1. Introduction & Notation

a) Definitions & Examples.

i) $\mathbb{R}^n_+$:

Def:- Let $(x_1, x_2 \cdots x_n)$ be a sequence of n non-negative real numbers. The set of all such sequences is denoted by $\mathbb{R}^n_+$.

Ex:- $(0, 1, 2 \cdots n) \in \mathbb{R}^n_+$
$(1, 1, \cdots 1) \in \mathbb{R}^n_+$

ii) Symmetric Polynomial:-

Let $X = (x_1, x_2 \cdots x_n)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n) \in \mathbb{Z}^n_+$.

Then $f(X) = \sum_{\alpha \in \mathbb{Z}^n_+} c_\alpha X^\alpha$ where $c_\alpha \in \mathbb{R}$ is a symm. polynomial if $f(X) = f(X_\sigma)$, where $\sigma \in \underset{\text{perm grp of } (1, 2, \cdots n)}{S_n}$

Ex:- Let $n = 2$.

$f(x_1, x_2) = f(x_2, x_1)$

$f(x_1, x_2) = x_1 + x_2 = f(x_2, x_1) = x_2 + x_1$

Let $\underline{n=3}$

$$f(x_1, x_2, x_3) = x_1^1 x_2^2 x_3^3 + x_1^1 x_3^2 x_2^3 + x_2^1 x_1^2 x_3^3 + x_2^1 x_3^2 x_1^3 + x_3^1 x_2^2 x_1^3 + x_3^1 x_1^2 x_2^3$$

For a symmetric polynomial with $n=3$,

$$f(x_1, x_2, x_3) = f(x_1, x_3, x_2)$$
$$= f(x_2, x_1, x_3) = f(x_2, x_3, x_1)$$
$$= f(x_3, x_1, x_2) = f(x_3, x_2, x_1)$$

Note :- Sum of symm. poly. of the same degree is also a symm. poly.

iii) Real Symmetric Polynomial Inequality (SPI) of degree d.

Def:- A real SPI of degree d is of the form

$$f(x) > 0, \; f(x) < 0, \; f(x) \geq 0, \; f(x) \leq 0$$

where $x \in \mathbb{R}_+^n$, and $f(x)$ is a symmetric polynomial of degree d.

Ex:- $d = 3$ $\qquad x_1 x_2 x_3 \geq 0$

$d = 2$ $\qquad x_1^2 + x_2^2 \leq 3 x_1 x_2$

iv) $\mathbb{R}[x_1, x_2, x_3 \cdots x_n]$:

Def:- It is the set of all real polynomials with n variables.

Example Elements in $\mathbb{R}[x_1, x_2 \cdots x_{10}]$:-

$x_1x_2x_3x_4x_5$, $x_6+x_7+x_8+x_9+x_{10}$, x_1x_{10}, x_1^2, $x_2^{10}x_{10}x_3 + x_4^5x_6^2$.

v) $\Sigma^{[n]}$:

Def:- $\Sigma^{[n]}$ is the set of all real symmetric poly. on $(x_1, x_2 \cdots x_n)$.

For $n = 2$
$$\Sigma^{[2]} = \mathbb{R}[x_1, x_2]^{S_2}$$

$\boxed{\text{Note}}$:- $S_2 = \{(1,2), (2,1)\}$

Example elements in $\Sigma^{[2]}$:-

x_1x_2, $x_1^2x_2^2$, x_1+x_2, $3(x_1+x_2)+x_1^{10}x_2^{10}$, 0, 1, 2, 3, 1000

$$\Sigma^{[2]} = \left\{ \sum_{\alpha \in \mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \;\middle|\; \sum_{\alpha \in \mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} = \sum_{\alpha \in \mathbb{Z}_+^2} c_\alpha x_{\sigma_1}^{\alpha_1} x_{\sigma_2}^{\alpha_2} \right\}$$

vi) $\Sigma_d^{[n]}$:- This is a subspace of the vector space $\Sigma^{[n]}$, where $d \leq n$ where $d, n \in \mathbb{N}$.

$$\Sigma_d^{[n]} := \{ f \in \Sigma^{[n]} \mid \deg(f) \leq d \}.$$

$$\mathcal{H}_d^{[n]} := \{ f \in \Sigma^{[n]} \mid f \text{ is } d\text{-homogenous} \} \subset \Sigma_d^{[n]}$$

Note :- d- homogenous means that every term in f is of degree d.

Example :- Let $n = 3, d = 2$

$$\Sigma^{[3]} = \left\{ \sum_{\alpha \in \mathbb{Z}_+^3} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \;\middle|\; \sum_{\alpha \in \mathbb{Z}_+^3} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = \sum_{\alpha \in \mathbb{Z}_+^3} c_\alpha x_{\sigma_1}^{\alpha_1} x_{\sigma_2}^{\alpha_2} x_{\sigma_3}^{\alpha_3} \right\}$$

$$\Sigma_2^{[3]} = \{ f \in \Sigma^{[3]} \mid \deg(f) \leq 2 \}$$

Example elements in $\Sigma_2^{[3]}$:- $x_1 x_3 + x_1 x_2 + x_2 x_3$, $10(x_3^2 + x_1^2 + x_2^2)$, 10, 1, 2, $5(x_1 + x_2 + x_3)$

$$\mathcal{H}_2^{[3]} := \{ f \in \Sigma_2^{[3]} \mid f \text{ is 2-homogenous} \}$$

Example Elements in $\mathcal{H}_2^{[3]}$:- $x_1 x_2 + x_2 x_3 + x_1 x_3$, $x_1^2 + x_2^2 + x_3^2$, $5(x_1^2 + x_2^2 + x_3^2) + 10(x_1 x_2 + x_2 x_3 + x_1 x_3)$

vii) $\forall\, a, b \in \mathbb{R}$,

$a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$,

$\overline{a, b} = \mathbb{Z} \cap [a, b]$, $\lfloor a \rfloor$ = Integer part of a.

Example:

a) Let $a = 0$, $b = -\sqrt{2}$

i) $a \wedge b = -\sqrt{2}$,

ii) $a \vee b = 0$,

iii) $\overline{b, a} = \{0, -1\} = \mathbb{Z} \cap [b, a]$

iv) $\lfloor a \rfloor = 0$.

b) Let $a = -\sqrt{3}$, $b = \frac{-\sqrt{3}}{1.01}$.

i) $a \wedge b = -\sqrt{3}$,

ii) $a \vee b = \frac{-\sqrt{3}}{1.01}$,

iii) $\overline{a, b} = \phi$,

iv) $\lfloor a \rfloor = -1$

viii) Let $x = (x_1, x_2 \cdots x_n)$

a) $\text{supp}(x) = \{ j \in \overline{1,n} \mid x_j \neq 0 \}$

b) $v(x) := \left| \{ x_j \mid j \in \overline{1,n} \} \right|$

c) $v^*(x) := \left| \{ x_j \mid x_j \neq 0 \} \right|$

<u>Example</u>:-

i) Let $x = (x_1, x_2, x_3) = (-3, 0, 2)$

a) $\text{supp}(x) = \{1, 3\}$

b) $v(x) = 3$,

c) $v^*(x) = 2$

ii) Let $x = (x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

a) $\text{supp}(x) = \phi$

b) $v(x) = 1$,

c) $v^*(x) = 0$.

Note:- $v(x) = n$, when x_j's are all distinct.

Q) What are semi-continuous functions?

Ans:- When a function is discontinuous as a whole, but is made up of several continuous pieces, it is called a semi-continuous function.

There are two types of semicontinuous functions:-

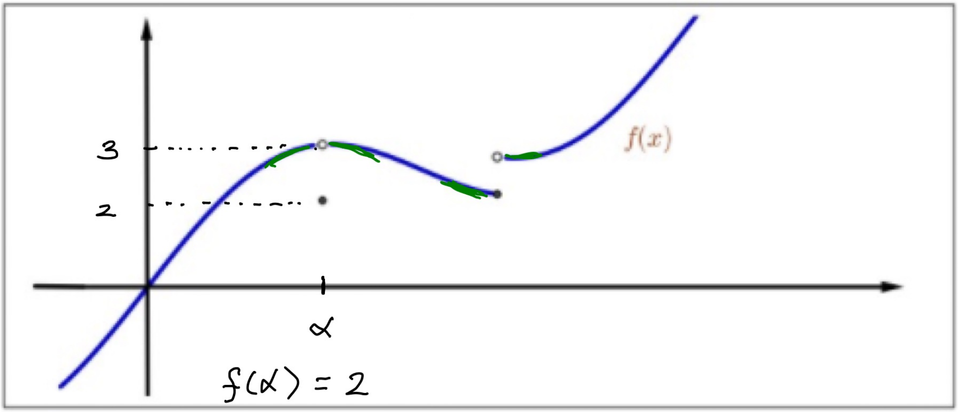
a) Lower semicontinuous

Def:- Let $f : D \to \mathbb{R}$, and let $\bar{x} \in D$. f is said to be lower semicontinuous at $\bar{x}$ if for $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that

$f(\bar{x}) - \varepsilon < f(x)$, for all $x \in B(\bar{x}; \delta) \cap D$ where $B(\bar{x}; \delta)$ is the δ-neighbourhood of $\bar{x}$.

Simple Def:- In other words, a function f is said to be lower semicontinuous if for any test input $\bar{x} \in D$, $f(\bar{x})$ is less than or equal to the neighbourhood values around the point $\bar{x}$.

Lower Semicontinuous

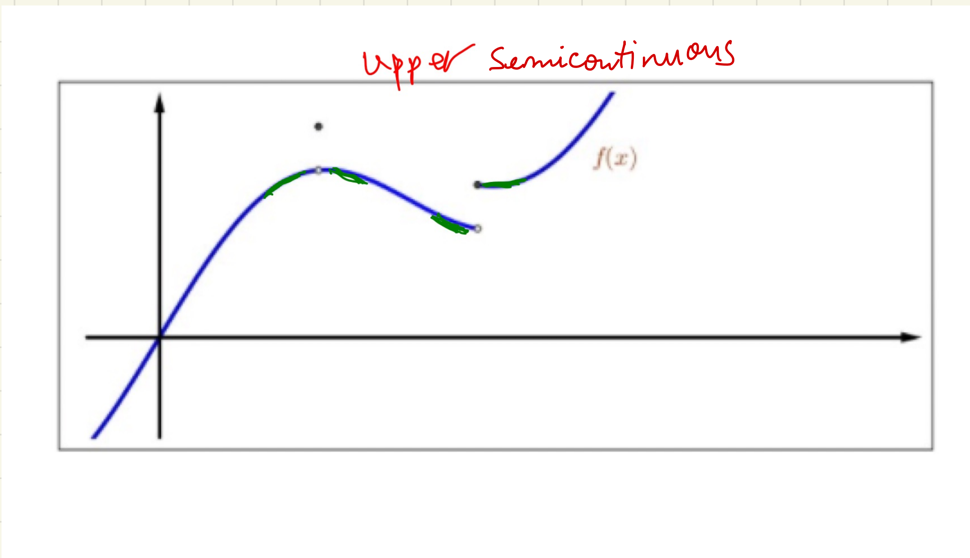


b) Upper semicontinuous

Def:- Let $f: D \to \mathbb{R}$, and let $\bar{x} \in D$. f is said to be upper semicontinuous at $\bar{x}$ if for $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) < f(\bar{x}) + \varepsilon$$, for all $x \in B(\bar{x}; \delta) \cap D$ where $B(\bar{x}; \delta)$ is the δ-neighbourhood of $\bar{x}$.

