(1) Comparison - type convergence tests

Direct comparison test:

If $0 \leq a_n \leq b_n$ for all $n > N$,
then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.
if $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

* Use when you have "extra terms" that are annoying
e.g. $\frac{3+\sin(n)}{n}$

Integral test:

If $f(x)$ is continuous, decreasing, and positive, and $a_n = f(n)$, then
$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_b^{\infty} f(x)\,dx$ converges, for some b.

* Use if things look like they might be integrable, e.g. $n \ln(n)$.

* Often Limit comp. test is just better!

Limit comparison test:

Suppose $0 \leq a_n$, and $0 < b_n$, for all $n > N$.
Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then:

$$L = \begin{cases} 0 & \sum_{n=1}^{\infty} a_n \text{ converges if } \sum_{n=1}^{\infty} b_n \text{ converges} \\ \infty & \sum_{n=1}^{\infty} a_n \text{ diverges if } \sum_{n=1}^{\infty} b_n \text{ diverges} \\ \text{non-zero number} & \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ \underline{both} converge or \underline{both} diverge.} \end{cases}$$

* Probably the most useful test!
Use when the terms "look like" geometric or p series.

Examples

Which of the following converge?

a) $\sum_{n=1}^{\infty} 2n e^{-n^2}$

b) $\sum_{n=1}^{\infty} \frac{2^n + \arctan(n)}{3^n}$

c) $\sum_{n=2}^{\infty} \frac{n}{(n-21.5)^3}$

(a) Integral test:

$f(x) = 2x e^{-x^2}$

is continuous, positive, and decreasing, for $x > 1$.

Moreover, $f(n) = 2n e^{-n^2}$, and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} 2x e^{-x^2}\,dx$$

$$= \left[-e^{-x^2}\right]_1^{\infty}$$

$$= e^{-1} \qquad \text{(this converges!)}$$

By the integral test, $\sum_{n=1}^{\infty} 2n e^{-n^2}$ converges.

(b) Comparison test: $\frac{2^n + \arctan(n)}{3^n}$ is a "geometric series with an extra bit". We can get rid of it like this:

$\arctan(n) \leq 2^n$ for all n.

$\therefore \sum_{n=1}^{\infty} \frac{2^n + \arctan(n)}{3^n}$ converges if $\sum_{n=1}^{\infty} \frac{2^n + 2^n}{3^n}$

$$= 2 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

converges.

This indeed converges by the geometric series test, so by direct comparison, $\sum_{n=1}^{\infty} \frac{2^n + \arctan(n)}{3^n}$ converges.

(c) $\sum_{n=1}^{\infty} \frac{n}{(n-21.5)^3}$.

This looks kinda like $\frac{n}{n^3} = \frac{1}{n^2}$.
Let's use limit comparison!

$$a_n = \frac{n}{(n-21.5)^3}$$

$$b_n = \frac{1}{n^2}.$$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Therefore $\sum_{n=1}^{\infty} \frac{n}{(n-21.5)^3}$ converges if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The latter converges by the p-test, so $\sum_{n=1}^{\infty} \frac{n}{(n-21.5)^3}$ converges.

(2) Series estimates from integrals

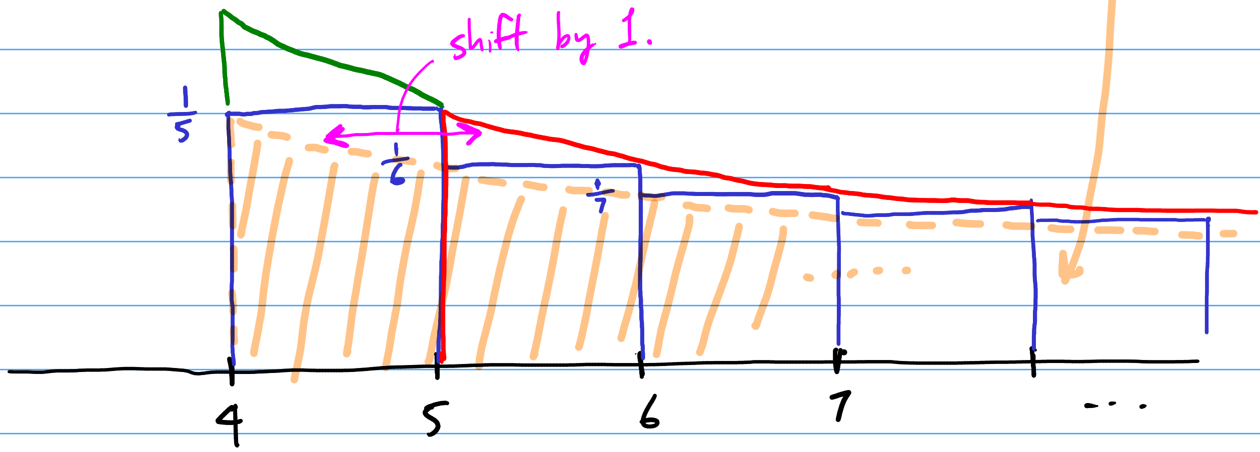
Let $a_n = \frac{1}{n}$, and $f(x) = \frac{1}{x}$.

