WhileCC-approximability and Acceptability of Elementary Functions

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Computability of Functions on $\mathbb{R}$

For total functions on $\mathbb{R}$, the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,testtesttesttesttest
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, ...

For partial functions on $\mathbb{R}$, Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

Problem

How general is this class of acceptable functions?

Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

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Definition (Acceptability)

A function $f : \mathbb{R} \to \mathbb{R}$ is **acceptable** if there exists a sequence X where:

1. X is an **effective open exhaustion** for $\mathbf{dom}(f)$, and
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3. for each $l \in \mathbb{N}$, $\overline{U_l} = \bigcup_{i=1}^{k_l} \overline{I_i^l} \subseteq U_{l+1}$.
4. for all l, the components I_i^l that are intervals building up the stage U_l, are *rational* and *ordered* i.e., $I_i^l = (a_i^l, b_i^l)$ for some $a_i^l, b_i^l \in \mathbb{Q}$ where $b_i^l < a_{i+1}^l$ for $i = 1, \ldots, k_l - 1$, and
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The sequence of open sets $(-1,1), (-2,2), \ldots, (-k,k), \ldots$ is the standard effective open exhaustion for $\mathbb{R}$.

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A function f on U is **effectively locally uniformly continuous** **w.r.t. an effective open exhaustion** $(U_n)_{n\in\mathbb{N}}$ of U, if there is a recursive function $M : \mathbb{N}^2 \twoheadrightarrow \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

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Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
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by applying (repeatedly) the basic operations below on elementary functions f, g:

- $(f+g)(x) = f(x) + g(x)$
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The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.

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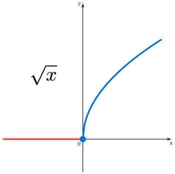
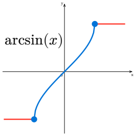
The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.





Background – Elementary Functions

Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
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by applying (repeatedly) the basic operations below on elementary functions f, g:

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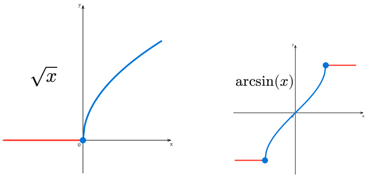

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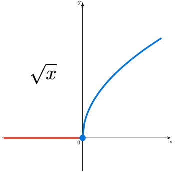

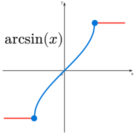

Contributions

Recall: Equivalence Theorem, [Fu and Zucker, 2014]

For any acceptable function $f : \mathbb{R} \to \mathbb{R}$ and any effective open exhaustion X for $\mathbf{dom}(f)$, the following are equivalent:

- f is an α-computable function.
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- f is GL-computable w.r.t. X.
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Theorem 1 (WhileCC-approximability Theorem)

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Theorem 2 (Acceptability Theorem)

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where $\mathbf{Nbd}(y, r)$ has the standard definition of neighborhood on $\mathbb{R}$ i.e.,

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Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

Challenge: Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate "piecewise" functions?

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$$\mathbf{Nbd}(y, r) = \{z \in \mathbb{R} \mid |y - z| < r\}.$$

Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

Challenge: Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate "piecewise" functions?

Result 1 - Background

Definition (**WhileCC**-approximability, [Fu and Zucker, 2014])

A **WhileCC**-procedure P of type real $\times$ nat $\to$ real on $\mathcal{R}$ is said to *approximate* a function $f : \mathbb{R} \to \mathbb{R}$ iff for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$:

- $x \in \mathbf{dom}(f) \implies \emptyset \neq P^{\mathcal{R}}(x, n) \subseteq \mathbf{Nbd}(f(x), 2^{-n})$, and
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Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

proc
  in x : real c : nat
begin
  if $x <_{\mathsf{R}} 0$ then
    return 0
  else
    return Root(x, c)
  fi
end

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\begin{array}{l}
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\quad \textsf{if } x <_{\mathsf{R}} 0 \textsf{ then} \\
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\quad \textsf{else} \\
\qquad \textsf{return } \text{Root}(x, c) \\
\quad \textsf{fi} \\
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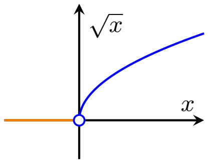
$$



Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of "choose"

```
proc
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                                     if k =N 1 then
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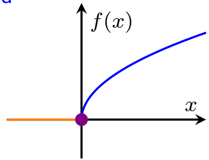

Result 2

Theorem 2.1 (Acceptability Theorem: Part 1)

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function. Then, $\mathbf{dom}(f)$ has an effective open exhaustion.

First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓(e.g. $\sin(x)$)
- Addition and multiplication ✓(e.g. $(f+g)(x)$)
- Composition case has a counterexample:

$$f(x) = id|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\mathbf{dom}(f)$ has an effective open exhaustion ✓
$\mathbf{dom}(g)$ has an effective open exhaustion ✓
$\mathbf{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

Strengthened to proving exhaustion reflection property

For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

Adding decomposition of $+$ and $\cdot$

$(f+g)(x) = f(x) + g(x)$ is composed of

- $Add(x, y) = x + y$,
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For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
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Adding decomposition of $+$ and $\cdot$

$(f+g)(x) = f(x) + g(x)$ is composed of

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Result 2 - Challenges

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Theorem (Reducing Exhaustion-reflection to to a Decision Procedure)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is exhaustion-reflecting, if we can decide for any m-cube Q_m whether an arbitrary rational closed n-cube is completely contained in $f^{-1}(Q_m)$.

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

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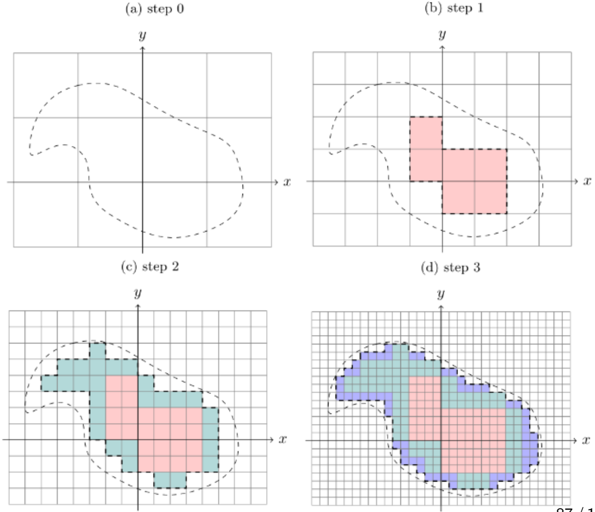

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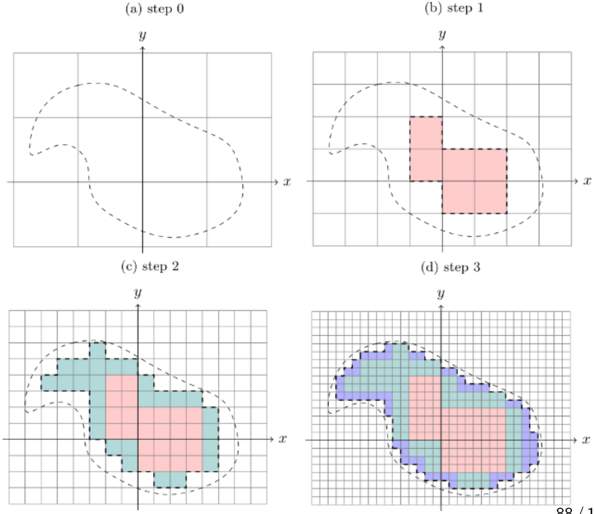

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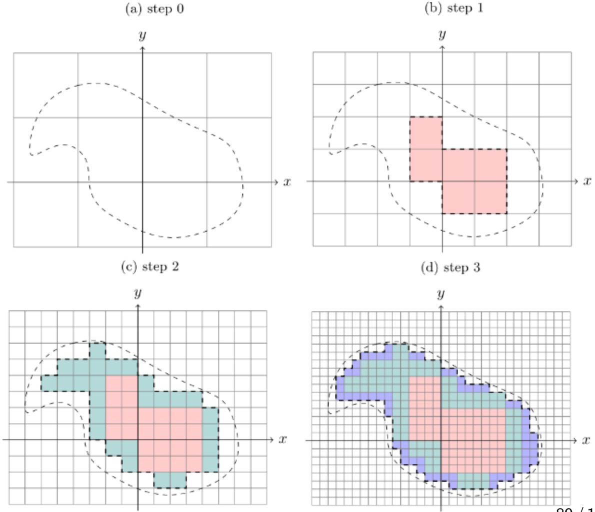

