Hollow channel plasma wakefield acceleration

MOTIVATION

We want to create a model of a hollow channel plasma that is relevant for future experiments at FACET. As far as we know, no such model exists.

QUESTIONS

1. How does the on-axis E_z field scale with
   (a) the channel radius a?
   (b) the beam density to plasma density n_b/n_0?
   (c) the bunch length σ_z?
2. Are there radial forces inside the channel, despite the fact that there are no ions? Are they linear?
3. How does the physics change for an electron driver versus a positron driver?
4. How does beam loading work in the hollow channel? Can positrons be effectively loaded in the channel wake?
5. How does the physics change for channel profiles that are flat, gaussian and bessel shaped? How does the width of the plasma layer effect the on-axis fields?
6. How do we describe the sheet crossing of the inner and outer layers of the plasma channel as they converge on the axis? This is especially important for positron beam loading in the second bubble.

STARTING POINT FOR ALL MODELS

We begin with Maxwell's equations in the Lorentz gauge:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = 4\pi\begin{pmatrix}\mathbf{J/c}\\ \rho\end{pmatrix} \tag{1}$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0 \tag{2}$$

Next, we make the change of coordinates from (x, y, z, t) to $(x, y, \xi \equiv ct - z, \tau \equiv t)$. In the new coordinates, the derivatives are:

$$\frac{\partial \phi(x, y, \xi, \tau)}{\partial t} = \frac{\partial \phi}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial \phi}{\partial \tau} + c\frac{\partial \phi}{\partial \xi} \tag{3}$$

$$\frac{\partial \phi(x, y, \xi, \tau)}{\partial z} = \frac{\partial \phi}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial z} = -\frac{\partial \phi}{\partial \xi} \tag{4}$$

Transforming the left hand side of equations 1 and 2 we have:

$$\left(\frac{1}{c^2}\left[\frac{\partial^2}{\partial \tau^2} + c^2\frac{\partial^2}{\partial \xi^2}\right] - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \xi^2}\right)\begin{pmatrix}\mathbf{A} \\ \phi\end{pmatrix} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \nabla_\perp^2\right)\begin{pmatrix}\mathbf{A} \\ \phi\end{pmatrix} \tag{5}$$

$$\frac{1}{c}\left[\frac{\partial}{\partial \tau} + c\frac{\partial}{\partial \xi}\right]\phi + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} - \frac{\partial A_z}{\partial \xi} = 0 \to \frac{1}{c}\frac{\partial \phi}{\partial \tau} + \nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial(\phi - A_z)}{\partial \xi} \tag{6}$$

where $\nabla_\perp = \partial_x \hat{x} + \partial_y \hat{y}$ and $\mathbf{A}_\perp = A_x \hat{x} + A_y \hat{y}$. Finally, we apply the quasistatic approximation $\partial_\tau \phi = \partial_\tau \mathbf{A} = 0$. The quasistatic approximation says that the fields change slowly in the co-moving frame. Defining $\psi \equiv \phi - A_z$ and setting $c = 1$, Maxwell's equations in the quasistatic approximation are:

$$-\nabla_\perp^2 \begin{pmatrix}\mathbf{A} \\ \phi\end{pmatrix} = 4\pi \begin{pmatrix}\mathbf{J} \\ \rho\end{pmatrix} \tag{7}$$

$$\nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial \psi}{\partial \xi} \tag{8}$$

The pseudopotential ψ obeys the Poisson-like equation

$$-\nabla_\perp^2 \psi = 4\pi(\rho - J_z) \tag{9}$$

The continuity equation is

$$\frac{\partial}{\partial \xi}(\rho - J_z) + \nabla_\perp \cdot J_\perp = 0 \tag{10}$$

The fields are

$$E_z = \frac{\partial \psi}{\partial \xi} \tag{11}$$

$$B_z = (\nabla_\perp \times A_\perp) \cdot \hat{z} \tag{12}$$

$$E_\perp = -\nabla_\perp \phi - \frac{\partial A_\perp}{\partial \xi} \tag{13}$$

$$B_\perp = \nabla_\perp \times (A_z \hat{z}) + \nabla_z \times A_\perp \tag{14}$$

