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Holographic free energy of finite temperature conformal field theory with defects

by

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“We might be the holographic image of a two-dimensional structure”

Brian Greene

Abstract

In this thesis we study a three dimensional field theory using the AdS/CFT correspondence. The free energy of the field theory at zero temperature is known exactly from the literature. We check the AdS/CFT correspondence by calculating the free energy on the gravity side of the duality and determine exact agreement with the result from the field theory.

We generalize the gravity solution found in the literature by showing that the equations of motion are still satisfied if we introduce a black hole into the space-time geometry. On the field theory side of the AdS/CFT correspondence this results in putting the field theory at finite temperature. Using this generalized solution we predict the free energy of the field theory at finite temperature in the strong coupling regime and a phase transition is also inferred.

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Contents

1 Introduction

Quantum field theory is a framework which has obtained great success, for example in the form of the Standard Model. This model has been able to reproduce experimentally measured quantities to astonishing accuracy, and is one of the cornerstones of modern physics. The Standard Model describes three of the four fundamental interactions, the strong interaction, the weak interaction and electromagnetism, by using a generalization of the theory of electromagnetism, called non-Abelian gauge theory or Yang-Mills theory.

The part describing the strong interaction is called quantum chromodynamics, or QCD. In electromagnetism, matter can be charged, and charged matter interacts with other charged matter through the exchange of photons. In QCD, there are three charges which are usually called colors. The fundamental particles which are charged under color are called the quarks. Like in electromagnetism, there are particles responsible for the mediation of the force, called the gluons. Quarks that are bound together by gluons will form the protons and neutrons which constitute the nucleus of atoms.

In Yang-Mills theories, a coupling constant occurs quite naturally. When the coupling constant is zero, the theory is free and is analytically solvable. If the coupling constant does not vanish, but is very small, one can use perturbation theory in order to calculate observable quantities. The term "coupling constant" is actually poorly chosen, since generally the coupling constant will not be a constant, but it will depend on what energy scale the theory is probed. It turns out that for QCD the coupling constant decreases as we go to higher energy scales. Therefore, QCD is a very good theory to explain high energy particle physics experiments. However at low energies, QCD becomes strongly coupled and hence very difficult to work with. It is conjectured that QCD undergoes a phase transition called confinement, where below a certain energy scale the quarks and gluons only exist in bound states which are colorless. These bound states are conjectured to be the protons and neutrons we detect. It is still an open problem to show that this confinement actually happens. In fact one of the famous Millenium problems[1] in mathematics is to show that confinement happens in pure Yang-Mills theory, hopefully an easier problem than showing that this happens in full QCD.

Brookhaven National Laboratory has constructed the Relativistic Heavy Ion Collider

[1] http://www.claymath.org/millennium/Yang-Mills_Theory/

(RHIC) in order to study QCD near the confinement point. Two heavy atomic nuclei are collided in order to produce quark-gluon plasma which is a gas of quarks and gluons just above the confinement scale. One of the surprising results was the measurement of the ratio of shear viscosity η to entropy density s. Perturbative calculations resulted in a very large ratio, but it was measured that $\eta/s \approx 1/8\pi$. One may wonder if this ratio is a universal feature of strongly coupled field theories, i.e. do all strongly coupled field theories have small values of η/s?

For a class of toy model gauge theories there is a duality, called the anti-de Sitter/conformal field theory correspondence, or AdS/CFT correspondence. This is a conjectured correspondence between a gauge theory and a gravity theory. One of the most exciting aspects of this correspondence is that it is a strong/weak duality. This means that both theories describe the same physics, and thus can both be used to calculate physical quantities, but when one theory is strongly coupled, the other one is weakly coupled. One of the best known gauge theories for which this correspondence exists, is for a theory called $\mathcal{N} = 4$ super Yang-Mills theory. This is a Yang-Mills theory where there exists an extra type of symmetry, called supersymmetry. The $\mathcal{N} = 4$ indicates that there are four different supersymmetries in this theory. This theory can be viewed as a deformation of QCD, and we may hope that it correctly captures certain aspects of QCD. It can be shown that at weak coupling it has a large η/s ratio, like QCD. Using the AdS/CFT correspondence it was calculated that at strong coupling $\mathcal{N} = 4$ super Yang-Mills has a viscosity to entropy density ratio of $\eta/s \approx \frac{1}{4\pi}$,[1][2] which is strikingly close to the value measured at RHIC. Thus we see that the AdS/CFT correspondence is able to capture certain features of the strong coupling regimo of QCD.

Besides applications to strongly coupled QCD, one may also hope to apply AdS/CFT to other strongly coupled systems. For example, in high T_c superconductors, it is conjectured that the strange metal phase is controlled by a strongly coupled conformal field theory. The common approach in this case is to use what is known as gravitational engineering (or the bottom up approach). In this case we do not start from a specific quantum field theory, like we started with QCD in the example above, but we define a QFT by its gravitational dual description. We can then use the AdS/CFT correspondence to calculate everything we wish to know about the QFT without ever needing an explicit expression for the QFT. This has been succesfully used to describe field theories which show behaviours similar to superconductors, including a critical temperature and evidence of pair formation.[3]

It is clear that the AdS/CFT is an exciting new tool to study strongly coupled field theories, even if the correspondence is not yet proven and only a conjecture. There is however evidence supporting the AdS/CFT correspondence in the form of examples in which certain quantities are computed on both sides of the correspondence, and compared. This is the first goal of this thesis, to check the AdS/CFT correspondence in a form which has not been verified yet. We will calculate the free energy of a field theory and its gravity dual at zero temperature, and confirm that these are indeed the same as required by the correspondence. The second goal will be to use AdS/CFT correspondence to predict the free energy of this field theory at finite temperature, which is not yet calculated in the literature.

To do this, we will first introduce the main features of the AdS/CFT correspondence in chapter 2. Here the original correspondence conjectured by Maldacena is stated. In chapter 3 we introduce a family superconformal field theories called $T^{\rho}_{\hat{\rho}}[SU(N)]$, which are defined by the partitions ρ and $\hat{\rho}$ of N. The field content and symmetries of this theory are reviewed as well as the restrictions on the partitions in order for the field theories to be well defined. We will discuss the brane construction in string theory, which led to the proposal of this family of field theories. These field theories are interesting since the free energy is known at any value of the coupling constant, allowing us to explicitly check the AdS/CFT correspondence.

In chapter 4 we will review type IIB supergravity. Using an Ansatz, we will find a class of solutions to the type IIB supergravity equations of motion. Solutions containing branes will be proposed, and using these we will be able to find the gravity dual to the field theories considered in this thesis. We will show that both theories satisfy the same symmetries and that the same partitions are obtained.

Before showing that both theories have the same free energy, we will first generalize the gravity solutions in chapter 5. Specifically, we will generalize the metric Ansatz, allowing for black holes to be introduced in the metric. Introducing a black hole in the gravity theory will correspond to taking the field theory to finite temperature. This will be the first part of this thesis which contains original work.

Chapter 6 will introduce the concept of holographic renormalization, which will ensure that we get a finite free energy from the gravity theory. The need for this is physically clear from the AdS/CFT correspondence. In field theories renormalization is a standard tool used to remove infinities from the theory, thus we can expect the need for a similar procedure in the gravity theory.

Finally in chapter 7 we will present the second part of the original work for this thesis. We will calculate the free energy of the supergravity theory and verify explicitly that the AdS/CFT conjecture is satisfied in this case, thus providing more evidence for the correctness of the correspondence. We will also predict the free energy of the field theory at finite temperature, a result which cannot yet be found in the literature.

2 AdS/CFT correspondence

We will discuss the original conjecture made by Maldacena in order to introduce the main ideas behind the AdS/CFT correspondence.[4] The conjecture is that there is a duality between the $\mathcal{N} = 4$ superconformal Yang-Mills theory in four dimensions, and type IIB string theory on the space $AdS_5 \times S_5$, explaining the name Anti-de Sitter/Conformal Field Theory correspondence. Because the conformal field theory lives on the boundary of the Anti-de Sitter space-time, and is conjectured to be a completely equivalent theory to the bulk theory, it is also said that the correspondence is holographic.

To clarify this conjecture, we will first introduce $\mathcal{N} = 4$ super Yang-Mills theory in section 2.1. After that we will discuss Anti-de Sitter space-time and gravity on this background in section 2.2. Finally we will explain the AdS/CFT conjecture and work out a very simple example of this conjecture in section 2.3.

2.1 $\mathcal{N} = 4$ Super Yang-Mills

Yang-Mills theory is a gauge theory. The simplest example is quantum electrodynamics. In QED we start with a Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2.1}$$

where ψ is a fermion field, in this case the electron. This Lagrangian density is invariant under transformations of the form $\psi \to e^{i\theta}\psi$, thus this is a global symmetry. We call it a global symmetry, since θ is a constant which is the same at every point in space-time.

A gauge theory is found by requiring that this symmetry is not just a global symmetry, but a local symmetry, meaning that we have transformations of the form $\psi \to e^{i\theta(x)}\psi$. This type of transformations is contained in the group $U(1)$. Requiring that the Lagrangian density remains invariant under this local symmetry forces us to introduce a vector field A_μ, and to replace ∂_μ with $D_\mu = \partial_\mu + iqA_\mu$. This extra field A_μ is the photon field, and will be the particle responsible for transmitting the electromagnetic force.

As mentioned in the introduction, quantum chromodynamics is also a gauge theory, and a generalization of QED. The main difference is that now we do not have just one charge, but three charges, called colors. The transformation leaving the Lagrangian density invariant will now be $\psi \to e^{i\alpha^i\lambda_i}\psi$, where ψ contains three spinor fields, corresponding to the three quarks of different color, and λ_i is no longer a scalar but a matrix. Transformations of this type are contained in the group $SU(3)$. Requiring that the Lagrangian remains invariant, we will now be forced to introduce not one but eight different vector fields, called the gluons. The difference with the photons is that these gluons will carry color themselves, which is a direct consequence of the fact that scalars commute, but matrices do not. Therefore QCD is more complex than QED, since the gluons will interact with themselves. The theory which describes the gluons is more generally called a non-Abelian gauge theory or Yang-Mills theory.

In $\mathcal{N} = 4$ super Yang-Mills theory we will have N different charges, corresponding to a gauge group of $SU(N)$. This is just a generalization from QCD, no qualitatively new features are introduced in going from three colors to N colors.

Now we have explained the Yang-Mills part of the name $\mathcal{N} = 4$ super Yang-Mills theory. To explain the first part we need to introduce the concept of supersymmetry. Supersymmetry is a symmetry between bosonic fields and fermionic fields, and will transform one into the other. To make sure the Lagrangian is invariant under supersymmetry, the fields are restricted to enter the Lagrangian in the form of multiplets. This is a group of fields which transform into the same group under supersymmetry, like a vector transforms into itself under Lorentz transformations. The $\mathcal{N} = 4$ stands for the fact that there are four different supersymmetries in this theory. Since Yang-Mills theory must contain gauge vector fields, the $\mathcal{N} = 4$ super Yang-Mills Lagrangian must consist of gauge multiplets, a multiplet containing a gauge vector, fermions and scalars. Thus $\mathcal{N} = 4$ super Yang-Mills, or $\mathcal{N} = 4$ SYM, may be thought of as a generalization or deformation of pure Yang-Mills theory where one introduces additional fermion and scalar degrees of freedom.

We have now explained the field content of $\mathcal{N} = 4$ SYM, the gauge symmetry and supersymmetry. A final symmetry which it should satisfy is ofcourse space-time symmetry. Usually field theories are invariant under transformations which leave the metric invariant. However, $\mathcal{N} = 4$ SYM is an example of a conformal field theory, as we could suspect from the name AdS/CFT. It is invariant under transformations which preserve the metric up to a scale factor,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = w(x)g_{\mu\nu}(x) \tag{2.2}$$

2.1.1 Correlators

What is it we would like to know about a field theory? For example, in quantum mechanics, we want to know all the possible states and the expectation values of observables in these states. In a conformal field theory every state will correspond to a certain operator in the theory. Some examples of operators are the Lagrangian itself, or the stress energy tensor. If we have a scalar field ϕ in our theory, another operator could be the scalar field itself, or its derivative $\partial_\mu\phi$. A correlator is then the expectation value of a combination of operators. A two-point correlator of operators $\mathcal{O}$ and $\mathcal{O}'$ is calculated using the path integral formalism by,

$$\langle \mathcal{O}\mathcal{O}' \rangle = \frac{\int \mathcal{D}[fields]\mathcal{O}\mathcal{O}' e^{iS}}{\int \mathcal{D}[fields] e^{iS}} \tag{2.3}$$

If we know all the correlators of all the operators, the theory is completely defined. Usually these correlators are derived from a functional known as the generating functional. This is defined by a path integral as follows,

$$Z[J] = \int \mathcal{D}[fields] e^{iS + \int d^d x J(x)\mathcal{O}(x)} \tag{2.4}$$

It is clear that

$$\langle \mathcal{O}\mathcal{O} \rangle = \frac{\delta}{\delta J}\frac{\delta}{\delta J} \ln Z[J] \tag{2.5}$$

One interesting feature of conformal field theories, is that the symmetry is so restrictive that the two-point correlation function must be of the form[5]

$$\langle \mathcal{O}(x_1)\mathcal{O}'(x_2) \rangle \sim \frac{1}{|x_1 - x_2|^{2\Delta}} \tag{2.6}$$

where Δ, called the dimension of the operator $\mathcal{O}$, is found by looking at the conformal transformation of the operator,

$$\mathcal{O} \to \mathcal{O}' \sim \left| \frac{\partial x'}{\partial x} \right|^{\Delta/d} \mathcal{O} \tag{2.7}$$

where d is the dimension of space-time. The Lagrangian is an example of a scalar operator of dimension d. We will use the form (2.6) of the two-point correlator when we compute a simple example of the AdS/CFT correspondence in section 2.3.

2.1.2 Thermal partition function from QFT methods

There is an interesting connection between the generating functional we just defined, and the partition function of statistical physics. In statistical physics the partition function for a certain temperature T is defined by

$$Z(T) = \sum_\alpha \exp\left(\frac{-E_\alpha}{k_B T}\right) \tag{2.8}$$

where k_B is the Boltzmann constant, and E_α is the energy of state α.

Now let us take the analytic continuation of the generating functional (2.4) without sources to Euclidean space, meaning that we will set $J = 0$ and $t \to i\tau$. If we now require that τ is a periodic variable with period $\frac{1}{k_B T}$, it can be shown that[6]

$$Z = \int \mathcal{D}[fields] e^{-S_E} = \sum_\alpha \exp\left(\frac{-E_\alpha}{k_B T}\right) \tag{2.9}$$

from which we see that the partition function of Euclidean QFT is the partition function of statistical mechanics. Using this identification we are able to define the free energy in the same way as in statistical physics,

$$F = -\ln Z \tag{2.10}$$

2.2 Anti-de Sitter space-time

Before discussing how AdS space-time is obtained in a gravitational theory, we will first discuss some of the usual coordinates used to describe AdS. An n-sphere can easily be described by its embedding in $\mathbb{R}^{n+1}$. Similarly, we can describe d dimensional AdS space, also written as AdS_d, by

$$\eta_{ab}x^a x^b = -L^2 \tag{2.11}$$

where L is called the radius of the space and $a = 0, \cdots, d$ and $\eta_{ab} = diag(- - + \cdots +)$. From this we see that AdS_d space is invariant under transformations of the group $SO(d-1, 2)$, a fact that will be important in the justification of the AdS/CFT correspondence.

