chapter 9

Sensitivity Analysis

9.1 INTRODUCTION

Suppose we take a practical problem and formulate it as the LP

$$\begin{cases} \text{Minimize} & z(x) = cx \\ \text{Subject to} & Ax = b \\ & x \geqq 0 \end{cases} \tag{9.1}$$

where A is a matrix of order $m \times n$ and rank m, and obtain an optimum feasible solution for it. In most real-life problems, the coefficients in A, c, and b are estimated from practical considerations. After the final optimal feasible solution has been obtained we may discover that some of the entries in b, c, or A have to be changed or that extra constraints or variables have to be introduced into the model. Solving the modified problem from scratch will be wasteful. *Sensitivity analysis* (also called *postoptimality analysis*) deals with the problem of obtaining an optimum feasible solution of the modified problem starting with the optimum feasible solution of the old problem. We consider the problem of introducing only one change at a time (e.g., how to get the new optimum feasible solution if the value of only one c_j has to be changed or if only one new constraint has to be added, etc.) If several changes have to be made, make them one at a time, or extend the methods discussed here in an obvious manner to take care of several simultaneous changes. In Chapter 8 we discussed some types of postoptimality analysis where the right-hand-side vector or the cost coefficient vector vary linearly as a function of a parameter as it ranges over the real line. Here we discuss various other types of postoptimality analyses. Some of the postoptimality analyses discussed here, for example, changes in cost coefficients or the right-hand-side constants, can be viewed as specializations of the parametric analysis applied to those cases. For an idea of the kind of questions that can be answered using the methods of sensitivity analysis, see Exercise 9.16. In most practical applications using a linear programming model, marginal and sensitivity analysis provide economic information that is very useful in planning.

Let **K** denote the set of feasible solutions of (9.1). Suppose the optimal basis obtained for (9.1) is $\tilde{B}$, associated with the basic vector $x_{\tilde{B}} = (x_1, \ldots, x_m)$, $c_{\tilde{B}} = (c_1, \ldots, c_m)$. Let $\tilde{\pi} = c_{\tilde{B}}\tilde{B}^{-1}$ and let $\tilde{x}$ be the BFS of (9.1) corresponding to the basis $\tilde{B}$. For illustration we will use the following problem, which has $x_B = (x_1, x_2, x_3)$ as an optimum basic vector. Therefore, $\tilde{x} = (3, 4, 2, 0, 0, 0)^T$ is an optimum feasible solution of this problem and the minimum objective value is $z(\tilde{x}) = 11$.

Tableau 9.1 Original Problem

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	b
1	2	0	1	0	−6	0	11
0	1	1	3	−2	−1	0	6
1	2	1	3	−1	−5	0	13
3	2	−3	−6	10	−5	1	0

$x_j \geqq 0$ for all j; z to be minimized.

Tableau 9.2 Optimum Inverse Tableau for the Problem in Tableau 9.1

Basic Variables	Inverse Tableau				Basic Values
x_1	−1	−2	2	0	3
x_2	1	1	−1	0	4
x_3	−1	0	1	0	2
$-z$	−2	4	−1	1	−11

9.2 INTRODUCING A NEW ACTIVITY

Suppose a new variable called x_{n+1} has to be introduced into the model (9.1). Let $A_{.n+1}$, c_{n+1} be the input-output coefficient vector and the cost coefficient, respectively, of this new variable. Let $X^T = (x^T, x_{n+1})$. The augmented problem is

$$\left\{\begin{array}{lr} \text{Minimize} \quad Z(X) = cx + c_{n+1}x_{n+1} & \\ \text{Subject to} \quad (A \vdots A_{.n+1})X = b & (9.2) \\ \qquad\qquad\qquad X \geqq 0 & \end{array}\right.$$

$\tilde{B}$ remains a feasible basis to the augmented problem, and the BFS of (8.2) corresponding to it is $\tilde{X}^T = (\tilde{x}^T, 0)$. From the optimality criterion, $\tilde{X}$ remains an optimum feasible solution of (9.2) if the relative cost coefficient of the new variable x_{n+1}, with respect to the basis $\tilde{B}$, is nonnegative; that is, if $\bar{c}_{n+1} = c_{n+1} - \tilde{\pi}A_{.n+1} \geqq 0$. On the other hand, if $\bar{c}_{n+1} < 0$, solve (9.2) by using the inverse tableau corresponding to the basis $\tilde{B}$ as an initial tableau. Bring x_{n+1} into the basic vector and complete the solution of (9.2) according to the revised simplex algorithm.

It often happens that one pivot (that of bringing the new variable x_{n+1} into the basic vector $x_{\tilde{B}}$) is all that is necessary to solve (9.2). However, there is no theoretical guarantee that this will be the general case. In some problems, it may be necessary to perform several pivots before the algorithm reaches a terminal basis for (9.2).

Example 9.1

Consider the LP in Tableau 9.1. Suppose a new variable x_7 corresponding to the column vector $A_{.7} = (1, 2, -3)^T$ and cost coefficient $c_7 = -7$ is introduced. The relative cost coefficient of x_7 with respect to the basis in Tableau 9.2 is $\bar{c}_7 = c_7 + (-\tilde{\pi})A_{.7} = -7 + (-2, 4, -1)(1, 2, -3)^T = 2$. Hence, the previous optimum solution with $x_7 = 0$ is still optimal. The optimum feasible solution of the new problem is $\tilde{X} = (3, 4, 2, 0, 0, 0, 0)^T$, with the optimal objective value $= \tilde{Z} = 11$. Thus, if x_7 is the level of some new activity that has become available, we conclude that it is optimal to perform the new activity at zero level.

Example 9.2

Consider the LP in Tableau 9.1 again. Suppose we have to include a new variable x_7 with $A_{.7} = (3, -1, 1)^T$ and $c_7 = 4$. The relative cost coefficient of x_7 with respect to the basis in Tableau 9.2 is $\bar{c}_7 = 4 + (-2, 4, -1)(3, -1, 1)^T = -7 < 0$. We conclude that it is optimal to include the new activity in the basic vector. The updated column vector of x_7 with respect to the basis in Tableau 9.2 is

$$\begin{pmatrix} \bar{A}_{.7} \\ \cdots \\ \bar{c}_7 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -7 \end{pmatrix}$$

This is the pivot column. The minimum ratio test indicates that x_1 drops out of the basic vector when x_7 enters.

Basic Variables	Inverse Tableau				Basic Values
x_7	-1	-2	2	0	3
x_2	2	3	-3	0	1
x_3	-3	-4	5	0	8
$-z$	-9	-10	13	1	10

Since the relative cost coefficient of x_6 here is -6, this basis is again not optimal to the augmented problem. Bring x_6 into the basic vector and continue the algorithm until a terminal basis for the augmented problem is obtained.

Exercises

9.1 Complete the solution of this numerical problem.

9.2 Prove that the optimum objective value in (9.2) is less than or equal to the optimum objective value in (9.1). Construct a numerical example to show that the objective function in (9.2) may be unbounded below even though (9.1) has an optimum feasible solution.

