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Multipolar social systems: Measuring polarization beyond dichotomous contexts

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ABSTRACT

Social polarization is a growing concern worldwide, as it strains social relations, erodes trust in institutions, and thus hurts democratic societies. Polarization has been traditionally studied in binary conflicts where two groups support opposite ideas. However, in many social systems, such as multi-party democracies, political conflicts involve multiple dissenting factions. Despite the prevalence of multipolar systems, there is still a lack of suitable analytical tools to study their polarization patterns. In this work, we introduce new polarization metrics for multidimensional scenarios and develop a methodology that extracts the ideological structure of multipolar contexts from social networks. We propose the trace of the covariance matrix (the total variation) of the multipolar opinion distribution as a measure of global polarization, and its eigendecomposition to identify the directions of maximum polarization. Instead of using a pre-conceived opinion space (conservative vs progressive, liberal vs authoritarian, etc.), our framework reveals the natural ideological axes of the system. We apply our methodology to quadripolar and pentapolar real-world democratic processes, finding non-trivial ideological structures with clear connections to the underlying social context. Our framework opens new avenues for understanding multilateral social tensions and facing the challenges of polarization in a unified way.

1. Introduction

Social polarization is a pervasive phenomenon [1] that has been observed in a wide variety of contexts like elections [2], referenda [3] and around controversial issues [4–7]. It has been argued that polarization exerts a negative influence over democracy [8], undermining social and economic relationships [9,10], reinforcing inequality [11,12], causing legislative gridlocks [13,14], and even posing health risks [4,15]. However, it has also been found that some level of polarization can have healthy effects, as it produces clearer party choices [16,17] and more constructive, focused and linguistically diverse debates [18]. In any case, to fully understand social dynamics we need tools to detect and measure polarization [19], as it is a core feature of many systems. In this work we develop a multidimensional opinion inference framework for networked systems and tools to measure and characterize the polarization patterns of the inferred opinion distributions.

Polarization is usually defined as the division of a population into two conflicting or contrasting groups [20,21]. This intuitive notion coincides with one of the aspects of polarization: *opinion extremeness*, which is quantified by measuring how concentrated is the distribution of attitudes of the population towards some issue at two polar opposite positions (for example, immigration, abortion, or cannabis legalization). Although one-dimensional (or bipolar) aspects of polarization like opinion extremeness are by far the most studied [22,23], there are other fundamental facets that can only be understood from a multidimensional perspective [20,24]. For instance, in a polarized context the policy preferences of the population on different issues are often correlated (a person that is in favor of strong immigration controls may also be against free abortion and cannabis legalization, and vice versa) [7,25]. This means that the individuals' opinions get aligned along multiple lines of potential disagreement, exacerbating tensions and leading to a new kind of polarization: *opinion alignment* [26]. If a system is highly polarized in the sense of opinion alignment, one can infer the ideological positions of an individual towards several issues by knowing her position on only one of them. A typical example is a highly partisan democracy [24,27], where party identification determines many of the ideological positions of individuals. Societies with strong partisanship can also present *affective polarization* [9], meaning that supporters of the different parties dislike or even loathe each other.

Opinion alignment is usually measured by analyzing the correlation between one-dimensional attitude distributions (one per issue or topic)

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built from opinion polls. These polls ask dichotomous questions about topics such as public health insurance, treatment of minorities or abortion [24]. The answers are typically organized in a Likert-like scale, ranging from complete agreement with something to complete disagreement. While this approach offers a multidimensional perspective, it is still fundamentally bipolar, as it is based in combining dichotomous issues. However, there are social contexts whose ideological space cannot be modeled as a combination of bipolar axes: scenarios where opinions about a given issue are divided into more than two qualitatively different positions [28]. We call these contexts *multipolar systems*. The most representative examples are multi-party democracies, in which more than two parties have a realistic chance of obtaining significant representation. We model these systems by considering each party as an opinion pole. The intrinsic complexity of reaching multilateral government agreements sometimes leads to political instability and gridlock, causing recurrent repetition of elections and long periods of interim executives. Recent examples are found in Belgium [29], Israel [30], or Spain [31]. Understanding *multipolar systems* [32] is critical to identify the main lines of disagreement and address these issues, but their multidimensional polarization patterns are still largely unexplored. In this paper we model the ideological space of multipolar systems with n opinion poles by placing each pole (for example, the parties in a multi-party democracy) at the vertex of a regular simplex of dimension $n-1$ (a multidimensional generalization of an equilateral triangle, which would correspond to a tripolar system). This way every pole is at the same distance of the others, avoiding the introduction of a priori biases.

The representation of opinions in an Euclidean space is a widespread practice in political science [25]. In spatial vote models, a voter's support for a party depends on the distance between the positions of the voter and the party in the opinion space [33]. The opinion points of the voters and the parties are usually obtained by applying techniques such as factor analysis [34] to the answers of surveys about a set of issues. However, a voter's support for a given party not only depends on ideological closeness, but also on other factors such as party valence (the perceived competence of the party), the attributes of the party leader, or the voter's sociodemographic features [35]. To get an estimation of the relative support of an individual to each opinion pole, we will use the information encoded in interaction networks. After all, the formation of social ties between individuals is often related to their ideological affinity [36], potentially leading to *network polarization* [11,37]; that is, the organization of a social network in highly connected subgroups (or clusters) with weak inter-group connectivity [21,38,39].

Therefore, to get a complete understanding of polarization, it is necessary to consider not only the ideological stances of the polarized set of individuals or parties, but also the relationships among them [40–42]. Taking this into account, we use an opinion inference framework based on networks of user interactions in online social media [43] (although the framework can be applied to any networked system). This methodology allows us to estimate the opinion distribution of the users and circumvents the common drawbacks of attitude inference techniques based on the content of user messages, such as the dependence on context and language [21].

To characterize and measure the polarization of the inferred opinion distribution we propose different metrics based on the covariance matrix, which is the multidimensional generalization of the variance, a quantity often adopted as a one-dimensional measure of polarization [22,23]. In particular, we use the trace of the covariance matrix (the total variation) as a global measure of *opinion extremeness*, and its eigendecomposition to quantify *pole alignment* (a multipolar analogue of opinion alignment), obtaining the direction of maximum polarization in the ideological space [44].

Summarizing, our multipolar framework is composed of three main elements: a multipolar ideological space, an opinion inference technique based on user interactions that yields a multidimensional opinion distribution and a set of polarization measures that are applied to that distribution. This approach synthesizes the ideological and networked perspectives of polarization and provides means to study multipolar analogues of both opinion extremeness and opinion alignment. To illustrate our new framework in practice, we apply it to Twitter data of two multipolar electoral contexts, the Spanish general elections of the 20 of December 2015, with four poles, and the Spanish general elections of the 28 of April 2019, with five poles.

