Firm Dynamics and Pricing under Customer Capital Accumulation

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Online Appendix

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A Proofs

A.1 Proof of Proposition 1: Joint Surplus Problem

Proof. Denote by $\overline{\boldsymbol{\omega}} = \Big\{\overline{p}, \overline{\mathbf{x}}'(n';\mathbf{s}')\Big\}$ a generic policy of the typical seller in state $(n;z,\varphi)$, where $\overline{p}$ is the price level,

$$\overline{\mathbf{x}}'(n';\mathbf{s}') = \Big\{\overline{x}'(n+1;z,\varphi), \overline{x}'(n-1;z,\varphi), \big\{\overline{x}'(n;z',\varphi) : z' \in \mathcal{Z}\big\}, \big\{\overline{x}'(n;z,\varphi') : \varphi' \in \Phi\big\}\Big\}$$

is the set of promised utilities, and $\overline{x}'(n+1;z,\varphi)$ and $\overline{x}'(n-1;z,\varphi)$ are the upsizing and downsizing choices, respectively. Recall that $\overline{x}'(n;z,\varphi) = x$ by stationarity. The value of the seller in equilibrium, $V^S(n,x;z,\varphi)$, can be written as the maximand on the right-hand side of (3a), evaluated at $\overline{\boldsymbol{\omega}}$. That is:

$$V^S(n,x;z,\varphi) = \max_{\overline{\boldsymbol{\omega}}\in\Omega} \quad \widetilde{V}^S(n;z,\varphi|\overline{\boldsymbol{\omega}}) \qquad \text{s.t.} \quad x \le V^B(n,\overline{\boldsymbol{\omega}};z,\varphi)$$

where $\widetilde{V}^S(n;z,\varphi|\overline{\boldsymbol{\omega}})$ is given by:

$$\begin{aligned}\widetilde{V}^S(n;z,\varphi|\overline{\boldsymbol{\omega}}) \equiv & \frac{1}{\rho(n;z,\varphi)}\Bigg[\overline{p}n - \mathcal{C}(n;z,\varphi) + \eta\Big(\theta\big(\overline{x}'(n+1;z,\varphi);\varphi\big)\Big)V^S\Big(n+1,\overline{x}'(n+1;z,\varphi);z,\varphi\Big) \\ & + n\delta_c V^S\Big(n-1,\overline{x}'(n-1;z,\varphi);z,\varphi\Big) + \sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)V^S\Big(n,\overline{x}'(n;z',\varphi);z',\varphi\Big) \\ & + \sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)V^S\Big(n,\overline{x}'(n;z,\varphi');z,\varphi'\Big)\Bigg] \end{aligned} \tag{A.1.1}$$

and we have defined $\rho(n;z,\varphi) \equiv r + \delta_f + n\delta_c + \eta\big(\theta(\overline{x}'(n+1;z,\varphi);\varphi)\big)$ as the *effective* discount rate of the firm. From (A.1.1), it is optimal to offer the highest possible price that is consistent with promise-keeping, for any given policy $\overline{\boldsymbol{\omega}}$. Indeed, the price has no bearing on the agents' incentives within the search market. Therefore, the promise-keeping constraint must bind with equality, and we can solve for the price $\overline{p}$ such that $x = V^B(n,\overline{\boldsymbol{\omega}};z,\varphi)$ using equation (2):

$$\begin{aligned} p^{PK} : \overline{\mathbf{x}}'(n';\mathbf{s}') \mapsto \; & \Big\{v(\varphi) - \rho(n;z,\varphi)x + \delta_f U^B(\varphi) + \eta\Big(\theta\big(\overline{x}'(n+1;z,\varphi);\varphi\big)\Big)\overline{x}'(n+1;z,\varphi) \\ & + \delta_c\Big(U^B(\varphi) + (n-1)\overline{x}'(n-1;z,\varphi)\Big) + \sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\overline{x}'(n;z',\varphi) + \sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\overline{x}'(n;z,\varphi')\Big\} \end{aligned} \tag{A.1.2}$$

Using the above notation, we can now substitute the price level $p^{PK}\big(\overline{\mathbf{x}}'(n';\mathbf{s}')\big)$ from (A.1.2) into the seller's value (A.1.1). After some straightforward algebra, we obtain:

$$\widetilde{W}(n,x;z,\varphi|\overline{\boldsymbol{\omega}}) = \frac{1}{\rho(n;z,\varphi)}\Bigg[n\Bigg(v(\varphi) + (\delta_f + \delta_c)U^B(\varphi)\Bigg)$$

$$
\begin{aligned}
&- \left(\mathcal{C}(n;z,\varphi) + \eta\Big(\theta\big(\overline{x}'(n+1;z,\varphi);\varphi\big)\Big)\overline{x}'(n+1;z,\varphi)\right) \\
&+ \eta\Big(\theta\big(\overline{x}'(n+1;z,\varphi);\varphi\big)\Big) W\Big(n+1, \overline{x}'(n+1;z,\varphi); z,\varphi\Big) + n\delta_c W\Big(n-1, \overline{x}'(n-1;z,\varphi); z,\varphi\Big) \\
&+ \sum_{z'\in\mathcal{Z}} \lambda_z(z'|z) W\Big(n, \overline{x}'(n;z',\varphi); z',\varphi\Big) + \sum_{\varphi'\in\Phi} \lambda_\varphi(\varphi'|\varphi) W\Big(n, \overline{x}'(n;z,\varphi'); z,\varphi'\Big)\Bigg]
\end{aligned} \tag{A.1.3}
$$

where we have defined:

$$
\widetilde{W}(n,x;z,\varphi|\overline{\boldsymbol{\omega}}) \equiv \widetilde{V}^S(n;z,\varphi|\overline{\boldsymbol{\omega}}) + nx \qquad \text{and} \qquad W(n,x;z,\varphi) \equiv \max_{\overline{\boldsymbol{\omega}}\in\Omega} \ \widetilde{W}(n,x;z,\varphi|\overline{\boldsymbol{\omega}})
$$

as the joint surplus under contract $\overline{\boldsymbol{\omega}}$, and the *maximized* joint surplus, respectively. Next, note that the right-hand side of equation (A.1.3) does not depend on x nor p, so we can write the joint surplus under a given policy as:

$$
\widetilde{W}(n,x;z,\varphi|\overline{\boldsymbol{\omega}}) = \widetilde{W}_n\Big(\overline{\mathbf{x}}'(n';\mathbf{s}'); z,\varphi\Big)
$$

The optimal contract is then $\boldsymbol{\omega}^* = \Big\{p^*, \mathbf{x}'^*(n';\mathbf{s}')\Big\}$, where:

$$
\mathbf{x}'^*(n';\mathbf{s}') \equiv \arg\max_{\mathbf{x}} \ \widetilde{W}_n\big(\mathbf{x};z,\varphi\big) \tag{A.1.4a}
$$

$$
p^* \equiv \ p^{PK}\Big(\overline{\mathbf{x}}'^*(n';\mathbf{s}')\Big) \tag{A.1.4b}
$$

By expressing the problem of the seller in terms of $\widetilde{W}$, we have just shown that the optimal contract, $\boldsymbol{\omega}^*$, must maximize the joint surplus. Conversely, for any vector $\overline{\mathbf{x}}'(n';\mathbf{s}')$ of continuation values that maximizes the joint surplus, there is a price level, given by $p^* = p^{PK}\big(\overline{\mathbf{x}}'^*(n';\mathbf{s}')\big)$, that maximizes the seller's value subject to the promise-keeping constraint. □

A.2 Proof of Proposition 2: Efficiency

Proof. Consider a benevolent planner that is constrained by the search frictions of the economy and seeks to maximize aggregate welfare subject to the resource constraints of the economy. The planner can allocate resources freely, so the problem does not feature contracts nor prices. Instead, we label each market segment directly by its tightness, θ. To simplify notation, it is understood that time subscripts embody the entire history of aggregate shocks, which is taken to be some arbitrary path $\varphi^t = (\varphi_j : j \leq t) \subseteq \Phi$.

The planner chooses:

- The tightness in each market segment, $\boldsymbol{\Theta}_t \equiv \big\{\theta_{n_t,t}(z_t) : (n_t,z_t) \in \mathbb{N}\times\mathcal{Z}\big\}$;
- Distributions of inactive and active buyers across markets, $\mathbf{B}^I_t \equiv \big\{B^I_{n_t,t}(z_t) : (n_t,z_t) \in \mathbb{N}\times\mathcal{Z}\big\}$ and $\mathbf{B}^A_t \equiv \big\{B^A_{n_t,t}(z_t) : (n_t,z_t) \in \mathbb{N}\times\mathcal{Z}\big\}$;

- A measure of potential entrants, $S_{0,t}$;
- A distribution of firms across states, $\mathbf{S}_t \equiv \big\{S_{n_t,t}(z_t) : (n_t, z_t) \in \mathbb{N} \times \mathcal{Z}\big\}$.

The planner's objective is:

$$\max_{\substack{\mathbf{\Theta}_t,\mathbf{B}_t^I,\mathbf{B}_t^A \\ S_{0,t},\mathbf{S}_t}} \quad \mathbb{E}_0 \int_0^{+\infty} e^{-rt}\mathbb{W}_t(\varphi_t)\mathrm{d}t \tag{A.2.1}$$

where

$$\mathbb{W}_t(\varphi_t) = -\kappa S_{0,t} + \sum_{n_t=1}^{+\infty}\sum_{z_t\in\mathcal{Z}} \Big[v(\varphi_t)B^A_{n_t,t}(z_t) - \mathcal{C}(n_t; z_t, \varphi_t)S_{n_t,t}(z_t) - cB^I_{n_t,t}(z_t)\Big]$$

The planner is subject to three sets of constraints. The first one concerns the evolution of the distribution of sellers:

$$\boldsymbol{\partial}_t S_{0,t} = \delta_f \sum_{n_t=1}^{+\infty}\sum_{z_t\in\mathcal{Z}} S_{n_t,t}(z_t) + \delta_c \sum_{z_t\in\mathcal{Z}} S_{1,t}(z_t) - \sum_{z^e\in\mathcal{Z}} \pi_z(z^e)\eta\big(\theta_{1,t}(z^e)\big)S_{0,t} \tag{A.2.2a}$$

$$\begin{aligned}\boldsymbol{\partial}_t S_{1,t}(z_t) = {} & \pi_z(z_t)\eta\big(\theta_{1,t}(z_t)\big)S_{0,t} + 2\delta_c S_{2,t}(z_t) + \sum_{\tilde{z}\neq z_t} \lambda_z(z_t|\tilde{z})S_{1,t}(\tilde{z}) \\ & - \Big(\delta_f + \delta_c + \eta\big(\theta_{2,t}(z_t)\big) + \sum_{\tilde{z}\neq z_t}\lambda_z(\tilde{z}|z_t)\Big)S_{1,t}(z_t)\end{aligned} \tag{A.2.2b}$$

$$\begin{aligned}\forall n_t \geq 2: \quad \boldsymbol{\partial}_t S_{n_t,t}(z_t) = {} & \eta\big(\theta_{n_t,t}(z_t)\big)S_{n_t-1,t}(z_t) + (n_t+1)\delta_c S_{n_t+1,t}(z_t) + \sum_{\tilde{z}\neq z_t}\lambda_z(z_t|\tilde{z})S_{n_t,t}(\tilde{z}) \\ & - \Big(\delta_f + n_t\delta_c + \eta\big(\theta_{n_t+1,t}(z_t)\big) + \sum_{\tilde{z}\neq z_t}\lambda_z(\tilde{z}|z_t)\Big)S_{n_t,t}(z_t);\end{aligned} \tag{A.2.2c}$$

for all $z_t \in \mathcal{Z}$, where z^e denotes the productivity draw upon entry. The second set of constraints describes the distribution of agents across states at any given time:

$$\forall (n_t, z_t) \in \mathbb{N}\times\mathcal{Z}: \quad B^A_{n_t,t}(z_t) = n_t S_{n_t,t}(z_t) \tag{A.2.3a}$$

$$\forall (n_t, z_t) \in \mathbb{N}\times\mathcal{Z}: \quad B^I_{n_t,t}(z_t) = \theta_{n_t,t}(z_t)S_{n_t-1,t}(z_t) \tag{A.2.3b}$$

$$1 = \sum_{n_t=1}^{+\infty}\sum_{z_t\in\mathcal{Z}}\Big(B^A_{n_t,t}(z_t) + B^I_{n_t,t}(z_t)\Big) \tag{A.2.3c}$$

Equation (A.2.3a) states that each customer consumes a single unit; equation (A.2.3b) states that each market segment is in equilibrium, in the sense that the measure of buyers who find a seller in any given market equals the measure of sellers within that market who find a new customer; equation (A.2.3c) says that every buyer in the economy is in either the active or the inactive state.