We will rarely make use of equation 12 because we assume that the plasma electrons only have radial motion which means that $\nabla_\perp \times A_\perp = 0$. This also means that $A_\perp = A_r$ and $\nabla_\perp = \partial_r \hat{r}$. Then equation 13 becomes

$$E_r = -\frac{\partial \phi}{\partial r} - \frac{\partial A_r}{\partial \xi} \tag{15}$$

and equation 14 becomes

$$B_\perp = \partial_r \hat{r} \times A_z \hat{z} - \partial_\xi \hat{z} \times A_r \hat{r} = -\left(\frac{\partial A_z}{\partial r} + \frac{\partial A_r}{\partial \xi} \right) \hat{\theta} \tag{16}$$

Using the components of $\vec{E}$ and $\vec{B}$ that we have solved for, we can write the Lorentz force as:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = q(E_r - v_z B_\theta)\hat{r} + q(E_z + v_r B_\theta)\hat{z} \tag{17}$$

We will make extensive use of the radial part of equation 17. Inserting the potentials, we have

$$F_r = q\left[-\frac{\partial \phi}{\partial r} - \frac{\partial A_r}{\partial \xi} + v_z \left(\frac{\partial A_z}{\partial r} + \frac{\partial A_r}{\partial \xi} \right) \right] = -q\left[\frac{\partial \phi}{\partial r} - v_z \frac{\partial A_z}{\partial r} + (1 - v_z) \frac{\partial A_r}{\partial \xi} \right] \tag{18}$$

We will want to work with ψ as much as possible so

$$F_r = -q\left[\frac{\partial \psi}{\partial r} + (1 - v_z) \frac{\partial A_z}{\partial r} + (1 - v_z) \frac{\partial A_r}{\partial \xi} \right] = -q\left[\frac{\partial \psi}{\partial r} + (1 - v_z)(\vec{\nabla} \times \vec{A})_\theta \right] \tag{19}$$

Lastly, we define E_g as the"gaussian" electric field

$$E_g = -\frac{\partial \psi}{\partial r} \tag{20}$$

This term is especially useful when describing the fields and potentials of the radial charge distribution since the source $\rho - J_z = \rho_0$ is constant in ξ.

THIN CYLINDER MODEL

Here we describe the response of an infinitely thin cylinder of plasma with radius a to a positively charged drive beam. The charge distribution of the electrons in the plasma is given by $\rho_0 = \sigma_0 \delta(r-a)$ where σ_0 is the surface charge density. We can also define a line charge density for the plasma as $\lambda_0 = 2\pi a \sigma_0$. We assume the beam is relativistic and much shorter than the wavelength of the plasma. In this model, the plasma electrons receive an initial kick due to the beam but we do not include the beam current in our description of the plasma response.

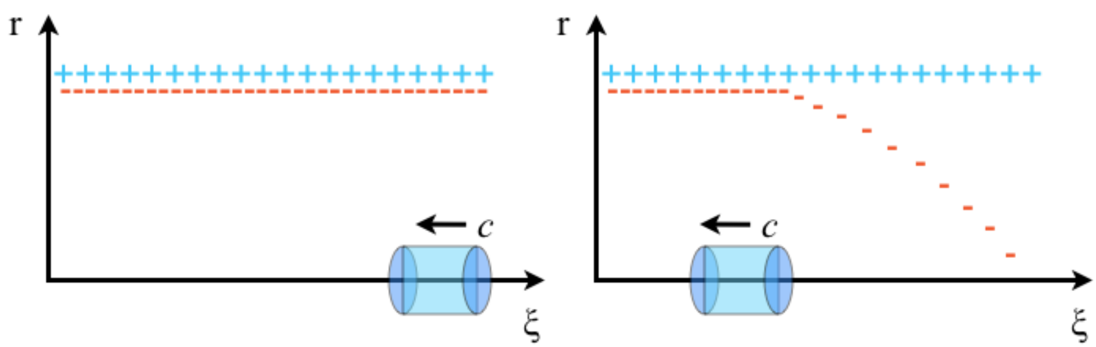

FIG. 1. A thin cylinder plasma with a flat top drive beam.

