PROJECT#2 Write-up

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Part I): Mass-Spring-Dashpot ODE model:

Description: In this part, we are required to compute scaled sensitivities of the response variable at a Nominal value of the parameters that are involved in the Mass-Spring-Dashpot system. The parameters involved are m(mass), k(spring constant), c(Damping constant) and A(initial displacement of mass). The response variable is: $y(t,\theta)$(displacement of the mass at time t). For this project, the Mass-Spring-Dashpot system is formulated as an initial value problem (IVP) for the following system of ODEs:

$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = -\frac{k}{m} * y_1 + \frac{c}{m} * y_2$$

along with the initial condition:$[y_1(0), y_2(0)]\text{^}T = [A, 0]\text{^}T$.
Here, $y_1(t) = y(t)$ is the displacement of the mass, and $y_2(t) = y'(t)$ is the velocity of the mass and A is its initial displacement. To solve the above system of ODE’s, we use the MATLAB ode solver ‘ode45’. And to solve for the scaled sensitivities, we use two different numerical approximation approaches, which are: (i)Forward Finite Difference approximation and (ii)Centered Finite Difference approximation.

Note: In our case, the scaled sensitivities are found by first perturbing the nominal values of our parameters(by $\alpha\%$), and then creating a linear mapping from the perturbed values to a unit hypercube. Then, we find the scaled sensitivities by taking derivatives of the appropriate response variable with respect to the appropriate parameter.

Scripts written by me:

- Dampedforwardfinite.m : Implements the Forward finite difference approximation technique for computing scaled sensitivities(with perturbation $\alpha = 0.1$) for $t \in [0,8]$ starting with h=0.1(in Forward finite difference formula), m = 1 kg, k = 1 N/m, A = 0.1 m and c = 2.3 kg/s.
- Dampedcenteredfinite.m : Implements the Centered finite difference approximation technique for computing scaled sensitivities(with perturbation $\alpha = 0.1$) for $t \in [0,8]$ starting with h=0.1(in Centered finite difference formula), m = 1 kg, k = 1 N/m, A = 0.1 m and c = 2.3 kg/s.

Additional Script:

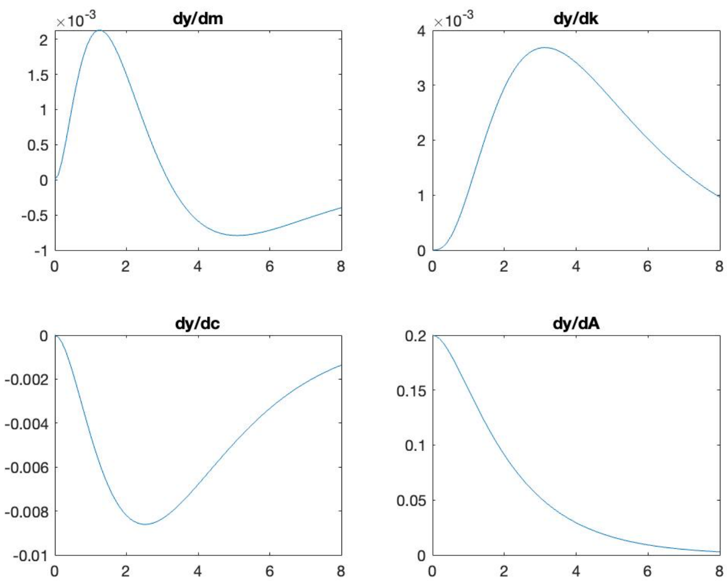
- fOD.m : Contains the explicit formula for the overdamped case of the Mass-Spring-Dashpot system.

Plots:

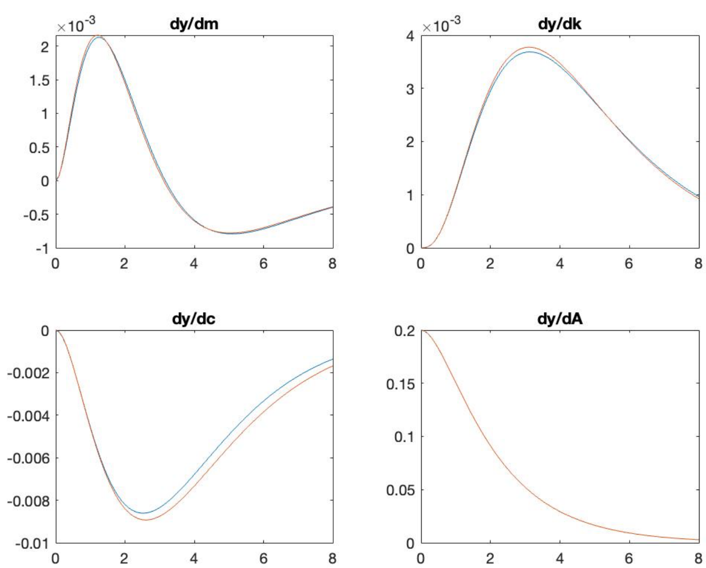
.. Plots of the Scaled sensitivities(y-axis) vs Time(x-axis) at different values of h using Forward Finite difference approximation:

Note: The plots are curves with different colors indicating how the scaled sensitivity plot changes until it converges.