Finally, the mass of potential entering sellers must be non-negative in any aggregate state:

$$S_{0,t} \geq 0 \tag{A.2.4}$$

To solve, first we use constraints (A.2.3a) and (A.2.3b) to rewrite (A.2.3c) as:

$$\sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} n_t S_{n_t,t}(z_t) + \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \theta_{n_t+1,t}(z_t) S_{n_t,t}(z_t) + S_{0,t} \sum_{z_t \in \mathcal{Z}} \theta_{1,t}(z_t) = 1 \tag{A.2.5}$$

Substituting constraints (A.2.3a) and (A.2.3b) into the objective function:

$$\begin{aligned}\max_{\boldsymbol{\Theta}_t, S_{0,t}, \mathbf{S}_t} \; \mathbb{E}_0 \int_0^{+\infty} e^{-rt} \Bigg\{ - \Bigg(\kappa + c \sum_{z_t \in \mathcal{Z}} \theta_1(z_t) \Bigg) S_{0,t} + v(\varphi_t) \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} n_t S_{n_t,t}(z_t) \\ - \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \mathcal{C}(n_t; z_t, \varphi_t) S_{n_t,t}(z_t) - c \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \theta_{n_t+1,t}(z_t) S_{n_t,t}(z_t) \Bigg\} \mathrm{d}t\end{aligned}$$

subject to (A.2.2a), (A.2.2b), (A.2.2c), (A.2.4), and (A.2.5). Conveniently, the variables $\mathbf{B}_t^I$ and $\mathbf{B}_t^A$ have disappeared from the problem. The state vector now only includes measures of sellers: $\mathbb{S}_t \equiv [S_{0,t}, \mathbf{S}_t]$. The current-value Hamiltonian of the simplified planning problem is:

$$\begin{aligned}\mathcal{H}_t(\boldsymbol{\Theta_t}; \mathbb{S}_t) \equiv\; & - \Bigg(\kappa + c \sum_{z_t \in \mathcal{Z}} \theta_1(z_t) \Bigg) S_{0,t} + v(\varphi_t) \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} n_t S_{n_t,t}(z_t) \\ & - \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \mathcal{C}(n_t; z_t, \varphi_t) S_{n_t,t}(z_t) - c \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \theta_{n_t+1,t}(z_t) S_{n_t,t}(z_t) \\ & + \phi_t \Bigg[1 - \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} n_t S_{n_t,t}(z_t) - \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} \theta_{n_t+1,t}(z_t) S_{n_t,t}(z_t) - S_{0,t} \sum_{z_t \in \mathcal{Z}} \theta_{1,t}(z_t) \Bigg] \\ & + \psi_{0,t} \Bigg[\delta_f \sum_{n_t=1}^{+\infty} \sum_{z_t \in \mathcal{Z}} S_{n_t,t}(z_t) + \delta_c \sum_{z_t \in \mathcal{Z}} S_{1,t}(z_t) - \sum_{z^e \in \mathcal{Z}} \pi_z(z^e) \eta\big(\theta_{1,t}(z^e)\big) S_{0,t} \Bigg] \\ & + \sum_{z_t \in \mathcal{Z}} \Bigg\{ \psi_{1,t}(z_t) \Bigg[\pi_z(z_t) \eta\big(\theta_{1,t}(z_t)\big) S_{0,t} + 2\delta_c S_{2,t}(z_t) + \sum_{\tilde{z} \neq z_t} \lambda_z(z_t|\tilde{z}) S_{1,t}(\tilde{z}) \\ & \qquad - \Bigg(\delta_f + \delta_c + \eta\big(\theta_{2,t}(z_t)\big) + \sum_{\tilde{z} \neq z_t} \lambda_z(\tilde{z}|z_t) \Bigg) S_{1,t}(z_t) \Bigg] \\ & \quad + \sum_{n_t=2}^{+\infty} \psi_{n_t,t}(z_t) \Bigg[\eta\big(\theta_{n_t,t}(z_t)\big) S_{n_t-1,t}(z_t) + (n_t+1)\delta_c S_{n_t+1,t}(z_t) + \sum_{\tilde{z} \neq z_t} \lambda_z(z_t|\tilde{z}) S_{n_t,t}(\tilde{z}) \\ & \qquad - \Bigg(\delta_f + n_t \delta_c + \eta\big(\theta_{n_t+1,t}(z_t)\big) + \sum_{\tilde{z} \neq z_t} \lambda_z(\tilde{z}|z_t) \Bigg) S_{n_t,t}(z_t) \Bigg] \Bigg\} + \vartheta_t S_{0,t}\end{aligned}$$

where $\psi_{n,t}(z) \geq 0$, $n \geq 1$ (respectively, $\psi_{0,t} \geq 0$) is the co-state variable on the flow equation for

$S_{n,t}(z)$ (respectively, $S_{0,t}$); $\phi_t \geq 0$ is the multiplier on (A.2.5); and $\vartheta_t \geq 0$ is the multiplier on the non-negative entry condition, where the corresponding complementary slackness hold. In vector notation, the necessary conditions for optimality are:

$$
\begin{aligned}
\nabla_{\boldsymbol{\Theta}} \mathcal{H}_t(\boldsymbol{\Theta_t}; \mathbb{S}_t) &= \mathbf{0} \\
\nabla_{\mathbb{S}} \mathcal{H}_t(\boldsymbol{\Theta_t}; \mathbb{S}_t) &= -\nabla_t \boldsymbol{\psi}_t + r\boldsymbol{\psi}_t
\end{aligned}
$$

where ∇ denotes the gradient operator, and $\boldsymbol{\psi}_t$ is a stacked vector of co-state variables. These conditions are also sufficient because the Hamiltonian is quasi-concave. Indeed, the objective function is linear in both control and state variables, and because of Assumption 1 establishing concavity of η, all the constraints are concave in the control and linear in the states.

Regarding the first set of optimality conditions, for given $z_t \in \mathcal{Z}$ we have:

$$
[\theta_1]: \qquad \phi_t + c = \Big(\psi_{1,t}(z_t) - \psi_{0,t}\Big)\pi_z(z_t)\frac{\partial \eta(\theta)}{\partial \theta}\Big|_{\theta=\theta_{1,t}(z_t)} \tag{A.2.6a}
$$

$$
[\theta_n : n \geq 2]: \qquad \phi_t + c = \Big(\psi_{n_t,t}(z_t) - \psi_{n_t-1,t}(z_t)\Big)\frac{\partial \eta(\theta)}{\partial \theta}\Big|_{\theta=\theta_{n_t,t}(z_t)} \tag{A.2.6b}
$$

As for the second set of conditions, we have:

$$
\begin{aligned}
[S_0]: \quad -\boldsymbol{\partial}_t \psi_{0,t} + r\psi_{0,t} &= -\kappa - \big(\phi_t + c\big)\sum_{z_t \in \mathcal{Z}} \theta_1(z_t) && \text{(A.2.7a)} \\
&\quad + \sum_{z^e \in \mathcal{Z}} \pi_z(z^e)\eta\big(\theta_{1,t}(z^e)\big)\psi_{1,t}(z^e) - \psi_{0,t}\sum_{z^e \in \mathcal{Z}} \pi_z(z^e)\eta\big(\theta_{1,t}(z^e)\big) + \vartheta_t \\
[S_{n_t}(z_t)]: \quad -\boldsymbol{\partial}_t \psi_{n_t,t}(z_t) + r\psi_{n_t,t}(z_t) &= n_t\big(v(\varphi_t) - \phi_t\big) - \big(\phi_t + c\big)\theta_{n_t+1,t}(z_t) - \mathcal{C}(n_t, z_t; \varphi_t) \\
&\quad + \delta_f\Big(\psi_{0,t} - \psi_{n_t,t}(z_t)\Big) + n_t\delta_c\Big(\psi_{n_t-1,t}(z_t) - \psi_{n_t,t}(z_t)\Big) && \text{(A.2.7b)} \\
&\quad + \eta\big(\theta_{n_t+1}(z_t)\big)\Big(\psi_{n_t+1,t}(z_t) - \psi_{n_t,t}(z_t)\Big) + \sum_{\tilde{z} \in \mathcal{Z}} \lambda_z(\tilde{z}|z_t)\Big(\psi_{n_t,t}(\tilde{z}) - \psi_{n_t,t}(z_t)\Big)
\end{aligned}
$$

for given $z_t \in \mathcal{Z}$, where in the last line we have used that $\lambda_z(z|z) = -\sum_{\tilde{z} \neq z} \lambda_z(\tilde{z}|z)$ for all $z \in \mathcal{Z}$, by the properties of the Markov chain. To show that a block-recursive equilibrium with non-negative entry of sellers satisfies the optimality conditions of the planner, we may choose the co-state variables of the planning problem appropriately. By equations (A.2.7a)-(A.2.7b), the co-state variables can be represented as HJB equations. Equations (A.2.6a)-(A.2.6b) are the corresponding first order conditions of those equations. Therefore, it suffices to find the values of the multipliers for which the HJB equations of the planner coincide with the joint surplus problem of the decentralized allocation.

Pick a decentralized equilibrium allocation $\big\{W_n(z, \varphi), x_n(z, \varphi), \theta_n(z, \varphi), U^B(\varphi) : (n, z, \varphi) \in \mathbb{N} \times \mathcal{Z} \times \Phi\big\}$, and consider the following realizations for each multiplier:

$$\begin{array}{rcl} \phi_t(\varphi^t) & = & rU^B(\varphi_t) \\ \psi_{0,t}(\varphi^t) & = & 0 \\ \forall n_t, z_t: \quad \psi_{n_t,t}(z_t,\varphi^t) & = & W_{n_t}(z_t,\varphi_t) - n_t U^B(\varphi_t) \end{array}$$

Under this guess, notice that $\boldsymbol{\partial}_t \psi_{0,t} = \boldsymbol{\partial}_t \psi_{n,t}(z_t) = 0$, $\forall n \geq 1$. Moreover, the multipliers depend only on the current realization of the aggregate state, and not the entire history. Further, for a sufficiently low value of κ, we can impose strictly positive entry and therefore $\vartheta_t = 0$, $\forall t$. Plugging these guesses into (A.2.7b), after some simple algebra we obtain:

$$\begin{aligned}(r+\delta_f)W_{n_t}(z_t,\varphi_t) = {} & n_t\Big(v(\varphi_t) + (\delta_f+\delta_c)U^B(\varphi_t)\Big) - \mathcal{C}(n_t,z_t;\varphi_t) \\ & - \Big[\big(rU^B(\varphi_t)+c\big)\theta_{n_t+1,t}(z_t) + \eta\big(\theta_{n_t+1}(z_t)\big)U^B(\varphi_t)\Big] \\ & + n_t\delta_c\Big(W_{n_t-1}(z_t,\varphi_t) - W_{n_t}(z_t,\varphi_t)\Big) \\ & + \eta\big(\theta_{n_t+1}(z_t)\big)\Big(W_{n_t+1}(z_t,\varphi_t) - W_{n_t}(z_t,\varphi_t)\Big) + \sum_{\tilde{z}\in\mathcal{Z}} \lambda_z(\tilde{z}|z_t)\Big(W_{n_t}(\tilde{z},\varphi_t) - W_{n_t}(z_t,\varphi_t)\Big)\end{aligned}$$

The last equation resembles the maximized HJB equation for the joint surplus (equation (5)) except for the term in square brackets on the second line. Using that $\eta(\theta) = \theta\mu(\theta)$ and $x_{n+1}(z,\varphi) = U^B(\varphi) + \frac{rU^B(\varphi)+c}{\mu(\theta_{n+1}(z,\varphi))}$ by inactive buyers' indifference, this term can be written as:

$$\big(rU^B(\varphi_t)+c\big)\theta_{n_t+1,t}(z_t) + \eta\big(\theta_{n_t+1}(z_t)\big)U^B(\varphi_t) = \eta\left(\theta_{n_t+1,t}(z_t,\varphi_t)\right)x_{n_t+1,t}(z_t,\varphi_t) \tag{A.2.8}$$

Grouping terms, we will then recognize the value of the joint surplus in the decentralized solution, equation (5). Similarly, plugging the guess for the multipliers into (A.2.7a), we obtain:

$$\kappa = -\big(rU^B(\varphi_t)+c\big)\sum_{z_t\in\mathcal{Z}}\theta_1(z_t) + \sum_{z^e\in\mathcal{Z}}\pi_z(z^e)\eta\big(\theta_{1,t}(z^e)\big)\Big(W_1(z^e) - U^B(\varphi_t)\Big)$$

A final manipulation using (A.2.8) again then allows us to obtain the free entry condition in the decentralized allocation, equation (8). Summing up, under an appropriate choice of the co-states, the planner's solution is equivalent to the problem of the decentralized economy. Hence, the equilibrium is constrained-efficient. □

A.3 Proof of Proposition 3: Joint Surplus Solution

Proof. The equilibrium allocation is composed of sequences:

$$\big\{W_n(z,\varphi), x_n(z,\varphi), \theta_n(z,\varphi), p_n(z,\varphi) : (n,z,\varphi) \in \mathbb{N}\times\mathcal{Z}\times\Phi\big\}$$

satisfying equations (5), (7), and (9), where the free entry condition (8) pins down x_1 and the first-order condition (6) pins down x_n given x_{n-1}, $n \geq 2$, for any $\varphi \in \Phi$. Using $\mu(\theta) = \theta^{\gamma-1}$, $\gamma \in (0,1)$, equation (7) defines the following equilibrium mapping:

$$\theta : (x; \varphi) \mapsto \left(\frac{x - U^B(\varphi)}{\Gamma^B(\varphi)} \right)^{\frac{1}{1-\gamma}} \tag{A.3.1}$$

Some algebra shows that equation (6) can be written as:

$$W_{n+1}(z,\varphi) - W_n(z,\varphi) - x_{n+1}(z,\varphi) = \frac{1-\gamma}{\gamma}\Big(x_{n+1}(z,\varphi) - U^B(\varphi)\Big) \tag{A.3.2}$$

