Diversification and Financial Stability

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Take-home Message

1. **Contribution to the debate on**
   1. double-edge properties of the financial system
   2. financial networks architecture and systemic risk
2. **Model layout**
   1. interconnected banks invest inf external asssets/projects
   2. External assets may generate positive or negative cash-flows
3. **Results**
   1. DP/Exp. Utiltiy increases (decrease) with diversification in case of down-turn (up-turn)
   2. There exists a subset of prob. of down-turn to which correspond an optimal level of risk diversification
   3. Banks'incentives favor a financial network that is over-diversified w.r.t. the diversification that is socially desirable

Introduction

- **Credit Crises 2007/08**: knife-edge properties of the financial system (too Homogeneous and Complex)*;
- **Classical view**: Diversification is always valuable†;
- **Reasearch Question**: Banks diversification in "external assets" is always desirable? Are individual incentives converging to social welfare ?

*See Haldane 2009, Rethinking the Financial Network, speech.

†E.g., See Markowitz H. 1952, Portfolio Slection, Jorunal of Finance.

Balance Sheet approach

- **Economy**:
  n financial risk-averse leveraged banks with **m** external investment opportunities;
- **Balance-Sheet identity**:

$$\mathbf{a}_i = \bar{\mathbf{p}}_i + \mathbf{e}_i \quad \forall i \in \{1, ..., n\}$$

where

- **a**: vector of banks assets' market values;
- $\bar{\mathbf{p}}$: vector of the book-face values of banks obligations;
- **e**: vector of equity values.

Two asset classes: m External and n Internal

Each bank assets are combination of external investment opportunities and claims towards other banks in the network

$$\mathbf{a}_i := \sum_j^m \mathbf{Z}_{ij}\mathbf{y}_j + \sum_j^n \mathbf{W}_{ij}\mathbf{p}_j$$

- $\mathbf{p}_j := \bar{\mathbf{p}}_j./\left[(1+r_j)\mathbf{1}_j\right]^{-t}$: vector of market value of bank-j's debts;
- $\mathbf{y}$: vector of external investments;
- $\mathbf{Z}_{n\times m}$: weighting matrix of external investments where each entries z_{ij} is the proportion of activity-j held by bank-i;
- $\mathbf{W}_{n\times n}$: adjacency matrix where each entries w_{ij} is the proportion of bank-j debt held by bank-i;
- r_j: rate of return used to discount the face value of obligor-j's debt.

Dynamics of external assets cash-flow

Each external asset/projects generates a log-normal cash flows process

$$d\log(Y_j)(t) := dy_j(t) = \mu_j dt + \sigma_j dB_j(t) \quad \forall j = 1, ..., m$$

- $B_j(t)$ standard Brownian;
- i.i.d. returns and $d(B_i, B_j) = \rho_{ij}$.

Leverage and balance sheet management

Banks Leverage: (debt/assets)

$$\begin{aligned}\phi_i &:= \bar{\mathbf{p}}_i./\mathbf{a}_i \quad \in [0,1] \\ &:= \bar{\mathbf{p}}_i./\left(\sum_j^m \mathbf{Z}_{ij}\mathbf{y}_j + \sum_j^n \mathbf{W}_{ij}\mathbf{p}_j\right)\end{aligned}$$

- $\mathbf{p}_j := \bar{\mathbf{p}}_j./\left[(1+r_j)\mathbf{1}_j\right]^{-t}$: vector of market value of bank-j's debts;
- $r_j := r_f + \beta\phi_j$: internal rate of return

Then,

$$\phi_i := \bar{\mathbf{p}}_i./\left(\sum_j^m \mathbf{Z}_{ij}\mathbf{y}_j + \sum_j^n \mathbf{W}_{ij}\frac{\bar{\mathbf{p}}_j}{(1+r_f+\beta\phi_j)}\right)$$

Mean Field Analysis

- $\bar{p}_i = \bar{p} \quad \forall i = 1, ...n;$
- $\phi_i = \phi \quad \forall i = 1, ...n;$
- For $z_{1j} \sim z_{2j} ... \sim z_{nj} \approx 1/m$, $\forall j = 1, ..., m$

$$\sum_j^m \mathbf{Z}_{ij} \mathbf{y}_j \sim \frac{1}{m} \sum_i^n \sum_j^m z_{ij} y_j = \frac{1}{m} \sum_j^m y_j := y$$