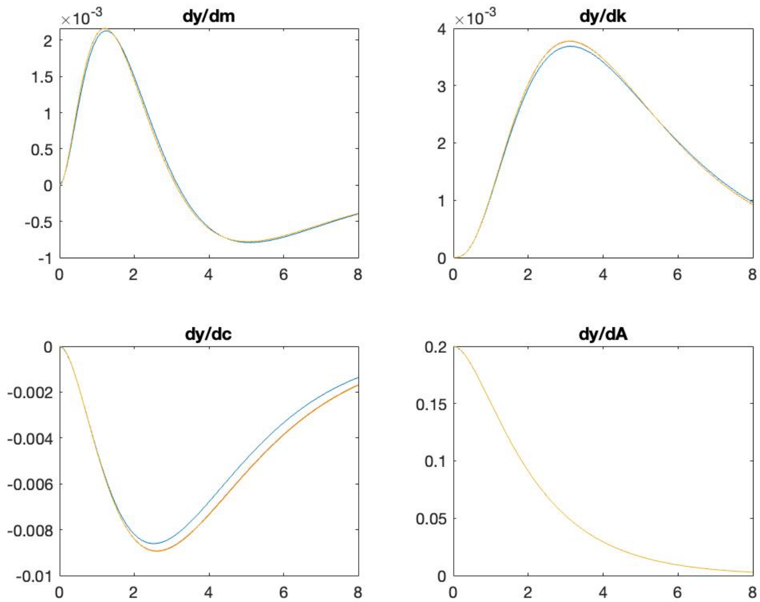
a)h=0.1:



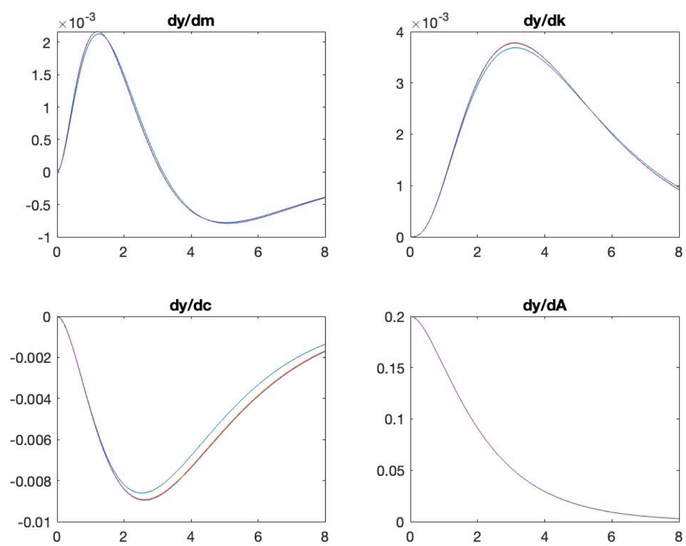
b)h=0.01



c)h=0.005(converged):



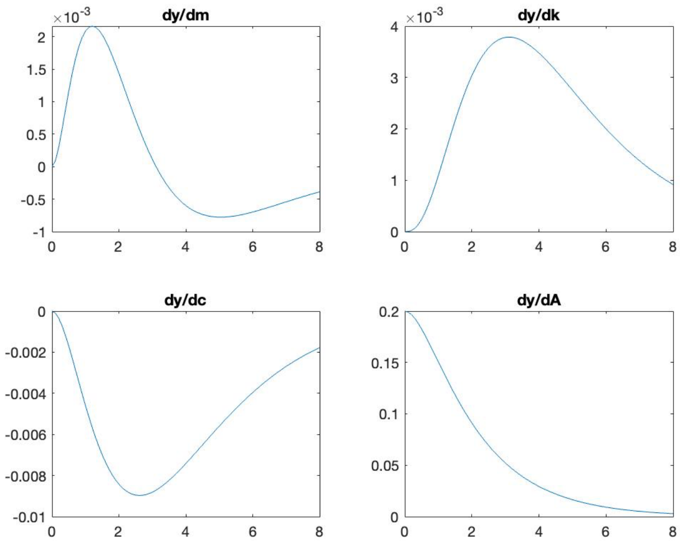
d)h=0.0025



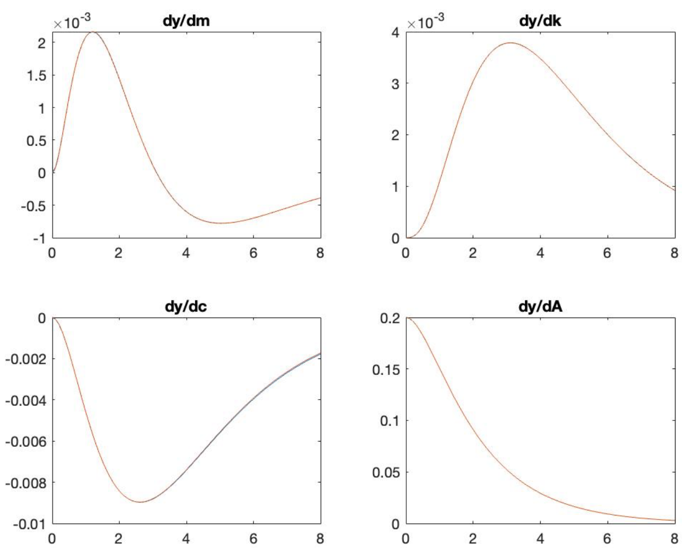
2. Plots of the Scaled sensitivities(y-axis) vs Time(x-axis) at different values of h using Centered Finite difference approximation:

<u>Note</u>: The plots are curves with different colors indicating how the scaled sensitivity plot changes until it converges.

