CAS 760

Logic for Practical Use

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Development Project: Continuity on Topological Partial Algebras

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Contents

1 Introduction

While-Computability is a fundamental model for imperative programming on any data type. Continuous data types are modeled by topological many-sorted algebras. The topology defines a concept of closeness and hence the process of approximation for data.

2 Topological algebra

Topological algebra is an algebra which is also a topological space at same time.

Definition 2.1 *A topological algebra over a topological field is a topological vector space together with a bilinear operation that is continuous in some definite sense.*

2.1 Topological algebra in Alonzo

Theory Definition 2.2 (Topological algebra)

Name: $\mathsf{TOP-ALG}$.

Base types: A

Constants: $\cup_{A\to A\to A}$, $\cap_{A\to A\to A}$.

Axioms:

1. $\emptyset_{\{A\}} \in \mathsf{ISOPEN}$ (Empty set is in the collection of open sets).
2. $U_{\{A\}} \in \mathsf{ISOPEN}$ (The base set itself is in the collection of open sets).
3. $\forall\, a_i : A\ .\ (a_i \in \mathsf{ISOPEN}) \Rightarrow (\bigcup a_i \in \mathsf{ISOPEN})$
4. $\forall\, a_i : A\ .\ ((a_i \in \mathsf{ISOPEN}) \wedge (i \neq \infty)) \Rightarrow (\bigcap a_i \in \mathsf{ISOPEN})$

3 What is a Topological Partial Algebra?

Definition 3.1 *A Topological Partial Algebra is a partial algebra with topologies on carriers such that each of the basic functions f is continuous on the product topology.*

Definition 3.2 *A function f is continuous if the pre-image of every open set in its range is open in its domain.*

Definition 3.3 *a function f is open if its domain is open implies the range of its pre-image is also open.*

3.1 Topological partial algebra in Alonzo

Theory Extension 3.4 (Topological partial algebra)

Name: TOP – PAR – ALG.

Extends TOP – ALG.

New base types: I

New constants: $(f_i : A^{\alpha_i} \to A^{\beta_i})_{i \in I}$,PREIMAGE$_{(A \to A) \to \{A\} \to \{A\}}$.

New axioms:

5. $\forall V : A \to o \,.\, V \in \mathsf{ISOPEN} \Rightarrow \mathsf{PREIMAGE(V)} \in \mathsf{ISOPEN}$ (Continuity).
6. $\forall i : I \,.\, \mathsf{PREIMAGE}(i, \alpha_i, \beta_i)\downarrow$
7. $\mathrm{PREIMAGE} = \lambda\, i : I \,.\, \lambda\, \alpha_i, \beta_i : N \,.\, \lambda\, f : A^{\alpha_i} \to A^{\beta_i} \,.\, \lambda\, V : A^{\beta_i} \to o \,.$ $\mathrm{I}\, K : A^{\alpha_i} \to o \,.\, \forall\, x : K \,.\, f(x)\downarrow \wedge \cup_{x \in K} f(x) = V$

Corollary 3.5 *Note that since ISOPEN is defined as a predicate, it is total, and hence is defined for any subset of the base set.*

References

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