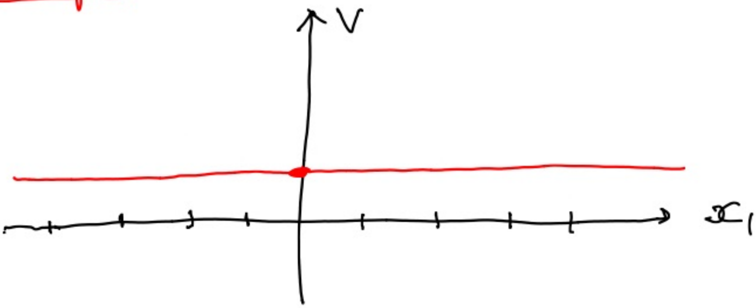
Simple Def:- In other words, a function f is said to be upper semicontinuous if for any test input $\bar{x} \in D$, $f(\bar{x})$ is greater than or equal to the neighbourhood values around the point $\bar{x}$.



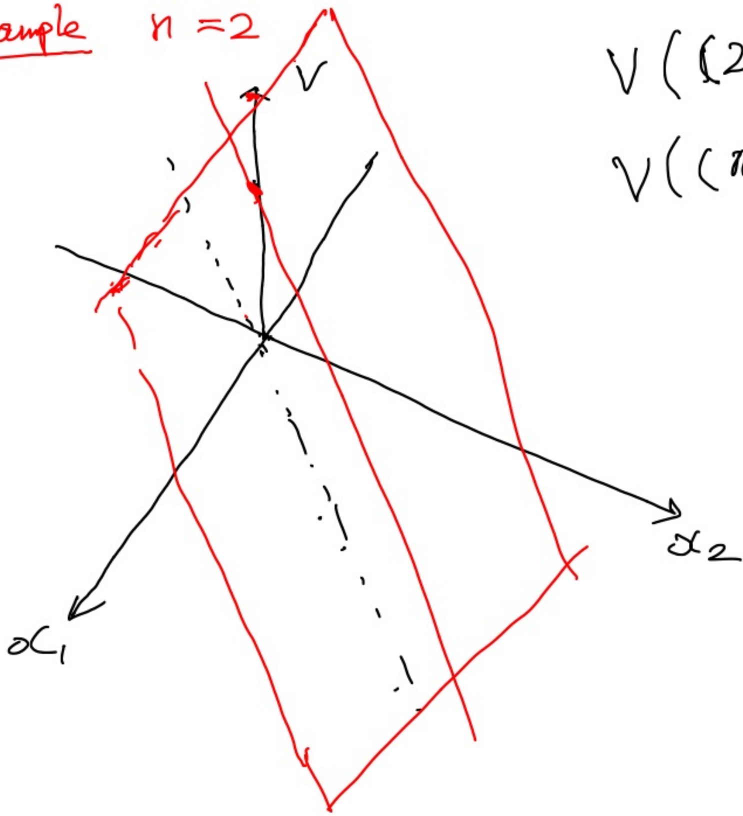
$\boxed{\text{Note:-}}$ $V, V^*: \mathbb{R}^n \to \overline{0, n}$ are lower semicontinuous functions.

$(*)\quad V : \mathbb{R}^n \longrightarrow \overline{0, n}$

Example $n = 1$



Example $n = 2$



$V(\langle 2, 2 \rangle) = 1$

$V(([illegible])) = 2$

ix) P_k, where $k \in \mathbb{N}^*$

<u>Def</u>:- P_k is the k^{th} symm. power sum defined by

$$P_k : \mathbb{R}^n \to \mathbb{R}, \quad P_k(x) = \sum_{j=1}^{n} x_j^k .$$

<u>Examples</u>:-

a) let $x = (0, 1, 2 \dots n-1)$, $k = 2$

$$P_2(x) = 0^2 + 1^2 + 2^2 \cdots (n-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

b) Let $n = 1$, $x = (-\sqrt{2})$, $k = 2$

$$P_2(x) = (-\sqrt{2})^2 = \boxed{2}$$

c) Let $n = 2$, $x = (1, 1)$, $k = 1$

$$P_1(x) = 1 + 1 = \boxed{2}$$

Important set Definitions:-

(A) Let $\sigma \in \mathbb{R}$ and $\sigma > 0$.

The symbol K_σ denotes the set $\{x \in \mathbb{R}_+^n \mid x_1 + x_2 \cdots x_n = \sigma\}$.

(B) Let $\sigma \in \mathbb{R}$ and $\sigma > 0$. Let $s \in \mathbb{N}^*$.

The symbol K_σ^s denotes the set $\{x \in K_\sigma \mid V^*(x) \leq s\}$.

(C) Let $\sigma \in \mathbb{R}$ and $\sigma > 0$. Let f be a continuous map such that

$f: \mathbb{R}_+^n \setminus \{0_n\} \longrightarrow \mathbb{R}$.

The symbol $M_\sigma(f)$ denotes the set $\{\xi \in K_\sigma \mid f(\xi) = \min_{x \in K_\sigma} f(x)\}$.

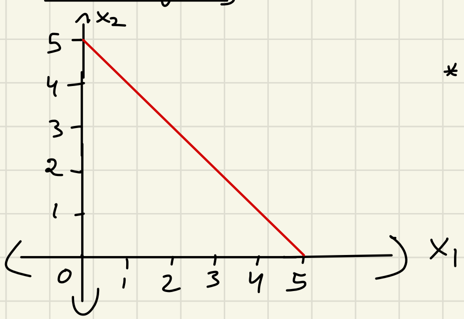
Examples:-

(A)

i) Let $n = 2$, $\sigma = 5$

$$K_5 = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 = 5\}$$

Graph of K_5:-

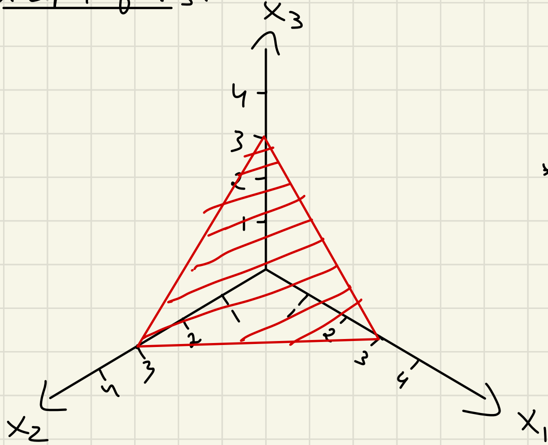


* The red line segment is the simplex K_5 in $n=2$.

ii) Let $n=3$, $\sigma=3$

$$K_3 = \{ x \in \mathbb{R}_+^3 \mid x_1 + x_2 + x_3 = 3 \}$$

<u>Graph of K_3</u> :-



* The red shaded region is the simplex K_3 in $n=3$.

(B)

(1)

K_σ^s

Def :- K_σ^s is the set of all n-tuples $x \in K_\sigma$ such that $v^*(x) \leq s$. $s \in \mathbb{N}^*$

Example :- Using previous example for K_σ,
Let $\sigma = 5$, $n = 3$.
we need to find $v^*(x)$ for all 21 elements in K_5.

$x = (1,2,2)$ & all its permutations :- $v^*(x) = 2$,

$x = (1,1,3)$ & " " :- $v^*(x) = 2$,

$x = (0,3,2)$ & " " :- $v^*(x) = 2$,

$x = (0,1,4)$ & " " :- $v^*(x) = 2$,

$x = (0,0,5)$ & " " :- $v^*(x) = 1$,

Now, let $s = 2$ (or any value greater than 2.)
Then $K_5^2 = K_5$.

But if $s = 1$, then $K_5^2 = \{(0,0,5), (0,5,0), (5,0,0)\}$

$M_\sigma(f)$

Def : $M_\sigma(f)$ is called the minimizer set. It is the set of all n-tuples $\xi \in K_\sigma$ s.t $\xi \neq 0_n$ & $f(\xi)$ outputs the minimum value of $f(x)$ when $x \in K_\sigma$.

$$M_\sigma(f) := \text{minimizer}(f|_{K_\sigma}) = \left\{ \xi \in K_\sigma \mid f(\xi) = \min_{x \in K_\sigma} f(x) \right\}.$$

Example :- Taking the prev. case where $n=3$, $\sigma=5$,

Let $f(x) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 x_3$

$\Rightarrow$ It is clear that f is cont. (infact its also symm.)

We want to find the lowest value of $f(x)$ when $x \in K_5$

$f(1,2,2)$ & all its permutations $= 13$

$f(1,1,3)$ & all its permutations $= 14$

$f(0,1,4)$ & all its permutations $= 17$

$f(0,3,2)$ & all its permutations $= 13$

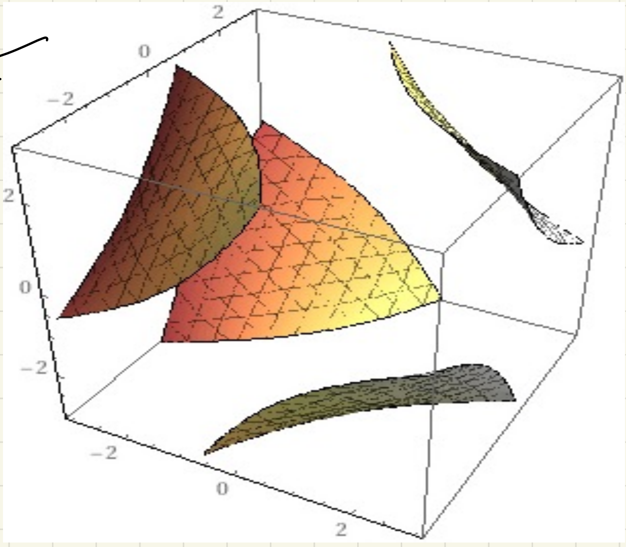
$f(0,0,5)$ & all its permutations $= 25$

Thus, $\min\limits_{x \in K_5} f(x) = 13$

This happens when x is equal to $(1,2,2)$, $(0,3,2)$ & their permutations

Thus, $M_5(f) = \{(1,2,2), (2,1,2), (2,2,1), (0,3,2), (0,2,3), (2,0,3), (2,3,0), (3,0,2), (3,2,0)\}$

Graph of $x^2+y^2+z^2+xyz=13$



$x \in K_\sigma$

The simplex $K_\sigma \subset \mathbf{R}^n_+ \setminus \{0_n\}$ is a compact set, $]0, \infty[\cdot K_\sigma = \mathbf{R}^n_+ \setminus \{0_n\}$, and the restriction $f|_{K_\sigma}$ attains its minimum on $M_\sigma(f) \neq \emptyset$.

