Ch. 8 Large Sample Theory (Asymptotic Theory)

► Goldberger, Ch. 9

Yale Note Ch. 10, 11
Wackerly et al. ch. 7

1. Asymptotics

(1) Basic Notions

- We know

ⓐ If $X_i \sim N(\mu, \sigma^2)$, then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

ⓑ If $X_i \sim Bernoulli(p)$, then $n\bar{X} = \sum_{i=1}^{n} X_i \sim B(n, p)$.

We will develop this story for all parent distribution when n is large.

- We will formalize

① <u>Laws of Large Number</u>: consistency of $\bar{X}$

► Probability limit of $\bar{X}$ is μ: $\bar{X} \xrightarrow{p} \mu$.

② <u>Central Limit Theorem</u>: asymptotic distribution of $\bar{X}$

► Limiting distribution of $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is $N(0,1)$: $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \xrightarrow{d} N(0,1)$.

► Asymptotic distribution of $\bar{X}$ is $N\left(\mu, \dfrac{\sigma^2}{n}\right)$: $\bar{X} \overset{a}{\sim} N\left(\mu, \dfrac{\sigma^2}{n}\right)$.

(2) Sequence of Sample Statistics

► Sequence of sample statistics indexed by sample size.

$\bar{X}_n$: sample mean in random sampling, sample size n.

(Example) $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{10}, \cdots, \bar{X}_n$

(3) <u>Modes of convergence</u>

Let T_n be a sequence of random variables, with cdf $G_n(t) = P(T_n \le t)$, expectations $E(T_n)$, and variance $V(T_n)$.

① T_n *<u>converges in probability</u>* to c $(T_n \xrightarrow{p} c)$,

<u>probability limit</u> of T_n is c ($\underset{n\to\infty}{p\lim} T_n = c$):

If for all $\varepsilon > 0$, $\lim_{n\to\infty} P\left(\left|T_n - c\right| > \varepsilon\right) = 0$ or $\lim_{n\to\infty} P\left(\left|T_n - c\right| < \varepsilon\right) = 1$.

(Note) θ: parameter, $\hat{\theta}_n$: estimator of θ.

If $\hat{\theta}_n \xrightarrow{p} \theta$ (**or** $p\lim \hat{\theta}_n = \theta$), $\hat{\theta}_n$ is a *consistent* estimator of θ.

(Note) Whether $\hat{\theta}_n$ is unbiased or not is NOT an issue in large sample theory.

② T_n <u>*converges in mean square*</u> to c ($T_n \xrightarrow{m.s.} c$):

If there is some constant c such that $\lim E\left[\left(T_n - c\right)^2\right] = 0$.

(Note)

☞ $E\left[\left(T_n - c\right)^2\right] = V(T_n) + \left\{Bias(T_n)\right\}^2$

If $c = E(T_n)$, $\lim V(T_n) = 0$, then $T_n \xrightarrow{m.s.} c$.

(Example) For $\overline{X}_n$, we know $E\left(\overline{X}_n\right) = \mu$, $Var(\overline{X}_n) = \dfrac{\sigma^2}{n} \to 0$. So, $\overline{X}_n \xrightarrow{m.s.} \mu$.

(Theorem) If $T_n \xrightarrow{m.s.} c$, then $T_n \xrightarrow{p} c$

(Proof)

③ T_n *<u>converges in distribution</u>* to T ($T_n \xrightarrow{d} T$):

If $T_n \sim G_n(\cdot)$, $T \sim G(\cdot)$, and $\lim_{n\to\infty} G_n(\cdot) = G(\cdot)$ at every continuity point of $G(\cdot)$,

then $T_n \xrightarrow{d} T$.

► We call $G(\cdot)$ the *limiting distribution* of T_n.

(Example) If $X \sim N\left(\mu, \sigma^2\right)$, then $T = \dfrac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$.

But, $T_n = \dfrac{\bar{X}_n - \mu}{s / \sqrt{n}} \sim t(n-1)$.

However, since $\lim_{n\to\infty} t(n-1) = N(0,1)$,

$$T_n \xrightarrow{d} T.$$

2. Asymptotics of Sample Mean

(1) Laws of Large Number (L.L.N.)

In random sampling from $E(X_i)=\mu$, $V(X_i)=\sigma^2$, then $\bar{X}_n \xrightarrow{p} \mu$.

(Proof)

(2) Central Limit Theorem (C.L.T.)