2. Results

We begin this section by laying out the foundations of the multidimensional opinion inference methodology, the ideological space built from pole vectors and the multi-polarization measures we use on the opinion distributions. Then, we apply this framework to two empirical systems, showing how our results can be easily interpreted in terms of the underlying social context and illustrating the usefulness of this methodology to characterize multipolar systems. To help build an intuition about the methodological framework and facilitate the analysis of the data, we have prepared interactive visualizations that can be found at https://vis.csh.ac.at/multipolar-viz.

2.1. Opinion inference and polarization measurement

2.1.1. The multidimensional opinion model

Our multidimensional opinion inference technique is a generalization of a bipolar (one-dimensional) methodology [43] based on models of opinion dynamics. But instead of using a model to understand how opinions are disseminated and adopted, we use it as an inference tool. The process consists in building a network of social interactions from empirical data, identifying the opinion leaders and their respective ideological positions [45]. Then, we use the model to propagate the leaders' opinions throughout the rest of the nodes. Finally, we take the model's outputs (the converged opinions) as the inferred opinions of the nodes. Therefore, the opinion inference technique requires:

- *A network of interactions*: we build it using data from social networking sites.
- *The elite*: n small disjoint subsets of nodes assigned to each of the n opinion poles to be used as opinion seeds. The nodes of each subset should have different and extreme opinions about the topic under study. We choose them using contextual information.
- *An opinion learning mechanism*: An iterative opinion updating procedure to infer the opinions of the rest of the nodes (*the listeners*).

For the learning mechanism, we have adopted a modified version of the DeGroot linear learning model [46]. In this model, each *elite* node is assigned a fixed opinion depending on the option they support. Then, the opinion of the remaining nodes of the network (the *listeners*) is computed by iteratively averaging the opinions of their neighbors. This modified version of the DeGroot model including nodes with fixed opinions was originally developed by Friedkin and Johnsen [47] and has been extensively applied in bipolar contexts [2,5,43]. A multidimensional extension of the Friedkin–Johnsen (FJ) model has been proposed and thoroughly characterized by Parsegov et al. [48], so we draw on their results for our inference technique.

In the FJ model a node's opinion is not only influenced by its neighbors, but also by the node's own prejudices. The relative influence of the prejudices and the neighbors' opinions is controlled by a *stubbornness parameter* that can be different for every node. We consider a particular version of the FJ model with only two kinds of nodes (two possible values for the stubbornness parameter): stubborn nodes that only trust their prejudices (*elite*) and nodes with no prejudices (*listeners*).

Formally, let G be a directed and weighted network with a set V of nodes, a set E of links and an adjacency matrix A with A_{ij} the weight

Table 1
Coordinates of the vertices of the opinion simplex ($\vec{v}_0, \vec{v}_1, \dots$) and distance between them for a regular k-simplex of different dimensions with vertices placed at distance $u = 1$ from its center.

Number of dimensions	Distance between poles	$\vec{v}_0$	$\vec{v}_1$	$\vec{v}_2$	$\vec{v}_3$
1	2	$+1$	-1		
2	$\sqrt{3}$	$(1, 0)$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	
3	$\sqrt{8/3}$	$(1, 0, 0)$	$(-\frac{1}{3}, \frac{\sqrt{8}}{3}, 0)$	$(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3})$	$(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3})$
k	$\sqrt{2\frac{k+1}{k}}$				

of the link $i \to j$. We will work with retweet data, and build retweet networks G by making a link $i \to j$ when user i retweets user j. A retweet usually implies that i agrees with j's ideas, so we can obtain the influence network $\hat{G}$ by transposing the adjacency matrix of G. A weighted link from j to i in $\hat{G}$ represents the weighted influence that j has on i. Let $S \subset V$ be the set of opinion seeds or *elite* nodes and $L = V \setminus S$, the set of *listeners*. Each elite node is assigned an opinion vector $\vec{x}_s$ that will remain constant throughout the iterative learning process. We call those extreme opinion vectors the *opinion poles*, we will define them below. The listener nodes are initially assigned a neutral opinion vector $\vec{x}_l = \vec{0}$ of the corresponding dimension. The opinion of a listener is inferred by iteratively averaging the opinion of their neighbors, such that at each time step t the opinion of a given listener $\vec{x}_i(t)$ is given by:

$$\vec{x}_i(t) = \frac{\sum_j A^*_{ij} \vec{x}_j(t-1)}{\sum_j A^*_{ij}} \tag{1}$$

With A^*_{ij} the i, j element of a modified retweet adjacency matrix defined in such a way that elite nodes do not change opinion throughout the learning process. This is done by replacing all their outgoing links (outgoing retweets) by a single self-loop (their rows are changed by vectors of 0s with a 1 in the diagonal, so we use the Kronecker delta δ_{ij}):

$$A^*_{ij} = \begin{cases} A_{ij} & if \quad i \in L \\ \delta_{ij} & if \quad i \in S \end{cases} \tag{2}$$

This averaging process is iterated until convergence, which is guaranteed (and the solution is unique) as long as every listener is connected to at least one elite node through a *directed path* [48]. If $X(t)$ is the matrix whose rows are the opinion vectors of the listeners, such that in a k-dimensional opinion space $\vec{x}_i(t) = [X_{i1}(t), X_{i2}(t), \dots, X_{ik}(t)]$, we consider that $X(t)$ has converged when $\|X(t) - X(t-1)\|_1 < tol$, where $\|M\|_1 = \sum_{i,j} |m_{ij}|$ is the entrywise 1-norm of the matrix M. We have set the tolerance to $tol = 10^{-6}$.

2.1.2. Pole vectors and the opinion simplex

The cornerstone of our multipolar methodology are the elite opinion vectors or *poles*, as they determine the structure of the ideological space. In bipolar settings [43], a scalar value of $x_s = +1$ or $x_s = -1$ is assigned to the nodes of the two opposing elite subsets, placing the poles at a symmetrical position around zero. To generalize this idea for multipolar systems, we need to place the poles at a fixed distance from the neutral ideological position (say $\vec{0}$) and in such a way that the distance between any two poles is the same. A geometrical object that fulfills these requirements is the regular simplex. A regular k-simplex is a generalization of the concept of equilateral triangle (which is a regular 2-simplex) to k dimensions. For a system with n poles we build a regular $(n-1)$ simplex and place the poles on its vertices (therefore, $\vec{x}_i \in \mathbb{R}^{n-1}$). In particular, we have chosen a regular simplex centered at the origin with the vertices at distance $u = 1$ from the origin. We detail the computational process to obtain the coordinates of the pole vectors for an opinion space of arbitrary dimension in Supplementary Section 1. In Fig. 1A we show the opinion spaces of dimension 1, 2 and 3. Notice that the distance between poles is different for opinion spaces of different dimension [49] (see Table 1).