?

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Definition 1.1.

A path $\gamma : [a,b] \to \mathbb{R}^n$ is said to be an (s)-path $(s \in \mathbb{N}^*)$ provided

i) $P_i \circ \gamma \equiv (P_i \circ \gamma)(a) \quad , \forall i \in \overline{1,s}$

ii) $\operatorname{supp}(\gamma(t)) = \operatorname{supp}(\gamma(a)) \quad , \forall t \in [a,b]$.

Example:-

I) $s = 2, \quad a = -2, \quad b = 2, \quad n = 3$

$$\begin{aligned} \gamma : [-2, 2] &\longrightarrow \mathbb{R}^3 \\ c &\longrightarrow (1, 1, 1) \end{aligned}$$

i) Claim:- γ is a 2-path.

Proof:-

① $\forall_{\substack{i \in \overline{1,2} \\ c \in [-2,2]}} (P_i \circ \gamma)(c) \equiv (P_i \circ \gamma)(-2)$

$$\begin{aligned} P_1 : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, \dots x_n) &\longrightarrow x_1 + \dots + x_n \end{aligned}$$

$$\begin{aligned} P_2 : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, \dots x_n) &\longrightarrow x_1^2 + \dots + x_n^2 \end{aligned}$$

$$\begin{aligned} \gamma : [-2, 2] &\longrightarrow \mathbb{R}^3 \\ c &\longrightarrow (1, 1, 1) \end{aligned}$$

$$\begin{aligned} P_1 \circ \gamma : [-2, 2] &\longrightarrow \mathbb{R} \\ [-2, 2] \xrightarrow{\gamma} &\ \mathbb{R}^3 \xrightarrow{P_1} \mathbb{R} \\ c \longrightarrow &\ (1,1,1) \longrightarrow 1+1+1 = \underline{\underline{3}} \end{aligned}$$

$$P_2 \circ \gamma : [-2, 2] \longrightarrow \mathbb{R}$$

$$\begin{array}{ccccc} [-2,2] & \xrightarrow{\gamma} & \mathbb{R}^3 & \xrightarrow{P_2} & \mathbb{R} \\ c & \longrightarrow & (1,1,1) & \longrightarrow & 1^2+1^2+1^2 = \underline{\underline{3}} \end{array}$$

Let $c \in [-2, 2]$.

Note that,

a) $(P_1 \circ \gamma)(c) = 3 = (P_1 \circ \gamma)(-2)$

b) $(P_2 \circ \gamma)(c) = 3 = (P_2 \circ \gamma)(-2)$

② $\forall_{t \in [-2,2]} \quad supp(\gamma(t)) = supp(\gamma(-2))$

Let $t \in [-2, 2]$

$$supp(\gamma(t)) = supp((1,1,1)) = \{1, 2, 3\} = supp(\gamma(-2))$$

Therefore, γ is a 2-path.

Note:-

An (n)-path $(n \in \mathbb{N}^*)$ will also be an $(n-1)$-path, $(n-2)$ path $\cdots$ (1)-path

~~II~~) $s = 1$, $a = -2$, $b = 2$, $n = 3$

$$\begin{array}{ccc} \gamma : [-2, 2] & \longrightarrow & \mathbb{R}^3 \\ c & \longrightarrow & (c, -c/2, -c/2) \end{array}$$

i) claim:- γ is not a 1-path

Proof :-

① $\forall_{c \in [-2,2]} \; (P_1 \circ \gamma)(c) \equiv (P_1 \circ \gamma)(-2)$

$$\begin{array}{ccc} P_1 : \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, \dots x_n) & \longrightarrow & x_1 + \cdots + x_n \end{array}$$

$$\gamma : [-2, 2] \longrightarrow \mathbb{R}^3$$
$$c \longmapsto (c, -c/_2, -c/_2)$$

$$p_1 \circ \gamma : [-2, 2] \longrightarrow \mathbb{R}$$
$$[-2, 2] \xrightarrow{\gamma} \mathbb{R}^3 \xrightarrow{p_1} \mathbb{R}$$
$$c \longmapsto (c, -c/_2, -c/_2) \longmapsto c - c/_2 - c/_2 = \underline{\underline{0}}$$

Let $c \in [-2, 2]$
Note that
$$(p_1 \circ \gamma)(c) = 0 = (p_1 \circ \gamma)(-2)$$

2) $\underset{t \in [-2,2]}{\exists}$ s.t $\text{supp}(\gamma(t)) \neq \text{supp}(\gamma(-2))$

Let $t \in [-2, 2]$
Note:-

a) $\text{supp}(\gamma(t)) = \text{supp}((t, -t/_2, -t/_2)) = \begin{cases} \phi &, \text{ if } t = 0 \\ \{1, 2, 3\}, & \text{ if } t \neq 0 \end{cases}$

b) $\text{supp}(\gamma(-2)) = \text{supp}((-2, 1, 1)) = \{1, 2, 3\}$

$\text{supp}(\gamma(t)) \neq \text{supp}(\gamma(-2))$ when $t = 0$.

Thus, γ is not a 1-path.

xi) h_k , $k \in \mathbb{N}^*$

Def:- h_k is called the k^{th} symmetric complete function. It is the sum of all monomials of total degree k on $\mathbb{R}^n$.

$$h_k := \sum_{\alpha : \sum \alpha_i = k} x^\alpha, \quad \text{where } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \text{ and } x = (x_1, x_2, \dots, x_n)$$

$\boxed{\text{Note}}$:- By convention $h_0 \equiv 1$.

Example:- $n=3$

i) $k=1$, $h_1 = x_1^0 x_2^0 x_3^1 + x_1^0 x_2^1 x_3^0 + x_1^1 x_2^0 x_3^0$

ii) $k=2$, $h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$

iii) $k=3$, $h_3 = x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_2^2x_3 + x_3^2x_1 + x_1^2x_3 + x_2^2x_1 + x_3^2x_2 + x_1x_2x_3$

$\boxed{\text{Note}}$:- $\forall_{x \in \mathbb{R}^n \setminus \{0_n\}} \ \forall_{k \in \mathbb{N}^*} \ h_{2k} > 0$.

xii) Let $i, j \in \mathbb{N}^*$ and $i \leq j$.

The symbol $P_{(i,j)}$ denotes a vector-valued function such that

$$P_{(i,j)}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^{j-i+1} \quad \text{and} \quad P_{(i,j)}(x := \left(P_i(x), \ldots\ldots, P_j(x)\right)$$

<u>Example</u>:- $i = 2,\ j = 4,\ x = (0.5, 1, 2)$

$$P_2(x) = 0.5^2 + 1^2 + 2^2 = 0.25 + 1 + 4 = 5.25$$
$$P_3(x) = 0.5^3 + 1^3 + 2^3 = 0.125 + 1 + 8 = 9.125$$
$$P_4(x) = 0.5^4 + 1^4 + 2^4 = 0.0625 + 1 + 16 = 17.0625$$

$$\boxed{P_{(2,4)}(x)} = \left(P_2(x), P_3(x), P_4(x)\right) = \boxed{(5.25, 9.125, 17.0625)}$$

xiii) Let J denote a subset of the set $\overline{1,n}$. Let $x \in \mathbb{R}^n$.

The symbol $\check{J}$ denotes the set $\overline{1,n} \setminus J$, and the symbol x_J denotes the element (x_j) where $j \in J$.

Example:- $n = 4$, $x = (1,1,3,4)$, $J = \{1,2,4\} \subset \overline{1,4}$.

i) $\check{J} := \{3\}$

ii) $x_J := (x_1, x_2, x_4) = (1,1,4)$

$\boxed{\text{Note:-}}$ Let $k \in \overline{1,n}$.

$0_k := (0,0,\ldots,0) \in \mathbb{R}^k$ and $1_k := (1,1,\ldots,1) \in \mathbb{R}^k$.

But $\forall_{x \in \mathbb{R}^n}$ x_k denotes the k^{th} element in x

$x = (x_1, x_2, \ldots x_k, \ldots x_n)$.

2. Main Results

Q) why do we not talk about $d < 2$ in this paper?

Ans) Let $f \in \Sigma_d^{[n]}$. $d \in \{0, 1\}$ is a trivial case.

$\Sigma_0^{[n]}$:= set of all symm polynomials in $R[x_1, x_2 \cdots x_n]$ with $d=0$

$= \{ a \mid a \in R \}$

$\Sigma_1^{[n]}$:= set of all symm polynomials in $R[x_1, x_2 \cdots x_n]$ with $d=1$

$= \{ b\, P_1(x) \mid b \in R \}$

Thus, $f = a + b P_1(x)$ s.t $a, b \in R$.

Therefore, the paper assumes $d \geq 2$.

