

# WhileCC-approximability and Acceptability of Elementary Functions

Fateme Ghasemi   Dr. Jeffery Zucker  
ghases5@mcmaster.ca   zucker@mcmaster.ca

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# Computability of Functions on $\mathbb{R}$

**For total functions on  $\mathbb{R}$** , the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions?  $1/x$ ,  $\sqrt[n]{x}$ ,  $\dots$

**For partial functions on  $\mathbb{R}$** , Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

## Problem

How general is this class of acceptable functions?

## Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

# Background – Acceptability

## Definition (**Acceptability**)

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **acceptable** if there exists a sequence  $X$  where:

- 1  $X$  is an **effective open exhaustion** for  $\text{dom}(f)$ , and
- 2  $f$  is **effectively locally uniformly continuous w.r.t.  $X$** .

# Background – Effective Open Exhaustions

Definition ([Fu and Zucker, 2014])

A sequence  $(U_1, U_2, \dots)$  of open sets is called an **effective open exhaustion** for an open  $U \subseteq \mathbb{R}$  if

- 1  $U = \bigcup_{l=0}^{\infty} U_l$ , and
- 2 for each  $l \in \mathbb{N}$ ,  $U_l$  is a finite union of non-empty open finite intervals  $I_1^l, I_2^l, \dots, I_{k_l}^l$  whose closures are pairwise disjoint, and
- 3 for each  $l \in \mathbb{N}$ ,  $\overline{U_l} = \bigcup_{i=1}^{k_l} \overline{I_i^l} \subseteq U_{l+1}$ .
- 4 for all  $l$ , the components  $I_i^l$  that are intervals building up the stage  $U_l$ , are *rational* and *ordered* i.e.,  $I_i^l = (a_i^l, b_i^l)$  for some  $a_i^l, b_i^l \in \mathbb{Q}$  where  $b_i^l < a_{i+1}^l$  for  $i = 1, \dots, k_l - 1$ , and
- 5 the map  $l \mapsto (a_1^l, b_1^l, \dots, a_{k_l}^l, b_{k_l}^l)$  which delivers the sequence of stages  $U_l = I_1^l \cup \dots \cup I_{k_l}^l$  is recursive.

## Example

The sequence of open sets  $(-1, 1), (-2, 2), \dots, (-k, k), \dots$  is the standard effective open exhaustion for  $\mathbb{R}$ .

## Background – Effective Local Uniform Continuity

Definition ([Fu and Zucker, 2014])

A function  $f$  on  $U$  is **effectively locally uniformly continuous w.r.t. an effective open exhaustion**  $(U_n)_{n \in \mathbb{N}}$  of  $U$ , if there is a recursive function  $M : \mathbb{N}^2 \rightarrow \mathbb{N}$  such that for all  $k, l \in \mathbb{N}$  and all  $x, y \in U_l$ :

$$|x - y| < 2^{-M(k,l)} \implies |f(x) - f(y)| < 2^{-k}$$

# Background – Elementary Functions

## Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on  $\mathbb{R}$  are partial functions defined by expressions built up from

- computable reals, and
- the variable  $x$ ,

by applying (repeatedly) the basic operations below on elementary functions  $f, g$ :

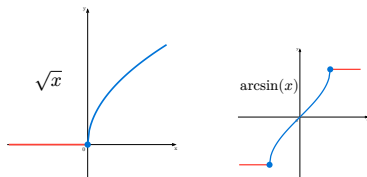
- $(f + g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\text{div}_f(x) = \frac{1}{f(x)}$  where  $\frac{1}{0} = \uparrow$
- $\text{root}_{n,f}(x) = \sqrt[n]{f(x)}$  where  $0 < n \in \mathbb{N}$
- $\ln_f(x) = \ln(f(x))$
- $\exp_f(x) = e^{f(x)}$
- $\sin_f(x) = \sin(f(x))$
- $\arcsin_f(x) = \arcsin(f(x))$

## Problem

The domains of elementary functions are not all open!

## Solution: Modifications

- We define  $\sqrt[n]{x} = 0$  for  $x < 0$  when  $n$  is even.
- We extend the definition of  $\arcsin(x)$  to be  $\frac{\pi}{2}$  for  $x > 1$  and to be  $-\frac{\pi}{2}$  for  $x < -1$ .



# Contributions

**Recall: Equivalence Theorem**, [Fu and Zucker, 2014]

For any **acceptable function**  $f : \mathbb{R} \rightarrow \mathbb{R}$  and any effective open exhaustion  $X$  for  $\text{dom}(f)$ , the following are equivalent:

- $f$  is an  $\alpha$ -computable function.
- $f$  is **WhileCC-approximable**.
- $f$  is GL-computable w.r.t.  $X$ .
- $f$  is effectively locally uniformly multipolynomially approximable w.r.t.  $X$ .

**Theorem 1 (WhileCC-approximability Theorem)**

All elementary functions are **WhileCC-approximable**.

