Theory *of* Finance

Solution Sheet on Problem Set 5

Derivatives

Deadline: 30.12.2021

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<table>
<tr><th>Task</th><th></th><th>Points Earned</th></tr>
<tr><td>1. Binomial Tree
a)
Difference between p and q
(5 points)</td><td>Risk neutral probability:

We calculate the risk-neutral probability with the following formula:

$$p = \frac{e^{yh} - d}{u - d}$$
Therefore, the risk neutral probability is 0.4785

Difference between risk neutral and true probabilities:

Risk neutral probabilities are the risk adjusted probabilities of potential future outcomes used to compute the expected asset values. Furthermore, to calculate the risk neutral prob. we assume that asset grow at the risk-free rate and that there is no arbitrage.

True probabilities are the actual probabilities which correspondent with the asset being in a good or bad state in period t+1.

The difference between risk neutral and true probabilities is that for the risk neutral probability we remove any trend component from the asset except for the growth given by the risk-free rate.</td><td></td></tr>
<tr><td>b)
Price of Euro Call
(10 points)</td><td>Price of Euro Call:

We can calculate the price of the call option by extending the formula LN 20.21 by one period:

$$f = e^{-yh}[p^3 f_{uuu} + 3p^2(1-p)f_{uud} + 3p(1-p)^2 f_{udd} + (1-p)^3 f_{ddd}]$$

Accordingly, the price of the Euro Call option with maturity of 3 month and strike rice at $5000 is $234.10.</td><td></td></tr>
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c) Price of exotic derivative (25 points)	**Exotic derivative:** We looked at all 8 possible price paths: $$uuu, uud, udd, ddd, ddu, duu, dud, udu$$ For each path we chose the highest price $\hat{S}$ and calculated the corresponding payoff according to: $$\bar{f}_{t+3} = \max\{\hat{S} - \$5'200\,, 0\}$$ The price of the exotic option is then given by $$f = e^{-yh}[p^3 \bar{f}_{uuu} + p^2(1-p)\bar{f}_{uud} + p(1-p)^2 \bar{f}_{udd} + (1-p)^3 \bar{f}_{ddd} + (1-p)^2 p \bar{f}_{ddu} + (1-p)p^2 \bar{f}_{duu} + (1-p)p(1-p)\bar{f}_{dud} + p(1-p)p\bar{f}_{udu}]$$ which yields: **$208.73.**	
2. Black-Scholes a) Forward price (8 points)	**Forward price:** We calculate the forward price with the following formula: $$F = Se^{m(y-\delta)}$$ Therefore, the fair price of a forward contract would be **$4763.30.**	
b) Put-Call parity (16 points)	**Put-Call parity:** We calculate the Put option price with the following formula for every Call and Strike price: $$C - P = e^{-my}(F - K),$$ Plotting the prices with yields the following graph:	

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We observe a downward sloping curve for the Call prices and an upward sloping curve for the Put price. Furthermore, the two-price curve intersect at the fair forward price (vertical line)
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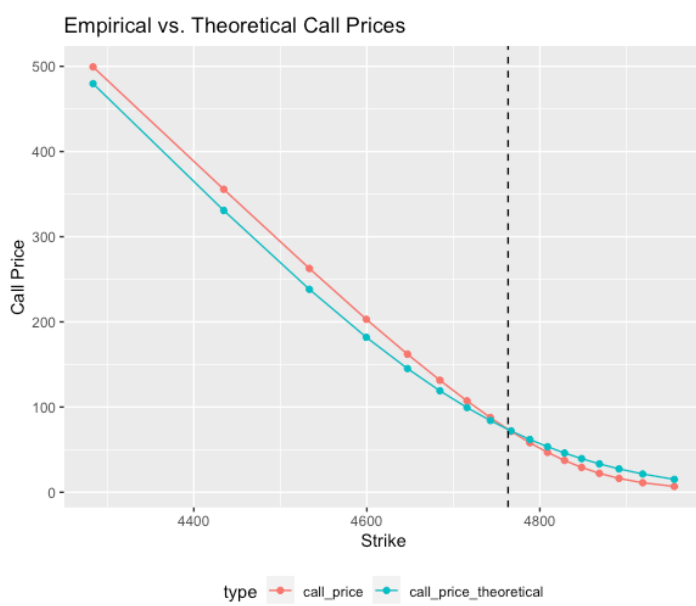
<td>c)
Black-Scholes Call prices
(16 points)</td>
<td>
Black-Scholes:

