(1) Differentiating, integrating, multiplying series.

Differentiating power series

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n (x-a)^n\right) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}.$$

Integrating power series

$$\int\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) dx = \sum_{n=0}^{\infty} \int a_n (x-a)^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} + C$$

Multiplying power series

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n (x-a)^n\right) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

Why is this true? We can write the product in a table:

	a_0 +	$a_1 x$ +	$a_2 x^2$ +	$a_3 x^3$ + ...
b_0 +	$a_0 b_0$	$a_1 b_0 x$	$a_2 b_0 x^2$	$a_3 b_0 x^3$
$b_1 x$ +	$a_0 b_1 x$	$a_1 b_1 x^2$	$a_2 b_1 x^3$	$a_3 b_1 x^4$
$b_2 x^2$ +	$a_0 b_2 x^2$	$a_1 b_2 x^3$	$a_2 b_2 x^4$	
$b_3 x^3$ + $\vdots$	$a_0 b_3 x^3$	$a_1 b_3 x^4$		$\ddots$

The product $\left(\sum a_n x^n\right)\left(\sum b_n x^n\right)$ is the sum of all the orange terms above. If we look at the diagonals, these have constant degrees! For example, the coefficient of x^2 is

$a_0 b_2 + a_1 b_1 + a_2 b_0$,

which is exactly as the theorem says.

① Examples:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n = \begin{cases} 3 & n=0 \\ 5 & n=1 \\ 8 & n=2 \\ 0 & n>2 \end{cases}$$

$$g(x) = \sum_{n=0}^{\infty} 3(x+1)^n$$

$$h(x) = \sum_{n=0}^{\infty} \frac{4}{n!}(x+1)^n$$

a) Write the first 4 terms of each series,
i.e. $\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

b) Differentiate each series

c) Integrate each series

d) Suppose $g(x)h(x) = \sum_{n=0}^{\infty} d_n (x+1)^n$.
what is d_0?
what is d_1?

(e) Suppose $f(x)\, g(x) = \sum_{n=0}^{\infty} c_n (x+1)^n$
What is c_0?
What is c_4?

a) $f(x) = 3 + 5x + 8x^2$

$g(x) = 3 + 3(x+1) + 3(x+1)^2 + 3(x+1)^3 + \cdots$

$h(x) = 4 + 4(x+1) + 2(x+1)^2 + \frac{2}{3}(x+1)^3 + \cdots$

b) $f'(x) = 5 + 16x$

$\left(= \sum_{n=0}^{\infty} n a_n x^{n-1} \right)$

$g'(x) = \sum_{n=0}^{\infty} 3n(x+1)^{n-1}$

$h'(x) = \sum_{n=0}^{\infty} \frac{4n}{n!} (x+1)^{n-1}$

c) $\int f(x)\,dx = 3x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + C$

etc.

d) Let

$g(x)h(x) = \left(\sum_{n=0}^{\infty} 3(x+1)^n\right)\left(\sum_{n=0}^{\infty} \frac{4}{n!}(x+1)^n\right)$

$= \sum_{n=0}^{\infty} d_n (x+1)^n$

Then from the general formula, we know that

$d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

where $a_n = 3$

$b_n = \frac{4}{n!}$

Therefore $d_0 = a_0 b_0 = 3 \cdot \frac{4}{1} = 12$

$d_1 = a_0 b_1 + a_1 b_0 = 3 \cdot \frac{4}{1} + 3 \cdot \frac{4}{1} = 24.$

e) We want to write $f(x)g(x)$ as a power series centered on $x = -1$, but $f(x)$ is written as

$f(x) = \sum_{n=0}^{\infty} a_n x^n.$

This is bad! We need to rewrite this in terms of $(x+1)$ to use the multiplication formula.

We have $f(x) = 3 + 5x + 8x^2$.

We replace every x with $((x+1)-1)$ to rewrite the function in terms of $x+1$:

$$
\begin{aligned}
f(x) &= 3 + 5((x+1)-1) + 8((x+1)-1)^2 \\
&= 3 + 5(x+1) - 5 + 8((x+1)^2 - 2(x+1) + 1) \\
&= 3 + 5(x+1) - 5 + 8(x+1)^2 - 16(x+1) + 8 \\
&= 6 - 11(x+1) + 8(x+1)^2.
\end{aligned}
$$

Thus $f(x) = \sum_{n=0}^{\infty} a_n'(x+1)^n$ where $a_n' = \begin{cases} 6 & n=0 \\ -11 & n=1 \\ 8 & n=2 \\ 0 & n>2. \end{cases}$

Now that f is in terms of $x+1$, we can use the product formula:

$$
\begin{aligned}
c_0 &= a_0' \cdot b_0 \\
&= 6 \cdot 3 = 18
\end{aligned}
$$

$$
\begin{aligned}
c_4 &= a_0' \cdot b_4 + a_1' \cdot b_3 + a_2' \cdot b_2 + a_3' \cdot b_1 + a_4' \cdot b_0 \\
&= 6 \cdot 3 + (-11) \cdot 3 + 8 \cdot 3 + 0 + 0 \\
&= 9.
\end{aligned}
$$

② Using the geometric series formula to understand power series.

