to think about $\cos(x)$. To use the p-test, we must approximate it by x^p.



$\cos x$ is less than $\frac{\pi}{2} - x$ from $\frac{3}{2}$ to $\frac{\pi}{2}$!

This means $\frac{1}{\cos(x)}$ is greater than $\frac{1}{\frac{\pi}{2}-x}$.

By the comparison test, if $\int_{\frac{3}{2}}^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx$ diverges, so does $\int_{\frac{3}{2}}^{\pi/2} \frac{1}{\cos(x)}\,dx$, and hence so does $\int_{\frac{3}{2}}^{\pi/2} \frac{\ln(x)}{\cos(x)}\,dx$.

Finally, by a change of variables to $x \longrightarrow t = \frac{\pi}{2}-x$,

$$\int_{\frac{3}{2}}^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx = \int_0^{\frac{\pi}{2}-\frac{3}{2}} \frac{1}{t}\,dt,$$

and $\int_0^{\frac{\pi}{2}-\frac{3}{2}} \frac{1}{t}\,dt$ diverges by the p-test.