Kempe's Universality Theorem

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Abstract

Kempe's Universality Theorem marks a significant milestone in the history of mechanical linkages, demonstrating that any algebraic curve can be traced by a system of rigid bars and pin joints. This blog post traces the evolution of mechanical design, from the 13th-century Elephant clock to the 18th-century automaton – 'The Writer,' leading up to Kempe's groundbreaking work. It explores the core mechanisms of Kempe's linkages, such as parallelograms, reversor, multiplicator, and additor linkages, and explains their mathematical foundations. Through examples, the post showcases the elegance and ingenuity of Kempe's linkages, emphasizing the creativity and innovation behind the theorem.

Introduction

The history of mechanical innovation is rich with examples of ingenuity and creativity, stretching back centuries. Our journey begins in the 13th century with the remarkable Elephant Clock, designed by the polymath Ismail al-Jazari. This clock was not just a device to measure time; it was an intricate blend of water clock technology and mechanical artistry, embodying the innovative spirit of its era. Jumping ahead to 1774, we encounter 'The Writer', an automaton crafted by the Swiss watchmaker Pierre Jaquet-Droz. Unlike any simple mechanical toy, this automaton could write letters with a quill, showcasing the potential for mechanisms to execute complex and precise tasks. Such inventions highlight the growing sophistication of mechanical designs as we move closer to the *industrial revolution*.

The industrial revolution (1760-1840) marked a period of unprecedented mechanical advancement. This era and what followed saw the development of linkage mechanisms, fundamental components in machinery that transformed various types of motion. Among these were the Peaucellier mechanism of 1864, the first to convert rotary motion into a perfect straight line, and the Chebyshev and Watt mechanisms, which could approximate a straight line. These innovations were pivotal in the functionality of early steam engines and other machines, setting the stage for more intricate mechanical designs. Building on these foundational advancements, the late 19th century saw a breakthrough in mechanisms with the introduction of Kempe's Universality Theorem [1]. Alfred Kempe, a lawyer by profession, was also a brilliant mathematician who demonstrated that any algebraic curve could be traced by a linkage — a system of rigid bars and pin joints. Kempe's theorem was famously paraphrased by the famous topologist William Thurston as "*One can design* *a linkage that will sign your name*", illustrating the complex capability that the theorem embodies. This theorem fundamentally transformed our understanding of mechanical linkages, proving that with the right configuration, linkages could replicate even the most complex curves.

(a) (b)

Figure 1: Historical Mechanical Innovations: 'The Elephant Clock' by Ismail al-Jazari and 'The Writer' by Pierre Jaquet-Droz

Kempe's Linkages

The fundamental building blocks of a Kempe linkage are the parallelogram and contra-parallelogram linkages that will feature extensively in the following sections. A parallelogram linkage consists of four bars forming a quadrilateral with opposite sides equal. Let ABCD be a parallelogram linkage; when the opposite side BC is flipped upside down, we get a contra-parallelogram as seen in Fig. 2. In the following sections, we shall see how Kempe leveraged these parallelogram and contra-parallelogram linkages to build complex mechanisms that can add, multiply, and preserve angles.

Reversor Linkage

The reversor linkage (shown in Fig. 3(a)), also known as the 'angle-doubler,' is a linkage designed to double an input angle. This mechanism consists of two contra-parallelograms as shown in Fig. 3, OACB and OBKF, made up of a total of six links. Both the contra-parallelograms are geometrically similar, meaning OBKF is essentially a scaled-down version of OACB. This similarity

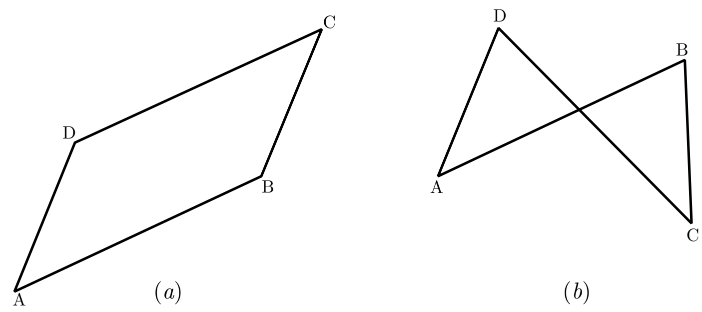


Figure 2: (a) Parallelogram, (b) contra-parallelogram

implies that the angles within these parallelograms maintain a the relationship: $\angle OBC = \angle OFK$ and $\angle BOA = \angle FOB$. To understand the operation of the reversor linkage, consider link OA as being fixed. When link OB is driven, it rotates and forms an angle θ with respect to OA, while causing the output link OF to rotate as well. Importantly, the design of the linkage ensures that the angle θ made by OB with OA is doubled, resulting in OF making an angle 2θ.