2.2.1 Global coordinates

A frequently used set of coordinates is obtained by the transformation

$$\begin{aligned}
x^i &= r\bar{x}^i & \sum_{i=2}^{d}(\bar{x}^i)^2 &= 1 \\
x^0 &= \sqrt{L^2+r^2}\sin\left(\frac{t}{L}\right) & x^1 &= \sqrt{L^2+r^2}\cos\left(\frac{t}{L}\right)
\end{aligned} \tag{2.12}$$

where the $\bar{x}^i$ are coordinates for S_{d-2}, $0 \leq r < \infty$ and $0 \leq t < 2\pi L$. The induced metric for these coordinates is

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \left(1+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega_{d-2}^2 \tag{2.13}$$

The coordinate t is interpreted as time. However, because $0 \leq t < 2\pi L$, time is a periodic variable. This is not a desired feature because if this was the case any theory on AdS space would not be a causal theory. Therefore we “unwrap” the time circle and let $-\infty \leq t \leq \infty$, and redefine AdS as the space generated by these coordinates, this is the covering space of the original AdS.

The boundary of this space is located at $r \to \infty$. From (2.13) we infer that the boundary metric will be conformally related to the metric describing the geometry $S_1 \times S_2$.

2.2.2 Poincaré patch coordinates

Another frequently used and useful coordinate system is found by the following coordinate transformation on the original x^a coordinates,

$$\begin{aligned}
x^0 &= \frac{Lx^0}{z} & x^1 &= \frac{z}{2}\left(1+\frac{1}{z^2}(L^2+x^2)\right) \\
x^i &= \frac{Lx^i}{z} & i &= 2, \cdots, d-1 \\
x^{d-1} &= \frac{z}{2}\left(-1+\frac{1}{z^2}(L^2-x^2)\right) & x^2 &= -(x^0)^2 + \sum_i (x^i)^2
\end{aligned} \tag{2.14}$$

and $z \geq 0$. This leads to the induced metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - (dx^0)^2 + \sum_i (dx^i)^2\right) \tag{2.15}$$

These coordinates are called the Poincaré patch coordinates. It is called the Poincaré patch coordinates since it does not cover the AdS covering space, as described by coordinates (2.13).

The boundary of this metric is conformally related to d-dimensional Minkowski space-time, which is why we say that the conformal field theory lives on the boundary of anti-de Sitter space-time.

2.2.3 Anti-de Sitter slicing coordinates

Anti-de Sitter space has a coordinate system in which constant radial slices are again anti-de Sitter spaces of dimension $d-1$. In order to see this, we rewrite (2.11) as follows

$$\eta_{ij} x^i x^j = -L^2 - (x^d)^2 \tag{2.16}$$

where $i = 0, \cdots, d-1$ and $\eta_{ij} = diag(-, -, +, \cdots, +)$. For constant x^d, this describes an AdS_{d-1} space. Let us introduce the new coordinate ρ by

$$1 + (x^d)^2 = \cosh^2(\rho) \Rightarrow x^d = \sinh\rho \tag{2.17}$$

Furthermore, we introduce the coordinates y^i by

$$x^i = y^i \cosh\rho \tag{2.18}$$

and

$$ds^2_{AdS_{(d-1)}} = \eta_{ij} y^i y^j \tag{2.19}$$

The induced metric is then given by

$$ds^2 = d\rho^2 + \cosh^2(\rho) ds^2_{AdS_{(d-1)}} \tag{2.20}$$

2.2.4 Scalar field on AdS

To be able to give a simple example of the AdS/CFT correspondence, we will consider a $d+1$ dimensional AdS theory that contains a massless scalar ϕ, given by the action

$$S = \int d^{d+1}x \sqrt{g}\,\partial_\mu\phi\,\partial^\mu\phi \tag{2.21}$$

and the metric is Euclidean AdS_{d+1} in Poincaré patch coordinates, thus

$$ds^2 = \frac{1}{x_0^2}\sum_{i=0}^{d}(dx_i)^2 \tag{2.22}$$

In the next section, it will be clarified that we wish to find a solution ϕ with boundary condition $\phi = \phi_0$ on the boundary $x_0 = \infty$. To accomplish this, we will look for a Green's function K of the equation of motion, whose boundary value is a delta function. Since K will be invariant under translations of the x_i, it can only be a function of x_0, and therefore it must be a solution to the equation

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0 \tag{2.23}$$

and thus

$$K(x_0) = c x_0^d \tag{2.24}$$

To see that this has a delta function on the boundary $x_0 = \infty$, we perform the transformation

$$x_i \to \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2} \tag{2.25}$$

which transforms K to

$$K(x) = c \frac{x_0^d}{\left(x_0^2 + \sum_{j=1}^d x_j^2\right)^d} \tag{2.26}$$

and the boundary at $x_0 = \infty$ is now at the origin. $K(x)$ is now zero everywhere except at the origin, from which we deduce that it is a delta function on the boundary.

We can now write the solution to the equation of motion with boundary ϕ_0 as

$$\phi(x_0, x_i) = c \int dx' \frac{x_0^d}{\left(x_0^2 + \sum_{i=1}^d |x_i - x_i'|^2\right)} \phi_0(x_i') \tag{2.27}$$

2.2.5 AdS gravity

Anti-de Sitter space-time is a solution of the Einstein-Hilbert action with a negative cosmological constant

$$S = -\frac{1}{2\kappa} \int (R - 2\Lambda) \sqrt{-g} d^d x \tag{2.28}$$

where κ is a constant related to the gravitational constant. The cosmological constant for d-dimensional AdS space-time is

$$\Lambda = -\frac{(d-1)(d-2)}{L^2} \tag{2.29}$$

where L is called the radius of the AdS space-time.

We could look at fluctuations in this solution, thus

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + \epsilon h_{\mu\nu} \tag{2.30}$$

where the field $h_{\mu\nu}$ is interpreted as the graviton. We could do the same analysis here that we did for the scalar field, but this would lead us too far. Instead we will continue with the statement of the AdS/CFT conjecture.

2.3 The AdS/CFT conjecture

There are three forms of the AdS/CFT conjecture of different "strength". The strongest form is the duality between a string theory and a conformal field theory. The string theory is type IIB string theory on $AdS_5 \times S_5$ with integer 5-form flux $N \sim \int_{S_5} F_5$ and with string coupling g_s. The radii of AdS_5 and S_5 are both equal to L. The conformal field theory is $\mathcal{N} = 4$ super Yang-Mills theory with coupling constant g_{YM} in four dimensions which is invariant under the $SU(N)$ gauge group. The parameters of both theories should be related by

$$g_s = g_{YM}^2 \qquad\qquad L^4 = 4\pi g_s N(\alpha')^2 \tag{2.31}$$

The relation between g_s and g_{YM} can be traced back to the fact that the Yang-Mills theory is a field theory describing open strings, while supergravity describes closed strings. It is known that $g_o^2 = g_c$ where g_o is the open string coupling and g_c is the closed string coupling.[7] The second part of (2.31) can be argued to be true by the supergravity equations of motion. These relate the scalar curvature of the space with the 5-form flux, $R \sim F_{(5)}^2$, implying that there should be this relation between the radius L and N. This correspondence is the strong form because it is conjectured to hold for all values of g_s and N.

A weaker form is obtained by taking the 't Hooft limit. The 't Hooft coupling is defined by

$$\lambda = g_s N = g_{YM}^2 N \tag{2.32}$$

The limit is then taken to be $N \to \infty$ while λ remains fixed. On the CFT side this limit is well defined.[8] On the string theory side we see that the string coupling is given by $g_s = \lambda/N \to 0$. Thus this means that there is weak coupling, implying that we can restrict ourselves to classical string theory.

Another limit can be taken. It can be shown that $\lambda^{-1/2}$ is proportional to the string length. Taking the limit $\lambda \to \infty$ the string length goes to zero, and we can describe string theory by type IIB supergravity. This implies that the large N and large λ limit of $\mathcal{N} = 4$ SYM is dual to type IIB supergravity.

Until now we talked about two theories being dual without explaining what is meant with this statement. First of all this means that the symmetries in both theories should be the same. Second, for every operator in the CFT, there is a dual degree of freedom in the gravity theory. The restriction of a field on the gravity side to the boundary of AdS spacetime determines the source in the Lagrangian for the dual CFT operator. For example, consider a scalar field ϕ in the gravity formulation. The restriction of ϕ to the boundary of AdS_5 is called ϕ_0 as in the previous section. According to the correspondence ϕ_0 sources an operator $\mathcal{O}$ on the field theory side. This operator $\mathcal{O}$ will be a scalar operator due to Lorentz symmetry. There is also a relation between the mass of the scalar field on the gravity side, and the dimension of the operator on the field theory side. For a scalar field for example, we have the relation

$$m^2 = \Delta(\Delta - 4) \tag{2.33}$$

Another good example for the duality between field theory operators and gravity fields is the stress energy tensor. The stress energy tensor in the conformal field theory is a rank two symmetric tensor, so its dual is naturally the graviton $h_{\mu\nu}$ we encountered in the previous section. From this it is clear that a theory dual to a gauge theory should contain gravity.

On the field theory side we are interested in the generating functional $Z[J]$ defined in (2.4). Define $Z_S(\phi_0)$ to be the supergravity partition function,

$$Z_S(\phi_0) = \int \mathcal{D}[fields]e^{-S_S[fields]} \tag{2.34}$$

with the condition that ϕ is equal to ϕ_0 on the boundary. The statement that both theories are dual will then imply that

$$Z[J] = Z_S(\phi_0) \tag{2.35}$$

The equality will also hold if we have particles of different spin and non-zero mass. This equivalence will allow us to calculate the free energy of a conformal field theory by calculating the free energy of the dual supergravity theory.

A simple example of this is found by considering the solution for the massless scalar ϕ on AdS_{d+1}, (2.27). We will consider the saddle-point approximation of the supergravity partition function, meaning that we will approximate the path integral by its largest contribution, which comes from evaluating the action using fields which are solutions to the equations of motion,

$$Z_S \approx e^{-S_S} \tag{2.36}$$

As mentioned before, the two point correlation function on the field theory side is calculated as functional derivatives of the logarithm of the generating functional,

$$\frac{\delta}{\delta J}\frac{\delta}{\delta J}\ln Z[J] = \langle \mathcal{O}\mathcal{O}\rangle \tag{2.37}$$

Using the AdS/CFT correspondence, we see that this should be equivalent to computing

$$\langle \mathcal{O}\mathcal{O}\rangle = -\frac{\delta}{\delta\phi_0}\frac{\delta}{\delta\phi_0}S_S \tag{2.38}$$

From (2.27) we are able to evaluate the scalar fied action,

$$S \sim \int dxdx'\frac{\phi_0(x)\phi_0(x')}{|x-x'|^{2d}} \tag{2.39}$$

Using this we see that

$$\frac{\delta}{\delta\phi_0}\frac{\delta}{\delta\phi_0}S \sim \frac{1}{|x-x'|^{2d}} \tag{2.40}$$

which is exactly what we expect for a two-point function of a conformal field theory, as shown in (2.6).

As an explicit example, we return to $\mathcal{N} = 4$ SYM and type IIB supergravity on $AdS_5 \times S_5$ and consider the $\mathcal{N} = 4$ SYM Lagrangian density. It can be shown that the dimension of the Lagrangian density will be $\Delta = 4$. The dual field in type IIB supergravity is the dilaton which is a massless scalar field. Taking $d = 4$ in (2.40) we see that the correlators in the CFT and gravity theory exactly match. This example is kind of trivial, since everything is determined by symmetry. In this thesis, we will give a non-trivial check which does not rely on the use of symmetries.

What indications do we have that the correspondence holds true? In the previous section we saw that the boundary of AdS_5 is four dimensional Minkowski space, the space on which $\mathcal{N} = 4$ SYM lives, as required by the correspondence. If the theories are to be dual we also expect their symmetries to be the same. The global symmetry in $\mathcal{N} = 4$ SYM can be shown to be $SU(2,2|4)$. The maximal bosonic subgroup is $SU(2,2) \times SU(4)_R \sim SO(2,4) \times SO(6)_R$. This is readily recognized as the isometry group of $AdS_5 \times S_5$, since we showed in the previous section that $SO(2,d)$ is an isometry group for AdS_d. It is shown that the complete symmetry groups are the same, and not just the bosonic subgroup, but this would lead us too far. It can also be shown that every operator of $\mathcal{N} = 4$ SYM theory can be mapped to supergravity fields.[9]

The AdS/CFT correspondence is not only applicable to $\mathcal{N} = 4$ SYM dual to type IIB supergravity on $AdS_5 \times S_5$, but this was the original correspondence first proposed in [4] and is understood best. In this thesis we will use the AdS/CFT correspondence between a conformal field theory called $T^{\rho}_{\hat{\rho}}[SU(N)]$, which will be introduced in chapter 3, and type IIB supergravity on a background of $AdS_4 \times S_2 \times S_2 \times \Sigma$, where Σ is a general two dimensional space with disk topology. We will explicitly compare the free energy on both sides of the correspondence as a check for the AdS/CFT correspondence in this case, before we use the correspondence to predict the free energy of the field theory at finite temperature.

3

$T^{\rho}_{\hat{\rho}}[SU(N)]$

In this chapter we introduce a family of field theories, called $T^{\rho}_{\hat{\rho}}[SU(N)]$. It is stated in [10] that these theories flow to three dimensional $\mathcal{N} = 4$ superconformal field theories in the infrared limit, to which the supergravity dual theory was obtained in [11]. In [12] the partition function for this field theory in the infrared limit is calculated and an explicit expression is obtained, which is used to calculate the free energy.

$T^{\rho}_{\hat{\rho}}[SU(N)]$ is determined by providing an integer N, and a pair of partitions, ρ and $\hat{\rho}$, of N. If the partitions

$$\rho : \; N = l_1 + l_2 + \cdots + l_k \tag{3.1}$$

$$\hat{\rho} : \; N = \hat{l}_1 + \hat{l}_2 + \cdots + \hat{l}_{\hat{k}} \tag{3.2}$$

satisfy

$$\rho^T > \hat{\rho} \tag{3.3}$$

then $T^{\rho}_{\hat{\rho}}[SU(N)]$ flows to a non-trivial infrared fixed point according to [10]. This inequality will be explained in more detail in (3.9). The l_i and the $\hat{l}_i$ are ordered as follows,

$$l_1 \geq l_2 \geq \cdots \geq l_k > 0 \tag{3.4}$$

$$\hat{l}_1 \geq \hat{l}_2 \geq \cdots \geq \hat{l}_{\hat{k}} > 0 \tag{3.5}$$

Next we define parameters describing the features of the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory. We define M_l to be the number of i for which $l = l_i$ and thus we have that $\sum l M_l = N$. m_l is the number of i for which $l_i \geq l$. Finally we define the numbers N_i by

$$N_1 = k - \hat{l}_1 \tag{3.6}$$

$$N_j = N_{j-1} + m_j - \hat{l}_j \qquad\qquad j = 2, \cdots, \hat{k} - 1 \tag{3.7}$$

Usually $T^{\rho}_{\hat{\rho}}[SU(N)]$ is presented as a linear quiver diagram, see Fig. 3.1. These quiver diagrams represent the (N_j, M_j). The set of numbers (N_j, M_j) determine the field theory. The N_i determine the gauge group G of $T^{\rho}_{\hat{\rho}}[SU(N)]$ by

$$G = U(N_1) \times U(N_2) \times \cdots \times U(N_{\hat{k}-1}) \tag{3.8}$$

This is represented in the linear quiver diagram by the circles. For every pair of neighbouring circles (every line between two circles) there is a hypermultiplet in the bifundamental representation of both groups. There are also M_i hypermultiplets in the fundamental representation of $U(N_i)$, represented by the squares.