In random sampling from $E(X_i)=\mu,\ V(X_i)=\sigma^2$, $\dfrac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma}\xrightarrow{d}N(0,1)$.

(Note)

① C.L.T. is an approximation procedure:

In fact, $\dfrac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}\sim H_n(\cdot)$.

We approximate $H_n(\cdot)$ by $\Phi(\cdot)$, where $\Phi(\cdot)$ is c.d.f. of $N(0,1)$.

This approximation is bad for small n, but good for large n.

② We do not write $\dfrac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma} \sim N(0,1)$ because of approximation.

Instead, we write $\dfrac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma} \xrightarrow{d} N(0,1)$ or $\bar{X}_n \overset{a}{\sim} N\left(\mu, \dfrac{\sigma^2}{n}\right)$.

(Example) In random sampling from $X \sim \chi^2(1)$, $P\left(\bar{X} \le 1.16\right) = ??$

① $n = 30$,

► We know $W \equiv n\bar{X} = \sum_{i=1}^{n} X_i \sim \chi^2(30)$.

$\Rightarrow P\left(W = n\bar{X} < 30 \times 1.16\right) = 0.75$.

► Since $E\left(X\right) = 1$, $V(X) = 2$, $\bar{X} \overset{a}{\sim} N\left(1, \frac{2}{30}\right)$,

$$P\left(\bar{X} \le 1.16\right) = P\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \le \frac{1.16 - 1}{\sqrt{2/30}} = 0.62\right) \approx 0.73.$$

② $n = 100$,

► We know $W \equiv n\bar{X} = \sum_{i=1}^{n} X_i \sim \chi^2(100)$.

$\Rightarrow P\left(W = n\bar{X} < 100 \times 1.16\right) = 0.87$

► Since $E(X) = 1,\ V(X) = 2,\ \bar{X} \overset{a}{\sim} N\left(1, \frac{2}{100}\right)$,

$P\left(\bar{X} \le 1.16\right) = P\left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \le \frac{1.16 - 1}{\sqrt{2/100}} = 1.13\right) \approx 0.87$.

3. Asymptotics of Sample Moment

The same logic can be applied into entire class of statistics that can be interpreted as sample mean (as sample mean theorem).

(1) Sample raw moment

$$M_r' = \frac{1}{n}\sum_{i=1}^{n} X_i^r \xrightarrow{p} E\left(X_i^r\right) = \mu_r'.$$

- The asymptotic distribution of sample raw moment can be explained similarly.

(Example) $r = 2$

$$M_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} E\left(X_i^2\right) = \mu_2' \quad \text{and} \quad \frac{\sqrt{n}\left(M_2' - \mu_2'\right)}{\sqrt{\mu_4' - \left(\mu_2'\right)^2}} \xrightarrow{d} N\left(0,1\right).$$

(2) Sample central moment

$$\left(M_r{*}=\right)\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^r \xrightarrow{p} E\left[\left(X-\mu\right)^r\right]\left(=\mu_r\right).$$

(Example) $r=2$

$$M_2{}^* = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \xrightarrow{p} E\left[\left(X-\mu\right)^2\right] = \mu_2 = \sigma^2$$

$$\frac{\sqrt{n}\left(M_2{}^* - \sigma^2\right)}{\sqrt{\mu_4-\sigma^4}} \xrightarrow{d} N(0,1)$$

4. Asymptotics of Functions of Sample Moment

► Asymptotics of linear functions of sample moment (ex) $T_n = a + b\bar{X}_n$).

► Asymptotics of nonlinear functions of sample moment (ex) $T_n = \dfrac{1}{\left(\bar{X}_n\right)^2}$).

(1) <u>Linear function</u>

Consider $T_n = a + b\overline{X}_n$.

Let $Y_i = a + bX_i : i.i.d.$.

$$T_n = \frac{1}{n}\sum_i (a + bX_i) = \xrightarrow{p} a + b\mu .$$

$$\frac{\sqrt{n}\left(T_n - \left(a + b\mu\right)\right)}{\left|b\right|\sigma_X} \xrightarrow{d} N(0,1) .$$

(2) Nonlinear function

(Example) $\dfrac{1}{\bar{X}_n}$, $S_X{}^2 = \dfrac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, sample t-ratio $\dfrac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{S_X}$.