Multiplicator Linkage

The multiplicator linkage (shown in Fig. 3(b)), as the name indicates, is used to multiply an input angle by an integer. The construction of this linkage is very similar to the reversor linkage. Suppose we want to multiply the input angle by a factor 3, we can add two more links to the reversor linkage, as shown in Fig. 3(b). In this figure, the multiplicator linkage has three contra-parallelograms: OBCA, OFKB, and OJMF. All of them are geometrically similar, and we can make the argument that $\angle BOA = \angle FOB = \angle JOF$. Therefore, fixing the link OA and turning the link OB by an angle θ will result in link OJ turning by an angle 3θ. Similarly, we can add more links to the reversor linkage and multiply the input angle by any integer.

Additor Linkage

The additor linkage is used to add or subtract angles, as shown in Fig. 3(c). This linkage primarily consists of two contra-parallelograms $OABC$ and $OAB'C'$. In this setup, the common link OA acts as the angular bisector for the angles $\angle E'OC'$ and $\angle EOC$, ensuring that the angles $\angle E'OE'$ and $\angle C'OC'$ are equal. If the link OC is fixed to the horizontal (the x-axis) and the link OC' makes an angle θ with the x-axis ($\angle C'OC' = \theta$), and the link OE' makes an angle φ with OC ($\angle E'OC = \varphi$), then the link OE will make an angle $\theta + \varphi$ with the horizontal. To subtract angles, if $\angle C'OC' = \theta$ and $\angle EOC = \varphi$, the link OE' will make an angle $\varphi - \theta$ with the x-axis.

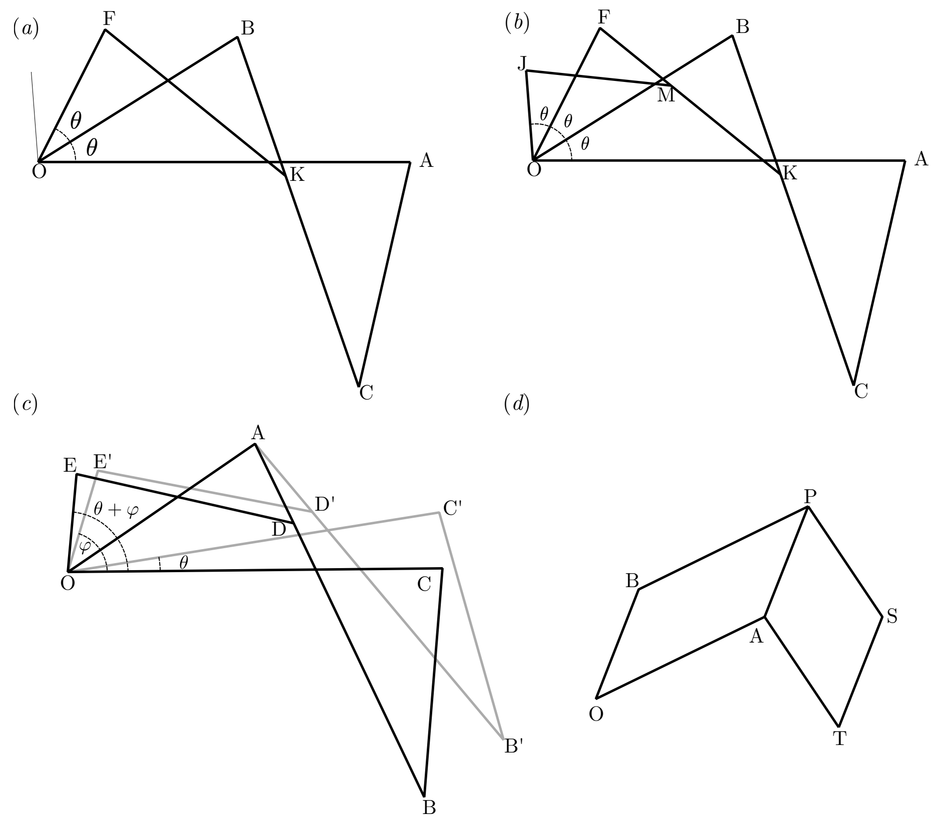


Figure 3: Kempe's Linkage: (a) Reversor linkage (angle-doubler) multiplying the input angle by a factor of 2 using two contra-parallelograms. (b) Multiplicator linkage extending the reversor linkage to multiply the input angle by 3 with additional links. (c) Additor linkage for adding or subtracting angles using two contra-parallelograms. (d) Translator linkage ensuring two links maintain the same angle with respect to the horizontal.

Translator Linkage

When two links are spaced apart and must maintain the same angle with the horizontal, a translator linkage can be employed. It consists of two parallelograms sharing a common link. When the input link rotates, the links at the other end of the parallelogram will maintain the same angle with respect to the horizontal as the input link. In Fig. 3(d), the links OB and ST are parallel to each other and maintain the same angle with the x-axis.

