

WhileCC-approximability and Acceptability of Elementary Functions

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CCC 2025
Swansea University
September 1st - 3rd, 2025



Computability of Functions on \mathbb{R}

For total functions on \mathbb{R} , the following models of computation are equivalent for all functions that are effectively locally uniformly continuous [Tucker and Zucker, 2005]:

- GL-computability,
- tracking computability,
- multipolynomial approximability, and
- WhileCC-approximability.

What about **partial** functions? $1/x$, $\sqrt[n]{x}$, \dots

For partial functions on \mathbb{R} , Fu and Zucker [2014] generalize effectively locally uniform continuity to **acceptability** to get an equivalence.

Problem

How general is this class of acceptable functions?

Useful First Step Towards Solution

Show that the elementary functions satisfy the acceptability conditions.

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Background – Acceptability

Definition (Acceptability)

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **acceptable** if there exists a sequence X where:

- 1 X is an **effective open exhaustion** for $\text{dom}(f)$, and
- 2 f is **effectively locally uniformly continuous w.r.t. X** .

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Example

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Background – Effective Local Uniform Continuity

Definition ([Fu and Zucker, 2014])

A function f on U is **effectively locally uniformly continuous w.r.t. an effective open exhaustion** $(U_n)_{n \in \mathbb{N}}$ of U , if there is a recursive function $M : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ and all $x, y \in U_l$:

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Background – Elementary Functions

Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on \mathbb{R} are partial functions defined by expressions built up from

- computable reals, and
- the variable x ,

by applying (repeatedly) the basic operations below on elementary functions f, g :

- $(f + g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x)g(x)$
- $\text{div}_f(x) = \frac{1}{f(x)}$ where $\frac{1}{0} = \uparrow$
- $\text{root}_{n,f}(x) = \sqrt[n]{f(x)}$ where $0 < n \in \mathbb{N}$
- $\ln_f(x) = \ln(f(x))$
- $\exp_f(x) = e^{f(x)}$
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The domains of elementary functions are not all open!

Solution: Modifications

- We define $\sqrt[n]{x} = 0$ for $x < 0$ when n is even.
- We extend the definition of $\arcsin(x)$ to be $\frac{\pi}{2}$ for $x > 1$ and to be $-\frac{\pi}{2}$ for $x < -1$.

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Background – Elementary Functions

Definition ([Tenenbaum and Pollard, 1985])

The **elementary functions** on \mathbb{R} are partial functions defined by expressions built up from

- computable reals, and
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by applying (repeatedly) the basic operations below on elementary functions f, g :

- $(f + g)(x) = f(x) + g(x)$
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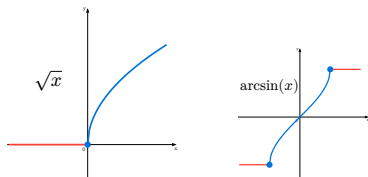
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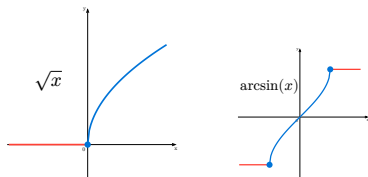
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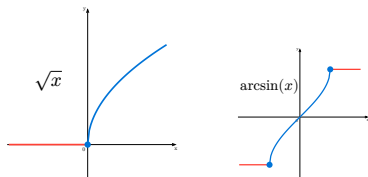
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Contributions

Recall: Equivalence Theorem, [Fu and Zucker, 2014]

For any **acceptable function** $f : \mathbb{R} \rightarrow \mathbb{R}$ and any effective open exhaustion X for $\text{dom}(f)$, the following are equivalent:

- f is an α -computable function.
- f is **WhileCC**-approximable.
- f is GL-computable w.r.t. X .
- f is effectively locally uniformly multipolynomially approximable w.r.t. X .

Theorem 1 (**WhileCC**-approximability Theorem)

All elementary functions are **WhileCC**-approximable.