Then $\sum_{n=5}^{\infty} a_n$ is the blue area

$\int_{n=5}^{\infty} f(x)\,dx$ is the red area

"Equal to this area"

$\int_{n=4}^{\infty} f(x)\,dx$ is the red + green area



Therefore

$$\int_{5}^{\infty} f(x)\,dx \le \sum_{n=5}^{\infty} a_n \le \int_{4}^{\infty} f(x)\,dx.$$

Error estimate:

The above works in general:

Let $f(x)$ be decreasing, positive, continuous, and $a_n = f(n)$. Then for any N,

$$\int_{N+1}^{\infty} f(x)\,dx \le \sum_{n=N+1}^{\infty} a_n \le \int_{N}^{\infty} f(x)\,dx$$

Called the error estimate because if we want to approximate $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{N} a_n$, $\sum_{n=N+1}^{\infty} a_n$ is left over.

Example

(A) List the following in ascending order.

(B) List the following in order of how good the estimate is.

We're estimating $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\boxed{\sum_{n=1}^{5} \frac{1}{n^2}}$$

$$\boxed{\sum_{n=1}^{100} \frac{1}{n^2} + \frac{\int_{100}^{\infty} \frac{1}{x^2}\,dx + \int_{101}^{\infty} \frac{1}{x^2}\,dx}{2}}$$

$$\boxed{\sum_{n=1}^{100} \frac{1}{n^2}}$$

$$\boxed{\int_{101}^{\infty} \frac{1}{x^2}\,dx}$$

$$\boxed{\sum_{n=1}^{100} \frac{1}{n^2} + \int_{100}^{\infty} \frac{1}{x^2}\,dx}$$

(A) ★ ★ ★ ★ ★

(B) ★ ★ ★ ★ ★

Try to think about these if you're not so sure!

Example question: "estimate $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{N} a_n$ to within 0.01".

Answer: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$.

We need $\sum_{n=N+1}^{\infty} a_n$ to be at most 0.01.

But $\sum_{n=N+1}^{\infty} a_n$ is at most $\int_{N}^{\infty} f(x)\,dx$.

Therefore we solve for N in

$$\int_{N}^{\infty} f(x)\,dx \le 0.01.$$

(3) Alternating series

Alternating series test:

Let $\sum a_n$ be a series such that it's

1. alternating, i.e. $a_n = (-1)^n b_n$ where $b_n \geq 0$
2. absolutely decreasing, i.e. $|a_n| = b_n$ is a decreasing sequence.

Then $\sum_{n=1}^{\infty} a_n$ converges.

Absolute convergence test:

If $\sum |a_n|$ converges, so does $\sum a_n$.

e.g. $\sum \frac{(-1)^n}{n}$ converges by the alternating series test.

However, it's <u>not</u> absolutely convergent.

<u>Example</u>

Some of the following series can be understood using the alternating series test, and some cannot. Figure out which ones are which!

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

b) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{-2}}$

c) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{2 + n\ln(n)}$

d) $\sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{3^n}$

e) $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{(-1)^n}{4^n}\right)$

f) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n}$

Which converge absolutely?

(a), (c), satisfy the premises of the alternating series test.

(a), (c) converge.

(b) fails because $\left|(-1)^n \frac{1}{n^{-2}}\right| = n^2$ is increasing.

(b) diverges.

(d) fails because it's not alternating, but every term is 0.

(d) converges to 0.

(e) fails because all terms are positive! However, converges by comparison test + geometric series test.

(f) fails because it isn't alternating! $\frac{\sin(n)}{n}$ might be positive or negative.

Diverges by the divergence test: $\lim_{n\to\infty} (-1)^n \frac{\sin(n)}{n}$ is not 1.

Next, which ones are absolutely convergent?

- Anything that already diverges, will diverge when we take absolute values.
  This leaves only a, c, d, e as candidates.
  - d and e are already positive! Taking absolute values does nothing, they already converge.
  - Only a and c remain to be checked.

a: absolute value gives

$$\sum_{n=1}^{\infty} \frac{1}{n!}.$$ This converges by comparison with $\sum \frac{2}{n^2}$.

(fun fact, $\sum \frac{1}{n!} = e$)

c: absolute value gives

$$\sum_{n=1}^{\infty} \frac{1}{2+n\ln(n)}.$$

Comparison test: compare to

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n\ln(n)},$$ because $\frac{1}{2}\frac{1}{n\ln(n)} \leq \frac{1}{2+n\ln(n)}$ for $n \geq 10$.

But now we use the integral test to study

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}.$$

Let $f(x) = \frac{1}{x\ln(x)}$.

$$\int_6^{\infty} \frac{1}{x\ln(x)}\,dx = \int_{e^6}^{\infty} \frac{1}{e^u u} e^u\,du$$

$$= \int_{e^6}^{\infty} \frac{1}{u}\,du$$

which <u>diverges</u>.

Therefore $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ <u>diverges</u> by integral test.

Therefore $\sum_{n=1}^{\infty} \frac{1}{2+n\ln(n)}$ <u>diverges</u> by direct comparison.