1. Taylor series and remainders

Defn The Taylor series of $f(x)$ centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Thm Let $I = (a-r, a+r)$ be an interval on which the Taylor series converges. Then for $x \in I$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Examples: Find the Taylor series of $\sin(x)$ about $a = \frac{\pi}{4}$?

Defn The degree N Taylor polynomial of $f(x)$ at a is

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

Defn The <u>remainder</u> is

$$R_N(x) = f(x) - P_N(x)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Intuitively, $P_N(x)$ is an "approximation" of $f(x)$ that we can compute, and $R_N(x)$ is the error in our approximation.

Thm Taylor's remainder theorem.

Suppose I is an interval containing "b" and "a".

We want to estimate how bad the error

$$R_N(b) = f(b) - P_N(b)$$

$$= \sum_{n=N+1}^{\infty} \frac{f^{(N)}(a)}{n!}(x-a)^n$$

is, given our approximation of $f(b)$ with $P_N(b)$.

Suppose that $|f^{(N+1)}(x)| \leq M$ for all x in I.

Then $$|R_N(b)| \leq \frac{M}{(N+1)!}|b-a|^{N+1}.$$

Example

"Let $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

For what N is $P_N(2)$ within 10^{-4} of $f(2)$?"

Strategy:

1. Choose an interval I containing "a" and "2". (The smaller the easier.)

2. Find some number M (in terms of N) so that for $x \in I$,

$$|f^{(N+1)}(x)| \leq M.$$

3. Now we know that

$$|R_N(2)| \leq \frac{M}{(N+1)!}|2-a|^{N+1}$$

by Taylor's theorem. Therefore we want to find N so that

$$\frac{M}{(N+1)!}|2-a|^{N+1} \leq 10^{-4}.$$

This "N" is the answer.

Example: We want to estimate e^2 using Taylor polynomials of e^x.

How many terms do we need to ensure that the error is at most 10^{-3}?

Example: We want to estimate e^{-1} using Taylor polynomials. How many terms do we need to ensure error $< 10^{-3}$?

2. Finding Taylor series.

Thm Uniqueness of Taylor series.

If $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(x-a)^n$, then

this <u>is</u> the Taylor series of $f(x)$.

i.e. $a_n = f^{(n)}(a)$.

<u>Strategy</u>:

- We know Taylor series for common functions: $\sin(x)$, $\cos(x)$, $e(x)$, $\arctan(x)$, $\ln(1+x)$, $\frac{1}{1-x}$.

  Try to write the given function in terms of "common" functions.

  - might require trig identities, integration / differentiation,

- Once this has been done, write the common functions as Taylor series, and add / multiply etc as needed.

<u>Strategy 2</u>:

If this fails, write out terms using the definition of Taylor series.

Examples:

1. Find Taylor series of the following functions

(a) $\sin(x) + \cos(x)$ centered at $a = 0$

(b) $e^x \sin(x)$ centered at $a = 0$

(c) $\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)$ centered at $a = 0$

(d) $\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)$ centered at $a = \frac{\pi}{4}$

2. Write the following as "function values"

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!}$

3. Find series converging to:

(a) $\arctan\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)$

(b) $\cos^2(3)$