1. MMSE estimation for Laplacian Source

Key Equations:

$$Y = X + Z$$

$$X \sim Laplacian(0, \sigma_X), \quad \sigma_X = 1$$

$$Z \sim Gaussian(0, \sigma_Z), \quad \sigma_Z^2 = 0.1$$

X : Original Signal
Z : Random Noise
Y : Noisy Signal

$$P_x(x) = \frac{1}{2\sigma_x} e^{-\frac{|x|^\alpha}{\sigma_x}}$$

$$0.5 < \alpha < 1$$

$$P_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_Z^2}}$$

$$MMSE\ Estimate: \quad \hat{X}(y) = \arg\min_{\hat{x}(y)} \mathbb{E}[(x - \hat{x}(y))^2]$$

Estimation of X

$$\hat{X}(y) = \mathbb{E}[X \mid Y] = \int x \cdot P_{x|y}(x \mid y)\, dx$$

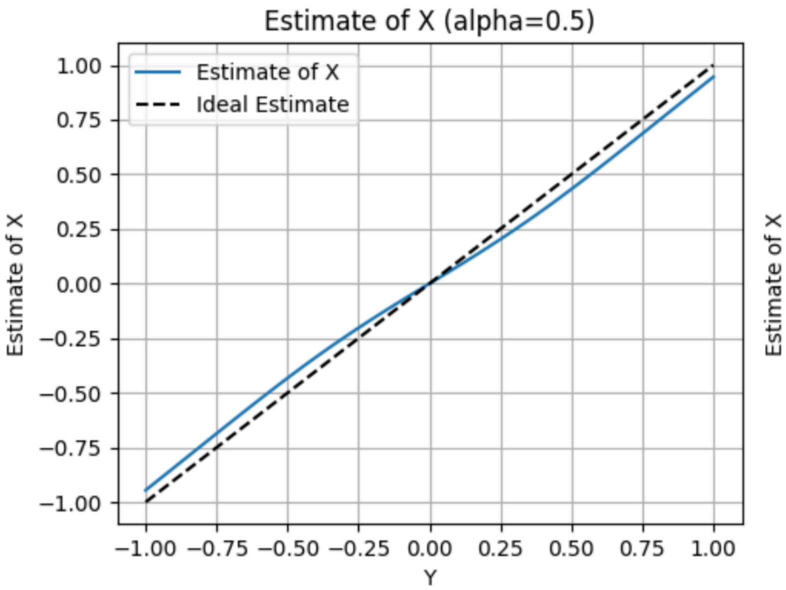
$$= \int \frac{x \cdot P_x(x) \cdot P_{y|x}(y \mid x)\, dx}{P_y}$$

$$= \frac{\int x \cdot P_x(x) \cdot P_{y|x}(y \mid x)\, dx}{\int P_x(x) \cdot P_{y|x}(y \mid x)\, dx}$$

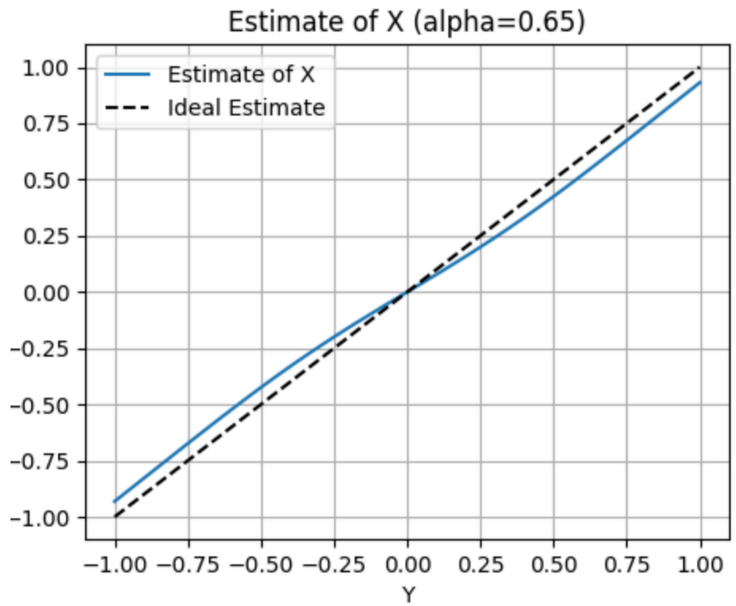
$$= \frac{\int x \cdot P_x(x) \cdot P_z(y - x)\, dx}{\int P_x(x) \cdot P_z(y - x)\, dx}$$