One can also write this condition as:

$$x_{n+1}(z,\varphi) - U^B(\varphi) = \gamma\Big(W_{n+1}(z,\varphi) - W_n(z,\varphi) - U^B(\varphi)\Big)$$

showing that the buyer absorbs a fraction γ of the marginal gains from matching. Next, define:

$$\Gamma^S_n(z,\varphi) \equiv (r+\delta_f)W_n(z,\varphi) - \pi_n(z,\varphi) + n\delta_c\Big(W_n(z,\varphi) - W_{n-1}(z,\varphi)\Big) - n(\delta_c+\delta_f)U^B(\varphi) - \Xi_n(z,\varphi) \tag{A.3.3}$$

where $\pi_n(z,\varphi) \equiv nv(\varphi) - \mathcal{C}(n; z, \varphi)$ is the flow joint surplus, and $\Xi_n(z,\varphi) \equiv \sum_{z' \in \mathcal{Z}} \lambda_z(z'|z)W_n(z',\varphi) + \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi)W_n(z,\varphi')$ is the expected value of the joint surplus across exogenous states. Next, note:

$$\begin{aligned}
\Gamma^S_n(z,\varphi) &= \eta\big(\theta_{n+1}(z,\varphi)\big)\Big(W_{n+1}(z,\varphi) - W_n(z,\varphi) - x_{n+1}(z,\varphi)\Big) \\
&= \left(\frac{1-\gamma}{\gamma}\right)\eta\big(\theta_{n+1}(z,\varphi)\big)\Big(x_{n+1}(z,\varphi) - U^B(\varphi)\Big) \\
&= \left(\frac{1-\gamma}{\gamma}\right)\theta_{n+1}(z,\varphi)\Gamma^B(\varphi)
\end{aligned}$$

where the first line uses the HJB equation for the joint surplus (equation (5)), the second line uses (A.3.2), and the third line uses (A.3.1) and $\eta(\theta) = \theta\mu(\theta)$. The right-hand side of the first equality allows us to interpret Γ^S as the expected match surplus for the seller (see main text). Using the last equality, we have found the market tightness:

$$\theta_{n+1}(z,\varphi) = \left(\frac{\gamma}{1-\gamma}\right)\frac{\Gamma^S_n(z,\varphi)}{\Gamma^B(\varphi)} \tag{A.3.4}$$

for all $n \geq 1$. Finally, we can write (A.3.2) as:

$$W_{n+1}(z,\varphi) - W_n(z,\varphi) = U^B(\varphi) + \frac{1}{\gamma}\big(x_{n+1}(z,\varphi) - U^B(\varphi)\big) = U^B(\varphi) + \frac{1}{\gamma}\Gamma^B(\varphi)\theta_{n+1}(z,\varphi)^{1-\gamma}$$

Using (A.3.4) and rearranging terms, we obtain:

$$W_{n+1}(z,\varphi) = W_n(z,\varphi) + U^B(\varphi) + \left(\frac{\Gamma^B(\varphi)}{\gamma}\right)^{\gamma}\left(\frac{\Gamma_n^S(z,\varphi)}{1-\gamma}\right)^{1-\gamma} \tag{A.3.5}$$

our desired result. □

A.4 Proof of Proposition 4: Invariant Distribution

Proof. Let $\{\theta_n(z,\varphi) : (n,z,\varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi\}$ be an equilibrium collection of market tightness levels, where $\mathcal{N} = \{1,\ldots,\overline{n}\}$, and $\overline{n} < +\infty$ is a large integer. In matrix notation, for each aggregate state $\varphi \in \Phi$, equations (12)-(13) can be written succinctly as:

$$\boldsymbol{\partial}_t \mathbf{S}_t(\varphi) = \mathbf{T}_\varphi \mathbf{S}_t(\varphi) \tag{A.4.1}$$

where $\mathbf{S}_t(\varphi) \equiv \left(S_{0,t}(\varphi), \mathbf{S}_{1,t}^\top, \ldots, \mathbf{S}_{\overline{n},t}^\top\right)^\top$, with $\mathbf{S}_{n,t} \equiv \left(S_{n,t}(z_1), \ldots, S_{n,t}(z_{k_z})\right)^\top$, $k_z \equiv |\mathcal{Z}|$, and $\mathbf{T}_\varphi$ is the partitioned matrix:

$$\mathbf{T}_\varphi \equiv \begin{pmatrix} t_{11} & \boldsymbol{\delta}_f^e + \boldsymbol{\delta}_c^e & \boldsymbol{\delta}_f^e & \boldsymbol{\delta}_f^e & \cdots & \boldsymbol{\delta}_f^e & \boldsymbol{\delta}_f^e & \boldsymbol{\delta}_f^e \\ \boldsymbol{\eta}_1^e(\varphi)^\top & \mathbf{D}_1(\varphi) & \boldsymbol{\delta}_{2,c} & \mathbf{0}_{k_z:k_z} & \cdots & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} \\ \mathbf{0}_{k_z:1} & \boldsymbol{\eta}_2(\varphi) & \mathbf{D}_2(\varphi) & \boldsymbol{\delta}_{3,c} & \cdots & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_{k_z:1} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \cdots & \mathbf{D}_{\overline{n}-2}(\varphi) & \boldsymbol{\delta}_{\overline{n}-1,c} & \mathbf{0}_{k_z:k_z} \\ \mathbf{0}_{k_z:1} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \cdots & \boldsymbol{\eta}_{\overline{n}-1}(\varphi) & \mathbf{D}_{\overline{n}-1}(\varphi) & \boldsymbol{\delta}_{\overline{n},c} \\ \mathbf{0}_{k_z:1} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \mathbf{0}_{k_z:k_z} & \cdots & \mathbf{0}_{k_z:k_z} & \boldsymbol{\eta}_{\overline{n}}(\varphi) & \mathbf{D}_{\overline{n}}(\varphi) \end{pmatrix}$$

where $t_{11} \equiv -\sum_z \pi_z(z)\eta(\theta_1(z,\varphi))$ is a scalar, $\mathbf{0}_{p:q}$ denotes a $p \times q$ matrix of zeros, and $\mathbf{T}_\varphi$ is a $K \times K$ square matrix, where $K \equiv 1 + \overline{n}k_z$. Further, we have defined the $1 \times k_z$ row vectors:

$$\boldsymbol{\delta}_f^e \equiv \Big(\delta_f, \ldots, \delta_f\Big); \qquad \boldsymbol{\delta}_c^e \equiv \Big(\delta_c, \ldots, \delta_c\Big); \qquad \boldsymbol{\eta}_1^e(\varphi) \equiv \Big(\pi_z(z_1)\eta\big(\theta_1(z_1,\varphi)\big), \ldots, \pi_z(z_{k_z})\eta\big(\theta_1(z_{k_z},\varphi)\big)\Big);$$

and the $k_z \times k_z$ matrices:

$$\forall n = 2,\ldots,\overline{n}: \qquad \boldsymbol{\delta}_{n,c} \equiv \operatorname{diag}\Big(n\delta_c, \ldots, n\delta_c\Big);$$

$$\boldsymbol{\eta}_n(\varphi) \equiv \operatorname{diag}\Big(\eta\big(\theta_n(z_1,\varphi)\big), \ldots, \eta\big(\theta_n(z_{k_z},\varphi)\big)\Big);$$

$$\forall n = 1,\ldots,\overline{n}: \quad \mathbf{D}_n(\varphi) \equiv \begin{pmatrix} d_n(z_1,\varphi) & \lambda_z(z_1|z_2) & \ldots & \lambda_z(z_1|z_{k_z}) \\ \lambda_z(z_2|z_1) & d_n(z_2,\varphi) & \ldots & \lambda_z(z_2|z_{k_z}) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_z(z_{k_z}|z_1) & \lambda_z(z_{k_z}|z_2) & \ldots & d_n(z_{k_z},\varphi) \end{pmatrix}$$

where diag(.) denotes a diagonal matrix, and the diagonal elements of $\mathbf{D}_n(\varphi)$ are given by:

$$d_n(z_j,\varphi) \equiv \begin{cases} -\Big(\delta_f + n\delta_c + \eta\big(\theta_{n+1}(z_j,\varphi)\big) + \sum_{\ell\neq j}\lambda_z(z_\ell|z_j)\Big) & \text{for } n = 1,\ldots,\overline{n}-1 \\ -\Big(\delta_f + \overline{n}\delta_c + \sum_{\ell\neq j}\lambda_z(z_\ell|z_j)\Big) & \text{for } n = \overline{n} \end{cases}$$

System (A.4.1) describes an *irreducible* Markov chain, as any state $(n', z') \in \mathcal{N}\times\mathcal{Z}$ can be reached almost surely from some $(n,z) \neq (n',z')$. Moreover, the Markov chain is *aperiodic*. These properties, plus the fact that the state space is finite, guarantee that the Markov chain is *ergodic*. Therefore, by Theorem 11.2 of Stokey and Lucas (1989), the system converges to a unique steady-state distribution $\mathbf{S}^*(\varphi)$, for each $\varphi \in \Phi$.

For the analytical characterization, note that the transition matrix $\mathbf{T}_\varphi$ is constant, so we can solve the differential equation (A.4.1) directly. The solution is:

$$\mathbf{S}_t(\varphi) = e^{\mathbf{T}_\varphi t}\mathbf{S}_0(\varphi)$$

where the initial distribution $\mathbf{S}_0(\varphi) \in \mathbb{R}^K_+$ is given. To compute $e^{\mathbf{T}_\varphi t}$, consider the eigenvalue decomposition $\mathbf{T}_\varphi = \mathbf{E}_\varphi\boldsymbol{\Lambda}_\varphi\mathbf{E}_\varphi^{-1}$, where $\boldsymbol{\Lambda}_\varphi \equiv \big(\lambda_1(\varphi),\ldots,\lambda_K(\varphi)\big)$ is the diagonal matrix of eigenvalues, and $\mathbf{E}_\varphi$ collects the corresponding eigenvectors. Defining $\mathbf{Z}_t(\varphi) \equiv \mathbf{E}_\varphi^{-1}\mathbf{S}_t(\varphi)$, then $\boldsymbol{\partial}_t\mathbf{Z}_t(\varphi) = \boldsymbol{\Lambda}_\varphi\mathbf{Z}_t(\varphi)$, and because $\boldsymbol{\Lambda}_\varphi$ is a diagonal matrix, we can solve this differential equation element-by-element, i.e. $\boldsymbol{\partial}_t Z_{i,t}(\varphi) = \lambda_i(\varphi)Z_{i,t}(\varphi)$ for each $i = 1,\ldots,K$. This is a simple system of ODEs with solution:

$$Z_{i,t}(\varphi) = c_i e^{\lambda_i(\varphi)t}, \qquad i = 1,\ldots,K$$

where $c_i \in \mathbb{R}$ is the constant of integration. Since $\mathbf{S}_t(\varphi) = \mathbf{E}_\varphi\mathbf{Z}_t(\varphi)$, we have obtained:

$$\mathbf{S}_t(\varphi) = \sum_{i=1}^{K} c_i e^{\lambda_i(\varphi)t}\boldsymbol{v}_i \tag{A.4.2}$$

where $\boldsymbol{v}_i$ is the $K\times 1$ eigenvector associated to the i-th eigenvalue. Therefore, the stability of system (A.4.2) as $t \to +\infty$ depends on the sign of the eigenvalues of $\mathbf{T}_\varphi$. The trace of $\mathbf{T}_\varphi$ is:

$$tr(\mathbf{T}_\varphi) = \sum_{i=1}^{K}\lambda_i(\varphi) = -\sum_{j=1}^{k_z}\pi_z(z_j)\eta(\theta_1(z_j,\varphi)) + \sum_{n=1}^{\overline{n}}\sum_{j=1}^{k_z} d_n(z_j) < 0$$

The trace being unambiguously negative means that there is at least one negative eigenvalue, if not more. Letting $1 \leq \ell \leq K$ denote the number of negative eigenvalues, and re-ordering the eigenvalues from small to large with no loss of generality, we can then impose $c_j = 0$, $\forall j \in \{\ell+1, \ell+2, \ldots, K\}$, on equation (A.4.2), and let $t \to +\infty$ to find the stable solution. That is:

$$\mathbf{S}^*(\varphi) = \lim_{t\to+\infty}\sum_{j=1}^{\ell} c_j e^{\lambda_j(\varphi)t}\boldsymbol{v}_j \in \mathbb{R}^K_+$$

is the unique invariant distribution of sellers in state $\varphi \in \Phi$. $\square$

B Price Discrimination

The assumption of no price discrimination across different customers is not key to generate firm dynamics. In this section, we show that, so long as we maintain the assumption of dynamic contracts with commitment, our model still generates these dynamics as well as cross-sectional dispersion.

If sellers were to use prices as their only instrument for customer attraction (instead of recursive contracts with price-utility pairs), an equilibrium with price discrimination across customers of different tenures would look similar to that of Gourio and Rudanko (2014): firms would attract customers by offering an instantaneous discount on the valuation v, and extract all surplus by charging v immediately after the customer joins the seller, and until separation. However, imposing price discrimination in our model does not lead to this prediction. This is because sellers must still trade off static payoffs coming from the current price with dynamic ones coming from the promised utilities.

Importantly, under price discrimination, tractability is preserved along several important dimensions:

(i.) The seller's and the join surplus problems are equivalent;

(ii.) The joint surplus is constant in the distribution of contracts across customers;

(iii.) As a novelty, there is equilibrium price indeterminacy.

Properties (i.) and (ii.) allow us to solve the model using a similar approach to the one used for the baseline model. Property (iii.) implies that there exist multiple equilibria in prices. These equilibria feature price dispersion both within and across sellers.