<u>a)h=0</u>.1:

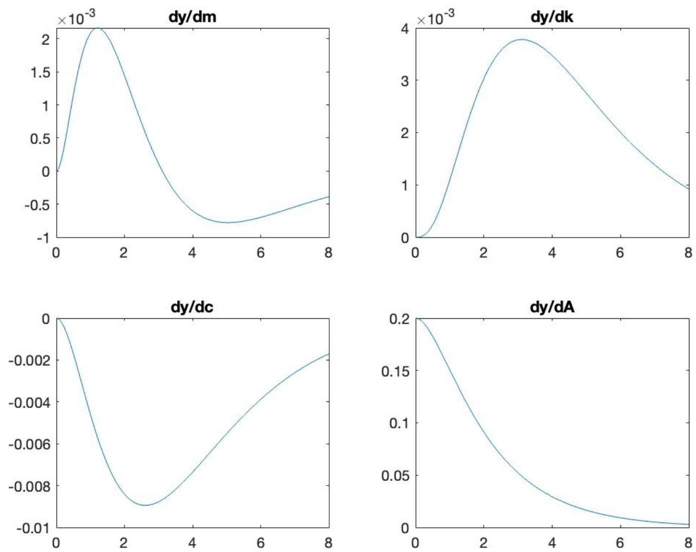


b)h=0.02(converged):

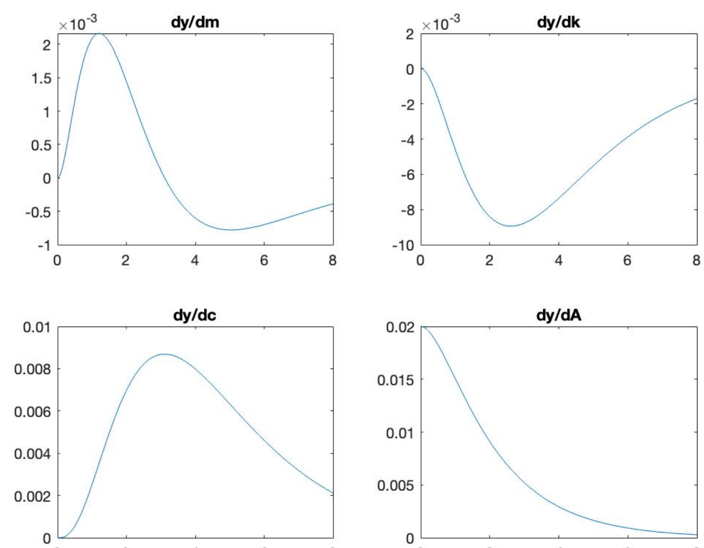


3. Plots to compare the solution of scaled sensitivities obtained using the ODE solver (for both Forward finite difference approximation and Centered finite difference approximation using our calibrated value of h) to those obtained using the explicit formula for the overdamped case given by the MATLAB file fOD.m.

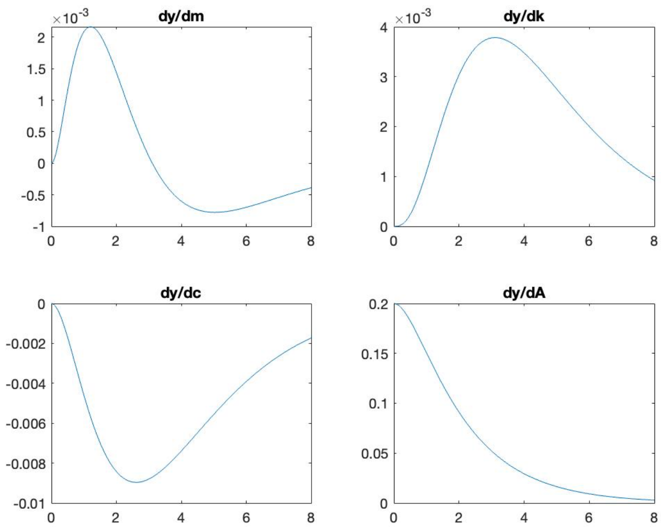
a) Using ODE solver: Plot of Scaled sensitivities given by forward finite difference method for h=0.005(which is our h value for when the plots visually converged).



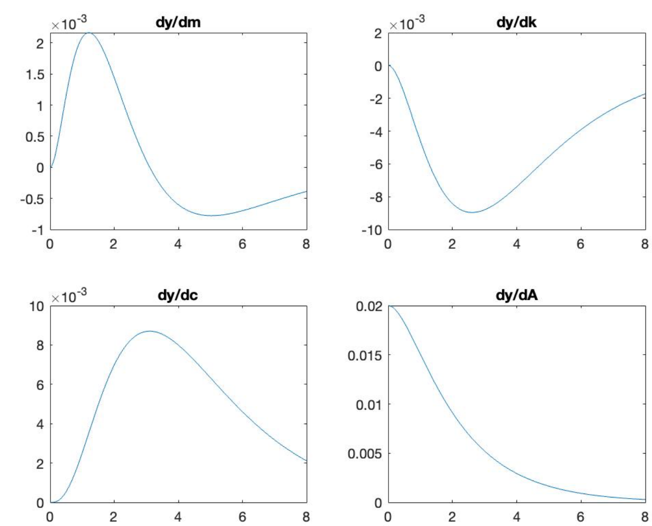
b) Using Explicit formula: Plot of scaled sensitivities given by forward finite difference method for h=0.005(which is our h value for when the plots visually converged).



c) Using ODE solver: Plot of Scaled sensitivities given by Centered finite difference method for h=0.02(which is our h value for when the plots visually converged).



d) Using Explicit formula: Plot of scaled sensitivities given by Centered finite difference method for h=0.02(which is our h value for when the plots visually converged).