Final Integral Solved by numerical integration

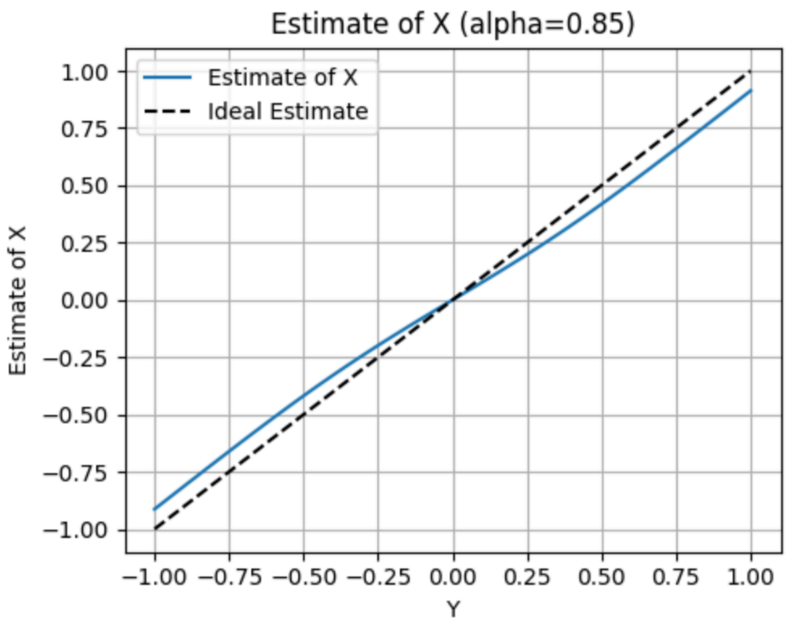
Plots of X vs estimation (varying alpha)



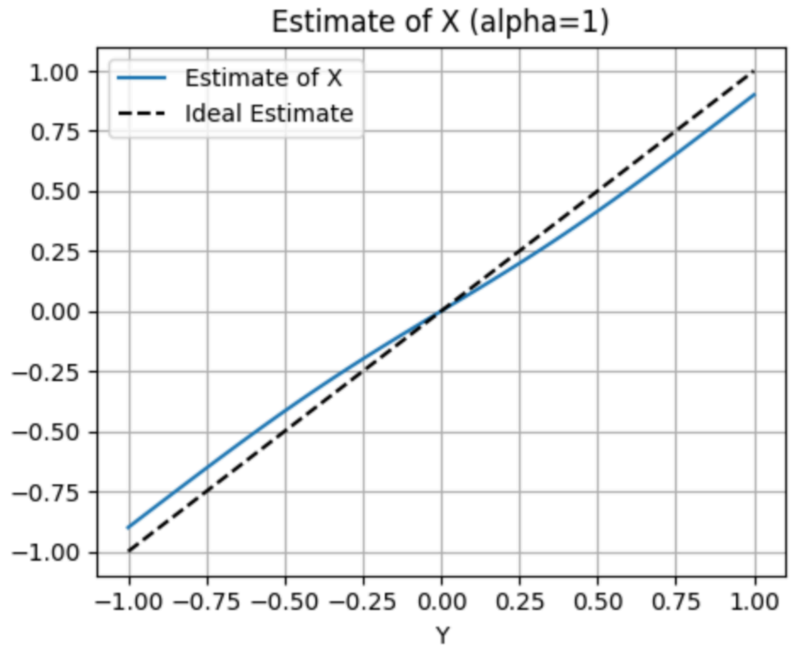
Alpha = 0.5



Alpha = 0.65



Alpha = 0.85



Alpha = 1

2.1. Image de-noising by Low pass gaussian filter

Key Equations

$$Y = X + Z$$

X : Original Image
Z : Random Noise
Y : Noisy Image

$$X = \mu_x + X_1$$

X : Original Image
μ_x: Loss pass Component
X_1: High Pass Component

$$\mu_Y = \mu_X + \mu_Z$$

$$where,\ \mu_y = Y(i,j) \circledast W(k,l)$$

W is a 2d gaussian kernel

$$\mu_z \approx 0 \quad and \quad \mu_X \approx \mu_Y$$

$$\therefore X \approx \mu_Y$$

At low frequencies noise is negligible compares to signal

Thus. the de-noised estimate of X is the low pass component of Noisy image

The low pass component is found by convolution of Noisy image with gaussian kernel

$$\hat{X}(y) = \mu_y$$

(Low pass de-noised estimate of X)

Input Image

X: Original Clean Image

Y: Noisy Image (Gaussian Noise added)