The beam is a flat top with radius $\sigma_r \ll a$, energy γ, and charge density n_b. The beam line charge is $\lambda_b = n_b \pi \sigma_r^2$ and the beam current is:

$$I_b = \frac{Q_b}{\sigma_z} c = \frac{n_b \pi \sigma_r^2 \sigma_z}{\sigma_z} c = \lambda_b c \tag{21}$$

We can normalize the beam current I_b to the Alfven Current $I_A = m_e c^3/e$ so $\bar{I} = eI_b/m_e c^3$. The quantity $\bar{I} = e\lambda_b/m_e c^2$ is also called the Budker parameter ν.

Next, we find the beam field at the plasma radius a using Gauss's law:

$$E_b = \frac{2\lambda_b}{a} = \frac{2m_e c^2}{e} \frac{\bar{I}}{a} \tag{22}$$

An electron at the plasma radius receives an inward kick $\Delta p = F\Delta t = -eE_b\sigma_z/c$ where σ_z is the bunch length. We want to describe the evolution of the plasma after the beam has passed and we also assume that the beam is short compared to the relevant length scale in this problem which is the plasma radius a. Let's take $\sigma_z = a/10$ to get:

$$\Delta p = -e\frac{2m_e c^2 \bar{I}}{e}\frac{\sigma_z/c}{a} = -\frac{m_e c \bar{I}}{5} \tag{23}$$

Of course it is completely arbitrary to choose $\sigma_z = a/10$, but this allows us to see that the strength of the kick is really controlled by the peak current and nothing else.

We assume the ions are stationary. Even though the plasma cylinder is infinitely thin, we can still introduce a notion of charge shielding by noting that the amount of plasma electrons that participate in the interaction is proportional to the line charge of the beam $\lambda_b = m_e c^2 \bar{I}/e$. Then the plasma line charge interacting with the beam is:

$$\lambda_i = \begin{cases} m_e c^2 \bar{I}/e & \lambda_b < \lambda_0 \\ \lambda_0 & \lambda_b > \lambda_0 \end{cases} \tag{24}$$

At this point our analysis branches. There are four regimes we will investigate in the thin cylinder model. They are

- $\lambda_b > \lambda_0$, small initial kick
- $\lambda_b > \lambda_0$, large initial kick
- $\lambda_b < \lambda_0$, small initial kick
- $\lambda_b < \lambda_0$, large initial kick

We will define "small initial kick" and "large initial kick" rigorously later on. Qualitatively, a small initial kick means that the plasma electrons do not cross the beam axis and instead undergo small oscillations about the plasma ions at the cylinder radius. For a large initial kick the plasma electrons cross the beam axis. The on-axis E_z field is much larger in this case.

The cases with $\lambda_b > \lambda_0$ are the simplest because in this scenario there is not enough charge in the plasma cylinder to shield the beam charge. All plasma electrons see the same force and behave coherently. We can use a particle model to describe the radial trajectory of the plasma electrons. The cases with $\lambda_b < \lambda_0$ are more complex because there is charge screening. The electrons do not all see the same force and the initially thin charge layer broadens. This case requires a fluid model to describe the full charge distribution $\rho(r, \xi)$.