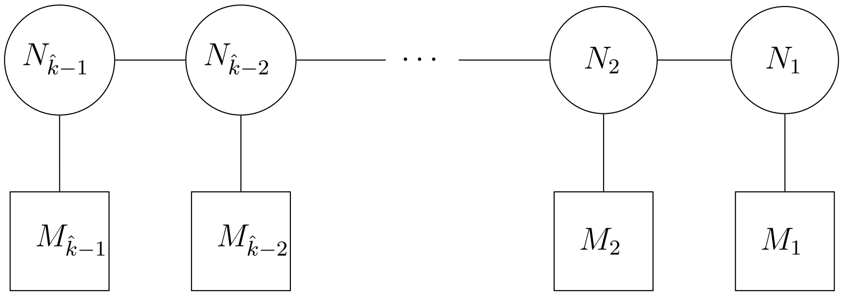

Figure 3.1: Linear quiver diagram for $T^{\rho}_{\hat{\rho}}[SU(N)]$

As mentioned above, $T^{\rho}_{\hat{\rho}}[SU(N)]$ is only conjectured to flow to a non-trivial infrared fixed point if (3.3) holds. We will show that this is a very natural constraint. The partitions ρ and $\hat{\rho}$ can be represented by Young tableaux, see Fig. 3.2. We observe that the m_i defined above can be interpreted as the l_i of ρ^T. This leads to the equivalence

$$\rho^T > \hat{\rho} \Leftrightarrow \sum_{s=1}^{i} m_s > \sum_{s=1}^{i} \hat{l}_s \qquad \forall i = 1, \cdots, l_1 \tag{3.9}$$

Since both ρ^T and $\hat{\rho}$ have N boxes, ρ^T has l_1 rows and $\hat{\rho}$ has $\hat{k}$ rows, $\hat{k} > l_1$ must hold. This guarantees that $N_1 > 0$, as we should have, because if $N_1 < 0$ then $U(N_1)$ is ill defined. This holds for all other N_i as well since

$$N_j = \sum_{i=1}^{j} (m_i - \hat{l}_i) > 0 \qquad \forall j = 1, \cdots, l_1 \tag{3.10}$$

and

$$\sum_{i=1}^{j} m_j = N > \sum_{i=1}^{j} \hat{l}_j \qquad \forall j = l_1 + 1, \cdots, \hat{k} - 1 \tag{3.11}$$

Because $\hat{k} > l_1 \geq l_i$ for all $i = 1, \cdots, l_k$, we must have that

$$M_i = 0 \qquad \forall i \geq \hat{k} \tag{3.12}$$

This means that there are no hypermultiplets corresponding to empty gauge groups. Thus the condition (3.3) is necessary to ensure that the field theory makes sense.

To get a rough physical picture of this system, we can think of every circle as representing a Yang-Mills or QCD like theory. The lines than represent degrees of freedom which are charged under these Yang-Mills theories. Even more roughly, one may think of a slab of material with charged 2+1 dimensional defects inserted into the material. The circles then describe the segments of material, while the lines describe the charged defects.

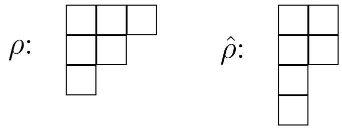

Figure 3.2: Young tableaux for $\rho = (3, 2, 1)$ and $\hat{\rho} = (2, 2, 1, 1)$

3.1 Brane construction

$T^{\rho}_{\hat{\rho}}[SU(N)]$ gauge theories where originally introduced as low-energy limits of string theory with certain brane configurations. It is known that supergravity is also a low energy limit of string theory, but a different limit than the one to obtain the gauge theories. Indeed, this is the idea behind the AdS/CFT correspondence. Since we are interested in the supergravity dual theory, we will describe the main features of this construction.

The branes necessary for our construction are k D5-branes living in the (012456) dimensions, $\hat{k}$ NS5-branes in the (012789) dimensions and D3-branes stretched between the D5- and NS5-branes in the (0123) dimensions, where the x_3 dimension corresponds to a finite interval. String theory on a stack of N coincidental D-branes will give rise to a gauge theory invariant under the gauge group $U(N)$.[7] Therefore, the vectormultiplets arise from open strings living on the D3-branes, and due to this finite extent in the x_3 dimension, the gauge theory will flow to a three dimensional field theory in the infrared limit. The hypermultiplets are due to strings stretching between D3-branes and D5-branes, or due to strings between two stacks of D3-branes ending on the same NS5-brane, one stack ending on the left and one on the right. There are no strings between a D3- and a NS5-brane, since open strings cannot end on NS-branes. Strings stretched between two D5-branes are not considered since they are believed to decouple in the infrared limit.

A theory is defined by placing the D5- and NS5-branes on the x_3 segment, and defining how many D3-branes end on each one. However we can still move the D5- and NS5-branes around on the segment, and this will not change the infrared limit theory, since this will only change the vacuum expectation value of the scalar fields in the field theory. It is known that when a D5-brane, which is left of a NS5-brane, moves right and passes the NS5-brane, a D3-brane stretching between the two is created.[13] Of course a D3-brane is destroyed, or an anti-D3-brane created, in moving in the opposite direction. D5-branes can be rearranged with respect to other D5-branes without any consequence, and the same holds for the NS5-branes.

This freedom in ordering the branes results in a redundancy of the initial configuration. In order to make a good initial configuration, we define the linking numbers, which have a suggestive symbol considering the previous section,

$$l_i = -n_i + R_i^{NS5} \qquad\qquad i = 1, \cdots, k \tag{3.13}$$

$$\hat{l}_i = \hat{n}_i + L_i^{D5} \qquad\qquad i = 1, \cdots, \hat{k} \tag{3.14}$$

n_i is defined as the number of D3-branes ending on the right side of the ith D5-brane minus the number of D3-branes ending on the left side. A completely analogous definition exists for the $\hat{n}_i$ with respect to the NS5-branes. R_i^{NS5} is the number of NS5-branes on the right of the ith D5-brane, and L_i^{D5} is the number of D5-branes on the left of the ith

NS5-brane. Suppose now that we have a set of linking numbers $(l_i, \hat{l}_i)$, and suppose that for the ith D5-brane, the jth NS5-brane is to its right. When we move the D5-brane to the right of the NS5-brane, $n_i \to n_i - 1$, $R_i^{NS5} \to R_i^{NS5} - 1$, $\hat{n}_j \to \hat{n}_j + 1$ and $L_j^{D5} \to L_j^{D5} - 1$. This leaves the linking numbers l_i and $\hat{l}_j$ invariant. Thus any configuration of the branes can always be rearranged into a configuration where all the NS5-branes are on the left, all the D5-branes are on the right, and the internal ordering of D5-branes and NS5-branes is such that the brane with the most D3-branes ending on it is the most in the middle, as shown in Fig. 3.3. In this convention the linking number is just the number of D3-branes ending on a given 5-brane, and we find two partitions of N,

$$N = l_1 + \cdots + l_k = \hat{l}_1 + \cdots + \hat{l}_{\hat{k}} \tag{3.15}$$

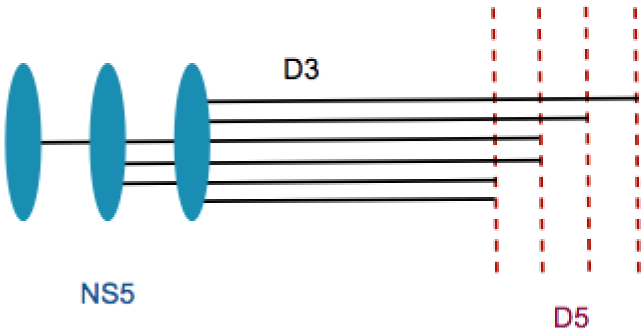

Figure 3.3: Brane configuration with $N = 6$, $\rho = (2, 2, 1, 1)$ and $\hat{\rho} = (3, 2, 1)$

To see why this brane configuration gives rise to $T^{\rho}_{\hat{\rho}}[SU(N)]$ we will first discuss some constraints on possible brane configurations. Starting from the configuration explained in the previous paragraph, we start moving the D5-brane in the middle, the one with the largest linking number, to the left until there are no more D3-branes ending on the D5-brane. There are three different possible endings. The first possibility is that $l_1 < \hat{k}$, i.e. the number of D3-branes ending on the D5-brane is less than the total number of NS5-branes. In this case the D5-brane will sit between the $(\hat{k} - l_1)$th and $(\hat{k} - l_1 - 1)$th NS5-brane, and it is no longer connected to any D3-branes. The second possibility is that $l_1 = \hat{k}$. In this case we can move the D5-brane past all the NS5-branes, and it will decouple from the system. Therefore we could just leave it out, because the configurations with or without the brane will describe the same field theory. The final case, $l_1 > \hat{k}$, can be shown to break supersymmetry [14] and therefore the gauge theory will not have the required supersymmetries. This exclusion is called the s-rule.

We now assume that $\hat{k} > l_1$ and we start moving the two D5-branes with the largest linking numbers, l_1 and l_2. There are $\hat{M}_1 = \hat{m}_1 - \hat{m}_2$ NS5-branes with only one D3-brane ending on them. Thus there are at least $l_1 + l_2 - (\hat{m}_1 - \hat{m}_2)$ D3-branes that are attached to the NS5-branes with two or more D3-branes. Unless

$$2\hat{m}_2 \geq l_1 + l_2 - (\hat{m}_1 - \hat{m}_2) \Leftrightarrow \hat{m}_1 + \hat{m}_2 \geq l_1 + l_2 \tag{3.16}$$

there will be some D5/NS5 pairs with more than one D3-brane between them, which is forbidden by the s-rule. The case for $\hat{m}_1 + \hat{m}_2 = l_1 + l_2$ will again lead to a decoupling of branes.

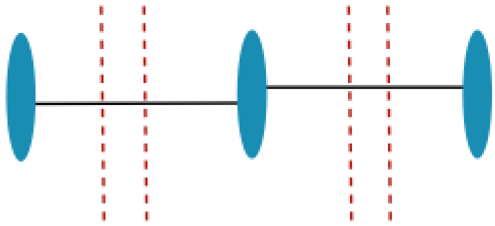

Figure 3.4: A brane configuration with an equivalent field theory as the brane configuration shown in Fig. 3.3

By repeating the same arguments for all D5-branes, we find a constraint on the partitions shown in (3.15),

$$\hat{\rho}^T > \rho \tag{3.17}$$

where $\hat{\rho} = (\hat{l}_1, \cdots, \hat{l}_{\hat{k}})$ and $\rho = (l_1, \cdots, l_k)$. It can be shown that

$$\hat{\rho}^T > \rho \Leftrightarrow \rho^T > \hat{\rho} \tag{3.18}$$

This constraint is the same as the constraint (3.3) if the partitions chosen in $T^{\rho}_{\hat{\rho}}[SU(N)]$ are the same.

All that remains in order to show that the gauge theory description of this brane construction leads to the $T^{\rho}_{\hat{\rho}}[SU(N)]$ gauge theory, is that we need to substantiate that if we choose the partitions defined in both theories to be the same, that both theories have the same particle content and symmetries. This is most easily seen in the brane configuration where all the D5-branes are moved so that there are no longer any D3-branes ending on them, as shown in Fig. 3.4. Define the numbers N_i, M_i and m_i in the same way as for the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory. The number of D3-branes stretched between the ith and $(i+1)$th NS5-brane is equal to N_i, and as in the $T^{\rho}_{\hat{\rho}}[SU(N)]$, N_i is only defined for $i = 1, \cdots, \hat{k}-1$. As mentioned before, a stack of N coincidental D3-branes results in a $U(N)$ gauge symmetry, with the corresponding vector multiplet. The number of D5-branes crossing the D3-branes between the ith and $(i+1)$th NS5-brane are exactly given by M_i, resulting in M_i hypermultiplets in the fundamental representation of the gauge group. Finally, strings between two stacks of D3-branes ending on the same NS5-brane will give a hypermultiplet in the bifundamental representation as stated before. From all this we conclude that both theories have exactly the same particle content.

3.2 $T[SU(N)]$

We have introduced the $T^{\rho}_{\hat{\rho}}[SU(N)]$ superconformal field theory which was defined by the choice of two partitions of the integer N, although the partitions have to satisfy the constraint (3.3). In this thesis we will not work with the general $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory, but we will make a particular choice for the partitions.

The choice we will make is

$$\rho = (1, 1, \cdots, 1) \tag{3.19}$$
$$\hat{\rho} = (1, 1, \cdots, 1) \tag{3.20}$$

It is easy to see that this partition choice satisfies (3.3), or the equivalent constraint shown in (3.9). In this case $l_1 = 1$, so we only need to consider one term of the sum. Since all $l_i = \hat{l}_j = 1$, it follows that $m_1 = N > \hat{l}_1$, and the constraint is satisfied.

As mentioned in the beginning of this chapter, the partition function of the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory was calculated for any value of the coupling strength on a S^3 background.[12] From this result one can obtain the free energy of $T[SU(N)]$ in the large N limit.[15]

$$F_{CFT} \approx \frac{1}{2} N^2 \log N \tag{3.21}$$

Can we reproduce this result on the supergravity side of the duality? To answer this question we will first look for the supergravity dual theory in the next chapter.

4

Supergravity dual

In chapter 3 we introduced a family of $\mathcal{N} = 4$ superconformal Yang-Mills field theories, we discussed the brane construction for this type of field theory and quoted the free energy for a specific type of field theory in this family. In this chapter we introduce the supergravity dual to these conformal field theories, as constructed in [11][16]. In this chapter and the following ones we will use differential forms. A short introduction is contained in appendix A.

4.1 Type IIB supergravity

First we review type IIB supergravity [17] in the case of vanishing fermion fields, since the solution constructed in [16] is a BPS solution, a concept which will be explained in section 4.2. Type IIB supergravity is a ten dimensional field theory in which supersymmetry is gauged. It is a low energy limit of type IIB string theory. The bosonic fields are the metric g_{MN}, the complex scalar B which contains the axion χ and the dilaton Φ, the complex two-form $\hat{B}_{(2)} = B_{(2)} + iC_{(2)}$ with $B_{(2)}$ and $C_{(2)}$ real two-forms, and the real four-form $C_{(4)}$. It will be useful to express these fields in terms of composite fields,

$$f^2 =(1 - |B|^2)^{-1} \tag{4.1}$$

$$P =f^2 dB \tag{4.2}$$

$$Q =f^2 Im(Bd\bar{B}) \tag{4.3}$$

$$\hat{F}_{(3)} =d\hat{B}_{(2)} \tag{4.4}$$

$$G =f(\hat{F}_{(3)} - B\bar{\hat{F}}_{(3)}) \tag{4.5}$$

$$\hat{F}_{(5)} =dC_{(4)} + \frac{i}{16}(B_{(2)} \wedge \bar{\hat{F}}_{(3)} - \bar{B}_{(2)} \wedge \hat{F}_{(3)}) \tag{4.6}$$

The scalar field B is related to the axion χ and the dilaton $\Phi = 2\phi$ by

$$B = \frac{1 + i\tau}{1 - i\tau} \tag{4.7}$$

with $\tau = \chi + ie^{2\phi}$.