We set the same distance between every pair of poles to adopt the most neutral, unbiased, or uninformative configuration. In some contexts it may seem more appropriate to place certain poles closer together; for example, in an election certain political parties may be more ideologically aligned with each other than with the rest. However, parties that share ideology also compete for the same electoral base, often leading to even greater antagonism between them. With our parsimonious approach, we impose no a priori bias and infer the affinity between poles from the opinion distribution obtained from interactions between individuals.

It is worth noting that the opinion assigned to any node will always be located within the convex hull of the opinion poles, as the opinion updating mechanism is a convex combination and the pole vectors form a regular simplex. Therefore, if a node updates its opinion by increasing its support to a given pole (getting closer to it), its support for the remaining poles will decrease in the same proportion (will get equally away from the others). This implies that each node has the same constant *opinion budget* that it allocates among the different poles (see Supplementary Section 2 for a formal derivation).

In Fig. 1B we illustrate how the opinion simplex works in a simple tripolar system consisting of a network with one listener node connected to three elite nodes, each associated to a different pole. If all the links have the same weight, the opinion of the listener node is neutral (see panel B.i); but if the weight of one of the links increases, the listener's opinion gets closer to the corresponding pole and, importantly, gets *equally away* from the others (see panel B.ii). Panel B.iii shows a situation with different weights for every pole and panel B.iv shows that when the listener is only connected to two of the poles (the remaining weight is 0), its opinion lies on the edge that joins the two poles.

The previous simple example only required one iteration of Eq. (1) to obtain the opinion vector of the listener node. In larger networks, it takes several iterations to achieve convergence. To better illustrate the process we use a tripolar synthetic network built with the Lancichinetti–Fortunato–Radichi (LFR) method [50]. This method generates scale-free modular networks with a given number of nodes (we have chosen 1300), a given number of communities (we have chosen 3, to simulate a tripolar system), and a certain probability of each node to be connected to nodes of other communities rather than their own. The higher this probability, the more mixed is the network and the less distinguishable are the modules. We have set this parameter to $\mu = 0.1$. For the opinion inference process, we have built the three subsets of elite nodes of the three poles by ranking the nodes according to their degrees and choosing the top 6 nodes of each community as the opinion seeds of each pole. In Fig. 1C we show the iterative opinion inference process on this network.

2.1.3. Measures of multi-polarization

To study polarization in multipolar contexts we propose several metrics based on the covariance matrix, which is the natural multidimensional extension of the variance, one of the most common measures of social (bi)polarization [51–53]. Firstly, we define a multipolar generalization of the notion of *opinion extremeness*; secondly, we propose a way to compute the directions along which polarization is maximal, allowing us to measure *pole alignment*.

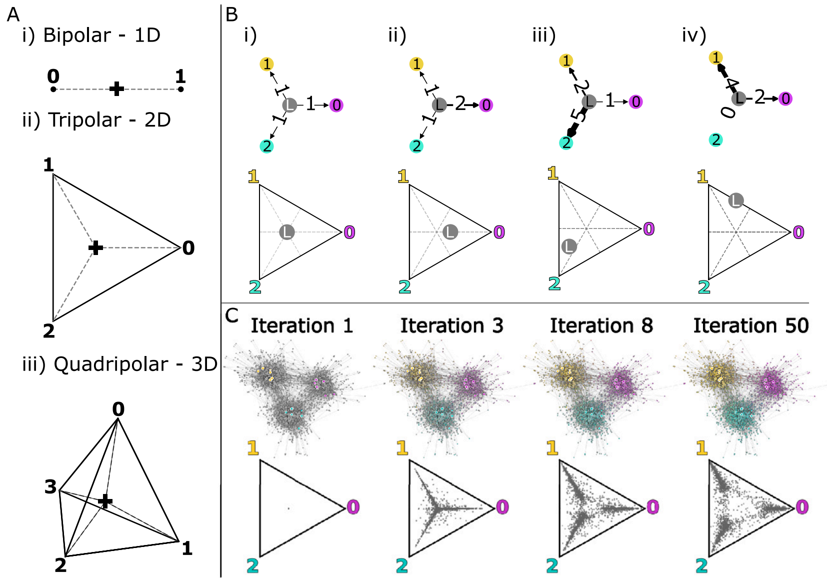

Fig. 1. Illustrating the ideological space and the opinion inference process. A: Diagram of the opinion spaces of dimension 1 (i), 2 (ii) and 3 (iii). While in 1D the poles can be intuitively labeled as against (−1) or in favor (+1) of something, in higher dimensions they are labeled with numbers from 0 to $n-1$ where n is the number of poles. The neutral point (the barycenter of the simplex's vertices) is marked with a black cross. B: Examples of the opinion position of a listener node for three different weighted networks of a tripolar system. Top: network of the tripolar system with one listener node in gray (L) connected to three elite nodes corresponding to the three poles. The numbers on the links are their weights. Bottom: the corresponding opinion simplex showing the opinion position of the listener node. C: Evolution of the opinions of the nodes driven by Eq. (1). The nodes of the networks in the top row are colored according to their closeness to each pole in the ideological space of the bottom row. Interactive versions of panels B and C can be found in https://vis.csh.ac.at/multipolar-viz.

The covariance matrix of a random vector $\vec{Y}$ with expected value $\mathbb{E}[\vec{Y}]$ is defined as follows:

$$Cov[\vec{Y},\vec{Y}] = \mathbb{E}[(\vec{Y}-\mathbb{E}[\vec{Y}])(\vec{Y}-\mathbb{E}[\vec{Y}])^T] \tag{3}$$

The trace of the covariance matrix can be interpreted as a measure of *multidimensional variance*, usually called *total variation* [54] (TV). Moreover, as we show in Section 4, when the distance between the barycenter of the opinion simplex and the poles is $u=1$, the maximum attainable TV is 1, so this metric is normalized by design. Crucially, the maximum TV is achieved when there are only extreme opinions and they are uniformly distributed among all the poles. Therefore, the TV can be used as a measure of global polarization combining the aspects of *opinion extremeness* and *community fragmentation* [52]; that is, it not only measures how extreme the opinions are but also how evenly is the population divided into the considered factions.

