Direct Calculation of Line Outage Distribution Factors

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***Abstract*—Line outage distribution factors (LODFs) are utilized to perform contingency analyses in power systems. Since each contingency requires separate LODFs, a quick calculation of LODFs, especially with multiple-line outages, could speed up contingency analyses and improve significantly the security analyses of power systems. This letter provides two direct proofs for expressing LODFs in terms of power transfer distribution factors (PTDFs) of pre-contingency network. Simple matrix calculations are required in the proposed approach for a fast evaluation of LODFs.**

***Index Terms*—Line outage distribution factor, power transfer distribution factor, shift factor.**

I. Introduction

LINE outage distribution factors (LODFs) defined by the following represent incremental real power flows on monitored transmission lines caused by lines on outage with a pre-contingency real power flow of one MW [1]–[3]:

$$\mathbf{PL}_{\mathbf{M}}^{c} = \mathbf{PL}_{\mathbf{M}}^{0} + \mathbf{LODF}_{\mathbf{M},\mathbf{O}}\mathbf{PL}_{\mathbf{O}}^{0}. \quad (1)$$

In (1), superscripts 0 and c represent base case and contingency cases, respectively, subscript $\mathbf{M}$ represents the set of u monitored lines, and subscript $\mathbf{O}$ represents the set of v lines on outage. $\mathbf{PL}_{\mathbf{M}}^{c}$ and $\mathbf{PL}_{\mathbf{M}}^{0}$ are $u \times 1$ vectors of post-contingency and pre-contingency power flows on monitored lines. $\mathbf{PL}_{\mathbf{O}}^{0}$ is a $v \times 1$ vector of pre-contingency power flows for lines on outage. $\mathbf{LODF}_{\mathbf{M},\mathbf{O}}$ is $u \times v$ matrix.

Fig. 1 illustrates (1) for the monitored line $m-n$ and line $p-q$ that is on outage. The post-contingency active network in Fig. 1(a) is equivalent to the superposition of pre-contingency active network in Fig. 1(b), and post-contingency passive network with nodal injection and withdrawal in Fig. 1(c). The incremental power flow on line $m-n$ is $LODF_{m-n,p-q}$ which is the same as $PTDF_{m-n,p-q}^{c}$. Extending the concept to multiple monitored lines when multiple lines are on outage, we have

$$\mathbf{LODF}_{\mathbf{M},\mathbf{O}} = \mathbf{PTDF}_{\mathbf{M},\mathbf{O}}^{c} \quad (2)$$

which is one way of computing LODFs. A more efficient approach is shown in (3) which utilizes PTDFs in the pre-contingency network [1]. Such PTDFs are easily calculated by applying shift factors (SFs) to a pre-contingency network:

$$\mathbf{LODF}_{\mathbf{M},\mathbf{O}} = \mathbf{PTDF}_{\mathbf{M},\mathbf{O}}^{0}(\mathbf{E} - \mathbf{PTDF}_{\mathbf{O},\mathbf{O}}^{0})^{-1}. \quad (3)$$

Here, $\mathbf{E}$ is an identity matrix of $v \times v$. Reference [4] has used the mathematical induction approach to prove (3). In this letter, we provide two direct proofs for (3) given in a matrix form.

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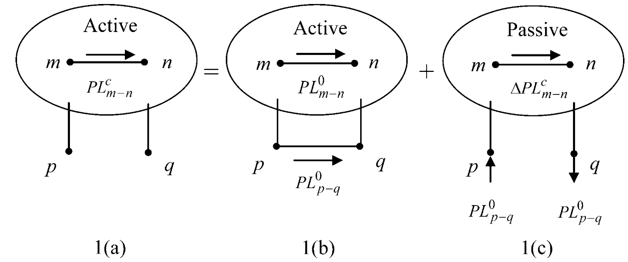

Fig. 1. Equivalent networks with contingencies.

II. Proposed Proof of LODF Calculation Method

Proof 1: Based on the definition of PTDFs, we have

$$\begin{aligned}
\mathbf{PTDF}_{\mathbf{M},\mathbf{O}}^{c} &= \mathbf{X}_{\mathbf{M}}^{-1}\mathbf{\Phi}^{T}[\mathbf{B}]^{c^{-1}}\mathbf{\Psi} \\
\mathbf{PTDF}_{\mathbf{M},\mathbf{O}}^{0} &= \mathbf{X}_{\mathbf{M}}^{-1}\mathbf{\Phi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi} \\
\mathbf{PTDF}_{\mathbf{O},\mathbf{O}}^{0} &= \mathbf{X}_{\mathbf{O}}^{-1}\mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi}
\end{aligned} \quad (4)$$

where $\mathbf{X}_{\mathbf{M}}$ and $\mathbf{X}_{\mathbf{O}}$ are diagonal matrices with elements representing reactances of monitored and outaged lines, respectively. $\mathbf{\Phi}$ is a bus-to-monitored line incident matrix and $\mathbf{\Psi}$ is a bus-to-tripped line incident matrix. For instance, if lines $i-j$ and $m-n$ are monitored when lines $a-b$ and $p-q$ are on outage, we have