9.3 How is the set of feasible solutions of the augmented problem related to **K**?

9.2.1 What Useful Planning Information Can Be Derived from This Type of Sensitivity Analysis?

The sensitivity analysis discussed above is very simple, and yet when a company has a linear programming model of its production operations, it can yield extremely useful planning information. For an example, consider the fertilizer problem (1.1) modeled in Section 1.1.3. From the discussion in Section 4.6.3, we know that the optimum dual solution associated with (1.1) is $\bar{\pi} = (5, 5, 0)$. Suppose a research chemist working for this manufacturer has

$M_1 - 2 \quad 1 \quad \square \quad C_i = t$

$M - 1 \quad X \quad \square \quad C_1 = t_1$

$m =>$

$m-$

come up with the formula for a new fertilizer called *Lushlawn*. The manufacture of Lushlawn requires as inputs $A_{.6} = (3, 2, 2)^T$ tons of raw materials 1, 2, and 3, respectively, per ton. Should the manufacturer produce this new fertilizer? How much profit (in dollars per ton) should 1 ton of Lushlawn fetch in the marketplace for the manufacturer to consider it worth producing? To answer these questions (1.1) can be transformed into standard form, and the analysis discussed above is applied. It can be verified that the breakeven profit for Lushlawn is $\bar{\pi}A_{.6} = (5, 5, 0)(3, 2, 2)^T = 25$ \$/ton. That is, Lushlawn is worth producing if it can be sold in the market at a price that leads to a profit $\geqq 25$ \$/ton manufactured. The fertilizer manufacturer can conduct a survey and determine whether the market would accept Lushlawn at a price greater than or equal to this breakeven level. Based on the results of the survey, the decision whether to produce Lushlawn or not can be taken very easily. Also, once the profit level for Lushlawn is set, the new optimum solution for the manufacturer can be determined easily, as discussed above, beginning with an optimum feasible basic vector for the standard form of (1.1).

Many companies use this type of sensitivity analysis to determine whether new products or processes would turn out to be profitable, to set prices on new products, and to estimate how sales volumes of existing products will be effected by the introduction of new products.

9.3 INTRODUCING AN ADDITIONAL INEQUALITY CONSTRAINT

Consider the LP (9.1) and the optimum basis $\bar{B}$ for it again. Suppose the additional constraint

$$\boxed{A_{m+1.}x \leqq b_{m+1}} \tag{9.3}$$

has to be introduced. Let $\mathbf{K}_1$ denote the set of feasible solutions of the augmented problem. $\mathbf{K}_1 \subseteq \mathbf{K}$ (see Figure 9.1). If the original problem has an optimum feasible solution, the augmented problem is either infeasible or it has an optimum feasible solution. Also $z(\bar{x}) = \text{minimum } \{z(x): x \in \mathbf{K}\} \leqq \text{minimum } \{z(x): x \in \mathbf{K}_1\}$. Hence, if $\bar{x} \in \mathbf{K}_1$, that is, $\bar{x}$ satisfies (9.3), $\bar{x}$ remains optimal to the augmented problem. On the other hand, if $\bar{x}$ does not satisfy (9.3), then

$$\bar{x}_{n+1} = -A_{m+1.}\bar{x} + b_{m+1} < 0 \tag{9.4}$$

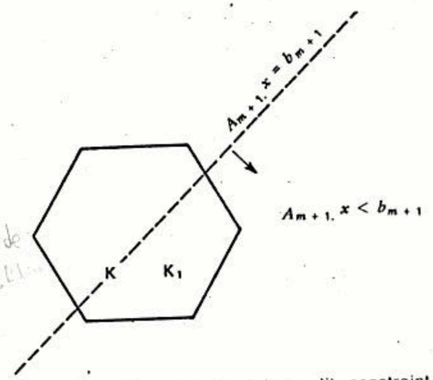

Figure 9.1 Introducing a new inequality constraint.

where x_{n+1} is the slack variable corresponding to (9.3). The augmented problem is

$$\text{Minimize} \quad Z(X) = \sum_{j=1}^{n} c_j x_j + 0x_{n+1}$$

$$\text{Subject to} \quad \begin{pmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ A_{n+1} & \vdots & 1 \end{pmatrix} \begin{pmatrix} x \\ \cdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} b \\ \cdots \\ b_{n+1} \end{pmatrix} \tag{9.5}$$

$$x_j \geqq 0 \qquad \text{for all } j$$

Let $X^T = (x^T, x_{n+1})$. By including x_{n+1} as an additional basic variable, we can enlarge $\bar{B}$ into a basis for (9.5), denoted by $\tilde{B}$. $X_{\tilde{B}} = (x_1, \ldots, x_m, x_{n+1})^T$. The basic solution of (9.5) corresponding to the basis $\tilde{B}$ is $\tilde{X}^T = (\bar{x}^T, \bar{x}_{n+1})$. By (9.4), $\tilde{B}$ is an infeasible basis for (9.5). Since $c_{n+1} = 0$, the relative cost coefficient of x_j with respect to the basis $\tilde{B}$ in (9.5) is equal to the relative cost coefficient of x_j with respect to the basis $\bar{B}$ in (9.1), which is $\bar{c}_j \geqq 0$ for all $j = 1$ to n. Thus, $\tilde{B}$ is a dual feasible but primal infeasible basis for (9.5). Using $\tilde{B}$ as an initial basis, we can solve (9.5) by using the dual simplex algorithm. We now discuss how to obtain the inverse tableau corresponding to the basis $\tilde{B}$ for (9.5).

$$\bar{B} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \quad \tilde{B} = \begin{bmatrix} a_{11} & \cdots & a_{1m} & 0 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} & 0 \\ a_{m+1,1} & \cdots & a_{m+1,m} & 1 \end{bmatrix}$$

Therefore

$$\tilde{B}^{-1} = \begin{pmatrix} \bar{B}^{-1} & \vdots & 0 \\ \cdots\cdots\cdots\cdots & & \cdots \\ -(a_{m+1,1}, \ldots, a_{m+1,m})\bar{B}^{-1} & \vdots & 1 \end{pmatrix}$$

Basic Variables	Inverse Tableau					Basic Values	Pivot Column (x_4)
x_1	−1	−2	2	0	0	3	−1
x_2	1	1	−1	0	0	4	1
x_3	−1	0	1	0	0	2	2
x_7	5	3	−6	1	0	−12	−4
$-z$	−2	4	−1	0	1	−11	1

Thus $\tilde{\mathbf{B}}^{-1}$ can easily be obtained from $\tilde{B}^{-1}$. Also obviously $\tilde{\Pi}$ = dual solution of (9.5) corresponding to $\tilde{\mathbf{B}}$ is equal to $(\tilde{\pi}, 0)$.

Thus the inverse tableau for (9.5) corresponding to the basis $\tilde{\mathbf{B}}$ is easily obtained from the inverse tableau for (9.1) corresponding to the basis $\tilde{B}$. On the other hand, if (9.1) was solved by computing the canonical tableaux after each pivot step, to obtain the canonical tableau of (9.5) with respect to the basis $\tilde{\mathbf{B}}$, introduce the $(m+1)$th constraint row at the bottom of the canonical tableau of (9.1) with respect to the basis $\tilde{B}$, include x_{n+1} as the basic variable in that row, and then price out all the basic column vectors of $\tilde{B}$ in it.