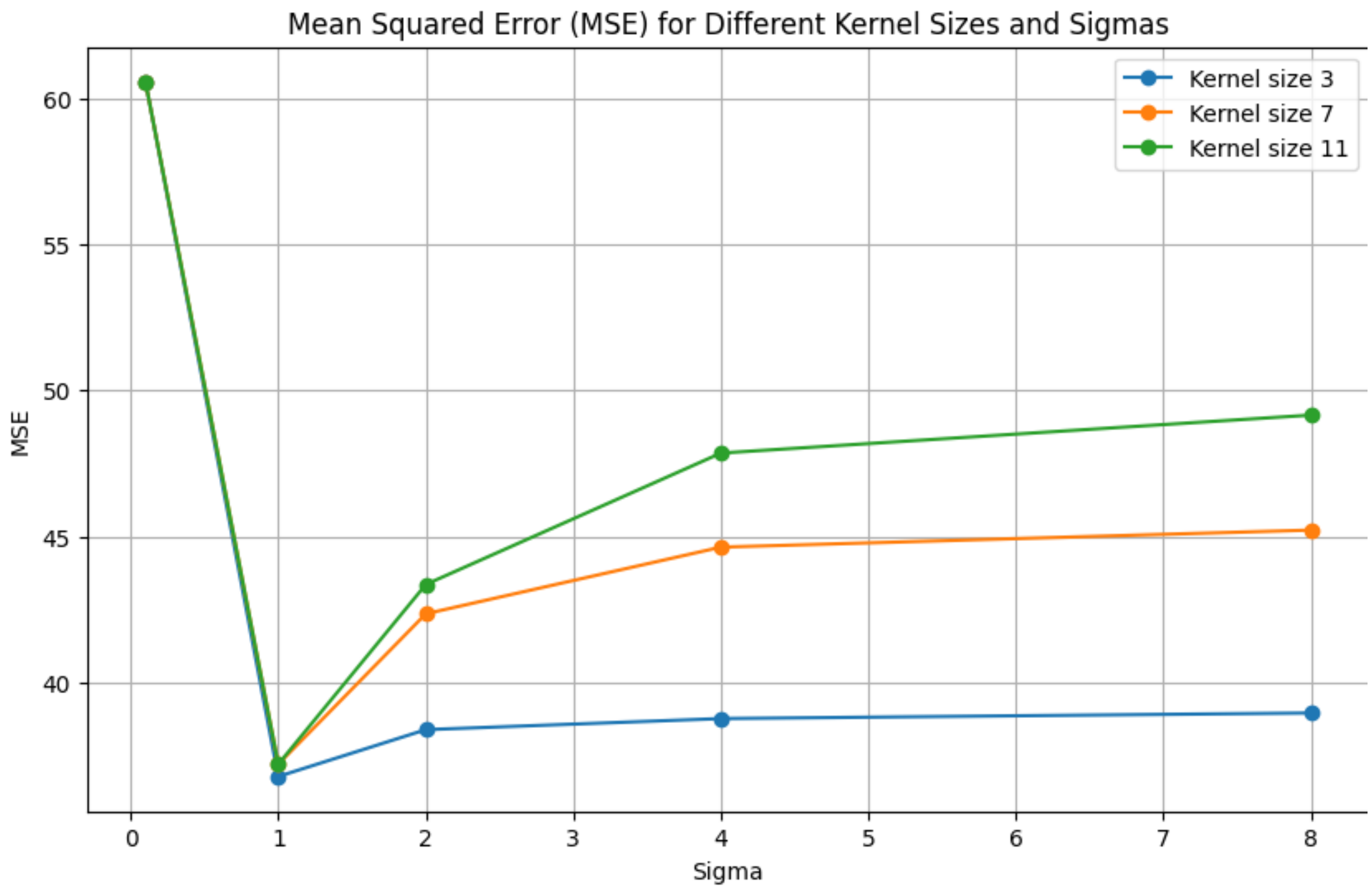
Note: In practical de-noising we don't have access to X and only work with Y

MSE for different filter parameters

$$MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\hat{X}(i,j) - X(i,j) \right)^2$$

Variance/Filter Size	[3X3]	[7X7]	[11X11]
0.1	60.48	60.48	60.48
1	**36.82**	**37.30**	**37.30**
2	38.44	42.27	43.44
4	38.79	44.54	47.72
8	38.97	45.33	49.14

MSE plot for different filter parameters



RESULT

BEST estimate of X

Low pass gaussian:
Estimates are computed by convolution of nosy image with a symmetric gaussian kernel

Best Estimate:
Filter Size: [3X3]
Variance: 1
MSE: 36.82

2.2. Image de-noising by MMSE estimate

Key Equations

$$Y = X + Z$$

X : Original Image
Z : Random Noise
Y : Noisy Image

$$Y \sim Gaussian(0, \sigma_Y)$$

$$X \sim Gaussian(0, \sigma_X)$$

$$Z \sim Gaussian(0, \sigma_Z), \quad \sigma_Z = 10$$

Gaussian PDF

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$X = \mu_x + X_1$$

X : Original Image
μ_x: Loss pass Component
X_1: High Pass Component

$$\mu_Y = \mu_X + \mu_Z$$

$$Y_1 = X_1 + Z_1$$

$$where,\ \mu_y = Y(i, j) \circledast W(k, l)$$

(W is a 2d gaussian kernel)

$$W(k, l) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{k^2 + l^2}{2\sigma^2}\right)$$

MMSE Estimate

$$X = \mu_x + X_1$$

$$\hat{X} = \hat{\mu}_x + \hat{X}_1(y)$$

$$\hat{\mu}_x \approx \mu_y$$

$$\hat{X}_1(Y_1) \approx \frac{\sigma_x^2}{\sigma_x^2 + \sigma_z^2} \cdot Y_1$$

$$\hat{X} = \mu_y + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_z^2} \cdot Y_1$$