In terms of these new fields, the Bianchi identities are

$$0 = dP - 2iQ \wedge P \tag{4.8}$$
$$0 = dQ + iP \wedge \bar{P} \tag{4.9}$$
$$0 = dG - iQ \wedge G + P \wedge \bar{G} \tag{4.10}$$
$$0 = d\hat{F}_{(5)} - \frac{i}{8} G \wedge \bar{G} \tag{4.11}$$

and the field equations are

$$0 = \nabla^M P_M - 2iQ^M P_M + \frac{1}{24} G_{MNP} G^{MNP} \tag{4.12}$$
$$0 = \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} + \frac{2}{3} i \hat{F}_{(5)MNPQR} G^{PQR} \tag{4.13}$$
$$\begin{aligned} 0 =& R_{MN} - P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6} \hat{F}_{(5)MP_1P_2P_3P_4} \hat{F}_{(5)N}{}^{P_1P_2P_3P_4} \\ &- \frac{1}{8}(G_M{}^{PQ} \bar{G}_{NPQ} \bar{G}_M{}^{PQ} G_{NPQ}) + \frac{1}{48} g_{MN} G^{PQR} \bar{G}_{PQR} \end{aligned} \tag{4.14}$$

where capital indices indicate ten dimensional space-time indices.

The bosonic part of the action for type IIB supergravity is given by

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \left(R - \frac{1}{2} \frac{\partial_M \tau \partial^M \bar{\tau}}{(\Im(\tau))^2} - \frac{1}{2} |G_{(3)}|^2 - \frac{1}{4} |\hat{F}_{(5)}|^2 \right) \\ &- \frac{i}{8\kappa_{10}^2} \int d^{10}x C_{(4)} \wedge G_{(3)} \wedge \bar{G}_{(3)} \end{aligned} \tag{4.15}$$
$$\begin{aligned} =& \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \left(R - 2\partial_M \phi \partial^M \phi - \frac{1}{2} e^{4\phi} \partial_M \chi \partial^M \chi - \frac{1}{2} e^{-2\phi} |H_{(3)}|^2 \right. \\ & \left. - \frac{1}{2} e^{2\phi} |F_{(3)} - \chi H_{(3)}|^2 - \frac{1}{4} |\hat{F}_{(5)}|^2 \right) - \frac{1}{4\kappa_{10}^2} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \end{aligned} \tag{4.16}$$

where $H_{(3)} = dB_{(2)}$ and $F_{(3)} = dC_{(2)}$ and κ_{10} is related to the ten dimensional gravitational coupling constant. There is one problem however, the variational principle is not well defined for this action, since the field equations are not found by the requirement that the action is extremal under arbitray variations of these fields, but we must impose a self-duality constraint on $\hat{F}_{(5)}$ after the variation in order to get the correct field equations. This will result in a complication in the free energy calculation in chapter 7.

In section 4.2 we will briefly discuss the concept of BPS solutions in supergravity. In order to find a BPS solution, we will need the linear supersymmetric variations of the fermionic fields. The fermionic fields present in type IIB supergravity are the dilatino λ and the gravitino ψ_M. Their supersymmetric variation is

$$\delta\lambda = i(\Gamma \cdot P)\mathcal{B}^{-1}\epsilon^\star - \frac{i}{24}(\Gamma \cdot G)\epsilon \tag{4.17}$$
$$\delta\psi_M = D_M \epsilon + \frac{i}{480}(\Gamma \cdot \hat{F}_{(5)})\Gamma_M \epsilon - \frac{1}{96}(\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M)\mathcal{B}^{-1}\epsilon^\star \tag{4.18}$$

where $\mathcal{B}$ is the charge conjugation matrix of the Clifford algebra, defined by $\mathcal{B}\mathcal{B}^\star = I$ and $\mathcal{B}\Gamma^M\mathcal{B}^{-1} = (\Gamma^M)^\star$, and the gamma matrix conventions are the same as [16].

4.2 BPS solution

Looking at the field equations of motion, it can be a daunting task trying to find a solution. If we are looking for classical solutions, a solution with no fermion fields, an alternative approach exists. We will search for solutions by setting the fermion fields equal to zero, and demanding that their supersymmetric variation vanishes, thus ensuring the fermion fields remain zero.

A great example of this is $\mathcal{N} = 1$, $D = 4$ supergravity. Here the action is

$$S = \frac{1}{2\kappa^2} \int d^D x \; e \left(e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \tag{4.19}$$

where the supersymmetric variation of the fields is

$$\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \tag{4.20}$$

$$\delta \psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon \tag{4.21}$$

The simplest solution is Minkowski spacetime with vanishing fermions. Because we have vanishing fermions, the variation of the vielbein vanishes automatically. The Minkowski metric implies $\omega_{\mu ab} = 0$, and therefore, $\delta\psi_\mu = \partial_\mu \epsilon$. Thus this solution is invariant under variation by constant spinors, which are called the Killing spinors.

More generally speaking, if we have a theory with boson fields B and fermion fields F. Then we can write the supersymmetric variations to first order as

$$\delta B = \bar{\epsilon} f(B) F + \mathcal{O}(F^3) \tag{4.22}$$

$$\delta F = g(B)\epsilon + \mathcal{O}(F^2) \tag{4.23}$$

Letting the fermionic fields vanish implies that the bosonic variation vanishes, and reduces the fermionic variations to a set of linear equations,

$$\delta F_{lin} = g(B)\epsilon = 0 \tag{4.24}$$

where we required that this variation vanishes to ensure that the fermionic fields remain zero. The solution to this equation, if it exists, is called a Killing spinor. If we find N linearly independent Killing spinors, we say that the BPS solution preserves N supercharges.

From the above we see that a BPS solution for type IIB supergravity is found by equating (4.17) and (4.18) to zero. However, we are interested in supergravity solutions dual to the conformal field theory described earlier, therefore the supergravity solution should have the same symmetries as the superconformal theory. This means the solution should have a $SO(2,3) \times SO(3) \times SO(3)$ symmetry.

Using this information, an Ansatz for the metric and the fields was made in [16], which will be explained in the following section. Using this Ansatz, the BPS equations were shown to form an integrable system, which was explicitly solved in [16] in terms of two holomorphic functions. In the same paper it was also proven that every solution of the BPS equations will be a solution to the equations of motion (4.12)-(4.14) and to the Bianchi identities (4.8)-(4.11).

4.3 Ansatz

4.3.1 Metric Ansatz

In [16] a supergravity solution was sought with $SO(2,3) \times SO(3) \times SO(3)$ symmetry. This led to a restriction of possible space-time metrics, namely, the most general metric considered was a product space of the form $AdS_4 \times S_1^2 \times S_2^2 \times \Sigma$ where Σ is a Riemann surface over which the products are warped. The metric Ansatz is

$$ds^2 = f_4^2 ds^2_{AdS_4} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2 dz d\bar{z} \tag{4.25}$$

with f_1, f_2, f_4 and ρ functions on the Σ space.

We can introduce an orthonormal frame

$$e^m = f_4 \hat{e}^m \qquad m = 0,1,2,3 \tag{4.26}$$

$$e^{i_1} = f_1 \hat{e}^{i_1} \qquad i_1 = 4,5 \tag{4.27}$$

$$e^{i_2} = f_2 \hat{e}^{i_2} \qquad i_2 = 6,7 \tag{4.28}$$

$$e^a \qquad a = 8,9 \tag{4.29}$$

The hatted frame fields $\hat{e}^m, \hat{e}^{i_1}, \hat{e}^{i_2}$ and the frame field e^a are frame fields for the manifolds AdS_4, S_1^2, S_2^2 and Σ.

Using the first Cartan equation $de^a + \omega^a{}_b \wedge e^b = 0$ we find the non-vanishing components of the spin connection which will be useful in the next chapter

$$\omega^{mn} = \hat{\omega}^{mn} \tag{4.30}$$

$$\omega^{i_1 j_1} = \hat{\omega}^{i_1 j_1} \tag{4.31}$$

$$\omega^{i_2 j_2} = \hat{\omega}^{i_2 j_2} \tag{4.32}$$

$$\omega^{ab} = \hat{\omega}^{ab} \tag{4.33}$$

$$\omega^i{}_a = \frac{1}{f_{[i]}} D_a f_{[i]} e^i \tag{4.34}$$

where $D_a f = e^\mu_a \partial_\mu f$ and $i \in 0, \cdots, 7$. $f_{[i]}$ is defined by

$$f_{[0]} = f_{[1]} = f_{[2]} = f_{[3]} = f_4 \tag{4.35}$$

$$f_{[4]} = f_{[5]} = f_1 \tag{4.36}$$

$$f_{[6]} = f_{[7]} = f_2 \tag{4.37}$$

and all $\hat{\omega}$ denote the spin connection of the submanifold itself. For example $\hat{\omega}^{i_1 j_1}$ is just the spin connection of a two dimensional sphere.

In chapter 5 we will try to extend the BPS solution to more general spaces allowing the introduction of black holes in the AdS space. Because of this it is important to note that the equations (4.30)-(4.34) do not depend on the fact that the first space is AdS_4. Changing this space with any other four dimensional space with the same conformal factor f_4 will only change $\hat{\omega}^{mn}$ in (4.30).

4.3.2 Field Ansatz

Again we follow the Ansatz proposed in [16]. The field Ansatz is

$$\begin{aligned} P =& p_a e^a \\ Q =& q_a e^a \\ G =& g_a e^{45a} + i h_a e^{67a} \\ \hat{F}_{(5)} =& f_a(-e^{0123a} + \epsilon^a{}_b e^{4567b}) \end{aligned} \tag{4.38}$$

with f_a, q_a real, and g_a, h_a, p_a complex.

This Ansatz can be simplified even more, because of the fact that type IIB supergravity is invariant under $SU(1,1)$ which leaves the metric and the real four-form $C_{(4)}$ invariant and acts by Möbius transformation on B and linearly on $B_{(2)}$ [18],

$$B \rightarrow \frac{uB+v}{\bar{v}B+\bar{u}} \tag{4.39}$$

$$B_{(2)} \rightarrow uB_{(2)} + v\bar{B}_{(2)} \tag{4.40}$$

Translated to the composite fields this becomes

$$P \rightarrow e^{2i\theta} P \tag{4.41}$$

$$Q \rightarrow Q + d\theta \tag{4.42}$$

$$G \rightarrow e^{i\theta} G \tag{4.43}$$

with

$$e^{i\theta} = \left(\frac{v\bar{B}+u}{\bar{v}B+\bar{u}}\right)^{1/2} \tag{4.44}$$

It follows from the BPS equations [16] that

$$\Im(ie^{-i\theta} g_a) = 0 \tag{4.45}$$

$$\Im(ie^{-i\theta} h_a) = 0 \tag{4.46}$$

$$\Im(p_a e^{-2i\theta}) = 0 \tag{4.47}$$

where $a = 8, 9$. It follows from (4.9) that $dQ = 0$. From this we see that we can always transform any solution to a solution where we have the reality conditions

$$q_a = 0 \tag{4.48}$$

$$\bar{p}_a = p_a \tag{4.49}$$

$$\bar{g}_a = g_a \tag{4.50}$$

$$\bar{h}_a = h_a \tag{4.51}$$

where again $a = 8, 9$. It is easy to see from (4.7) that a solution with $Q = 0$ corresponds to a solution with a vanishing axion.

4.4 General solution

The BPS equations were solved explicitly in terms of two real harmonic functions, h_1 and h_2 which depend only on the Σ coordinates z and $\bar{z}$. All the fields and metric factors can be explicitly expressed using only these two functions. We only quote the results obtained in [16].

The harmonic functions h_1 and h_2 are defined in function of two holomorphic functions, $\mathcal{A}_1$ and $\mathcal{A}_2$. Using this relation we define the dual harmonic functions,

$$h_1 = -i(\mathcal{A}_1 - \bar{\mathcal{A}}_1) \rightarrow h_1^D = \mathcal{A}_1 + \bar{\mathcal{A}}_1 \tag{4.52}$$
$$h_2 = \mathcal{A}_2 + \bar{\mathcal{A}}_2 \rightarrow h_2^D = i(\mathcal{A}_2 - \bar{\mathcal{A}}_2) \tag{4.53}$$

The metric factors and the dilaton are most easily expressed in terms of

$$W = \partial h_2 \bar{\partial} h_2 + \bar{\partial} h_1 \partial h_2 \tag{4.54}$$
$$N_1 = 2h_1 h_2 |\partial h_1|^2 - h_1^2 W \tag{4.55}$$
$$N_2 = 2h_1 h_2 |\partial h_2|^2 - h_2^2 W \tag{4.56}$$

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. The metric factors are then given by

$$f_4^8 = 16 \frac{N_1 N_2}{W^2} \tag{4.57}$$
$$f_1^8 = 16 h_1^8 \frac{N_2 W^2}{N_1^3} \tag{4.58}$$
$$f_2^8 = 16 h_2^8 \frac{N_1 W^2}{N_2^3} \tag{4.59}$$
$$\rho^8 = \frac{N_1 N_2 W^2}{h_1^4 h_2^4} \tag{4.60}$$

and the dilaton is obtained by

$$e^{4\phi} = \frac{N_2}{N_1} \tag{4.61}$$

The three form field strengths are

$$H_{(3)} = \hat{e}^{45} \wedge db_1 \tag{4.62}$$
$$F_{(3)} = \hat{e}^{67} \wedge db_2 \tag{4.63}$$

where $\hat{e}^{45}$ and $\hat{e}^{67}$ are the volume forms of S_1^2 respectively S_2^2. b_1 and b_2 are defined by

$$b_1 = 2ih_1 \frac{h_1 h_2 (\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_1} + 2h_2^D \tag{4.64}$$
$$b_2 = 2ih_2 \frac{h_1 h_2 (\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2)}{N_2} - 2h_1^D \tag{4.65}$$

Finally, the five form $\hat{F}_{(5)}$ is written as

$$\hat{F}_{(5)} = -4f_4^4\hat{e}^{0123} \wedge \mathcal{F} + 4f_1^2 f_2^2 \hat{e}^{45} \wedge \hat{e}^{67} \wedge (\star_2\mathcal{F}) \tag{4.66}$$

where $\hat{e}^{0123}$ is the volume form of AdS_4, $\mathcal{F}$ is a one form on Σ and $\star_2$ denotes the Hodge dual with respect to the Σ space. $\mathcal{F}$ is obtained from the expression

$$f_4^4\mathcal{F} = dj_1 \tag{4.67}$$

where

$$j_1 = 3\mathcal{C} + 3\bar{\mathcal{C}} - 3\mathcal{D} + i\frac{h_1 h_2}{W}(\partial h_1 \bar{\partial} h_2 - \bar{\partial} h_1 \partial h_2) \tag{4.68}$$

and

$$\partial\mathcal{C} = \mathcal{A}_1\partial\mathcal{A}_2 - \mathcal{A}_2\partial\mathcal{A}_1 \tag{4.69}$$

$$\mathcal{D} = \bar{\mathcal{A}}_1\mathcal{A}_2 + \mathcal{A}_1\bar{\mathcal{A}}_2 \tag{4.70}$$