**Theorem 2 (Acceptability Theorem)**

All elementary functions are acceptable.

# Contributions

**Recall: Equivalence Theorem**, [Fu and Zucker, 2014]

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- $f$  is effectively locally uniformly multipolynomially approximable w.r.t.  $X$ .

**Theorem 1 (WhileCC-approximability Theorem)**

All elementary functions are **WhileCC**-approximable.

**Theorem 2.1 (Acceptability Theorem: Part 1)**

The domain of any elementary function has an effective open exhaustion.

**Theorem 2.2 (Acceptability Theorem: Part 2)**

Any elementary function is effectively locally uniformly continuous w.r.t. an effective open exhaustion for its domain.



# Result 1

Theorem 1 (**WhileCC**-approximability Theorem)

All elementary functions are **WhileCC**-approximable.

This is the easiest part, yet occupies about 30 pages of my master's thesis ... ☺

# Result 1 - Background - WhileCC Programming Language

## Syntax

### • Terms

$$t^s ::= x^s \mid F(t_1^{s_1}, \dots, t_m^{s_m})$$

### • Statements

$S ::=$

skip | div |  $\bar{x} := \bar{t}$  |  $S_1 \ S_2$

| if  $b$  then  $S_1$  else  $S_2$  fi

| while  $b$  do  $S_0$  od

|  $n := \text{choose } (z : \text{nat}) : P(z, \bar{t})$

### • Procedures

$P ::= \text{proc } D \text{ begin } S \text{ end}$

## Algebra $\mathcal{R}$

$0_{\mathbb{R}}, 1_{\mathbb{R}}, -1_{\mathbb{R}} : \rightarrow \mathbb{R}$   
 $+_{\mathbb{R}}, \times_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$   
 $+_{\mathbb{N}}, \times_{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$   
 $\text{inv}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$   
 $0_{\mathbb{N}} : \rightarrow \mathbb{N}$   
 $\text{suc}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}$   
 $\text{tt}, \text{ff} : \rightarrow \mathbb{B}$   
 $\text{and}, \text{or} : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$   
 $\text{not} : \mathbb{B} \rightarrow \mathbb{B}$   
 $=_{\mathbb{N}}, <_{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{B}$   
 $=_{\text{real}}, <_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{B}$

## Semantics

$$\text{inv}_{\mathbb{R}}(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$$

$$=_{\text{real}}(x, y) = \begin{cases} \text{ff} & \text{if } x \neq y \\ \uparrow & \text{otherwise} \end{cases}$$

$$<_{\mathbb{R}}(x, y) = \begin{cases} \text{tt} & \text{if } x < y \\ \text{ff} & \text{if } x > y \\ \uparrow & \text{if } x = y \end{cases}$$

The Semantics of a program  $P$  is denoted by a many-valued function  $P^{\mathcal{R}}$ .

Statement	Possible Values for $n$
$n := \text{choose } (k : \text{nat}) : k < 0$	$\{\uparrow\}$
$n := \text{choose } (k : \text{nat}) : \text{toReal}(k) = 0$	$\{\uparrow\}$
$n := \text{choose } (k : \text{nat}) : k < k + 1$	$\{0, 1, 2, \dots\}$
$n := \text{choose } (k : \text{nat}) : k > 2 \text{ and } k < 4$	$\{3\}$

# Result 1 - Background

Definition (**WhileCC**-approximability, [Fu and Zucker, 2014])

A **WhileCC**-procedure  $P$  of type  $\text{real} \times \text{nat} \rightarrow \text{real}$  on  $\mathcal{R}$  is said to *approximate* a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  iff for all  $n \in \mathbb{N}$  and all  $x \in \mathbb{R}$ :

- $x \in \text{dom}(f) \implies \uparrow \notin P^{\mathcal{R}}(x, n) \subseteq \text{Nbd}(f(x), 2^{-n})$ , and
- $x \notin \text{dom}(f) \implies P^{\mathcal{R}}(x) = \{\uparrow\}$

where  $\text{Nbd}(y, r)$  has the standard definition of neighborhood on  $\mathbb{R}$  i.e.,

$$\text{Nbd}(y, r) = \{z \in \mathbb{R} \mid |y - z| < r\}.$$

## Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

**Challenge:** Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate “piecewise” functions?

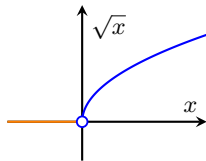
# Result 1 - Challenges

## Problem

When defining piecewise functions, comparison makes a hole!

## Example: Even Root - First Attempt

```
proc
  in  $x$  : real  $c$  : nat
begin
  if  $x <_{\mathbb{R}} 0$  then
    return 0
  else
    return Root( $x, c$ )
  fi
end
```



# Result 1 - Challenges

## Problem

When defining piecewise functions, comparison makes a hole!

