WhileCC-approximability and Acceptability of Elementary Functions

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Computability of Functions on $\mathbb{R}$

For total functions on $\mathbb{R}$, the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, . . .

For partial functions on $\mathbb{R}$, Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

Problem

How general is this class of acceptable functions?

Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

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Definition (Acceptability)

A function $f : \mathbb{R} \to \mathbb{R}$ is **acceptable** if there exists a sequence X where:

1. X is an **effective open exhaustion** for $\mathbf{dom}(f)$, and
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2. for each $l \in \mathbb{N}$, U_l is a finite union of non-empty open finite intervals $I_1^l, I_2^l, ..., I_{k_l}^l$ whose closures are pairwise disjoint, and
3. for each $l \in \mathbb{N}$, $\overline{U_l} = \bigcup_{i=1}^{k_l} \overline{I_i^l} \subseteq U_{l+1}$.
4. for all l, the components I_i^l that are intervals building up the stage U_l, are *rational* and *ordered* i.e., $I_i^l = (a_i^l, b_i^l)$ for some $a_i^l, b_i^l \in \mathbb{Q}$ where $b_i^l < a_{i+1}^l$ for $i = 1, ..., k_l - 1$, and
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The sequence of open sets $(-1, 1), (-2, 2), \ldots, (-k, k), \ldots$ is the standard effective open exhaustion for $\mathbb{R}$.

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Definition ([Fu and Zucker, 2014])

A function f on U is **effectively locally uniformly continuous w.r.t. an effective open exhaustion** $(U_n)_{n\in\mathbb{N}}$ of U, if there is a recursive function $M : \mathbb{N}^2 \twoheadrightarrow \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

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Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

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by applying (repeatedly) the basic operations below on elementary functions f, g:

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The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.

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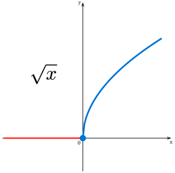
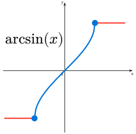
Problem

The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.





Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
- the variable x,

by applying (repeatedly) the basic operations below on elementary functions f, g:

- $(f+g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\mathrm{div}_f(x) = \frac{1}{f(x)}$ where $\frac{1}{0} = \uparrow$
- $\mathrm{root}_{n,f}(x) = \sqrt[n]{f(x)}$ where $0 < n \in \mathbb{N}$
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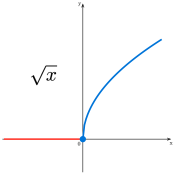

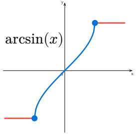

Background – Elementary Functions

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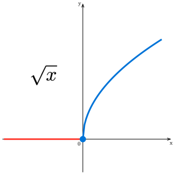

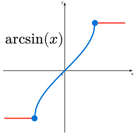

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- f is an α-computable function.
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- f is GL-computable w.r.t. X.
- f is effectively locally uniformly multipolynomially approximable w.r.t. X.

Theorem (WhileCC-approximability Theorem)
All elementary functions are **WhileCC**-approximable.

Theorem (Acceptability Theorem)
All elementary functions are acceptable.

WhileCC-programming Language:

- Variables from $\mathbb{R}$, $\mathbb{N}$, $\mathbb{B}$
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Theorem (Acceptability Theorem: Part 1)

The domain of any elementary function has an effective open exhaustion.

Theorem (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous w.r.t. an effective open exhaustion for its domain.

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Theorem (Acceptability Theorem: Part 1)

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function. Then, $\mathbf{dom}(f)$ has an effective open exhaustion.

First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓(e.g. $\sin(x)$)
- Addition and multiplication ✓(e.g. $(f+g)(x)$)
- Composition case has a counterexample:

$$f(x) = id|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \le x \le 1 \\ 1 & \text{otherwise} \end{cases}$$

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$\mathbf{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

Strengthened to proving exhaustion reflection property

For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

Adding decomposition of $+$ and $\cdot$

$(f+g)(x) = f(x) + g(x)$ is composed of

- $Add(x, y) = x + y$,
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For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

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Adding decomposition of $+$ and $\cdot$

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Theorem (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

Any elementary function has a local continuity witness.

Definition (Local continuity witness)

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

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Summary

We proved that:

- all elementary functions are **WhileCC**-approximable.
- all elementary functions are acceptable and hence computable in the other three models as well.

We also presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept.

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Questions left unanswered:

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  A generalization of acceptability in arbitrary metric spaces is given by Tucker and Zucker [2004].
- Can we extend the equivalence theorem in Fu and Zucker [2014] to acceptable partial functions of type $\mathbb{R}^m \to \mathbb{R}$?
- What functions are **WhileCC**-approximable but not While*-approximable [Tucker and Zucker, 1999]?

Conjecture:

- All partial unary **WhileCC**-approximable functions are acceptable.
  - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
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Ming Quan Fu and Jeffery Zucker. Models of computation for partial functions on the reals. *Journal of Logical and Algebraic Methods in Programming*, 84(2):218–237, 11 2014. ISSN 2352-2208. doi:10.1016/j.jlamp.2014.11.001.

M. Tenenbaum and H. Pollard. *Ordinary Differential Equations: An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences*. Dover Books on Mathematics. Dover Publications, 1985. ISBN 9780486649405. URL `https://books.google.ca/books?id=iU4zDAAAQBAJ`.

John V. Tucker and Jeffery I. Zucker. Abstract versus concrete computation on metric partial algebras. *ACM Trans. Comput. Log.*, 5(4):611–668, 2004. doi:10.1145/1024922.1024924.

J.V. Tucker and J.I. Zucker. Computation by 'While' programs on topological partial algebras. *Theoretical Computer Science*, 219(1):379–420, 1999. ISSN 0304-3975. doi:10.1016/S0304-3975(98)00297-7.

J.V. Tucker and J.I. Zucker. Computable total functions on metric algebras, universal algebraic specifications and dynamical systems. *The Journal of Logic and Algebraic Programming*, 62(1): 71–108, 2005. ISSN 1567-8326. doi:10.1016/j.jlap.2003.10.001.

A HUGE Thank you!