WhileCC-approximability and Acceptability of Elementary Functions

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CCC 2025
Swansea University
September 1st - 3rd, 2025

McMaster University

Computability of Functions on $\mathbb{R}$

For total functions on $\mathbb{R}$, the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, ...

For partial functions on $\mathbb{R}$, Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

Problem

How general is this class of acceptable functions?

Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

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Definition (Acceptability)

A function $f : \mathbb{R} \to \mathbb{R}$ is **acceptable** if there exists a sequence X where:

1. X is an **effective open exhaustion** for $\mathbf{dom}(f)$, and
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Definition ([Fu and Zucker, 2014])

A function f on U is **effectively locally uniformly continuous** **w.r.t. an effective open exhaustion** $(U_n)_{n\in\mathbb{N}}$ of U, if there is a recursive function $M : \mathbb{N}^2 \to \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

$$|x - y| < 2^{-M(k,l)} \implies |f(x) - f(y)| < 2^{-k}$$

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Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
- the variable x,

by applying (repeatedly) the basic operations below on elementary functions f, g:

- $(f+g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\mathrm{div}_f(x) = \frac{1}{f(x)}$ where $\frac{1}{0} = \uparrow$
- $\mathrm{root}_{n,f}(x) = \sqrt[n]{f(x)}$ where $0 < n \in \mathbb{N}$
- $\ln_f(x) = \ln(f(x))$
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The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.

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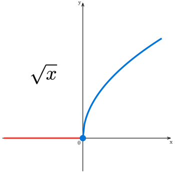

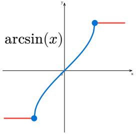

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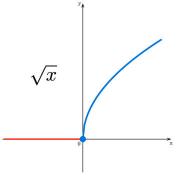

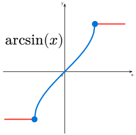

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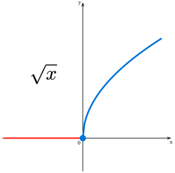

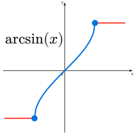

Contributions

Recall: Equivalence Theorem, [Fu and Zucker, 2014]
For any acceptable function $f : \mathbb{R} \to \mathbb{R}$ and any effective open exhaustion X for $\mathbf{dom}(f)$, the following are equivalent:

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Theorem 2 (Acceptability Theorem)

All elementary functions are acceptable.

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Theorem 2.1 (Acceptability Theorem: Part 1)

The domain of any elementary function has an effective open exhaustion.

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous w.r.t. an effective open exhaustion for its domain.

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- Statements

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where $\mathbf{Nbd}(y, r)$ has the standard definition of neighborhood on $\mathbb{R}$ i.e.,

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Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

Challenge: Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate "piecewise" functions?

Result 1 - Background

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Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

```
proc
  in  x : real c : nat
begin
  if x <_R 0 then
    return 0
  else
    return Root(x, c)
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end
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Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of "choose"

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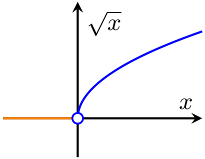

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   else
      return Root(x, c)
   fi
end
```

Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of “choose”

Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

```
proc
   in  x : real c : nat
begin
   if x <_R 0 then
      return 0
   else
      return Root(x, c)
   fi
end
```

Solution

- Find an overlapping interval where both pieces are defined
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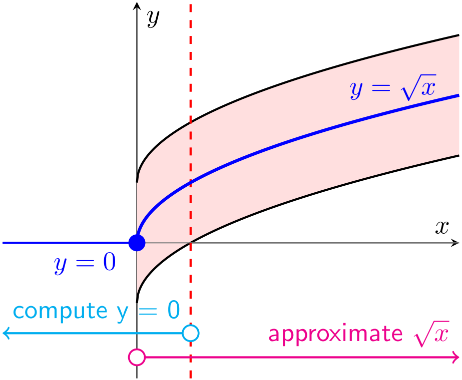

Result 2

Theorem 2.1 (Acceptability Theorem: Part 1)

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function. Then, $\mathbf{dom}(f)$ has an effective open exhaustion.

First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓(e.g. $\sin(x)$)
- Addition and multiplication ✓(e.g. $(f+g)(x)$)
- Composition case has a counterexample:

$$f(x) = id|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\mathbf{dom}(f)$ has an effective open exhaustion ✓
$\mathbf{dom}(g)$ has an effective open exhaustion ✓
$\mathbf{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

Strengthened to proving exhaustion reflection property

For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

Adding decomposition of $+$ and $\cdot$

$(f+g)(x) = f(x) + g(x)$ is composed of

- $Add(x, y) = x + y$,
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Theorem (Reducing Exhaustion-reflection to to a Decision Procedure)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is exhaustion-reflecting, if we can decide for any m-cube Q_m whether an arbitrary rational closed n-cube is completely contained in $f^{-1}(Q_m)$.

