# **WhileCC**-approximability and Acceptability of Elementary Functions


Fateme Ghasemi
Supervisor: Dr. Jeffery Zucker


April 2025

# Computability of Functions on $\mathbb{R}$

**For total functions on** $\mathbb{R}$, the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, ...

**For partial functions on** $\mathbb{R}$, Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

**Problem**

How general is this class of acceptable functions?

**Useful First Step Towards Solution**

Show that the elementary functions satisfy the acceptability conditions.

# Background – Acceptability

**Definition (Acceptability)**

A function $f : \mathbb{R} \to \mathbb{R}$ is **acceptable** if there exists a sequence $X$ where:

1. $X$ is an **effective open exhaustion** for $\mathbf{dom}(f)$, and
2. $f$ is **effectively locally uniformly continuous w.r.t.** $X$.

# Background – Effective Open Exhaustions

### Definition ([Fu and Zucker, 2014])

A sequence $(U_1, U_2, \ldots)$ of open sets is called an **effective open exhaustion** for an open $U \subseteq \mathbb{R}$ if

1. $U = \bigcup_{l=0}^{\infty} U_l$, and
2. for each $l \in \mathbb{N}$, $U_l$ is a finite union of non-empty open finite intervals $I_1^l, I_2^l, ..., I_{k_l}^l$ whose closures are pairwise disjoint, and
3. for each $l \in \mathbb{N}$, $\overline{U_l} = \bigcup_{i=1}^{k_l} \overline{I_i^l} \subseteq U_{l+1}$.
4. for all $l$, the components $I_i^l$ that are intervals building up the stage $U_l$, are *rational* and *ordered* i.e., $I_i^l = (a_i^l, b_i^l)$ for some $a_i^l, b_i^l \in \mathbb{Q}$ where $b_i^l < a_{i+1}^l$ for $i = 1, ..., k_l - 1$, and
5. the map $l \mapsto (a_1^l, b_1^l, ..., a_{k_l}^l, b_{k_l}^l)$ which delivers the sequence of stages $U_l = I_1^l \cup ... \cup I_{k_l}^l$ is recursive.

### Example

The sequence of open sets $(-1,1), (-2,2), \ldots, (-k,k), \ldots$ is the standard effective open exhaustion for $\mathbb{R}$.

# Background – Effective Local Uniform Continuity

**Definition ([Fu and Zucker, 2014])**

A function $f$ on $U$ is **effectively locally uniformly continuous** **w.r.t. an effective open exhaustion** $(U_n)_{n\in\mathbb{N}}$ of $U$, if there is a recursive function $M : \mathbb{N}^2 \twoheadrightarrow \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

$$|x - y| < 2^{-M(k,l)} \implies |f(x) - f(y)| < 2^{-k}$$

# Background – Elementary Functions

### Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
- the variable $x$,

by applying (repeatedly) the basic operations below on elementary functions $f, g$:

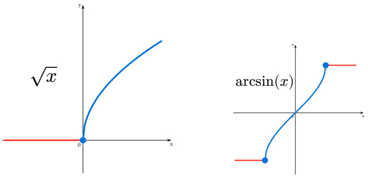
- $(f+g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\mathrm{div}_f(x) = \frac{1}{f(x)}$ where $\frac{1}{0} = \uparrow$
- $\mathrm{root}_{n,f}(x) = \sqrt[n]{f(x)}$ where $0 < n \in \mathbb{N}$
- $\ln_f(x) = \ln(f(x))$
- $\exp_f(x) = e^{f(x)}$
- $\sin_f(x) = \sin(f(x))$
- $\arcsin_f(x) = \arcsin(f(x))$

### Problem

The domains of elementary functions are not all open!

### Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when $n$ is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.



