Hollow channel plasma wakefield acceleration

MOTIVATION

We want to create a model of a hollow channel plasma that is relevant for future experiments at FACET. As far as we know, no such model exists.

QUESTIONS

1. How does the on-axis E_z field scale with
   (a) the channel radius a?
   (b) the beam density to plasma density n_b/n_0?
   (c) the bunch length σ_z?
2. Are there radial forces inside the channel, despite the fact that there are no ions? Are they linear?
3. How does the physics change for an electron driver versus a positron driver?
4. How does beam loading work in the hollow channel? Can positrons be effectively loaded in the channel wake?
5. How does the physics change for channel profiles that are flat, gaussian and bessel shaped? How does the width of the plasma layer effect the on-axis fields?
6. How do we describe the sheet crossing of the inner and outer layers of the plasma channel as they converge on the axis? This is especially important for positron beam loading in the second bubble.

STARTING POINT FOR ALL MODELS

We begin with Maxwell's equations in the Lorentz gauge:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = 4\pi\begin{pmatrix}\mathbf{J/c}\\ \rho\end{pmatrix} \tag{1}$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0 \tag{2}$$

Next, we make the change of coordinates from (x, y, z, t) to $(x, y, \xi \equiv ct - z, \tau \equiv t)$. In the new coordinates, the derivatives are:

$$\frac{\partial \phi(x,y,\xi,\tau)}{\partial t} = \frac{\partial \phi}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial \phi}{\partial \tau} + c\frac{\partial \phi}{\partial \xi} \tag{3}$$

$$\frac{\partial \phi(x,y,\xi,\tau)}{\partial z} = \frac{\partial \phi}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial z} = -\frac{\partial \phi}{\partial \xi} \tag{4}$$

Transforming the left hand side of equations 1 and 2 we have:

$$\left(\frac{1}{c^2}\left[\frac{\partial^2}{\partial \tau^2} + c^2\frac{\partial^2}{\partial \xi^2}\right] - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \xi^2}\right)\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \nabla_\perp^2\right)\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} \tag{5}$$

$$\frac{1}{c}\left[\frac{\partial}{\partial \tau} + c\frac{\partial}{\partial \xi}\right]\phi + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial \xi} = 0 \rightarrow \frac{1}{c}\frac{\partial \phi}{\partial \tau} + \nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial(\phi - A_z)}{\partial \xi} \tag{6}$$

where $\nabla_\perp = \partial_x \hat{x} + \partial_y \hat{y}$ and $\mathbf{A}_\perp = A_x\hat{x} + A_y\hat{y}$. Finally, we apply the quasistatic approximation $\partial_\tau \phi = \partial_\tau \mathbf{A} = 0$. The quasistatic approximation says that the fields change slowly in the co-moving frame. Defining $\psi \equiv \phi - A_z$ and setting $c = 1$, Maxwell's equations in the quasistatic approximation are:

$$-\nabla_\perp^2\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = \begin{pmatrix}\mathbf{J}\\ \rho\end{pmatrix} \tag{7}$$

$$\nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial \psi}{\partial \xi} \tag{8}$$

That's enough for now.

THIN CYLINDER MODEL

No longitudinal plasma motion

Here we describe the response of an infinitely thin cylinder of plasma with radius a to a positively charged drive beam. We assume the beam is relativistic and much shorter than the wavelength of the plasma. In this model, the plasma electrons receive an initial kick due to the beam but we do not include the beam current in our description of the plasma response. We also assume that the plasma electrons only move radially. Note that after the beam has passed, $J_z = 0$ and therefore $A_z = 0$ so $\psi = \phi$.

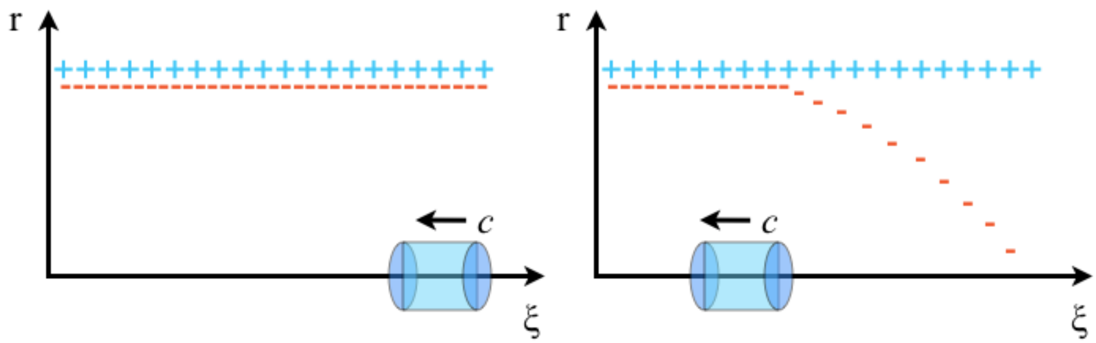

FIG. 1. A thin cylinder plasma with a flat top drive beam.

