Seminar_HW

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Theorem. *Suppose X_1, X_2, ... are i.i.d with a distribution that satisfies*

(i) $lim_{x \longrightarrow \infty} P(X_1 > x)/P(|X_1| > x) = \theta \in [0,1]$

(ii) $P(|X_1| > x) = x^{-\alpha} L(x)$

where $\alpha < 2$ and L is slowly varying. Let $S_n = X_1 + ... + X_n$

$$a_n = inf\{x : P(|X_1| > x) \leq n^{-1}\} \text{ and } b_n = nE(X_1 1_{(|X_1| < a_n)})$$

$$As\, n \longrightarrow \infty,\ (S_n - b_n)/a_n \Rightarrow Y \text{ where } Y \text{ has a nondegenerate distribution.}$$

Remark. *This is not much of a generalization of the example, but the conditions are necessary for the existence of constants a_n and b_n so that $(S_n - b_n)/a_n \Rightarrow Y$ where Y is nondegenerate. Proofs of necessity can be found in Chapter 9 of Brieman (1968) or in Gnendenko and Kolmogorov (1954). (3.8.11) gives the ch.f. of Y. The reader has seen the main ideas in the second proof of (3.8.3) and so can skip to that point without much loss.*

Proof. It is not hard to see that (ii) implies

$$nP(|X_1| > a_n) \longrightarrow 1$$

To prove this, not that $nP(|X_1| > a_n) \leq 1$ and let $\epsilon > 0$. Taking $x = a_n/(1+\epsilon)$ and $t = 1 + 2\epsilon$, (ii) implies

$$(1+2\epsilon)^(-\alpha) = \lim_{n\to\infty} \frac{P(|X_1| > (1+2\epsilon)a_n\ (1+\epsilon))}{P(|X_1| > a_n/(1+\epsilon))} \leq \liminf_{x\to\infty} \frac{P(|X_1| > a_n)}{1/n}$$

proving (3.8.6) since ϵ is arbitrary. Combining (3.8.6) with (i) and (ii) gives

$$nP(X_1 > xa_n) \to \theta x^{-\alpha} \text{ for } x > 0$$

so $|\{m \le n : X_m > xa_n\}| \Rightarrow \text{Poisson}(\theta x^{-\alpha})$. The last result leads, as before, to the conclusion that $\mathbf{X}_n = \{X_m/a_n : 1 \le m \le n\}$ converges to a Poisson process on a $(-\infty, \infty)$ with mean measure

$$\mu(A) = \int_{A\bigcap(0,\infty)} \theta\alpha|x|^{-(\alpha+1)}dx + \int_{A\bigcap(-\infty,0)} (1-\theta)\alpha|x|^{-(\alpha+1)}dx$$

To sum up the points, let $I_n(\epsilon) = \{m \le n : |X_m| > \epsilon a_n\}$

$$\hat{\mu}(\epsilon) = EX_m 1_{(\epsilon a_n < |X_m| \le a_n)} \qquad \hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m$$

$$\overline{\mu}(\epsilon) = EX_m 1_{(|X_m| \le \epsilon a_n)}$$

$$\overline{S}_n(\epsilon) = (S_n - b_n) - (\hat{S}_n(\epsilon) - n\hat{\mu}(\epsilon)) = \sum_{m=1}^{n} \{X_m 1_{(|X_m| \le \epsilon a_n)} - \overline{\mu}(\epsilon)\}$$

If we let $\overline{X}_m(\epsilon) = X_m 1_{(|X_m| \le \epsilon a_n)}$ then

$$\begin{aligned} E(\overline{S}_n(\epsilon/a_n)^2 &= n\text{var}(\overline{X}_1(\epsilon)/a_n) \le nE(\overline{X}_1(\epsilon)/a_n)^2 \\ E(\overline{X}_1(\epsilon/a_n)^2 &\le \int_0^\epsilon 2yP(|X_1| > ya_n)dy \\ &= P(|X_1| > a_n) \int_0^\epsilon 2y \frac{P(|X_1| > ya_n)}{P(|X_1| > ya_n)} dy \end{aligned}$$

We would like to use (3.8.7) and (ii) to conclude

$$nE(\overline{X}_1(\epsilon)/a_n)^2 \to \int_0^\epsilon 2yy^{-\alpha}dy = \frac{2}{2-\alpha}\epsilon^{2-\alpha}$$

and hence

$$\limsup_{n\to\infty} E(\overline{S}_n(\epsilon/a_n)^2 \le \frac{2\epsilon^{2-\alpha}}{2-\alpha}$$

To justify interchanging the limit and the integral and complete the proof of (3.8.8), we show the following (take $\delta < 2 - \alpha$).

□