A low dimensional introduction to TQFT

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April 29, 2020

Abstract

We introduce the notion of a topological quantum field theory (TQFT), motivated by Feynman's path integrals. We then classify TQFTs in dimensions 1 and 2. Finally we describe some applications of TQFT to other areas of maths.

1 Physics

A turning point in quantum field theory was Feynman's introduction of *path integrals*, which we've been introduced to in earlier talks. Let φ_0 and φ_1 represent *states*, i.e. (normalised) vectors in some Hilbert space $\mathcal{H}$. The *path integral* from φ_0 to φ_1 is represented by

$$A(\varphi_0, t_0; \varphi_1, t_1) = \int_{\Phi \in \Gamma(E) \text{ s.t. } \Phi|_{t_i} = \varphi_i} D[\Phi] e^{iS[\Phi]}.$$

The left side is a *quantum propagator*; a map which computes the likelihood of state φ_0 evolving to state φ_1. We are integrating over all of the field configurations on our spacetime with prescribed boundary data (φ_0, t_0) and (φ_1, t_1). S is some action corresponding to a weight for each path, and $D[\Phi]$ is some sort of measure.

A big result is that the "path integral" develops into a whole formulation of quantum field theory. Feynman showed that from the single notion of a path integral one can recover the Schrödinger equation with the Hamiltonian corresponding to the action, and it's all equivalent to the *canonical quantisation* approach to quantum physics. Therefore it's reasonable to *define* a quantum field theory by the corresponding path integral.

Unfortunately the right side (provably) doesn't make sense in general (Cameron 1960). (However In the case where each φ is a path $\Phi : [0, 1] \to \mathcal{H}$ we can use the Weiner measure.) In general, we'll integrate over $\Gamma(E)$ where E is some fibre bundle $E \to \Sigma$, where Σ is the spacetime. In particular the Weiner measure doesn't apply. Therefore the first aim of topological quantum field theory is to axiomatise the properties of path integrals to circumvent the inability to define them.

What are the main properties we want for a notion of path integral?

1. At t_0, we obtain an $n-1$ dimensional time-constant slice V_0, or hypersurface, of spacetime. Every such hypersurface has a corresponding Hilbert space $\mathcal{H}_{V_0} = Z(V_0)$ of states. (For TQFTs, these spaces are finite dimensional.)
2. At t_0 and t_1, we have hypersurfaces V_0 and V_1. An n dimensional submanifold M sandwiched between V_0 and V_1 (i.e. a cobordism M between V_0 and V_1) determines all possible field configurations from V_0 to V_1. Therefore M should correspond to the path integral

$$Z(M) = \int_{\Phi|_{t_i} = \ ?} D[\Phi] e^{iS[\Phi]}$$

where the boundary conditions are determined by $\varphi_i \in Z(V_i)$. More suggestively, we should obtain a propagator $Z(M) : \mathcal{H}_{V_0} \otimes \mathcal{H}_{V_1} \to \mathbb{C}$.

3. Suppose V_0 and V_0' are two disjoint components of a time-slice. Then they don't interact with each other, so we expect

$$\mathcal{H}_{V_0 \sqcup V_0'} = \mathcal{H}_{V_0} \otimes \mathcal{H}_{V_0'}.$$

4. Suppose t_0 and t_1 are "close together", and we have a thin cylindrical cobordism $M = V_0 \times [t_0, t_1]$. This corresponds to the propagator $Z(M) : \mathcal{H}_{V_0} \otimes \mathcal{H}_{V_0} \to \mathbb{C}$. In particular, we expect $A(\varphi_0 \otimes \varphi_0) \sim 1$ for φ_0 normalised. In this case $Z(M)$ is a non-degenerate map, and induces an isomorphism $Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_0}^*$.