Clearly, the plots of scaled sensitivities found using the ode solver and the Explicit formula for the overdamped case are matching(almost exactly). Therefore, we can conclude that our implementation is correct.

Part II): SEIR Epidemiological Model

Description: The SEIR epidemiological model is time-dependent disease model that is used to observe how certain response variables(that will be list below) change with respect to time. The response variables are:

(i) S: The count of the sub-population that is Susceptible to the disease.

(ii) E: The count of the sub-population that is Exposed to the disease but are not yet infected with the disease.

(iii) I: The count of the sub-population that is Infected by the disease.

(iv) R: The count of the sub-population that has Recovered from the disease.

The SEIR model can be formulated as an initial value problem (IVP) for the following system of ODEs:

$$\frac{dS}{dt} = -k_1 SI$$

$$\frac{dE}{dt} = k_1 SI - k_2 E$$

$$\frac{dI}{dt} = k_2 E - k_3 I$$

$$\frac{dR}{dt} = k_3 I$$

along with the initial conditions:

$S(0) = S_0,\ E(0) = E_0,\ I(0) = I_0,\ R(0) = 0$

The parameters for this model are: $k_1, k_2, k_3, S_0, E_0, I_0$

In this part of the project, we will mainly be answering some questions regarding the SEIR model that will help us better understand the properties of the model.

Scripts written by me:

- SEIRmodel_B.m: Plots the four response variable(S,E,I, and R) versus Time(t) based on the Nominal values of the parameters:$k_1 = 10^{-5}, k_2 = 0.03, k_3 = 0.01, S_0 = 10^6, E_0 = 0, I_0 = 10^3$ for t=0 days to t=100 days(part(b) of the project instructions for Part (II)).
- SEIRPartCandD.m : Plots all the Scaled Sensitivities(for each response variable) versus Time(t) based on the Nominal values of the parameters:$k_1 = 10^{-5}, k_2 = 0.03, k_3 = 0.01, S_0 = 10^6, E_0 = 0, I_0 = 10^3$ for t=0 days to t=100 days using Centered finite difference approximation(part(c) of the project instructions for Part (II)). This MATLAB file is also used to plot the response variables for a new set of perturbed parameter values(part (d) of the project instructions for Part (II)).

Analysis of the SEIR Epidemiological model:

a) Question: Explain the significance of each term on the right-hand side of equations (3-6). In other words, explain the effect that the term is modeling and why the given mathematical form does so. Also, describe a state in an infectious disease outbreak that the initial conditions are modeling.

Answer: (i)The $\frac{dS}{dt}$ term is used to model the rate of change of the number of people who are susceptible to an infectious disease(for which the SEIR model is used) over time. In our SEIR model, $\frac{dS}{dt} = -k_1SI$. Here, k_1 denotes the rate at which people who are susceptible to the disease are encountering the people who are already infected with the disease. Therefore, k_1SI denotes the number of susceptible people who have encountered infected people. We have $\frac{dS}{dt} = -k_1SI$ because, as the k_1SI increases the number of susceptible people decreases as they are now part of the exposed group.

(ii) The $\frac{dE}{dt}$ term is used to model the rate of change of the number of people who are exposed(not yet showing symptoms of the disease) to an infectious disease(for which the SEIR model is used) over time. In our SEIR model, $\frac{dE}{dt} = k_1SI - k_2E$. Here, k_2 denotes the rate at which the sub-population that is exposed to the disease is becoming part of the infected population. Therefore, k_2E denotes the proportion of the exposed population that is now infected. We have $\frac{dE}{dt} = k_1SI - k_2E$ because, the number of people who are exposed to the disease rises when a part of the susceptible population is becoming exposed(k_1SI term) and it falls when a part of the exposed population becomes infected(k_2E term).

(iii) The $\frac{dI}{dt}$ term is used to model the rate of change of the number of people who are infected(showing symptoms of the disease) to an infectious disease(for which the SEIR model is used) over time. In our SEIR model, $\frac{dI}{dt} = k_2E - k_3I$. Here, k_3 denotes the rate at which the infected population is recovering(this can mean either death of the person or a healthy recovery from the disease). Therefore, k_3I denotes the proportion of the Infected population that has now recovered. We have $\frac{dI}{dt} = k_2E - k_3I$ because, the number of people who are infected by the disease rises when a part of the Exposed population is becoming infected(k_2E term) and it falls when a part of the Infected population recovers(k_3I term).

(iv) The $\frac{dR}{dt}$ term is used to model the rate of change of the number of people who have recovered(this can mean either death of the person or a healthy recovery from the disease) from an infectious disease(for which the SEIR model is used) over time. In our model, $\frac{dR}{dt} = k_3I$. This is because, as the number of infected people rises a proportion of them(k_3I term) will recover from the disease.

The initial conditions of the above SEIR model are representing the following states in a disease outbreak:

Let us consider a case where a city(or country) has just encountered a case of the infectious disease(time t=0).

(i) $S(0) = S_0$ represents any population that is within a certain radius of that city(or country) is susceptible to the disease.

