Math. Econ. Statistics

Answer Key 10

1.

(1) Consider $\hat{\mu} = c_1\overline{Y}_1 + c_2\overline{Y}_2$. Then $E(\hat{\mu}) = (c_1 + c_2)\mu$. For $\hat{\mu}$ to be unbiased, we require $c_1 + c_2 = 1$. $V(\hat{\mu}) = \frac{\sigma_1^{\,2}}{n_1}c_1^{\,2} + \frac{\sigma_2^{\,2}}{n_2}c_2^{\,2}$..

For $\hat{\mu}$ to be MVUE, minimize $V(\hat{\mu})$ $s.t.$ $c_1 + c_2 = 1$.

The answer will be $c_1{*} = \dfrac{\sigma_2^{\,2}/n_2}{\sigma_1^{\,2}/n_1 + \sigma_2^{\,2}/n_2}$, $c_1{*} = \dfrac{\sigma_1^{\,2}/n_1}{\sigma_1^{\,2}/n_1 + \sigma_2^{\,2}/n_2}$.

(2) Let $A = \frac{\sigma_1^{\,2}}{n_1}$, $B = \frac{\sigma_2^{\,2}}{n_2}$. Then $A, B > 0$, and

$$V(T) = \left(\frac{B}{A+B}\right)^2 A + \left(\frac{A}{A+B}\right)^2 B = \frac{AB}{A+B} < A = \frac{A(A+B)}{(A+B)} = V(\overline{Y}_1).$$

And $V(T) = \dfrac{AB}{A+B} < B = \dfrac{B(A+B)}{(A+B)} = V(\overline{Y}_2)$.

2. $T = \hat{P}(Y \le c) = \dfrac{\#\, of\ Y_i \le c}{n}$.

(1) $X = \#\, of\ Y_i \le c = \sum_{i=1}^{n} Z_i$, where $Z_i = \begin{cases} 1 & for\ Y_i \le c \\ 0 & otherwise \end{cases}$.

Then $X \sim B(n, \theta)$ and $Z_i \sim Bernoulli(\theta)$.
Note that $E(X) = n\theta$, $V(X) = n\theta(1-\theta)$.

Therefore, $E(T) = \frac{1}{n}E(X) = \theta$; unbiased.

(2) $V(T) = \dfrac{1}{n^2}V(X) = \dfrac{\theta(1-\theta)}{n}$.

(3) Note that $T = \frac{1}{n}\sum_{i=1}^{n} Z_i = \overline{Z} \xrightarrow{p} E(Z) = \theta$, since $V(T) \to 0$.

(4) From Central Limit Theorem, $T \overset{a}{\sim} N(\theta, \theta(1-\theta)/n)$.

(5) From (4), $Asy.V(T) = \dfrac{\theta(1-\theta)}{n}$.

Since T is a consistent estimator of θ, propose $\hat{A}sy.V(T) = \dfrac{T(1-T)}{n}$.

Then, since $p\lim T = \theta$, $\hat{A}sy.V(T) = \dfrac{T(1-T)}{n} \xrightarrow{p} \dfrac{\theta(1-\theta)}{n} = Asy.V(T)$.

(6) From (4) and (5), 95% asymptotic confidence interval of θ is $\left[T \pm 1.96\sqrt{\dfrac{T(1-T)}{n}} \right]$.