Let us discuss these results more formally. For this, we must extend our baseline framework to allow for discrimination across buyers within the seller. Let $\boldsymbol{\omega}_i = \left\{p_i, \mathbf{x}_i'(n'; \mathbf{s}')\right\}$ be the contract offered to the typical customer $i = 1, \ldots, n$, which is composed of an individual-specific price level p_i, and a personalized menu of continuation utilities $\mathbf{x}_i'(n'; \mathbf{s}')$, one for each $n' \in \{n-1, n, n+1\}$ and $\mathbf{s}' \in \{(z', \varphi), (z, \varphi')\}$. A seller is characterized by the collection $\{x_i\}_{i=1}^n$ of outstanding promises, and must choose: (i) a menu of contracts $\{\boldsymbol{\omega}_i\}_{i=1}^n$ for the n current customers; and (ii) a starting promised utility $x_0' \in \mathbb{R}$ for the new $(n+1)$-th incoming customer (if there is any). The HJB equation for the seller now reads:

$$
\begin{aligned}
rV^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big) = \max_{x_0', \{\boldsymbol{\omega}_i\}_{i=1}^n} \Bigg\{ & \sum_{i=1}^n p_i - \mathcal{C}(n; z, \varphi) + \delta_f \Big(V_0^S(\varphi) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \delta_c \sum_{j=1}^n \Big(V^S\Big(n-1, \{x_i'(n-1; z, \varphi)\}_{i=1}^n \backslash_- \{x_j'(n-1; z, \varphi)\}; z, \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \eta\big(\theta(x_0'; \varphi)\big) \Big(V^S\Big(n+1, \{x_i'(n+1; z, \varphi)\}_{i=1}^n \cup_+ \{x_0'\}; z, \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{z' \in \mathcal{Z}} \lambda_z(z'|z) \Big(V^S\Big(n, \{x_i'(n; z', \varphi)\}_{i=1}^n; z', \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi) \Big(V^S\Big(n, \{x_i'(n; z, \varphi')\}_{i=1}^n; z, \varphi'\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \Bigg\}
\end{aligned}
$$

where $\setminus_-$ and $\cup_+$ are multiset difference and union operators.[1] The most important differences relative to the baseline model (equation (3a)) have been highlighted in blue. Now, when a customer $i = 1, \ldots, n$ separates, the vector of promised utilities shrinks in cardinality and the customers that remain matched obtain the new promise $x_i'(n-1; z, \varphi)$. The seller attracts new buyers by offering a starting utility x_0' to the entering customer. The promise-keeping constraint now reads:

$$\forall i = 1, \ldots, n: \quad x_i \leq V^B(n, \boldsymbol{\omega}_i; z, \varphi)$$

for all $(z, \varphi) \in \mathcal{Z} \times \Phi$, establishing that the seller commits to each and every customer. We then solve for the optimal menu of contracts by solving for the joint surplus problem:

Proposition B.1 (Joint Surplus Problem with Price Discrimination) *In the economy with price discrimination, the seller's and the joint surplus problems are* equivalent*, in that:*

(i) Given a menu of contracts $\boldsymbol{\omega}_i = \{p_i, \mathbf{x}_i'(n'; \mathbf{s}')\}$ for $i = 1, \ldots, n$ that maximize the seller's value subject to the promise-keeping constraint, $\{\mathbf{x}_i'(n'; \mathbf{s}')\}_{i=1}^n$ maximizes:

$$W\big(n, \{x_i\}_{i=1}^n; z, \varphi\big) \equiv V^S\big(n, \{x_i\}_{i=1}^n; z, \varphi\big) + \sum_{i=1}^n x_i;$$

(ii) Conversely, for every $\{\mathbf{x}_i'(n'; \mathbf{s}')\}_{i=1}^n$ that maximizes $W\big(n, \{x_i\}_{i=1}^n; z, \varphi\big)$, there exists a menu of personalized price levels $\{p_i\}_{i=1}^n$ such that the collection $\{p_i, \mathbf{x}_i'(n'; \mathbf{s}')\}_{i=1}^n$ constitutes a solution to the seller's problem.

Proof. The argument is conceptually similar to that of the baseline model (see Appendix A.1). Let $\{\overline{x}_0', \{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n\}$ be a generic policy for the seller, with $\overline{\boldsymbol{\omega}}_i \equiv \{\overline{p}_i, \overline{\mathbf{x}}_i'(n'; \mathbf{s}')\}$ and $\overline{\mathbf{x}}_i'(n'; \mathbf{s}') = \{\overline{x}_i'(n+1; z, \varphi), \overline{x}_i'(n-1; z, \varphi), \{\overline{x}_i'(n; z', \varphi) : z' \in \mathcal{Z}\}, \{\overline{x}_i'(n; z, \varphi') : \varphi' \in \Phi\}\}$, for $i = 1, \ldots, n$. The seller's problem can be written as:

$$V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big) \equiv \max_{\overline{x}_0', \{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n} \widetilde{V}^S\Big(n; \overline{x}_0', \{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n; z, \varphi\Big) \ \text{ s.t. } x_i \leq V^B(n, \overline{\boldsymbol{\omega}}_i; z, \varphi), \forall i = 1, \ldots, n$$

where:

$$\begin{aligned}\widetilde{V}^S\Big(n; \overline{x}_0', \{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n; z, \varphi\Big) \equiv \frac{1}{\rho(n; z, \varphi)} \Bigg[& \sum_{i=1}^n \overline{p}_i - \mathcal{C}(n; z, \varphi) \\ & + \delta_c \sum_{j=1}^n V^S\Big(n-1, \{\overline{x}_i'(n-1; z, \varphi)\}_{i=1}^n \setminus_- \{\overline{x}_j'(n-1; z, \varphi)\}; z, \varphi\Big) \\ & + \eta\big(\theta(\overline{x}_0'; \varphi)\big) V^S\Big(n+1, \{\overline{x}_i'(n+1; z, \varphi)\}_{i=1}^n \cup_+ \{\overline{x}_0'\}; z, \varphi\Big)\end{aligned} \tag{B.1}$$

[1] These operators are defined by $\{a, b, b\} \setminus_- \{b\} = \{a, b\}$ and $\{a, b\} \cup_+ \{b\} = \{a, b, b\}$, and they are needed here because different customers may be promised the same valuation.

$$+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)V^S\Big(n,\{\overline{x}_i'(n;z',\varphi)\}_{i=1}^n;z',\varphi\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)V^S\Big(n,\{\overline{x}_i'(n;z,\varphi')\}_{i=1}^n;z,\varphi'\Big)\Bigg]$$

is the value of the seller, with $\rho(n;z,\varphi)\equiv r+\delta_f+n\delta_c+\eta\big(\theta(\overline{x}_0';\varphi)\big)$ being the effective discount rate. The value of buyer $i=1,\ldots,n$ under this policy is:

$$\begin{aligned}rV^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)&=v(\varphi)-p_i+(\delta_f+\delta_c)\Big(U^B(\varphi)-V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)\Big)\\&+(n-1)\delta_c\Big(\overline{x}_i'(n-1;z,\varphi)-V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)\Big)+\eta\big(\theta(\overline{x}_0';\varphi)\big)\Big(\overline{x}_i'(n+1;z,\varphi)-V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)\Big)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\Big(\overline{x}_i'(n;z',\varphi)-V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\Big(\overline{x}_i'(n;z,\varphi')-V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi)\Big)\end{aligned}$$

Notice that the seller is re-optimizing after each size change. By monotonicity of preferences, the promise-keeping constraint will bind for every customer:

$$x_i=V^B(n,\overline{\boldsymbol{\omega}}_i;z,\varphi),\qquad\forall i=1,\ldots,n$$

From this equation, we can solve for the promise-compatible price level to be charged to each customer under the policy $\Big\{\overline{x}_0',\{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n\Big\}$:

$$\begin{aligned}p_i^{PK}\Big(\big\{\overline{x}_0',\{\mathbf{x}_j'(n';\mathbf{s}')\}_{j=1}^n\big\}\Big)&=v(\varphi)-\rho(n;z,\varphi)x_i+\delta_fU^B(\varphi)\\&+\delta_c\Big(U^B(\varphi)+(n-1)\overline{x}_i'(n-1;z,\varphi)\Big)+\eta\big(\theta(\overline{x}_0');\varphi\big)\overline{x}_i'(n+1;z,\varphi)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\overline{x}_i'(n;z',\varphi)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\overline{x}_i'(n;z,\varphi')\end{aligned}\tag{B.2}$$

Importantly, note that the price level for a specific customer i is independent of the distribution of utilities for all the *other* customers, that is:

$$p_i^{PK}\Big(\big\{\overline{x}_0',\{\mathbf{x}_j'(n';\mathbf{s}')\}_{j\neq i}\big\}\cup_+\{\mathbf{x}_i'(n';\mathbf{s}')\}\Big)=p_i^{PK}\Big(\big\{\overline{x}_0',\{\mathbf{x}_{\phi(j)}'(n';\mathbf{s}')\}_{\phi(j)\neq i}\big\}\cup_+\{\mathbf{x}_i'(n';\mathbf{s}')\}\Big)$$

for an arbitrary bisective function $\phi:\{1,\ldots,n\}\to\{1,\ldots,n\}$. Therefore, since the seller's problem internalizes the price level, the resulting maximization will be independent of the initial distribution of utilities. Indeed, plugging (B.2) into (B.1) and performing some straightforward algebra we obtain:

$$\begin{aligned}\widetilde{W}\Big(n,\{x_i\}_{i=1}^n;\overline{x}_0',\{\overline{\boldsymbol{\omega}}_i\}_{i=1}^n;z,\varphi\Big)&\equiv\frac{1}{\rho(n;z,\varphi)}\Bigg[n\Big(v(\varphi)+(\delta_f+\delta_c)U^B(\varphi)\Big)-\Big(\mathcal{C}(n;z,\varphi)+\eta\big(\theta(\overline{x}_0';\varphi)\big)\overline{x}_0'\Big)\\&+\delta_c\sum_{j=1}^nW\Big(n-1,\{\overline{x}_i'(n-1;z,\varphi)\}_{i=1}^n\setminus_-\{\overline{x}_j'(n-1;z,\varphi)\};z,\varphi\Big)\end{aligned}$$

$$+\,\eta\big(\theta(\overline{x}_0';\varphi)\big)W\Big(n+1,\big\{\overline{x}_i'(n+1)\big\}_{i=1}^{n}\cup_{+}\big\{\overline{x}_0'\big\};z,\varphi\Big)$$
$$+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)W\Big(n,\big\{\overline{x}_i'(n;z',\varphi)\big\}_{i=1}^{n};z',\varphi\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)W\Big(n,\big\{\overline{x}_i'(n;z,\varphi')\big\}_{i=1}^{n};z,\varphi'\Big)\Bigg] \qquad \text{(B.3)}$$

where we have defined:

$$\widetilde{W}\Big(n,\big\{x_i\big\}_{i=1}^{n};\overline{x}_0',\big\{\overline{\boldsymbol{\omega}}_i\big\}_{i=1}^{n};z,\varphi\Big)\equiv\widetilde{V}^S\Big(n;\overline{x}_0',\big\{\overline{\boldsymbol{\omega}}_i\big\}_{i=1}^{n};z,\varphi\Big)+\sum_{i=1}^{n}x_i$$

as the joint surplus under policy $\big\{\overline{x}_0',\{\overline{\boldsymbol{\omega}}_i\}_{i=1}^{n}\big\}$, and:

$$W\Big(n,\big\{x_i\big\}_{i=1}^{n};z,\varphi\Big)\equiv\max_{\overline{x}_0',\{\overline{\boldsymbol{\omega}}_i\}_{i=1}^{n}}\widetilde{W}\Big(n,\big\{x_i\big\}_{i=1}^{n};\overline{x}_0',\big\{\overline{\boldsymbol{\omega}}_i\big\}_{i=1}^{n};z,\varphi\Big)$$

as the *maximized* joint surplus. Importantly, it is still the case that the right-hand side of (B.3) does not depend on the initial distribution of utilities $\{x_i\}_{i=1}^{n}$, nor the distribution of prices $\{p_i\}_{i=1}^{n}$, so we can write the joint surplus under a given policy as:

$$\widetilde{W}\Big(n,\big\{x_i\big\}_{i=1}^{n};\overline{x}_0',\big\{\overline{\boldsymbol{\omega}}_i\big\}_{i=1}^{n};z,\varphi\Big)=\widetilde{W}_n\Big(\overline{x}_0',\big\{\overline{\mathbf{x}}_i'(n';\mathbf{s}')\big\}_{i=1}^{n};z,\varphi\Big)$$

This allows us to break up the optimal contracting problem into two separate stages. Where we denote an optimal contract by $\big\{x_0'^{*},\{p_i^{*},\mathbf{x}_i'^{*}(n';\mathbf{s}')\}_{i=1}^{n}\big\}$, we have:

$$\begin{aligned}\big\{x_0'^{*},\{\mathbf{x}_i'^{*}(n';\mathbf{s}')\}_{i=1}^{n}\big\}&=\arg\max\ \widetilde{W}_n\Big(\overline{x}_0',\big\{\overline{\mathbf{x}}_i'(n';\mathbf{s}')\big\}_{i=1}^{n};z,\varphi\Big)\\ p_i^{*}&=\ p_i^{PK}\Big(\Big\{\overline{x}_0'^{*},\{\mathbf{x}_j'^{*}(n';\mathbf{s}')\}_{j=1}^{n}\Big\}\Big),\qquad\forall i=1,\dots,n\end{aligned}$$

Thus, the joint surplus and the seller's problems are equivalent. □

The characterization of the equilibrium is also similar to the baseline model. First, by utility-invariance of the joint surplus we can write $W_n(z,\varphi)=W\Big(n,\big\{x_i\big\}_{i=1}^{n};z,\varphi\Big)$, $\forall(n,z,\varphi)\in\mathbb{N}\times\mathcal{Z}\times\Phi$. Under the optimal policy, the joint surplus problem can then be written as follows:

$$\begin{aligned}(r+\delta_f)W_n(z,\varphi)=\max_{x_0',\{x_j'(n-1;z,\varphi)\}_{j=1}^{n}}&\Bigg\{n\Big(v(\varphi)+(\delta_f+\delta_c)U^B(\varphi)\Big)-\Big(\mathcal{C}(n;z,\varphi)+\eta\big(\theta(x_0';\varphi)\big)x_0'\Big)\\&+n\delta_c\Big(W_{n-1}(z,\varphi)-W_n(z,\varphi)\Big)+\eta\big(\theta(x_0';\varphi)\big)\Big(W_{n+1}(z,\varphi)-W_n(z,\varphi)\Big)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\Big(W_n(z',\varphi)-W_n(z,\varphi)\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\Big(W_n(z,\varphi')-W_n(z,\varphi)\Big)\Bigg\}\end{aligned}$$