$\lambda_b > \lambda_0$, Small initial kick

For the case $\lambda_b > \lambda_0$, all plasma electrons have the same radial position in the plasma sheath for all ξ. We seek an equation of motion for the position of the plasma electron sheath $r_s(\xi)$. To solve for r_s, we first need to solve for the potential ψ. The source term for $\psi(r,\xi)$ is $\rho - J_z$. Although the plasma sheath moves radially, no charge moves through the sheath so the radial current density $J_r = 0$. The continuity equation gives:

$$\frac{\partial}{\partial \xi}(\rho - J_z) = 0 \rightarrow \rho - J_z = const = \rho_0 \tag{25}$$

The constant is ρ_0; before the beam arrives $J_z = 0$ and $\rho = \rho_0$. Rewriting ρ_0 in terms of the plasma electron line charge density we have

$$\rho_0 = \frac{\lambda_0}{2\pi r}\delta(r - r_s) \tag{26}$$

and the ion line charge density is

$$\rho_i = -\frac{\lambda_0}{2\pi r}\delta(r - a) \tag{27}$$

If $r_s < a$, Gauss's law gives $E_g(r > a) = 0$. Inside the electron sheath there is no charge either, so $E_g(r < r_s) = 0$. Between the electron sheath and the ion layer there is a net charge and Gauss's law gives $E_g(r) = 2\lambda_0/r$ for $r_s < r < a$.

If we want to consider oscillations of the plasma electrons about the ion cylinder we should also describe the field for $a < r_s$. The field is $E_g(r) = -2\lambda_0/r$ for $a < r < r_s$ which is the same as in the previous case but the sign is flipped. We can accommodate both $r_s > a$ and $r_s < a$ in a single equation if we write:

$$E_g(r) = \frac{2\lambda_0}{r}\Theta(r - r_s) - \frac{2\lambda_0}{r}\Theta(r - a) \tag{28}$$

$\Theta(r)$ is the Heaviside step function. Note that $\lambda_0 < 0$ refers to the plasma electron charge, so the first term in equation 28 is the (negative) field due to the plasma electrons and the second term is the (positive) field due to the ions.

We can now find the potential by integrating the electric field:

$$\begin{aligned} \psi(r) &= -\int_0^r E(r')dr' = \psi_0 - \int_{\min(r_s,a)}^r \frac{2\lambda_0}{r'}\left(\Theta(r' - r_s) - \Theta(r' - a)\right)dr' \\ &= \psi_0 - 2\lambda_0 \log\left(\frac{r}{r_s}\right)\Theta(r - r_s) + 2\lambda_0 \log\left(\frac{r}{a}\right)\Theta(r - a) \end{aligned} \tag{29}$$

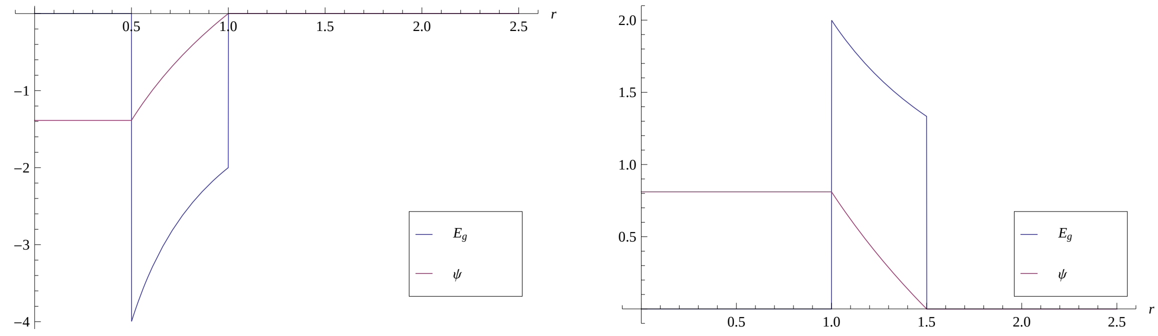

FIG. 2. Left: $r_s = 0.5$, $a = 1$. Right: $r_s = 1.5$, $a = 1$.

Using the boundary condition $\psi(\max(r_s, a)) = 0$ because $\psi = 0$ at infinity, we find that $\psi_0 = 2\lambda_0 \log\left(\frac{a}{r_s}\right)$ for both $r_s < a$ and $r_s > a$. The sign of ψ_0 flips between cases.