Example 9.3

Returning to the LP in Tableau 9.1, suppose the new constraint, $x_1 - x_2 + 3x_3 \leqq -7$, has to be imposed. The optimum solution $\tilde{x} = (3, 4, 2, 0, 0, 0)^T$ violates this new constraint. Let x_7 be the slack variable. $x_7 = -x_1 + x_2 - 3x_3 - 7$. The inverse tableau for the augmented problem corresponding to the basic vector (x_1, x_2, x_3, x_7) is given at the top. x_7 is the only negative-valued basic variable here. The updated row vector in which x_7 is the basic variable is given at the bottom. The pivot column is the updated column vector of x_4, which is entered as the last column in the inverse tableau at the top. The pivot is performed and the dual simplex algorithm applied in a similar manner until termination.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	b
Updated 4th Row	0	0	0	−4	0	−3	1	0	−12
Updated Cost Row	0	0	0	1	3	8	0	1	−11
$-\dfrac{\bar{c}_j}{\bar{a}_{4j}}$ for $\bar{a}_{4j} < 0$				$\frac{1}{4}$		$\frac{8}{3}$			

$\pi = C_B B^{-1}$ $\quad \pi = (3\ 2\ -3\ 0)\begin{pmatrix} -1 & -2 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 5 & 3 & -6 & 1 \end{pmatrix} = (2\ 4\ 1\ 0)$

Exercises

9.4 Complete the solution of this augmented problem.

9.5 Show that the type of sensitivity analysis discussed here (introducing an additional inequality constraint) is the dual of the one discussed in Section 9.2 (introducing a new sign restricted variable).

9.6 Prove that the set of extreme points of $\mathbf{K}_1$ consists of:

(a) All extreme points of $\mathbf{K}$ that satisfy (9.3);

(b) Points of intersection of edges of $\mathbf{K}$ that do not completely lie on the hyperplane $\mathbf{H}$, with $\mathbf{H}$, where $\mathbf{H} = \{x : A_{m+1.}x = b_{m+1}\}$.

—K. G. Murty [3.35]

9.7 If every optimum feasible solution of (9.1) violates (9.3), and if $\mathbf{K}_1 \neq \emptyset$, prove that every optimum feasible solution of the augmented problem satisfies (9.3) as an equation, that is, lies on the hyperplane $\mathbf{H}$ defined in Exercise 9.6.

9.4 INTRODUCING AN ADDITIONAL EQUALITY CONSTRAINT

Returning to the LP (9.1), suppose an additional equality constraint has to be introduced: $A_{m+1.}x = b_{m+1}$. The problem (9.1) together with this additional equality constraint is called the *augmented problem*.

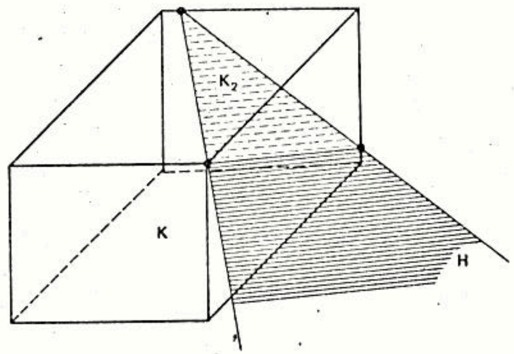

Figure 9.2 Introducing a new equality constraint.

Let $\mathbf{H} = \{x : A_{m+1.}x = b_{m+1}\}$. The set of feasible solutions of the augmented problem is $\mathbf{K}_2 = \mathbf{K} \cap \mathbf{H}$. See Figure 9.2 for an example. If the optimum solution of (9.1), $\bar{x} \in \mathbf{H}$, obviously $\bar{x}$ is optimal to the augmented problem. Suppose $\bar{x} \notin \mathbf{H}$, then $A_{m+1.}\bar{x} \neq b_{m+1}$. Suppose $A_{m+1.}\bar{x} > b_{m+1}$, then change the original problem by adding the constraints $A_{m+1.}x - x_{n+1} = b_{m+1}$, $x_{n+1} \geqq 0$. (If, on the other hand, it turned out that $A_{m+1.}\bar{x} < b_{m+1}$, then the coefficient of x_{n+1} should be $+1$ instead.) Here is the new problem:

Minimize $\quad Z(X) = cx + Mx_{n+1}$

Subject to $\quad \begin{pmatrix} A & \vdots & 0 \\ \cdots & \cdots & \cdots \\ A_{m+1.} & \vdots & -1 \end{pmatrix} \begin{pmatrix} x \\ \cdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} b \\ \cdots \\ b_{m+1} \end{pmatrix}$

$$X = \begin{pmatrix} x \\ \cdots \\ x_{n+1} \end{pmatrix} \geqq 0 \qquad (9.6)$$

where $\boldsymbol{M}$ is an arbitrarily large positive number. We will refer to (9.6) as the *new problem*. The variable x_{n+1} is an artificial variable in the new problem, and hence it is included in the objective function in (9.6) with a coefficient of $\boldsymbol{M}$, as in the Big-$\boldsymbol{M}$ method. Clearly, $X_{\tilde{\mathbf{B}}} = (x_1, \ldots, x_m, x_{n+1})$ is a basic vector for the new problem, and the basic solution of the new problem corresponding to this basic vector can be verified to be $\tilde{X} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{x}_{n+1})^{\mathrm{T}}$, where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is the BFS of the original problem (9.1) corresponding to the optimal basic vector $x_{\bar{B}}$ for it and $\bar{x}_{n+1} = A_{m+1.}\bar{x} - b_{m+1}$. Since $\bar{x}_{n+1} > 0$ by our assumptions, $\tilde{X} \geqq 0$, and hence $X_{\tilde{\mathbf{B}}}$ is a feasible basic vector for (9.6). The associated basis for (9.6), $\tilde{\boldsymbol{B}}$, is

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} a_{11} & \cdots & a_{1m} & 0 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} & 0 \\ a_{m+1,1} & \cdots & a_{m+1,m} & -1 \end{bmatrix}$$

$$\therefore \quad \tilde{\boldsymbol{B}}^{-1} = \begin{pmatrix} \bar{B}^{-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ (a_{m+1,1}, \ldots, a_{m+1,m})\bar{B}^{-1} & \vdots & -1 \end{pmatrix}$$