The beam is a flat top with radius $\sigma_r \ll a$, energy γ, and charge density n_b. The beam line charge is $\lambda_b = n_b \pi \sigma_r^2$ and the beam current is:

$$I_b = \frac{Q_b}{\sigma_z} c = \frac{n_b \pi \sigma_r^2 \sigma_z}{\sigma_z} c = \lambda_b c \tag{9}$$

We can normalize the beam current to I_b to the Alfven Current $I_A = m_e c^3/e$ so $\bar{I} = eI_b/m_e c^3$. The quantity $\bar{I} = e\lambda_b/m_e c^2$ is also called the Budker parameter ν.

Next, we find the beam field at the plasma radius a using Gauss's law:

$$E_b = \frac{2\lambda_b}{a} = \frac{2m_e c^2}{e} \frac{\bar{I}}{a} \tag{10}$$

An electron at the plasma radius receives an inward kick $\Delta p = F\Delta t = -eE_b \sigma_z/c$ where σ_z is the bunch length. We want to describe the evolution of the plasma after the beam has passed and we also assume that the beam is short compared to the relevant length scale in this problem which is the plasma radius a. Let's take $\sigma_z = a/10$ to get:

$$\Delta p = -e \frac{2m_e c^2 \bar{I}}{e} \frac{\sigma_z/c}{a} = -\frac{m_e c \bar{I}}{5} \tag{11}$$

Of course it is completely arbitrary to choose $\sigma_z = a/10$, but this allows us to see that the strength of the kick is really controlled by the peak current and nothing else. We also note that our model does not include radiative effects, so for our model to be reasonably accurate, we should require that the normalized kick $\Delta p/m_e c \ll 1$ which means $\bar{I} \ll 5$. At FACET, $\bar{I} \approx 1$ and $\sigma_z/\lambda_p \approx 1/5$.

We now seek an equation of motion for the position of the plasma electron sheath r_s. We assume the ions are stationary. First, we solve for the potential ψ and we take $\psi(\infty) = 0$. Outside the cylinder, there is zero net charge (after the drive beam has passed), so Gauss's law gives $E(r > a) = 0$ and therefore $\psi(r > a) = const = 0$. Inside the electron sheath there is no charge either, so $E(r < r_s) = 0$ and $\psi(r < r_s) = \psi_0$. Between the electron sheath and the ion layer there is a net charge and Gauss's law gives $E(r) = 2\bar{I}/r$ for $r_s < r < a$. Note that we normalize E and ψ to $m_e c^2/e$. We can now find the potential by integrating the electric field:

$$\psi(r) = -\int_0^r E(r')dr' = \psi_0 - \int_{r_s}^r \frac{2\bar{I}}{r'}dr' = \psi_0 + 2\bar{I}\log\left(\frac{r_s}{r}\right) \tag{12}$$

Using the boundary condition $\psi(a) = 0$ we find that $\psi_0 = 2\bar{I}\log\left(\frac{a}{r_s}\right)$ so:

$$\psi(r) = \begin{cases} 2\bar{I}\log\left(\frac{a}{r_s}\right) & 0 < r < r_s \\ 2\bar{I}\log\left(\frac{a}{r_s}\right) + 2\bar{I}\log\left(\frac{r_s}{r}\right) & r_s < r < a \\ 0 & a < r \end{cases} \tag{13}$$

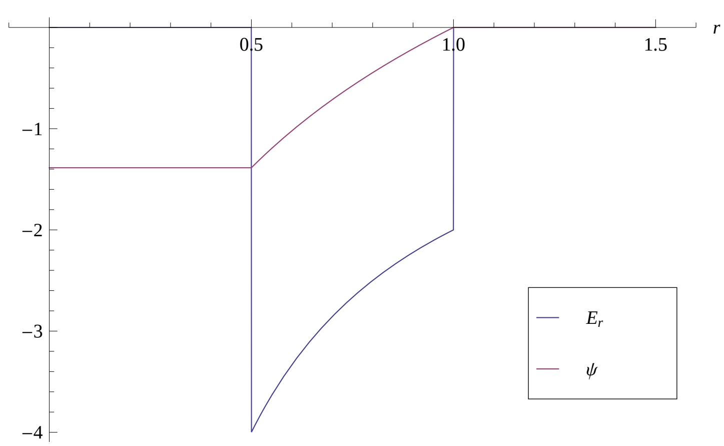

FIG. 2. E_r and ψ for the thin cylinder model. $\bar{I} = 1$ and distances are normalized to $a = 1$.

