Answer Key 7

1. Since $(X,Y) \sim BVN$, then $Y|X \sim N(E(Y|X), V(Y|X))$ where $E(Y|X) = \alpha + \beta X$ with $\beta = \sigma_{YX} / \sigma_X{}^2 = 6/9 = 2/3, \alpha = \mu_Y - \beta\mu_X = 4 - \frac{2}{3} \times 3 = 2$.

Therefore, $E(Y|X) = 2 + \frac{2}{3}X$.

And $V(Y|X) = \sigma_Y{}^2(1-\rho^2) = 20(1-1/5) = 16$ with $\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{6}{3 \times \sqrt{20}} = 1/\sqrt{5}$.

(1) $E(Y|x=3) = 4$ (2) $E(Y|x=6) = 6$

(3) $V(Y|x=3) = 16$ (4) $V(Y|x=6) = 16$

(5) Since $Y|x=3 \sim N(4,16)$,

$P(Y \le 8|x=3) = P\left(\frac{Y-4}{4} \le \frac{8-4}{4}\right) = P(Z \le 1) = \Phi(1) = 0.8413$, where $\Phi(.)$ denotes CDF of $N(0,1)$.

(6) Since $Y|x=6 \sim N(6,16)$, $P(Y \le 8|x=6) = P\left(\frac{Y-6}{4} \le \frac{8-6}{4}\right) = \Phi(0.5) = 0.6915$.

(7) Since $Y \sim N(4, 20)$, $P(Y \le 8) = P\left(\frac{Y-4}{\sqrt{20}} \le \frac{8-4}{\sqrt{20}}\right) = \Phi(0.89) = 0.8133$.

2.

(1) Since $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$ and $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \ne 0$, U and V are bivariate normal.

$E(U) = \mu_X + \mu_Y$, $V(U) = \sigma_X{}^2 + \sigma_Y{}^2 + 2\rho\sigma_X\sigma_Y$.

$E(V) = \mu_X - \mu_Y$, $V(V) = \sigma_X{}^2 + \sigma_Y{}^2 - 2\rho\sigma_X\sigma_Y$

and $Cov(U,V) = \sigma_X{}^2 - \sigma_Y{}^2$, $\rho_{U,V} = \frac{\sigma_X{}^2 - \sigma_Y{}^2}{\sigma_U \sigma_V}$.

(2) $U \sim N\left(\mathrm{E}(U), \mathrm{V}(U)\right)$, $V \sim N\left(\mathrm{E}(V), \mathrm{V}(V)\right)$.

3. Note $E(X) = E(Z) + E(W) = 42$, $V(X) = V(Z) + V(W) = 3000$.

Since $(Z,W) \sim BVN(42,0, 2500,500, 0)$ and

$Z = Z + 0 \times W (= b_1 Z + c_1 W), X = Z + W (= b_2 Z + c_2 W)$ with $b_1 c_2 - b_2 c_1 = 1 - 0 = 1 \neq 0$, therefore $(Z, X) \sim BVN(42,42,2500,3000,2500)$.

Then, from theorem on bivariate normal distribution, $Z \mid X \sim N\big(E(Z \mid X), V(Z \mid X)\big)$,

where $E(Z \mid X) = \alpha + \beta X$ with $\beta = \sigma_{ZX} / \sigma_X{}^2 = 2500/3000 = 5/6, \alpha = \mu_Z - \beta \mu_X = 7$.

Therefore, $E(Z \mid X) = 7 + \frac{5}{6} X$.

We know that the CEF is linear since $(Z, X) \sim BVN$.

4.

(1) $E(\underline{z}) = \underline{g}_1 + \underline{H}_1 \underline{\mu} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

$$V(\underline{z}) = \underline{H}_1 \Sigma \underline{H}_1{}' = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 9 \\ 9 & 26 \end{pmatrix}.$$

(2) $\underline{H}_2$ matrix has two rows and two rows are linearly independent(since they do not have same information), $rank(\underline{H}_2) = 2 = \#\text{ of rows}$. So $\underline{w}$ is bivariate inormally distributed.

$$E(\underline{w}) = \underline{g}_2 + \underline{H}_2 \underline{\mu} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

$$V(\underline{w}) = \underline{H}_2 \Sigma \underline{H}_2{}' = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 21 \end{pmatrix}.$$

(3) $Cov(\underline{z}, \underline{w}) = \mathrm{H}_1 \Sigma H_2{}' = \begin{pmatrix} 4 & 13 \\ 2 & -13 \end{pmatrix}$.

(4) Let $\underline{y} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$, where $\mathbf{x}_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\mathbf{x}_2 = y_3$.

$$\boldsymbol{\mu}_1 = \begin{pmatrix} E(y_1) \\ E(\mathrm{y}_2) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \boldsymbol{\mu}_2 = E(y_3) = 3$$

$$\Sigma_{11} = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}, \Sigma_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Sigma_{22} = 3$$

Then,

apply $E(y_3 \mid y_1, y_2) = E(\mathrm{x}_2 \mid \mathrm{x}_1) = \alpha + B'\mathbf{x}_1$

with $B = \Sigma_{11}^{-1}\Sigma_{12} = \frac{1}{9}\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \alpha = \boldsymbol{\mu}_2 - B'\boldsymbol{\mu}_1 = 3 - \frac{1}{3}\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{5}{3}$

So, $E\left(y_3 \middle| y_1, y_2\right) = \frac{5}{3} + \frac{2}{3}y_1 + \frac{1}{3}y_2$

$V(y_3 \mid y_1, y_2) = V\left(\mathbf{X}_2 \middle| \mathbf{X}_1\right) = \Sigma_{22} - B'\Sigma_{11}B = 3 - 1 = 2$

(5) $E\left(y_3 \middle| y_1 = 1, y_2 = 1\right) = \frac{8}{3}$