Theorem 2.1 (Of Engagement) :-

$$\forall\, \begin{matrix} n,d \in \mathbb{N}^* \\ d \geq 2 \end{matrix} \quad \forall\, f \in \Sigma_d^{[n]} \quad \forall\, \begin{matrix} \sigma > 0 \\ \sigma \in \mathbb{R} \end{matrix} \quad \forall\, \begin{matrix} \xi \in M_\sigma(f) \\ v^*(\xi) > \lfloor d/2 \rfloor \end{matrix} \quad \forall\, \varepsilon > 0 \quad \exists\, \begin{matrix} \gamma : \text{Injective} \\ (\lfloor d/2 \rfloor)\text{-path} \\ \text{in} \\ M_\sigma(f) \cap B(\xi, \varepsilon) \end{matrix} \quad \exists\, \begin{matrix} \zeta \neq \xi \\ v^*(\zeta) = |\mathrm{supp}(\zeta)| \\ \mathrm{supp}(\zeta) = \mathrm{supp}(\xi) \end{matrix} \quad \forall\, \begin{matrix} i \in \overline{1, \lfloor d/2 \rfloor} \\ p_i(\zeta) = p_i(\xi) \end{matrix} \quad \gamma \text{ connects } \zeta \text{ to } \xi.$$

Theorem 2.2 (Of Reduction)

$$\forall\, \begin{matrix} n,d \in \mathbb{N}^* \\ d \geq 2 \end{matrix} \quad \forall\, f \in \Sigma_d^{[n]} \quad \forall\, \begin{matrix} \sigma > 0 \\ \sigma \in \mathbb{R} \end{matrix} \quad \forall\, \xi \in M_\sigma(f) \quad \exists\, \begin{matrix} \gamma : (\lfloor d/2 \rfloor)\text{-path} \\ \text{in} \\ M_\sigma(f) \end{matrix} \quad \exists\, \begin{matrix} \zeta \\ v^*(\zeta) \leq \lfloor d/2 \rfloor \\ \mathrm{supp}(\zeta) = \mathrm{supp}(\xi) \end{matrix} \quad \exists\, \begin{matrix} x \in K_\sigma \\ y \in K_\sigma^{\lfloor d/2 \rfloor} \\ \min_{x \in K_\sigma} f(x) = \min_{y \in K_\sigma^{\lfloor d/2 \rfloor}} f(x) \end{matrix} \quad \forall\, \begin{matrix} i \in \overline{1, \lfloor d/2 \rfloor} \\ p_i(\zeta) = p_i(\xi) \end{matrix} \quad \gamma \text{ connects } \zeta \text{ to } \xi.$$

Notation

$\Sigma_d^{[n]} = \{ f \in \mathbb{R}[x_1, \dots, x_n] \mid f \text{ is symmetric}, \deg f \le d \}$

$P_i(x) = x_1^i + \dots + x_n^i$

$B(\xi, \varepsilon) = \{ x : \|x - \xi\| \le \varepsilon \}$

$\mathrm{supp}(x) = \{ i : x_i \ne 0 \}$

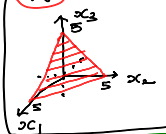
$v^*(x) = \#$ of distinct non-zeros in x.

$K_\sigma = \{ x \in \mathbb{R}^n_{\ge 0} \mid x_1 + \dots + x_n = \sigma \}$

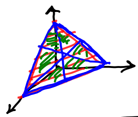
$M_\sigma(f) = \underset{x \in K_\sigma}{\arg\min}\ f(x)$

$n = 3 \quad \sigma = 5 \quad d = 3 \quad f = x_1^3 + x_2^3 + x_3^3$

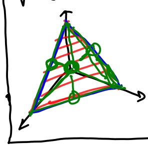
K_σ



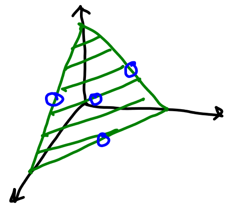
$v^*(x) = 3$



$v^*(x) = 2$



$v^*(x) \ge 2$



Definition

$\gamma : [a,b] \to \mathbb{R}^n$ is "s-path" if

1. $\forall_{t \in [a,b]} \ \forall_{1 \le i \le s} \ (P_i \circ \gamma)(t) = (P_i \circ \gamma)(a)$

2. $\forall_{t \in [a,b]} \ \mathrm{supp}(\gamma(t)) = \mathrm{supp}(\gamma(a))$

Theorem (Enlargement)

$\forall_{\substack{n \ge 1 \\ d \ge 2}} \ \forall_{f \in \Sigma_d^{[n]}} \ \forall_{\sigma > 0} \ \forall_{\xi \in M_\sigma(f)}$, $v^*(\xi) > \lfloor d/2 \rfloor$ $\quad \forall_{\varepsilon > 0} \ \exists \ \gamma : \lfloor d/2 \rfloor$-path injective in $M_\sigma(f) \cap B(\xi, \varepsilon)$ $\quad \exists \ \zeta \ne \xi$, $v^*(\zeta) = \#\mathrm{supp}(\zeta)$ $\quad \gamma$ connects ζ to ξ.

Explaination

Corollary 2.1:-

(1) A symmetric polynomial inequality of degree $d \in \mathbb{N}^*$ holds on $\mathbb{R}^n_+$ if and only if it holds on $\{x \in \mathbb{R}^n_+ \mid v^*(x) \leq \lfloor d/2 \rfloor \vee 1\}$.

(2) A symmetric Polynomial inequality of even degree $d \in \mathbb{N}^*$ holds on $\mathbb{R}^n$ if and only if it holds on $\{x \in \mathbb{R}^n \mid v(x) \leq \lfloor d/2 \rfloor \vee 2\}$.

Examples :- (1) a) Let $d = 4$, $n = 2$.

symmetric polynomial Inequality of degree $d=4$ & $n=2$:-

$$2\left(x_1 x_2^3 + x_1^3 x_2\right) \geq -\left(x_1^4 + x_2^4\right)$$

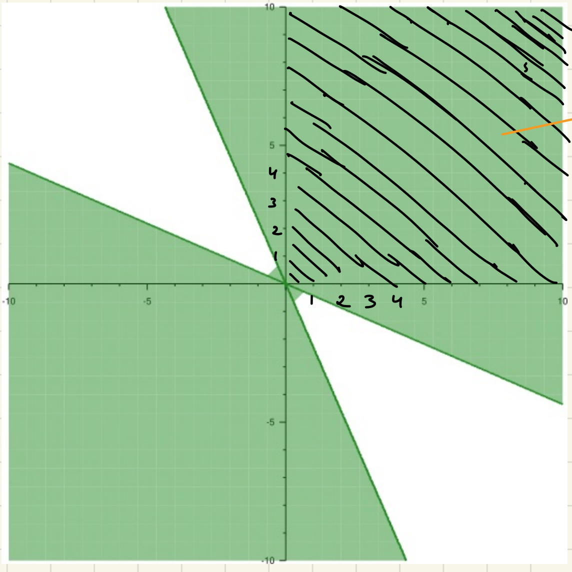
$$\lfloor d/2 \rfloor \vee 1 = 2$$

Assume $A := \{x \in \mathbb{R}^2_+ \mid v^*(x) \leq 2\}$

Example elements in A:

$(1,2)$	$(5,10)$	$(2,0)$	$(2,2)$	$(0,0)$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$v^*(x)=2$	$v^*(x)=2$	$v^*(x)=1$	$v^*(x)=1$	$v^*(x)=0$

Graph of the Inequality



The inequality holds on all $x \in \mathbb{R}^2_+$. Thus, it holds on $\mathbb{R}^+_2$.

b) Let $n=2$, $d=2$

Symmetric Polynomial inequality of degree $d=2$ & $n=2$:-

$$x_1 + x_2 - x_1 x_2 < 0$$

$$\lfloor d/2 \rfloor \vee 1 = \lfloor 1 \rfloor \vee 1 = 1$$

Assume $A := \{ x \in \mathbb{R}^2_+ \mid v^*(x) \leq 1 \}$

Example elements in A:-

$(1,1) \rightarrow v^*(x) = 1$

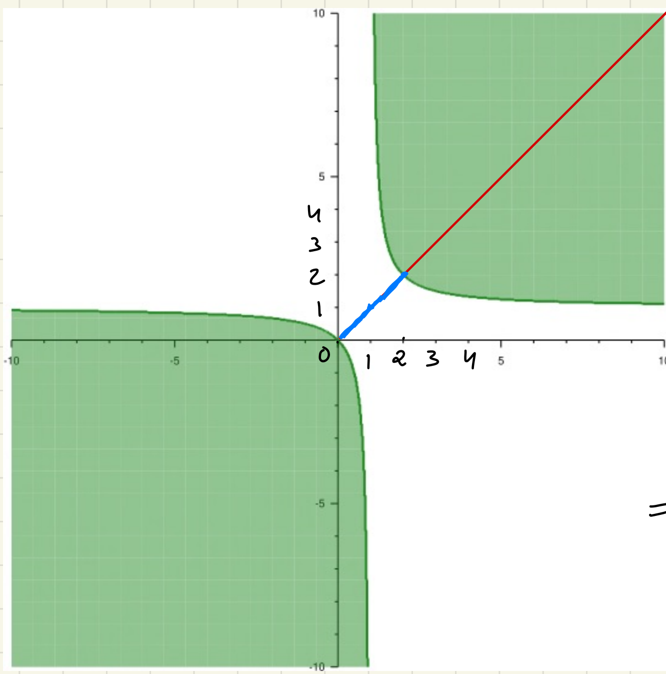
$(1,0) \rightarrow v^*(x) = 1$

$(\sqrt{2}, \sqrt{2}) \rightarrow v^*(x) = 1$

$(0, 1.5) \rightarrow v^*(x) = 1$

$(0,0) \rightarrow v^*(x) = 0$

Graph of the inequality:-



*Since the inequality doesn't hold on the points $(0,0)$, $(1,1)$, $(1.5,2)$, etc. (basically points on the blue line), the inequality does not hold for all elements of set A.

$\Rightarrow$ Therefore, the inequality does not hold on $\mathbb{R}^2_+$.

② a) Let $n = 3$, $d = 2$

Symmetric Polynomial inequality of degree $d=2$ & $n=3$:-

$$x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_2x_3 + x_3x_1) < 0$$

$$\lfloor d/2 \rfloor \vee \quad = \lfloor 1 \rfloor \vee 2 = \underline{2} \text{ (even degree)}.$$

Assume $A := \{ x \in \mathbb{R}^3 \mid v^*(x) \leq 2 \}$

<u>Example elements in A</u> :-

$(1,2,1)$ $\downarrow$ $v^*(x) = 2$

$(0,-1,5)$ $\downarrow$ $v^*(x) = 2$

$(0,0,1.5)$ $\downarrow$ $v^*(x) = 1$

$(-1,-1,-1)$ $\downarrow$ $v^*(x) = 1$

$(-1,-1,0)$ $\downarrow$ $v^*(x) = 1$

$(0,0,0)$ $\downarrow$ $v^*(x) = 0$

<u>Example Points where the inequality fails</u> :-

a) At $x = (1,1,1)$ $\quad (1)^2 + (1)^2 + (1)^2 - (1\cdot1 + 1\cdot1 + 1\cdot1) = 0 \not< 0.$

b) At $x = (0,-1,5)$ $\quad (0)^2 + (-1)^2 + (5)^2 - (0 + (-5) + 0) = 31 \not< 0.$

c) At $x = (0,0,0)$ $\quad 0^2 + 0^2 + 0^2 - (0+0+0) = 0 \not< 0.$

It is hard to draw the inequality graphically; But it is clear, from above, that the inequality doesn't hold on $\mathbb{R}^n$.