$$\mathbf{\Phi}^{T} = \begin{bmatrix} 0,\ldots,1,\ldots,-1,\ldots,\ldots,\ldots,0 \\ 0,\ldots,\ldots,\ldots,1,\ldots,-1,\ldots,0 \end{bmatrix}, \quad \mathbf{X}_{\mathbf{M}} = \begin{bmatrix} x_{ij} & \\ & x_{mn} \end{bmatrix}$$
$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ i & m & j & n \end{array}$$

$$\mathbf{\Psi}^{T} = \begin{bmatrix} 0,\ldots,1,\ldots,-1,\ldots,\ldots,\ldots,0 \\ 0,\ldots,\ldots,\ldots,1,\ldots,-1,\ldots,0 \end{bmatrix}, \quad \mathbf{X}_{\mathbf{O}} = \begin{bmatrix} x_{ab} & \\ & x_{pq} \end{bmatrix}.$$
$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ a & p & b & q \end{array}$$

$[\mathbf{B}]^{c}$ and $[\mathbf{B}]^{0}$ are susceptance matrices representing contingency and base case conditions, respectively. $[\mathbf{B}]^{c}$ is expressed as $[\mathbf{B}]^{c} = [\mathbf{B}]^{0} + \mathbf{\Psi}(-\mathbf{X}_{\mathbf{O}})^{-1}\mathbf{\Psi}^{T}$. According to the inverse matrix modification lemma (IMML) [5]

$$[\mathbf{B}]^{c^{-1}} = [\mathbf{B}]^{0^{-1}} - [\mathbf{B}]^{0^{-1}}\mathbf{\Psi}\mathbf{C}\mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}} \quad (5)$$

where $\mathbf{C} = (-\mathbf{X}_{\mathbf{O}} + \mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})^{-1}$. The existence of $[\mathbf{B}]^{c^{-1}}$ requires $(-\mathbf{X}_{\mathbf{O}} + \mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})$ to be nonsingular. Note that

$$\begin{aligned}
\mathbf{C} &= (-\mathbf{X}_{\mathbf{O}} + \mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})^{-1} \\
&= (-\mathbf{X}_{\mathbf{O}} + \mathbf{X}_{\mathbf{O}}\mathbf{X}_{\mathbf{O}}^{-1}\mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})^{-1} \\
&= (-\mathbf{E} + \mathbf{X}_{\mathbf{O}}^{-1}\mathbf{\Psi}^{T}[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})^{-1}\mathbf{X}_{\mathbf{O}}^{-1} \\
&= (-\mathbf{E} + \mathbf{PTDF}_{\mathbf{O},\mathbf{O}}^{0})^{-1}\mathbf{X}_{\mathbf{O}}^{-1} \\
&= -(\mathbf{E} - \mathbf{PTDF}_{\mathbf{O},\mathbf{O}}^{0})^{-1}\mathbf{X}_{\mathbf{O}}^{-1}.
\end{aligned} \quad (6)$$

The nonsingularity of $(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})$ provides an alternative for checking the network connectivity in contingencies. Accordingly

$$\begin{aligned}
&\mathbf{LODF}_{\mathbf{M,O}} = \mathbf{PTDF}^c_{\mathbf{M,O}} = \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{c^{-1}}\mathbf{\Psi}\\
&= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T([\mathbf{B}]^{0^{-1}} - [\mathbf{B}]^{0^{-1}}\mathbf{\Psi}\mathbf{C}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}})\mathbf{\Psi}\\
&= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T([\mathbf{B}]^{0^{-1}}\mathbf{\Psi} - [\mathbf{B}]^{0^{-1}}\mathbf{\Psi}\mathbf{C}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})\\
&= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi}(\mathbf{E} - \mathbf{C}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi})\\
&= \mathbf{PTDF}^0_{\mathbf{M,O}}[\mathbf{E} + (\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1}\mathbf{X}^{-1}_{\mathbf{O}}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi}]\\
&= \mathbf{PTDF}^0_{\mathbf{M,O}}[\mathbf{E} + (\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1}\mathbf{PTDF}^0_{\mathbf{O,O}}]\\
&= \mathbf{PTDF}^0_{\mathbf{M,O}}(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1}[(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})\\
&\qquad\qquad + \mathbf{PTDF}^0_{\mathbf{O,O}}]\\
&= \mathbf{PTDF}^0_{\mathbf{M,O}}(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1} \quad \text{Q.E.D.}
\end{aligned} \tag{7}$$

Proof 2: Transmission flows on lines that are monitored and those on outage are represented as

$$\begin{aligned}
\mathbf{PL}^c_{\mathbf{M}} &= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{c^{-1}}\mathbf{P}^{inj}\\
\mathbf{PL}^0_{\mathbf{M}} &= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{0^{-1}}\mathbf{P}^{inj}\\
\mathbf{PL}^0_{\mathbf{O}} &= \mathbf{X}^{-1}_{\mathbf{O}}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{P}^{inj}
\end{aligned} \tag{8}$$

where $\mathbf{P}^{inj}$ represents the nodal injection. Substituting (8) into (1), we have

$$\begin{aligned}
\mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{c^{-1}}\mathbf{P}^{inj} &= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{0^{-1}}\mathbf{P}^{inj}\\
&\quad + \mathbf{LODF}_{\mathbf{M,O}}\mathbf{X}^{-1}_{\mathbf{O}}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{P}^{inj}.
\end{aligned} \tag{9}$$