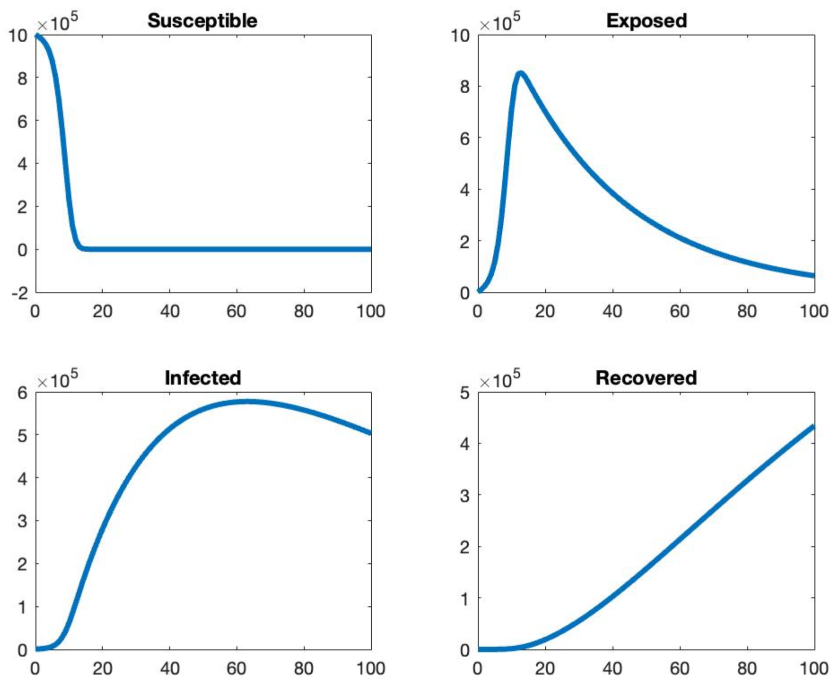
(ii) $E(0) = E_0$ represents any proportion of population that has encountered an infected case but is not showing symptoms of the disease at time t=0. In our case, $E_0 = 0$ as we cannot have any exposed cases at time t=0.

(iii) $I(0) = I_0$ represents the part of the population that has been infected by the disease at t=0.

(iv) $R(0) = 0$ represents the number of recovered people in the city(or country) at t=0. $R(0) = 0$ because we cannot have any recovered cases at the start of the disease outbreak.

b) Plot the four response variable(S,E,I, and R) versus Time(t) based on the Nominal values of the parameters: $k_1 = 10^{-5}, k_2 = 0.03, k_3 = 0.01, S_0 = 10^6, E_0 = 0, I_0 = 10^3$ for t=0 days to t=100 days. Qualitatively explain your simulation results over the time course of the simulation. Also state some limitations of the SEIR model above.

Plot of the response variables:



Qualitative explanation of the above simulation:

Note: While observing the above plot, it might seem that I(0) = 0. But this is only because the y-axis of the above plots is scaled to 10^5.

a)Time t=0: Initially, the count of susceptible people is high(10^6 cases), while the counts of exposed group(0 cases),infected group(10^3 cases), and recovered group(0 cases) are low.

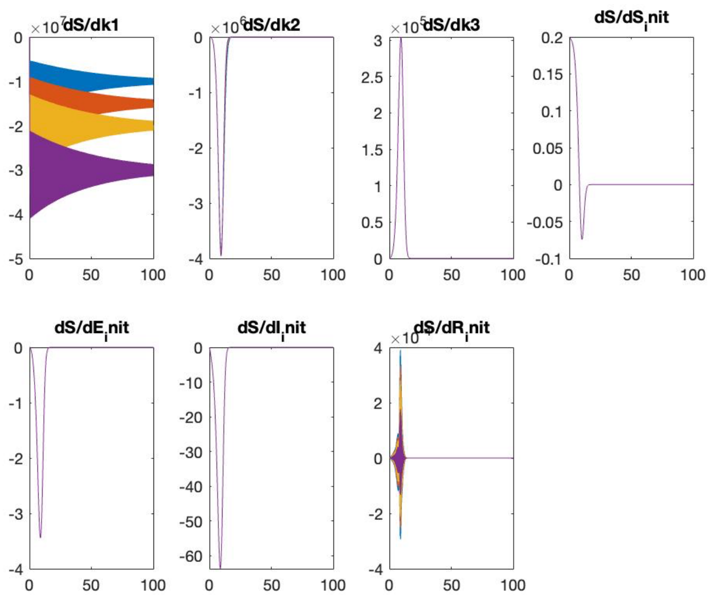
b)Time 0≤t≤15 days: During this time, we can observe that most of the susceptible population is now moving to the exposed group. This might be because the susceptible population has encountered several infected cases. We can observe that, during this time, the absolute values of slopes of the (S vs t) and the (E vs t) plots are almost vertical, and the slope of (I vs t) plot is quite low(although it is still exponential) compared to slopes of (S vs t) and the (E vs t) plots(this is most likely due to a low value of k_2 compared to k_1). This signifies that a majority of the susceptible group has quickly shifted to the exposed group and the exposed group is shifting to the infected group at a lower rate.
c)Time 15≤t≤100 days: During this time, the population of the susceptible group is 0, the population of the exposed group is decreasing at an exponential rate and the population of the infected group is rising almost exponentially. We can also observe that, during this time, the population of the recovered group is rising linearly(this is because the population of the infected is reducing in a linear fashion).

