Homework I

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1 Proof for Properties of Common Functions

The **logistic sigmoid** funcition is

$$\sigma(x) = \frac{1}{1 + \exp(-x)}. \tag{3.30}$$

The **softplus** function is

$$\zeta(x) = \log(1 + \exp(x)). \tag{3.31}$$

Multiply both the denominator and numerator of the fraction on the right side of Equation 3.30 with $\exp(x)$, we have

$$\sigma(x) = \frac{\exp(x)}{\exp(x) + 1} = \frac{\exp(x)}{\exp(x) + \exp(0)}. \tag{3.33}$$

The derivative of $\sigma(x)$ is

$$\begin{aligned} \frac{d}{dx}\sigma(x) &= \frac{d}{dx}\left(1 + e^{-x}\right)^{-1} \\ &= \left(1 + e^{-x}\right)^{-2} e^{-x} \\ &= \frac{1}{1 + e^{-x}} \cdot \frac{e^{-x}}{1 + e^{-x}} \\ &= \sigma(x)(1 - \sigma(x)). \end{aligned} \tag{3.34}$$

Also, we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + \exp(-x)} = \frac{\exp(-x)}{1 + \exp(-x)} = \frac{1}{\exp(x) + 1} = \sigma(-x) \tag{3.35}$$

$$\log \sigma(x) = \log \frac{1}{1 + \exp(-x)} = -\log(1 + \exp(-x)) = -\zeta(-x). \tag{3.36}$$

The derivative of $\zeta(x)$ is

$$\frac{d}{dx}\zeta(x) = \frac{d}{dx}\log(1 + \exp(x)) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}} = \sigma(x). \tag{3.37}$$

Let $y = \sigma(x)$, we have

$$\begin{aligned}
& y = \frac{1}{1+\exp(-x)} \\
\iff & \exp(-x) = \frac{1}{y} - 1 \\
\iff & x = -\log\frac{1-y}{y} = \log\frac{y}{1-y} \\
\iff & \sigma^{-1}(x) = \log\frac{x}{1-x}, \quad \forall x \in (0,1).
\end{aligned} \tag{3.38}$$

Let $y = \zeta(x)$, we have

$$\begin{aligned}
& y = \log(1+\exp(x)) \\
\iff & \exp(x) = \exp(y) - 1 \\
\iff & x = \log(\exp(y) - 1) \\
\iff & \zeta^{-1}(x) = \log(\exp(x) - 1), \quad \forall x > 0.
\end{aligned} \tag{3.39}$$

Also, we have

$$\begin{aligned}
\int_{-\infty}^{x} \sigma(y)dy &= \int_{-\infty}^{x} \frac{1}{1+\exp(-y)}dy \\
&= \int_{-\infty}^{x} \frac{e^y}{e^y+1}dy \\
&\xlongequal{z \triangleq e^y} \int_{0}^{e^x} \frac{1}{z+1}dz \\
&= \log(z+1)|_0^{e^x} \\
&= \log(\exp(x)+1) \\
&= \zeta(x).
\end{aligned} \tag{3.40}$$

Finally, we also have

$$\begin{aligned}
\zeta(x) - \zeta(-x) &= \log(1+e^x) - \log(1+e^{-x}) \\
&= \log\frac{1+e^x}{1+e^{-x}} \\
&= \log\frac{e^x(e^{-x}+1)}{1+e^{-x}} \\
&= \log e^x \\
&= x.
\end{aligned} \tag{3.41}$$