# Contributions

**Recall: Equivalence Theorem**, [Fu and Zucker, 2014]

For any acceptable function $f : \mathbb{R} \to \mathbb{R}$ and any effective open exhaustion $X$ for $\mathbf{dom}(f)$, the following are equivalent:

- $f$ is an $\alpha$-computable function.
- $f$ is **WhileCC**-approximable.
- $f$ is GL-computable w.r.t. $X$.
- $f$ is effectively locally uniformly multipolynomially approximable w.r.t. $X$.

**Theorem (WhileCC-approximability Theorem)**

All elementary functions are **WhileCC**-approximable.

**Theorem (Acceptability Theorem)**

All elementary functions are acceptable.

**WhileCC-programming Language:**

- Variables from $\mathbb{R}$, $\mathbb{N}$, $\mathbb{B}$
- Terms
  $t^s ::= x^s \mid F(t_1^{s_1}, \ldots, t_m^{s_m})$
- Statements

$$\begin{aligned} S ::= \text{skip} \mid \text{div} \mid \bar{x} := \bar{t} \mid S_1\ S_2 \\ \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \text{ fi} \\ \mid \text{while } b \text{ do } S_0 \text{ od} \\ \mid n := \text{choose } (z : \text{nat}) : P(z, \bar{t}) \end{aligned}$$

- Procedures

$$P ::= \text{proc } D \text{ begin } S \text{ end}$$

# Contributions

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**Theorem (WhileCC-approximability Theorem)**

All elementary functions are **WhileCC**-approximable.

**Theorem (Acceptability Theorem: Part 1)**

The domain of any elementary function has an effective open exhaustion.

**Theorem (Acceptability Theorem: Part 2)**

Any elementary function is effectively locally uniformly continuous w.r.t. an effective open exhaustion for its domain.

# Challenges

**Theorem (WhileCC-approximability Theorem)**

All elementary functions are **WhileCC**-approximable.

This is the easiest part, yet occupies about 30 pages of my thesis . . . ☺

# Challenges

**Theorem (Acceptability Theorem: Part 1)**

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function. Then, $\mathbf{dom}(f)$ has an effective open exhaustion.

**First attempt – Strengthening:**

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓(e.g. $\sin(x)$)
- Addition and multiplication ✓(e.g. $(f+g)(x)$)
- Composition case has a counterexample:

$$f(x) = id|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\mathbf{dom}(f)$ has an effective open exhaustion ✓
$\mathbf{dom}(g)$ has an effective open exhaustion ✓
$\mathbf{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

**Strengthened to proving exhaustion reflection property**

For any open set $U$ with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

**Adding decomposition of $+$ and $\cdot$**

$(f+g)(x) = f(x) + g(x)$ is composed of

- $Add(x, y) = x + y$,
- $(f \times g)(x, y) = (f(x), g(y))$,
- $Diag(x) = (x, x)$ ✓✓

# Challenges

**Theorem (Acceptability Theorem: Part 2)**

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

**(We prove that this is) equivalent to proving:**

Any elementary function has a local continuity witness.

**Definition (Local continuity witness)**

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for $f$ iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

**Proof by induction**

- Base cases: Using **WhileCC**-approximability theorem ✓
- Addition, Multiplication, and Composition: Using **WhileCC**-approximability theorem ✓

# Summary

We proved that:

- all elementary functions are **WhileCC**-approximable.
- all elementary functions are acceptable and hence computable in the other three models as well.

We also presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept.

# Future Work

**Questions left unanswered:**

- Are non-unary elementary functions acceptable?
  A generalization of acceptability in arbitrary metric spaces is given by Tucker and Zucker [2004].
- Can we extend the equivalence theorem in Fu and Zucker [2014] to acceptable partial functions of type $\mathbb{R}^m \to \mathbb{R}$?
- What functions are **WhileCC**-approximable but not While*-approximable [Tucker and Zucker, 1999]?

**Conjecture:**

- All partial unary **WhileCC**-approximable functions are acceptable.
    - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
    - **If not**, what is a model of computation that characterizes exactly the class of acceptable functions?

# References

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A HUGE Thank you!