3. Symmetric Functions and Curved Simplices

Theorem 3.1:

$$\forall_{f \in \Sigma_d^{[n]}} \; \exists!_{\tilde{f}: \mathbb{R}^{\bar{d}} \to \mathbb{R}} \text{ s.t } \Big[\big(f = \tilde{f} \circ P_{(1,\bar{d})} = \tilde{f}(P_1, \dots, P_{\bar{d}}) \big) \wedge \Big(\exists_{g, g_i \in \mathbb{R}[x]} \text{ s.t } f = g(P_1, \dots, P_{\lfloor d/2 \rfloor}) + \sum_{i=\lfloor d/2 \rfloor + 1}^{d} g_i(P_1, \dots, P_{d-i}) \cdot P_i \Big) \Big]$$

$\bar{d} := d \wedge n$

Note:- $g_i \equiv 0$ if $i > \bar{d}$

Note:- f depends only affinely on each powersum P_i with $i > \lfloor d/2 \rfloor$.

Note:- An Affine function is basically a linear map plus a constant vector.

Definition 3.1:

(A) Let $n, s \in \mathbb{N}$.

$G_s^{[n]}$ is the set $\{ g : \mathbb{R}_+^n \setminus \{0_n\} \to \mathbb{R} \mid g = \bar{g} \circ P_{(1,s)} \}$ where $\bar{g}$ is a continuous function and $\bar{g} : (0, \infty)^s \to \mathbb{R}$.

By convention, $G_0^{[n]} = \mathbb{R}$.

(B) Let $n, d, s \in \mathbb{N}$, and $d \leq s$.

$F_{d,s}^{[n]}$ is the set $\{ f : \mathbb{R}_+^n \setminus \{0_n\} \to \mathbb{R} \mid f = g_s + \sum_{i=s+1}^{d} g_i P_i \}$

where $g_i, g_s \in G_s^{[n]}$, $\forall i \in \overline{s+1, d}$.

(C) Let $s \in \mathbb{N}^*$ and $n \in \mathbb{N}$. Assume $x \in \mathbb{R}^n_+ \setminus \{0_n\}$.

$\Omega_s^{[n]}$ denotes the set

$$\{ \omega : \mathbb{R}^n_+ \setminus \{0_n\} \longrightarrow (0,\infty) \,\big|\, (\omega \in G_s^{[n]}) \wedge (\omega_x(t) \text{ is increasing}) \wedge (\lim_{t \to 0} \omega_x(t) = 0) \wedge (\lim_{t \to \infty} \omega_x(t) = \infty) \}$$

where $\omega_x : (0,\infty) \longrightarrow (0,\infty)$, and $\omega_x(t) = \omega(tx)$.

$\boxed{\text{Note:-}}$

i) $\Omega_s^{[n]} \subset G_s^{[n]}$

ii) $\forall_{i \in \overline{1,s}} \; P_i \in \Omega_s^{[n]}$

iii) g, ω, f need not necessarily be polynomials.

Example 3.1

i) $g = 2^{P_2} + P_1^2 \log(1 + P_1^4 + P_2) \in G_2^{[n]} \setminus \Omega_2^{[n]}$

$\bar{g}(a,b) = 2^b + a^2 \log(1 + a^4 + b)$, where $a,b \in [0,\infty)$

$g = \bar{g} \circ P_{(1,2)} = \bar{g}(P_1, P_2)$

a) Let $n=2$, $x=(0,1)$

$P_1(x) = 0+1 = \underline{1}$
$P_2(x) = 0^2+1^2 = \underline{\underline{1}}$

$g(x) = \bar{g}(P_1(x), P_2(x)) = \bar{g}(1,1)$

$= 2^1 + 1^2 \log(1 + 1^4 + 1) = 2 + \log(3) = \underline{\underline{2.477}}$

ii) $f = \sqrt[3]{P_1 - P_2} \cdot P_4 + \log(P_1^2 + P_2) \cdot P_3 + P_1 \sin(P_1 P_2) \in F_{4,2}^{[n]}$.

$= g_2 + \sum_{i=3}^{4} g_i P_i$, where $g_2 = P_1 \sin(P_1 P_2) \in G_2^{[n]}$,

$g_3 = \log(P_1^2 + P_2) \in G_2^{[n]}$,

$g_4 = \sqrt[3]{P_1 - P_2} \in G_2^{[n]}$

a) Let $n=3$, $x = (1,2,0)$

$P_1(x) = 1+2+0 = 3$
$P_2(x) = 1^2+2^2+0^2 = 5$

$g_2(x) = 3\sin(3*5) = 3\sin(15) = \underline{\underline{0.777}}$

$g_3(x) = \log(3^2+5) = \log(14) = \underline{\underline{1.146}}$

$g_4(x) = \sqrt[3]{3-5} = -\sqrt[3]{2} = \underline{\underline{-1.26}}$

$P_3(x) = 1^3 + 2^3 + 0^3 = \underline{\underline{9}}$
$P_4(x) = 1^4 + 2^4 + 0^4 = \underline{\underline{17}}$

$$f(x) = \underbrace{0.777}_{g_2(x)} + (\underbrace{1.146}_{g_3(x)} * \underbrace{9}_{P_3(x)}) + (\underbrace{-1.26}_{g_4(x)} * \underbrace{17}_{P_4(x)}) = \underline{\underline{-10.329}}$$

$\boxed{\text{Observe:-}}$

a) $F_{d,d}^{[n]} = G_d^{[n]} \supset F_{d,s}^{[n]}$.

b) $\forall_{f \in F_{d,s}^{[n]}} \exists$ $\bar{f} : [0,\infty)^s \times \mathbb{R}_+^{d-s} \longrightarrow \mathbb{R}$ $\qquad f = \bar{f} \circ P_{(1,d)}$

$$\bar{f}(y) = \bar{g}_s(y_1, \dots, y_s) + \sum_{i=s+1}^{d} \bar{g}_i(y_1, \dots, y_s) y_i$$

iii) $w = 2^{P_2} - 1 + P_1^2 \log(1 + P_1^4 + P_2) \in \Omega_2^{[n]} \subset G_2^{[n]}$.

a) Let $n = 3$, $x = (2, 0, 0)$

$P_1(x) = \underline{\underline{2}}$
$P_2(x) = 2^2 = \underline{\underline{4}}$

$$w(x) = 2^4 - 1 + 2^2 \log(1 + 2^4 + 4) = 15 + 4\log(21) = \underline{\underline{20.29}}$$

$w_x(t) = w(tx) = w((2t, 0, 0))$.

$P_1((2t, 0, 0)) = 2t$
$P_2((2t, 0, 0)) = 4t^2$

$$w((2t, 0, 0)) = 2^{4t^2} - 1 + 4t^2 \log(1 + 16t^4 + 4t^2)$$

$w(x) \in \Omega_2^{[3]}$ as it satisfies the conditions:-

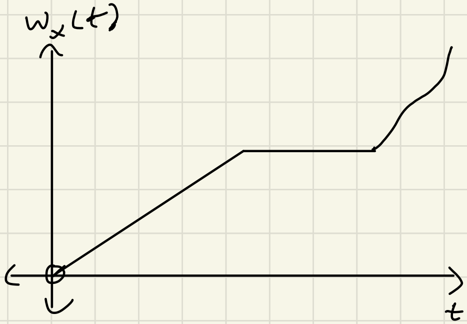
i) $\lim_{t \to 0} w((2t, 0, 0)) = 2^0 - 1 + 0 = \underline{\underline{0}}$

ii) $\lim_{t \to \infty} w(2t, 0, 0) = \underline{\underline{\infty}}$

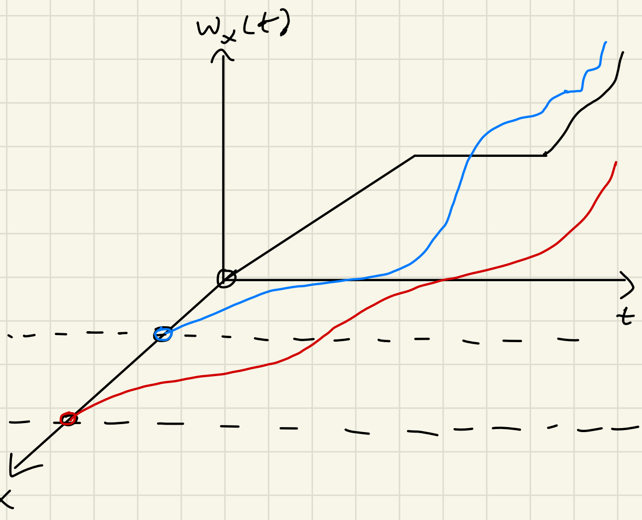
iii) $w_x(t)$ increases as t increases.

General Behavior of $w_x(t)$ and $w(x)$

(A) Graph of $w_x(t)$ for a fixed x value, and varying t value.



(B) Graph of $w_x(t)$ for varying x value, and varying t value.

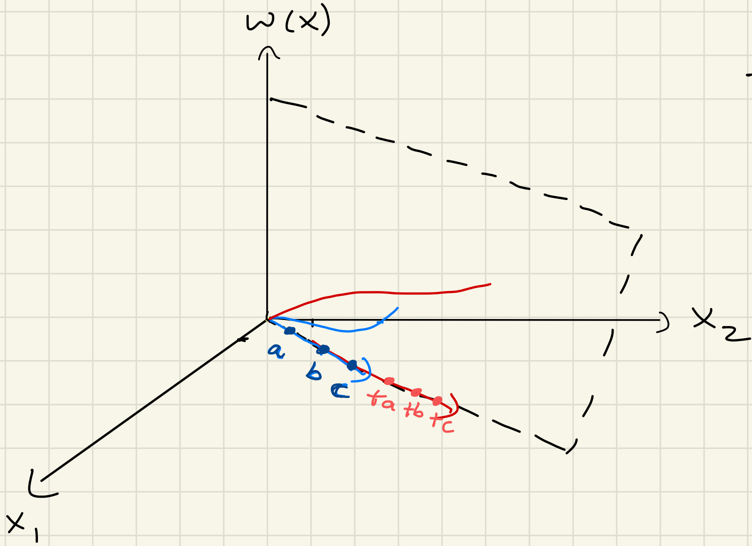


$\longrightarrow x \in \mathbb{R}$.

$\boxed{\text{Note:-}}$ $w_x(t) = w(tx)$ is increasing $\Rightarrow$ $w(x)$ is increasing.

when $t \geq 1$.

(C) Graph of $w(x)$ for fixed x values and fixed t value.



→ $a, b, c \in \mathbb{R}^2$.

→ $t > 1$

Remark 3.1 :-

(1) The family of all the sets $F_{d,s}^{[n]}$ is increasing with respect to d and s.

Every $G_s^{[n]}$ is an $\mathbb{R}$- algebra and every $F_{d,s}^{[n]}$ is a $G_s^{[n]}$-module.