Limitations of the above SEIR model:

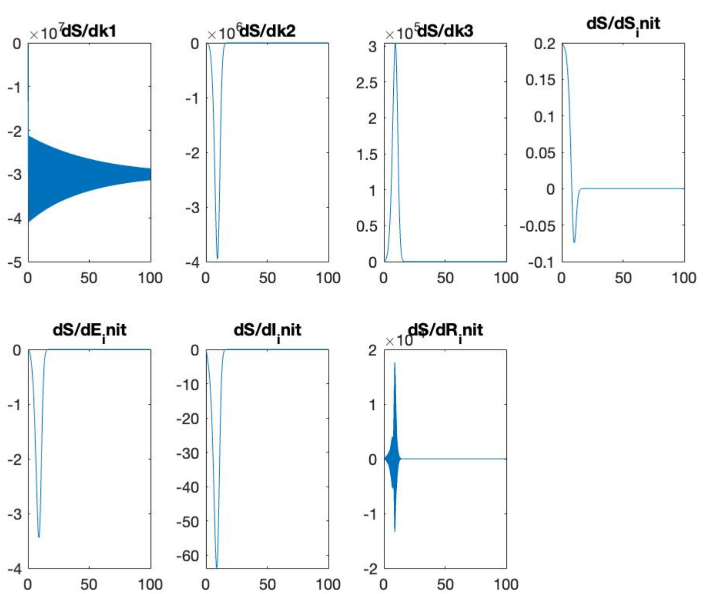
1. The above SEIR model does not consider the decrease in the population of the susceptible group due to encounters with exposed cases. Considering this, would modify the system of ODE's so that :
$$\frac{dS}{dt} = -k_1SI - k_4SE$$
$$\frac{dE}{dt} = k_1SI - k_2E + k_4SE$$
This modification in the system of ODE's can change the plots of our response variables dramatically.
2. The above SEIR model also does not account for different categories of infected cases(mild, severe, chronic). This is necessary for the model because the rate of change in the population of susceptible and exposed groups could be different if the susceptible group encountered a mild case of infection compared to a chronic case of infection(there can be other scenarios as well). Incorporating a different response variable for each level of infected cases will make the model more comprehensive.
3. Finally, the above SEIR model does not categorize cases of Deaths, and cases of healthily recovered individuals into two separate categories. Instead, the Recovered group(R) consists of cases that are both Dead, and those that have healthily recovered. This can lead to several misinterpretations of the plots of the response variable R over time.

c) Plot all the Scaled Sensitivities(for each response variable) versus Time(t) based on the Nominal values of the parameters: $k_1 = 10^{-5}, k_2 = 0.03, k_3 = 0.01, S_0 = 10^6, E_0 = 0, I_0 = 10^3$ for t=0 days to t=100 days using Centered finite difference approximation. Be sure to demonstrate your calibration of the parameter h via a plot. Interpret the sensitivities in the context of the application, indicating reasons why one or more variables are more or less sensitive to specific parameters in a particular time window of the disease outbreak.

(A)Plots of Scaled sensitivities for S(Susceptible group):
(i)Demonstration of calibration of h(h=0.01,0.007,0.005,0.003)until convergence(h=0.003):

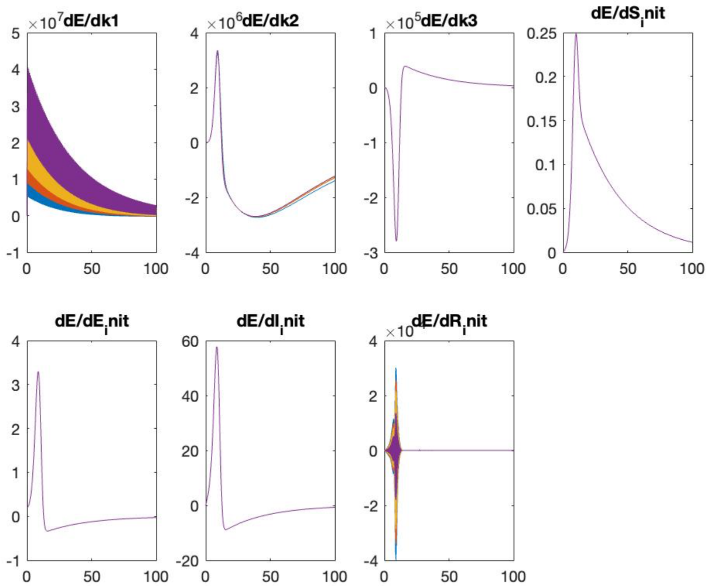


(ii) Plot of Scaled sensitivities at convergent h value(0.003):

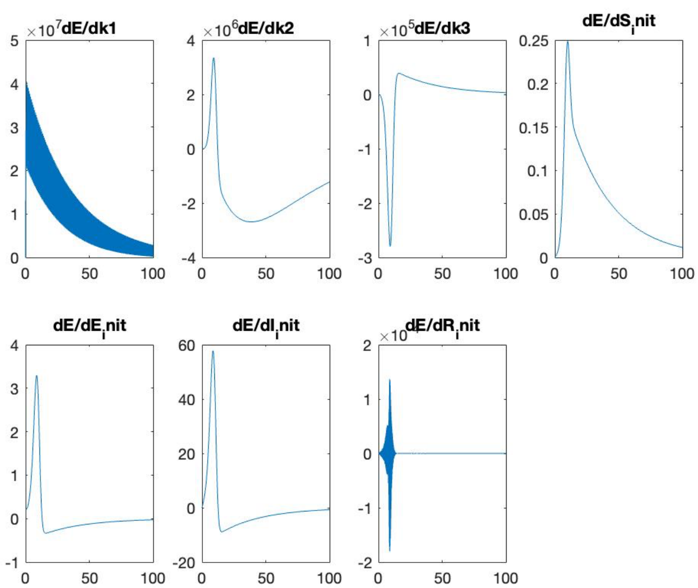


Interpretation: We can know from the SEIR system of ODE's that S is dependent on k1, S, and I. This is shown clearly by the above sensitivity plots where S is highly sensitive to changes in k1, k2 and k3(because S depends on I, and I is affected by k2 and k3). S is sensitive to S_init, E_init, I_init and R_init only during 0≤t≤15 because these are only initial conditions(time t=0).