MMSE Estimate: Linear Estimate for high pass component

$$where:$$

$$\sigma_Y = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(Y(i,j) - \bar{Y} \right)^2$$

$$\bar{Y} = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} Y(i,j)$$

$$\sigma_Z = \left\{ [1 - W(0,0)]^2 + \sum_{i=1, i \neq 0}^{M} \sum_{j=1, j \neq 0}^{N} \left(Y(i,j) - \bar{Y} \right)^2 \right\} \sigma_Z^2$$

$$\sigma_X^2 = \sigma_Y^2 - \sigma_Z^2$$

(Noise and Image variance)

2 Methods

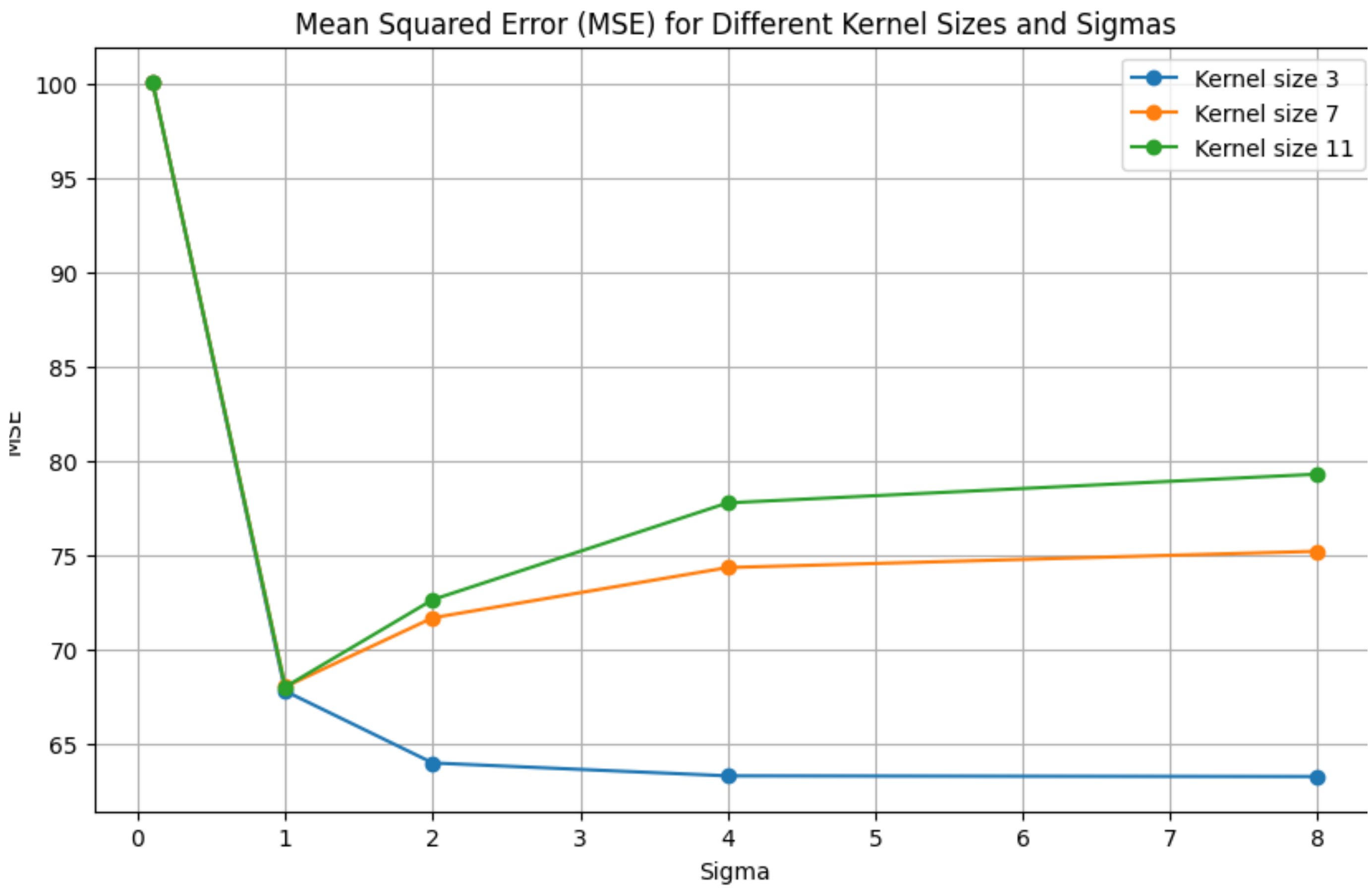
- **Ordinary :** De-noise the entire image at once
- **Adaptive :** De-noise image in patches of [32X32] with an overlap of 16

MSE for different filter parameters(Ordinary)

$$MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\hat{X}(i,j) - X(i,j) \right)^2$$

Variance/Filter Size	[3X3]	[7X7]	[11X11]
0.1	100.12	100.12	100.12
1	67.76	**67.99**	**67.98**
2	63.94	71.65	72.61
4	63.26	74.32	77.75
8	**63.22**	75.18	79.28

MSE plot for different filter parameters(ordinary)



RESULT (ordinary)

