Section notes 9th Feb

You find a series $\sum_{n=1}^{\infty} a_n$ in the wild! what do you do?

1. Write the first few terms as
$$a_1 + a_2 + a_3 + a_4 + \cdots$$

2. Compute the first few partial sums
$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$

3. Does the sequence of partial sums look familiar?

★ Does it look like the terms aren't converging?

<u>Divergence test</u>

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

★ Does it look like a p-series?

<u>p-Series test</u>

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

★ Does it look like a geometric series?

<u>Geometric series test</u>

$\sum_{n=1}^{\infty} ar^n$ converges if and only if $|r| < 1$.

In this case, the sum is given by

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$$

For each of the following series,

1. Write it as $a_1 + a_2 + a_3 + \cdots$
2. Compute the first few partial sums
3. Does it look familiar? Can you write a formula for S_n? Does the series look like it will converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3^n}$

(a) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

$\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$, so by the divergence test, $\sum_{n=1}^{\infty} \frac{1}{2}$ diverges.

(b) $\ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \cdots$

$= \ln\left(\frac{\cancel{2}\cdot\cancel{3}\cdot 4\cdot\cdots}{1\cdot\cancel{2}\cdot\cancel{3}\cdots}\right)$

Therefore the partial sums are given by

$$S_1 = \ln(2)$$
$$S_2 = \ln\left(\frac{2\cdot 3}{1\cdot 2}\right) = \ln(3)$$
$$S_3 = \ln\left(\frac{2\cdot 3\cdot 4}{1\cdot 2\cdot 3}\right) = \ln(4)$$
$$\vdots$$
$$S_n = \ln(n+1).$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty.$

(c) $2^{2n} = 4^n$, so

$\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$, diverges by geometric series test.

Re-indexing and related issues.

1. Super important theorem!
A series converges if and only if its tail does.

$\sum_{n=1}^{\infty} a_n$ converges if and only if for some N, $\sum_{n=N}^{\infty} a_n$ converges.

Which of the following converge?

$\boxed{\sum_{n=1}^{\infty} \frac{3}{n^3}}$ ✓ $\qquad \boxed{\sum_{n=15}^{\infty} \frac{3}{n^3}}$ ✓ $\qquad \sum_{n=1000}^{\infty} \frac{1}{n}$:(no

Sometimes when we encounter a series, it'll look weird, like this:

$$\sum_{n=5}^{\infty} \frac{3^{n+2}(n+1)}{4^n + 3^{\frac{n}{2}}}$$

What do we do?

1. Use algebra to make all of the "n" terms as similar as possible

2. Re-index to simplify the problem

$$\sum_{n=i}^{\infty} a_{n+j} = \sum_{n=i+j}^{\infty} a_n$$

$$\parallel \qquad\qquad \parallel$$

$$a_{i+j} + a_{i+j+1} + a_{i+j+2} + \cdots$$

* If needed!

e.g.

$$\sum_{n=10}^{\infty} \frac{1}{n-4} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots = \sum_{n=6}^{\infty} \frac{1}{n}.$$

3. Use the "tail theorem" to test for convergence. * If needed!

e.g.

$$\sum_{n=5}^{\infty} \frac{3^{n+2}(n+1)}{4^n + 9^{\frac{n}{2}}}$$

1. Make all the "n bits" look the same:

$3^{n+2} = 3 \cdot 3^{n+1}$ $\quad 4^n = \frac{1}{4} 4^{n+1}$, etc

$$\rightsquigarrow \sum_{n=5}^{\infty} \frac{3 \cdot 3^{n+1}(n+1)}{\frac{1}{4} 4^{n+1} + \frac{1}{3} 3^{n+1}}$$

2. Re-index

$$\sum_{n=6}^{\infty} \frac{3n \, 3^n}{\frac{1}{4}4^n + \frac{1}{3}3^n}$$

Now it's all just n terms! Maybe we can compare it to a geometric series, etc.

In breakout rooms:

Which of the following series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{2n+1} = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{1}{25}\right)^n$, converges - geometric series test

(b) $\displaystyle\sum_{n=30}^{\infty} \frac{1}{\sqrt{n-2}\,(n-2)} = \sum_{n=28}^{\infty} \frac{1}{n^{3/2}}$, converges - p test

(c) $\displaystyle\sum_{n=5}^{\infty} (-1)^n \frac{3^{n+1}(-5)^{n+3}}{2^{n+2}\, 7^{n-1}} = \sum_{n=5}^{\infty} \frac{3(-5)^3 7}{4}\left(\frac{15}{14}\right)^n$,

diverges - geometric series test

Some more general convergence tests:

Integral comparison test

Suppose $f(x)$ is
- continuous
- decreasing
- positive

and that $a_n = f(n)$.

Then
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \text{for some } b, \int_b^{\infty} f(x)\,dx \text{ converges.}$$

* The "b" corresponds to sums of tails, $\sum_{n=N}^{\infty} a_n$.

Direct comparison test

If $0 \leq a_n \leq b_n$, then if

$\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Examples

(a) $$\sum_{n=4}^{\infty} \frac{3^n}{5^n+4}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{(n-100.1)^2}$$

(a) $\frac{3^n}{5^n+4} \le \frac{3^n}{5^n}$ for all n,
but $\sum_{n=4}^{\infty} \left(\frac{3}{5}\right)^n$ converges
by the geometric series test. Therefore by direct comparison,
$$\sum_{n=4}^{\infty} \frac{3^n}{5^n+4} \text{ converges.}$$

(b) Consider $f(x) = \frac{1}{(x-100.1)^2}$.
This is:

- Continuous for $x > 200$
- Decreasing for $x > 200$
- Positive
- $a_n = f(n)$ for $n \ge 200$.

Moreover, by the p-test,
$\int_{200}^{\infty} \frac{1}{(x-100.1)^2}\,dx$ converges. By integral comparison, so does $\sum \frac{1}{(n-100.1)^2}$.

In breakout rooms:
Determine which of the following converge

You might need to use more than one tool!

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^3-n+1}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{5}\right)^n$$

(c) $$\sum_{n=8}^{\infty} \frac{n(1+\cos(n))}{n^3+2}$$

See next page.

(a) $\dfrac{1}{n^3-n+1} < \dfrac{1}{n^3-n}$.

For $n \geq 2$, we have $\dfrac{n^3}{2} \geq n$.

Thus $\dfrac{1}{n^3-n} < \dfrac{1}{n^3-\left(\frac{n^3}{2}\right)}$.

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-\left(\frac{n^3}{2}\right)} = \sum_{n=1}^{\infty} \frac{2}{n^3}$ converges by the p test.

Now by direct comparison,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-n+1}$ converges.

(b) $\dfrac{1}{n}\left(\dfrac{1}{5}\right)^n < \left(\dfrac{1}{5}\right)^n$.

But $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ converges by the geometric series test.

Therefore $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{5}\right)^n$ converges by direct comparison.

(c) $\dfrac{n(1+\cos(n))}{n^3+2} \leq \dfrac{2n}{n^3+2} \leq \dfrac{2n}{n^3} = \dfrac{2}{n^2}$

But $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges by the p-test.

Therefore by direct comparison, so does

$\displaystyle\sum_{n=1}^{\infty}\frac{n(1+\cos(n))}{n^3+2}$.