Answer Key 5

1.

(1)

$$M_t(X) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$= \int_0^{\infty} e^{tx}e^{-x}dx = \int_0^{\infty} e^{(t-1)x}dx$ (finite when t<1, so we require t<1)

$$= \frac{1}{t-1} e^{(t-1)x} \Big|_0^{\infty} = \frac{1}{1-t}$$

(2)

$$M_t'(X) = \frac{1}{(1-t)^2},\ M_t''(X) = \frac{2}{(1-t)^3}$$

$\Rightarrow$

$\mu_1' = E(X) = M_t'(0) = 1$

$\mu_2' = E(X^2) = M_t''(0) = 2$

So, $V(X) = 1$

2.

(1) $f_1(x) = \int_0^1 f(x,y)dy = \dfrac{(6x^2+3)}{22}$ *for* $0 \le x \le 2$ *with* $f_1(x) = 0$ *elsewhere.*

(2) $f_2(y) = \dfrac{(8+6y)}{11}$ *for* $0 \le y \le 1$ *with* $f_2(y) = 0$ *elsewhere.*

(3) For $0 \le x \le 2$,

$g_2(y \mid x) = \dfrac{f(x,y)}{f_2(x)} = \dfrac{2x^2+2y}{2x^2+1}$ *for* $0 \le y \le 1$ *with* $g_2(y \mid x) = 0$ *elsewhere*.

(4)

$$\Pr(A \mid x) = \int_0^{1/2} g_2(y \mid x)dy = \int_0^{1/2} \frac{2x^2+2y}{2x^2+1}dy = \frac{(2x^2y+y^2)}{(2x^2+1)}\Bigg|_0^{1/2} = \frac{1}{(2x^2+1)}(x^2+1/4) = \frac{4x^2+1}{8x^2+4}.$$

Therefore, $\Pr(A \mid x=0) = 1/4,\ \Pr(A \mid x=1) = 5/12,\ \Pr(A \mid x=2) = 17/36$.

3..

(1) $f_2(y) = \int_0^1 \frac{3}{2}(x^2 + y^2)dx = \left[\frac{x^3 + 3y^2 x}{2}\right]_0^1 = \frac{1+3y^2}{2}$.

(2) For $0 < y < 1$,

$$f(x \mid y) = \frac{f(x,y)}{f_2(y)} = \frac{3x^2 + 3y^2}{1+3y^2} \quad \textit{for } 0 < x < 1 \quad \textit{with } f(x \mid y) = 0 \textit{ elsewhere.}$$

(3) $f(x \mid Y = 0.5) = \frac{12x^2 + 3}{7}$ *for* $0 < x < 1$ *with* 0 *elsewhere*.

4.

(1) $f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_2^4 (1/8)(6 - x - y)dy = (1/8)(6 - 2x) \qquad 0 < x < 2$

(2) $g(y|x) = \frac{f(x,y)}{f(x)} = \frac{(1/8)(6-x-y)}{(1/8)(6-2x)} = \frac{6-x-y}{6-2x} \qquad 0 < x < 2, 2 < y < 4$

(3) $P(2 < Y < 3|x = 1) = \int_2^3 g(y|x=1)dy = \int_2^3 \frac{5-y}{4}dy = \frac{5}{8}$