We know: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r|<1$.

This works for <u>literally any</u> a and r, <u>provided</u> $|r|<1$, and it's in the form $\frac{a}{1-r}$.
(the "1 −" on the bottom left is the key.)

② Examples:

a) Write the following as series. What are their intervals of convergence?

i. $\frac{1}{1+x}$ ii. $\frac{1}{1-3x^2}$ iii. $\frac{x^2}{2-x^2}$

b) Rewrite the following power series so that they're centered on $x=\frac{1}{3}$.
(Hint: First convert to $\frac{a}{1-r}$ form)

i. $\sum_{n=0}^{\infty} 2x^n$ ii. $\sum_{n=0}^{\infty} 2^{-n}\left(x+\frac{1}{3}\right)^n$

a) When writing $\frac{a}{1-f(x)}$ as $\sum a(f(x))^n$, the radius of convergence is exactly the values of x that guarantee that $|f(x)|<1$.

Therefore we have:

i. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$

Converges $\Leftrightarrow$ $|x|<1$,
so interval of convergence is $(-1, 1)$.

ii. $\dfrac{1}{1-3x^2} = \sum_{n=0}^{\infty} (3x^2)^n$

interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

iii. $\dfrac{x^2}{2-x^2} = \dfrac{x^2/2}{1-x^2/2}$ So that it's in the right form.

Now, this is $\sum_{n=0}^{\infty} \left(\frac{x^2}{2}\right)\left(\frac{x^2}{2}\right)^n$

$$= \sum_{n=0}^{\infty} \left(\frac{x^2}{2}\right)^{n+1}$$

interval of convergence is $(-\sqrt{2}, \sqrt{2})$.

(b) $\sum_{n=0}^{\infty} 2x^n = \dfrac{2}{1-x}$ for $|x|<1$.

We want to write a power series in terms of $(x-\frac{1}{3})$. Therefore we replace x with $((x-\frac{1}{3})+\frac{1}{3})$. This gives:

$$\frac{2}{1-x} = \frac{2}{1-((x-\frac{1}{3})+\frac{1}{3})} = \frac{2}{\frac{2}{3}-(x-\frac{1}{3})}.$$

Next, we multiply everything by $\frac{3}{2}$ to get the fraction in the form $\dfrac{a}{1-r}$.

$$\frac{2}{\frac{2}{3}-(x-\frac{1}{3})} = \frac{3}{1-\left(\frac{3}{2}(x-\frac{1}{3})\right)}.$$

Now for $\left|\frac{3}{2}(x-\frac{1}{3})\right| < 1$, this is

$$\sum_{n=0}^{\infty} 3\left(\frac{3}{2}(x-\frac{1}{3})\right)^n.$$

$$\sum_{n=0}^{\infty} 2^{-n}(x+\tfrac{1}{3})^n = \sum_{n=0}^{\infty}\left(\frac{x+\frac{1}{3}}{2}\right)^n.$$

For $\left|\frac{x+\frac{1}{3}}{2}\right| < 1$, this is

$$\frac{1}{1-\frac{x+\frac{1}{3}}{2}}.$$

To write this in terms of $x-\frac{1}{3}$, we use

$$x+\tfrac{1}{3} = (x-\tfrac{1}{3})+\tfrac{2}{3}.$$

Then

$$\frac{1}{1-\frac{x+\frac{1}{3}}{2}} = \frac{1}{1-\frac{(x-\frac{1}{3})+\frac{2}{3}}{2}}$$

$$= \frac{1}{1-\left(\frac{x-\frac{1}{3}}{2}\right)-\frac{1}{3}}$$

$$= \frac{1}{\frac{2}{3}-\left(\frac{x-\frac{1}{3}}{2}\right)}$$

Multiplying everything by $3/2$ to remove the $2/3$,

$$\frac{1}{\frac{2}{3} - \left(\frac{x-\frac{1}{3}}{2}\right)} = \frac{3/2}{1 - \frac{3}{4}\left(x-\frac{1}{3}\right)}$$

$$= \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{3}{4}\left(x-\frac{1}{3}\right)\right)^n,$$

for $\left|\frac{3}{4}\left(x-\frac{1}{3}\right)\right| < 1.$