5. *Sewing law.* We expect the integral over all field configurations with boundary data φ_0 at t_0 and φ_1 at t_1 to agree with the integral over intermediate states. That is, if we have some $t_0 < t' < t_1$, then

$$\int_{\Phi|_{t_i}=\varphi_i} D[\varphi] e^{iS[\varphi]} = \int_{\varphi' \text{ at } t'} D[\varphi'] \left(\int_{\Phi|_{t_i}=\varphi_i, \Phi|_{t'}=\varphi'} D[\Phi] e^{iS[\Phi]} \right).$$

Unwrapping the integrals, this is saying that

$$Z(M) = Z(M_1) Z(M')^{-1} Z(M_0)$$

where we've identified $Z(N) : \mathcal{H}_V \otimes \mathcal{H}_W \to \mathbb{C}$ with $Z(N) : \mathcal{H}_V \to \mathcal{H}_W^*$. M is a cobordism from V_0 to V_1, with $M = M_0 M' M_1$ where M' is a thin cylindrical cobordism from V' to V' (and V' is the constant-time slice of M at t').

With appropriate formalism we can remove $Z(M')^{-1}$ so that the Sewing law is really just functoriality of Z.

2 Category theory

Definition 2.1. A *braided monoidal category* is a category $\mathcal{C}$ equipped with a "tensor product". More precisely, $\mathcal{C}$ is equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which

- has a unit ($1 \in \mathcal{C}$ such that $x \otimes 1 \cong 1 \otimes x \cong x$),
- is associative ($x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$),
- and has a *braiding* $B_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$.

(Here $\cong$ denotes a natural isomorphism). There are additional "coherence conditions" for the natural isomorphisms (requiring that certain diagrams commute). For example, the following diagram commutes:

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes w) \arrow[dr] & \\
((x \otimes y) \otimes z) \otimes w \arrow[ur] \arrow[d] & \text{pentagon identity} & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z) \otimes w \arrow[rr] & & x \otimes ((y \otimes z) \otimes w). \arrow[u]
\end{tikzcd}$$

Definition 2.2. A braided monoidal category is called a *symmetric monoidal category* if the braiding is involutive:

$$B_{x,y} \circ B_{y,x} = \mathrm{id}_{x \otimes y}.$$

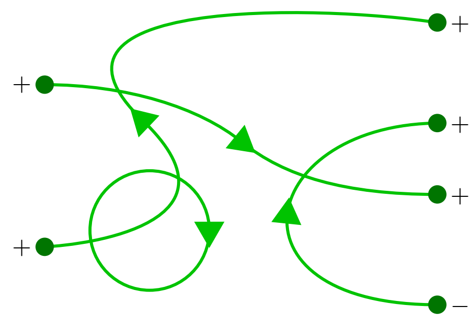
Example. Let $\mathbf{Vect}_k$ be the category of vector spaces over a field k. Then $\mathbf{Vect}_k$ is a symmetric monoidal category, with the product given by the usual tensor product $\otimes_k$.

Example. Let $\mathbf{Cob}_n$ be the category of oriented n-dimensional cobordisms. That is, the objects are closed oriented n-manifolds, and a morphism $M \to N$ is a cobordism with boundary $-M \sqcup N$. ($-M$ denotes M with the opposite orientation.) The product of two closed n-manifolds is given by their *disjoint union*. The empty manifold is a unit for this product.

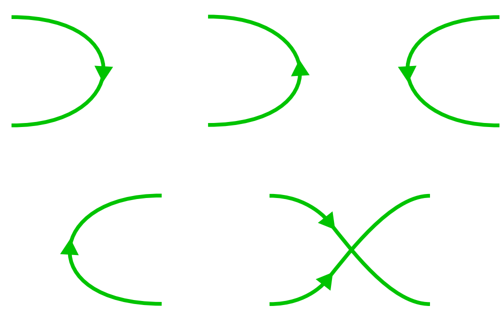
For example, the objects of $\mathbf{Cob}_1$ are oriented 0-manifolds, i.e. finite disjoint unions of signed points:

$$\varnothing, \quad +, \quad + \sqcup - \sqcup -, \quad +^n \sqcup -^m.$$

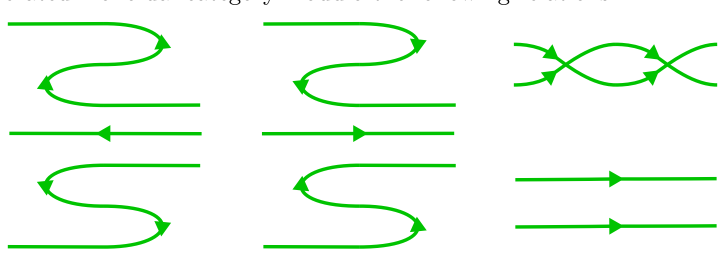
The morphisms are (oriented) 1-manifolds with these points as boundaries. For example,