(2) We have the natural inclusion $\Sigma_d^{[n]} \subset F_{d,\lfloor d/2 \rfloor}^{[n]}$. If $d \leq 2s+1$, then

$$\mathcal{H}_d^{[n]} \subset \Sigma_d^{[n]} \subset F_{d,s}^{[n]}.$$

Proposition 3.1 :-

$$\forall_{\substack{d \in \mathbb{N} \\ d \leq 2s+1}} \left(d \wedge n \leq 2(s \wedge n)+1 \right) \wedge \left(F^{[n]}_{d,s} = F^{[n]}_{d\wedge n, s \wedge n} \right) \wedge \left(G^{[n]}_{s} = G^{[n]}_{s \wedge n} \right) \wedge \left(\Omega^{[n]}_{s} = \Omega^{[n]}_{s \wedge n} \right)$$

Definition 3.2 :- Let $w \in \Omega^{[n]}_{s}$. Let $\sigma \in \mathbb{R}$ and $\sigma > 0$. Let f be a continuous function where

$f : \mathbb{R}^n_+ \setminus \{0_n\} \longrightarrow \mathbb{R}$. Let $s \in \mathbb{N}^*$.

(A) $K_\sigma(w) := \{ x \in \mathbb{R}^n_+ \mid w(x) = \sigma \}$

(B) $K^s_\sigma(w) := \{ x \in K_\sigma(w) \mid v^*(x) \leq s \}$

(C) $M_\sigma(f, w) := \{ \xi \in K_\sigma(w) \mid f(\xi) = \min_{x \in K_\sigma(w)} f(x) \}$

Example 3.2 :-

The following are the given information:-

i) $w(x) = P_1(x) + P_2(x)$

$w(x)$ belongs to $\Omega_2^{[2]}$ as it satisfies the following conditions:-

a) $w(x) = P_1(x) + P_2(x) \in G_2^{[2]}$

Proof:- $w(x)$ is of the form $w(x) = \bar{g} \circ P_{(1,2)}$
$= \bar{g}(P_1, P_2)$

where $\bar{g} : (0,\infty)^2 \to \mathbb{R}$, and

$\bar{g}(a,b) = a+b.$

$w(x) = P_1(x) + P_2(x) = \bar{g}(P_1(x), P_2(x)).$

∎

b) $w_x(t)$ is increasing, where $w_x : (0,\infty) \to (0,\infty)$ and $w_x(t) = w(tx)$

Proof:- $w_x(t) = w(tx) = P_1(tx) + P_2(tx)$

$$= tx_1 + tx_2 + t^2x_1^2 + t^2x_2^2$$

since $t \in (0,\infty)$ and $(x_1, x_2) \in \mathbb{R}^2$,

$w_x(t)$ is an increasing function.

c) $\lim_{t \to 0} w_x(t) = 0$, where $w_x : (0,\infty) \to (0,\infty)$ and $w_x(t) = w(tx)$

<u>Proof</u> :- $w_x(t) = tx_1 + tx_2 + t^2x_1^2 + t^2x_2^2$

$$\lim_{t \to 0} w_x(t) = \underline{\underline{0}}$$

d) $\lim_{t \to \infty} w_x(t) = \infty$, where $w_x : (0,\infty) \to (0,\infty)$ and $w_x(t) = w(tx)$

<u>Proof</u> :- $w_x(t) = tx_1 + tx_2 + t^2x_1^2 + t^2x_2^2$

$$\lim_{t \to \infty} w_x(t) = \underline{\underline{\infty}}$$

Therefore, $w(x) \in \Omega_2^{[2]}$.

ii) $\sigma = 2$

iii) $s = 2$

iv) $f(x) = P_1(x) + P_2(x) + P_1(x) \cdot P_3(x)$

$f(x)$ belongs to $F_{3,2}^{[2]}$ as it satisfies the following condition:-

a) $f(x) = g_2(x) + \sum_{i=3}^{3} g_i(x) P_i(x)$

$$= g_2(x) + g_3(x) P_3(x)$$

$$= \bar{g}_2(P_1(x), P_2(x)) + \bar{g}_3(P_1(x), P_2(x)) \cdot P_3(x)$$

where $\bar{g}_2 : (0,\infty)^2 \to R$ and $\bar{g}_2(a,b) = a+b$

$\bar{g}_3 : (0,\infty)^2 \to R$, and $\bar{g}_3(a,b) = a$

Therefore,

$$f(x) = \left(P_1(x) + P_2(x)\right) + P_1(x)\cdot P_3(x) = g_2(x) + g_3(x) P_3(x).$$

Using the above given information let us define the sets $K_2(w)$, $K_2^2(w)$ and $M_2(f,w)$.

(A) $K_2(w) = \{ x \in R_+^2 \mid w(x) = 2 \}$

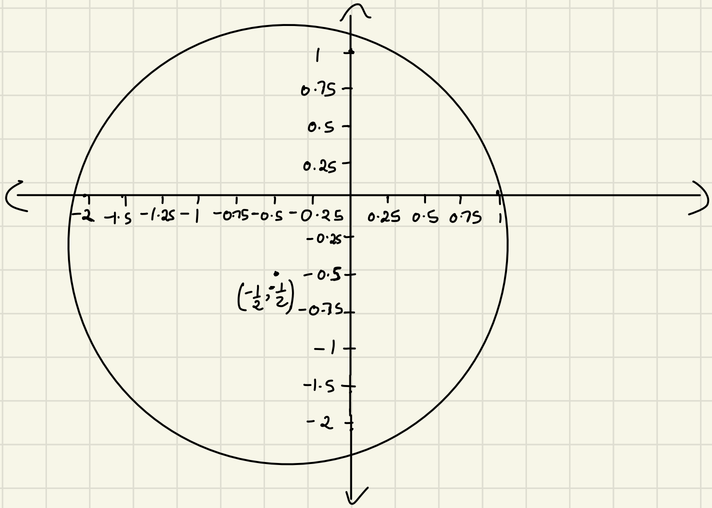
$$= \{ x \in R_+^2 \mid x_1 + x_2 + x_1^2 + x_2^2 = 2 \}$$

The equation $x_1 + x_2 + x_1^2 + x_2^2 = 2$ can be written as,

$$\left(x_1 + \frac{1}{2}\right)^2 + \left(x_2 + \frac{1}{2}\right)^2 = \frac{5}{2}$$

The above equation is a circle equation with center $(h,k) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$ and radius $r = \sqrt{\frac{5}{2}}$.

Graph of the circle equation :-



All the points on the above circle that lie in I^{st} Quadrant belong to $K_2(w)$.

(B) $K_2^2(w) := \{x \in K_2(w) \mid v^*(x) \leq 2\}$

$$= K_2(w).$$

Example elements in $K_2^2(w)$:-

$(1.08, 0)$, $(0.5, 0.73)$

$\downarrow$ $\downarrow$

$v^*(x) = 1$ $v^*(x) = 2$

(C) $M_2(f, w) := \{\xi \in K_2(w) \mid f(\xi) = \min_{x \in K_2(w)} f(x)\}$

To find out all the elements in $M_2(f, w)$, we need to find out $\min_{x \in K_2(w)} f(x)$.

$$f(x) = P_1(x) + P_2(x) + P_1(x) \cdot P_3(x)$$
$$= x_1 + x_2 + x_1^2 + x_2^2 + (x_1 + x_2) \cdot (x_1^3 + x_2^3)$$
$$= x_1 + x_2 + x_1^2 + x_2^2 + x_1^4 + x_1 x_2^3 + x_2 x_1^3 + x_2^4.$$

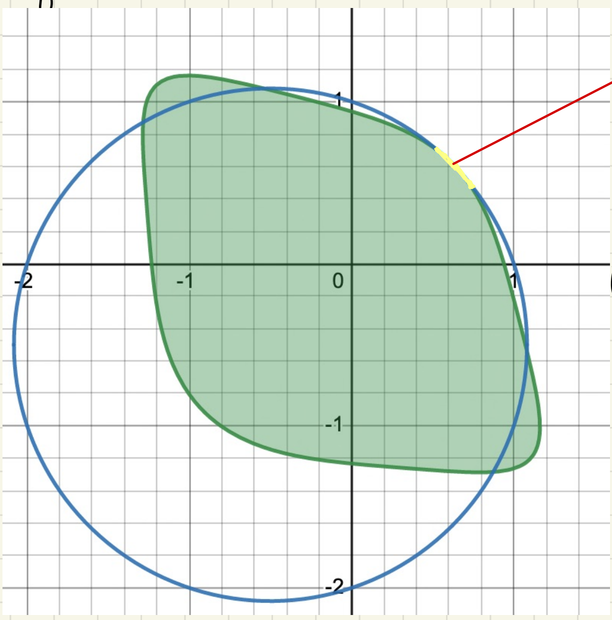
Evaluating $f(x)$ at $(0.62, 0.62)$,

$f((0.62, 0.62)) = \underline{\underline{2.599}}$

To check if $f((0.62, 0.62)) = \min_{x \in K_2(w)} f(x)$, let us look at the intersection of i) The graph of the circle equation and ii) The graph of the inequality

$$x_1 + x_2 + x_1^2 + x_2^2 + x_1^4 + x_1 x_2^3 + x_2 x_1^3 + x_2^4 \leq 2.599$$

<u>Graph of the intersection:-</u>



The yellow line denotes the set of value where $f(x)$ is minimum.

$\boxed{\text{Note}}$:- $\min_{x \in K_2(w)} f(x) = \underline{\underline{2.599}}$

Thus,
$M_2(f, w) = \{ x \in \mathbb{R}_2^+ \mid x \text{ lies on the yellow line} \}$

4. Construction of s-paths.

Lemma 4.1 := $\forall$ $\xi \in (0,\infty)^n$, $r := v^*(\xi) \geq 2$ $\exists$ $I =]\alpha, \beta[$ $\exists$ $\varphi : \overline{I} \to \mathbb{R}^n_+$ satisfying the following:-

i) $t_0 := P_r(\xi) \in I$ and $\varphi(t_0) = \xi$.

ii) φ is continuous on $\overline{I}$ and $\varphi(I) \subset (0,\infty)^n$.

iii) $P_i \circ \varphi \equiv P_i(\xi)$ for each $i < r$, but $P_r(\varphi(t)) = t$ for every $t \in \overline{I}$.

iv) $v^*(\varphi(t)) = r$ for every $t \in I$, but $v^*(\varphi(\alpha)) < r$ and $v^*(\varphi(\beta)) < r$.

v) $\varphi(a) \in (0,\infty)^n$ or $\varphi(\beta) \in (0,\infty)^n$.

vi) $\varphi \in C^\infty(I, \mathbb{R}^n)$, $\varphi_j' \neq 0$ on I, $\forall j \in \overline{1,n}$ and $(P_K \circ \varphi)' > 0$, $\forall K \geq r$

vii) If $r = n$, then $P_K \circ \varphi$ is an affine function for each $K < 2n$ and

$$(P_K \circ \varphi)' \equiv \frac{K}{n} h_{K-n}(\xi), \ \forall K \in \overline{n, 2n-1}.$$

Example for Lemma 4.1 :-

I) Let $n = 4$, $\xi = (1, 0.5, 0.5, 1)$

$\boxed{\text{Note :-}}$ $r := v^{*}(\xi) = 2 \geq 2$.