To characterize and quantify *pole alignment* we compute the eigendecomposition of the covariance matrix, a technique commonly referred to as *Principal Component Analysis* (PCA) [54]. The eigenvectors (or Principal Components — PCs) form an orthogonal basis of the opinion space and their corresponding eigenvalues are the variances of the opinion distribution projected onto the direction of their associated eigenvector. Therefore, the eigenvector with the largest eigenvalue (or first Principal Component) corresponds to the direction of maximum variance; that is, the direction of maximum polarization [34,55]. We will show that the projections of the opinion distribution onto the directions of maximum polarization reveal the latent ideological space of the system. We propose the proportion of variance explained by the first PCs as a measure of *pole alignment*. If it is high, it means that the system's ideological structure can be approximated by a space of lower dimension (most of the tensions lie in a subspace). PCA allows us to extract the natural ideological dimensions that organize the social tensions of the system.

To help build an intuition of these polarization metrics, in Fig. 2 we present synthetic opinion distributions on a tripolar system with different levels of opinion extremeness and pole alignment, respectively measured by their TV and the relative amount of variance explained by the PCs.

2.2. Empirical study of multipolar systems

In this section, we illustrate the application of our opinion inference and multi-polarization measurement methodology to two real-world scenarios, one with four poles (3D opinion space) and another with five poles (4D opinion space).

We analyze these systems using Twitter data. We use the retweet networks of the Twitter users who participate in the conversation as the *interaction networks* on which the opinion inference process is performed. Retweets can be considered as a proxy of influence because they are a broadcasting mechanism that usually implies that the retweeting user agrees with the original tweet and has enough interest to perform the retweet action [56–59]. Hence, whenever user i retweets a message originally posted by user j, i is being influenced by j's ideas and we make a directed link from i to j. The weight of the $i \to j$ link is the number of times i has retweeted j within a given time interval.

In our analyses, we will focus on the multipolar patterns of the multidimensional opinion distributions, but to obtain these distributions we first need to choose the set of elite nodes. To this end, we look for users with a high retweet count that participate frequently in the conversation. Then, we compute the communities of the complete retweet network using the Nested Stochastic Block Model [60] and identify the engaged and influential users that belong to communities whose members show strong support for one of the poles. Since we analyze electoral processes, we look for communities with politicians of the contending parties. Finally, we can only infer the opinions of users that have been influenced (directly or indirectly) by elite nodes. Therefore, the listeners are the subset of users that can reach at least one elite node following a path of directed links. We cover the technical details of these preliminary computations in Supplementary Sections 4 and 5.

Throughout the analysis, we will show that the emergent patterns are consistent with the underlying sociopolitical reality and can be interpreted taking into account their context, highlighting the usefulness of the methodology for the study of multipolar social systems.

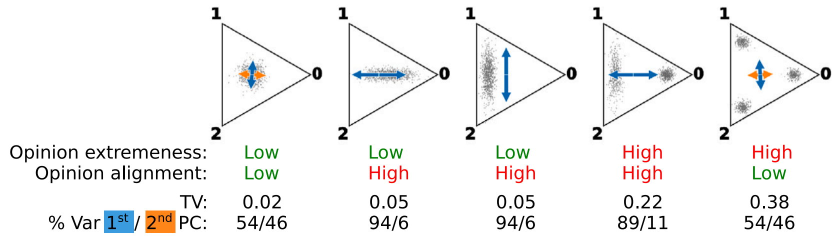

Fig. 2. Visualization of multi-polarization measures. Synthetic opinion distributions of tripolar systems and the direction of their corresponding first (blue) and second (orange) PCs shown as arrows with a length proportional to the fraction of the variance they explain. In the three central panels the second PC explains so little variance that is not visible in the plot. The TV of the system (measure of opinion extremeness) and the fraction of explained variance by each PC (measure of pole alignment) is also shown. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2.2.1. Quadripolar system: Spanish General Elections of 2015

We begin by analyzing the Spanish general elections of the 20 of December 2015, with four poles corresponding to the four main political parties (PP, PSOE, Podemos, and Cs). For this study we have worked with Twitter data retrieved during the electoral campaign of 4/12/2015 - 21/12/2015 by searching for messages that include a set of keywords relevant to the context, like *EleccionesGenerales2015* or *EleccionesGenerales28A*. The details of the dataset can be found in Supplementary Section 6, and in Supplementary Section 7 we provide an overview of the sociopolitical context. The elite of this electoral context are politicians and supporters of PP, PSOE, Podemos and Cs. An elite node is required to have at least 100 retweets and a participation rate (proportion of days she is active in the conversation) of at least 50%. Additionally, it must belong to one of the communities associated with one of the poles (see Supplementary Section 4): 0 (PP — conservatism and Christian-democracy), 1 (PSOE — social-democracy), 2 (Podemos — left-wing populism), or 3 (Cs — national-liberalism).

In a quadripolar context, the opinion space is a three-dimensional simplex (a tetrahedron). Since the original 3D opinion distribution is difficult to visualize in detail (see Fig. 3A), we have projected it onto the faces of the simplex, as shown in Fig. 3B. To avoid noise, we have filtered out users with low activity (tweets < 10). Additionally, in each projection, we only show the opinions of the users that are closer to the poles of interest (the vertices of the face in 2D projections and the vertices of the edge in 1D projections), i.e., opinions within the Voronoi cells of the corresponding poles. As explained in Supplementary Section 3, we do this to provide the most relevant information for each projection.

Fig. 3 shows that users' opinions are mainly concentrated near the poles, with *filaments* emerging from them and stretching towards the point of highest density. This maximum is found halfway between the Podemos pole (2) and the barycenter (marked in the projections with a cross), with the center of mass (marked with a square) very close to that maximum. The location of this maximum also causes the direction of maximum polarization (the first Principal Component, PC 1) to be mostly oriented with Podemos on one side and the other parties on the other. This can be explained both by the ideology (which will be discussed below) and the popularity of Podemos. In 2015 it was an emergent party that had achieved a moderate success in the previous European elections of 2014 (their first election), and had even reached first place in some electoral polls [61,62]. Podemos' popularity in Twitter has been also reported in other studies [63]. Since our multi-polarization measures are based on the variance of the opinion distribution, it is natural that a pole with high support (which concentrates a lot of opinion points in its vicinity) drives polarization.

The white double arrows shown in the projections mark the direction of maximum polarization projected onto each face. PC 1 explains 46% of the total variation, which is as a sign of moderate to high *pole alignment*, meaning that opinions are aligned along a 1D axis to a certain extent. The total variation is $TV = 0.23$, so this system presents moderate overall polarization in the sense of *opinion extremeness*.