Ordinary MMSE:
Estimates is computed at once for entire image

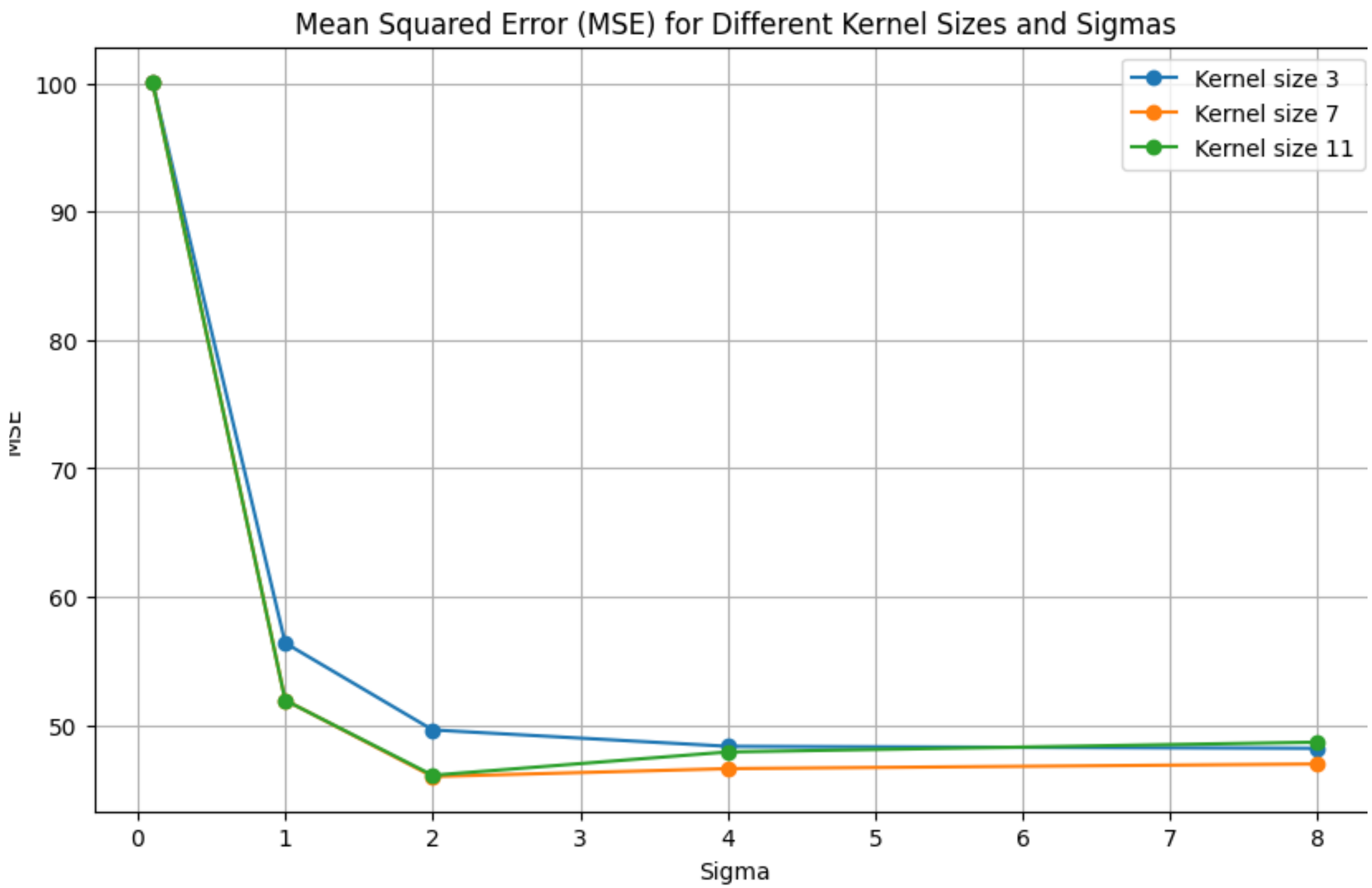
Best Estimate:
Filter Size: [3X3]
Standard dev: 8
MSE: 63.22

MSE for different filter parameters(Adaptive)