(B) Plots of Scaled sensitivities for E(Exposed group):
(i)Demonstration of calibration of h(h=0.01,0.007,0.005,0.003)until convergence(h=0.003):

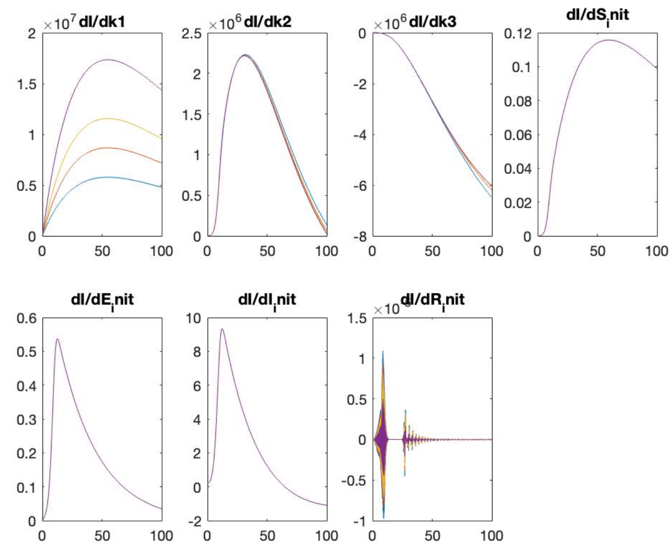


(ii) Plot of Scaled sensitivities at convergent h value(0.003):

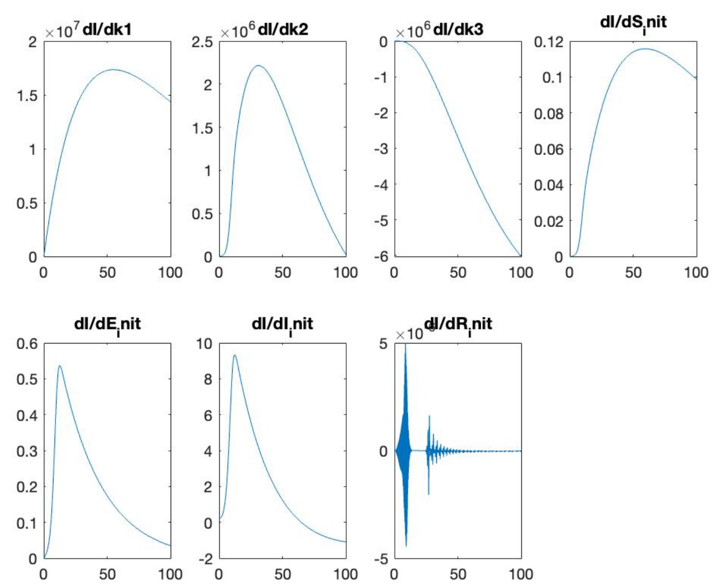


Interpretation: We can know from the SEIR system of ODE’s that E is dependent on k1, S, I, k2 and E. This is shown clearly by the above sensitivity plots where E is highly sensitive to changes in k1, k2 and k3. E is sensitive to k3 because E depends on I, and I is affected by k3. E is sensitive to S_init, E_init, I_init and R_init only during 0≤t≤15 because these are only initial conditions(time t=0).

Plots of Scaled sensitivities for I(Infected group):
(i)Demonstration of calibration of h(h=0.01,0.007,0.005,0.003)until convergence(h=0.003):



(ii) Plot of Scaled sensitivities at convergent h value(0.003):

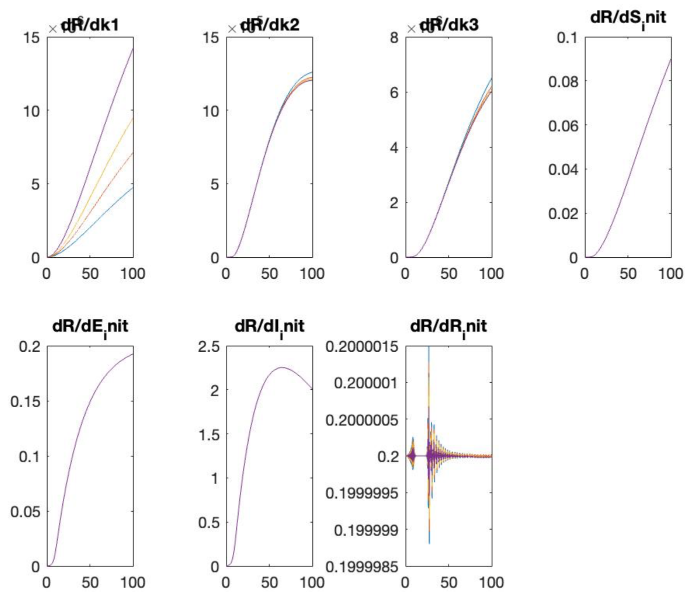


Interpretation: We can know from the SEIR system of ODE’s that I is dependent on k2, E, k3, and I. This is shown clearly by the above sensitivity plots where I is highly sensitive to changes in k2 and k3. I is also highly sensitive to k1 because