⦿ <u>Slutsky Theorem</u>

① If $T_n \xrightarrow{p} c$ and $g(\cdot)$ is continuous at c, then $g(T_n) \xrightarrow{p} g(c)$.

► $\boxed{p\lim T_n = c \quad \Rightarrow \quad p\lim g(T_n) = g(c)}$

(Note) $p\lim g(T_n) = g\left(p\lim T_n\right) = g(c)$ but $E\left[g(T_n)\right] \neq g\left[E(T_n)\right] = g(c)$.

(Example) If $S^2 \xrightarrow{p} \sigma^2$(will be shown at 5-(1)),

$S \xrightarrow{p} \sigma$ ($p\lim S^2 = \sigma^2 \Rightarrow p\lim S = \sigma$).

▶ However, $E(S^2) = \sigma^2 \quad \not\Rightarrow \quad E(S) = \sigma$.

② If $X_n \xrightarrow{p} c_1$, $Y_n \xrightarrow{p} c_2$ and $g(\cdot)$ is continuous at (c_1, c_2),

then $g(X_n, Y_n) \xrightarrow{p} g(c_1, c_2)$.

⦿ Some implications of the Slutsky theorem

If $p\lim X_n = c_1,\ \ p\lim Y_n = c_2$, then

① $p\lim\left(X_n \pm Y_n\right) = c_1 \pm c_2$.

② $p\lim(X_n Y_n) = c_1 c_2$.

③ $p\lim\left(\dfrac{X_n}{Y_n}\right) = \dfrac{c_1}{c_2}$ provided $c_2 \neq 0$.

⦿ Combined Theorem

If $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{p} c$, then

① $X_n \pm Y_n \xrightarrow{d} X \pm c$.

② $X_n \cdot Y_n \xrightarrow{d} cX$.

③ $\dfrac{X_n}{Y_n} \xrightarrow{d} \dfrac{X}{c}$.

(Example) $X_n \xrightarrow{d} N\left(0, \sigma^2\right)$, $Y_n \xrightarrow{p} c$,

ⓐ $X_n + Y_n \xrightarrow{d} N\left(c,\ \sigma^2\right)$: location.

ⓑ $X_n \cdot Y_n \xrightarrow{d} N\left(0,\ c^2\sigma^2\right)$: scale.

5. Asymptotics of Sample Variance

(1) Sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \frac{n}{n-1}\ \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \frac{n}{n-1}\ M_2$$

where $M_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 - \left(\mu - \bar{X}\right)^2$$

① Consistency

Since $$\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\mu\right)^2 \xrightarrow{p} E\left(\left(X_i-\mu\right)^2\right)=\sigma^2,$$

$$\left(\mu-\overline{X}\right)^2 \xrightarrow{p} 0,$$

so, $M_2 \xrightarrow{p} \sigma^2$.

$$s^2=\frac{n}{n-1}M_2 \xrightarrow{p} \sigma^2.$$

Therefore, s^2 is consistent for σ^2.

② Asymptotic distribution

$$M_2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 - \left(\mu - \bar{X}\right)^2 \xrightarrow{p} \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2$$

$$\overset{a}{\sim} N\left(\sigma^2, \frac{1}{n}\left[E(X_i - \mu)^4 - \sigma^4\right]\right).$$

(2) Sample t-ratio: $\dfrac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{s}$

① Consistency

Since $s^2 \xrightarrow{p} \sigma^2$ and $s \xrightarrow{p} \sigma$,

$$\bar{X}_n - \mu \xrightarrow{p} 0.$$

Thus, $\dfrac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{s} \xrightarrow{p} 0.$

② <u>Asymptotic distribution</u>

Since $s \xrightarrow{p} \sigma$,

$$\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{s}=\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma}\frac{\sigma}{s}$$

$$\xrightarrow{p}\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma}\xrightarrow{d}N\left(0,1\right).$$

6. Delta Method

▶ Limiting distribution of $g\left(\bar{X}_n\right)$.

• Suppose $\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right)$ and

$T_n = g\left(\bar{X}_n\right)$ is continuously differentiable at μ,

then $\sqrt{n}\left(T_n - g(\mu)\right) \xrightarrow{d} N\left(0,\ \left\{g'(\mu)\right\}^2 \sigma^2\right)$.