Next, we find $\mathbf{A}_\perp$ using equation 8 and assuming that the motion of the plasma electrons is purely radial so $A_z = 0$ and $\psi = \phi$. Then

$$\nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial \phi}{\partial \xi} = -\frac{\partial \phi}{\partial r_s}\frac{\partial r_s}{\partial \xi} = \frac{2\bar{I}}{r_s}r_s' \tag{14}$$

In cylindrical coordinates, the divergence is

$$\nabla_\perp \cdot \mathbf{A}_\perp = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} \tag{15}$$

but we drop the θ term due to the cylindrical symmetry. We note that r and r_s are independent variables in this equation so we can easily integrate the equation in r to find A_r

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) = \frac{2\bar{I}}{r_s}r_s' \to A_r = \frac{\bar{I}rr_s'}{r_s} \tag{16}$$

At this point, we'd like to start determining some physical quantities of interest, like E_z

$$E_z = \frac{\partial \psi}{\partial \xi} = -\frac{2\bar{I}}{r_s}r_s' \tag{17}$$

so we need an equation of motion for r_s. We can get the EOM by finding the force on the plasma electrons in the sheath:

$$F = -(E_r - V_z B_\theta) = -E_r \tag{18}$$

because we have assumed that $V_z = 0$. The radial field is

$$E_r = -\nabla_\perp \phi - \frac{\partial A_r}{\partial \xi} = -\frac{\bar{I}r}{r_s^2}[r_s r_s'' - r_s'^2] \tag{19}$$

with $\nabla_\perp \phi = 0$ for the sheath electrons because there is zero charge within the sheath layer. The EOM $\partial_\xi P_\perp = -E_r$ is rewritten with the "particle tracking" trajectory $\xi = t - z \to \partial_\xi = (1 - v_z)\partial_t$:

$$\frac{\partial P_\perp}{\partial \xi} = \frac{\partial \gamma v_\perp}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\gamma \frac{\partial r_\perp}{\partial t}\right) = \frac{\partial}{\partial \xi}\left[\gamma(1 - v_z)\frac{\partial r_\perp}{\partial \xi}\right] \tag{20}$$

and using the integral of motion $\gamma - P_z = 1 + \psi$

$$\frac{\partial P_\perp}{\partial \xi} = \frac{\partial}{\partial \xi}[(1+\psi)r_s'] = \frac{\bar{I}r}{r_s^2}[r_s r_s'' - r_s'^2] \tag{21}$$

Plugging in $\psi = \phi$ we have:

$$\left(1 + 2\bar{I}\log\left(\frac{a}{r_s}\right) - \bar{I}\right)r_s'' - \bar{I}\frac{r_s'^2}{r_s} = 0 \tag{22}$$

In order to study this equation, let's set $a = 1$:

$$[1 + \bar{I}(1 + 2\log(r_s))]r_s r_s'' + \bar{I}r_s'^2 = 0 \tag{23}$$

Immediately we see that we might run into trouble when $r_s = 0$ due to the divergence in the log term (although $x \log x = 0$ as $x \to 0$).

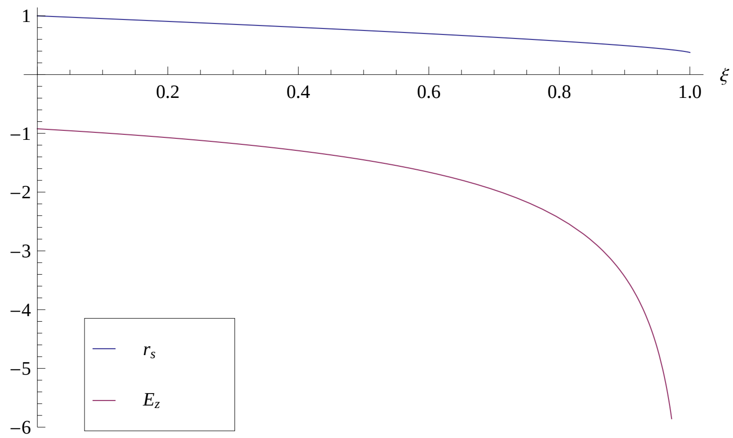

FIG. 3. r_s and E_z for the thin cylinder model. $\bar{I} = 1$ and distances are normalized to $a = 1$. The initial kick $r_s'(0) = -0.46$. This is the largest kick that Mathematica can use to solve the equation for the given plot range.

Adding in longitudinal plasma motion