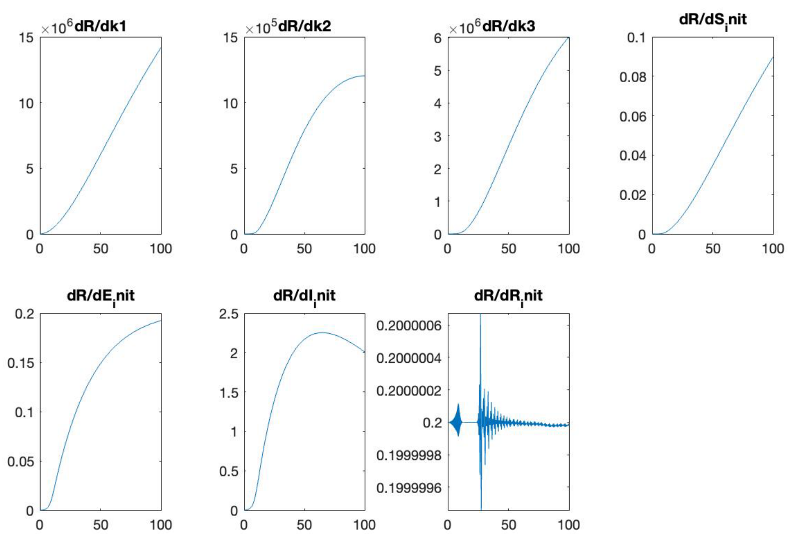
I depends on E, and E is affected by k1. I is sensitive to S_init, E_init, I_init and R_init only during 0≤t≤15 because these are only initial conditions(time t=0).

<u>Plots of Scaled sensitivities for R(Recovered group):</u>

<u>(i)Demonstration of calibration of h(h=0.01,0.007,0.005,0.003)until convergence(h=0.003):</u>



<u>(ii) Plot of Scaled sensitivities at convergent h value(0.003):</u>



<u>Interpretation:</u> We can know from the SEIR system of ODE's that R is dependent on k3 and I. This is shown clearly by the above sensitivity plots where R is highly sensitive to changes in k3. R is also highly sensitive to k1 because R depends on I, and I is affected by E which depends on k1. R is also highly sensitive to k2 because R depends on I, and I is affected by k2. R is sensitive to S_init, E_init,

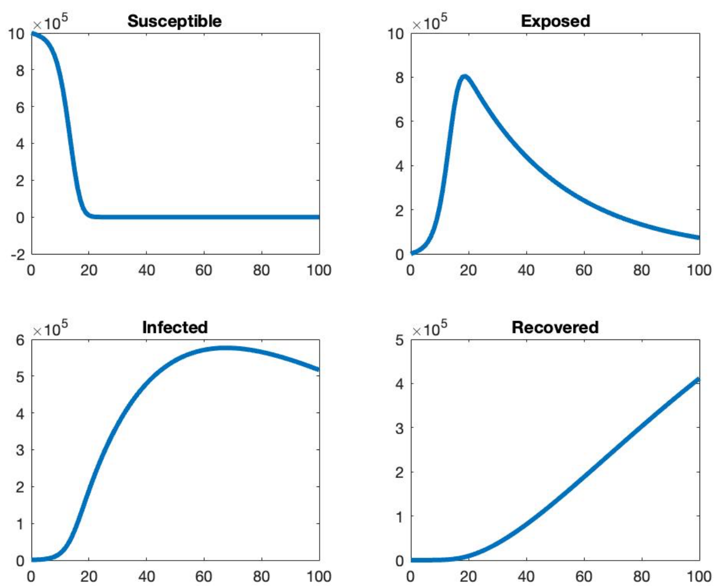
I_init and R_init only during 0≤t≤15 because these are only initial conditions(time t=0).

d) Explain which parameters in the model are most likely to be altered due to medical or public health interventions. Simulate one example of such an intervention by perturbing one (or more) parameters in the model, justifying your new parameter value and interpreting the effects on your new simulated results. Your script should plot all four dependent variables on the same graph.

- One of the parameters that can be affected by a public health intervention is : k1.
- Following is an example of a scenario when the above statement would be true: Let us create an SEIR model for COVID-19. During COVID-19, the government made it compulsory to weak masks. This public health intervention can significantly change our SEIR model because, now the population of susceptible people will decrease at a much lower rate(or the population of exposed people will increase at a much lower rate). In other words, the value of k1 is reduced due to this public health intervention.
- Let us simulate an example of the above scenario. Assume that because of wearing masks, 50% of the susceptible people are protected from exposed cases of COVID - 19. Based on the above information, our new perturbed nominal values of parameters are:

$$k_1 = 0.5 * 10^{-5}, k_2 = 0.03, k_3 = 0.01, S_0 = 10^6, E_0 = 0, I_0 = 10^3$$

Plots of the response variables(S,E,I,R) vs Time(t) for the new nominal values:



- We can observe that, with our reduce value of k1, the population of the susceptible group becomes 0 only after t=20 days(as opposed to t=15 days for our original k1 value). This is due to a reduction in the slope(k1 value) of the Susceptible population plot. In addition, we can observe that the slopes of all the other plots have also reduced due to a reduction in k1 value. This is because E depends on S which in turn depends on k1, I depends on E(which we now know depends on k1), and R depends on I(which we now know is affected by k1).