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

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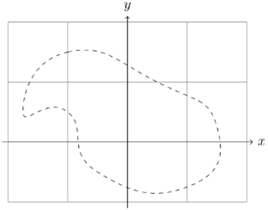

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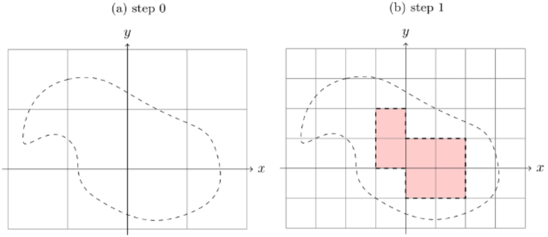

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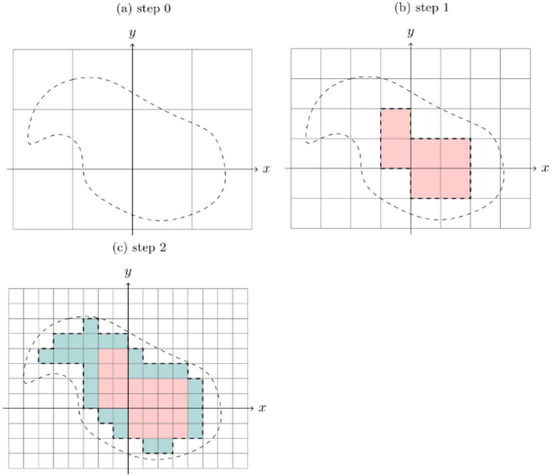

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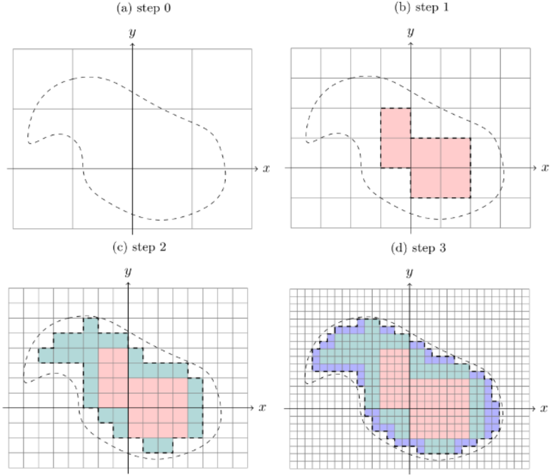

Result 2

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

Any elementary function has a local continuity witness.

Definition (Local continuity witness)

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

The notion of effective local uniform continuity is independent of effective open exhaustion.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be **WhileCC**-approximable and monotone on its domain. Then f has a local continuity witness.

Result 2

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

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Definition (Local continuity witness)

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

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Questions left unanswered:

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M. Q. Fu and J. Zucker. Models of computation for partial functions on the reals. *Journal of Logical and Algebraic Methods in Programming*, 84(2):218–237, 11 2014. ISSN 2352-2208. doi:10.1016/j.jlamp.2014.11.001.

M. Tenenbaum and H. Pollard. *Ordinary Differential Equations: An Elementary Textbook for Students of Mathematics, Engineering, and the Sciences*. Dover Books on Mathematics. Dover Publications, 1985. ISBN 9780486649405. URL https://books.google.ca/books?id=iU4zDAAAQBAJ.

J. Tucker and J. Zucker. Computation by 'While' programs on topological partial algebras. *Theoretical Computer Science*, 219(1):379–420, 1999. ISSN 0304-3975. doi:10.1016/S0304-3975(98)00297-7.

J. Tucker and J. Zucker. Computable total functions on metric algebras, universal algebraic specifications and dynamical systems. *The Journal of Logic and Algebraic Programming*, 62(1): 71–108, 2005. ISSN 1567-8326. doi:10.1016/j.jlap.2003.10.001.

J. V. Tucker and J. I. Zucker. Abstract versus concrete computation on metric partial algebras. *ACM Trans. Comput. Log.*, 5(4):611–668, 2004. doi:10.1145/1024922.1024924.

A HUGE Thank you!