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All elementary functions are acceptable.

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Theorem 2.1 (Acceptability Theorem: Part 1)

The domain of any elementary function has an effective open exhaustion.

Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous w.r.t. an effective open exhaustion for its domain.

Result 1

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Result 1 - Background - WhileCC Programming Language

Syntax

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Definition (**WhileCC**-approximability, [Fu and Zucker, 2014])

A **WhileCC**-procedure P of type $\text{real} \times \text{nat} \rightarrow \text{real}$ on \mathcal{R} is said to *approximate* a function $f : \mathbb{R} \rightarrow \mathbb{R}$ iff for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$:

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where $\text{Nbd}(y, r)$ has the standard definition of neighborhood on \mathbb{R} i.e.,

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Goal

Construct **WhileCC**-procedures approximating elementary functions by induction

Challenge: Using comparison operators introduces undefinedness.

How do we **WhileCC**-approximate “piecewise” functions?

Result 1 - Background

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Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

```
proc
  in  $x$  : real  $c$  : nat
begin
  if  $x <_{\mathbb{R}} 0$  then
    return 0
  else
    return Root( $x, c$ )
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end
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Solution

- Find an overlapping interval where both pieces are defined
- Use the nondeterminism of “choose”

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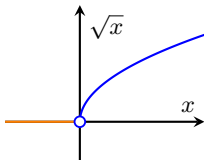
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Result 1 - Challenges

Problem

When defining piecewise functions, comparison makes a hole!

Example: Even Root - First Attempt

```
proc
  in  $x$  : real  $c$  : nat
begin
  if  $x <_{\mathbb{R}} 0$  then
    return 0
  else
    return Root( $x, c$ )
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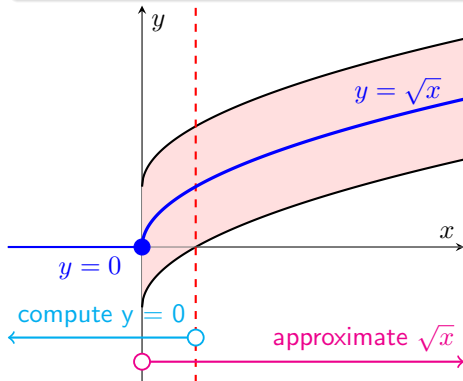
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Result 2

Theorem 2.1 (Acceptability Theorem: Part 1)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an elementary function. Then, $\text{dom}(f)$ has an effective open exhaustion.

First attempt – Strengthening:

Elementary function constructions preserve the property that the domain has an effective open exhaustion.

- Base cases ✓ (e.g. $\sin(x)$)
- Addition and multiplication ✓ (e.g. $(f + g)(x)$)
- Composition case has a counterexample:

$$f(x) = \text{id}|_{(-1,1)} \text{ and } g(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\text{dom}(f)$ has an effective open exhaustion ✓

$\text{dom}(g)$ has an effective open exhaustion ✓

$\text{dom}(f \circ g) = [-1, 1]$ has no open exhaustion ✗

Strengthened to proving exhaustion reflection property

For any open set U with an effective open exhaustion, $f^{-1}(U)$ has an effective open exhaustion.

Proof: By induction

- Base cases ✓
- Composition ✓
- Addition and multiplication ✗

Adding decomposition of $+$ and \cdot

$(f + g)(x) = f(x) + g(x)$ is composed of

- $\text{Add}(x, y) = x + y$,
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Theorem (Reducing Exhaustion-reflection to a Decision Procedure)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is exhaustion-reflecting, if we can decide for any m -cube Q_m whether an arbitrary rational closed n -cube is completely contained in $f^{-1}(Q_m)$.

- This is very useful for proving the exhaustion-reflection property for the addition and multiplication case.