▶ If $\bar{X}_n \overset{a}{\sim} N\left(\mu, \dfrac{\sigma^2}{n}\right)$, then $g\left(\bar{X}_n\right) \overset{a}{\sim} N\left(g\left(\mu\right),\ \dfrac{\left\{g'(\mu)\right\}^2 \sigma^2}{n}\right)$.

(proof)

(Example) We know $\sqrt{n}\left(\bar{X}_n-\mu\right)\xrightarrow{d}N\left(0,\sigma^2\right)$

① What is asymptotic distribution of $\left(\bar{X}_n\right)^2$?

$$\left(\bar{X}_n\right)^2\overset{a}{\sim}N\left(\mu^2,\frac{4\mu^2\sigma^2}{n}\right)\text{ or }\sqrt{n}\left(\left(\bar{X}_n\right)^2-\mu^2\right)\xrightarrow{d}N\left(0,4\mu^2\sigma^2\right).$$

(Cf) $$\frac{n\left(\bar{X}_n-\mu\right)^2}{\sigma^2}=\left(\frac{\sqrt{n}\left(\bar{X}_n-\mu\right)}{\sigma}\right)^2\xrightarrow{d}\chi^2(1).$$

② $$\frac{1}{\bar{X}_n}\overset{a}{\sim}N\left(\frac{1}{\mu},\frac{\sigma^2}{n\mu^4}\right)\text{ or }\sqrt{n}\left(\frac{1}{\bar{X}_n}-\frac{1}{\mu}\right)\xrightarrow{d}N\left(0,\frac{\sigma^2}{\mu^4}\right).$$

7. Sampling Distribution of Sample Covariance

① <u>Bivariate Ranom Sample</u>

$(X_1,Y_1),(X_2,Y_2),\cdots,(X_n,Y_n)$ from $f(x,y)$,

with $E(X)=\mu_X$, $E(Y)=\mu_Y$, $V(X)=\sigma_X^2,\ Var(Y)=\sigma_Y^2,\ Cov(X,Y)=\sigma_{XY}$.

► From $i.i.d.$,

Independence $\Rightarrow$ (X_i,Y_i) is independent of (X_j,Y_j) for $i \neq j$.

$\not\Rightarrow$ X_i is independent of Y_i.

• The joint p.d.f. is $g\left((x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\right)=\prod_{i=1}^{n} f(x_i,y_i)$.

ⓐ $C\left(\bar{X},\bar{Y}\right)=\frac{1}{n}\sigma_{XY}$

ⓑ $C\left(\bar{X},\bar{X}\right)=V\left(\bar{X}\right)=\frac{\sigma_X^2}{n}$

(2) Sample Covariance

$$s_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) = \frac{n}{n-1}s^*_{XY},$$

where

$$\begin{aligned} s^*_{XY} &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) \\ &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_x)(Y_i - \mu_y) - (\bar{X} - \mu_x)(\bar{Y} - \mu_y) \end{aligned}.$$

Denote $M_{11} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)(Y_i - \mu_Y)$.

① Small Sample Case

For $M_{11} = \dfrac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)(Y_i - \mu_Y)$, and let $U_i = (X_i - \mu_X)(Y_i - \mu_Y):\ i.i.d.$

$$M_{11} = \frac{1}{n}\sum_{i=1}^{n} U_i = \overline{U}\,.$$

By Sample mean theorem,

$$E\left(M_{11}\right) = E\left(U_i\right) = E\left((X_i - \mu_X)(Y_i - \mu_Y)\right) = \sigma_{XY}\,.$$

$$E\left(s^*_{XY}\right) = \sigma_{XY} - E\left((\overline{X} - \mu_X)(\overline{Y} - \mu_Y)\right)$$

$$= \sigma_{XY} - \frac{1}{n}\sigma_{XY} = \frac{n-1}{n}\sigma_{XY}$$

$$E\left(s_{XY}\right) = \frac{n}{n-1}E\left(s^*_{XY}\right) = \frac{n}{n-1}\frac{n-1}{n}\sigma_{XY} = \sigma_{XY}\,.$$

② Large Sample Case

$$M_{11} \xrightarrow{p} E(U) = \sigma_{XY} \text{.}$$

$$s_{XY}^{*} = M_{11} - (\bar{X} - \mu_X)(\bar{Y} - \mu_Y) \xrightarrow{p} \sigma_{XY} \text{.}$$

$$s_{XY} = \frac{n}{n-1} s_{XY}^{*} \xrightarrow{p} \sigma_{XY} \text{.}$$