Define $c_{n+1} = 0$, $c_j^* = 0$, for $j = 1$ to n, and $c_{n+1}^* = 1$. Then the objective function in (9.6) is $Z(X) = \sum_{j=1}^{n+1} (c_j + Mc_j^*)x_j$. In the original tableau for (9.6) the cost coefficients can be entered in two separate rows, as under the Big-M method discussed in Section 2.7. One row contains the c_j's, the cost coefficients in $z(x)$ from the original objective function. The second row contains c_j^*'s, the coefficients of $\boldsymbol{M}$ in $Z(X)$. Compute $\Pi_{\tilde{\mathbf{B}}} = (\Pi_1, \ldots, \Pi_{m+1}) = c_{\tilde{B}}\tilde{B}^{-1}$, $\Pi_{\tilde{\mathbf{B}}}^* = (\Pi_1^*, \ldots, \Pi_{m+1}^*) = \mathbf{c}_{\tilde{\mathbf{B}}}^*\tilde{\boldsymbol{B}}^{-1}$. The inverse tableau for (9.6), corresponding to the basic vector $X_{\tilde{\mathbf{B}}}$ is:

Basic Variables	*Inverse Tableau*		*Basic Values*
x_1 $\vdots$ x_{n+1}	$\tilde{\boldsymbol{B}}^{-1}$	0	b_1 $\vdots$ b_{m+1}
$-Z$	$-\Pi_1 \cdots -\Pi_{m+1}$ $-\Pi_1^* \cdots -\Pi_{m+1}^*$	1 0 0 1	$-Z$ $-Z^*$

Letting $\bar{c}_j = c_j - \Pi_{\mathbf{B}}A_{.j}$, $\bar{c}_j^* = c_j^* - \Pi_{\mathbf{B}}^*A_{.j}$, the actual relative cost coefficient of x_j in (9.6) with respect to the current basic vector is $\bar{c}_j + M\bar{c}_j^*$. If $\bar{c}_j + M\bar{c}_j^* \geqq 0$ for each j, the algorithm terminates. If these termination conditions are not satisfied, let $\mathbf{J} = \{j : j$, such that $\bar{c}_j^* < 0$, or $\bar{c}_j^* = 0$ and $\bar{c}_j < 0\}$. The *eligible variables* at this stage are those x_j for $j \in \mathbf{J}$. Select one of the eligible variables as the entering variable, bring it into the basic vector, and continue in this manner until a basic vector satisfying the above termination conditions is obtained. Suppose $\hat{X} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{x}_{n+1})^{\mathrm{T}}$ is the final optimum solution obtained for (9.6). If $\hat{x}_{n+1} > 0$, $\mathbf{K}_2 = \varnothing$, in this case the additional equality constraint has made the original problem infeasible. On the other hand, if $\hat{x}_{n+1} = 0$, $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^{\mathrm{T}}$ is an optimum feasible solution of the augmented problem.

Basic Variables	Inverse Tableau						Basic Values	Pivot Column x_6	Ratios
x_1	−1	−2	2	0	0	0	3	−2	
x_2	1	1	−1	0	0	0	4	−2	
x_3	−1	0	1	0	0	0	2	1	2
x_7	−3	−4	4	−1	0	0	4	2	4/2 minimum
$-Z$	−2	4	−1	0	1	0	−11	8	
	3	4	−4	1	0	1	−4	−2	

Example 9.4

Consider the LP in Tableau 9.1. Suppose the additional constraint $x_1 - 2x_2 + x_4 + 3x_5 = -9$ is to be included. $\bar{x} = (3, 4, 2, 0, 0, 0)^T$, the optimum solution of the original problem, violates this new constraint. Let $\bar{x}_7 = \bar{x}_1 - 2\bar{x}_2 + \bar{x}_4 + 3\bar{x}_5 - (-9) = 4$. The new problem as discussed above, is:

New Problem

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-Z$	b
	1	2	0	1	0	−6	0	0	11
	0	1	1	3	−2	−1	0	0	6
	1	2	1	3	−1	−5	0	0	13
	1	−2	0	1	3	0	−1	0	−9
c	3	2	−3	−6	10	−5	0	1	0
c^*	0	0	0	0	0	0	0		0

$x_j \geq 0$ for all j, minimize Z, x_7 is the artificial variable.

Let $\tilde{B}$ be the basis for the new problem corresponding to the basic vector (x_1, x_2, x_3, x_7). The inverse tableau for the new problem with respect to the basis $\tilde{B}$ is given at the top.

The updated cost rows are $\bar{c} = (0, 0, 0, 1, 3, 8, 0)$ and $\bar{c}^* = (0, 0, 0, 4, -1, -2, 0)$. So x_5 and x_6 are eligible to enter the basic vector. Suppose we choose x_6 as the entering variable. Continuing the application of the Big-M method, it can be verified that the optimum solution of the new problem is $X^T = (0, 8, 7/2, 0, 7/3, 5/6, 0)^T$. Since $x_7 = 0$ in this solution, $x = (0, 8, 7/2, 0, 7/3, 5/6)^T$ is an optimum solution of the augmented problem.

Exercises

9.8 Consider the LP in Tableau 9.1. Suppose the additional constraint $-x_1 - x_2 - x_3 + x_4 = 300$ has to be introduced. Apply the algorithm discussed here to solve the augmented problem. Show that the augmented problem is infeasible.

9.9 Returning to the LP in Tableau 9.1, suppose the additional constraint $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$ has to be imposed. Obtain an optimum feasible solution of the augmented problem.

9.10 Discuss an approach for solving the augmented problem with the additional equality constraint, using the dual simplex algorithm instead of the Big-M approach described here.

9.11 Prove that the set of extreme points of $\mathbf{K}_2$ consists of (a) all extreme points of $\mathbf{K}$ that are in $\mathbf{H}$, and (b) points of intersection of edges of $\mathbf{K}$ that do not completely lie in $\mathbf{H}$ with $\mathbf{H}$.

9.12 Let $\mathbf{H}$, $\mathbf{K}_2$ be defined as in the beginning of this section. Let $\mathbf{K}_1 = \mathbf{K} \cap \{x : \mathbf{A}_{m+1.}x \leqq b_{m+1}\}$, $\mathbf{K}_3 = \mathbf{K} \cap \{x : \mathbf{A}_{m+1.}x \geqq b_{m+1}\}$. If $\mathbf{K}_2 \neq \emptyset$, prove that $\mathbf{K}_2$ contains an optimum solution of at least one of the following two problems.

(a) Minimize $z(x) = cx$
$x \in \mathbf{K}_1$

(b) Minimize $z(x) = cx$
$x \in \mathbf{K}_3$

9.5 COST RANGING OF A NONBASIC COST COEFFICIENT

Consider again the LP (9.1). Suppose x_r is a variable that is not in the optimal basic vector $x_{\tilde{B}}$. Assuming that all the other cost coefficients except c_r remain fixed at their specified values, determine the range of values of c_r within which $\tilde{B}$ remains an optimal basis. For this, treat c_r as a parameter. $\tilde{B}$ remains an optimal basis as long as $\bar{c}_r = c_r - \bar{\pi}A_{.r} \geqq 0$, that is

$c_r \geqq \bar{\pi}A_{.r}$. The relative cost coefficients of all the other variables are independent of the value of c_r and, hence, they remain nonnegative. If the new value of c_r is not in the closed interval $[\bar{\pi}A_{.r}, \infty]$, x_r is the only nonbasic variable that has a negative relative cost coefficient with respect to the basis $\bar{B}$ in the modified problem. Bring x_r into the basic vector and continue the application of the simplex algorithm until a terminal basis for the modified problem is obtained.