For the above value of ξ, we take

a) $\alpha = 0$, $\beta = 5$

b) $I =]\alpha, \beta[=]0, 5[$

c) $\bar{I} = [0, 5]$

d) $\varphi : \bar{I} \to \mathbb{R}_+^4$ where $\varphi(x) = \begin{cases} \varphi(x) = (0.5, 0.5, 0.5, 0.5) & \text{for } x = \alpha = 0. \\ \varphi(x) = (1, 0.5, 0.5, 1) & \text{for } 0 < x < 5. \\ \varphi(x) = (1, 1, 1, 1) & \text{for } x = \beta = 5. \end{cases}$

Now let us observe if the conditions of Lemma 4.1 are satisfied :-

Condition i) :- a) $P_r(\xi) \in I$

b) $\varphi(P_r(\xi)) = \xi$

Result :- SATISFIED

Proof :- a) $P_r(\xi) = P_2((1, 0.5, 0.5, 1)) = 1^2 + \frac{1}{2}^2 + \frac{1}{2}^2 + 1^2 = \underline{\underline{2.5}} \in]0, 5[= I$

b) $\varphi(P_r(\xi)) = \varphi(2.5) = (1, 0.5, 0.5, 1)$.

$\blacksquare$

Condition ii) :- a) φ is continuous on $\bar{I}$
b) $\varphi(I) \in (0,\infty)^n$

Result :- SATISFIED

Proof :- a) Based on the definition of φ, $\varphi(x)$ is defined $\forall x \in [0,5]$. Therefore φ is continuous on $\bar{I}$.

b) $\forall x \in [0,5] = \bar{I}$, $\varphi(x) \in (0,\infty)^4$.

$I \subset \bar{I} \implies \varphi(I) \in (0,\infty)^4$.

$\blacksquare$

Condition iii) :- a) $P_i \circ \varphi \equiv P_i(\xi)$, $\forall i < r$
b) $P_r(\varphi(t)) = t$, $\forall t \in \bar{I}$

Result :- NOT SATISFIED

Reason :- a) $P_1(\varphi(0)) = 0.5 + 0.5 + 0.5 + 0.5 = \boxed{2}$

$$\neq$$

$P_1(\xi) = 1 + 0.5 + 0.5 + 1 = \boxed{3}$

Condition iv) :- a) $v^*(\varphi(t)) = r$, $\forall t \in I$

b) $[v^*(\varphi(\alpha)) < r] \wedge [v^*(\varphi(\beta)) < r]$

Result :- SATISFIED

Proof :- a) $\forall t \in I$, $v^*(\varphi(t)) = v^*((1, 0.5, 0.5, 1)) = \underline{\underline{2 = r}}$

b) $v^*(\varphi(\alpha)) = v^*((0.5, 0.5, 0.5, 0.5)) = \underline{\underline{1 < 2 = r}}$

$v^*(\varphi(\beta)) = v^*((1,1,1,1)) = \underline{\underline{1 < 2 = r}}$

$\blacksquare$

Condition v) :- a) $[\varphi(\alpha) \in (0,\infty)^n] \vee [\varphi(\beta) \in (0,\infty)^n]$

Result :- SATISFIED

Proof :- a) $\varphi(\alpha) = \varphi(0) = (0.5, 0.5, 0.5, 0.5) \in (0, \infty)^4$.

▨

Condition vi) :- a) $\varphi \in C^\infty(I, \mathbb{R}^n)$
b) $\varphi_j' \neq 0$ on I, $\forall j \in \overline{1,n}$
c) $(P_k \circ \varphi)' > 0$, $\forall k \geq r$

$\boxed{\text{Note}}$:- $C^\infty(I, \mathbb{R}^n)$ is the set of all functions whose $1^{st}, 2^{nd} \dots \infty^{th}$ derivatives are continuous functions mapped from interval I to $\mathbb{R}^n$.

Result :- NOT SATISFIED

Reason :- a) Since φ is always equal to $(1, 0.5, 0.5, 1)$ on the interval $I = (0, 5)$, all the derivatives of φ are equal to $(0,0,0,0)$. Therefore $\varphi \in C^\infty(I, \mathbb{R}^4)$.

b) Based on the definition of φ, $\varphi_j' = 0$, $\forall j \in \overline{1,4}$.

c) $(P_k \circ \varphi)' = 0$, $\forall k \geq 2$.

Condition vii) :- If $r = n$, then $P_k \circ \varphi$ is an affine function for each $k < 2n$ and

$$(P_k \circ \varphi)' \equiv \frac{k}{n} h_{k-n}(\xi_0), \quad \forall k \in \overline{n, 2n-1}$$

Result :- NOT APPLICABLE

Reason :- Hypothesis $(r = n)$ is not satisfied.

II) Let $n=2$, $\xi_0 = (1,2)$

$\boxed{\text{Note}}$:- $r := v^*(\xi_0) = 2 \geq 2$

For the above value of ξ_0, we need to find a bounded interval $\bar{I} = (\alpha, \beta)$ and a function $\varphi : \bar{I} \to \mathbb{R}^n_+$.

a) $\varphi(t) = (\varphi_1(t), \varphi_2(t)) = \left(\dfrac{6 - \sqrt{8t-36}}{4}, \dfrac{6 + \sqrt{8t-36}}{4} \right)$

b) $\bar{I} = (\frac{9}{2}, 11)$

c) $\alpha = \frac{9}{2}$, $\beta = 10$

d) $I = (\alpha, \beta) = (\frac{9}{2}, 10) \subset \bar{I} = (\frac{9}{2}, 11)$.

<u>Question</u>:- How did you find the function $\varphi : \bar{I} \to \mathbb{R}^2_+$?

<u>Answer</u> :- I needed a function $\varphi : \bar{I} \to \mathbb{R}^2_+$ which satisfied the conditions:-

i) $\forall_{t \in \bar{I}} \; (P_1 \circ \varphi)(t) = 3$

ii) $\forall_{t \in \bar{I}} \; P_2(\varphi(t)) = t$

Let $\varphi(t) = (\varphi_1(t), \varphi_2(t)) = (z_1, z_2)$, $\forall\, t \in \bar{I}$.
The equations are:-

i) $z_1 + z_2 = 3$
ii) $z_1^2 + z_2^2 = t$

Solving the above equations, we get

$$(3 - z_2)^2 + z_2^2 = t \Rightarrow 2z_2^2 - 6z_2 + (9-t) = 0$$

$$z_2 = \frac{6 \pm \sqrt{36 - 8(9-t)}}{4} = \frac{6 \pm \sqrt{8t - 36}}{4}, \quad z_1 = 3 - z_2.$$

we take,

$$z_1 = \frac{6 - \sqrt{8t-36}}{4}, \quad z_2 = \frac{6 + \sqrt{8t-36}}{4}$$

Thus, we define $\varphi : I \to \mathbb{R}_+^2$ as $\varphi(t) = \left(\frac{6 - \sqrt{8t-36}}{4}, \frac{6 + \sqrt{8t-36}}{4} \right)$

$\boxed{\text{Note}}$:- z_1 & z_2 are defined only when $8t - 36 \geq 0 \Rightarrow t \geq \frac{9}{2}$.

Now let us observe if the conditions of Lemma 4.1 are satisfied:-

Condition i) :- a) $P_r(\xi_0) \in I$

b) $\varphi(P_r(\xi_0)) = \xi_0$

Result :- SATISFIED

Proof :- a) $P_2((1,2)) = 1^2 + 2^2 = 5 \in (\frac{9}{2}, 10) = I$

b) $\varphi(P_r((1,2))) = \varphi(5) = \left(\frac{6 - \sqrt{40-36}}{4}, \frac{6 + \sqrt{40-36}}{4} \right)$

$= (1,2) = \xi_0$.

$\boxed{\text{Note}}$:- If we defined $\varphi(t)$ as $\varphi(t) = \left(\frac{6 + \sqrt{8t-36}}{4}, \frac{6 - \sqrt{8t-36}}{4} \right)$,

then $\varphi(5)$ would be $(2,1) \neq \xi_0$. In that case, condition i) would not have been satisfied.

Condition ii) :- a) φ is continuous on $\bar{I}$
b) $\varphi(I) \in (0,\infty)^n$

Result :- SATISFIED

Proof :- a) Based on the definition of φ, $\varphi(t)$ is defined $\forall\, t \in (\frac{9}{2}, 11) = \bar{I}$. Therefore φ is continuous on $\bar{I}$.

b) $\forall x \in (\frac{9}{2}, 11) = \bar{I}$, $\varphi(x) \in (0,\infty)^2$.

$$I \subset \bar{I} \Rightarrow \varphi(I) \in (0,\infty)^2.$$

$\blacksquare$

Condition iii) :- a) $P_i \circ \varphi \equiv P_i(\xi)$, for each $i < r$.
b) $P_r(\varphi(t)) = t$, $\forall\, t \in \bar{I}$.

Result :- SATISFIED

Proof :- We know that $r := \nu^*(\xi) = \underline{\underline{2}}$
Let $t \in \bar{I} = (\frac{9}{2}, 11)$.

i) For $i = 1 < 2 = r$,

$$(P_1 \circ \varphi)(t) = \varphi_1(t) + \varphi_2(t) = \left(\frac{6 - \sqrt{8t-36}}{4}\right) + \left(\frac{6 + \sqrt{8t-36}}{4}\right)$$
$$= 3$$
$$= \underline{\underline{P_1(\xi)}}$$

ii) $$P_2(\varphi(t)) = \varphi_1(t)^2 + \varphi_2(t)^2 = (\varphi_1(t) + \varphi_2(t))^2 - 2\varphi_1(t)\varphi_2(t)$$
$$= 9 - 2\left(\frac{6 - \sqrt{8t-36}}{4}\right)\left(\frac{6 + \sqrt{8t-36}}{4}\right)$$
$$= 9 - \frac{(36 - (8t - 36))}{8}$$
$$= \frac{72 - (72 - 8t)}{8} = \underline{\underline{t}}$$

$\blacksquare$

Condition iv) :- a) $v^*(\varphi(t)) = r \quad , \forall\ t \in I =]\alpha, \beta[$

b) $v^*(\varphi(\alpha)) < r$ and $v^*(\varphi(\beta)) < r$

Result :- NOT SATISFIFD

Reason :- b) $I =]\alpha, \beta[= (\frac{9}{2}, 10)$

$$\varphi(t) = \left(\frac{6 - \sqrt{8t-36}}{4}, \frac{6 + \sqrt{8t-36}}{4}\right)$$

$$v^*(\varphi(\beta)) = v^*\left(\left(\frac{6 - \sqrt{80-36}}{4}, \frac{6 + \sqrt{80-36}}{4}\right)\right)$$

$$= v^*\left(\left(\frac{6 - \sqrt{44}}{4}, \frac{6 + \sqrt{44}}{4}\right)\right) = 2 \not< 2 = r$$

Therefore condition iv) is not satisfied.