- For $w_{1j} \sim w_{2j} ... \sim w_{nj} \approx 1/n$, $\forall i = 1, ..., n$

$$\sum_j^n \mathbf{W}_{ij} \mathbf{p}_j \sim \frac{1}{n} \sum_j^n p_j := p$$

From M-F approximation, the fragility becomes

$$\phi = \frac{\bar{p}}{y + p} = \frac{\bar{p}}{y + \frac{\bar{p}}{1 + r_f + \beta\phi}}$$

Solving for ϕ

$$\phi\,(y + r_f y + \beta\phi y + \bar{p}) = \bar{p}(1 + r_f + \beta\phi)$$
$$\ldots$$
$$\phi^2 \beta y + \phi\,(yR + \bar{p}(1 - \beta)) - \bar{p}R = 0$$

Then

$$\phi = \frac{1}{2\beta y}\left[\bar{p}(\beta - 1) - Ry + \left(4\beta\bar{p}Ry + (\bar{p}(1 - \beta) + Ry)^2\right)^{1/2}\right]$$

where $R = 1 + r_f$.

- The fragility is a solution of a quadratic equation in ϕ;
- There exist a precise dependence of ϕ from y

Dynamics of external assets cash-flow

- Each external asset/projects generates a log-normal cash flows process

$$d\log(Y_j)(t) := dy_j(t) = \mu_j dt + \sigma_j dB_j(t) \quad \forall j = 1, ..., m$$

  - $B_j(t)$ is standard Brownian;
  - i.i.d. returns and $d(B_i, B_j) = \rho_{ij}$.
- By the M-F approx., $y = \frac{1}{m}\sum_j^m y_j$. Then,

$$dy = \mu_y dt + \sigma_y dB \quad \forall i = 1, ..., n$$

where

  - $\mu_y := \frac{1}{m}\sum_{j=1}^m \mu_j$;
  - $\sigma_y := \sqrt{\frac{\sigma^2}{m} + \frac{m-1}{m}\bar{\rho}\sigma^2}$

Default Probability

Proposition

- **Default Probability:** $\mathbb{P}(\phi = b_\phi)$.
  It is the probability that the fragility ϕ, initially starting at an arbitrary level $\phi_0 \in (0 \leq a_\phi, b_\phi \leq 1)$ exits through b_ϕ;
- **Default Probability:** $\mathbb{P}(y = a_y)$.
  - *explict form*

$$\mathbb{P}(y = a_y) := \frac{\left(\int_{y_0}^{b_y} dy\psi(y)\right)}{\left(\int_{a_y}^{b_y} \mathrm{d}y\psi(y)\right)}; \quad \psi(x) = \exp\left(\int_0^x -\frac{2\mu_y}{\sigma_y^2}\,\mathrm{d}y\right)$$

  - *closed form solution*

$$\mathbb{P}(y = a_y) = \left(\exp\left(-\frac{2\mu_y y_0}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right) / \left(\exp\left(-\frac{2\mu_y a}{\sigma_y^2}\right) - \exp\left(-\frac{2\mu_y b}{\sigma_y^2}\right)\right)$$

Let define
$p := \mathbb{P}(\mu_y \leq x)$;
$q := \mathbb{P}(y = a_y \mid \mu_y < 0)$;
$g := \mathbb{P}(y = a_y \mid \mu_y > 0)$.

Proposition (3)

The definition of default probability $\mathbb{P}(y = a_y)$ in Prop.2, implies

$$\exists \quad m^* > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > m^*$$

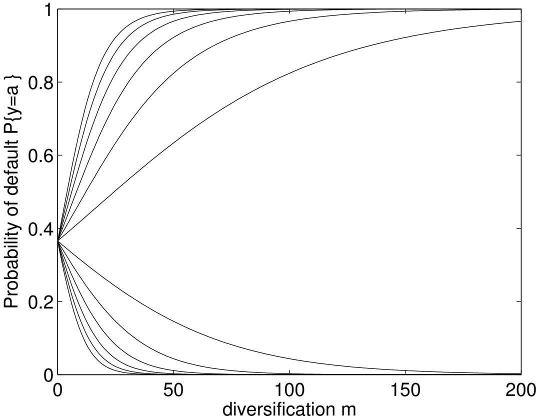


Figure: The lower curves shows the variation of $\mathbb{P}(y = a_y)$ when $\mu_y > 0$ for increasing degree of diversification m. The upper curves shows the variation of $\mathbb{P}(y = a_y)$ when $\mu_y < 0$ for increasing degree of diversification m.