In fact, by the classification of 1-manifolds, one can show that any morphism in $\mathbf{Cob}_1$ is obtained by finitely many compositions and disjoint unions of the following *generators* (and identity maps):



The last generator corresponds to the natural transformation $x \otimes y \mapsto y \otimes x$. In fact, $\mathbf{Cob}_1$ is the freely generated monoidal category modulo the following relations:



The last relation is exactly what makes $\mathbf{Cob}_1$ a *symmetric* monoidal category; it states that $B_{+,+} \circ B_{+,+} = \mathrm{id}_{+\sqcup+}$. (No standard terminology exists), we call the first four relations the *wave relations*, and the last the *symmetry relation*.

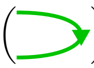
Definition 2.3. A *symmetric monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories which preserves the product and the symmetry. More precisely, the following diagram commutes:

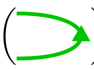
$$\begin{tikzcd}
F(x \otimes y) \arrow[r] \arrow[d] & F(y \otimes x) \arrow[d] \\
F(x) \otimes F(y) \arrow[r] & F(y) \otimes F(x)
\end{tikzcd}$$

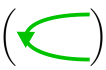
Example. We construct a symmetric monoidal functor $Z : \mathbf{Cob}_1 \to \mathbf{Vect}_k$. It suffices to establish the following data:

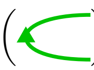
- $Z(+) = V \in \mathbf{Vect}_k$.
- $Z(-) = W \in \mathbf{Vect}_k$.
- $Z(\varphi)$ for each generator φ.

From the wave relations and functoriality, we can conclude that $Z(-)$ is necessarily equal to $Z(+)^*$. Moreover, the same derivation shows that V is finite dimensional, and moreover shows the following:

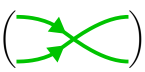
Z () $: V \otimes V^* \to k, \quad v \otimes \varphi \mapsto \varphi(v)$

Z () $: V^* \otimes V \to k, \quad \varphi \otimes v \mapsto \varphi(v)$

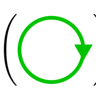
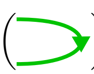
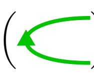
Z () $: k \to V^* \otimes V, \quad \lambda \mapsto \lambda \sum e_i^* \otimes e_i$

Z () $: k \to V \otimes V^*, \quad \lambda \mapsto \lambda \sum e_i \otimes e_i^*.$

Finally the fact that Z is a *symmetric* monoidal functor forces the following:

Z () $: V \otimes V \to V \otimes V, \quad (v, w) \mapsto (w, v).$

In summary, any symmetric monoidal functor $Z : \mathbf{Cob}_1 \to \mathbf{Vect}_k$ is completely determined by $Z(+)$. Since we didn't actually compute anything and just made claims, as an example we now compute the image of a circle under Z:

Z () $= Z$ () $\circ\, Z$ ()

$$= \lambda \mapsto \lambda \sum e_i \otimes e_i^* \mapsto \lambda \sum 1 = (\dim V)\lambda.$$

But finite dimensional vector spaces over a field k are determined up to isomorphism by dimension, so we have the following correspondences:

$$\{\text{symmetric monoidal functors } Z : \mathbf{Cob}_1 \to \mathbf{Vect}_k\} \longleftrightarrow \mathbf{Vect}_k$$

$$\{\text{symmetric monoidal functors } Z : \mathbf{Cob}_1 \to \mathbf{Vect}_k \text{ up to natural isomorphism}\} \longleftrightarrow \mathbb{N}.$$

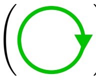
3 Topological quantum field theory

Definition 3.1. An n-dimensional *topological quantum field theory* is a symmetric monoidal functor

$$Z : \mathbf{Cob}_n \to \mathbf{Vect}_k,$$

for some fixed $n \in \mathbb{N}$ and field k.

Theorem 3.2. *Up to natural isomorphism, one dimensional topological quantum field theories are in bijective correspondence with* $\mathbb{N}$. *The correspondence is given by*

$Z \longmapsto Z\Big($  $\Big).$

Why is this a suitable definition?