4.5 $AdS_5 \times S_5$ as a solution

A simple choice for the real harmonic functions h_1 and h_2 is

$$h_1 = -i\alpha\sinh(z) + c.c. \tag{4.71}$$

$$h_2 = \hat{\alpha}\cosh(z) + c.c. \tag{4.72}$$

with α and $\hat{\alpha}$ real arbitrary real parameters. The Σ space is defined by

$$\Sigma = \left\{ z \in \mathbb{C} : 0 \leq \Im(z) \leq \frac{\pi}{2} \right\} \tag{4.73}$$

We calculate the metric factors using the equations (4.57)-(4.60), which gives

$$\rho^4 = |\alpha\hat{\alpha}| \tag{4.74}$$

$$f_4^2 = 4\rho^2\cosh^2\left(\frac{z+\bar{z}}{2}\right) \tag{4.75}$$

$$f_1^2 = 4\rho^2\sin^2\left(\frac{z-\bar{z}}{2i}\right) \tag{4.76}$$

$$f_2^2 = 4\rho^2\cos^2\left(\frac{z-\bar{z}}{2i}\right) \tag{4.77}$$

Using $z = x + iy$, the metric is given by

$$ds^2 = f_4^2 ds^2_{AdS_4} + f_1^2 ds_{S_1^2} + f_2^2 ds_{S_2^2} + 4\rho^2(dx^2 + dy^2) \tag{4.78}$$

$$= 4|\alpha\hat{\alpha}|^{1/2}\left(dx^2 + \cosh^2(x)ds^2_{AdS_4} + dy^2 + \sin^2(y)ds^2_{S_1^2} + \cos^2(y)ds_{S_2^2}\right) \tag{4.79}$$

Remembering the anti-de Sitter slicing coordinates (2.20), we see that this is $AdS_5 \times S_5$ with radius $L^2 = 4|\alpha\hat{\alpha}|^{1/2}$. The dilaton is constant and equal to

$$e^{2\phi} = \left|\frac{\hat{\alpha}}{\alpha}\right| \tag{4.80}$$

The five form field strength is

$$\hat{F}_{(5)} = 4L^4(1+\star)e^{4567y} \tag{4.81}$$

where e^{4567y} is the volume form of the unit radius five sphere S^5, and the G-field vanishes. The five form field strength is sourced by $D3$ branes, and thus we can compute the $D3$ brane charge by

$$\int_{S_5} \hat{F}_{(5)} = 4\pi^3 L^4 = (2\pi)^4(\alpha')^2 N_{D3} \tag{4.82}$$

where N_{D3} is the number of D3 branes. This gives a relation between the radius L and the number of D3 branes N,

$$L^4 = 4\pi(\alpha')^2 N_{D3} \tag{4.83}$$

This equals (2.31), apart from the g_s factor. However, it can be shown that our choice of coupling strength, $\frac{1}{2\kappa_2^{10}}$, corresponds with $g_s = 1$. Thus we can conclude that this simple solution is dual to $\mathcal{N} = 4$ superconformal Yang-Mills theory, because of the AdS/CFT conjecture originally proposed by Maldacena.

4.6 Regularity conditions

We now have solutions to the Bianchi identities (4.8)-(4.11) and to the equations of motion (4.12)-(4.14) in terms of the real harmonic functions h_1 and h_2 which vary on Σ. Choosing an arbitrary Σ space, and arbitrary real harmonic functions for h_1 and h_2 will usually lead to singularities in the dilaton ϕ or singularities in the metric factors f_4, f_1, f_2 and ρ, and thus a singular supergravity solution. In order to make sure that we have regular solutions, we will require that the solutions satisfy certain regularity conditions introduced in [19]. We will only quote the results obtained in this paper.

The first regularity conditions are

- The dilaton and the metric factors are non-singular in the interior of Σ
- and non-singular on the boundary $\partial\Sigma$ except possibly at isolated points.

It can also be shown that the boundary $\partial\Sigma$ of the Σ space must be partitioned into two open sets $\partial\Sigma_+$ and $\partial\Sigma_-$. The closures of $\partial\Sigma_+$ and $\partial\Sigma_-$ intersect at isolated points and their union is $\partial\Sigma$. h_1 and h_2 satisfy certain boundary conditions on this partition,

$$\begin{cases} f_1 = h_1 = 0 \\ \partial_n h_2 = 0 \end{cases} \qquad \text{on } \partial\Sigma_+ \tag{4.84}$$

$$\begin{cases} f_2 = h_2 = 0 \\ \partial_n h_1 = 0 \end{cases} \qquad \text{on } \partial\Sigma_- \tag{4.85}$$

where ∂_n indicates the derivative normal to the boundary $\partial\Sigma$. Note that these conditions imply that $W = 0$ on all of $\partial\Sigma$.

We see that the $AdS_5 \times S_5$ solution described in the previous section satisfies all these properties. The boundary $\partial\Sigma_+$ is equal to $\{z \in \mathbb{C} : \Im(z) = 0\}$ for this solution, and $\partial\Sigma_- = \left\{z \in \mathbb{C} : \Im(z) = \frac{\pi}{2}\right\}$. The $AdS_5 \times S_5$ solution also has $h_1 > 0$, $h_2 > 0$ and $W < 0$ in the interior of Σ. Since we are interested in solutions which are connected to the $AdS_5 \times S_5$ solution, we will require that these conditions continue to hold for other solutions.

4.7 Solutions with five-branes

The brane construction of $T^{\rho}_{\hat{\rho}}[SU(N)]$ included D5 and NS5 branes. Therefore, if we want to construct a supergravity dual theory it is logical to include five-branes into the supergravity solution.

4.7.1 Stack of NS5-branes

To do this, we consider the harmonic functions

$$h_1 = -\, i\alpha \sinh(z) + c.c. \tag{4.86}$$
$$h_2 = \hat{\alpha} \cosh(z) - \gamma \log\left(\tanh\left(\frac{z}{2}\right)\right) + c.c. \tag{4.87}$$

where again all parameters are real, and we still work with the Σ space

$$\Sigma = \left\{z \in \mathbb{C} : 0 \leq \Im(z) \leq \frac{\pi}{2}\right\} \tag{4.88}$$

The function $\log\left(\tanh\left(\frac{z}{2}\right)\right)$ is purely imaginary for $\Im(z) = \frac{\pi}{2}$. Therefore the regularity condition $h_2 = 0$ on $\partial\Sigma_-$ still holds.

This solution will have NS5-brane charge and no D5-brane charge. To show this, we need to identify a non-contractible 3-cycle that can support 3-form flux. Consider an curve $\mathcal{I}$ starting and ending on the lower boundary of the Σ space, as shown in figure 4.1. As stated in the previous section on the regularity conditions, f_1^2 will vanish on the lower boundary Σ, which implies that the S_1^2 sphere will shrink to zero size. Because of this we can view $\mathcal{I}$ as a three dimensional sphere.

NS5 charge can be measured as the NS-NS flux through $S_1^2 \times \mathcal{I}$. This flux is due to the $H_{(3)}$ three-form. Using this, we can calculate the NS5 charge contained inside the region enclosed by $S_1^2 \times \mathcal{I}$.

$$Q_{NS5} = \int_{S_1^2 \times \mathcal{I}} H_{(3)} = \int_{\mathcal{I}} db_1 \int_{S_1^2} \omega^{45} = 4\pi b_1|_{\partial\mathcal{I}} \tag{4.89}$$

Since h_1 vanishes on both ends of $\mathcal{I}$, we see using (4.64),

$$b_1|_{\partial\mathcal{I}} = 2h_2^D\big|_{\partial\mathcal{I}} \tag{4.90}$$
$$= 2\left(\left[i\hat{\alpha}\cosh(z) - i\gamma \log\left(\tanh\left(\frac{z}{2}\right)\right)\right] + c.c.\right)\Big|_{\partial\mathcal{I}} \tag{4.91}$$

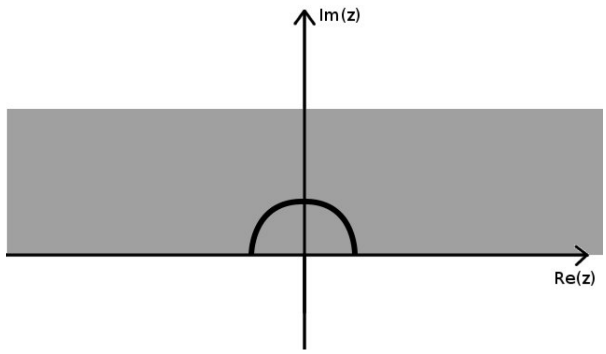

Figure 4.1: Σ space (grey area) with 3-cycle $\mathcal{I}$

The cosh will give no contribution, which is why the $AdS_5 \times S_5$ did not contain any five-branes, but the tanh part will give a contribution, from which it follows that

$$b_1|_{\partial\mathcal{I}} = 4\pi\gamma \tag{4.92}$$

Thus the discontinuity in the origin contains NS5 charge,

$$Q_{NS5} = 16\pi^2\gamma \tag{4.93}$$

What is the meaning of the remaining parameters α and $\hat{\alpha}$? Because of our experience with the $AdS_5 \times S_5$ solution we expect them to be related to the dilaton. To see this, we will look at the asymptotic behaviour of this solution, in the two limits $x \to \pm\infty$.

$$-\gamma \log\left(\tanh\left(\frac{z}{2}\right)\right) \approx \begin{cases} 2\gamma e^{-z} & x \to \infty \\ -i\gamma\pi + 2\gamma e^{z} & x \to -\infty \end{cases} \tag{4.94}$$

Now we can rewrite h_2 in the asymptotic regions as

$$h_2 \approx \hat{\alpha}^{\pm}\left(\cosh(z - \beta^{\pm}) + \cosh(\bar{z} - \beta^{\pm})\right) \tag{4.95}$$

where

$$\hat{\alpha}^{\pm} = \hat{\alpha}\sqrt{1 + \frac{4\gamma}{\hat{\alpha}}} \tag{4.96}$$

$$e^{\beta^{\pm}} = \left(1 + \frac{4\gamma}{\hat{\alpha}}\right)^{\pm 1/2} \tag{4.97}$$

Since $\hat{\alpha}^{\pm}$ and $\beta^{\pm}$ are real (it can be shown that $\frac{\gamma}{\hat{\alpha}}$ must be positive [19]), this will asymptotically go to the basic $AdS_5 \times S_5$ geometry, however, the radius and the value of the dilaton will be different in the two different limits. Repeating the calculations of section 4.5, it is easy to verify that

$$L_{\pm}^4 = 16|\alpha\hat{\alpha}^{\pm}|\cosh(-\beta^{\pm}) \tag{4.98}$$

$$e^{2\phi_{\pm}} = \left|\frac{\hat{\alpha}^{\pm}}{\alpha}\right| e^{\mp\beta^{\pm}} \tag{4.99}$$

4.7.2 Stack of D5-branes

The previous solution did not contain any D5 charge. To obtain this type of charge, we consider the solution

$$h_1 = - i\alpha \sinh(z) - \gamma \log\left(\tanh\left(\frac{i\pi}{4} - \frac{z}{2}\right)\right) + c.c. \tag{4.100}$$

$$h_2 = \hat{\alpha}\cosh(z) + c.c. \tag{4.101}$$

The extra term does not change the regularity condition $h_1 = 0$ on $\partial\Sigma_+$, allowing us to define the non-contractible 3-cycle $\mathcal{I}$. This time, $\mathcal{I}$ will start and end on the upper boundary of Σ, again defining a three dimensional sphere. Repeating the same calculations as we did for the first solution of this section, we will find

$$Q_{D5} = 16\pi^2\gamma \tag{4.102}$$

and vanishing NS5 charge.

The geometry of this solution is the same of that of stack of NS5 branes, since this solution can be obtained by an $SL(2,\mathbb{Z})$ symmetry and a reparametrization of the Σ space.[20] However, the sign of the dilaton will be flipped, and $H_{(3)}$ and $F_{(3)}$ will be exhanged.

Just as the $AdS_5 \times S_5$ solution, both these solutions with five-branes will contain three branes. It can be shown that D3 branes can intersect or end on NS5 branes and D5 branes. [21] We have not discussed yet the D3 brane charge for these five brane solutions, since in this case the definition of D3 brane charge will be more subtle. Before doing this, we will first generalize to solutions with many stacks of D5 branes and NS5 branes.

4.7.3 Many stacks of five-branes

We showed two solutions that contain five-brane charge. These solutions can be generalized to a solution with many stacks of D5 and NS5 branes,

$$h_1 = - i\alpha \sinh(z) - \sum_{a=1}^{q} \gamma_a \log\left(\tanh\left(\frac{i\pi}{4} - \frac{z-\delta_a}{2}\right)\right) + c.c. \tag{4.103}$$

$$h_2 = \hat{\alpha}\cosh(z) - \sum_{b=1}^{\hat{q}} \hat{\gamma}_b \log\left(\tanh\left(\frac{z-\hat{\delta}_b}{2}\right)\right) + c.c. \tag{4.104}$$

where α, $\hat{\alpha}$, γ_a, $\hat{\gamma}_b$, δ_a and $\hat{\delta}_b$ are all real parameters. Furthermore, the δ_a and $\hat{\delta}_b$ are ordered, $\delta_1 < \delta_2 < \cdots < \delta_q$ and $\hat{\delta}_1 > \hat{\delta}_2 > \cdots > \hat{\delta}_{\hat{q}}$. It is obvious from our previous brane charge calculations, that this solution corresponds to q stacks of D5 branes and $\hat{q}$ stacks of NS5 branes. The position of the ath D5 brane stack is at $z = \delta_a + i\frac{\pi}{2}$ and the position of the bth NS5 brane stack is at $z = \hat{\delta}_b$. The five-brane charges of the stacks are

$$Q_{D5}^{(a)} = (4\pi)^2\gamma_a = 4\pi^2\alpha' N_{D5}^{(a)} \tag{4.105}$$

$$Q_{NS5}^{(b)} = (4\pi)^2\hat{\gamma}_b = 4\pi^2\alpha' \hat{N}_{NS5}^{(b)} \tag{4.106}$$

where $N_{D5}^{(a)}$ is the number of D5 branes in the ath stack, and $\hat{N}_{NS5}^{(b)}$ is the number of NS5 branes in the bth stack. This equality with the number of branes in the stack is a result obtained from string theory.

As promised, we will now continue with the D3 brane charges. In the $AdS_5 \times S_5$ solution it was fairly easy to calculate the D3 charge, the S_5 part of the metric gave us a non-contractible 5-cycle which supported the D3 charge. Our first task is to identify non-contractible 5-cycles capable of supporting D3 charge in this new geometry. Note that in order to calculate the D5 and NS5 charges in (4.105) and (4.106), we used non-contractable 3-cycles like the $\mathcal{I}$ defined previously. Using these, we can define non-contractible 5-cycles. For example, let's say $\mathcal{I}$ starts on the lower boundary of Σ, between $\hat{\delta}_1$ and $\hat{\delta}_2$, and ends between $\hat{\delta}_2$ and $\hat{\delta}_3$. As explained before, together with S_1^2 this forms a three dimensional sphere. Now consider the warped product $\hat{\mathcal{C}}_5^2 = S_2^2 \times \mathcal{I} \times S_1^2$. Topologically this is $S^2 \times S^3$. Completely analogously we can define $\hat{\mathcal{C}}_5^b$ and $\mathcal{C}_5^a$. These are the 5-cycles we seek.

