ECO 4004: Math. Econ. Stat.

Problem Set 9: Large Sample Theory

1. Let the independent random variables $X_1, \cdots, X_n$ be distributed as U(0, 1).
(1) Use the CLT to find an approximate value for the probability $P(a \le \overline{X} \le b)$.
(2) What is the numerical value of this probability for $n = 12,\ a = 7/16,\ b = 9/16$?

2. Let $X_i \{i = 1, 2, \cdots, n\}$ and $Y_i \{i = 1, 2, \cdots, n\}$ be independent random variables such that the $X_i's$ are identically distributed with $E(X_i) = \mu_X, V(X_i) = \sigma^2$ and the $Y_i's$ are identically distributed with $E(Y_i) = \mu_Y, V(Y_i) = \sigma^2$. If $\overline{X}_n$ and $\overline{Y}_n$ are the respective sample means of the $X_i's$ and $Y_i's$.

(1) Find $E(\overline{X}_n - \overline{Y}_n), V(\overline{X}_n - \overline{Y}_n)$.
(2) Is $\overline{X}_n - \overline{Y}_n$ a consistent estimator of $\mu_X - \mu_Y$? Why?
(3) Use the CLT to find the asymptotic distribution of $\overline{X}_n - \overline{Y}_n$. Why?
Write down the limiting distribution of the above statistics.
(4) What is a consistent estimator of μ_X / μ_Y if $\mu_Y \neq 0$? Why?

3. Let $\overline{X}$ denote the sample mean in random sampling, sample size n, from a population in which $X \sim \exp onential(\lambda)$, that is, $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f(x) = 0$ elsewhere.
For convenience, let $\theta = E(X) = 1/\lambda$. So, $E(\overline{X}) = \theta, V(\overline{X}) = \theta^2 / n$ and the limiting distribution of $\sqrt{n}(\overline{X} - \theta)$ is $N(0, \theta^2)$. Consider the sample statistic $T = 1/\overline{X}$.

(1) Use a Slutsky theorem to show that $p\lim T = \lambda$.

(2) Use the Delta method to find the limiting distribution of $\sqrt{n}(T - \lambda)$.

(3) Use your result to approximate $P(T \le 5/2)$ in random sampling sample size 16, from an exponential population with $\lambda = 2$.

4. Let $\hat{\theta}$ be a consistent estimator of $\theta > 0$ and its limiting distribution is as follows: $\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow{d} N(0, \sigma^2)$. Let $\hat{\gamma} = \log(\hat{\theta})$ be an estimator of $\gamma = \log(\theta)$.

(1) Is $\hat{\gamma}$ a consistent estimator of γ? Justify your answer.
(2) Find <u>asymptotic</u> variance of $\hat{\gamma}$.