Answer Key 12

1.

(1) $L(y_1, y_2, \cdots, y_n : \theta) = \prod_{i=1}^{n} f(y_i;\theta) = \theta^{-n} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} y_i \right)$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \ln L(y_1, y_2, \cdots, y_n : \theta)$$

$$= \underset{\theta}{\operatorname{argmax}} \left(-n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} y_i \right)$$

(2)

$$F.O.C.: \frac{d\ln L}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} y_i}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \overline{y}$$

(3) First, find $E(y_i)$ and $E(y_i^{\,2})$.

Since $\ln f(y_i;\theta) = -\ln\theta - \frac{y_i}{\theta}$

$\frac{d\ln f(y_i;\theta)}{d\theta} = -\frac{1}{\theta} + \frac{y_i}{\theta^2}$ and $\frac{d^2 \ln f(y_i;\theta)}{d\theta^2} = \frac{1}{\theta^2} - 2\frac{y_i}{\theta^3}$.

$$E\left(\frac{d^2 \ln f(y_i;\theta)}{d\theta^2} \right) = \frac{1}{\theta^2} - 2\frac{E(y_i)}{\theta^3} = -\frac{1}{\theta^2}$$

Or $\left(\begin{aligned} & E\left(\frac{d\ln f(y_i;\theta)}{d\theta} \right)^2 = E\left(-\frac{1}{\theta} + \frac{y_i}{\theta^2} \right) = E\left(\frac{-\theta + y_i}{\theta^2} \right)^2 \\ & = \frac{1}{\theta^4}\left(\theta^2 - 2\theta E(y_i) + E(y_i^{\,2}) \right) \\ & = \frac{1}{\theta^2} \end{aligned} \right)$

Therefore, $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \quad \theta_0^{\,2})$ or $\hat{\theta} \overset{a}{\sim} N(\theta_0, \frac{1}{n}\theta_0^{\,2})$

(4) From exponential distribution, $E(y) = \theta_0$, $E(\overline{y}) = E(\hat{\theta}) = \theta_0$.

(5) From (2), $\dfrac{d^2 \ln L}{d\theta^2} = \dfrac{n}{\theta^2} - 2\dfrac{\sum_i y_i}{\theta^3}$,

Therefore, $\dfrac{d^2 \ln L(\hat{\theta})}{d\theta^2} = -\dfrac{n}{\bar{y}^2} < 0$.

2.

(1) Done in the class. $\hat{\theta} = \bar{X}$.

(2) Since $\bar{X} \xrightarrow{p} \theta,\ E(\bar{X}) = \theta,$ and $V(\bar{X}) = \theta(1-\theta)$,

$\hat{\theta} \overset{a}{\sim} N\left(\theta, \dfrac{1}{n}\theta(1-\theta)\right)$.

(3) Natural estimator for γ, $\hat{\gamma} = \dfrac{1}{\hat{\theta}}$.

(4) By delta method and $\dfrac{d\gamma}{d\theta} = -\theta^{-2}$, therefore $\hat{\gamma} \overset{a}{\sim} N\left(\gamma, \dfrac{1}{n}\theta(1-\theta)\theta^{-4}\right)$

Or $\sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} N\left(0,\ \theta(1-\theta)\theta^{-4}\right)$.

Note that consistent estimator of $A.V.(\hat{\gamma}) = \dfrac{1}{n}(1-\hat{\theta})\hat{\theta}^{-3}$