Optimal Diversification Policy

Definition (Max Profit/Loss)

Ceteris paribus, the profit of the financial system is bounded between a lower barrier ($y = a_y$) an an upper barrier ($y = b_y$). Then,

Max Gain: $\pi^+ := b_y - y_0$;

Max Loss: $\pi^- := y_0 - a_y$

With y_0: aggregate value of external assets when financed at time $t = 0$ by the banking sector.

Definition (Naive Diversification Strategy)

Select the number m of external investment (assets and/or projects) in which to invest an equal dollar amount.

Definition (**Bank's Maximization Problem**)

banks are mean-variance decision makers, such that the utility function $\mathbb{E}U(\Pi_m)$ may be written as a smooth function $V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right)$ of the mean $\mathbb{E}(\Pi_m)$ and the variance $\sigma^2(\Pi_m)$ of Π_m or

$$V\left(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)\right) := \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2}$$

Then,

$$\max_m \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2} \tag{1}$$

$$\text{s.t.:} \quad m \geq 1\,;\, \frac{1}{n}\sum_i^n \sum_j^m z_{ij} = 1$$

Expected Profit:

$$\mathbb{E}(\Pi_m) := p\left[q\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right] \quad (2)$$

Variance:

$$\begin{aligned}\sigma^2(\Pi_m) := p&\left[q\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] \\ &+ (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right]\end{aligned} \quad (3)$$

Remind Proposition 3:

$$\exists \quad m^* > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > m^*$$

Corollary (Increasing/Decreasing $\mathbb{E}U(\Pi_m)$ w.r.t. m and μ_y)

- *If $\mu_y < 0$ with probability one ($p = 1$) $\Rightarrow$*
  (i) for a fixed μ_y, $\mathbb{E}U(\Pi_m)$ is **monotonically decreasing** *in m;*
  (ii) for a fixed m, $\mathbb{E}U(\Pi_m)$ is **monotonically decreasing** *in μ_y;*
- *If $\mu_y > 0$ with probability one ($p = 0$) $\Rightarrow$*
  (i) for a fixed μ_y, $\mathbb{E}U(\Pi_m)$ is **monotonically increasing** *in m;*
  (ii) for a fixed m, $\mathbb{E}U(\Pi_m)$ is **monotonically increasing** *in μ_y.*

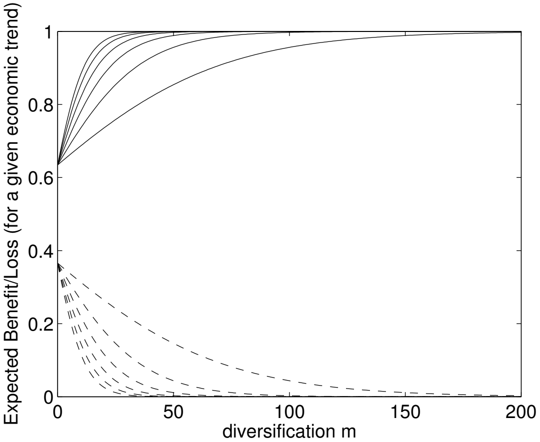


Figure: The $\mathbb{E}U(\Pi_m)$ is increasing in m when the trend of external assets is positive (upper curves). While it is decreasing in m when the trend of external assets is negative (lower curves). Parameters: $\sigma^2 = 0.35, \quad r_f = 0.03, \quad \pi^- = \pi^+ = 1, \bar{\rho} = 0.1, \; m \in [0, 200]$.