Example 9.5

Consider the LP in Tableau 9.1. The range of values of c_5 (whose present value is 10) for which the basic vector $x_B = (x_1, x_2, x_3)$ remains optimal is determined by $c_5 + (-2, 4, -1)(0, -2, -1)^T \geqq 0$, that is, $c_5 \geqq 7$. Suppose the value of c_5 has to be modified to 6. When c_5 is changed from 10 to 6, the updated column vector of x_5 becomes $(2, -1, -1, -1)^T$. With this as the pivot column, bring x_5 into the basic vector. This leads to the following tableau.

Basic Variables	Inverse Tableau				Basic Values
x_5	$-\frac{1}{2}$	-1	1	0	$\frac{3}{2}$
x_2	$\frac{1}{2}$	0	0	0	$\frac{11}{2}$
x_3	$-\frac{3}{2}$	-1	2	0	$\frac{7}{2}$
$-z$	$-\frac{5}{2}$	3	0	1	$-\frac{19}{2}$

It can be verified that this tableau displays an optimum tableau for the modified problem.

9.6 COST RANGING OF A BASIC COST COEFFICIENT

Consider the LP (9.1). Suppose all the cost coefficients except c_1, which is a basic cost coefficient, remain fixed at their present values. Treating c_1 as a parameter, determine the range of values of c_1 for which $\bar{B}$ remains an optimal basis. Since c_1 is a basic cost coefficient, any change in c_1 changes the dual solution corresponding to the basis $\bar{B}$, and all the relative cost coefficients. For simplicity let γ_1 denote the parameter and c_1 its present value. Let $\pi(\gamma_1)$, $\bar{c}_j(\gamma_1)$, etc., denote the dual solution and the relative cost coefficient of x_j corresponding to the basis $\bar{B}$ as functions of the parameter γ_1. Then $\pi(\gamma_1) = (\gamma_1, c_2, \ldots, c_m)\bar{B}^{-1}$. Hence, the dual vector is an affine function of γ_1. If we use this dual solution, the relative cost coefficient of x_j is $\bar{c}_j(\gamma_1) = c_j - \pi(\gamma_1)A_{.j}$, for $j = 1$ to n. Hence, each $\bar{c}_j(\gamma_1)$ is itself an affine function of γ_1. The range of values of γ_1 for which $\bar{B}$ remains an optimum basis is the range of γ_1 within which every $\bar{c}_j(\gamma_1)$ remains nonnegative. This range will turn out to be a nonempty closed interval. If it is necessary to modify the cost coefficient of x_1 to some value c'_1 outside this closed interval, fix $\gamma_1 = c'_1$ and compute the relative cost coefficients $\bar{c}_j(c'_1)$ using the formulas already developed. Bring one of the variables for which $\bar{c}_j(c'_1) < 0$ into the basic vector and continue the applications of the simplex algorithm until a terminal basis for the modified problem is obtained.

Example 9.6

In the LP in Tableau 9.1, let γ_1 represent the cost coefficient of x_1 whose present value is 3. The dual solution corresponding to the basic vector (x_1, x_2, x_3) as a function of γ_1 is $\pi(\gamma_1) = (\gamma_1, 2, -3)\bar{B}^{-1} = (-\gamma_1 + 5, -2\gamma_1 + 2, 2\gamma_1 - 5)$. Using $\bar{c}_j(\gamma_1) = c_j - \pi(\gamma_1)A_{.j}$, the row of relative cost coefficients is $\bar{c}(\gamma_1) = (0, 0, 0, \gamma_1 - 2, -2\gamma_1 + 9, 2\gamma_1 + 2) \geqq 0$ iff $2 \leqq \gamma_1 \leqq 9/2$. So $\bar{B}$ is an optimum basis whenever the cost coefficient of x_1 is in the interval $[2, 9/2]$, assuming that all the other cost coefficients remain at their present values.

Suppose γ_1 has to be changed to 5. Changing the cost coefficient of x_1 from the present value of 3 to 5, the modified dual solution is $\pi(\gamma_1 = 5) = (0, -8, 5)$. With this change, the inverse tableau becomes

Basic Variables	Modified Inverse Tableau for Basis B				Basic Values	Pivot Column (x_5)
x_1	-1	-2	2	0	3	② Pivot Row
x_2	1	1	-1	0	4	-1
x_3	-1	0	1	0	2	-1
$-z$	0	8	-5	1	-17	-1

The modified relative cost coefficient of x_5 is -1. Hence, bring x_5 into the basic vector. The updated column vector of x_5 is entered as the pivot column in the inverse tableau. Remember that the cost coefficient of x_1 is now 5, and continue the algorithm.

Exercises

9.13 Construct a counter example to the following: "If B is an optimum basis for (9.1), then it remains an optimum basis when some of the entries in c_B are decreased."

9.14 Is the following true? "x_1 is a basic variable in an optimum basic vector for (9.1). If c_1 is decreased and if the modified problem has an optimum feasible solution, then it has an optimum basic vector containing x_1."

9.7 RIGHT-HAND-SIDE RANGING

In the LP (9.1) suppose we wish to determine the range of values of one of the right-hand-side constants, say b_1, for which the basis $\tilde{B}$ remains optimal. Treat this right-hand-side constant as a parameter and denote it by β_1; b_1 is the present value of β_1. We assume that all the other right-hand-side constants stay fixed at their present values. $\tilde{B}$ is an optimum basis when $\beta_1 = b_1$. Hence, $\tilde{B}$ is dual feasible. So $\tilde{B}$ is optimal for all values of β_1 for which it is primal feasible. The values of the basic variables $x_{\tilde{B}}$ are all functions of the parameter β_1, and $\tilde{B}$ is primal feasible as long as all these values are nonnegative. $x_{\tilde{B}}(\beta_1) = \tilde{B}^{-1}(\beta_1, b_2, \ldots, b_m)^T \geqq 0$. Since all these inequalities are linear in β_1, this will determine a closed interval of the form $\underline{\lambda} \leqq \beta_1 \leqq \bar{\lambda}$. For all values of β_1 in this closed interval, $\tilde{B}$ remains an optimal basis. If it is necessary to modify the value of β_1 from its present value b_1 to a value b'_1 outside the closed interval $[\underline{\lambda}, \bar{\lambda}]$, fix β_1 at b'_1 and obtain the modified values of the basic variables. $\tilde{B}$ is still dual feasible, but primal infeasible. Starting with $\tilde{B}$, apply the dual simplex routine until a new terminal basis is obtained.