Next, we find $\mathbf{A}_\perp$ using equation 8:

$$\nabla_\perp \cdot \mathbf{A}_\perp = -\frac{\partial \psi}{\partial \xi} = -\frac{\partial \psi}{\partial r_s}\frac{\partial r_s}{\partial \xi} \tag{30}$$

$$\frac{\partial \psi}{\partial r_s} = \frac{2\lambda_0}{r_s}\Theta(r - r_s) - 2\lambda_0 \log\left(\frac{r_s}{r}\right)\delta(r - r_s) - \frac{2\lambda_0}{r_s} \tag{31}$$

In cylindrical coordinates, the divergence is

$$\nabla_\perp \cdot \mathbf{A}_\perp = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} \tag{32}$$

but we drop the θ term due to the cylindrical symmetry. We note that r and r_s are independent variables in this equation so we can easily integrate the equation in r to find A_r

$$A_r = -\frac{1}{r}\int_0^r r' \left[\frac{2\lambda_0}{r_s}\Theta(r' - r_s) + 2\lambda_0 \log\left(\frac{r_s}{r'}\right)\delta(r' - r_s) - \frac{2\lambda_0}{r_s}\right]\frac{\partial r_s}{\partial \xi} dr' \tag{33}$$

The second term in the integral drops out because $\log(r_s/r) = 0$ at $r = r_s$. Call the derivative of r_s with respect to ξ r_s'. Being careful with the integral of the Heaviside function, we find:

$$A_r = -\frac{\lambda_0 r_s'}{r_s}\left[\frac{r^2 - r_s^2}{r}\Theta(r - r_s) - r\right] \tag{34}$$

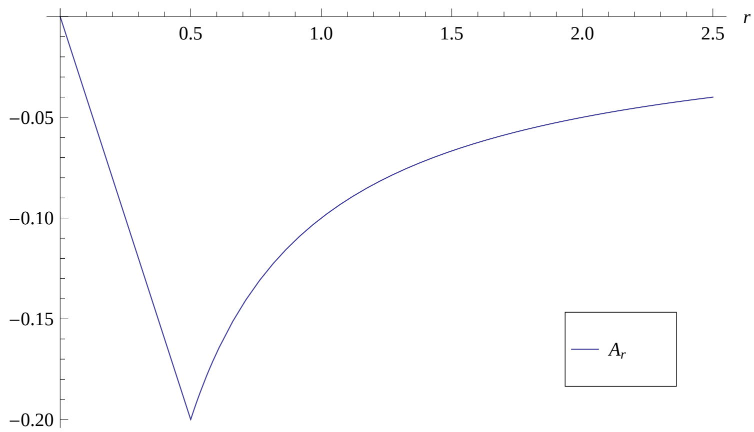

FIG. 3. Radial vector potential A_r for $r_s = 0.5$ and $r_s' = 0.2$.

We are almost at the point where we can solve the radial force equation 19. We need an expression for the axial vector potential A_z. However, we can express A_z in terms of ψ by noting that $J_z = v_z\rho_0/(1-v_z)$. In our fluid description of the plasma sheath, v_z is not a function of r. Then

$$A_z = \frac{v_z}{1-v_z}\psi \tag{35}$$

Plugging into equation 19 and setting $q = -1$ for the electron charge, we have:

$$F_r = \left[(1+v_z)\frac{\partial\psi}{\partial r} + (1-v_z)\frac{\partial A_r}{\partial\xi}\right] \tag{36}$$

The equation of motion is:

$$\frac{\partial P_\perp}{\partial\xi} = \frac{1}{1-v_z}F_r = \frac{1+v_z}{1-v_z}\frac{\partial\psi}{\partial r} + \frac{\partial A_r}{\partial\xi} \tag{37}$$