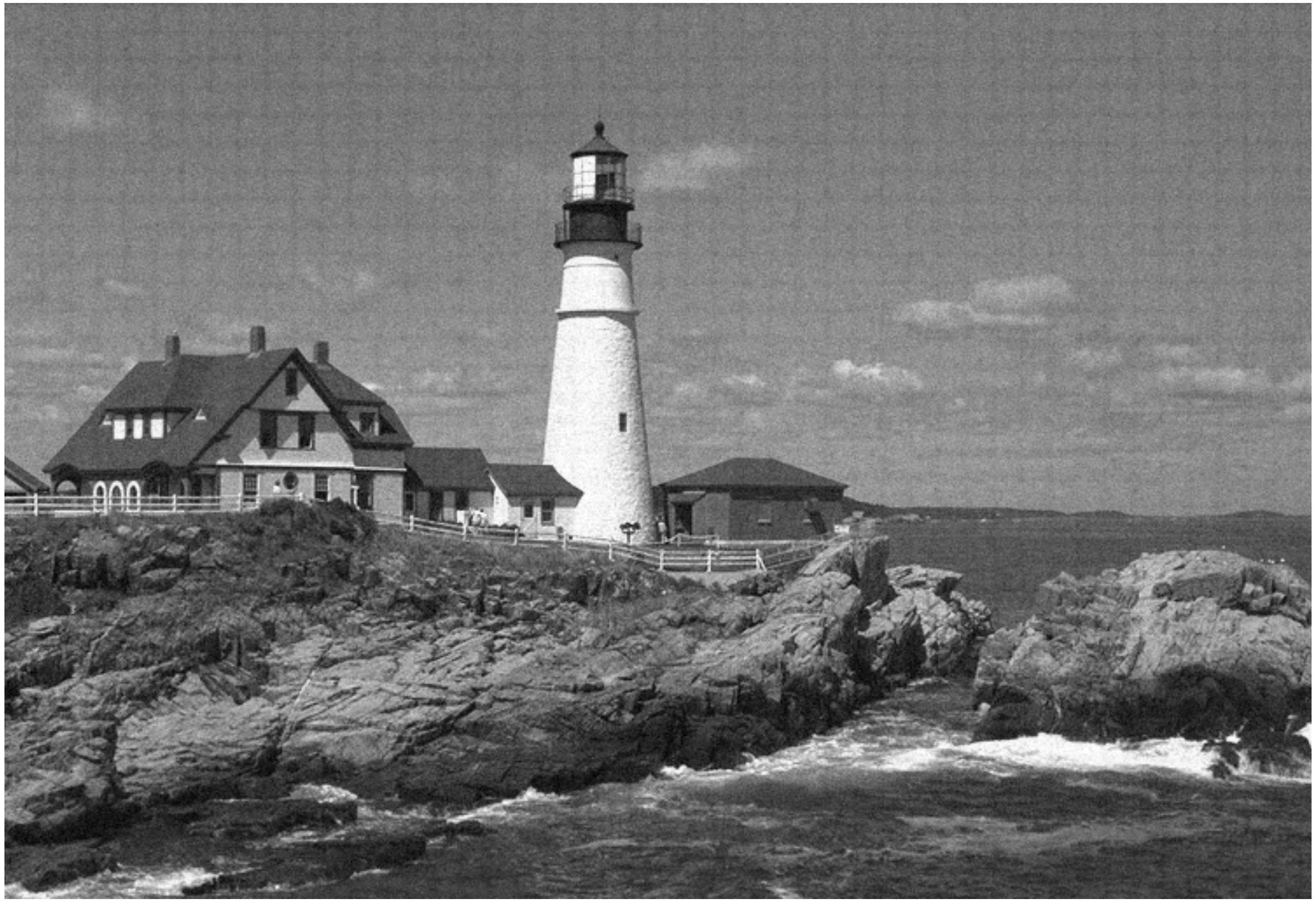
$$MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\hat{X}(i,j) - X(i,j) \right)^2$$

Variance/Filter Size	[3X3]	[7X7]	[11X11]
0.1	100.12	100.12	100.12
1	56.39	51.94	51.93
2	**49.62**	**45.99**	**46.08**
4	48.34	46.604	47.90
8	48.18	46.96	48.66

MSE plot for different filter parameters(Adaptive)



RESULTS (Adaptive)



Adaptive MMSE:
Estimates are computed separately for patches of size [32X32] with an overlap of 16

Best Estimate:
Filter Size: [7X7]
Standard dev: 2
MSE: 45.99

2.2. Image de-noising by adaptive Shrinkage estimate

Key Equations

$$Y = X + Z$$

X : Original Image
Z : Random Noise
Y : Noisy Image

$$Y \sim Gaussian(0, \sigma_Y)$$

$$X \sim Gaussian(0, \sigma_X)$$

$$Z \sim Gaussian(0, \sigma_Z), \quad \sigma_Z = 10$$

Gaussian PDF

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$X = \mu_x + X_1$$

X : Original Image
 : Loss pass Component
 : High Pass Component

$$\mu_Y = \mu_X + \mu_Z$$

$$Y_1 = X_1 + Z_1$$

$$where,\ \mu_y = Y(i,j) \circledast W(k,l)$$

(W is a 2d gaussian kernel)

$$W(k,l) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{k^2 + l^2}{2\sigma^2}\right)$$