Next we want to calculate the three brane charge. In the $AdS_5 \times S_5$ solution, this was just the integral of $\hat{F}_{(5)}$ over the 5-cycle. To see why we can not do this this time, consider the definition of brane charge. Brane charge can be defined as the failure of the Bianchi identity for the gauge-invariant field strength,

$$d\hat{F}_{(5)} - H_{(3)} \wedge F_{(3)} = \star j_{D3}^{bs} \tag{4.107}$$

The problem with this charge is that it is not conserved in the presence of D5 or NS5 branes,

$$d(\star j_{D3}^{bs}) = -(\star j_{NS5}) \wedge F_{(3)} + H_{(3)} \wedge (- \star j_{D5}) \tag{4.108}$$

It is possible to fix this, and define a conserved charge, but we will have to sacrifice gauge invariance. Define the Page charge by

$$\star j_{D3}^{Page} = \star j_{D3}^{bs} + (\star j_{NS5}) \wedge C_{(2)} - B_{(2)} \wedge (\star j_{D5}) \tag{4.109}$$

From now on all D3 brane charges will be Page charges. Now it can be shown that the D3 charge on the $\hat{\mathcal{C}}_5^b$ 5-cycle is equal to

$$\hat{Q}_{D3}^{(b)} = \int_{\hat{\mathcal{C}}_5^b} \hat{F}_{(5)} + C_{(2)} \wedge H_{(3)} \tag{4.110}$$

while on the $\mathcal{C}_5^a$ 5-cycle it is

$$Q_{D3}^{(a)} = \int_{\mathcal{C}_5^a} \hat{F}_{(5)} - B_{(2)} \wedge F_{(3)} \tag{4.111}$$

We will not go into the details of the D3 brane charge calculation, but the result will be that

$$Q_{D3}^{(a)} = 2^8 \pi^3 \left(\hat{\alpha} \gamma_a \sinh(\delta_a) - 2\gamma_a \sum_{b=1}^{\hat{q}} \hat{\gamma}_b \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \right) \tag{4.112}$$

$$\hat{Q}_{D3}^{(b)} = 2^8 \pi^3 \left(\alpha \hat{\gamma}_b \sinh(\hat{\delta}_b) + 2\hat{\gamma}_b \sum_{a=1}^{q} \gamma_a \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \right) \tag{4.113}$$

4.7.4 Closing the $AdS_5 \times S_5$ regions

Before we continue with identifying the supergravity theory which is dual to the field theory described in chapter 3, we note that for the previous solutions, the geometry was asymptotically $AdS_5 \times S_5$ in the limit $x \to \pm\infty$. Since the field theory is three dimensional in the infrared limit, we expect that the geometry of the supergravity dual theory is a warped product $AdS_4 \times K$. Therefore, we would like to change this asymptotical behaviour of the geometry.

To see how we can do this, consider the solution of one stack of D5-branes at $z = \delta + \frac{i\pi}{2}$, and one stack of NS5-branes at $z = \hat{\delta}$,

$$h_1 = -\gamma \log\left(\tanh\left(\frac{i\pi}{4} - \frac{z-\delta}{2}\right)\right) + c.c. \tag{4.114}$$

$$h_2 = -\hat{\gamma} \log\left(\tanh\left(\frac{z-\hat{\delta}}{2}\right)\right) + c.c. \tag{4.115}$$

Performing a change of coordinates for the real part of $z = x + iy$,

$$r^2 = 2\left(e^{2\delta} + e^{2\hat{\delta}}\right) e^{-2x} \tag{4.116}$$

will lead for $x \to \infty$ to the metric

$$ds^2 \approx \frac{16\gamma\hat{\gamma}}{\cosh(\delta - \hat{\delta})}\left[ds^2_{AdS_4} + dr^2 + r^2\left(\sin^2 y ds^2_{S_1^2} + \cos^2 y ds^2_{S_2^2} + dy^2\right)\right] \tag{4.117}$$

The asymptotic geometry for $x \to -\infty$ is exactly the same. Thus we now have a solution with $AdS_4 \times K$ geometry.

4.7.5 Quantization of parameters

In order to find a supergravity dual to $T^\rho_{\hat{\rho}}[SU(N)]$ we needed five-branes, and a geometry which is a warped product $AdS_4 \times K$. We were able to find a solution which satisfies this in the previous subsections. The most general solution we consider now is

$$h_1 = -\sum_{a=1}^{q} \gamma_a \log\left(\tanh\left(\frac{i\pi}{4} - \frac{z-\delta_a}{2}\right)\right) + c.c. \tag{4.118}$$

$$h_2 = -\sum_{b=1}^{\hat{q}} \hat{\gamma}_b \log\left(\tanh\left(\frac{z-\hat{\delta}_b}{2}\right)\right) + c.c. \tag{4.119}$$

The brane charge of this solution is the same as those found in 4.7.3,

$$Q^{(a)}_{D5} = (4\pi)^2\gamma_a = 4\pi^2\alpha' N^{(a)}_{D5} \tag{4.120}$$

$$Q^{(b)}_{NS5} = -(4\pi)^2\hat{\gamma}_b = -4\pi^2\alpha'\hat{N}^{(b)}_{NS5} \tag{4.121}$$

$$Q^{(a)}_{D3} = -2^9\pi^3\gamma_a \sum_{b=1}^{\hat{q}} \hat{\gamma}_b \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) = (2\pi)^4(\alpha')^2 N^{(a)}_{D3} \tag{4.122}$$

$$Q^{(b)}_{D3} = 2^9\pi^3\hat{\gamma}_b \sum_{a=1}^{q} \gamma_a \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) = (2\pi)^4(\alpha')^2 \hat{N}^{(b)}_{D3} \tag{4.123}$$

where $N_{D3}^{(a)}$ is the number of D3-branes ending on the ath stack of D5-branes and $\hat{N}_{D3}^{(b)}$ is the number of D3-branes on the bth stack of NS5-branes. The second equality in the last two equations is again a result from string theory which we will not explain further in this thesis. Equations (4.120)-(4.121) imply the quantization of γ_a and $\hat{\gamma}_b$, while equations (4.120)-(4.123) allow us to relate the number of D3-branes, D5-branes and NS5-branes,

$$N_{D3}^{(a)} = - N_{D5}^{(a)} \sum_{b=1}^{\hat{q}} \hat{N}_{NS5}^{(b)} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.124}$$

$$\hat{N}_{D3}^{(b)} = \hat{N}_{NS5}^{(b)} \sum_{a=1}^{q} N_{D5}^{(a)} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.125}$$

These equations will imply the quantization of δ_a and $\hat{\delta}_b$. Note that this is possible since the δ_a and $\hat{\delta}_b$ contain $q + \hat{q} - 1$ relevant parameters, since the system is unchanged by an overal shift. There are $q + \hat{q}$ different D3-brane numbers, but they are subject to the constraint

$$-\sum_{a=1}^{q} N_{D3}^{(a)} = \sum_{b=1}^{\hat{q}} \hat{N}_{D3}^{(b)} = \sum_{a=1}^{q} \sum_{b=1}^{\hat{q}} N_{D5}^{(a)} \hat{N}_{NS5}^{(b)} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.126}$$

Another interesting feature of (4.124) and (4.125) follows by recognizing that the arctan function is bounded between $-\pi/2$ and $\pi/2$. This leads to

$$|N_{D3}^{(a)}| \leq N_{D5}^{(a)} \sum_{b=1}^{\hat{q}} \hat{N}_{NS5}^{(b)} \tag{4.127}$$

$$|\hat{N}_{D3}^{(b)}| \leq \hat{N}_{NS5}^{(b)} \sum_{a=1}^{q} N_{D5}^{(a)} \tag{4.128}$$

This inequality is precisely the s-rule introduced in 3.1. For example, the first inequality tells us that the number of D3-branes emanating from a stack D5-branes cannot exceed the number of D5-branes in this stack multiplied with the total number of NS5-branes. Should it exceed this number, then there must be at least one D5/NS5-brane pair which is connected by more than one D3-brane, violating the s-rule.

4.8 Duality map

In this section we will establish an explicit correspondence between the three dimensional superconformal field theory $T^{\rho}_{\hat{\rho}}[SU(N)]$ and the supergravity theory defined by the real harmonic functions (4.118) and (4.119). We already know that the symmetries of both theories will be exactly the same, since we constructed our supergravity solution by requiring that it satisfies the same symmetry.

Recall that all information about the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory was derivable from two partitions of N, called $\rho = (l_1, \cdots, l_i)$ and $\hat{\rho} = (\hat{l}_1, \cdots, \hat{l}_j)$. The numbers l_i and $\hat{l}_j$ were called the linking numbers, and they were ordered such that $l_1 \geq \cdots \geq l_i > 0$ and $\hat{l}_1 \geq \cdots \geq \hat{l}_j > 0$. On the supergravity side we will now define linking numbers by

$$l^{(a)} = -\frac{N^{(a)}_{D3}}{N^{(a)}_{D5}} \tag{4.129}$$

$$\hat{l}^{(b)} = \frac{\hat{N}^{(b)}_{D3}}{\hat{N}^{(b)}_{NS5}} \tag{4.130}$$

and the partitions as

$$\rho = (\underbrace{l^{(1)}, \cdots, l^{(1)}}_{N^{(1)}_{D5}}, \cdots, \underbrace{l^{(a)}, \cdots, l^{(a)}}_{N^{(a)}_{D5}}, \cdots, \underbrace{l^{(q)}, \cdots, l^{(q)}}_{N^{(q)}_{D5}}) \tag{4.131}$$

$$\hat{\rho} = (\underbrace{\hat{l}^{(1)}, \cdots, \hat{l}^{(1)}}_{\hat{N}^{(1)}_{NS5}}, \cdots, \underbrace{\hat{l}^{(b)}, \cdots, \hat{l}^{(b)}}_{\hat{N}^{(b)}_{NS5}}, \cdots, \underbrace{\hat{l}^{(\hat{q})}, \cdots, \hat{l}^{(\hat{q})}}_{\hat{N}^{(\hat{q})}_{NS5}}) \tag{4.132}$$

Remembering the ordering of the δ_a and $\hat{\delta}_b$ parameters, and the expressions for the D3-brane charges (4.122) and (4.123), it follows immediately from the monoticity of the arctan function that we also have an ordering for the supergravity linking numbers: $l^{(1)} \geq \cdots \geq l^{(q)} > 0$ and $\hat{l}^{(1)} \geq \cdots \geq \hat{l}^{(\hat{q})} > 0$. Furthermore, we define

$$N = \sum_{a=1}^{q} \sum_{b=1}^{\hat{q}} N^{(a)}_{D5} \hat{N}^{(b)}_{NS5} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.133}$$

Combining this with the charge conservation condition of (4.126) we see that

$$\sum_{a=1}^{q} N^{(a)}_{D5} l^{(a)} = \sum_{b=1}^{\hat{q}} \hat{N}^{(b)}_{NS5} \hat{l}^{(b)} = N \tag{4.134}$$

We also see that the numbers $M_{l^{(a)}}$ defined in chapter 3 are given by the number of D5-branes in the ath stack of D5-branes, thus $M_{l^{(a)}} = N^{(a)}_{D5}$ and analogously $\hat{M}_{\hat{l}^{(b)}} = \hat{N}^{(b)}_{NS5}$.

This completes duality map between both theories with respect to the free parameters, since we now are now able to choose the parameters of the supergravity theory in order to get the exact same partitions as in the $T^{\rho}_{\hat{\rho}}[SU(N)]$ theory. However, there were constraints on the possible partitions ρ and $\hat{\rho}$ in $T^{\rho}_{\hat{\rho}}[SU(N)]$, which had to satisfy

$$\rho^T > \hat{\rho} \Leftrightarrow \sum_{s=1}^{r} m_s > \sum_{s=1}^{r} \hat{l}_s \qquad \forall r = 1, \cdots, l_1 \tag{4.135}$$

We will show that the partitions of the supergravity theory we just defined must satisfy the same constraints.

Plugging equations (4.124) and (4.125) into the definition of the linking numbers (4.129) and (4.130), we see that we can alternatively write these as

$$l^{(a)} = \sum_{b=1}^{\hat{q}} \hat{N}^{(b)}_{NS5} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.136}$$

$$\hat{l}^{(b)} = \sum_{a=1}^{q} N^{(a)}_{D5} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_a}\right) \tag{4.137}$$

If we want to prove the inequality $\rho^T > \hat{\rho}$, it is obvious that we need an expression for ρ^T. From (4.131) we see that

$$\rho^T = \Bigg(\underbrace{\sum_{a=1}^{q} N_{D5}^{(a)}, \cdots \sum_{a=1}^{q} N_{D5}^{(a)}}_{l^{(q)}}, \cdots, \underbrace{\sum_{a=1}^{A} N_{D5}^{(a)}, \cdots, \sum_{a=1}^{A} N_{D5}^{(a)}}_{l^{(A)}-l^{(A+1)}}, \cdots, \underbrace{N_{D5}^{(1)}, \cdots, N_{D5}^{(1)}}_{l^{(1)}-l^{(2)}} \Bigg) \tag{4.138}$$

where in the ith block $A = q - i + 1$.

We recall from chapter 3 that equation (4.135) implied that $l_1 < \hat{k}$. Rewriting this in the supergravity partition notation this becomes

$$\sum_{b=1}^{\hat{q}} \hat{N}_{NS5}^{(b)} \frac{2}{\pi} \arctan\left(e^{\hat{\delta}_b - \delta_1}\right) < \sum_{b=1}^{\hat{q}} \hat{N}_{NS5}^{(b)} \tag{4.139}$$

Since $\frac{2}{\pi}\arctan(x) < 1$ for all finite x, this is trivially true.

Before we try to prove the general inequality, we show that it is sufficient to prove this for a r which satisfies

$$r = \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)} \qquad\qquad J = 1, \cdots, \hat{q} - 1 \tag{4.140}$$

To see this, take r in the range $\sum_{b=1}^{J-1} \hat{N}_{NS5}^{(b)} < r \leq \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}$. If the inequality is not satisfied for r, it will also not be satisfied for $r' = \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}$. This is because $\hat{l}_s$ is constant for $\sum_{b=1}^{J-1} \hat{N}_{NS5}^{(b)} < s \leq \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}$, while m_s on the other hand is non-increasing in this interval. Thus we can restrict ourselves to the r defined in (4.140).