Example 9.7

In the LP in Tableau 9.1 the values of the basic variables in the basic vector (x_1, x_2, x_3) as functions of β_1 are

$$\tilde{B}^{-1}\begin{pmatrix}\beta_1\\6\\13\end{pmatrix} = \begin{pmatrix}-1 & -2 & 2\\1 & 1 & -1\\-1 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta_1\\6\\13\end{pmatrix} = \begin{pmatrix}-\beta_1 + 14\\\beta_1 - 7\\-\beta_1 + 13\end{pmatrix}$$

All these are nonnegative iff $7 \leqq \beta_1 \leqq 13$. Hence, this is the interval within which the basic vector (x_1, x_2, x_3) remains optimal. In this range, an optimum solution of the problem is $x = (14 - \beta_1, -7 + \beta_1, 13 - \beta_1, 0, 0, 0)^T$, with an optimum objective value of $-11 + 2\beta_1$ and an associated optimum dual solution of $\pi = (2, -4, 1)$.

Now, suppose it is required to solve the problem for $\beta_1 = 15$. The inverse tableau for the modified problem corresponding to the present basic vector (x_1, x_2, x_3) is:

Basic Variables	Inverse Tableau				Basic Values	Pivot Column
x_1	-1	-2	2	0	-1	2
x_2	1	1	-1	0	8	-1
x_3	-1	0	1	0	-2	-1
$-z$	-2	4	-1	1	-19	3

Applying the dual simplex algorithm starting with this inverse tableau, it can be verified that the modified problem is infeasible.

Exercises

9.15 Obtain the new optimum feasible solution of the LP in Tableau 9.1 after b_1 is changed from its present value of 11 to 6.

9.8 CHANGES IN THE INPUT-OUTPUT COEFFICIENTS IN A NONBASIC COLUMN VECTOR

Let x_j be a variable that is not in the optimum basic vector of $x_{\tilde{B}}$ of (9.1). And a_{ij} is one of the input-output coefficients in the column vector of x_j. If all the other coefficients in the problem except a_{ij} remain fixed at their present values, what is the range of values of a_{ij} within which $\tilde{B}$ remains an optimal basis?

Treat a_{ij} as a parameter. To avoid confusion, call this parameter, α_{ij}, and let a_{ij} be its present value. Since x_j is a nonbasic variable, a change in α_{ij} does not affect the primal feasibility of $\tilde{B}$. A change in α_{ij} can only change the relative cost coefficient of x_j

and this is obtained by $\bar{c}_j(\alpha_{ij}) = c_j + (\sum_{r \neq i}(-\bar{\pi}_r)a_{rj}) - \bar{\pi}_i\alpha_{ij}$. As long as $\bar{c}_j(\alpha_{ij}) \geqq 0$, $\tilde{B}$ remains an optimal basis. This determines a closed interval for α_{ij}, and as long as α_{ij} is in this interval $\tilde{B}$ remains an optimal basis. If α_{ij} has to be changed to a value a'_{ij} outside this interval, make the change, bring x_j into the basic vector, and continue with the application of the simplex algorithm until a new terminal basis is obtained.

Example 9.8

In the LP in Tableau 9.1, the present value of α_{25} in the column vector of x_5 is -2. The relative cost coefficient $\bar{c}_5(\alpha_{25}) = 10 + (-2, 4, -1)(0, \alpha_{25} - 1)^T = 11 + 4\alpha_{25} \geqq 0$ iff $\alpha_{25} \geqq -11/4$. Thus $\tilde{B}$ remains an optimal basis for $\alpha_{25} \geqq -11/4$. Suppose we change α_{25} from its present value of -2 to -3. The updated column vector of x_5 changes to $(4, -2, -1, -1)^T$. With this pivot column bring x_5 into the basic vector and continue until termination is reached.

9.9 CHANGES IN A BASIC INPUT-OUTPUT COEFFICIENT

Let x_1 be a basic variable in the optimum basic vector $x_{\tilde{B}}$ for (9.1). Suppose we have to modify one input-output coefficient, say a_{11}, in the column vector of x_1 to a'_{11}. The modified column vector of x_1 will be $A'_{.1} = (a'_{11}, a_{21}, \ldots, a_{m1})^T$. Let x'_1 indicate the level of this activity corresponding to the new column vector $A'_{.1}$. The previous column vector $A_{.1}$ is no longer a part of the problem and it should be eliminated. Physically x'_1 replaces x_1 in the original tableau. We will refer to the altered problem by the name *modified problem*.

Construct a new problem by augmenting the present original tableau with the new variable x'_1 with its column vector $A'_{.1}$ and cost coefficient c_1, and changing the cost coefficient of x_1 to M, where M is a very large positive number. This leads to the problem, which we call the *new problem*. In this problem x_1 plays the role of an artificial variable, associated with a cost coefficient of M, a very large positive number. The objective function Z, in the new problem is $\sum_{j=2}^{n} c_j x_j + c_1 x'_1 + Mx_1$. As in the Big-$M$ method, the cost coefficients are entered in two rows. $\tilde{B}$ is still a feasible basis to this new problem. However, in the inverse tableau corresponding to the basis $\tilde{B}$, the dual vector has to be recomputed, since the cost coefficient of x_1 has been changed to M. The new dual solution corresponding to $\tilde{B}$ is $\pi(M) = (M, c_2, \ldots, c_m)\tilde{B}^{-1} = \pi + M\pi^*$, where $\pi = (0, c_2, \ldots, c_m)\tilde{B}^{-1}$, $\pi^* = (1, 0, \ldots, 0)\tilde{B}^{-1}$. Construct the inverse tableau for the new problem, corresponding to the basic vector $x_{\tilde{B}}$ and solve it as discussed in Section 9.4, and the normal termination conclusions of the big-M method apply.

9.10 PRACTICAL APPLICATIONS OF SENSITIVITY ANALYSIS

When studying a system using a linear programming model, techniques of sensitivity analysis can be used to evaluate new products (as in Section 9.2), to determine the breakeven selling price of a new product, at which point it becomes competitive with the existing list of products in terms of profitability (as in Sections 9.2 and 9.5); to evaluate new technologies or processes for making products (as in Sections 9.8 and 9.9); to assess how profitable it is to acquire additional resources and to determine which resources to acquire in what quantities (using the ideas discussed in Sections 4.6.3, 9.7, and Chapter 8); to evaluate the effects of changes in the costs (as in Sections 9.5, 9.6, and Chapter 8); and to determine optimal policies for handling new constraints that might arise. When used in this manner, the linear programming model not only determines an optimum solution to implement but becomes a tool for optimal planning. Examples of some of these uses are discussed in the problems that follow.

Optimization models other than linear programming models (e.g., integer programming models and nonlinear programming models) do not lend themselves that readily to a marginal analysis or sensitivity analysis as the linear programming model. That's why if a practical problem can be approximated reasonably closely by a linear programming model, it is so much easier to study it than otherwise.