Shrinkage Estimate

$$X = \mu_x + X_1$$

$$\hat{X} = \hat{\mu}_x + \hat{X}_1(y)$$

$$\hat{\mu}_x = \mu_y$$

$$\hat{X}_1(Y_1) = \mathrm{sgn}(Y_1) \cdot \{\,|Y_1| - t\}_+$$

Shrinkage Estimate:
Linear Estimate for high pass component

$$\hat{X} = \mu_y + \mathrm{sgn}(Y_1) \cdot \{\,|Y_1| - t\}_+ \cdot Y_1$$

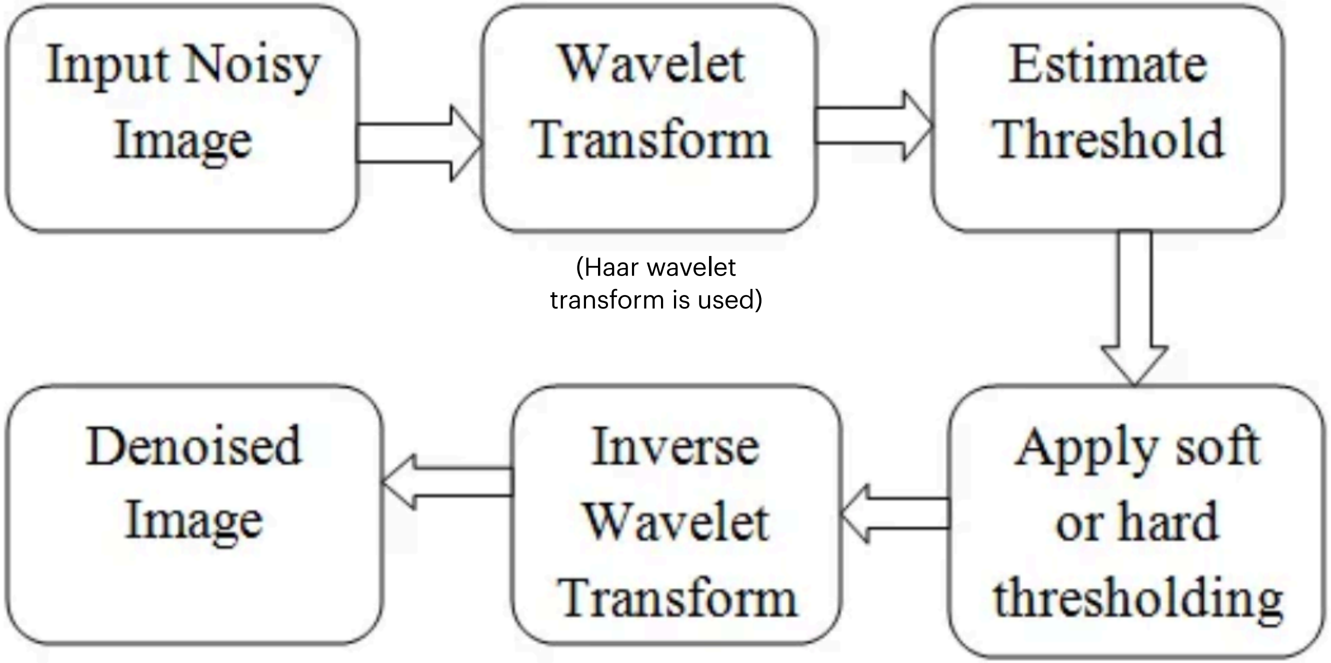
SUREshrink

$$SURE(t,\ y_1(i,j)) : \quad MN\sigma_{z_1}^2 + \sum_{i=1}^{M}\sum_{j=1}^{N} g[y_1(i,j)]^2 + 2\sigma_{z_1}^2 \sum_{i=1}^{M}\sum_{j=1}^{N} g'[y_1(i,j)]$$

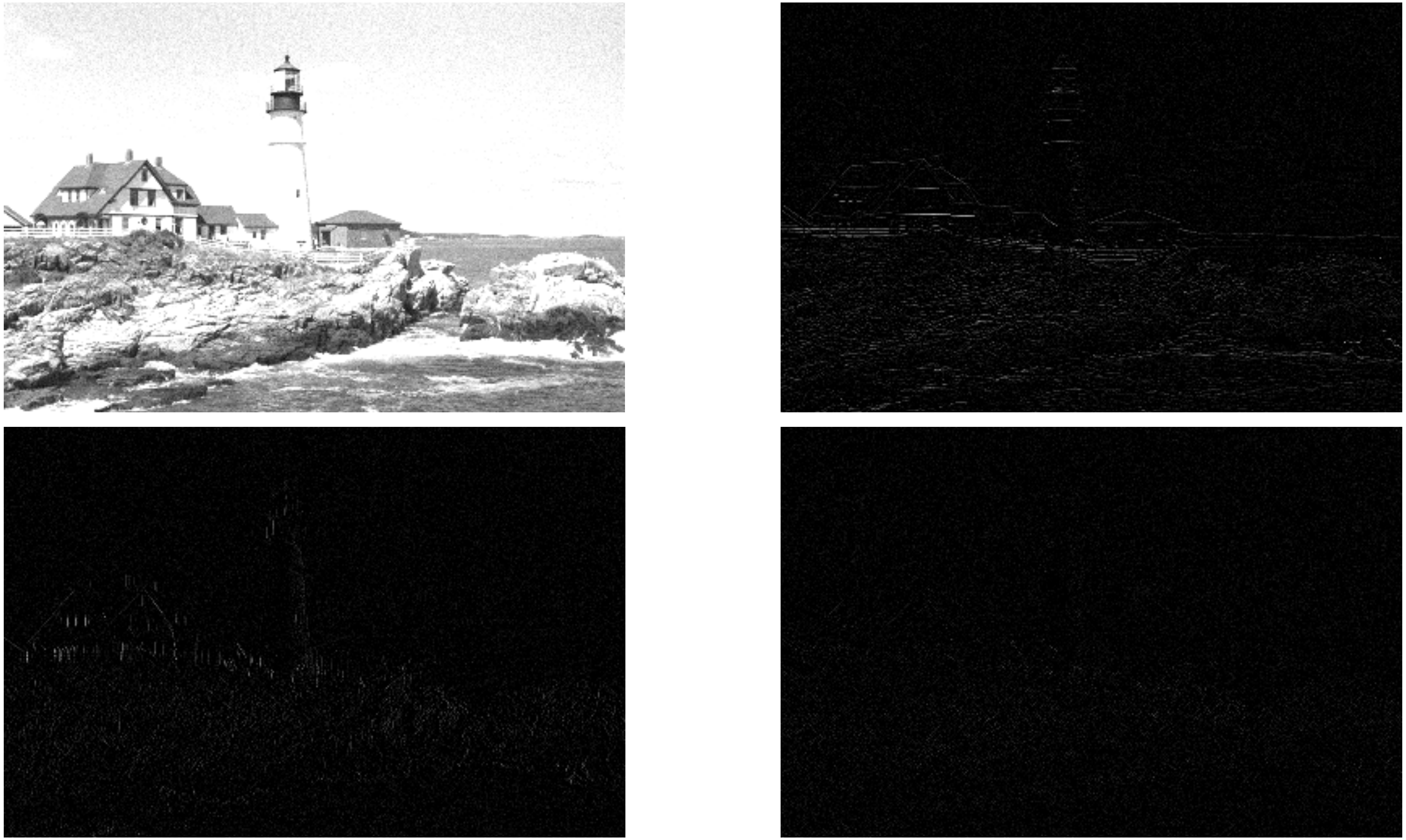
$$\sum_{i=1}^{M}\sum_{j=1}^{N} g[y_1(i,j)] = \sum_{i=1}^{M}\sum_{j=1}^{N} min(y_1(i,j),\ t)^2$$