By dropping $\mathbf{P}^{inj}$, we have

$$\begin{aligned}
\mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{c^{-1}} &= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{0^{-1}}\\
&\quad + \mathbf{LODF}_{\mathbf{M,O}}\mathbf{X}^{-1}_{\mathbf{O}}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}.
\end{aligned} \tag{10}$$

By multiplying both sides of (10) by $\mathbf{\Psi}$, we have

$$\begin{aligned}
\mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{c^{-1}}\mathbf{\Psi} &= \mathbf{X}^{-1}_{\mathbf{M}}\mathbf{\Phi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi}\\
&\quad + \mathbf{LODF}_{\mathbf{M,O}}\mathbf{X}^{-1}_{\mathbf{O}}\mathbf{\Psi}^T[\mathbf{B}]^{0^{-1}}\mathbf{\Psi}.
\end{aligned} \tag{11}$$

By substituting (2) and (4) into (11), we have

$$\begin{aligned}
&\mathbf{LODF}_{\mathbf{M,O}} = \mathbf{PTDF}^0_{\mathbf{M,O}} + \mathbf{LODF}_{\mathbf{M,O}}\mathbf{PTDF}^0_{\mathbf{O,O}} \Rightarrow\\
&\mathbf{LODF}_{\mathbf{M,O}}(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}}) = \mathbf{PTDF}^0_{\mathbf{M,O}} \Rightarrow\\
&\mathbf{LODF}_{\mathbf{M,O}} = \mathbf{PTDF}^0_{\mathbf{M,O}}(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1} \quad \text{Q.E.D.}
\end{aligned} \tag{12}$$

III. Case Studies

The IEEE 14-bus system listed at http://motor.ece.iit.edu/LODF/IEEE14.txt is used to demonstrate the application of (3) and (1) to obtain post-contingency flows on monitored lines 2–5 and 6–13 when lines 1–5, 3–4, and 6–11 are on outage. We calculate PTDFs using SFs of base case network:

$$\begin{aligned}
&\mathbf{PTDF}^0_{\mathbf{O,O}}\\
&= \begin{bmatrix} PTDF^0_{1-5,1-5} & PTDF^0_{1-5,3-4} & PTDF^0_{1-5,6-11}\\ PTDF^0_{3-4,1-5} & PTDF^0_{3-4,3-4} & PTDF^0_{3-4,6-11}\\ PTDF^0_{6-11,1-5} & PTDF^0_{6-11,3-4} & PTDF^0_{6-11,6-11}\end{bmatrix}\\
&= \begin{bmatrix} 0.3894 & 0.0790 & -0.0092\\ 0.1031 & 0.6193 & 0.0078\\ -0.0103 & 0.0067 & 0.7407\end{bmatrix}\\
&\mathbf{PTDF}^0_{\mathbf{M,O}}\\
&= \begin{bmatrix} PTDF^0_{2-5,1-5} & PTDF^0_{2-5,3-4} & PTDF^0_{2-5,6-11}\\ PTDF^0_{6-13,1-5} & PTDF^0_{6-13,3-4} & PTDF^0_{6-13,6-11}\end{bmatrix}\\
&= \begin{bmatrix} 0.2918 & 0.1283 & -0.0149\\ -0.0053 & 0.0034 & 0.0817\end{bmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{LODF}_{\mathbf{M,O}} &= \mathbf{PTDF}^0_{\mathbf{M,O}}(\mathbf{E}-\mathbf{PTDF}^0_{\mathbf{O,O}})^{-1}\\
&= \begin{bmatrix} 0.5551 & 0.4511 & -0.0637\\ -0.0120 & 0.0121 & 0.3159\end{bmatrix}.
\end{aligned}$$

According to (1)

$$\begin{aligned}
&\mathbf{PL}^c_{\mathbf{M}} = \mathbf{PL}^0_{\mathbf{M}} + \mathbf{LODF}_{\mathbf{M,O}}\cdot\mathbf{PL}^0_{\mathbf{O}} \Rightarrow\\
&\begin{bmatrix} PL^c_{2-5}\\ PL^c_{6-13}\end{bmatrix} = \begin{bmatrix} PL^0_{2-5}\\ PL^0_{6-13}\end{bmatrix} + \mathbf{LODF}_{\mathbf{M,O}}\cdot\begin{bmatrix} PL^0_{1-5}\\ PL^0_{3-4}\\ PL^0_{6-11}\end{bmatrix}\\
&= \begin{bmatrix} 40.9040\\ 17.0337\end{bmatrix} + \mathbf{LODF}_{\mathbf{M,O}}\\
&\cdot\begin{bmatrix} 71.1194\\ -24.1498\\ 6.3048\end{bmatrix} = \begin{bmatrix} 69.0880\\ 17.8806\end{bmatrix}.
\end{aligned}$$

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