To prove the general case (4.135), we will first rewrite this inequality. Define two auxiliary linking numbers, $l^{(0)} = \infty$ and $l^{(q+1)} = 0$. Then for a random J with $1 \leq J \leq \hat{q} - 1$ and $r = \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}$, we are always able to find an integer I such that

$$l^{(I)} > r \geq l^{(I+1)} \tag{4.141}$$

Using (4.138) we see that we can write

$$\sum_{s=1}^{r} m_s = \sum_{a=I+1}^{q} l^{(a)} N_{D5}^{(a)} + r \sum_{a=1}^{I} N_{D5}^{(a)} \tag{4.142}$$

$$= \sum_{a=I+1}^{q} l^{(a)} N_{D5}^{(a)} + \left(\sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}\right)\left(\sum_{a=1}^{I} N_{D5}^{(a)}\right) \tag{4.143}$$

This is most easily explained using an example. Consider the partition $\rho = (8, 8, 7, 7, 2, 1)$, which is shown in figure 4.2. Let us choose $r = 3$. The figure shows us which boxes are included in the sum $\sum_{s=1}^{3} m_s$, namely, the first three columns of the Young tableau. The figure shows a division of the first three columns into three seperate blocks. This corresponds to three seperate terms in (4.142). We see this by first searching the integer

I, it is easy to verify that $I = 2$, since $l^{(2)} = 7$ and $l^{(3)} = 2$. Using this we can evaluate (4.142) as

$$\sum_{s=1}^{3} m_s = \sum_{a=3}^{4} l^{(a)} N_{D5}^{(a)} + 3 \left(\sum_{a=1}^{2} N_{D5}^{(a)} \right) \tag{4.144}$$

$$= l^{(4)} N_{D5}^{(4)} + l^{(3)} N_{D5}^{(3)} + 3 \left(N_{D5}^{(1)} + N_{D5}^{(2)} \right) \tag{4.145}$$

The first term is just the single block at the bottom which is filled with a stripe pattern. The second term corresponds to the two black blocks. The last term is the large block of striped blocks. Generelazing this example, we see that (4.142) must be true, and by replacing $r = \sum_{b=1}^{J} \hat{N}_{NS5}^{(b)}$, we get the second equality.

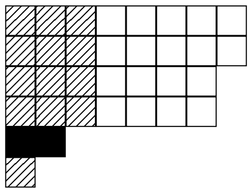

Figure 4.2: Young tableau for $\rho = (8, 8, 7, 7, 2, 1)$.

Using the above, we see that (4.135) becomes

$$\sum_{b=1}^{J} \hat{l}^{(b)} \hat{N}_{NS5}^{(b)} < \sum_{a=I+1}^{q} l^{(a)} N_{D5}^{(a)} + \left(\sum_{a=1}^{I} N_{D5}^{(a)} \right) \left(\sum_{b=1}^{J} \hat{N}_{NS5}^{(b)} \right) \tag{4.146}$$

Plugging in (4.136) and (4.137) transforms the inequality to

$$\sum_{a=1}^{q} \sum_{b=1}^{J} N_{D5}^{(a)} \hat{N}_{NS5}^{(b)} \frac{2}{\pi} \arctan \left(e^{\hat{\delta}_b - \delta_a} \right) < \sum_{a=I+1}^{q} \sum_{b=1}^{\hat{q}} N_{D5}^{(a)} \hat{N}_{NS5}^{(b)} \frac{2}{\pi} \arctan \left(e^{\hat{\delta}_b \delta_a} \right)$$
$$+ \sum_{a=1}^{I} \sum_{b=1}^{J} N_{D5}^{(a)} \hat{N}_{NS5}^{(b)} \tag{4.147}$$

Due to the fact that $\frac{2}{\pi} \arctan(x) < 1$ for all finite x, this inequality must be satisfied.

Thus we have now shown that the supergravity solution defined by the two real harmonic functions (4.118) and (4.119) exhibits the same symmetry as $T_{\hat{\rho}}^{\rho}[SU(N)]$, and that the partitions ρ and $\hat{\rho}$ can be explicitly obtained in the supergravity solution. These partitions must also satisfy the same constraints in both theories. This is why we call the supergravity solution with real harmonic functions (4.118) and (4.119) the holographic dual theory of $T_{\hat{\rho}}^{\rho}[SU(N)]$.

4.9 $T[SU(N)]$ dual

In the previous section we found the supergravity solution dual to $T_{\hat{\rho}}^{\rho}[SU(N)]$. The AdS/CFT conjecture implies that these two theories contain the same physics, implying that they should have the same free energy. We will check this explicitly in chapter 7, but only for the special case of $T[SU(N)]$. Therefore we seek the real harmonic functions that define the holographic dual theory of $T[SU(N)]$.

Remember that $T[SU(N)]$ was found by choosing the partitions

$$\rho =(1,\cdots,1) \tag{4.148}$$
$$\hat{\rho} =(1,\cdots,1) \tag{4.149}$$

Comparing with (4.131) and (4.132) we see that that we have one stack of N D5-branes and one stack of N NS5-branes, leading us to consider the supergravity solution defined by

$$h_1 =-\gamma\log\left(\tanh\left(\frac{i\pi}{4}-\frac{z-\delta}{2}\right)\right)+c.c. \tag{4.150}$$
$$h_2 =-\hat{\gamma}\log\left(\tanh\left(\frac{z-\hat{\delta}}{2}\right)\right)+c.c. \tag{4.151}$$

The five-brane charges of the stacks (4.105) and (4.106) allow us to express γ and $\hat{\gamma}$ in function of N,

$$\gamma=\hat{\gamma}=\frac{\alpha' N}{4} \tag{4.152}$$

From the definitions of the linking numbers (4.129) and (4.130) we see that there should be N D3-branes spanning between both stacks. From (4.124) and (4.125) we find a quantization relation for the parameters δ and $\hat{\delta}$,

$$\hat{\delta}-\delta=\log\left(\tanh\left(\frac{\pi}{2N}\right)\right) \tag{4.153}$$

Since we can perform a translation in the x direction, we can choose $\hat{\delta}=-\delta=\frac{1}{2}\log\left(\tanh\left(\frac{\pi}{2N}\right)\right)$ and the final result of this chapter is that the supergravity theory dual to $T[SU(N)]$ is given by the real harmonic functions

$$h_1 =-\frac{\alpha' N}{4}\log\left(\tanh\left(\frac{i\pi}{4}-\frac{z-\delta}{2}\right)\right)+c.c. \tag{4.154}$$
$$h_2 =-\frac{\alpha' N}{4}\log\left(\tanh\left(\frac{z+\delta}{2}\right)\right)+c.c. \tag{4.155}$$

5 Generalized supergravity solutions

In chapter 4 we obtained the supergravity dual of the $T^{\rho}_{\hat{\rho}}[SU(N)]$ superconformal field theories introduced in chapter 3. As stated before we wish to compare the free energy on both sides of the AdS/CFT correspondence in order to explicitly check the correspondence. In principle we could do this now, we quoted the free energy in the superconformal field theory and we have the supergravity dual solution from which we can deduce the free energy, but we postpone this to chapter 7. The reason for this is that if we assume the AdS/CFT correspondence to hold, we can calculate the free energy of a certain supergravity solution and the dual field theory should have the same free energy. As the title of this thesis suggests, we are interested in the free energy of finite temperature field theories. A temperature can be introduced in the supergravity theory by adding a black hole in the space. To make sure that we can introduce a black hole in the AdS_4 space, we explicitly check that we can generalize the Ansatz of chapter 4 to a general space A_4 which is only required to satisfy the same Einstein equations as AdS_4. Thus the new metric Ansatz is

$$ds^2 = f_4^2 ds^2_{A_4} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2 dz d\bar{z} \tag{5.1}$$

while the rest of the Ansatz from chapter 4 remains the same. The results of this chapter constitute the first part of the original work done for this thesis.

5.1 Dilaton field equation

The dilaton field equation is given by:

$$0 = \nabla_M P^M - 2iQ^M P_M + \frac{1}{24} G_{MNP} G^{MNP} \tag{5.2}$$

where as shown in (4.48) we can transform any solution to a solution where $Q = 0$. This reduces the dilaton equation to

$$0 = \nabla_M P^M + \frac{1}{24} G_{MNP} G^{MNP} \tag{5.3}$$

Using the ansatz for the G field we calculate

$$\frac{1}{24}G_{MNP}G^{MNP} = \frac{1}{4}(g_a g^a - h_a h^a) \tag{5.4}$$

The first term in the dilaton equation can be rewritten as

$$\begin{aligned}\nabla_M P^M =& g^{MN}\nabla_M\left(e^a_N P_a\right) &(5.5)\\ =& g^{MN}\left(\nabla_M(e^a_N)P_a + e^a_N\partial_M P_a\right) &(5.6)\\ =& g^{MN}(-\omega_M{}^p{}_q e^q_N P_p + e^p_N\partial_M P_p) &(5.7)\end{aligned}$$

We remember from chapter 4 that

$$P = f^2 dB \tag{5.8}$$

where

$$B = \frac{1-e^{2\phi}}{1+e^{2\phi}} \tag{5.9}$$

since we have a vanishing axion, and

$$\begin{aligned}f^2 =& \left(1-|B|^2\right)^{-1} &(5.10)\\ =& \frac{1}{1-\left(\frac{1-e^{2\phi}}{1+e^{2\phi}}\right)^2} &(5.11)\end{aligned}$$

Using this we find that

$$\begin{aligned}P_\mu =& \frac{1}{1-\left(\frac{1-e^{2\phi}}{1+e^{2\phi}}\right)^2}\partial_\mu\left(\frac{1-e^{2\phi}}{1+e^{2\phi}}\right) &(5.12)\\ =& \partial_\mu\phi &(5.13)\\ \Rightarrow P_a =& D_a\phi &(5.14)\end{aligned}$$

where $D_a = e^\mu_a\partial_\mu$. Since ϕ depends only on the Σ space coordinates, we know that $P_p \neq 0$ only for $p \in \Sigma$. We see from section 4.3 that the only contributing spin connection terms are $\omega^p{}_m, \omega^p{}_{i_1}, \omega^p{}_{i_2}$ and $\omega^p{}_a$. We already have explicit expressions for all of the spin connections, except for $\omega^p{}_a$. Writing the Σ metric in real coordinates x and y,

$$ds^2_\Sigma = 4\rho^2\left(dx^2 + dy^2\right) \tag{5.15}$$

it is easy to verify that

$$\omega^8{}_9 = \frac{1}{\rho}\partial_y\rho dx - \frac{1}{\rho}\partial_x\rho dy \tag{5.16}$$

We first look at the terms in (5.7) coming from $q \in \Sigma$. These are

$$\begin{aligned}&\frac{1}{4\rho^2}\left(-\omega_x{}^9{}_8 e^8_x D_9\phi + e^8_x\partial_x D_8\phi - \omega_y{}^8{}_9 e^9_y D_8\phi + e^9_y\partial_y\phi\right) &(5.17)\\ =&\frac{1}{4\rho^2}\left(-\left(\frac{1}{\rho}\partial_y\rho\right)\partial_y\phi + 2\rho\partial_x\left(\frac{1}{2\rho}\partial_x\phi\right) - \left(\frac{1}{\rho}\partial_x\rho\right)\partial_x\phi + 2\rho\partial_y\left(\frac{1}{2\rho}\partial_y\phi\right)\right) &(5.18)\\ =&\frac{1}{4\rho^2}\left(\partial_x\partial_x\phi + \partial_y\partial_y\phi\right) &(5.19)\end{aligned}$$

The terms with $q \notin \Sigma$ are

$$g^{MN}\left(-\omega_M{}^p{}_m e^m_N P_p - \omega_M{}^p{}_{i_1} e^{i_1} N P_p - \omega_M{}^p{}_{i_2} e^{i_2}_N P_p\right) \tag{5.20}$$

The first of these terms is

$$\begin{aligned} -\omega_M{}^p{}_m e^m_N P_p g^{MN} =& \frac{D_a f_4}{f_4} e^n_M e^m_N \eta_{nm} D^a \phi g^{MN} && (5.21)\\ =& D^a \phi D_a \ln(f_4^4) && (5.22) \end{aligned}$$

The other two terms will give very similar results, allowing us to write

$$g^{MN}\left(-\omega_M{}^p{}_m e^m_N P_p - \omega_M{}^p{}_{i_1} e^{i_1} N P_p - \omega_M{}^p{}_{i_2} e^{i_2}_N P_p\right) = D^a \phi D_a \ln(f_1^2 f_2^2 f_4^4) \tag{5.23}$$

Collecting all the results leaves us with

$$0 = \frac{1}{4\rho^2}\left(\partial_x \partial_x \phi + \partial_y \partial_y \phi\right) + D^a \phi D_a \ln(f_1^2 f_2^2 f_4^2) + \frac{1}{4}(g_a g^a - h_a h^a) \tag{5.24}$$

This is the same dilaton field equation as found in [16]. It was shown in this paper that the general solution quoted in section 4.4 will automatically satisfy (5.24). Thus the solution with a more general metric Ansatz will still be a solution to the dilaton field equations.

5.2 G-field equation

The G-field equation is

$$0 = \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} + \frac{2}{3} i F_{(5)MNPQR} G^{PQR} \tag{5.25}$$

As stated above we can put $Q^P = 0$, eliminating the second term. The third is easily verified to be

$$P^P \bar{G}_{MNP} = p^a g_a e^4 \wedge e^5 - i p^a h_a e^6 \wedge e^7 \tag{5.26}$$

The last term can be written as

$$\frac{2}{3} i F_{(5)MNPQR} G^{PQR} = 4(i f_a \epsilon^a{}_b g^b e^6 \wedge e^7 - f_a \epsilon^{ab} h_b e^4 \wedge e^5) \tag{5.27}$$

The first term we rewrite as follows,

$$\begin{aligned} \nabla^P G_{MNP} =& \nabla^P \left[e^a_M e^b_N e^c_P G_{abc}\right) && (5.28)\\ =& g^{PQ} \nabla_Q \left(e^a_M e^b_N e^c_P G_{abc}\right) && (5.29)\\ =& g^{PQ}\left(\left(\nabla_Q e^a_M\right) e^b_N e^c_P G_{abc} + e^a_M \left(\nabla_Q e^b_N\right) e^c_P G_{abc}\right.\\ &\left. + e^a_M e^b_N \left(\nabla_Q e^c_P\right) G_{abc} + e^a_M e^b_N e^c_P \partial_Q G_{abc}\right] && (5.30)\\ =& g^{PQ}\left[-\omega_Q{}^a{}_d e^d_M e^b_N e^c_P G_{abc} - e^a_M \omega_Q{}^b{}_d e^d_N e^c_P\right.\\ &\left. - e^a_M e^b_N \omega_Q{}^c{}_d e^d_P G_{abc} + e^a_M e^b_N e^c_P \partial_Q G_{abc}\right] && (5.31) \end{aligned}$$

It is important to note here that nowhere in this sum a spin connection is needed with two AdS_4 indices. The only spin connections we will need are found in (5.16) and (4.34). Using this, we eventually find that

$$\begin{aligned}\nabla^P G_{MNp} = &\left(D_a(\rho g^a) + 2g_a D^a \ln(f_2 f_4^2)\right) e^4 \wedge e^5 \\ &+ i\left(D_a(\rho h^a) + 2h_a D^a \ln(f_1 f_4^2)\right) e^6 \wedge e^7\end{aligned} \tag{5.32}$$

Thus we find that the G field equation leads to two equations

$$\left(D_a(\rho g^a) + 2g_a D^a \ln(f_2 f_4^2)\right) - p^a g_a - 4f_a \epsilon^{ab} h_b = 0 \tag{5.33}$$

$$\left(D_a(\rho h^a) + 2h_a D^a \ln(f_1 f_4^2)\right) + p^a h_a + 4f_a \epsilon^{ab} g_b = 0 \tag{5.34}$$

As was the case with the dilaton equation, this is the same G field equation found in [16]. Thus the solution of chapter 4 will still be a solution to the G field equation with the generalized metric Ansatz.