The length of the segment delimited by the two short lines that crosses each white arrow is proportional to the size of the projection of PC 1 in the corresponding face. So when the segment is long, PC 1 is mostly parallel to the face, while when it is short, PC 1 is mostly orthogonal to it. The most orthogonal face to the direction of maximum polarization is $(0, 1, 3)$, corresponding to PP, PSOE and Cs. This result can be explained because, since Podemos is on the left of PSOE (center-left) and Cs was defined as a center party [64], $(0, 1, 3)$ is the pole triplet with highest ideological affinity. This can also be observed in Supplementary Figure 7 A, which shows the ideological location of the parties according to Spanish citizens' perceptions.

Since the principal components reveal the directions along which social tensions are organized, we can also use them to extract the structure of the latent ideological space. In that space, the distances between the parties will no longer be fixed, but determined by their respective affinity according to the opinion distribution. We expose this latent space by projecting the opinion distribution and the positions of the poles onto the PCs. For the sake of visualization we do this for PC1 and PC2, which together explain 78% of the variance. Fig. 3C schematizes the projection process and Fig. 3D shows the resulting distribution.

By looking at the projection along PC 1, we see that the parties are spontaneously divided into left-wing and right-wing. Therefore, even without imposing any such structure on the ideological space a priori, this information is encoded in the opinion distribution. But ideology is not the only driver of antagonism; for instance, parties that support similar policies also compete for the same electoral base, so they tend to emphasize their doctrinal differences [16,65]. This *secondary* source of conflict is manifested in the distance between Podemos and PSOE. These two parties have some ideological affinity: they negotiated to reach a government agreement in 2015 and formed a joint executive after the elections of the 10 of November 2019. However, they also competed to be the leading party on the left, which triggered frictions between them [66]. In the case of the right-wing parties, this phenomenon does not manifest itself in PC 1, but it appears very clearly in PC 2, where they are located at opposite extremes.

2.2.2. Pentapolar system: Spanish General Elections of 2019

In this section, we analyze the Spanish General Elections of the 28 of April 2019 as a pentapolar case study. During this election, the support for the four main parties (PP, PSOE, Podemos and Cs) remained high, but a previously minority far-right party called Vox had attracted much attention, causing a transition towards a pentapolar system. In Supplementary Section 7 we give more details about the context in which this transition took place.

For this analysis we have worked with Twitter data retrieved during the period 11/4/2019 - 29/4/2019 (see Supplementary Section 6 for more details). The elite sets for this electoral context have been selected from the communities of influential and engaged users (at least 1000 retweets and a participation rate equal or greater than 70%) associated to the five relevant parties (see Supplementary Section 4): 0 (PP), 1 (PSOE), 2 (Podemos), 3 (Cs), or 4 (Vox).

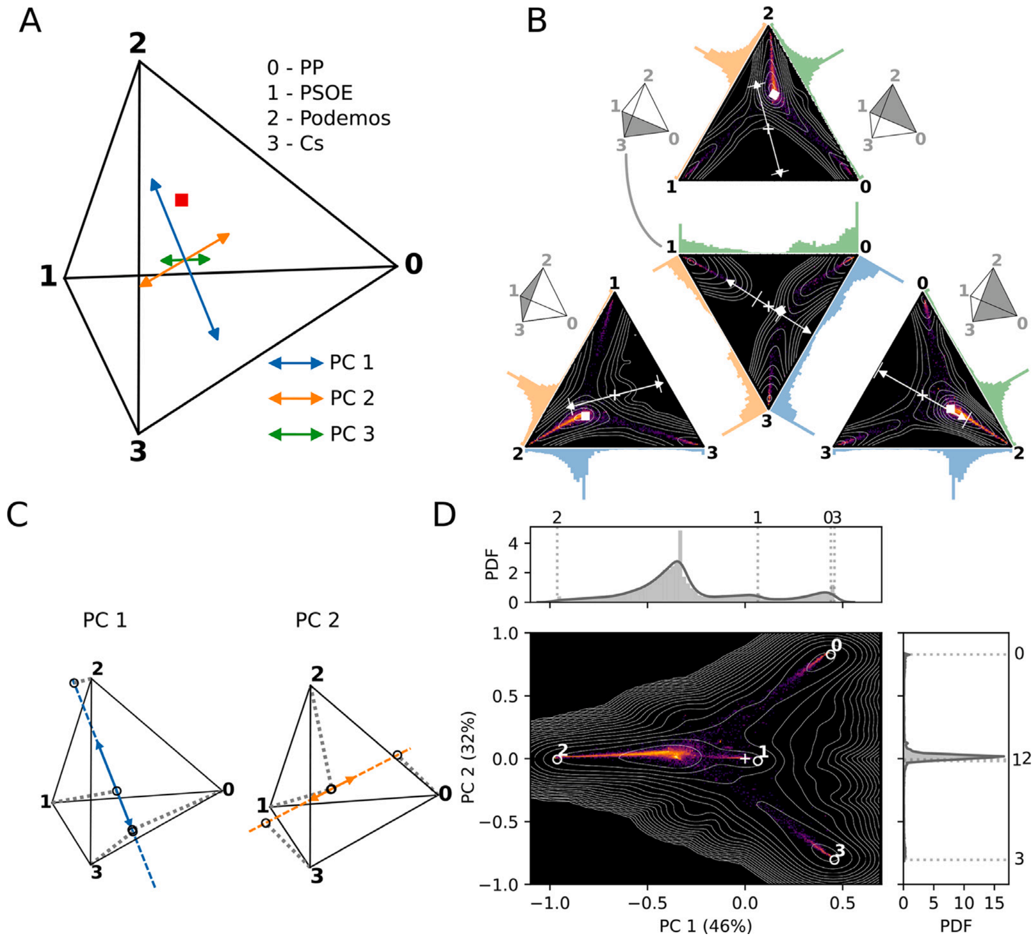

Fig. 3. Quadripolar opinion distribution of the 2015 Spanish elections. A: Cloud of users' opinions with its center of mass marked as a red square and the principal components as arrows with length proportional to their respective explained variance. An interactive version of this panel can be found in https://vis.csh.ac.at/multipolar-viz. B: Projections of the distribution onto the simplex faces shown as heat maps and contour plots. The centers of mass of the projected opinion distributions are represented as white squares and the projection of the direction of maximum polarization (PC 1), as a double headed arrow. The 1D projections onto each edge of the simplex are shown on the sides of the triangles. C: Diagram showing the projection of the poles onto the first two principal components. D: Opinion distribution projected onto PC1 and PC2, with the proportion of explained variance included in the axes labels. Each point has been projected as in the diagram of panel C, which shows the poles' projections.