Using Black-Scholes to calculate the theoretical call prices yields the following plot:


We observe that the empirical call prices and theoretical call prices are downward slopping curves. Furthermore, we observe that the theoretical call price is below the actual call price for any price below the fair forward price. The two curves intersect at the fair forward price (vertical line).
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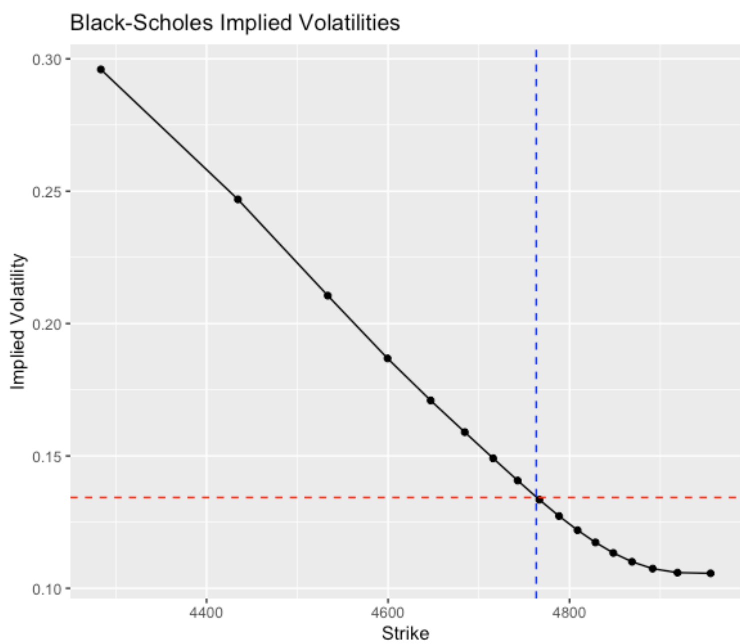
	The differences between the actual prices and Black-Scholes (BS) prices can be explained by the volatility assumption when applying the BS to derive the prices. In Black-Scholes formula we assume the volatility to be at the constant level of 0.1343. Meanwhile, we often observe a volatility “smirk” pattern for the equity markets where the volatility is very high for very low strike prices. Intuitively, investors are more willing to pay a lot for put option to protect themselves against a sharp decline in the stock market. This behavior indicates that the distribution has more mass at the lower price spectrum than a normal distribution (negative skewness). Therefore, the BS will slightly underprice the Call option at the lower end of the strike prices and overestimate for higher prices.	
d) Implied volatility surface (20 points)	**Implied Volatility:** To calculate the implied volatility, we inserted the empirical option prices and all other given parameters in the Black-Scholes formula and solved it for volatility. If we plot the implied volatilities against the strike levels, we obtain the following graph:  We observe that the implied volatility decreases in strike price which unlike assumed in the BS-model turned out to be a non-constant. Furthermore, we observe that the volatility we used in exercise 2c (red horizontal line), lies on the curve at the fair forward price level (blue vertical line).	