1. The functor Z sends each time slice to a space of states; i.e. a vector space.
2. Z sends a cobordism (M, V_0, V_1) to a linear map $Z(M) : \mathcal{H}_{V_0} \to \mathcal{H}_{V_1}$. (In the earlier motivation, our maps were from $\mathcal{H}_{V_0} \to \mathcal{H}^*_{V_1}$, but this is a cleaner construction e.g. for functoriality.)
3. Since Z is a symmetric monoidal functor, it indeed sends $Z(V \sqcup V') = Z(V) \otimes Z(V')$.
4. By functoriality, $Z(M) = \mathrm{id}$ whenever M is a cylinder (trivial cobordism).
5. By functoriality, $Z(M_0 M_1) = Z(M_1) \circ Z(M_0)$, verifying the *sewing law*.

This shows that our definition of a topological quantum field theory satisfies all five of the essential properties of path integrals outlined at the start.

Although we've established that 1-dimensional TQFTs are in bijective correspondence with finite dimensional vector spaces, what we really want is a correspondence that preserves more of the structure of the collection of all TQFTs. Since a TQFT is a functor, we can canonically turn the collection of TQFTs into a category by assigning the morphisms to be natural transformations. In fact, one can show that every natural transformation is an isomorphism, so the set of TQFTs forms a groupoid.

Theorem 3.3. *There is an equivalence of groupoids*

$$\{\mathit{TQFTs}\ \mathbf{Cob}_1 \to \mathbf{Vect}_k\} \longleftrightarrow \mathcal{DP}_k$$

$\mathcal{DP}_k$ is the category of "dual pairs" over k:

- The objects are "dual pairs" (U, V, b, d), where $b : k \to U \otimes V$, $d : V \otimes U \to k$ are maps satisfying the *Zorro moves*

  $$\mathrm{id}_V = (\mathrm{id}_V \otimes b) \circ (d \otimes \mathrm{id}_V), \quad \mathrm{id}_U = (b \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes d).$$

  The idea is that this is the simplest way of saying that U and V are dual to one another, without giving a preference to either U or V, and without referencing dual spaces. The maps b (birth) and d (death) are also dual to each other.

- The morphisms are

  $$(f, g) : (U, V, b, d) \to (U', V', b', d')$$

  such that

  $$f : U \to U', \quad g : V \to V'; \quad d = d' \circ (g \otimes f), \quad (f \otimes g) \circ b = b'.$$

  The idea is that if f maps from X to Y, then g maps from X^* to Y^*. It almost feels like g is going backwards - but the point is that the dual of f maps in the opposite direction so by declaring g to map "backwards" the category of dual pairs forms a groupoid.

What about TQFTs from higher dimensional categories of cobordisms? In dimension 2 we have a simple classification of manifolds, so with some luck we'll be able to classify TQFTs in terms of some algebraic category. In dimension 3 we also have a classification of manifolds because the geometrisation conjecture is true, but it turns out that knot theory comes into play. Since we don't know what all of the knots are, we can't classify 3 dimensional TQFTs. In dimensions at least 4, it's even worse because we can't classify 4-manifolds, let alone having to keep track of knotted submanifolds.

Theorem 3.4. *There is an equivalence of groupoids*

$$\{\mathit{TQFTs}\ \mathbf{Cob}_2 \to \mathbf{Vect}_k\} \longleftrightarrow \mathbf{comFrob}_k$$

Definition 3.5. A *Frobenius algebra* is an algebra A over a field equipped with a non-degenerate bilinear form

$$\sigma : A \times A \to k, \quad \sigma(ab, c) = \sigma(a, bc).$$

- $\mathrm{Mat}_{n\times n}$ equipped with $\sigma(A, B) = \mathrm{tr}(AB)$.
- $k[G]$ equipped with $\sigma(a, b) =$ coefficient of e in ab.