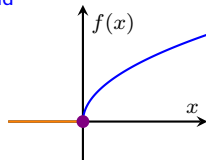
## Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of “choose”

```
proc
  in  $x : \text{real } c : \text{nat}$ 
  aux  $\text{chosenVal} : \text{nat } l : \text{real}$ 
begin
   $l := \text{choose } (q : \text{real}) : \text{isCloseEnough}(q, c, n)$ 
```

```
   $\text{chosenVal} := \text{choose } (k : \text{nat}) : \text{proc}$ 
    in  $k : \text{nat } x : \text{real}$ 
  begin
    if  $k =_{\text{N}} 1$  then
      return  $0 < x$ 
    else if  $k =_{\text{N}} 2$  then
      return  $x < l$ 
    else
      return ff
    fi
  end
```

```
  if  $\text{chosenVal} =_{\text{N}} 1$  then
    return  $\text{Root}(x, c)$ 
  else if  $\text{chosenVal} =_{\text{N}} 2$  then
    return 0
  fi
end
```



## Result 2 - Challenges

### Theorem 2.1 (Acceptability Theorem: Part 1)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an elementary function. Then,  $\text{dom}(f)$  has an effective open exhaustion.

#### First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓ (e.g.  $\sin(x)$ )
- Addition and multiplication ✓ (e.g.  $(f + g)(x)$ )
- Composition case has a counterexample:

$$f(x) = \text{id}|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\text{dom}(f)$  has an effective open exhaustion ✓

$\text{dom}(g)$  has an effective open exhaustion ✓

$\text{dom}(f \circ g) = [-1, 1]$  has no open exhaustion ✗

#### Strengthened to proving exhaustion reflection property

For any open set  $U$  with an effective open exhaustion,  $f^{-1}(U)$  has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

#### Adding decomposition of $+$ and $\cdot$

$(f + g)(x) = f(x) + g(x)$  is composed of

- $\text{Add}(x, y) = x + y$ ,
- $(f \times g)(x, y) = (f(x), g(y))$ ,
- $\text{Diag}(x) = (x, x)$

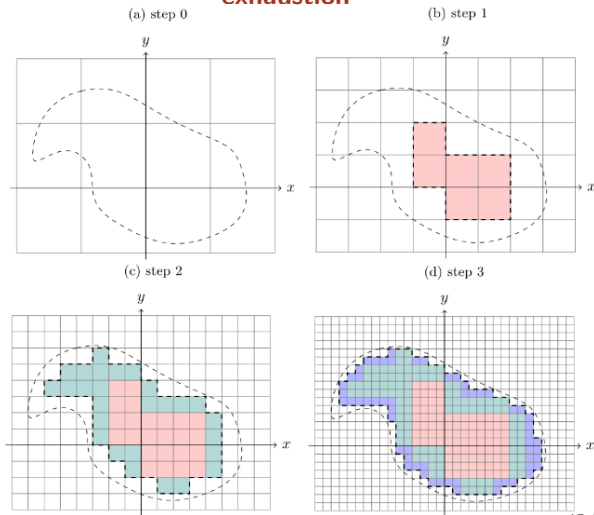
# Interesting results - Acceptability Theorem

Theorem (Reducing Exhaustion-reflection to a Decision Procedure)

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is exhaustion-reflecting, if we can decide for any  $m$ -cube  $Q_m$  whether an arbitrary rational closed  $n$ -cube is completely contained in  $f^{-1}(Q_m)$ .

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

## The process of building an effective open exhaustion



# Challenges

## Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

**(We prove that this is) equivalent to proving:**

Any elementary function has a local continuity witness.

## Definition (Local continuity witness)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . A recursive function  $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{N}$  is called a **local continuity witness** for  $f$  iff for any  $a, b \in \mathbb{Q}$  with  $[a, b] \subseteq \text{dom}(f)$  and  $k \in \mathbb{N}$ , we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

## Proof by induction

- Base cases: Using **WhileCC**-approximability theorem ✓
- Addition, Multiplication, and Composition: Using **WhileCC**-approximability theorem ✓



# Summary

We proved that:

- all elementary functions are **WhileCC**-approximable.
- all elementary functions are acceptable and hence computable in the other three models as well.

We also

- presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept, and
- found a few useful tricks along the way for implementing approximations of piecewise functions.

# Future Work

## Questions left unanswered:

- Are non-unary elementary functions acceptable?  
A generalization of acceptability in arbitrary metric spaces is given by [Tucker and Zucker \[2004\]](#).
- Can we extend the equivalence theorem in [Fu and Zucker \[2014\]](#) to acceptable partial functions of type  $\mathbb{R}^m \rightarrow \mathbb{R}$ ?
- What functions are **WhileCC**-approximable but not **While\***-approximable [[Tucker and Zucker, 1999](#)]?

## Conjecture:

- All partial unary **WhileCC**-approximable functions are acceptable.
  - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
  - **If not**, what is a model of computation that characterizes exactly the class of acceptable functions?

## Currently working on:

- Formalizing the concept of **WhileCC**-approximability and the aforementioned proofs in Lean.

## References

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A HUGE Thank you!