Exercises

9.16 Consider the family's diet problem discussed in Chapter 4. The original tableau for it is given at the top of page 319. Here x_1 to x_6 are the kilograms of the primary foods 1 to 6 in the family's diet, and x_7, x_8 are the slack variables representing the excess amounts of the nutrients, vitamins A and C, in the diet over the minimum requirements. The basis B_1, Original Tableau for the Family's Diet Problem

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$-z$	b
1	0	2	2	1	2	−1	0	0	9
0	1	3	1	3	2	0	−1	0	19
35	30	60	50	27	22	0	0	1	0 (minimize z)

$x_j \geqq 0$ for all j.

associated with the basic vector (x_5, x_6) is optimal to this problem. The optimum inverse tableau is:

Basic Variables	*Inverse Tableau*			*Basic Values*
x_5	$-\frac{1}{2}$	$\frac{1}{2}$	0	5
x_6	$\frac{3}{4}$	$-\frac{1}{4}$	0	2
$-z$	−3	−8	1	−179

Answer each of the following questions with respect to this original problem:

(a) Suppose a new primary food, food 7, is available in the market at 88 cents/kg. One kilogram of this food contains two units of vitamin A and four units of vitamin C. Should the family include this new food in its diet? If not, how much should the cost of this food decrease before the family can consider including it in its diet?

At this breakeven cost show that there is an optimum diet that includes food 7 and another that does not.

What is the optimum diet if food 7 is actually available at 32 cents/kg?

(b) Consider the original family's diet problem again. The family has just read an article in a health magazine that says that another nutrient, vitamin E, is very important for the family's health. The vitamin E content of the six primary foods are 2, 3, 5, 2, 1, and 1 units/kg, respectively. The minimum requirement of vitamin E is 10 units. The family wants to include this additional constraint in the problem. How does this change its optimal diet?

(c) Referring to the original problem, in addition to meeting the minimum requirements on vitamins A and C, suppose the family decides to have the diet consist of a total of 2000 calories exactly. The calorie contents of the six primary foods are 160, 20, 500, 280, 300, and 360/kg, respectively. How does this additional requirement change its optimal diet?

(d) What is the marginal effect of increasing the minimum requirement of vitamin C on the cost of the optimal diet in the original family's diet problem? How much can the minimal requirement of vitamin C increase before the basis B_1 becomes nonoptimal to the problem? What is an optimal diet when this requirement is 39 units?

(e) In the original family's diet problem, suppose that each additional unit of vitamin A in the diet is expected to bring an average savings of 10 cents in medical expenses. How does this alter the optimal diet?

(f) For what range of cost per kilogram of primary foods 4 and 5 does the basis B_1 remain optimal to the original family's diet problem?

(g) What happens to the optimal diet in the original family's diet problem if the vitamin C content of food 4 changes? For what range of values of this quantity does the basis B_1 remain optimal? Suppose a richer version of food 4 containing $1 + \alpha$ units of vitamin C per kilogram is available at a cost of $50 + 4\alpha$ cents/kg for any $\alpha \geqq 0$. What is the minimum value of α at which it becomes attractive for the family to include it in its diet?

9.17 Data on different solvents are given in the table at the top of page 320. Let x_j be the

	Solvent				Chemical Requirement in Blend per kg
	1	2	3	4	
Chemical 1 content (units/kg)	180	120	90	60	$\geqq 90$
Chemical 2 content (units/kg)	3	2	6	5	$\leqq 4$
Cost (cent/kg)	16	12	10	11	

proportion of solvent type j in the blend, $j = 1$ to 4. Variables x_5 and x_6 are the slack variables associated with chemicals 1 and 2 requirements, respectively. The basis B_1 associated with the basic vector (x_2, x_3, x_5) is an optimum basis for the problem. Compute B_1^{-1}, and using it, answer the following parts, each of which is independent of the others.

(a) Write down the dual problem and the complementary slackness conditions for optimality. Obtain an optimum feasible solution for the dual problem from the above information.

(b) How much does the optimum objective value change if the minimal chemical 1 requirement is changed to 88 units? Why?

(c) Let β_1 be the minimal chemical 1 requirement. Its present value is 90. For what range of values of β_1 does B_1 remain optimal to the problem? When β_1 is 114, what is an optimum feasible solution to the problem?

(d) How much can the cost per kilogram of solvent 3 change before B_1 becomes nonoptimal to the problem? When the cost per kilogram of solvent 3 becomes 11 cents, what is an optimum solution?

9.18 Consider the LP given at the bottom. Variables x_5, x_6, x_7 are the slack variables corresponding to the various inequalities. The basis B_1 corresponding to the basic vector (x_1, x_3, x_2) is optimal to the problem. Compute B_1^{-1}

(a) If the availability of only one of the raw materials can be marginally increased, which one should be picked? Why?

(b) For what range of values of b_1 (the amount of raw material 1 available) does the basis B_1 remain optimal? What is an optimal solution to the problem if $b_1 = 20$?

(c) If seven more units of raw material 1 can be made available (over the present 8 units), what is the maximum you can afford to pay for it? Why?

(d) The company has an option to produce a new product. Let x_8 be the number of units of this product manufactured. The input-output vector of x_8 will be $(10, 20, 24 - 3\lambda, -25, + \lambda/4)^T$, where λ is a parameter that can be set anywhere from 0 to 6. What is the minimum value of λ at which it becomes profitable to produce the product? What is an optimum solution when $\lambda = 4$?

9.19 Consider the diet problem with data given at the top of page 321. Let x_1, x_2, and x_3 be the amounts of greens, potatoes, and corn included in the diet, respectively. Let x_4, x_5, and x_6 be the slack variables representing the excess of vitamins A, C, and D, respectively, in the diet. The basis B_1 associated with the basic vector (x_4, x_1, x_6) is optimal to this problem.

(a) Find the optimum primal and dual solutions associated with the basis B_1.

$$
\begin{array}{lrl}
\text{Minimize} & z(x) = -2x_1 - 4x_2 - x_3 - x_4 & \\
\text{Subject to} & x_1 + 3x_2 \qquad + x_4 \leqq 8 & = \text{available amount of raw material 1} \\
& 2x_1 + x_2 \qquad\qquad \leqq 6 & = \text{available amount of raw material 2} \\
& x_2 + 4x_3 + x_4 \leqq 6 & = \text{available amount of raw material 3} \\
& x_j \geqq 0 \quad \text{for all } j &
\end{array}
$$

Nutrient	Nutrient Content in Foods Available (units/kg)			Minimum Daily Requirement (MDR) for Nutrient
	Greens	Potatoes	Corn	
Vitamin A	10	1	9	5
C	10	10	10	50
D	10	11	11	10
Cost (cent/kg)	50	100	51	

(b) A new food (milk) has become available. One liter of milk contains 0, 10, and 20 units, respectively, of vitamins A, C, and D and costs 40 cents. Would you recommend including it in the diet? Why? What is the highest price of milk at which it is still attractive to include it in the diet?

(c) Consider the original problem again. A nutrition specialist claims that the actual MDRs for vitamins A, C, and D should really be 5, $50 + 10\lambda$, and $10 + 15\lambda$, where λ is a nonnegative parameter; and an experiment is proposed to estimate λ. Assuming that the claim is correct, find $\bar{\lambda}$, the maximum value of λ at which B_1 is still an optimum basis to the problem. What is an optimum solution if $\lambda = \bar{\lambda} + 1$?

9.20 Consider the LP (9.1) and its dual. Discuss what effects the following have on the primal and dual feasible solution sets and the respective optimal objective values.