In a pentapolar system the analysis grows considerably in complexity, as the opinion simplex has 10 faces. The 2D projections of the opinion distribution onto those faces are shown in Fig. 4A.

The highest concentration of users can be found near Podemos and Vox, although Podemos is more popular according to the position of the center of mass (see the projections where poles 2 or 4 appear). Additionally, high density areas appear near the edges shared by the two left-wing parties (Podemos and PSOE), as can be appreciated in projections $(0, 1, 2)$, $(1, 2, 3)$ and $(1, 2, 4)$. Similar concentrations are also found near the edges shared by two right-wing parties (Cs, PP and Vox) in the projections of 28 A, like $(0, 2, 4)$, $(0, 1, 4)$, $(1, 3, 4)$ or $(0, 1, 3)$.

With respect to the behavior of PC 1, faces $(0, 1, 3)$ and $(0, 3, 4)$ are the most orthogonal to the direction of maximum polarization, which can be explained by ideological affinity, as poles $(3, 0)$-(Cs, PP), $(3, 1)$-(Cs, PSOE) and $(0, 4)$-(PP, Vox) have formed coalitions in some regional governments. PC 1 explains 57% of the total variance, indicating a high polarization in the sense of pole alignment. Notice that, if the variance was evenly distributed between the four PCs, each would account for 25%, and PC 1 is responsible for more than double that value in this case. With a total variation of $TV = 0.34$, this system is moderately polarized in the sense of opinion extremeness.

In Fig. 4B we show the projection of the opinion distribution onto the first two principal components, which account for 75% of the variance. As in the 2015 elections, in PC 1 the poles are spontaneously divided into left-wing and right-wing parties. And beyond that, their order strikingly matches their ideological positions according to the citizens' perceptions (see Fig. 4C). Therefore, in this context the left-wing/right-wing axis completely controls the direction of maximum polarization. The polarization along this axis is so strong that the projected distribution presents a gap (a global density minimum) between the two groups. These results are a quantitative validation of the classical notion of left/right ideological axis that has been extensively used to study party systems [16].

Once the effects of the left/right tensions have been discounted, in PC 2 the parties get split into extremist and moderate, with poles Podemos (2) and Vox (4) on one side and the rest on the other. Moreover, the relative distances of the poles to the endogenous ideological center matches their order in PC 2. We define the endogenous ideological center as the middle point between the two extreme parties; that is, the intermediate point between Podemos' and Vox's averages in Fig. 4C, which is $z_c = 5.83$. The average position ($\bar{z}$) and the distance of each party to the center ($|\bar{z} - z_c|$) are presented in Table 2.

2.2.3. Robustness of the results

As a robustness check, we have verified the sensitivity of our method to the choice of the elite. As stated above, we choose the elite by first selecting engaged users (high participation) with a lot of retweets (highly influential). Then, we perform a community analysis of the full retweet network and identify communities with users associated to only one of the opinion poles (the political parties).

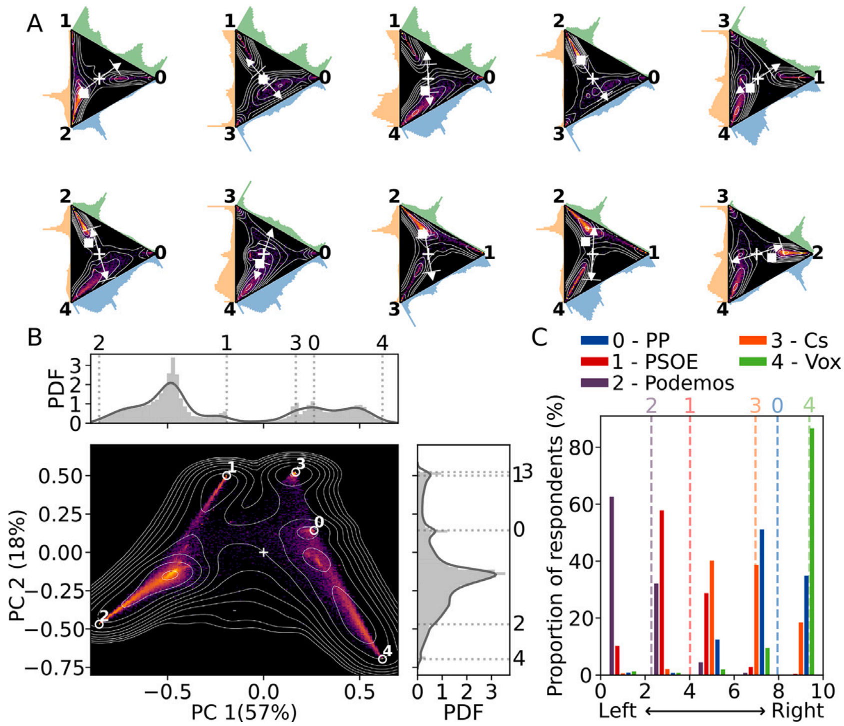

Fig. 4. Pentapolar opinion distribution of the 2019 Spanish elections. A: Contour plots corresponding to the 2D projections of the opinion distribution of the 2019 Spanish elections onto the faces of the simplex. The centers of mass of the projected opinion distributions are represented as white squares and the direction of maximum polarization as a double-headed arrow. The 1D projections onto each edge of the simplex are shown on the sides of the triangles. As in Fig. 3, we have filtered out users with low activity (tweets < 30) to avoid noise, and in each projection we have only considered the opinions of the users that are closer to the poles of interest. B: Projection of the opinion distribution and the poles onto the first two principal components. C: Citizen placement of parties on the left/right scale according to a nationwide survey carried out in May 2019 [67]. The distribution of the answers is shown for every party as a bar plot and the average as a vertical dashed line.

Table 2
Perceived level of extremism of political parties.

	Vox (4)	Podemos (2)	PP (0)	PSOE (1)	Cs (3)
$\bar{z}$	9.38	2.28	7.95	4.02	6.96
$\lvert\bar{z} - z_c\rvert$	3.55	3.55	2.12	1.81	1.13

Average values of the ideological positions of the parties ($\bar{z}$) according to Spanish citizens' perceptions (shown in Fig. 4C as dashed lines). The parties' extremism is measured as the distance of these averages to the endogenous ideological center z_c (middle point between the averages of Vox and Podemos).

The engagement threshold (proportion of days a user participates in the conversation), and influence threshold (number of received retweets) are modeling choices. For our analyses, we use thresholds that represent a balance between focusing only on popular users or only on engaged users. We have compared our results to two alternative cases where we select our elite only among popular users and only among engaged users, disregarding engagement in the former case and popularity in the latter. When choosing the thresholds in these extreme cases, we aimed at obtaining an elite set of comparable size (similar order of magnitude), and we required the two main accounts of each party (the official party account and the account of the party leader) to be present in the selected user sets. Since the community analysis is performed in the complete retweet network, it does not depend on the chosen thresholds, so we use the same communities for each pole. We present the thresholds and the elite sizes in Supplementary Table 4.