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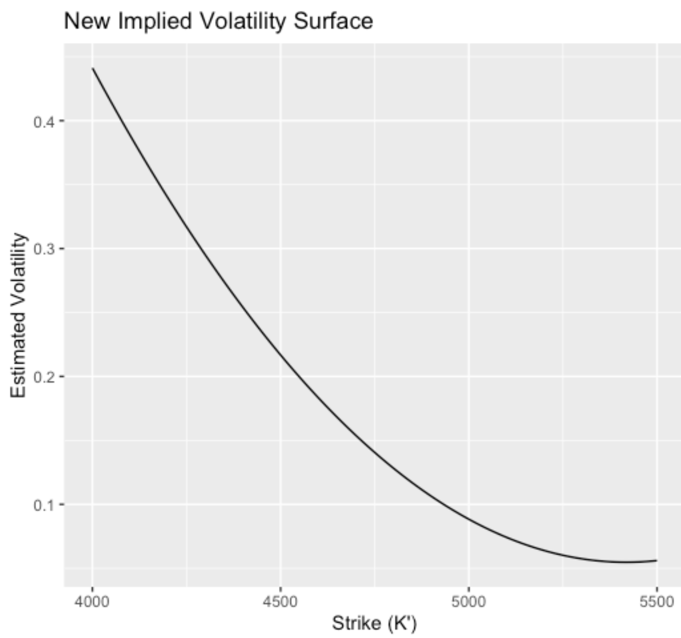
<tr><td>3. Extra Exercise
a)
New volatility surface
(5 points)</td><td>New implied volatility surface:

Interpolating and extrapolating the implied volatility surface using a quadratic regression gives us the following graph: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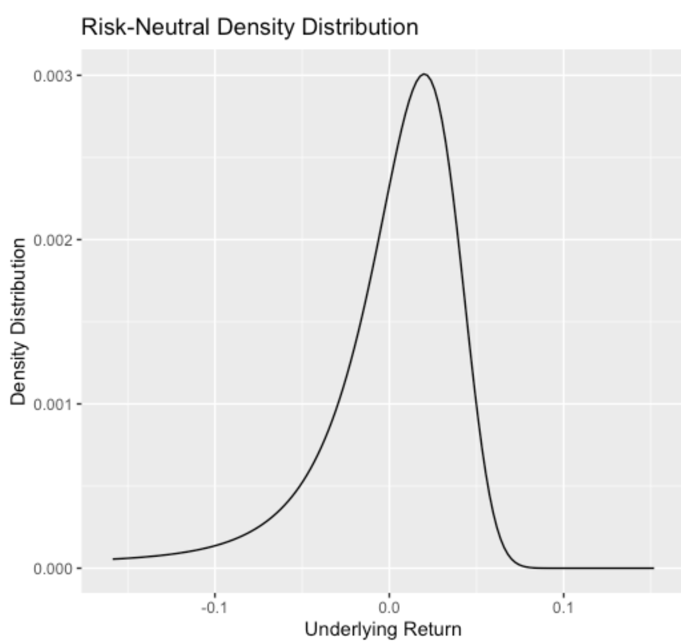
<tr><td>b)
Risk-Neutral density distribution
(5 points)</td><td>Risk neutral density distribution:

Plotting the density distribution function yields the following graph:



We observe that the density distribution of the return has a bell-shaped curve which is left-skewed. As already briefly explained in 2c) the fat tail relates to the empirical smirk pattern for volatility in</td><td></td></tr>
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	equity markets where the volatility is very high for very low strike prices. We recite form 2c:” Intuitively, investors are more willing to pay a lot for put option to protect themselves against a sharp decline in the stock market. This behavior indicates that the distribution has more mass at the lower price spectrum than a normal distribution (negative skewness).” While we observe different volatilities in the market, the Black Scholes model assumes that the price of the assets evolves according to the geometric Brownian motion with constant implied volatility across all strike prices with the same underlying asset. However, as we can see above, the implied volatility observed in the market is usually different as used in the BS-model. Participants price options higher for low strike prices than assumed by the BS-model which causes the mentioned smirk pattern. Accordingly, the smirk pattern, will also have more mass at the lower end of the return which can be observed in the graph above.	