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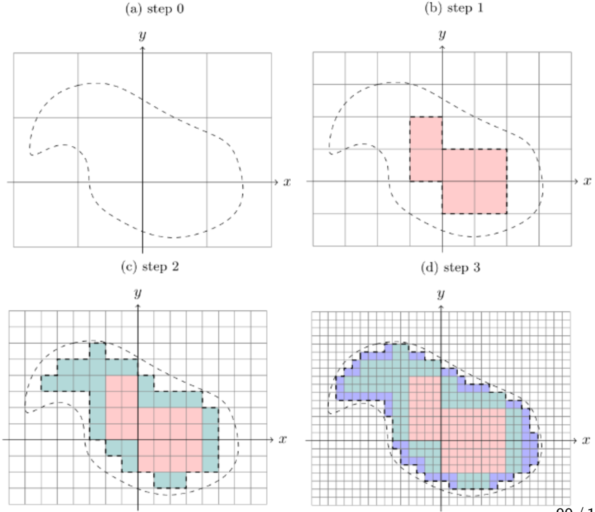

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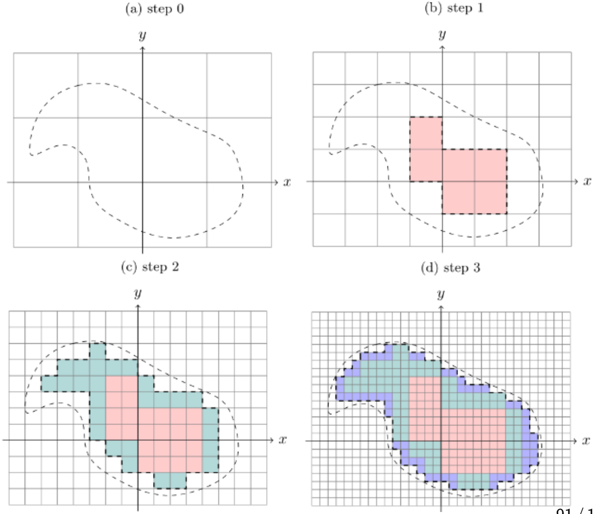

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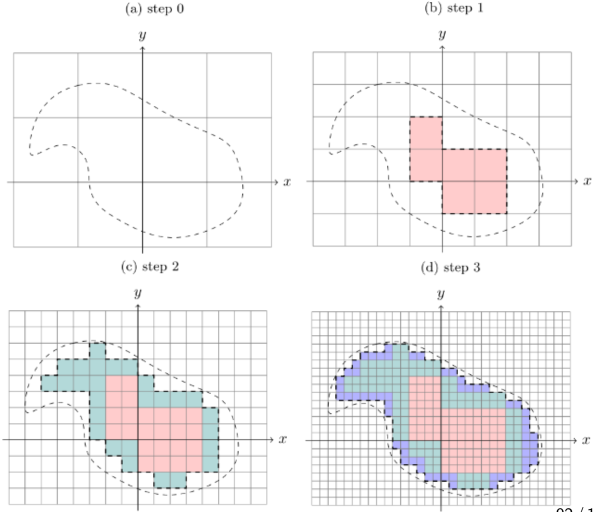

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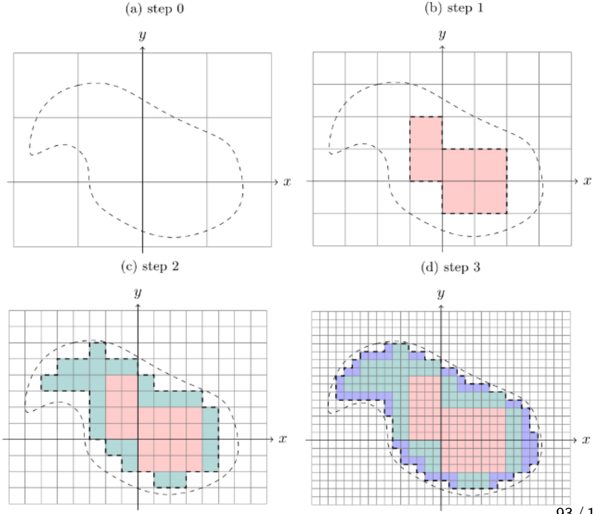

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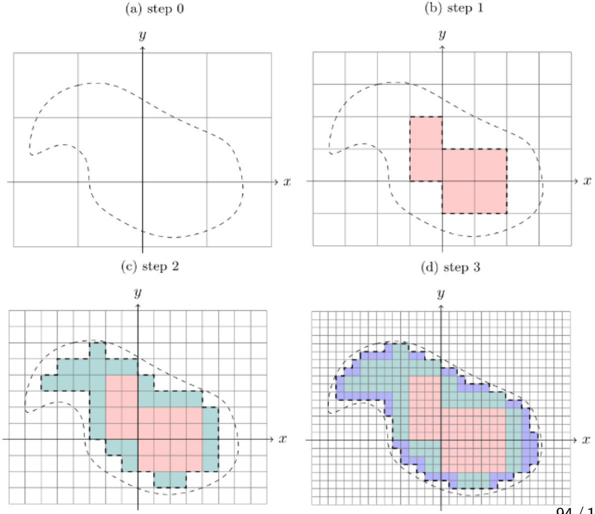

Challenges

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

Any elementary function has a local continuity witness.

Definition (Local continuity witness)

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

Proof by induction

- Base cases: Using **WhileCC**-approximability theorem ✓
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We proved that:

- all elementary functions are **WhileCC**-approximable.
- all elementary functions are acceptable and hence computable in the other three models as well.

We presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept. We also found a few useful tricks along the way for implementing approximations of piecewise functions.

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Questions left unanswered:

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  A generalization of acceptability in arbitrary metric spaces is given by Tucker and Zucker [2004].
- Can we extend the equivalence theorem in Fu and Zucker [2014] to acceptable partial functions of type $\mathbb{R}^m \to \mathbb{R}$?
- What functions are **WhileCC**-approximable but not While*-approximable [Tucker and Zucker, 1999]?

Conjecture:

- All partial unary **WhileCC**-approximable functions are acceptable.
  - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
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- Formalizing the concept of **WhileCC**-approximability and the aforementioned proofs in Lean.

Future Work

Questions left unanswered:

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A HUGE Thank you!