Chapter 1 Brief Review of Probability Theory

Read Yale Note 1(pp.1-6), 2(pp.1-7), 3

Wackerly et al. Ch. 2

- Basic Probability Theory
- Basic Properties of Probability
- Conditional Probability
- Statistical Independence

1. Introduction

• *Probability*: Measure of one's belief in the occurrence of an uncertain future event.

► Randomness: The fact that an event may occur, but may not.

• *Experimentation*: A scientist performs a controlled (repeatable) experiment and observes an outcome, which cannot be predicted with certainty prior to the experiment ⇒ Random experiment.

(Example) Outcome of tossing a coin or rolling a dice.

(Examples of deterministic events)

① When you drop something, it will hit the floor.

② The sun rises every day.

③ A student showing up for all classes.

- *Sample point*: each conceivable outcome.
- *Sample space*(S): set of all possible outcome.

(Examples)

① coin tossing: $S = \{H, T\}$.

② dice rolling: $S = \{1, 2, 3, 4, 5, 6\}$.

③ drawing a random point from the $[0, 1]$ interval: $S = \{w : 0 \le w \le 1\}$.

► *Finite sample space*: a space with a finite number of outcomes(①, ②).

► *Infinite sample space*: a space with a infinite number of outcomes(③).

(Example) Let the experiment be the roll of a single die.

$S = \{1,2,3,4,5,6\}$

A=event of obtaining an even outcomes=$\{2,4,6\}$.

(Example) Consider the experiment of flipping a coin until a head appears.

$S = \{H, TH, TTH, TTTH, \cdots\}$: infinite # of elements.

► The sample space is *countable* or *countably infinite*.

• If a sample space contains *a finite # of elements* or an *infinite though countable # of elements*, it is said to be *discrete*.

◎ Interpretations of Probabilites

① *Classical Approach*: notion of mutually exclusive and equally likely experiment outcomes.

► The prob. of m equally likely outcomes of a random experiment is $\frac{1}{m}$.

► Logical probability.

► No actual experiment is needed for this interpretation.

► Random sampling.

② *Frequentist Approach*: relative frequency interpretation of probability.

► Let A be an event defined on sample space S.

Suppose we repeat the experiment N times and observe that the event A occurs f times, then $P(A) = \lim_{N \to \infty} \frac{f}{N}$.

► This does not require equally likely sample points.

► Based on empirical probability.

► It is called objective probability.

(Example) Rolling a dice in many repetitions, the relative frequency of each number is 1/6.

③ *Subjective Approach*: the probability of an outcome is ascribed by one's knowledge, general agreements etc.

(Example) Probability of champion of the World Cup.

► Bayesian.

► There might not e agreement among people as to the probabilities.

2. Some Basic Properties of Probability

◎ *Axioms of Probability*

① For any event $\mathrm{A} \subset S$, it must hold that $\mathrm{P(A)} \geq 0$.

② $\mathrm{P(S)} = 1$.

③ $\mathrm{P}(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

For every sequence of disjoint events $A_1, A_2, \cdots$.

(Theorem 1) For each $\mathrm{A} \subset \mathrm{S}$, $\mathrm{P(A)} = 1 - \mathrm{P}(\bar{A})$.

(Theorem 2) $\mathrm{P}(\emptyset) = 0$.

(Theorem 3) If A and B are subsets of S such that $\mathrm{A} \subset \mathrm{B}$, then $\mathrm{P(A)} \leq \mathrm{P(B)}$.

(Theorem 4) For each $\mathrm{A} \subset \mathrm{S}$, $0 \leq \mathrm{P}(\mathrm{A}) \leq 1$.

(Theorem 5) For $\mathrm{A} \subset \mathrm{S}$ and $\mathrm{B} \subset \mathrm{S}$,

it holds that $\mathrm{P}(\mathrm{A} \cup \mathrm{B}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) - \mathrm{P}(\mathrm{A} \cap \mathrm{B})$.

3. Conditional Probability

► Interested in an event(A) given that some other event(B).

► Compute probability given that some additional information is known.

(Example) Suppose that you are interested in the stock price of Samsung.

$$P(\text{Samsung stock price increased}) \overset{?}{=} P(\text{Samsung stock price increased} \mid KOSPI \text{ increased}).$$

(Example) A single die is tossed. Let $A=\{1\}$ and $B=\{1, 3, 5\}$.

Unconditional probability: $P(A)=1/6$, $P(B)=3/6$.

Conditional probability: $P(A \mid B)=1/3$.

- From probability theory,

① $P(B|B) = 1,$

② $P(A|B) = P(A \cap B|B)$

③ $\dfrac{P(A \cap B|B)}{P(B|B)} = \dfrac{P(A \cap B)}{P(B)}$ by proportion argument.

- If A and B are events in S, then the conditional probability of event A given that event B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then $P(A|B) = 0$.

(Example) If $P(A) = 0.4,\quad P(B) = 0.5,\quad P(A \cap B) = 0.3$, then

$$P(A|B) = 0.3/0.5 = 0.6$$

$$P(B|A) = 0.3/0.4 = 0.75$$

- ***Axioms of Conditional Probability***

① $P(A|B) \geq 0$ for any event A.

② $P(A|B) = 1$ for any event $B \subset A$.

③ If $\{A_i \cap B\},\ i = 1, 2, \cdots$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots | B) = P(A_1|B) + P(A_2|B) + \cdots$$

► $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.

4. Statistical Independence

- Intuition: Two events are independent if the probability of one of them occurring is not influenced by the occurrence of the other, $P(A|B) = P(A)$.

  ► Therefore, $P(A|B) = \dfrac{P(A \cap B)}{P(B)} == P(A)$.

(Definition) Two events A and B are *independent* iff $P(A \cap B) = P(A) \cdot P(B)$.

Otherwise, A and B are *dependent*.

(Example) Toss a coin and roll a dice.

The events $\{H,1\}, \{H,2\}, \cdots, \{T,6\}$ all have probability equal to 1/12.

Define $A = \{H\}, B = \{1\}$. Then A and B are independent because

$P(A) = 1/2,\ P(B) = 1/6$ and $P(A \cap B) = 1/12 = P(A) \cdot P(B)$.

▶There are many economic questions that can be posed in terms of whether two events are independent, e.g. money and income, education and earnings, etc.

(Example) Consider the experiment of tossing a coin three times.

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Let

$$A = \{\text{Head on each of the first 2 tosses}\} = \{HHH, HHT\}$$

$$B = \{\text{Tail occurs on third toss}\} = \{HHT, HTT, THT, TTT\}$$

$$C = \{\text{Exactly 2 tails on 3 tosses}\} = \{HHT, THT, TTH\}$$

► Events A and B are independent, since

$$P(A) = 2/8,\ P(B) = 4/8,\ P(A \cap B) = 1/8 = P(A) \cdot P(B)$$

Note that $P(A|B) = P(A),\ P(B|A) = P(B)$.

► Events B and C are dependent, since

$$P(B) = 4/8,\ P(C) = 3/8,\ P(B \cap C) = 2/8 \neq P(B) \cdot P(C) = 3/16.$$

- This definition extends to more than two events.

For example, events A, B and C are independent

iff $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$.

(Note) Events that are independent are sometimes called *statistically independent*, *stochastically independent* or *independent in a probability sense*.

(Note) We are going to omit Bayes Rule.