Construction of Kempe's Linkages

In this section, we will mathematically explore how the linkages discussed earlier can be used to trace any curve. According to Kempe, we begin with a parallelogram, say $OAPB$, where $OA = BP = a$ and $OB = AP = b$. Our goal is to have the point P of the parallelogram trace out a prescribed curve. Let the angles $AOX = \theta$ and $BOX = \varphi$ (see Fig. 4). To achieve this, we start with the general equation of a 2D plane curve in polynomial form and convert it to polar form. Assuming $\phi(x, y) = 0$ represents any plane curve of degree n, the coordinates x and y of point P can be expressed as,

$$x = a\cos\theta + b\cos\varphi, \tag{1}$$

$$y = a\sin\theta + b\sin\varphi. \tag{2}$$

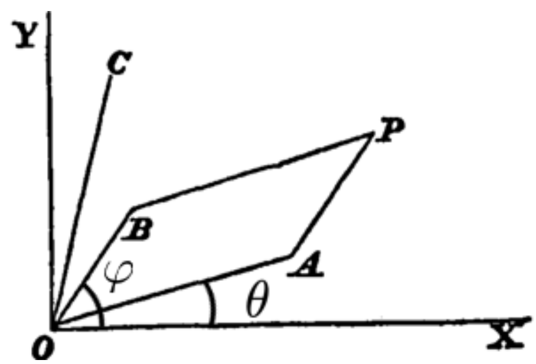


Figure 4: Construction of the parallelogram $OAPB$ where sides OA and OB make angles θ and φ with the horizontal, respectively.

Substituting the expressions for x and y into $\phi(x, y)$ and expanding the result in terms of cosines and sines yields,

$$\phi(x, y) = \sum_{i=1}^{n} \left[A_i \cos(r_i\theta \pm s_i\varphi \pm \alpha_i)\right] + C. \tag{3}$$

We work out the algebra involved for such a representation in the box 1.

To perform these algebraic operations—adding, subtracting, and multiplying angles, we use the linkages discussed earlier. Using the multiplicator linkage, we can construct a mechanism for the i-th term where the mechanism multiplies the angles θ by r_i and φ by s_i. Then, employing the additor linkage, we can add the angles $r_i\theta$, $s_i\varphi$, and α_i to construct a new linkage OL_i, which will form an angle $r_i\theta \pm s_i\varphi \pm \alpha_i$ with the horizontal axis OX. The length of the linkage corresponds to the coefficient A_i. The distance from this linkage OL_i to the y-axis is $A_i \cos(r_i\theta \pm s_i\varphi \pm \alpha_i)$.

In essence, we have taken the i-th term from the polar form expansion of the algebraic function $\phi(x, y)$ and built a linkage OL_i that represents this term.

Similarly, we construct linkages $OL_1, OL_2, \ldots OL_n$, where each linkage makes an angle with the x-axis equal to the angles in the polar form expansion. The length of each linkage corresponds to the coefficients $A_1, A_2, \ldots A_n$. The sum of the distances from $OL_1, OL_2, \ldots OL_n$ to the y-axis will equal the sum of all the cosine terms in the expansion. By constructing a chain of links $OP_1, P_2P_3, \ldots$, which are serially connected and parallel to the linkages $OL_1, OL_2, \ldots OL_n$, the extremity point P' will be located at a distance of $\sum_{i=1}^{n} [A_i \cos(r_i\theta \pm s_i\varphi \pm \alpha_i)]$ from the y-axis.

Let's quickly recap: Our goal is to make the point P in the parallelogram $OAPB$ trace a particular path defined by $\phi(x, y) = 0$. Given P can be moved freely in space by changing θ, φ, we want to control the motion of θ, φ such that P moves along $\phi(x, y) = 0$. To do that, we connect $OAPB$ to linkages that add and multiply angles $(r_i\theta \pm s_i\varphi \pm \alpha_i)$, which correspond to the individual terms in the summation from Eqn. 3. This process generates links $OL_1, OL_2, \ldots OL_n$. A separate translator linkage is needed to connect these linkages serially, forming new linkages $OP_1, P_1P_2, \ldots$ The extremity of this chain is point P', located at a distance of $\sum_{i=1}^{n} [A_i \cos(r_i\theta \pm s_i\varphi \pm \alpha_i)]$ from the y-axis. The reason why it is at a distance from the y-axis is because we have written the target function $\phi(x, y) = 0$ as a sum of cosines, which are projections of links $OL_1, OL_2, \ldots OL_n$ along the horizontal direction i.e. a fixed distance from the y-axis. When P' moves vertically along the y-axis or the line $x + C = 0$, the angles φ and θ will adjust to satisfy Eqn. 3. In this way, we have constructed a physical mechanism that embodies Eqn. 3, constrained to move on the line $x+C = 0$, thereby generating the full polar expansion of $\phi(x, y)$. When all the linkages work together, the mechanism has only one degree of freedom, and as point P' moves vertically, the angles φ and θ change to satisfy the equation $\phi(x, y) = 0$, thus allowing point P of the parallelogram $OAPB$ to trace the curve $\phi(x, y)$.

