WhileCC-approximability and Acceptability of Elementary Functions

Fateme Ghasemi   Dr. Jeffery Zucker
ghases5@mcmaster.ca   zucker@mcmaster.ca

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Computability of Functions on $\mathbb{R}$

For total functions on $\mathbb{R}$, the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- **WhileCC**-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, ...

For partial functions on $\mathbb{R}$, Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

Problem

How general is this class of acceptable functions?

Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

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Definition (Acceptability)

A function $f : \mathbb{R} \to \mathbb{R}$ is **acceptable** if there exists a sequence X where:

1. X is an **effective open exhaustion** for $\mathbf{dom}(f)$, and
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A function f on U is **effectively locally uniformly continuous** **w.r.t. an effective open exhaustion** $(U_n)_{n\in\mathbb{N}}$ of U, if there is a recursive function $M : \mathbb{N}^2 \twoheadrightarrow \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

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Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on $\mathbb{R}$ are partial functions defined by expressions built up from

- computable reals, and
- the variable x,

by applying (repeatedly) the basic operations below on elementary functions f, g:

- $(f + g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\mathrm{div}_f(x) = \frac{1}{f(x)}$ where $\frac{1}{0} = \uparrow$
- $\mathrm{root}_{n,f}(x) = \sqrt[n]{f(x)}$ where $0 < n \in \mathbb{N}$
- $\ln_f(x) = \ln(f(x))$
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The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.

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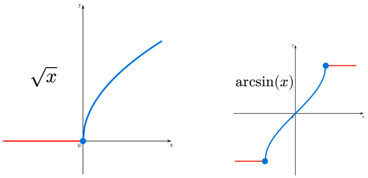

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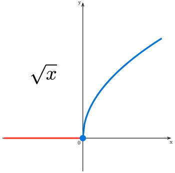

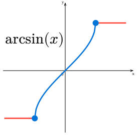

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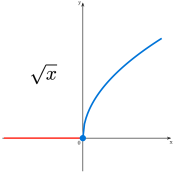

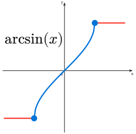

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Recall: Equivalence Theorem, [Fu and Zucker, 2014]

For any acceptable function $f : \mathbb{R} \to \mathbb{R}$ and any effective open exhaustion X for $\mathbf{dom}(f)$, the following are equivalent:

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Result 1 - Background - **WhileCC** Programming Language

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- Statements

  $S ::=$

  skip $\mid$ div $\mid \bar{x} := \bar{t} \mid S_1\ S_2$

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- Procedures

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Result 1 - Background - **WhileCC** Programming Language

Syntax

- Terms

  $t^s ::= x^s \mid F(t_1^{s1}, \ldots, t_m^{sm})$
- Statements

  $S ::=$

  skip $\mid$ div $\mid \bar{x} := \bar{t} \mid S_1 \; S_2$

  $\mid$ if b then S_1 else S_2 fi

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$0_{\mathsf{R}}, 1_{\mathsf{R}}, -1_{\mathsf{R}} : \; \twoheadrightarrow \mathbb{R}$

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Result 1 - Background

Definition (WhileCC-approximability, [Fu and Zucker, 2014])

A **WhileCC**-procedure P of type real $\times$ nat $\to$ real on $\mathcal{R}$ is said to *approximate* a function $f : \mathbb{R} \to \mathbb{R}$ iff for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$:

- $x \in \mathbf{dom}(f) \implies \uparrow \notin P^{\mathcal{R}}(x, n) \subseteq \mathbf{Nbd}(f(x), 2^{-n})$, and
- $x \notin \mathbf{dom}(f) \implies P^{\mathcal{R}}(x) = \{\uparrow\}$

where $\mathbf{Nbd}(y, r)$ has the standard definition of neighborhood on $\mathbb{R}$ i.e.,

$$\mathbf{Nbd}(y, r) = \{z \in \mathbb{R} \mid |y - z| < r\}.$$

Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

Challenge: Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate “piecewise” functions?

Result 1 - Background

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Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

```
proc
   in  x : real c : nat
begin
   if x <_R 0 then
      return 0
   else
      return Root(x, c)
   fi
end
```

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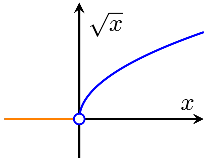

Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of "choose"

```
proc
  in  x : real c : nat
  aux chosenVal : nat l : real
begin
  l := choose (q : real) : isCloseEnough(q, c, n)

  chosenVal := choose (k : nat) : proc
                                    in k : nat x : real
                                  begin
                                    if k =_N 1 then
                                      return 0 < x
                                    else if k =_N 2 then
                                      return x < l
                                    else
                                      return ff
                                    fi
                                  end
  if chosenVal =_N 1 then
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  chosenVal := choose (k : nat) : proc
                                    in k : nat x : real
                                  begin
                                    if k =_N 1 then
                                      return 0 < x
                                    else if k =_N 2 then
                                      return x < l
                                    else
                                      return ff
                                    fi
                                  end
  if chosenVal =_N 1 then
    return Root(x, c)
  else if chosenVal =_N 2 then
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Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of "choose"

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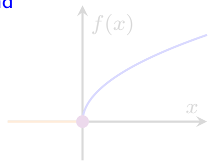

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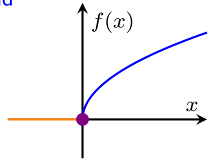

Result 2

Theorem 2.1 (Acceptability Theorem: Part 1)

Let $f : \mathbb{R} \to \mathbb{R}$ be an elementary function. Then, $\mathbf{dom}(f)$ has an effective open exhaustion.

First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓(e.g. $\sin(x)$)
- Addition and multiplication ✓(e.g. $(f+g)(x)$)
- Composition case has a counterexample:

$$f(x) = id|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\mathbf{dom}(f)$ has an effective open exhaustion ✓
$\mathbf{dom}(g)$ has an effective open exhaustion ✓
$\mathbf{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

Strengthened to proving exhaustion reflection property

For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

Adding decomposition of $+$ and $\cdot$

$(f+g)(x) = f(x) + g(x)$ is composed of

- $Add(x, y) = x + y$,
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Theorem (Reducing Exhaustion-reflection to to a Decision Procedure)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is exhaustion-reflecting, if we can decide for any m-cube Q_m whether an arbitrary rational closed n-cube is completely contained in $f^{-1}(Q_m)$.

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

Building an effective open exhaustion

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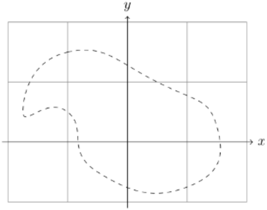
(a) step 0



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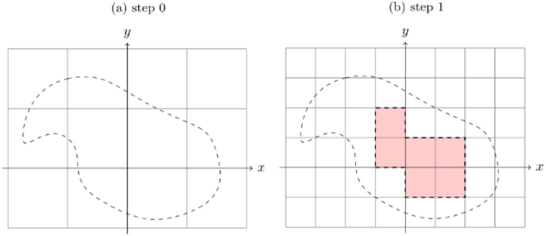

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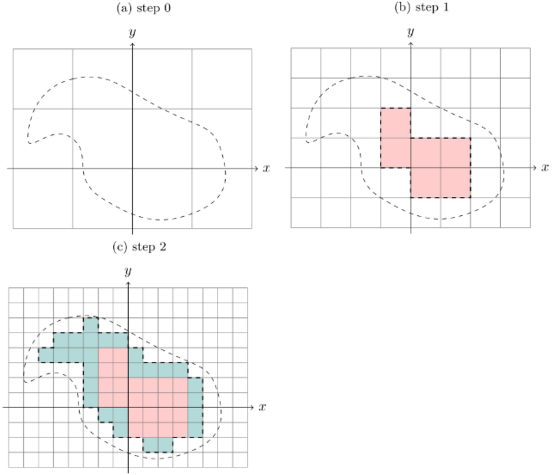

Result 2.1 - Takeaways

Theorem (Reducing Exhaustion-reflection to to a Decision Procedure)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is exhaustion-reflecting, if we can decide for any m-cube Q_m whether an arbitrary rational closed n-cube is completely contained in $f^{-1}(Q_m)$.

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

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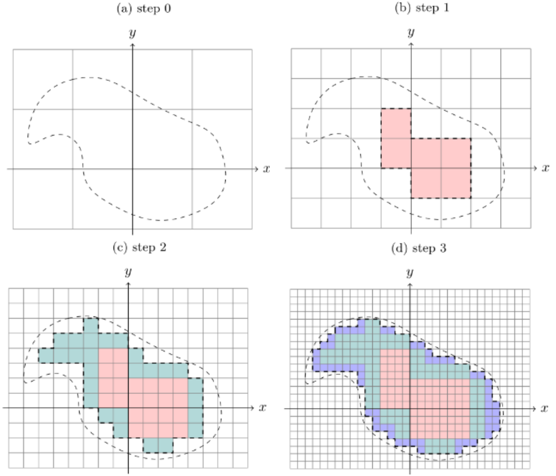

Result 2

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

Any elementary function has a local continuity witness.

Definition (Local continuity witness)

Let $f : \mathbb{R} \to \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \mathbf{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a,b,k)} \implies |f(x) - f(y)| < 2^{-k}.$$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be **WhileCC**-approximable and monotone on its domain. Then f has a local continuity witness.

The notion of effective local uniform continuity is independent of effective open exhaustion.

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Summary

We proved that:

- all elementary functions are **WhileCC**-approximable.
- all elementary functions are acceptable.

We also

- presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept, and
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Questions left unanswered:

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  A generalization of acceptability in arbitrary metric spaces is given by Tucker and Zucker [2004].
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Conjecture:

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  - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
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A HUGE Thank you!