Building an effective open exhaustion

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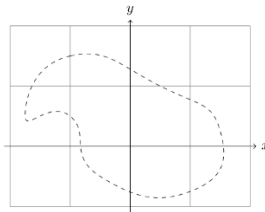
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(a) step 0



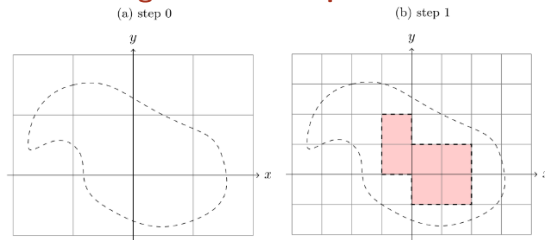
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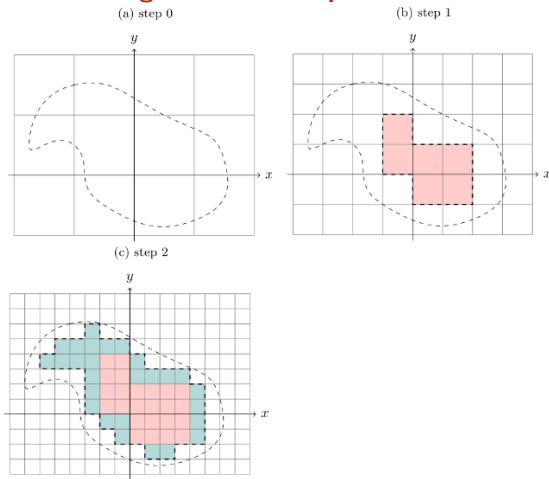
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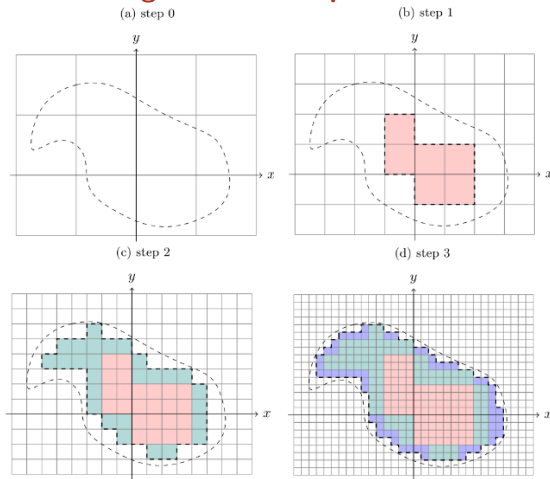
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Theorem 2.2 (Acceptability Theorem: Part 2)

Any elementary function is effectively locally uniformly continuous **w.r.t. an effective open exhaustion** for its domain.

(We prove that this is) equivalent to proving:

Any elementary function has a local continuity witness.

Definition (Local continuity witness)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$. A recursive function $N : \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \rightarrow \mathbb{N}$ is called a **local continuity witness** for f iff for any $a, b \in \mathbb{Q}$ with $[a, b] \subseteq \text{dom}(f)$ and $k \in \mathbb{N}$, we have

$$\forall x, y \in (a, b) \quad |x - y| < 2^{-N(a, b, k)} \implies |f(x) - f(y)| < 2^{-k}.$$

The notion of effective local uniform continuity is independent of effective open exhaustion.

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Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be **WhileCC**-approximable and monotone on its domain. Then f has a local continuity witness.

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We proved that:

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- all elementary functions are acceptable.

We also

- presented an **alternative characterization** of acceptable functions using the **local continuity witness** concept, and
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Questions left unanswered:

- Are non-unary elementary functions acceptable?
A generalization of acceptability in arbitrary metric spaces is given by [Tucker and Zucker \[2004\]](#).
- Can we extend the equivalence theorem in [Fu and Zucker \[2014\]](#) to acceptable partial functions of type $\mathbb{R}^m \rightarrow \mathbb{R}$?
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Conjecture:

- All partial unary **WhileCC**-approximable functions are acceptable.
 - **If the conjecture holds**, are *non-unary* **WhileCC**-approximable functions acceptable?
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A HUGE Thank you!