Answer Key 3

1. The function of interest is $\big(U(X)=\big)Y=3X-1$, which is increasing function.

Since $\big(U^{-1}(Y)=\big)X=\dfrac{Y+1}{3}$ and $\dfrac{dX}{dY}=\dfrac{1}{3}$.

Thus, $f_Y(y)=f_X(u^{-1}(y))=\begin{cases} 2\left(\dfrac{y+1}{3}\right)\left(\dfrac{1}{3}\right) & \text{for} \quad 0\le \dfrac{y+1}{3}\le 1 \\ 0 & \text{elsewhere} \end{cases}$

Or equivalently,

$$f_Y(y)=\begin{cases} \dfrac{2(y+1)}{9} & \text{for} \quad -1\le y\le 2 \\ 0 & \text{elsewhere} \end{cases}$$

2. The transformation $Y=aX+b$ gives $\big(U^{-1}(Y)=\big)X=\dfrac{Y-b}{a}$, so that $\dfrac{dX}{dY}=\dfrac{1}{a}$.

Therefore, $f_Y(y)=f_X\left(\dfrac{y-b}{a}\right)\dfrac{1}{|a|}$.

That is,

$$f_Y(y)=\frac{1}{|a|}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{\left(\dfrac{y-b}{a}-\mu\right)^2}{2\sigma^2}\right]$$

$$=\frac{1}{|a|\sigma\sqrt{2\pi}}\exp\left[-\frac{\big(y-(a\mu+b)\big)^2}{2(a\sigma)^2}\right]$$

Thus, if $X\sim N(\mu,\sigma^2)$, then $Y=aX+b\sim N\big(a\mu+b,(a\sigma)^2\big)$

3.

(1)

Since $f(x)=2e^{-2x}$ for $x>0$ and Jacobian $|dx/dy|=1/2$,

$g(y) = f(y/2)(1/2) = e^{-y}$.

Therefore, $g(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}$.

In general, if $X \sim \exp onential(\theta)$ and $Y = kX$, then $Y \sim \exp onential(\theta / k)$.

(2) Since $f(x) = 1$ for $0 < x < 1$ and $|dx/dy| = e^{-y}$, $g(y) = f(e^{-y})e^{-y} = e^{-y}$.

Therefore, $g(y) = \begin{cases} e^{-y} & \text{for } 0 < y \\ 0 & \text{elsewhere} \end{cases}$.

(3) $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ for $-\infty < x < \infty$.

$$
\begin{aligned}
G(y) &= P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\
&= 2\int_0^{\sqrt{y}} (2\pi)^{-1/2} e^{-\frac{1}{2}x^2} dx \qquad let\ y = x^2, \sqrt{y} = x, dx = (1/2)y^{-1/2}dy \\
&= \int_0^{y} (2\pi)^{-1/2} e^{-\frac{1}{2}y} y^{-1/2} dy
\end{aligned}
$$

Therefore,

$$g(y) = G'(y) = \begin{cases} (2\pi)^{-1/2} e^{-\frac{1}{2}y} y^{-1/2} & \text{for } 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases},$$

which is a chi-square distribution with 1 degree of freedom.

- You are NOT responsible to memorize the density function of chi-square distribution.

4.

x	-1	0	1
$f(x)$	1/8	2/8	5/8
y	1	0	1

Therefore,

$$g(y) = \begin{cases} 1/4 & \text{for } y = 0 \\ 3/4 & \text{for } y = 1 \\ 0 & \text{elsewhere} \end{cases}.$$