The optimality condition for x_0' is:

$$\frac{\partial\eta\big(\theta(x_0';\varphi)\big)}{\partial x_0'}\Big(W_{n+1}(z,\varphi)-W_n(z,\varphi)\Big)=\frac{\partial\eta\big(\theta(x_0';\varphi)\big)}{\partial x_0'}x_0'+\eta\big(\theta(x_0';\varphi)\big)$$

similar to equation (6). By the usual arguments, the solution to this equation is *unique*. Denoting the solution by $x_{0,n}'(z,\varphi)$, the law of motion for the seller of size n_t at time t is then:

$$n_{t+\Delta}-n_t=\begin{cases}1 & \text{w/prob.} \quad \eta\Big(\theta\big(x_{0,n_t}'(z,\varphi)\big)\Big)\Delta+o(\Delta)\\ -1 & \text{w/prob.} \quad n_t\delta_c\Delta+o(\Delta)\\ -n_t & \text{w/prob.} \quad \delta_f\Delta+o(\Delta)\\ 0 & \text{else}\end{cases}$$

Therefore, we have found:

Corollary B.1 (Firm dynamics with price discrimination) *In the model with price discrimination, the equilibrium features unique predictions for firm dynamics.*

Although seller growth is pinned down uniquely, prices are not. Since any distribution of utilities (and thus prices) among incumbents is compatible with optimality, there is now a multiplicity of contracts that can be sustained in the optimal allocation.[2] This is stated formally in the following proposition:

Proposition B.2 (Price Indeterminacy) *There is a continuum of contracts* $\big\{p_i^*,\mathbf{x}_i'^*(n';\mathbf{s}')\big\}_{i=1}^n$ *that: (i) maximize the joint surplus, and (ii) leave both the buyers and the seller indifferent.*

Proof. Pick $\varepsilon\in\mathbb{R}$ arbitrarily. The goal of the proof is to show that there is some $\beta_n(\varphi)>0$ (possibly a function of size and the aggregate state) for which, if a given contract with $\boldsymbol{\omega}^b=\big\{p_i+\varepsilon\beta_n(\varphi),\mathbf{x}_i'(n';\mathbf{s}')+\varepsilon\big\}_{i=1}^n$ is optimal, then each customer and the seller maximize their value under contract $\boldsymbol{\omega}^a=\big\{p_i,\mathbf{x}_i'(n';\mathbf{s}')\big\}_{i=1}^n$. The value of contract $\boldsymbol{\omega}_i^b$ for customer $i=1,\dots,n$ is:

$$\begin{aligned}rV^B(n,\boldsymbol{\omega}_i^b;z,\varphi)=&\ v(\varphi)-p_i-\varepsilon\beta_n(\varphi)+(\delta_f+\delta_c)\Big(U^B(\varphi)-V^B(n,\boldsymbol{\omega}_i;z,\varphi)\Big)\\&+(n-1)\delta_c\Big(x_i'(n-1;z,\varphi)+\varepsilon-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)+\eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big(x_i'(n+1;z,\varphi)+\varepsilon-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\Big(x_i'(n;z',\varphi)+\varepsilon-V^B(n,\boldsymbol{\omega}_i;z,\varphi)\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\Big(x_i'(n;z,\varphi')+\varepsilon-V^B(n,\boldsymbol{\omega}_i;z,\varphi)\Big)\\=&\ v(\varphi)-p_i-\varepsilon\Big(\beta_n(\varphi)-(n-1)\delta_c-\eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big)+(\delta_f+\delta_c)\Big(U^B(\varphi)-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)\\&+(n-1)\delta_c\Big(x_i'(n-1;z,\varphi)-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)+\eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big(x_i'(n+1;z,\varphi)-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\Big(x_i'(n;z',\varphi)-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\Big(x_i'(n;z,\varphi')-V^B(n,\boldsymbol{\omega}_i^b;z,\varphi)\Big)\\=&\ rV^B(n,\boldsymbol{\omega}_i^a;z,\varphi)-\varepsilon\Big(\beta_n(\varphi)-(n-1)\delta_c-\eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big)\end{aligned}$$

[2] Additional equilibria may exist outside of the Markov-perfect equilibrium class. Here we only point out that equilibrium uniqueness is lost in Markov Perfect equilibria when sellers can price-discriminate.

where we have used $\sum_{z'} \lambda_z(z'|z) = \sum_{\varphi'} \lambda_\varphi(\varphi'|\varphi) = 0$. Thus, $V^B(n, \boldsymbol{\omega}_i^a) = V^B(n, \boldsymbol{\omega}_i^b)$ if, and only if:

$$\beta_n(\varphi) = (n-1)\delta_c + \eta\big(\theta(x_0'(\varphi);\varphi)\big) \tag{B.4}$$

As for the seller's value, note that:

$$\begin{aligned}
rV^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big) = \max_{x_0'(\varphi), \{\boldsymbol{\omega}_i\}_{i=1}^n} \Bigg\{ & \sum_{i=1}^n p_i + n\varepsilon\beta_n(\varphi) - \mathcal{C}(n; z, \varphi) + \delta_f\Big(V_0^S(\varphi) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \delta_c \sum_{j=1}^n \Big(V^S\Big(n-1, \{x_i'(n-1; z, \varphi) + \varepsilon\}_{i=1}^n \backslash_- \{x_j'(n-1; z, \varphi) + \varepsilon\}; z, \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big(V^S\Big(n+1, \{x_i'(n+1; z, \varphi) + \varepsilon\}_{i=1}^n \cup_+ \{x_0'(\varphi)\}; z, \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{z' \in \mathcal{Z}} \lambda_z(z'|z)\Big(V^S\Big(n, \{x_i'(n; z', \varphi) + \varepsilon\}_{i=1}^n; z', \varphi\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi)\Big(V^S\Big(n, \{x_i'(n; z, \varphi') + \varepsilon\}_{i=1}^n; z, \varphi'\Big) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big)\Bigg\} \\
= \max_{x_0'(\varphi), \{\boldsymbol{\omega}_i\}_{i=1}^n} \Bigg\{ & \sum_{i=1}^n p_i + n\varepsilon\beta_n(\varphi) - \mathcal{C}(n; z, \varphi) + \delta_f\Big(V_0^S(\varphi) - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \delta_c \sum_{j=1}^n \Big(W_{n-1}(z, \varphi) - \sum_{i \neq j} x_i'(n-1; z, \varphi) - (n-1)\varepsilon - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big(W_{n+1}(z, \varphi) - \sum_{i=1}^n x_i'(n+1; z, \varphi) - x_0' - n\varepsilon - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{z' \in \mathcal{Z}} \lambda_z(z'|z)\Big(W_n(z', \varphi) - \sum_{i=1}^n x_i'(n; z', \varphi) - n\varepsilon - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big) \\
& + \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi)\Big(W_n(z, \varphi') - \sum_{i=1}^n x_i'(n; z, \varphi') - n\varepsilon - V^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big)\Big)\Bigg\} \\
= rV^S\Big(n, \{x_i\}_{i=1}^n; z, \varphi\Big) & + n\varepsilon\Big(\beta_n(\varphi) - (n-1)\delta_c - \eta\big(\theta(x_0'(\varphi);\varphi)\big)\Big)
\end{aligned}$$

where we have used the definition of W in the second equality. Thus, by equation (B.4), the seller is also indifferent. In sum, contract $\{\boldsymbol{\omega}_i^a\}_{i=1}^n$ is optimal if, and only if, $\{\boldsymbol{\omega}_i^b\}_{i=1}^n$ is optimal. Generally, there is a continuum of optimal contracts, indexed by ε. $\square$

Note that the Gourio and Rudanko (2014) pricing strategy can be seen as one of the multiple equilibria of the model, in which:

$$x_i'(n; z, \varphi) = U^B(\varphi), \quad \forall i = 1, \ldots, n, \ \forall (n, z, \varphi) \in \mathbb{N} \times \mathcal{Z} \times \Phi$$

which, by equation (B.2), implies that $p_i = v$, $\forall i = 1, \ldots, n$. In this case, x_0' plays the role of a price discount relative to valuation for the incoming customer, who will be charged a fixed price of v thereafter.

C Numerical Appendix

C.1 Numerical Implementation of the Exogenous Processes

This appendix shows how to parametrize and estimate (z, φ) as continuous-time Markov chain (CTMC) processes. The same structure applies to both shocks, so let us consider just the idiosyncratic shock (z).

The $k_z \times k_z$ infinitesimal generator matrix $\mathbf{\Lambda}_z$ to be estimated is:

$$\mathbf{\Lambda}_z = \begin{pmatrix} -\sum_{j\neq 1}\lambda_{1j} & \lambda_{12} & \dots & \lambda_{1k_z} \\ \lambda_{21} & -\sum_{j\neq 2}\lambda_{2j} & \dots & \lambda_{2k_z} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k_z 1} & \lambda_{k_z 2} & \dots & -\sum_{j\neq k_z}\lambda_{k_z j} \end{pmatrix}$$

where $\lambda_{ij} > 0$ is short-hand for $\lambda_z(z_j|z_i)$, $z_i, z_j \in \mathcal{Z}$. This level of generality would require the estimation of $k_z(k_z-1)$ transition rates, which is potentially a very large number. Therefore, we reduce the parameter space by specializing the CTMC as follows:

- First, we assume z follows a driftless Ornstein-Uhlenbeck process in logs. This process is a type of mean-reverting and autoregressive CTMC which can be loosely viewed as the continuous-time analogue of an AR(1). Formally:

$$\mathrm{d}\log(z_t) = -\rho_z \log(z_t)\mathrm{d}t + \sigma_z \mathrm{d}B_t$$

  where B_t is a standard Brownian motion, and $\rho_z, \sigma_z > 0$ are parameters.

- Operationally, in the numerical version of the model in which time is partitioned and takes values in a grid $\mathbb{T} = \{\Delta, 2\Delta, 3\Delta, \dots\}$, we use the *Euler-Maruyama method*, that is:

$$\log(z_k) = (1-\rho_z\Delta)\log(z_{k-1}) + \sigma_z\sqrt{\Delta}\varepsilon_k^z, \qquad \varepsilon_k^z \sim iid\ \mathcal{N}(0,1) \tag{C.1}$$

  for each $k \in \mathbb{T}$. This is an AR(1) processes with autocorrelation $\widetilde{\rho}_z \equiv 1-\rho_z\Delta$ and variance $\dfrac{\sigma_z^2}{\rho_z(1+\widetilde{\rho}_z)}$. Thus, $\rho_z > 0$ controls for the degree of mean-reversion, with lower values corresponding to higher persistence.

- The discrete-time process (C.1) is, in turn, estimated using the Tauchen (1986) method, with a discrete-state Markov chain that we define on the theoretical grid, $\mathcal{Z}$. The outcome of this method are estimates for (ρ_z, σ_z), and a transition probability matrix $\mathbf{\Pi}_z = (\pi_{ij})$, where π_{ij} denotes the probability of a z_i-to-z_j transition in the $\mathbb{T}$ space.

- For the mapping back into continuous time, we use the fact that, for small enough $\Delta > 0$, transition *probabilities* (the π's) are well approximated by transition *rates* (the λ's) in the following sense:

$$\forall i = 1, \ldots, k_z: \qquad \pi_{ij} \approx \Delta\lambda_{ij}, \forall j \neq i \quad \text{and} \quad \pi_{ii} \approx 1 - \Delta \sum_{j \neq i} \lambda_{ij}$$

C.2 Stationary Solution Algorithm

To solve for the stationary equilibrium, we implement the following procedure:

- First, we solve the maximization of the joint surplus function using a value function iteration (VFI) algorithm, under a guess for U^B.
- To update U^B, we check that the free entry condition is satisfied. Combining equations (1) and (8), we can write the free entry condition as:

$$\kappa = \sum_{z_0 \in \mathcal{Z}} \pi_z(z_0) \left\{ \eta \circ \mu^{-1} \left(\frac{\Gamma^B(\varphi)}{x_1'(z_0, \varphi) - U^B(\varphi)} \right) \Big(W_1(z_0, \varphi) - x_1'(z_0, \varphi) \Big) \right\}$$

  To find U^B, we use a bisection method: increase (or decrease) U^B if there are too many (or too few) entering sellers.

Throughout, the state space grid is fixed at $\mathcal{N} \times \mathcal{Z} \times \Phi$, where $\mathcal{N} = \{1, \ldots, \bar{n}\}$, $\mathcal{Z} = \{z_i\}_{i=1}^{k_z}$, and $\Phi = \{\varphi_j\}_{j=1}^{k_\varphi}$. The following describes the steps of the algorithm:

Step 1. Set the counter to $k = 0$. Choose guesses $\underline{U}^{(0)}(\varphi)$ and $\overline{U}^{(0)}(\varphi) \gg \underline{U}^{(0)}(\varphi)$ for each $\varphi \in \Phi$. Set the value of inactivity to:

$$U^{B(0)}(\varphi) = \frac{1}{2} \Big(\underline{U}^{(0)}(\varphi) + \overline{U}^{(0)}(\varphi) \Big)$$

Step 2. For any given $k \in \mathbb{N}$ and $n \in \mathcal{N}$, use VFI to find the fixed point $W_n^{(k)}(z, \varphi)$ of:

$$\begin{aligned}
(r + \delta_f) W_n^{(k)}(z, \varphi) = \; & n \Big(v(\varphi) + (\delta_f + \delta_c) U^{B(k)}(\varphi) \Big) - \mathcal{C}(n; z, \varphi) + n\delta_c \Big(W_{n-1}^{(k)}(z, \varphi) - W_n^{(k)}(z, \varphi) \Big) \\
& + \max_{x_{n+1}'} \left\{ \eta \circ \mu^{-1} \left(\frac{\Gamma^{B(k)}(\varphi)}{x_{n+1}' - U^{B(k)}(\varphi)} \right) \Big(W_{n+1}^{(k)}(z, \varphi) - W_n^{(k)}(z, \varphi) - x_{n+1}' \Big) \right\} \\
& + \sum_{z' \in \mathcal{Z}} \lambda_z(z'|z) \Big(W_n^{(k)}(z', \varphi) - W_n^{(k)}(z, \varphi) \Big) + \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi) \Big(W_n^{(k)}(z, \varphi') - W_n^{(k)}(z, \varphi) \Big)
\end{aligned}$$

where $\Gamma^{B(k)} = c + rU^{B(k)}(\varphi) - \sum_{\varphi' \in \Phi} \lambda_\varphi(\varphi'|\varphi) \Big(U^{B(k)}(\varphi') - U^{B(k)}(\varphi) \Big)$. Store the corresponding policy functions as $\Big\{ x_{n+1}'^{(k)}(z, \varphi) : (n, z, \varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi \Big\}$.