$\boxed{\text{Note}}$:- $v^*(\varphi(t)) < 2$ only when $\sqrt{8t-36} = 0$.

Condition v) :- $\varphi(\alpha) \in]0, \infty[^n$ and $\varphi(\beta) \in]0, \infty[^n$

Result :- SATISFIFD

Proof :- $I =]\alpha, \beta[= (\frac{9}{2}, 10)$

Based on the definition of φ,

i) $\varphi(\frac{9}{2}) \in (0, \infty)^2$

ii) $\varphi(10) \in (0, \infty)^2$

$\blacksquare$

Condition vi) :- a) $\varphi \in C^{\infty}(I, \mathbb{R}^n)$

b) $\varphi_j' \neq 0$ on I, $\forall j \in \overline{1,n}$

c) $(P_k \circ \varphi)' > 0$, $\forall k \geq r$

<u>Result</u> :- SATISFIFD

<u>Proof</u> :- a) $$\varphi(t) = \left(\frac{6 - \sqrt{8t-36}}{4}, \frac{6 + \sqrt{8t-36}}{4} \right) \in C(I, \mathbb{R}^2)$$

$$\varphi'(t) = \left(\frac{-1}{\sqrt{8t-36}}, \frac{1}{\sqrt{8t-36}} \right) \in C(I, \mathbb{R}^2)$$

$$\varphi''(t) = \left(\frac{4}{(8t-36)^{3/2}}, \frac{-4}{(8t-36)^{3/2}} \right) \in C(I, \mathbb{R}^2)$$

$$\vdots$$

Thus, $\varphi \in C^{\infty}(I, \mathbb{R}^2)$.

b) $$\varphi(t) = (\varphi_1(t), \varphi_2(t))$$

$$= \left(\frac{6 - \sqrt{8t-36}}{4}, \frac{6 + \sqrt{8t-36}}{4} \right)$$

$$\varphi_1'(t) = \frac{-1}{\sqrt{8t-36}} \neq 0, \ \forall t \in I = \left(\tfrac{9}{2}, 10\right)$$

$$\varphi_2'(t) = \frac{1}{\sqrt{8t-36}} \neq 0, \ \forall t \in I = \left(\tfrac{9}{2}, 10\right)$$

Thus, $\varphi_j' \neq 0$, $\forall j \in \overline{1,2}$.

c) $\forall k \geq r = 2$,

$$(P_k \circ \varphi)(t) = \varphi_1^k(t) + \varphi_2^k(t)$$

$$(P_k \circ \varphi)'(t) = \left[k\, \varphi_1^{k-1}(t) \times (\varphi_1'(t)) \right] + \left[k\, \varphi_2^{k-1}(t) \times (\varphi_2'(t)) \right]$$

Since $\varphi_1'(t) = -\varphi_2'(t)$,

$$(P_k \circ \varphi)'(t) = \left[-k\,\varphi_1^{k-1}(t) \times (\varphi_2'(t))\right] + \left[k\,\varphi_2^{k-1}(t) \times (\varphi_2'(t))\right]$$

Since $\varphi_2(t) > \varphi_1(t)$, $\forall\, t \in I$, it can be concluded that $(P_k \circ \varphi)' > 0$, for every $k \geq r$.

$\blacksquare$

Condition vii) :- If $r = n$,

$$\left(\forall_{k \in \overline{n, 2n-1}} \quad (P_k \circ \varphi)' \equiv \frac{k}{n} h_{k-n}(\xi_0) \right)$$

$$\wedge$$

$\left(\forall_{k < 2n} \quad P_k \circ \varphi \text{ can be expressed as a linear map plus a constant vector.} \right)$

Result :- NOT SATISFIED

Reason :- i) $(P_k \circ \varphi)'$ is not a constant whereas $\frac{k}{n} h_{k-n}(\xi_0)$ is a a constant.

ii) $P_k \circ \varphi$ cannot be expressed as a linear map because of the term $\sqrt{8t-36}$ in $\varphi_1(t)$ and $\varphi_2(t)$.

5. Minimizers on ω-curved Simplices

5.1 Symmetric Inequalities on $\mathbb{R}_+^n$

Context :- Unless otherwise stated, we will always consider the following setting for subsection 5.1 :-

i) $f \in \mathcal{F}_{d,s}^{[n]}$

ii) $\omega \in \Omega_s^{[n]}$

iii) $\sigma > 0$

iv) $\xi \in M_\sigma(f, \omega)$

Furthermore, other than for Lemma 5.1, we require $d \le 2s+1$.

Important Note:- i) The PURPOSE of Section 5 is to investigate the set $M_\sigma(f, \omega)$.

ii) Important cases that we will observe are:-

a) $f \in \Sigma_d^{[n]}$ and $\omega = P_1$.

b) $f \in \mathcal{H}_n^{[d]}$ and $\omega \in \mathcal{H}_d^{[n]}$.

Remark 5.1 :- Let $\gamma : [a,b] \to \mathbb{R}_+^n \setminus \{0_n\}$ be an (s)-path.

Then, every function from $G_s^{[n]} \supset \Omega_s^{[n]}$ is constant on $\gamma([a,b])$.

Also, if $f \in \mathcal{F}_{d,s}^{[n]}$,

$$\forall_{\zeta \in \gamma([a,b])} \quad f \circ \gamma = g_s(\zeta) + \sum_{i=s+1}^{d} g_i(\zeta)(P_i \circ \gamma).$$

Question :- How do you construct an (s)-path ?

(Answer) :- To construct an (s)-path you just need to construct a path $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ which satisfies the following 2 conditions:-

i) $P_i \circ \gamma \equiv (P_i \circ \gamma)(a)$, $\forall\, i \in \overline{1,s}$.

ii) $\mathrm{supp}(\gamma(t)) = \mathrm{supp}(\gamma(a)), \forall\, t \in [a,b]$.

$\boxed{\text{Note}}$:- see Example 5.1 for an example of (s)-path.

Theorem 5.1 (Of Enlargement) :-

$\forall_{v^*(\xi) > s}$ $\forall_{\varepsilon > 0}$ ξ is connected by an injective (s)-path in $M_\sigma(f,w) \cap B(\xi, \varepsilon)$ to a point $\zeta \neq \xi$ satifying $v^*(\zeta) = |\mathrm{supp}(\zeta)|$.

Example 5.1 :- i) Let us take the following initial conditions:-

a) $n = 2$

b) $s = 1$

c) $d = 3 \leq 3 = 2s+1$

d) $f = x_1^3 + x_2^3 + 3x_1 x_2^2 + 3x_2 x_1^2$

$\boxed{\text{Note}}$:- $f \in H_3^{[2]} \subset \Sigma_3^{[2]} \subset F_{3,1}^{[2]}$.

$\boxed{\text{Note}}$:- The above inclusion is true only because $d \leq 2s+1$.

e) $w = x_1^2 + x_2^2 + 2x_1x_2$

$$= P_1^2$$

$$= g_w \circ P_{(1,1)} \in G_1^{[2]}, \text{ where } g_w(t) = t^2.$$

$\Rightarrow$ Also, note that w belongs to $\Omega_1^{[2]}$ as it satisfies all the required conditions to belong to $\Omega_1^{[2]}$.

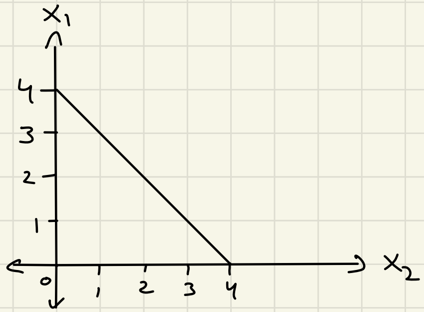
$\boxed{\text{Note}}$:- $w \in \mathcal{H}_2^{[2]}$.

f) $\sigma = 16$

g) $K_{16}(w) = \{ x \in \mathbb{R}_+^2 \mid w = 16 \}$

$$= \{ x \in \mathbb{R}_+^2 \mid P_1 = 4 \}$$

Graph of $K_{16}(w)$:-



h) $M_{16}(f, w) = \{ \xi \in K_{16}(w) \mid f(\xi) = \min_{x \in K_{16}(w)} f(x) \}$

we know that $f = x_1^3 + x_2^3 + 3x_1x_2^2 + 3x_1^2x_2$

$$= P_1^3$$

$$= (x_1 + x_2)^3$$

since x_1+x_2 is always equal to 4, $M_{16}(f,w)=K_{16}(w)$.

i) $\gamma:[5,10]\longrightarrow\mathbb{R}^2$, where γ is a (1)-path.

For γ to be a 1-path it needs to satisfy the following conditions:-

i) $P_1\circ\gamma\equiv(P_1\circ\gamma)(5)$

ii) $\text{supp}(\gamma(t))=\text{supp}(\gamma(5))$, $\forall\, t\in[5,10]$.

Let $\gamma(t)=(\gamma_1(t),\gamma_2(t))$. Thus, we have

i) $\gamma_1(t)+\gamma_2(t)=\gamma_1(5)+\gamma_2(5)$, $\forall\, t\in[5,10]$.

ii) $\text{supp}((\gamma_1(t),\gamma_2(t)))=\text{supp}((\gamma_1(5),\gamma_2(5)))$, $\forall\, t\in[5,10]$.

Let $\gamma(t)=\left(\varphi_1(t)-\frac{3}{2},\ \varphi_2(t)-\frac{3}{2}\right)=\left(-\frac{\sqrt{8t-36}}{4},\ \frac{\sqrt{8t-36}}{4}\right)$

$\boxed{\text{Note}}$:- $\varphi(t)$ has been defined in Example II for Lemma 4.1.

To check if $\gamma(t)$ is a (1)-path,

i) $\gamma_1(t)+\gamma_2(t)=0=\gamma_1(5)+\gamma_2(5)$, $\forall\, t\in[5,10]$.

ii) $\text{supp}(\gamma(t))=2=\text{supp}(\gamma(5))$, $\forall\, t\in[5,10]$.