(a) Introducing a new nonnegative primal variable.

(b) Introducing a new inequality constraint in the primal problem.

(c) Introducing a new equality constraint in the primal problem.

9.21 Consider the LP given at the bottom. Construct the inverse tableau for this problem corresponding to the basic vector (x_1, x_2, x_3) and verify that it is an optimum basic vector. Using it, answer the following questions. Each of these questions is independent of the others, and each refers to the original problem given above.

(a) Find the range of values of the parameter c_2 for which the solution obtained above remains optimal to the problem. If the new value of c_2 is slightly greater than the upper bound of the optimality range computed above, which variables become eligible to enter the basic vector? Why?

(b) Find the range of values of b_1 for which the basic vector (x_1, x_2, x_3) remains optimal to the problem. What is the slope of the optimum objective value as a function of b_1 in this optimality range? Why?

(c) Consider the original LP back again. Suppose a new activity has become available. This activity leads to a new nonnegative variable x_7, with the data $A_{.7} = (0, 2, 3)^T$, $c_7 = 18$, in the original tableau for the problem. Is it worthwhile for the decision maker to perform this new activity? Why? If not, determine the value to which c_7 should decrease (assuming that all the other data, including the entries in $A_{.7}$, remain unaltered) before this new activity becomes economically competitive. Find an optimum solution to the augmented problem assuming that $A_{.7}$ is as given above, but that $c_7 = 12$, using the methods of sensitivity analysis.

$$
\begin{array}{lrl}
\text{Minimize} & z(x) = & -4x_1 \qquad\qquad + 9x_3 + 12x_4 - 3x_5 + 7x_6 \\
\text{Subject to} & & x_1 + 2x_2 - x_3 + x_4 + 2x_5 + 2x_6 = 3 \\
& & \qquad\quad x_2 + x_3 + 2x_4 + x_5 + 2x_6 = 6 \\
& & -x_1 \qquad\quad + 2x_3 + 2x_4 - 2x_5 + x_6 = 5 \\
& & x_j \geqq 0 \quad \text{for all } j = 1 \text{ to } 6.
\end{array}
$$

$$\begin{array}{lll} \text{Minimize Cost} = 28x_1 + 67x_2 + 12x_3 + 35x_4 & & \\ \text{Subject to} & \text{Product 1 output} = & x_1 + 2x_2 \qquad\quad + x_4 \geqq 17 \\ & \text{Product 2 output} = & 2x_1 + 5x_2 + x_3 + 2x_4 \geqq 36 \\ & \text{Product 3 output} = & x_1 + x_2 \qquad\quad + 3x_4 \geqq 8 \\ & x_j \geqq 0 & \text{for all } j \end{array}$$

9.22 In the LP given at the top, x_5, x_6, x_7 are the slacks in that order. Vector (x_1, x_2, x_7) is an optimum basic vector. (1) What is the most critical product for the company? Why? (2) Determine the optimality range of c_1 for the present basis. (3) Determine the optimality range of b_3. (4) Find the new optimum when b_3 changes to 16 from 8.

9.23 In (9.1), let $g(b_1)$ denote the optimum objective value as a function of b_1 when all other data remain fixed. Prove that $g(b_1)$ is piecewise linear convex. Discuss how to minimize $g(b_1)$ over b_1. From this, discuss how to find the new optimum when a constraint from (9.1) is eliminated.

9.24 Let $\mathbf{K} = \{x: Ax = b, x \geqq 0\}$. Let $f_1(x) = cx$, $f_2(x) = dx$. If both $f_1(x)$ and $f_2(x)$ are >0 on $\mathbf{K}$, develop an algorithm for minimizing $f_1(x)f_2(x)$ over $x \in \mathbf{K}$, using a parametric right-hand-side LP. Generalize to the case where $f_1(x)$, $f_2(x)$ may not be positive on $\mathbf{K}$.

—[*Y. P. Aneja, V. Aggarwal, and K. P. K. Nair, "On a class of quadratic programs," University of New Brunswick, Fredericton, N. B., Canada*]

9.25 Let (P) be the LP: minimize cx, subject to $A_{i.}x \geqq b_i$, $i = 1$ to m. Let x^1 be optimal to the LP: minimize cx, subject to $A_{i.}x \geqq b_i$, $i = 2$ to m. Let x^2 be optimal to the LP: minimize cx, subject to $A_{1.}x = b_1$, $A_{i.}x \geqq b_i$, $i = 2$ to m. Prove that one of the points x^1 or x^2 must be optimal to (P).

9.26 Let $\mathbf{K}$ be the set of feasible solutions of (9.1), and $\mathbf{K}_1 = \{x: x \in \mathbf{K}, x_n = 0\}$. Assume that (9.1) is nondegenerate and that $\mathbf{K}$, $\mathbf{K}_1$, are both nonempty and bounded. All data are fixed except c_n. Prove that there exists a γ such that for all $c_n > \gamma$, x_n is a nonbasic variable in all optimum bases for (9.1), and for all $c_n < \gamma$, x_n is a basic variable in all optimum bases for (9.1). How can γ be computed?

9.27 Consider the following variant of the simplex algorithm for solving the LP (9.1). It begins with a feasible basic vector x_B for (9.1). Let $\bar{x}$ be the associated BFS, and let $\bar{c} = (\bar{c}_j)$ be the vector of relative cost coefficients with respect to x_B. If $\bar{c} \geqq 0$, $\bar{x}$ is optimal to (9.1), terminate. If $\bar{c} \ngeqq 0$, let $\mathbf{J} = \{j: \bar{c}_j < 0\}$. Let $A_{.n+1} = \sum_{j \in \mathbf{J}} A_{.j}w_j$, $c_{n+1} = \sum_{j \in \mathbf{J}} c_j w_j$, where $w_j > 0$ for each $j \in \mathbf{J}$ are positive weights. Introduce a new variable x_{n+1} into (9.1) with its column vector $A_{.n+1}$ and cost coefficient c_{n+1}. Let $X = (x_1, \ldots, x_n; x_{n+1})$. Bring x_{n+1} into the basic vector x_B for this augmented problem and suppose this leads to the new solution $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n; \tilde{x}_{n+1})$. Remembering that $x_{n+1} = \sum_{j \in \mathbf{J}} w_j x_j$, obtain the feasible solution $\hat{x}$ of the original problem (9.1) corresponding to $\tilde{X}$ for the augmented problem. In general, $\hat{x}$ may not be a BFS for (9.1). Using the method discussed in the proof of Theorem 3.3 of Section 3.5.5, obtain a BFS x^* for (9.1) whose objective values is $\leqq$ the objective value at $\hat{x}$. Repeat the procedure, starting with the new BFS x^*.

Possible choices for the weights in this procedure are either $w_j = 1$ for all $j \in \mathbf{J}$, or $w_j = -\bar{c}_j$ for each $j \in \mathbf{J}$, or some other positive values.

Compare this procedure for solving (9.1), starting with a feasible basic vector for it, with the usual simplex algorithm, in terms of computational efficiency. How does this procedure differ from the simplex algorithm geometrically?