Step 3. For each $\varphi \in \Phi$, compute the object:

$$\Delta^{(k)}(\varphi) \equiv \kappa - \sum_{z_0 \in \mathcal{Z}} \pi_z(z_0) \left\{ \eta \circ \mu^{-1} \left(\frac{\Gamma^{B(k)}(\varphi)}{x_1'^{(k)}(z_0, \varphi) - U^{B(k)}(\varphi)} \right) \Big(W_1^{(k)}(z_0, \varphi) - x_1'^{(k)}(z_0, \varphi) \Big) \right\}$$

Stop if $\Delta^{(k)}(\varphi) \in [-\varepsilon, \varepsilon]$, $\forall \varphi \in \Phi$, for some small $\varepsilon > 0$. Otherwise, set:

$$U^{B(k+1)}(\varphi) = \frac{1}{2}\left(\underline{U}^{(k+1)}(\varphi) + \overline{U}^{(k+1)}(\varphi)\right)$$

for each $\varphi \in \Phi$, where:

(a) If $\Delta^{(k)}(\varphi) > \varepsilon$, then $\underline{U}^{(k+1)}(\varphi) = \underline{U}^{(k)}(\varphi)$ and $\overline{U}^{(k+1)}(\varphi) = U^{B(k)}(\varphi)$;

(b) If $\Delta^{(k)}(\varphi) < -\varepsilon$, then $\underline{U}^{(k+1)}(\varphi) = U^{B(k)}(\varphi)$ and $\overline{U}^{(k+1)}(\varphi) = \overline{U}^{(k)}(\varphi)$;

and go back to Step 2. with $[k] \leftarrow [k+1]$.

The VFI algorithm of Step 2 is guaranteed to converge because, given a U^B, the joint surplus is a contraction and therefore has a unique fixed point. (For a proof of this statement, see Appendix F.2).

D Additional Quantitative Results

D.1 Identification

To pin down the 7 parameters of the model that we calibrate internally, $\boldsymbol{p} = (\kappa, \delta_c, \psi, w, c, \rho_z, \sigma_z)$, first we use a Sobol sequence to uniformly draw parameter combinations in a seven-dimensional hypercube, and then use a minimum-distance criterion function to minimize the discrepancy between model-implied moments and their data counterpart.[3] This method is useful because it can help inform identification in models in which, like in our case, all moments are determined jointly by all parameters. Specifically, we use the identification strategy laid out in Daruich (2019), which proceeds as follows:

- First, for a given hypercube of the parameter space, we draw over 1,000,000 candidate parameter vectors generated from a Sobol sequence.
- Second, for each parameter in $\boldsymbol{p}$, we associate a target moment.
- Third, for each case, we partition the parameter vector into quantiles, and within each quantile we compute the median and inter-quantile range of the associated moment from the underlying distribution implied by random variation in the 6 remaining parameters.

Using this method, a moment is well-identified by the chosen associated parameter if the distribution moves (i.e. if median and 25th and 75th percentiles shift) across different quantiles. The slope of each curve is a measure of how well-identified the parameter is by each moment, with steeper curves indicating a better identification. Moreover, the smaller the inter-quartile range for each quantile, the relatively less important the remaining 6 parameters are for the corresponding moment. This method is powerful because it does not fix all the remaining parameters to their estimated values. In this sense, it is superior to identification techniques based on local derivatives, as in those methods the other parameters do not affect the chosen moment in any way, by construction.

[3] By drawing parameter vectors using a Sobol sequence, we successively form finer uniform partitions of the parameter space, ensuring that we explore all the potentially relevant regions.

Figure D.1 shows the identification results for our estimated set of parameters seen in Table 1. The central array of black dots denotes the median of the distribution in that quantile, while the arrays of gray dots are the 25th and 75th percentiles. The moments that are best identified are the entry rate, which is informed by our choice of δ_c, and the average size, which is related to the curvature in technology ψ. Indeed, these two moments change considerably with these two parameters over different quantiles, and the distribution is narrowly distributed around the median, indicating that other parameters do not play a major role in pinning down the specific moment.

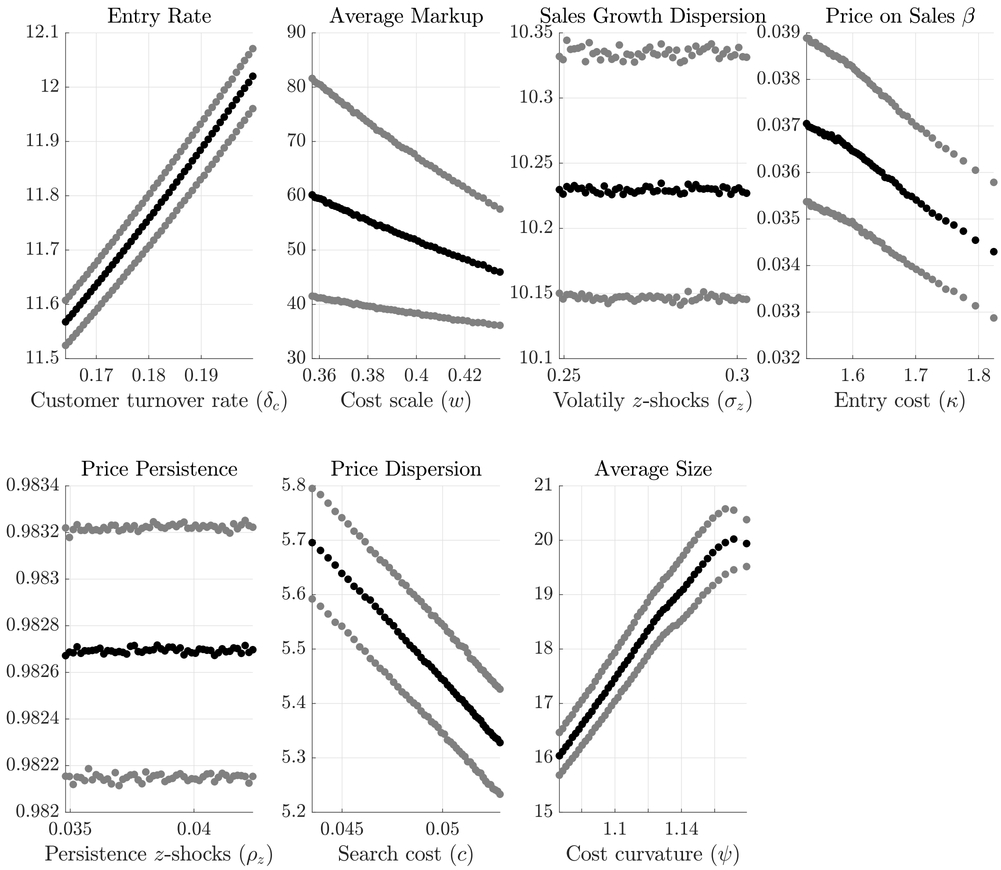

Figure D.1: Identification of the internally estimated parameters. *Notes:* Each figure plots the median (black dots) and inter-quartile range (gray dots) for 50 different quartiles of the distribution of each moment across different parameter values, keeping the chosen associated parameter fixed at each given value. A better identification corresponds to (i) stronger correlation between parameter and median; (ii) narrower inter-quartile ranges.

The parameters (w, κ, c) determine three moments jointly: the average markup, the price on sales regression coefficient, and the amount of price dispersion. In particular, a higher cost scale, entry cost, and search cost, imply a lower markup, a lower coefficient of price on sales, and a lower degree of dispersion in prices, respectively. Finally, the parameters related to the persistence and volatility in the idiosyncratic productivity process z do not uniquely identify sales growth dispersion and price persistence, as the other parameters play a similarly important role in pinning down these moments. This is likely due to the fact that most dispersion and persistence in the estimated model comes from the size mechanism, and not from productivity differences. We therefore see this as not a major problem of the estimation, as the main objective for the estimated model is to explain lifecycle dynamics through demand-driven differences and not through productivity differences, motivated by the empirical literature cited in the Introduction.

D.2 Shocks to Marginal Costs

In this Appendix, we show that the model delivers procyclicality in markups in response to shocks to the marginal cost of sellers, i.e. the exogenous cost scale parameter w.

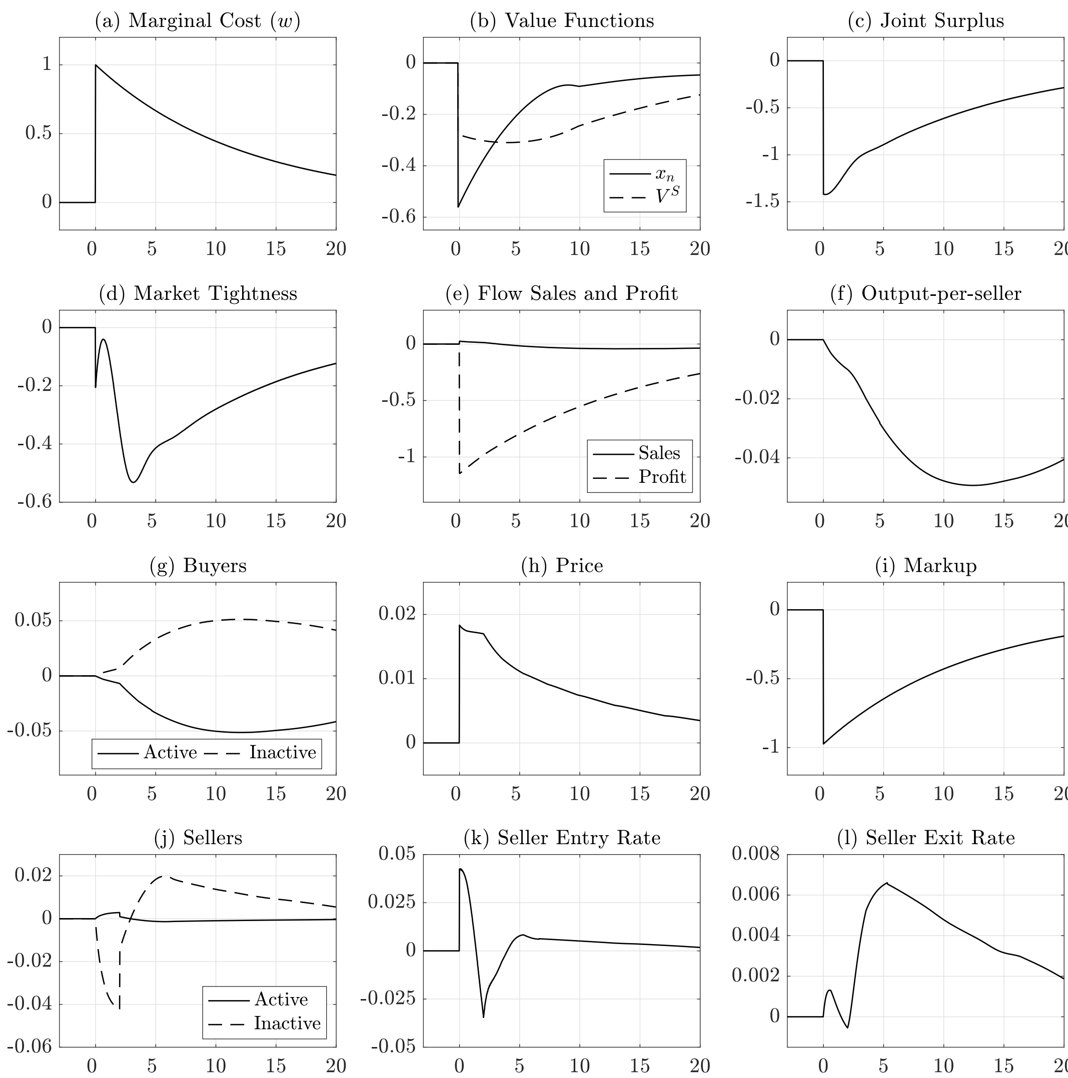

Figure D.2: Impulse responses of selected variables to a one-time –1% shock to marginal costs (w). *Notes:* Series presented in %-deviations from the steady state where $w(\varphi)$ equals its calibrated value (from Table 1). The shock hits at date $t_0 = 0$ and the time period is a year. All response functions have been computed using the theoretical distribution of sellers over states, and later interpolated using cubic splines.

Figure D.2 shows the response of various aggregate variables to this shock. The intuitions for the effects on the joint surplus, the value functions, market tightness, and seller dynamics (entry and exit) are similar to the ones developed in the main text for the demand shock. The key difference between the two shocks is the behavior of prices. After a shock to marginal costs, prices experience an increase on impact. Note that the magnitude of the increase is only about 2% of the size of the shock. The reason why sellers exhibit this incomplete pass-through is that, in response to the shock, they inter-temporally transfer the cost of the shock between current and future buyers by slightly increasing prices today and lowering continuation promises going forward. Since the price level reacts less than one for one to the increase in marginal costs, the markup is procyclical.