We rewrite the EOM with the "particle tracking" trajectory $\xi = t - z \to \partial_\xi = (1-v_z)\partial_t$:

$$\frac{\partial P_\perp}{\partial\xi} = \frac{\partial\gamma v_\perp}{\partial\xi} = \frac{\partial}{\partial\xi}\left(\gamma\frac{\partial r_\perp}{\partial t}\right) = \frac{\partial}{\partial\xi}\left[\gamma(1-v_z)\frac{\partial r_\perp}{\partial\xi}\right] \tag{38}$$

and using the integral of motion $\gamma - P_z = 1 + \psi$

$$\frac{\partial}{\partial\xi}[(1+\psi)r_s'] = \frac{1+v_z}{1-v_z}\frac{\partial\psi}{\partial r} + \frac{\partial A_r}{\partial\xi} \tag{39}$$

v_z is also determined by the integral of motion:

$$1 - v_z = \frac{2(1+\psi)^2}{1 + P_\perp^2 + (1+\psi)^2} \tag{40}$$

$$1 + v_z = \frac{2(1+P_\perp)^2}{1 + P_\perp^2 + (1+\psi)^2} \tag{41}$$

$$\frac{1+v_z}{1-v_z} = \frac{(1+P_\perp)^2}{(1+\psi)^2} = \frac{\left[1 + (1+\psi)^2 r_s'^2\right]^2}{(1+\psi)^2} \tag{42}$$

The derivative $\partial_r \psi$ is just $-E_g$. The A_r derivative is:

$$\frac{\partial A_r}{\partial \xi} = -\frac{\lambda_0}{r_s}\left(r_s'' - \frac{r_s'^2}{r_s}\right)\left[\frac{r^2 - r_s^2}{r}\Theta(r - r_s) - r\right] - \frac{\lambda_0 r_s'}{r_s}\left[-\frac{r^2 - r_s^2}{r} r_s' \delta(r - r_s) - \frac{2 r_s r_s'}{r}\Theta(r - r_s)\right] \tag{43}$$

This derivative looks terrible. Fortunately, we evaluate the derivative at $r = r_s$ and it simplifies considerably:

$$\left.\frac{\partial A_r}{\partial \xi}\right|_{r=r_s} = \lambda_0\left(r_s'' - \frac{r_s'^2}{r_s}\right) + \lambda_0 \frac{2 r_s'^2}{r_s}\Theta(0) \tag{44}$$

Plugging into equation 39 and evaluating at $r = r_s$, we have:

$$(1 + \psi(r_s))\, r_s'' - \frac{\lambda_0 r_s'^2}{r_s} - \lambda_0 r_s = -\frac{\left[1 + (1+\psi(r_s))^2 r_s'^2\right]^2}{(1+\psi(r_s))^2} E_g(r_s) \tag{45}$$

Ugh! This isn't quite right but getting closer. . .

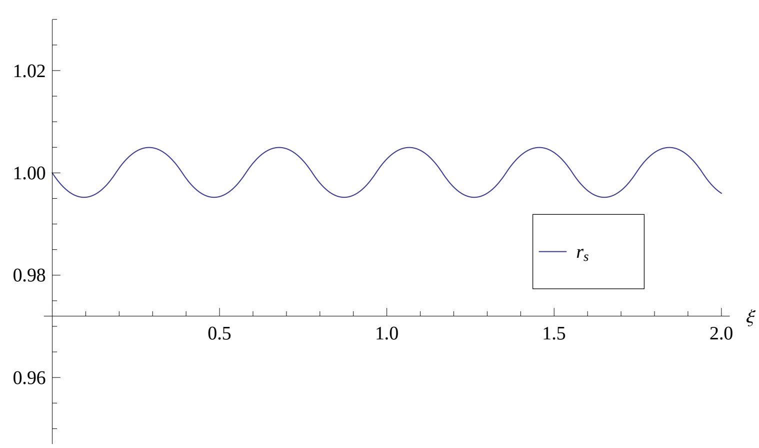

FIG. 4. r_s as a function of ξ. $r_s'(0) = -0.1$