Answer Key 8

1. (1) From the definite integral formula, $E(X^j)=\dfrac{j!}{\theta^j}$. Now, substitute $\theta=2$, and $j=1,2,3,4$ to get the numbers shown.

(2) Let $W=X^2$, so $E(W)=1/2$ and $V(W)=3/2-1/4=5/4$.

Now $\overline{W}$ is sample mean of 20 independent drawing of random variable W, so $E(\overline{W})=E(W)=1/2$.

(3) Similarly, $V(\overline{W})=V(W)/20=(5/4)/20=1/16$.

2. From $X_i \sim N(\mu,\sigma^2)$ $\left(X_i \sim N(10,80)\right)$, $n=20$,

we have

① $\overline{X}\sim N(\mu,\sigma^2/n)$ $\quad\Rightarrow\ \overline{X}\sim N(10,4)$ and $Z\equiv\dfrac{(\overline{X}-10)}{2}\sim N(0,1)$.

② $\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$ $\quad\Rightarrow\ W\equiv\dfrac{19S^2}{80}\sim\chi^2(19)$,

③ Z and W: independent

④ $\dfrac{Z}{\sqrt{W/(n-1)}}\sim t(n-1)$ $\quad\Rightarrow\ U\equiv\dfrac{(\overline{X}-10)}{S/\sqrt{20}}\sim t(19)$,

Let $F(.), G(.)$ and $H(.)$ denote the cdf of $N(0,1)$, $\chi^2(19)$ and $t(19)$ respectively.

$P(A)=P(Z\le 2)=F(2)=0.9773$ using ①.

$P(B)=P(0\le Z\le 1)=F(1)-F(0)=0.3413$ using ①.

$P(C)=P(W\le 103.36)\approx 1$ using ②.

$P(D)=P(B)*P(C)=0.3413$ by independence

$P(E)=P(U\le 1.066)=H(1.066)=0.85$ using ④

$P(F)=P(U\le 1.363)=H(1.328)\approx 0.910$.

3. Since $W=\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$ and we know that $E(W)=(n-1)$, $V(W)=2(n-1)$.

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right)=\frac{(n-1)^2}{\sigma^4}V(S^2)=2(n-1)\ \Rightarrow\ V(S^2)=\frac{2\sigma^4}{(n-1)}.$$