$$\sum_{i=1}^{M}\sum_{j=1}^{N} g'(y_1(i,j)) = -\left|\{(i,j) : |Y_1(i,j)| < t\}\right|$$

Overview of process



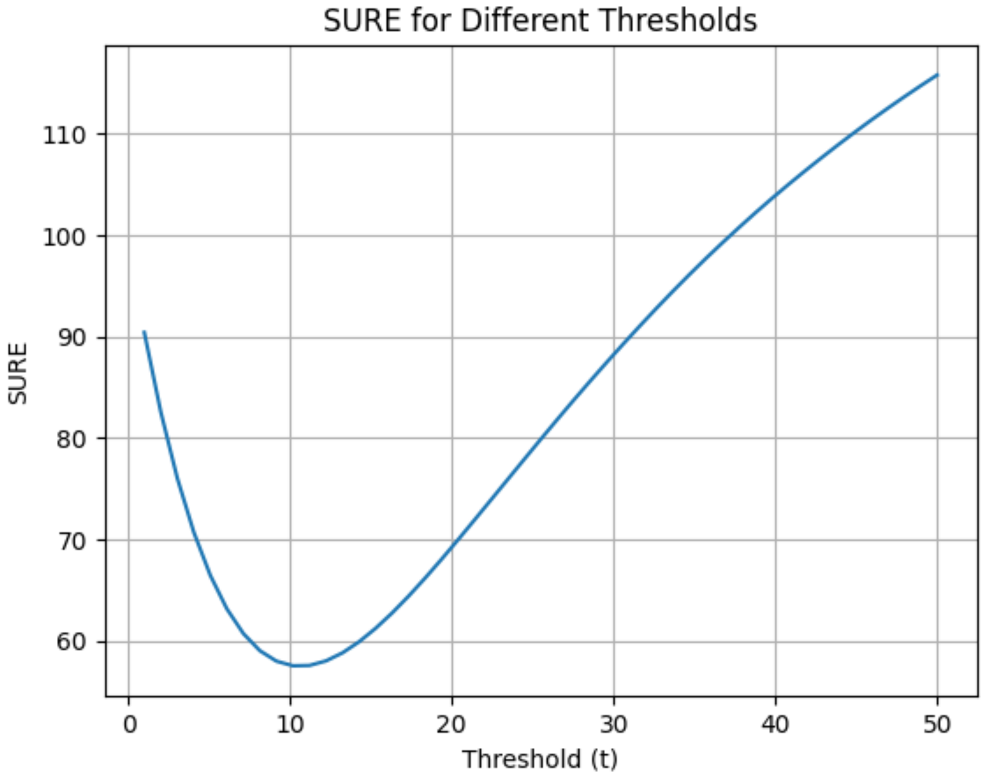
Haar wavelet transform of image



SURE-shrink threshold

SURE is an unbiased estimate and is often used to find optimal threshold t when only information about the noise variance is available.

SURE function value is repeatedly called for different values of t to find the optimal threshold.



RESULTS

Adaptive shrinkage:
Estimates are computed separately for patches of size [32X32] with an overlap of 16

Best Estimate:
Optimal t : 10
MSE: 50.75