CAS 760

Logic for Practical Use

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Development Project: Continuity on Topological Partial Algebras

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Contents

1 Introduction

While-Computability is a fundamental model for imperative programming on any data type. Continuous data types are modeled by topological many-sorted algebras. The topology defines a concept of closeness and hence the process of approximation for data.

1.1 What is a Topological Partial Algebra?

Definition 1.1 *A Topological Partial Algebra* $\mathbb{A} = (A, (f_i)_{i \in I})$ *is a partial algebra with topologies on carriers such that each of the basic functions f_i is* continuous *on the product topology.*

Definition 1.2 *A function $f : X \to Y$ is continuous if the pre-image of every open set in Y is* open *in X.*

Definition 1.3 *a function f is open if $\forall V : Y \to o \cdot V \in$* ISOPEN *$\Rightarrow$* (PREIMAGE$(f))(V) \in$ *ISOPEN.*

Theory Definition 1.4 (Topological partial algebra)

   Name: TOP − PAR − ALG.

   Base types: I, A

   Constants: $(f_i : A^{\alpha_i} \to A^{\beta_i})_{i \in I}$, PREIMAGE$_{(A \to A) \to \{A\} \to \{A\}}$.

   Axioms:

1. $\emptyset_{\{A\}} \in$ ISOPEN   (Empty set is in the collection of open sets).

2. $U_{\{A\}} \in$ ISOPEN   (The base set itself is in the collection of open sets).

3. Any (finite or infinite) union of sets that are open is itself open

4. Any finite intersection of sets that are open is itself open

5. $\forall V : A \to o \cdot V \in$ ISOPEN $\Rightarrow$ PREIMAGE(V) $\in$ ISOPEN   (Continuity).

6. $\forall i : I$ PREIMAGE$(i, \alpha_i, \beta_i) \downarrow$

7. PREIMAGE $= \lambda i : I \lambda \alpha_i, \beta_i : N \cdot \ \lambda f : A^{\alpha_i} \to A^{\beta_i} \cdot$
   $\lambda V : A^{\beta_i} \to o \cdot I \ K : A^{\alpha_i} \to o \cdot$
   $\forall x \in K \ f(x) \downarrow \wedge \cup_{x \in K} f(x) = V$

Corollary 1.5 *Note that since ISOPEN is defined as a predicate, it is total, and hence is defined for any subset of the base set.*

References

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[2] Farmer, W, 'Simple Type Theory A Practical Logic for Expressing and Reasoning about Mathematical Ideas'

[3] J.V. Tucker, J.I. Zucker, 'Computation by 'While' programs on topological partial algebras', Theoretical Computer Science, Volume 219, Issues 1–2, 1999, Pages 379-420, ISSN 0304-3975, `https://doi.org/10.1016/S0304-3975(98)00297-7`