Answer Key 9

1. From $X \sim U(0,1)$, $E(X)=\frac{1}{2}, V(X)=\frac{1}{12}$.

(1) Since $\bar{X} \overset{a}{\sim} \mathrm{N}\left(\frac{1}{2}, \frac{1}{12n}\right)$, $P(a \le \bar{X} \le b) = P\left(\frac{a-0.5}{\sqrt{1/12n}} \le \frac{\bar{X}-0.5}{\sqrt{1/12n}} \le \frac{b-0.5}{\sqrt{1/12n}}\right)$.

(2)

$$
\begin{aligned}
&P\left(\frac{7}{16} \le \bar{X} \le \frac{9}{16}\right) = P\left(\frac{7/16-0.5}{\sqrt{1/12^2}} \le \frac{\bar{X}-0.5}{\sqrt{1/12^2}} \le \frac{9/16-0.5}{\sqrt{1/12^2}}\right) \\
&= P\left(\frac{-0.0625}{1/12} \le Z \le \frac{0.0625}{1/12}\right) = P(-0.75 \le Z \le 0.75) \\
&= \Phi(0.75) - \Phi(-0.75) = 0.547
\end{aligned}
$$

2.

(1) $E(\bar{X}_n - \bar{Y}_n) = \mu_X - \mu_Y$, $V(\bar{X}_n - \bar{Y}_n) = \frac{2\sigma^2}{n}$.

(2) By L.L.N., $\bar{X}_n \xrightarrow{p} \mu_X$, $\bar{Y}_n \xrightarrow{p} \mu_Y$, then by Slustsky theorem, $\bar{X}_n - \bar{Y}_n \xrightarrow{p} \mu_X - \mu_Y$.

(3) $\bar{X}_n - \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i)$ and

$E(X_i - Y_i) = \mu_X - \mu_Y$, $V(X_i - Y_i) = 2\sigma^2$ and $X_i - Y_i : i.i.d$

$$\bar{X}_n - \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i) \overset{a}{\sim} N\left(\mu_X - \mu_Y, \frac{2\sigma^2}{n}\right)$$

(4) Consistent estimator of $\mu_X / \mu_Y = \bar{X}/\bar{Y}$.

3. (1) Since $\bar{X} \xrightarrow{p} E(X) = \theta = 1/\lambda$ by LLN, $p\lim T = p\lim(1/\bar{X}) = \frac{1}{p\lim \bar{X}} = \lambda$.

(2) $\sqrt{n}(T-\lambda) = \sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\theta}\right) \xrightarrow{d} N\left(0, \left(-\theta^{-2}\right)^2 \theta^2\right) = N\left(0, \theta^{-2}\right)$.

Therefore, $\sqrt{n}(T-\lambda) = \xrightarrow{d} N\left(0, \lambda^2\right)$.

(3) For $n = 16, \lambda = 2$, $T \overset{a}{\sim} N(2, 1/4)$.

So, $P(T \le 5/2) = P\left(Z \le \frac{5/2-2}{1/2}\right) = P(Z \le 1) = 0.8413$.

4. (1) Since $\hat{\theta} \xrightarrow{p} \theta$ and $\log x$ is continuous at $x>0$,

$\log(\hat{\theta}) \xrightarrow{p} \log(\theta)$ by Slutsky theorem.

(2) Since $\log x$ is continuously differentiable at $x>0$ ($\because \dfrac{d \log x}{dx} = \dfrac{1}{x}$ is continuous at $x>0$),

$\sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} N\left(0, \dfrac{\sigma^2}{\theta^2}\right)$ by delta method.

So, Asy.Var($\hat{\gamma}$)= $\dfrac{\sigma^2}{n\theta^2}$.