A categorical definition is as follows:

Definition 3.6. A *Frobenius algebra* is a vector space A equipped with four morphisms $\mu : A \otimes A \to A, \eta : k \to A, \delta : A \to A \otimes A$, and $\varepsilon : A \to k$, such that (A, μ, η) is a monoid, (A, δ, ε) is a comonoid, and the "Frobenius conditions" are satisfied by δ and μ:

$$\delta \circ \mu = (\mathrm{id}_A \otimes \mu) \circ (\delta \otimes \mathrm{id}_A) = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \delta).$$

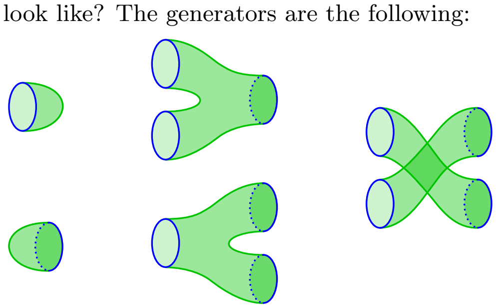
(A, μ, η) is a *monoid* if $\mu : A \otimes A \to A$ corresponds to "multiplication" and $\mu : 1 \to A$ gives the "unit" in A. Therefore A really corresponds to a monoid in the classical sense. Since A is a vector space, this is equivalent to requiring that (A, μ, η) is a unital algebra over k. A *comonoid* is the formal dual of a monoid.

Definition 3.7. A *morphism of Frobenius algebras* is a morphism which simultaneously preserves both the monoid and comonoid structures.

Translating this back to the more intuitive definition of a Frobenius algebra, a morphism is a k-algebra isomorphism between Frobenius algebras which preserves the Frobenius form. (This is as opposed to a k-algebra *homomorphism* preserving the Frobenius form, as one might expect.) This means that the category of Frobenius algebras truly forms a groupoid as opposed to just a category.

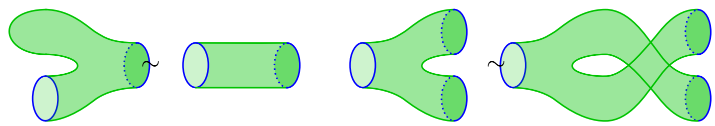
Recalling that we claimed that the groupoid of two dimensional TQFTs are equivalent to the groupoid of commutative Frobenius algebras, we have the following hierarchy: A commutative Frobenius algebra is a Frobenius algebras whose underlying algebra structure is commutative. On the other hand, every Frobenius algebra is a vector space and so on.

So what does $\mathbf{Cob}_2$ look like? The generators are the following:



This again follows from the classification of one manifolds and two manifolds. The only closed one manifolds are finite disjoint unions of circles. Compact 2-manifolds are classified by genus and boundary component. For example, a famous fact is that every closed surface with genus at least 2 admits a "pants decomposition".

What are some of the relations? We have the following.



We now attempt to understand symmetric monoidal functors $\mathbf{Cob}_2 \to \mathbf{Vect}_k$. Suppose $Z(\mathbb{S}^1) = A$. Then $\mathbb{Z}(\mathbb{S}^1 \sqcup \mathbb{S}^1) = A \otimes A$, so the pairs of pants correspond to maps from $A \otimes A \to A$ and $A \to A \otimes A$ respectively. Similarly the "caps" are maps from $k \to A$ and $A \to k$ respectively. Again the "X" cobordism maps to the braiding.

To prove that the TQFT corresponds to a Frobenius algebra, it remains to show that the maps obtained from these pants are exactly the multiplication, unit, comultiplication, and counit maps in the definition of a Frobenius algebra. We won't prove this, but we'll observe the meaning of the previous two relations.

The first one says that composing $k \to A$ (on the left) with $A \otimes A \to A$ gives the identity map. This is exactly saying that $k \to A$ is a left unit. Since the analogous relation holds (with the bottom left circle capped), $k \to A$ is genuinely the unit.

What about the relation involving the cross? it exactly says that comultiplication is commutative, so this is the relation (together with its dual) that guarantees that our Frobenius algebra is a commutative Frobenius algebra.

4 Applications

Topological quantum field theories $n\mathbf{Cob} \to \mathbf{Vect}_k$ are a starting point for other "functorial QFTs".