E Model Extensions

E.1 Endogenous Customer Separations

To introduce customer seller-to-seller transitions, we can model customer search explicitly. While we assume that there is still an exogenous risk $\delta_c > 0$ of separation for each customer, additionally we now add the possibility that customers search, and potentially endogenously separate, while on the match. We assume that active buyers do not face a cost of search, as they do not discontinue their consumption when transitioning from one seller to the another.

Introducing this additional dimension into our full model with aggregate shocks is not straightforward. Endogenous buyer transitions across sellers would break the ex-ante indifference condition among inactive buyers, which pins down the equilibrium market tightness in the baseline model. In order to preserve the block-recursive structure, one remedy would be to assume free entry across all markets on the seller side. This would change the environment substantially, so we leave it for future work.

Thus, suppose there are no aggregate shocks. The problem of an active buyer with value V^B is:

$$\max_{x\in[V^B,\overline{x}]} \ \mu(\theta(x))\big(x - V^B\big)$$

The matched buyer only considers offers that deliver an expected value that weakly dominates the current perceived utility, V^B. Let $\widehat{x}(n,\boldsymbol{\omega};z)$ be the corresponding policy. The first-order condition reads:

$$\Big(\widehat{x}(n,\boldsymbol{\omega};z) - V^B(n,\boldsymbol{\omega};z)\Big)\frac{\partial\mu(\theta(x))}{\partial x}\bigg|_{x=\widehat{x}(n,\boldsymbol{\omega};z)} = -\mu\Big(\theta\big(\widehat{x}(n,\boldsymbol{\omega};z)\big)\Big) \tag{E.1.1}$$

Intuitively, the inactive buyer trades off the expected option value of transitioning (left-hand side) to the rate at which this offer can be obtained (right-hand side). It is not difficult to show (e.g. Shi (2009)) that $\widehat{x}(n,\boldsymbol{\omega};z)$ is increasing in $V^B(n,\boldsymbol{\omega};z)$. In words, the more profitable a match is ex-post, the higher the offer for which the customer will apply next. Therefore, customers separate according to their initial state, and climb up the utility ladder. This effect tends to shift the mass of customers (and therefore sellers) toward higher promised utilities, and thus acts as a countervailing force to the equilibrium dynamics of the baseline model: when the sellers offering the worst terms of trade lose customers, they need to start setting up more favorable contracts.

The risk of endogenously losing customers must now be incorporated into the pricing decisions of sellers. The buyers' and seller's HJB equations are then identical to (2) and (3a), respectively, except that we now must replace δ_c by an "effective" customer separation rate, given by:

$$\widehat{\delta}_c(n, \boldsymbol{\omega}; z) \equiv \delta_c + \mu\Big(\theta\big(\widehat{x}(n, \boldsymbol{\omega}; z)\big)\Big)$$

Likewise, the market tightness must incorporate that the pool of searching buyers is composed of both inactive as well as active buyers:

$$\theta_n(z) = \frac{1}{S_{n-1}(z)} \left(B_n^I(z) + B_{\iota(n)}^A(z)\right)$$

for any $n \geq 1$, where $\iota(n) \in \mathbb{N}$ is the size of the seller that a customer seeking to transition to a size-n seller is currently matched with, i.e. the solution to $x_n(z) = \widehat{x}(\iota(n), \boldsymbol{\omega}; z)$.

E.2 Micro-Foundation for Marginal Utility (v)

Relative to the baseline model, we assume that there exists a representative household, comprised of a measure-one continuum of identical buyers, with quasilinear preferences:

$$U_t(\varphi_t) = \mathbb{E}_t \left\{ \int_t^{+\infty} e^{-r(t-\tau)} \Big(a_\tau + \ln C_\tau(\varphi_\tau)\Big) \mathrm{d}\tau \right\}$$

where $r > 0$ is the discount rate, and a_τ is consumption of a numeraire good at time $\tau \geq t$. Household consumption C is a bundle of output levels of a continuum of sellers via the CES aggregator:

$$C(\varphi) = \int q(\varphi) c(z, \varphi) \mathrm{d}\lambda_f(z) \tag{E.2.1}$$

where $\lambda_f(z)$ is the distribution of active sellers over the space of idiosyncratic states $z \sim \lambda_z(z'|z)$, and $c(z, \varphi)$ is consumption from seller z in aggregate state $\varphi \sim \lambda_\varphi(\varphi'|\varphi)$. Equation (E.2.1) assumes that the goods of the different sellers are perfect substitutes, so we can interpret the continuum of sellers as effectively selling the same product. The shifter $q(\varphi)$ is the aggregate demand shock, acting as a shock to preferences. As the numeraire is separable, the marginal utility of buyers is simply $v(\varphi) \equiv \frac{1}{C(\varphi)}$.

Aggregate consumption C is obtained by the household at a price of P. Cost minimization implies:

$$c(z, \varphi) = \frac{q(\varphi)}{p(z, \varphi)} P(\varphi) C(\varphi), \qquad \text{with } P(\varphi) = \left(\int q(\varphi) \left(\frac{p(z, \varphi)}{q(\varphi)} \right)^{-1} \mathrm{d}\lambda_f(z) \right)^{-1}$$

where $p(z, \varphi)$ is the price that seller z sets for its product in aggregate state φ. Sellers are exactly as in the baseline model, making instantaneous profits $\pi(n; z, \varphi) = p(n; z, \varphi)n - \mathcal{C}(n; z, \varphi)$, where $n \in \mathbb{Z}_+$ is the number of customers and $p(n; z, \varphi)$ is the price policy function. Therefore, the measure of sellers in state z is $\lambda_f(z) = \sum_{n \in \mathbb{N}} S_n(z)$. By feasibility, we have $C(\varphi) = q(\varphi) \sum_{n \in \mathbb{N}} \sum_{z \in \mathcal{Z}} c_n(z, \varphi) S_n(z)$. As sellers sell an indivisible product, $c_n(z, \varphi) = n$, so:

$$C(\varphi) = q(\varphi) \sum_{n \in \mathbb{N}} \sum_{z \in \mathcal{Z}} n S_n(z, \varphi) = q(\varphi) B^A$$

where B^A is the aggregate measure of active buyers. Therefore, the individual valuation of buyers is $v(\varphi) = \frac{1}{q(\varphi)B^A}$. Intuitively, for a given aggregate shock, individual consumers value the product less when there are more active buyers, because that means that the product market is more congested.

F Additional Theoretical Results

F.1 Aggregate Measures of Agents

To derive aggregate measures, we first must derive the equilibrium *shares* of agent types. Throughout, we fix $\varphi \in \Phi$. Let $g_{n,t}(z) \equiv \frac{S_{n,t}(z)}{\mathcal{S}_t}$, where $\mathcal{S}_t \equiv \sum_{n\geq 1}\sum_z S_{n,t}(z)$ is the total measure of incumbents. After a period of size $\Delta > 0$, the share of sellers of size $n \geq 2$ becomes:

$$g_{n,t+\Delta}(z) = \Big[\eta\big(\theta_{n,t+\Delta}(z,\varphi)\big)\Delta + o(\Delta)\Big]g_{n-1,t}(z) + (n+1)\Big[\delta_c\Delta + o(\Delta)\Big]g_{n+1,t}(z) + \sum_{\tilde{z}\neq z}\Big[\lambda_z(z|\tilde{z})\Delta + o(\Delta)\Big]g_{n,t}(\tilde{z})$$
$$+ \Big[1 - \delta_f\Delta - n\delta_c\Delta - \eta\big(\theta_{n+1,t+\Delta}(z,\varphi)\big)\Delta - \sum_{\tilde{z}\neq z}\lambda_z(\tilde{z}|z)\Delta + o(\Delta)\Big]g_{n,t}(z)$$

Subtracting $g_{n,t}(z)$ from both sides of the equation, dividing by Δ, and taking the limit as $\Delta \to 0$:

$$\begin{aligned}\boldsymbol{\partial}_t g_{n,t}(z) &= \eta\big(\theta_{n,t}(z,\varphi)\big)g_{n-1,t}(z) + (n+1)\delta_c g_{n+1,t}(z)\\ &\quad + \sum_{\tilde{z}\neq z}\lambda_z(z|\tilde{z})g_{n,t}(\tilde{z}) - \Big(\delta_f + n\delta_c + \eta\big(\theta_{n+1,t}(z,\varphi)\big) + \sum_{\tilde{z}\neq z}\lambda_z(\tilde{z}|z)\Big)g_{n,t}(z)\end{aligned}$$

The derivation is similar for $n = 1$. For potential entrants, we have:

$$S_{0,t+\Delta}(\varphi) = \Big[\delta_f\Delta + o(\Delta)\Big]\mathcal{S}_t + \Big[\delta_c\Delta + o(\Delta)\Big]\sum_z S_{1,t}(z) + \Big[1 - \sum_{z_0}\pi_z(z_0)\eta\big(\theta_{1,t+\Delta}(z_0,\varphi)\big)\Delta + o(\Delta)\Big]S_{0,t}(\varphi)$$

for given φ. Taking the continuous-time limit in the usual way, we arrive at:

$$\boldsymbol{\partial}_t S_{0,t}(\varphi) = \Big(\delta_f + \delta_c\sum_z g_{1,t}(z)\Big)\mathcal{S}_t - \sum_{z_0}\pi_z(z_0)\eta\big(\theta_{1,t}(z_0,\varphi)\big)S_{0,t}(\varphi)$$

In the stationary solution, $\boldsymbol{\partial}_t g_{n,t}(z) = 0$ and $\boldsymbol{\partial}_t S_{0,t}(\varphi) = 0$. This yields a system of second-order equations which can be solved numerically on the state-space grid, $\mathcal{N} \times \mathcal{Z} \times \Phi$. The solution is a matrix $\{g_n(z)\}_{n,z}$, and the share of potential entrants per incumbent seller, $h_0(\varphi) \equiv S_0(\varphi)/\mathcal{S}$. Then, to compute the aggregate measures $B^A \equiv \sum_{n,z} B^A_n(z)$ and $B^I = 1 - B^A$, we use equation (11) to obtain $b^A_n(z) \equiv \frac{B^A_n(z)}{\mathcal{S}}$ by:

$$b^A_n(z) = ng_n(z)$$

Then, $b^A \equiv B^A/\mathcal{S} = \sum_{n=1}^{+\infty}\sum_z ng_n(z)$. On the other hand, from equation (10) we know that $B^I_n(z,\varphi) = \mathcal{S}\theta_n(z,\varphi)g_{n-1}(z)$. Therefore, adding across $n \geq 2$ yields:

$$\mathcal{S}\sum_{n=2}^{+\infty}\sum_{z}\theta_n(z,\varphi)g_{n-1}(z)=\sum_{n=2}^{+\infty}\sum_{z}B_n^I(z,\varphi)=B^I-\sum_{z}B_1^I(z,\varphi)=1-B^A-\sum_{z}\theta_1(z,\varphi)S_0(\varphi)$$

Using the definitions above, we can then write:

$$\mathcal{S}=\frac{1-\Big(b^A+h_0(\varphi)\sum_z\theta_1(z,\varphi)\Big)\mathcal{S}}{\sum_{n\geq 2}\sum_z\theta_n(z,\varphi)g_{n-1}(z)}$$

Solving for $\mathcal{S}$, we finally obtain the stationary measure of active sellers:

$$\mathcal{S}=\left(b^A+h_0(\varphi)\sum_{z}\theta_1(z,\varphi)+\sum_{n=1}^{+\infty}\sum_{z}\theta_{n+1}(z,\varphi)g_n(z)\right)^{-1}$$

Computing the remaining aggregate measures is straightforward: the mass of potential entrants is $S_0(\varphi)=\mathcal{S}h_0(\varphi)$, the measure of incumbent sellers is $S_n=\mathcal{S}g_n$, the measure of active buyers is $B^A=\mathcal{S}b^A$, and that of inactive buyers is $B^I=1-B^A$.

F.2 Existence of the Joint Surplus Function, given U^B

This section shows that there exists a unique joint surplus W, for each given U^B. First, we define the relevant functional space. Recall that the state space is $\mathcal{N}\times\mathcal{Z}\times\Phi=\{1,\dots,\bar{n}\}\times\{z_i\}_{i=1}^{k_z}\times\{\varphi_j\}_{j=1}^{k_\varphi}$.

Definition F.1 *Let $\mathcal{W}$ be a closed and bounded subspace of vector-valued functions $\mathbf{W}:\mathcal{N}\times\mathcal{Z}\times\Phi\to\mathbb{R}$, with the following properties:*

1. *Increasing in n, i.e. $W_{n+1}(z,\varphi)>W_n(z,\varphi)$, $\forall n\in\mathcal{N}$.*

2. *Constant at the upper bound of $\mathcal{N}$, i.e. $W_{\overline{n}}(z,\varphi)=W_{\overline{n}+1}(z,\varphi)$.*

For a given U^B, the joint surplus can be written as follows:

$$\begin{aligned}(r+\delta_f)W_n(z,\varphi)=\max_{\mathbf{x}'(n',z',\varphi')}\Bigg\{&n\Big(v(\varphi)+(\delta_f+\delta_c)U^B(\varphi)\Big)-\Big(\mathcal{C}(n;z,\varphi)+\psi\Big(x'(n+1,z,\varphi);\varphi\Big)x'(n+1,z,\varphi)\Big)\\&+n\delta_c\Big(W_{n-1}(z,\varphi)-W_n(z,\varphi)\Big)+\psi\Big(x'(n+1;z,\varphi);\varphi\Big)\Big(W_{n+1}(z,\varphi)-W_n(z,\varphi)\Big)\\&+\sum_{z'\in\mathcal{Z}}\lambda_z(z'|z)\Big(W_n(z',\varphi)-W_n(z,\varphi)\Big)+\sum_{\varphi'\in\Phi}\lambda_\varphi(\varphi'|\varphi)\Big(W_n(z,\varphi')-W_n(z,\varphi)\Big)\Bigg\}\end{aligned}\tag{F.2.1}$$

where we have used the short-hand notation:

$$\psi(x;\varphi)\equiv\eta\circ\mu^{-1}\left(\frac{\Gamma^B(\varphi)}{x-U^B(\varphi)}\right)$$

Equation (F.2.1) is a continuous-time recursive problem. In order to use dynamic programming methods, we first need to transform the problem into an amenable form that allows us to use Blackwell's Theorem. We do this by a so-called *uniformization method*.[4] In short, the objective of this method is to construct a set of transition probabilities that mimic those of the continuous-time specification.