To compare the computations performed with the alternative thresholds to our previous results, we have projected the opinion distributions onto the first two principal components, as this is the visualization we use for the core of our analyses. As shown in Fig. 5, the results are almost indistinguishable from the ones we report in the manuscript; even in panel B, where the alternative elite includes less than 50% of the nodes of the original one and more than 50% new nodes (see Supplementary Table 4). These results are consistent with a previous study where we performed analogous tests on a bipolar system [5].

Additionally, we have explored the effect of the size of the elite. To do so, we used progressively smaller elite sets until the resulting distributions were no longer visually comparable. In the case of the 2015 elections, we managed to obtain remarkably similar distributions for elites of as few as 16 nodes. In the case of the 2019 elections, as we discuss in Supplementary Section 4, even an extremely small elite of 10 nodes still yields similar results.

As a final test, we have considered an extreme scenario with a minimal elite including only the two main user accounts of each party (the official party account and the account of the party candidate). The resulting opinion distributions remain quite similar and our qualitative analyses hold reasonably well (see Supplementary Section 4).

3. Discussion

In this work, we have developed a framework to infer multidimensional opinions and measure different aspects of polarization in multipolar systems. These kinds of systems are common (the most representative example being multi-party democracies), but despite the increasing interest in modeling multidimensional polarization dynamics [7,32,55,68], an analytical framework for characterizing real-world multipolar systems was lacking.

One caveat about the datasets used to illustrate the application of this framework is that Twitter data may contain some biases due to the type of audience using it. Two potential de-biasing strategies could be (i) using complementary data from other online social networks and/or (ii) calibrating the results using voter surveys and expert surveys on the parties' positions on collections of issues [69]. However, we have shown that even without applying any de-biasing, we obtain meaningful results with a clear sociological interpretation.

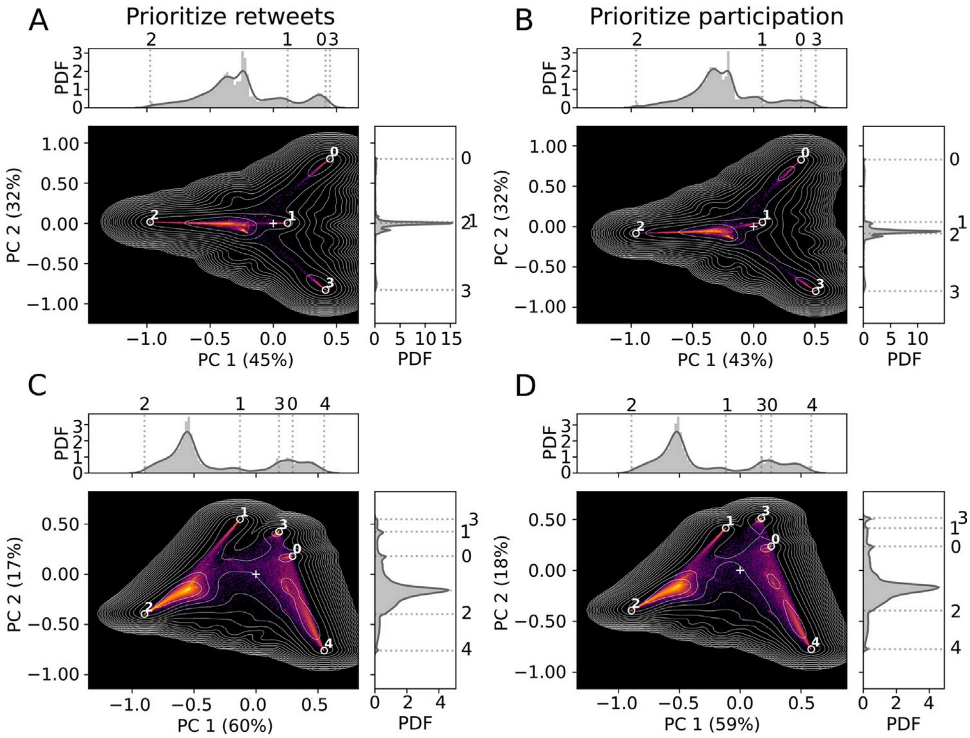

Fig. 5. Consistent results with two alternative elite choices. Opinion distributions for the 2015 (A, B) and 2019 (C, D) elections computed with alternative elite sets. In panels A and C the elite is built from popular users and in panels B and D with engaged users. Panels A and B are analogous to Fig. 3D, and panels C and D to Fig. 4B.

As for the method itself, it is worth emphasizing that opinion inference and polarization measurement are two independent processes. Therefore, the opinion inference technique can be used on its own in any networked system to obtain a cloud of multidimensional opinion points. This opens the door to the application of any multivariate statistical techniques and machine learning models. In our case studies, for example, clustering techniques like Gaussian mixture models could be used to identify probability density maxima, which can be interpreted as *endogenous* opinion poles. On the other hand, the multipolar analysis based on the covariance matrix is of course not restricted to opinion distributions obtained with our method, but can be used for any opinion distribution [55]. The flexibility of our framework makes it suitable to address pressing societal issues.

De-escalation of social tensions and working towards consensus. As discussed in the Introduction, there is a growing concern about the pernicious influence of polarization on democracy. It has been argued that multipolarity could be a possible pathway towards reducing polarization through a more complex information space and greater diversity [70]. But does the diversity of multipolar societies actually alleviate opinion extremeness? Does it help cross-cutting cleavages? As our empirical results show, our analytical framework can significantly contribute to answer these questions. Beyond their use as descriptive tools, our polarization measures could be used to find the main lines of disagreement in a given context. This could help policymakers devise de-escalation interventions by addressing the issues that cause the strongest tensions.

Mis- and disinformation spreading by bots and trolls. It has been argued that the pervasive use of (semi)automatic accounts to spread disinformation is undermining democratic processes by fostering doubt and destabilizing democratic societies [71]. In some cases, a majority of users, unaware of their role, amplify messages that polarize communities. Our opinion inference technique could help to tackle this issue by evaluating the influence of bots and trolls. Once these accounts have been identified, they can be used as the opinion seeds (the elite) of the inference process. Then, their influence over other users can be compared to that of human actors [72].

From a wider perspective, our results have implications not only for the measurement of polarization, but also for the characterization of party systems.