Additional structure on M	Corresponding FQFT
Conformal	Conformal field theory
pseudo-Riemannian	Relativistic QFT
Submanifolds	Defect TQFT
Spin	Spin TQFT
Framing	Framed TQFT

"Conformal field theory" is a very famous theory. The idea is that every cobordism is equipped with a conformal structure, i.e. an equivalence class of metrics with the same angles. It turns out that conformal field theory is richest in two dimensions. (In this case the conformal group has the most interesting representations.) Two major applications of 2-dimensional conformal field theory are in condensed matter physics and string theory. The former is because thermodynamic critical points in condensed matter systems are often conformally invariant, and conformal field theory can explain some critical phenomena. The latter is because 2-d CFTs are literally used as building blocks of string theory. You might recall that in string theory a big idea is to add dimensions but make them small via compactifications. These compactifications are exactly chosen to be conformal - the underlying geometry in string theory is conformal geometry.

The setting of general relativity is to have an ambient "space-time" manifold, which is a smooth manifold equipped with a pseud-Riemannian metric. By changing our domain from n**Cob** to n-cobordisms equipped with psuedo-Riemannian metrics, we're representing space-time in the relativistic sense.

What about defect TQFT? The idea is that to realistically model the universe we might require singularities. Zero dimensional singularities are often called *monopoles* and codimension one singularities are called *domain walls*. These singularities can be any dimension less than that of the ambient space, so they're exactly the *branes* that Sarah introduced us to in the first string theory talk.

That's the end of this aside: back to TQFT. We'll look at two genuinely applicable TQFTs.

- 3d TQFT: Chern-Simons theory

- 4d TQFT: Topological Yang-Mills theory.

Example. Chern-Simons theory is a *Schwarz-type* topological quantum field theory.

While TQFTs can be *defined* in terms of functors, sophisticated TQFTs are generally constructed using Lagrangians and path integrals with something physical in mind. A *Schwarz-type* TQFT is a theory which considers a manifestly metric-independent functional. The resulting path integral is certainly "topological" in the sense that it can only detect topological properties since any space-time metric is completely irrelevant.

Specifically in our case, the corresponding path integral is determined by the *action*

$$S[A] = \frac{k}{4\pi}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A).$$

The initial data that we need is a choice of 3-manifold M, equipped with a principal G-bundle $P \to M$. G is called the *Gauge group* of M. (Chern-Simons theory is also a Gauge theory, but unfortunately I don't even know the definition of a Gauge theory.) In the original notion of a path integral we integrated over field configurations. Here A takes the role of a field configuration, so A should be a section of some bundle. We take A to be a Lie-algebra valued connection 1-form $A \in \Omega^1(M, \mathfrak{g})$. Then the integrand is automatically a 3-form, so we don't even need a measure to integrate it! It's clear that the action doesn't depend on any metric, so this gives a topological quantum field theory.

(A *Lie algebra connection 1-form* is a 1-form in the sense that it's a section of the tensor product bundle of the cotangent bundle and principal G-bundle. To be a *connection form* it needs to satisfy a couple of additional properties like G-equivariance.)

Can we formalise this as a functorial QFT? For the case where G is an abelian Lie group, this has been done by Freed, Hopkins, Lurie, and Teleman. The general case isn't completely worked out.

What are some applications of Chern-Simons theory?

For example, consider a link L in a 3-manifold M. Suppose the group G defining the Chern-Simons theory is $U(n)$. We can define a "path integral" for (M, L) by

$$A(M, L) = \int_{\Omega^1(M,\mathfrak{g})} e^{iS[A]} \prod \chi_{L_i}(A)\, dA.$$

The additional terms $\chi_{L_i}(A)$ are all *observables* of the Chern-Simons theory, so they are functionals from the Hilbert space to $\mathbb{C}$. Precisely $\chi_{L_i}(A)$ is the holonomy of A around the ith component of the link L. It turns out that when M is the 3-sphere, these integrals literally calculate the HOMFLY polynomial of the link L. In the special case of $n = 2$, they give the Jones polynomial, and if $G = SO(n)$, one obtains the Kauffman polynomial.

This means that a special case of Chern-Simons theory gives a generalisation of the HOMFLY polynomial! We can extend the definition of all of these knot polynomials for links embedded in any 3-manifold M. This is an example of why the elevator pitch for TQFT is that it's a "quantum field theory that computes topological invariants".

5 References

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Topological quantum field theories from compact Lie groups Dan Freed, Michael Hopkins, Jacob Lurie, Constantin Teleman,

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