For a given vector of current states $\boldsymbol{\gamma} \equiv (n, z, \varphi)$, define $\Gamma' \equiv \{0, n-1, n+1\} \times \mathcal{Z} \times \Phi$ as the set of possible future states. Let $\boldsymbol{\zeta} \equiv \{\mathbf{x}'(n', z', \varphi') : (n', z', \varphi') \in \Gamma'\} \subseteq \mathcal{X}$ denote a set of policies, and $\mathcal{P}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta})$ denote the probability of a $\boldsymbol{\gamma}$-to-$\boldsymbol{\gamma}'$ transition under policy $\boldsymbol{\zeta}$. Moreover, let $q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta})$ be the vector of Markov transition rates for a fixed $\boldsymbol{\gamma}$. Then, we have:

$$\mathcal{P}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta}) \equiv \frac{1}{q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta})} \cdot \begin{cases} \psi\Big(x'(n+1, z, \varphi); \varphi\Big) & \text{for } \boldsymbol{\gamma}' = (n+1, z, \varphi) \\ n\delta_c & \text{for } \boldsymbol{\gamma}' = (n-1, z, \varphi) \\ \delta_f & \text{for } \boldsymbol{\gamma}' = (0, z, \varphi) \\ \lambda_z(z'|z) & \text{for } \boldsymbol{\gamma}' = (n, z', \varphi), \text{ any } z' \neq z \\ \lambda_\varphi(\varphi'|\varphi) & \text{for } \boldsymbol{\gamma}' = (n, z, \varphi'), \text{ any } \varphi' \neq \varphi \end{cases}$$

and

$$q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta}) \equiv \psi\Big(x'(n+1, z, \varphi); \varphi\Big) + n\delta_c + \delta_f + \sum_{z' \neq z} \lambda_z(z'|z) + \sum_{\varphi' \neq \varphi} \lambda_\varphi(\varphi'|\varphi)$$

Since the state space is bounded, there exists a $\overline{q}^S < +\infty$ for which $q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta}) < \overline{q}^S$, for all states $\boldsymbol{\gamma}$, given $\boldsymbol{\zeta}$. Therefore, we can think of transitions actually occurring at rate $\overline{q}$, with a fraction $q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta})/\overline{q}$ of them being actual transitions out of state $\boldsymbol{\gamma}$, and the remainder being "fictitious" transitions of the state $\boldsymbol{\gamma}$ into itself. Thus, we may represent the Markov chain with the following transition probabilities:

$$\widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\zeta) \equiv \begin{cases} \dfrac{q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta})}{\overline{q}} \mathcal{P}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta}) & \text{for } \boldsymbol{\gamma}' \neq \boldsymbol{\gamma} \\ 1 - \dfrac{q_{\boldsymbol{\gamma}}(\boldsymbol{\zeta})}{\overline{q}} & \text{otherwise} \end{cases}$$

Finally, define the discount factor as $\beta \equiv \frac{\overline{q}}{r+\overline{q}}$, and the per-period payoff function in state (n, z, φ) as:

$$\widetilde{\Pi}_n(z, \varphi; \boldsymbol{\zeta}) \equiv \frac{1}{\overline{q}} \left[n\Big(v(\varphi) + (\delta_f + \delta_c)U^B(\varphi)\Big) - \Big(\mathcal{C}(n; z, \varphi) + \psi\big(x'(n+1, z, \varphi); \varphi\big)x'(n+1, z, \varphi)\Big)\right]$$

We can now state the dynamic optimization problem (F.2.1) in discretized form:

$$W_n(z, \varphi) = \max_{\boldsymbol{\zeta} \subseteq \mathcal{X}} \left\{ \widetilde{\Pi}_n(z, \varphi; \boldsymbol{\zeta}) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta}) W_{n'}(z', \varphi') \right\} \tag{F.2.2}$$

We are now ready to prove the main result:

[4] See Ross (1996), Section 5.8. For an application in economics, see Acemoglu and Akcigit (2012).

Lemma F.1 *For any $(n, z, \varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi$, the joint surplus problem (F.2.1) admits a unique solution. That is, the mapping $T : \mathcal{W} \to \mathcal{W}$ defined by:*

$$T.W_n(z, \varphi) = \max_{\boldsymbol{\zeta} \subseteq \mathcal{X}} \left\{ \widetilde{\Pi}_n(z, \varphi; \boldsymbol{\zeta}) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta}) W_{n'}(z', \varphi') \right\}$$

has a fixed point $T.W_n(z, \varphi) = W_n(z, \varphi)$.

Proof. T is a well-defined mapping from $\mathcal{W}$ to $\mathcal{W}$. We want to show that it defines a contraction. Since $\mathcal{W}$ is closed and $\boldsymbol{\zeta}$ takes values in a compact set, the contraction property will be enough to invoke Banach's Fixed Point theorem. Hence, we check that T satisfies monotonicity and discounting.

- **Monotonicity:** Take $\mathbf{W}^a, \mathbf{W}^b \in \mathcal{W}$ such that $W_n^a(z, \varphi) \leq W_n^b(z, \varphi)$, $\forall (n, z, \varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi$. Denote the corresponding optimal policies by:

$$\hat{\boldsymbol{\zeta}}^i \equiv \arg\max_{\boldsymbol{\zeta}} \left\{ \widetilde{\Pi}_n(z, \varphi; \boldsymbol{\zeta}) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\boldsymbol{\zeta}) W^i_{n'}(z', \varphi') \right\}$$

  for each $i = a, b$. Then:

$$\begin{aligned} T.W_n^b(z, \varphi) &\geq \widetilde{\Pi}_n(z, \varphi; \hat{\boldsymbol{\zeta}}^a) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\hat{\boldsymbol{\zeta}}^a) W^b_{n'}(z', \varphi') \\ &\geq \widetilde{\Pi}_n(z, \varphi; \hat{\boldsymbol{\zeta}}^a) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\hat{\boldsymbol{\zeta}}^a) W^a_{n'}(z', \varphi') \\ &= T.W_n^a(z, \varphi) \end{aligned}$$

  for any $(n, z, \varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi$, where the first inequality follows by optimality, and the second one follows from $\mathbf{W}^a \leq \mathbf{W}^b$.

- **Discounting:** Let $a \geq 0$ and $\mathbf{W} \in \mathcal{W}$, and denote the optimal policy by $\hat{\boldsymbol{\zeta}}$. Since a is a constant, we have that:

$$\begin{aligned} T.\big[W + a\big]_n(z, \varphi) &= \widetilde{\Pi}_n(z, \varphi; \hat{\boldsymbol{\zeta}}) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\hat{\boldsymbol{\zeta}}) \Big(W_{n'}(z', \varphi') + a \Big) \\ &= \widetilde{\Pi}_n(z, \varphi; \hat{\boldsymbol{\zeta}}) + \beta \sum_{\boldsymbol{\gamma}' \in \Gamma'} \widetilde{\mathcal{P}}_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}(\hat{\boldsymbol{\zeta}}) W_{n'}(z', \varphi') + a\beta \\ &= T.W_n(z, \varphi) + a\beta \end{aligned}$$

  for any $(n, z, \varphi) \in \mathcal{N} \times \mathcal{Z} \times \Phi$. Since $\beta < 1$, discounting obtains.

Therefore, for a given U^B, T defines a contraction in $\mathcal{W}$ with modulus β, and by Banach's fixed-point theorem there exists a unique value function $W_n(z, \varphi)$ such that $T.W_n(z, \varphi) = W_n(z, \varphi)$. $\square$

G Additional Figures

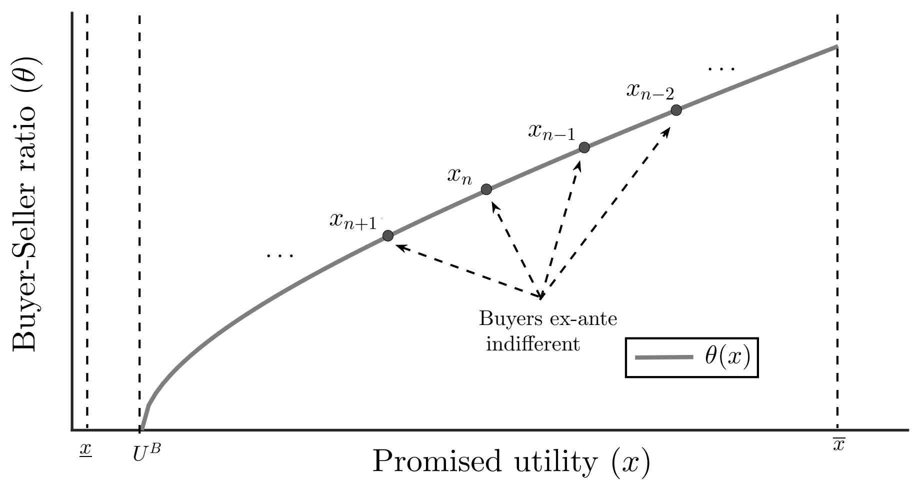

Figure G.1: Equilibrium tightness $\theta : x \mapsto \mu^{-1}\left(\frac{\Gamma^B}{x-U^B}\right)$, and set of equilibrium markets.

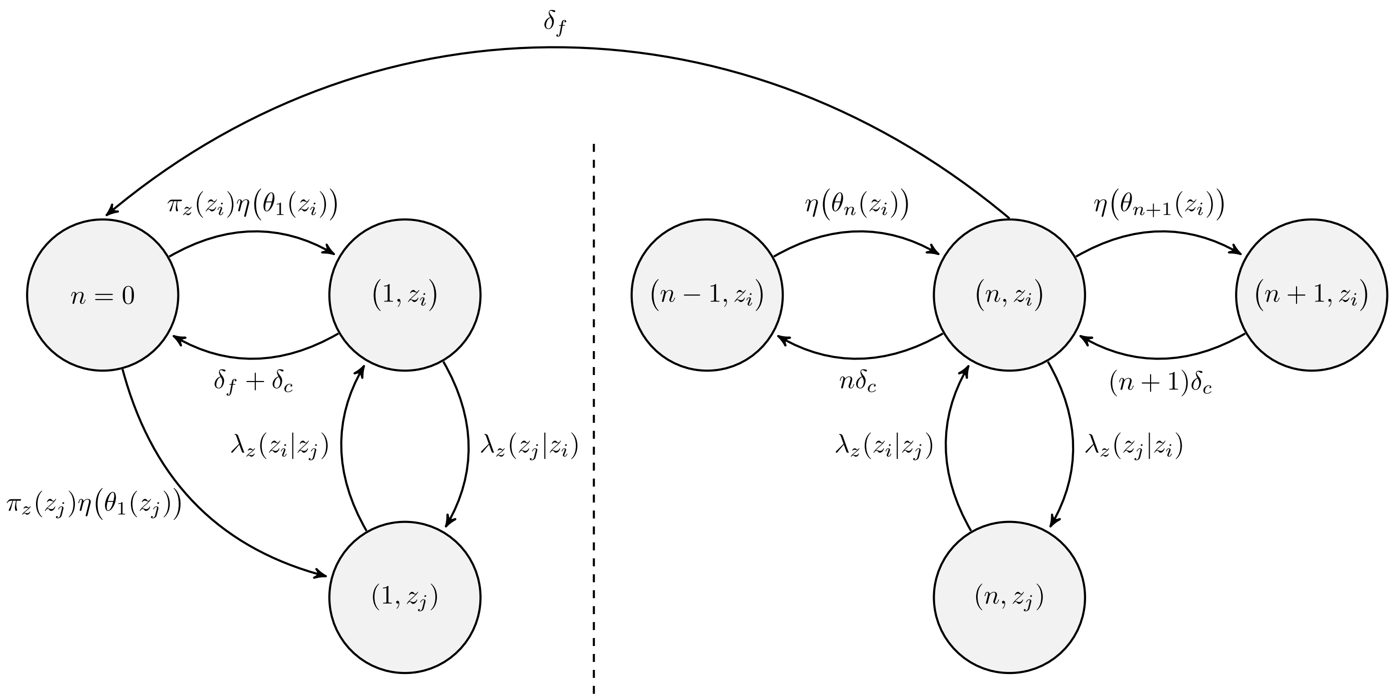

Figure G.2: Seller transitional dynamics for a typical incumbent (right-hand side block) and for entrants (left-hand side block). The arrow labels indicate flow rates.

References

Acemoglu, D. and Akcigit, U. (2012). Intellectual property rights policy, competition and innovation. *Journal of the European Economic Association*, **10** (1), 1 – 42.

Daruich, D. (2019). The macroeconomic consequences of early childhood development policies. *Mimeo, Federal Reserve Bank of St. Louis.*

Gourio, F. and Rudanko, L. (2014). Customer capital. *Review of Economic Studies*, **81** (3), 1102 – 1136.

Ross, S. (1996). *Stochastic Processes.* John Wiley & Sons, Inc., New York.

Shi, S. (2009). Directed search for equilibrium wage-tenure contracts. *Econometrica*, **77** (2), 561 – 584.

Stokey, N. L. and Lucas, R. E. (1989). *Recursive Methods in Economic Dynamics.* Harvard University Press.

Tauchen, G. (1986). Finite state Markov-Chain approximations to univariate and vector autoregressions. *Economics Letters*, **20** (2), 177 – 181.