Comprehensive measurement of polarization. In line with the literature [20,52], we argue that polarization is a multi-faceted phenomenon. Its multiple potential meanings complicate its adequate characterization and measurement. Therefore, specifying the notion of polarization under study and its particular operationalization is of utmost importance. Additionally, to fully characterize a system, different measures for complementary aspects of polarization should be used. As we have shown, sometimes it is the less intuitive facets that provide the most relevant information.

Characterization of party systems. The overlapping collections of policy preferences and ideological identities of parties shape the multipolar opinion distribution. As our analyses of real-world multipolar contexts show, this distribution can be used to infer the structure of the underlying ideological space, including, but not being limited to, the usual left-wing/right-wing axis. This is one of the most commonly used notions to characterize party systems [16]. Although we have validated the employment of the left/right axis for such analyses, our findings demonstrate that a rigid classification of parties into pre-defined dimensions (left/right, authoritarian/libertarian, etc.) may not paint the most complete or accurate picture. This finding is aligned with the political science literature, which shows that the dimensions along which social cleavages emerge are highly context-dependent [33–35]. Each context has its own structure, and even within the same democratic system, this structure may and do change. Instead of imposing any structure, our parsimonious approach is flexible and adaptive, finding the most significant dimensions for each context in a principled way.

To summarize, we have presented an opinion inference technique, an ideological space model, and a set of multi-polarization measures to characterize multipolar systems. We perform the opinion inference using networks of social interactions. After building the network, we select a small set of elite nodes (opinion leaders) divided into n disjoint subsets, each with a strong affinity for one of the n poles of the system. Then, we use the elite nodes as opinion seeds to estimate the opinion of the remaining nodes (the listeners) with the multidimensional FJ model (an extension of the DeGroot model).

A key contribution of this work is the abstraction of opinions as $(n-1)$ dimensional vectors, with the opinion poles placed at the vertices of an $(n-1)$ regular simplex. The pole vectors defined this way endow the opinion points with the properties of unbiasedness and interdependence. They are unbiased because the poles are placed at a fixed distance from the neutral point and the distance between any two poles is the same. They are interdependent because for an opinion point to get closer to a given pole, it simultaneously must get away from the others.

From the application of the opinion inference technique we get an $(n-1)$ dimensional opinion distribution, which we analyze with multidimensional polarization measures based on the covariance matrix. We use the trace of the covariance matrix as a multidimensional metric for *opinion extremeness* and its eigendecomposition to get the direction of maximum polarization and quantify *pole alignment*.

We have applied this framework to empirical data from multi-party elections, revealing meaningful and complex properties of the systems with direct links to their sociological background. We have found that the ideological differences and affinities between the parties are clearly reflected in the obtained opinion distribution. Those parties with higher affinity tend to form subspaces that are orthogonal to the direction of maximum polarization. Furthermore, the parties' positions along that direction coincide with their location in the left-wing/right-wing axis. However, despite its central role in shaping the opinion space, ideology is not the only driver of polarization. In one of the studied systems, competition between parties of similar ideology was another relevant source of tension. In the other, the perceived extremism of the parties acted as a secondary differentiating axis. Therefore, our approach extracts the drivers of polarization specific to each system. This adaptability makes our framework suitable for the study of democratic systems undergoing political transitions, where the traditional methods based on qualitative sociopolitical analysis may fail to reveal all the relevant information.

4. Materials and methods

4.1. Total variation as aggregate multidimensional variance

To show that the trace of the covariance matrix (the total variation) can be interpreted as a measure of aggregate multidimensional variance, let us take the definition of variance:

$$Var(Y) = \mathbb{E}[(Y - \mathbb{E}[Y])^2] \tag{4}$$

and replace the random variable Y by a random vector $\vec{Y} = (Y_1, Y_2, \dots, Y_k)$, such that the difference between the variable and its expected value is replaced by the Euclidean distance $\|\cdot\|_2$. Then, define the total variation TV as:

$$\begin{aligned} TV(\vec{Y}) &= \mathbb{E}[\|\vec{Y} - \mathbb{E}[\vec{Y}]\|_2^2] \\ &= \mathbb{E}[\sum_{i=1}^{k}(Y_i - \mathbb{E}[Y_i])^2] \\ &= \sum_{i=1}^{k}\mathbb{E}[(Y_i - \mathbb{E}[Y_i])^2] \\ &= \sum_{i=1}^{k} Var(Y_i) = \sum_{i=1}^{k} Cov[\vec{Y},\vec{Y}]_{ii} = tr(Cov[\vec{Y},\vec{Y}]) \end{aligned} \tag{5}$$

4.2. Maximum variance of an opinion distribution

Our opinion distributions are embedded in a regular k-simplex with distance u between the barycenter of the simplex and the vertices. This means that the diameter of an opinion distribution, which coincides with the distance between any two vertices of the simplex, is:

$$d = \sqrt{u^2 \frac{2k+2}{k}} \tag{6}$$

Now, let us take into account this theorem proven in a recent work by Lim and McCann [73], where they use the term variance instead of total variation for the quantity of Eq. (5):

> **Theorem: Isodiametric variance bound.** If the support of a Borel probability measure μ on $\mathbb{R}^k$ has diameter no greater than d, then $Var(\mu) \leq \frac{k}{2k+2}d^2$. Equality holds if and only if μ assigns mass $1/(k+1)$ to each vertex of a regular k-simplex having diameter d.

That is, the maximum variance of a probability distribution embedded in a regular k-simplex is:

$$Var_{max} = \frac{k}{2k+2}d^2 = u^2 \tag{7}$$

So, since we have set $u = 1$ for our computations, $Var_{max} = 1$. And this value is achieved when there are only extreme opinions (at the vertices of the simplex) and they are uniformly distributed among the poles.

CRediT authorship contribution statement

Samuel Martin-Gutierrez: Conceptualization, Methodology, Software, Validation, Formal analysis, Data curation, Writing – original draft, Visualization, Funding acquisition. **Juan C. Losada:** Conceptualization, Methodology, Writing – review & editing. **Rosa M. Benito:** Conceptualization, Methodology, Writing – review & editing, Resources, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The interactive visualizations elaborated by Liuhuaying Yang and Johannes Sorger from the visualization team of the Complexity Science Hub Vienna are hosted in https://vis.csh.ac.at/multipolar-viz.

The data and python codes to reproduce the results of the paper are hosted in https://github.com/samuel-mg/multipolar_paper. The data have been pseudonymized and protected with the code `multipolarpaper_data122021`.

We have also prepared easy-to-use python scripts so that researchers can apply the multipolar framework to analyze their own datasets https://github.com/samuel-mg/multipolar.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.chaos.2023.113244.

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