Corollary (Optimal Level of Diversification m^*)

Let the event down-turn occurs with probability p, and the event up-turn occurs with probability $1-p$ where $p \in \Omega_P := [0,1]$. Then, there exists $\Omega_{P^*} \subset \Omega_P$ s.t., for a given $p^* \in \Omega_{P^*}$,

$$m^* = \left[(g^{-1}; q^{-1}) \circ f^{-1}\right](p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{m^*}) \geq \mathbb{E}U(\Pi_m) \quad \forall m \gtrless m^*$$

with $f := 1/\left(1 + \frac{q(m^*)}{g(m^*)}\left(\frac{\partial q(m^*)}{\partial g(m^*)}\right)\right)$.

For a fixed absolute value of $|\mu_y|$ we assign a probability p to the event down-turn and $1-p$ to the up-turn. Then, for certain values of p^*, there exist a specific optimal level of diversification m^* which is a function of p^* and maximizes $\mathbb{E}U(\Pi_m)$.

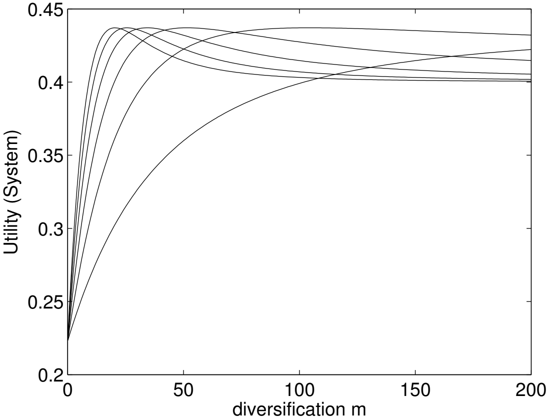


Figure: $\mathbb{E}U(\Pi_m)$ exhibits a maximum w.r.t. m for different levels of $p = 0.4$. Parameters: $\sigma^2 = 0.35$, $r_f = 0.03$, $\pi^- = \pi^+ = 1, \bar{\rho} = 0.1$, $m \in [0, 200]$, $|\mu_y| = 0.005, 0.01, 0.015, 0.020, 0.025, 0.030$.

Private Incentives Vs. Social Welfare

A policy maker has to include some **negative externalities** which might be generated by losses occurred in the bad states of world. In this scenario, the regulator it is plausibly to includes **social costs** (e.g., unemployment) that might emerge due to the losses suffered by the financial system. Then for the regulator standpoint of view, the total loss to be accounted in bad states, is a monotonically increasing cost function of the number k of losses

$$f(k, \pi^{-}) := k\pi^{-} \quad \text{with} k > 1$$

Treating the policy-maker as an expected utility maximizer, it has an objective function $\mathbb{E}U_R(\Pi_m)$ which is differently expressed w.r.t. $\mathbb{E}U(\Pi_m)$. Explicitly, eq.(14) and (15) become

Expected Profit Regulator:

$$\mathbb{E}_R(\Pi_m) = p\left[qk\pi^- + (1-q)\pi^+\right] + (1-p)\left[g\pi^- + (1-g)\pi^+\right]$$

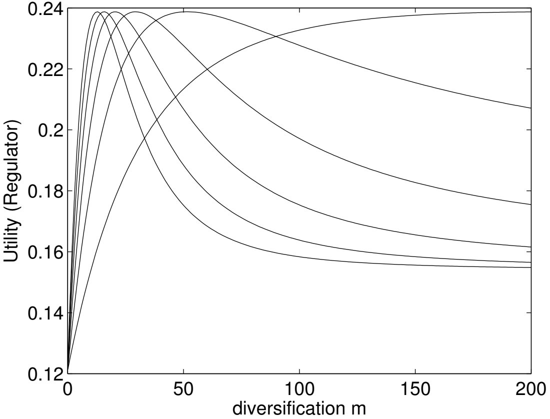
Variance Regulator:

$$\begin{aligned}\sigma_R^2(\Pi_m) = p &\left[q\left(k\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-q)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right] \\ &+ (1-p)\left[g\left(\pi^- - \mathbb{E}(\Pi_m)\right)^2 + (1-g)\left(\pi^+ - \mathbb{E}(\Pi_m)\right)^2\right]\end{aligned}$$

Corollary

Individual banks' incentives favor a financial network that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable

$$m^* \geq m^R$$



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