Box 1: Constructing a Kempe linkage that traces a simple function

In this example, we illustrate how to construct the physical linkages that enable the tracing of a curve based on the polar form representation of an algebraic function. Let us consider a simple function $\phi(x, y) = xy$ as the curve the mechanism must trace. First, we express x and y in terms of the given substitutions,

$$x = a\cos\theta + b\cos\varphi, \quad y = a\sin\theta + b\sin\varphi.$$

Substituting these into $\phi(x, y)$, we get:

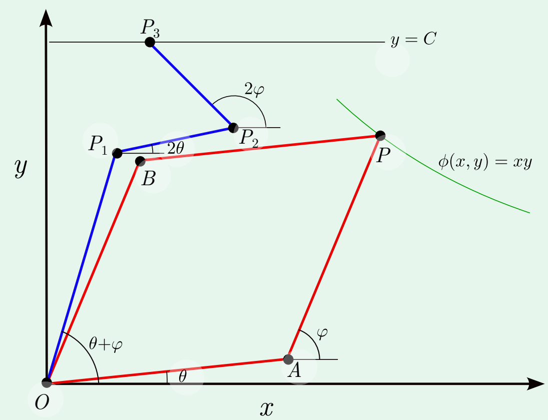
$$\phi(x, y) = xy = (a\cos\theta + b\cos\varphi)(a\sin\theta + b\sin\varphi).$$

Expanding this we get,

$$\phi(x, y) = a^2\cos\theta\sin\theta + ab\cos\theta\sin\varphi + ab\cos\varphi\sin\theta + b^2\cos\varphi\sin\varphi.$$

Next we use standard trignometric identites to simplify the expression to get,

$$\phi(x, y) = \frac{1}{2}a^2\sin 2\theta + ab\sin(\theta + \varphi) + \frac{1}{2}b^2\sin 2\varphi.$$



We proceed to construct Kempe linkages based on this polar expansion. We need linkages to multiply the angles θ and φ by 2, add θ and φ. We can set the length of the sides OA and AP of the parallelogram $OAPB$ to 1 ($a = b = 1$). We then continue with the following steps:

1. **Multiplicator Linkages**: The terms $\sin 2\theta$ and $\sin 2\varphi$ require multiplicator linkages OL_1 and OL_2, which multiply the input angles by 2.
2. **Addition Linkage**: The term $\sin(\theta + \varphi)$ requires an additor linkage OL_3, which sums the angles θ and φ also this linkage is twice the length of Multiplicator linkages.
3. **Serial Linkage Construction**: Once the multiplicator and additor linkages are constructed, a translator linkage is used to serially connect them. This forms the linkages OP_1, P_1P_2, P_2P_3 as seen in the figure, which allow the overall mechanism to adjust the angles θ and φ as P_3 moves along the $y = 0$ line to make make the point P trace out the curve $\phi(x, y) = xy$ seen in green.

A minor detail to be noted here is that earlier we had the polar expansion in terms of cosine but now we have written it in terms of sine so now the point of extremity $P' \equiv P_3$ in the above figure travels along the $y = 0$ line. Finally, with the full linkage mechanism in place, the point P of the parallelogram $OAPB$ traces the curve defined by $\phi(x, y) = xy$.

Conclusion

Kempe's Universality Theorem, while a monumental achievement in showcasing the capabilities of mechanical linkages, comes with a caveat directly acknowledged by Kempe himself: "*It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork employed, a necessary consequence of the perfect generality of the demonstration.*" Despite its impracticality in real-world applications due to the intricate and extensive linkages required, it remains a valuable read. The ingenuity and simplicity in Kempe's approach offer inspiration, encouraging us to think creatively and explore innovative solutions in our own work.

We encourage the curious reader towards the excellent pedagogical article by Prof. Anupam Saxena where he has worked out the Kempe Linkages for a complex function, $\phi(x, y) = (x - y)(x + y + \frac{1}{2})$: Kempe's Linkages and the Universality Theorem.

References

[1] A. B. Kempe. "On a General Method of Describing Plane Curves of the nth Degree by Linkwork". In: *Proceedings of the London Mathematical Society* s1-7 (1875), pp. 213–216. DOI: 10.1112/plms/s1-7.1.213.