5.3 Einstein equations

The Einstein equation is

$$\begin{aligned}0 = &R_{MN} - P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6}\hat{F}_{(5)MP_1P_2P_3P_4}\hat{F}_{(5)N}{}^{P_1P_2P_3P_4} \\ &- \frac{1}{8}(G_M{}^{PQ}\bar{G}_{NPQ} + \bar{G}_M{}^{PQ}G_{NPQ}) + \frac{1}{48}g_{MN}G^{PQR}\bar{G}_{PQR}\end{aligned} \tag{5.35}$$

For convenience we define T_{MN} by

$$\begin{aligned}T_{MN} = &- P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6}\hat{F}_{(5)MP_1P_2P_3P_4}\hat{F}_{(5)N}{}^{P_1P_2P_3P_4} \\ &- \frac{1}{8}(G_M{}^{PQ}\bar{G}_{NPQ} + \bar{G}_M{}^{PQ}G_{NPQ}) + \frac{1}{48}g_{MN}G^{PQR}\bar{G}_{PQR}\end{aligned} \tag{5.36}$$

The only non vanishing components of the Einstein equation will be mn, i_1j_1, i_2j_2 and ab, which we will discuss now.

5.3.1 A_4 components

It is readily verified that in this case,

$$T_{mn} = 4\eta_{mn} f_a f^a + \frac{1}{8}\eta_{mn}(g_a g^a + h_a h^a) \tag{5.37}$$

To calculate the R_{MN} term we use the following equation,

$$\Omega^r{}_s = \frac{1}{2}R^r{}_{spq} e^p \wedge e^q \tag{5.38}$$

$$\Rightarrow e_p \Omega^p{}_r = R_{rq} e^q \tag{5.39}$$

The curvature 2-form Ω is defined by

$$\Omega^r{}_s = d\omega^r{}_s + \omega^r{}_s + \omega^r{}_p \wedge \omega^p{}_s \tag{5.40}$$

In order to calculate R_{mn} we need the $\Omega^m{}_n, \Omega^{i_1}{}_n, \Omega^{i_2}{}_n$ and $\Omega^a{}_n$ components of the curvature 2-form. Using Eq. 5.40 we find that

$$\begin{aligned}
\Omega^m{}_n =& d\omega^m{}_n + \omega^m{}_r \wedge \omega^r{}_m && (5.41)\\
=& d\hat{\omega}^m{}_n + \hat{\omega}^m{}_{n_1} \wedge \hat{\omega}^{n_1}{}_n + \omega^m{}_a \wedge \omega^a{}_n && (5.42)\\
=& \hat{\Omega}^m{}_n - \frac{D_a f_4 D^a f_4}{f_4^2} e^m \wedge e_n && (5.43)
\end{aligned}$$

where

$$\hat{\Omega}^m{}_n = -\frac{e^m \wedge e_n}{f_4^2} \tag{5.44}$$

because A_4 satisfies the same Einstein equations as AdS_4, which is a maximally symmetric space. This implies that the curvature 2-form is of this form. To see this, we start from the Riemann tensor of AdS space-time,

$$R_{\mu\nu\rho\sigma} = -\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right) \tag{5.45}$$

Thus the curvature two-form is

$$\begin{aligned}
\hat{\Omega}^m{}_n =& \frac{1}{2} R^\lambda{}_{\nu\kappa\mu} \hat{e}^m_\lambda \hat{e}^\nu_n dx^\kappa \wedge dx^\mu && (5.46)\\
=& -\frac{1}{2} g^{\alpha\lambda} \hat{e}^m_\lambda \hat{e}^\nu_n \left(g_{\alpha\kappa} g_{\nu\mu} - g_{\alpha\mu} g_{\nu\kappa}\right) dx^\kappa \wedge dx^\mu && (5.47)\\
=& -\frac{1}{2} \hat{e}^m_\lambda \hat{e}^\nu_n \hat{e}^\kappa_c \hat{e}^\mu_d \left(\delta^\lambda_\kappa g_{\nu\mu} - \delta^\lambda_\mu g_{\nu\kappa}\right) \hat{e}^c \wedge \hat{e}^d && (5.48)\\
=& -\hat{e}^m_\kappa \hat{e}^\nu_n \hat{e}^\kappa_c \hat{e}^\mu_d g_{\nu\mu} \hat{e}^c \wedge \hat{e}^d && (5.49)\\
=& -\hat{e}^m \wedge \hat{e}_n && (5.50)\\
=& -\frac{e^m \wedge e_n}{f_4^2} && (5.51)
\end{aligned}$$

The curvature 2-forms with one sphere index and one A_4 index are

$$\begin{aligned}
\Omega^{i_1}{}_n =& \omega^{i_1}{}_a \wedge \omega^a{}_n && (5.52)\\
=& -\frac{D_a f_1}{f_1} \frac{D^a f_4}{f_4} e^{i_1} \wedge e_n && (5.53)\\
\Omega^{i_2}{}_n =& \omega^{i_2}{}_a \wedge \omega^a{}_n && (5.54)\\
=& -\frac{D_a f_2}{f_2} \frac{D^a f_4}{f_4} e^{i_2} \wedge e_n && (5.55)
\end{aligned}$$

The last non vanishing part of the curvature 2-form is

$$\Omega^a{}_n = d\omega^a{}_n + \omega^a{}_m \wedge \omega^m{}_n + \omega^a{}_b \wedge \omega^b{}_n \tag{5.56}$$

$$= d\left(-D^a f_4 \hat{e}_n\right) - D^a f_4 \hat{e}_m \wedge \hat{\omega}^m{}_n + \omega^a{}_b \wedge \left(-\frac{D^b f_4}{f_4}\right) e_n \tag{5.57}$$

$$= -\frac{D_b D^a f_4}{f_4} e^b \wedge e_n - D^a f_4 d\hat{e}_n - D^a f_4 \hat{e}_m \wedge \hat{\omega}^m{}_n + \omega^a{}_b \wedge \left(-\frac{D^b f_4}{f_4}\right) e_n \tag{5.58}$$

$$= -\frac{D_b D^a f_4}{f_4} e^b \wedge e_n - \frac{D^b f_4}{f_4} \omega^a{}_b \wedge e_n - D^a f_4 \left(d\hat{e}_n + \hat{e}_m \wedge \hat{\omega}^m{}_n\right) \tag{5.59}$$

The last term of this equation will vanish due to the first Cartan structure equation $de_n + \omega_n{}^m \wedge e_m$ and the anti-symmetry of the spin connection. Thus finally we have

$$\Omega^a{}_n = -\frac{D_b D^a f_4}{f_4} e^b \wedge e_n - \frac{D^b f_4}{f_4} \omega^a{}_b \wedge e_n \tag{5.60}$$

Looking at (5.39) we see that we need to calculate $e_M \Omega^M{}_n$ in order to calculate the Ricci tensor R_{NQ}. We know that $e_m \hat{\Omega}^m{}_n = -\frac{3}{f_4^2} e_n$, because of (5.44). Thus we find that

$$e_m \Omega^m{}_n = -\frac{3}{f_4^2} e_n - 3\frac{D_a f_4 D^a f_4}{f_4^2} e_n \tag{5.61}$$

The curvature 2-forms with sphere indices are easily calculated to be

$$e_{i_1} \Omega^{i_1}{}_n = -2\frac{D_a f_1 D^a f_4}{f_1 f_4} e_n \tag{5.62}$$

$$e_{i_2} \Omega^{i_2}{}_n = -2\frac{D_a f_2 D^a f_4}{f_2 f_4} e_n \tag{5.63}$$

The final term is slightly more complicated

$$e_a \Omega^a{}_n = -\frac{D_a D^a f_4}{f_4} e_n - \frac{D^b f_4}{f_4} e_a \omega^a{}_b e_n \tag{5.64}$$

$$= -\frac{1}{f_4}\left(e_a^\alpha \partial_\alpha \left(e_b^\beta \partial_\beta f_4\right) \eta^{ab} - \left(\eta^{cb} e_c^\alpha \partial_\alpha f_4\right)\left(e_d^\alpha \omega_\alpha{}^d{}_b\right)\right) e_n \tag{5.65}$$

$$\begin{aligned} = -\frac{1}{f_4}\Bigg[\frac{1}{2\rho}\left(-\frac{1}{2\rho^2}\partial_x\rho\partial_x f_4 + \frac{1}{2\rho}\partial_x\partial_x f_4 - \frac{1}{2\rho^2}\partial_y\rho\partial_y f_4 \frac{1}{2\rho}\partial_y\partial_y f_4\right) \\ + \left(\frac{1}{4\rho^3}\partial_x f_4 \partial_x \rho + \frac{1}{4\rho^3}\partial_y f_4 \partial_y \rho\right)\Bigg] e_n \end{aligned} \tag{5.66}$$

$$= -\frac{1}{f_4}\frac{1}{4\rho^2}\left(\partial_x\partial_x f_4 + \partial_y\partial_y f_4\right) \tag{5.67}$$

$$= -\frac{1}{f_4}\partial_a\partial^a f_4 \tag{5.68}$$

Collecting all the results, we find that

$$R_{nm} e^m = \left(-\frac{3}{f_4^2} - 3\frac{D_a f_4 D^a f_4}{f_4^2} - 2\frac{D^a f_4 D_a(f_1 f_2)}{f_1 f_2 f_4} - \frac{\partial_a \partial^a f_4}{f_4}\right)\eta_{mn} e^m \tag{5.69}$$

from which we easily derive

$$R_{mn} = \left(-\frac{3}{f_4^2} - 3\frac{D_a f_4 D^a f_4}{f_4^2} - 2\frac{D^a f_4 D_a(f_1 f_2)}{f_1 f_2 f_4} - \frac{\partial_a \partial^a f_4}{f_4}\right)\eta_{mn} \tag{5.70}$$

Combining with the expression we found for T_{mn}, we get the Einstein equation in the A_4 components,

$$\begin{aligned}0 =& -\frac{3}{f_4^2} - 3\frac{D_a f_4 D^a f_4}{f_4^2} - 2\frac{D^a f_4 D_a(f_1 f_2)}{f_1 f_2 f_4} - \frac{\partial_a \partial^a f_4}{f_4} \\ &+ 4 f_a f^a + \frac{1}{8} g_a g^a + \frac{1}{8} h_a h^a\end{aligned} \tag{5.71}$$

This equation is exactly the same as found in [16]. Thus again, we know that the general solution of chapter 4 will remain a solution if we extend the metric Ansatz.

5.3.2 S_1 and S_2 components

The calculations for the S_1 and S_2 components are completely analogous, thus we will only do them for the S_1 components.

Again we start with $T_{i_1 j_1}$. By plugging in the field Ansatz in Eq. 5.36 we find

$$T_{i_1 j_1} = \eta_{i_1 j_1}\left(-4 f_a f^a - \frac{3}{8} g_a g^a + \frac{1}{8} h_a h^a\right) \tag{5.72}$$

As mentioned above, the calculation for the S_2 components will be completely analogous. However the result will be slightly different,

$$T_{i_2 j_2} = \eta_{i_2 j_2}\left(-4 f_a f^a + \frac{1}{8} g_a g^a - \frac{3}{8} h_a h^a\right) \tag{5.73}$$

The relevant curvature two-form terms for the calculation of $R_{i_1 j_1}$ are calculated in exactly the same way as in the previous subsection, therefore we only quote the results,

$$\Omega^m{}_{i_1} = -\frac{D_a f^4}{f_4}\frac{D^a f_1}{f_1} e^m \wedge e_{i_1} \tag{5.74}$$

$$\Omega^{j_1}{}_{i_1} = \hat{\Omega}^{j_1}{}_{i_1} - \frac{D_a f_1}{f_1}\frac{D^a f_1}{f_1} e^{j_1} \wedge e_{i_1} \tag{5.75}$$

$$\Omega^{j_2}{}_{i_1} = -\frac{D_a f_2}{f_2}\frac{D^a f_1}{f_1} e^{j_2} \wedge e_{i_1} \tag{5.76}$$

$$\Omega^a{}_{i_1} = -\frac{D_b D^a f_1}{f_1} e^b \wedge e_{i_1} + \omega^a{}_b \wedge -\frac{D_b f_1}{f_1} e_{i_1} \tag{5.77}$$

Again we use 5.39 to calculate the Ricci tensor, where all necessary calculations are again the same as the previous subsection. Thus we find that the Ricci tensor is given by

$$R_{i_1 j_1} = \eta_{i_1 j_1}\left(\frac{1}{f_1^2} - \frac{D_a f_1 D^a f_1}{f_1^2} - 4\frac{D_a f_4 D^a f_1}{f_1 f_4} - 2\frac{D_a f_2 D^a f_1}{f_1 f_2} - \frac{\partial_a \partial^a f_1}{f_1}\right) \tag{5.78}$$

Adding $R_{i_1j_1}$ and $T_{i_1j_1}$ gives the Einstein equation in the S_1 components,

$$\begin{aligned}0 =&\frac{1}{f_1^2} - \frac{D_a f_1 D^a f_1}{f_1^2} - 4\frac{D_a f_4 D^a f_1}{f_1 f_4} - 2\frac{D_a f_2 D^a f_1}{f_1 f_2}\\ &- \frac{\partial_a \partial^a f_1}{f_1} - 4 f_a f^a - \frac{3}{8} g_a g^a + \frac{1}{8} h_a h^a\end{aligned} \tag{5.79}$$

Analogously we find the Einstein equation in the S_2 components,

$$\begin{aligned}0 =&\frac{1}{f_2^2} - \frac{D_a f_2 D^a f_2}{f_2^2} - 4\frac{D_a f_4 D^a f_2}{f_2 f_4} - 2\frac{D_a f_2 D^a f_1}{f_1 f_2}\\ &- \frac{\partial_a \partial^a f_2}{f_2} - 4 f_a f^a + \frac{1}{8} g_a g^a - \frac{3}{8} h_a h^a\end{aligned} \tag{5.80}$$

Again, this equation is proven to follow from the general solution in chapter 4.[16]

5.3.3 Σ components

In this case it was verified that

$$T_{ab} = -2 D_a\phi D_b\phi - 4\delta_{ab} f_c f^c + 8 f_a f_b + \frac{1}{8}\delta_{ab}(g_c g^c + h_c h^c) - \frac{1}{2} g_a g_b - \frac{1}{2} h_a h_b \tag{5.81}$$

The relevant curvature two-form components are

$$\Omega^m{}_a =\frac{D_b D_a f_4}{f_4} e^b \wedge e^m + \frac{D_b f_4}{f_4} e^m \wedge \omega^b{}_a \tag{5.82}$$

$$\Omega^{i_1}{}_a =\frac{D_b D_a f_1}{f_1} e^b \wedge e^{i_1} + \frac{D_b f_1}{f_1} e^{i_1} \wedge \omega^b{}_a \tag{5.83}$$

$$\Omega^{i_2}{}_a =\frac{D_b D_a f_2}{f_2} e^b \wedge e^{i_2} + \frac{D_b f_1}{f_1} e^{i_2} \wedge \omega^b{}_a \tag{5.84}$$

$$\Omega^b{}_a =d\hat{\omega}^b{}_a \tag{5.85}$$

To extract R_{ab}, we note that

$$e_b \Omega^b{}_a =R^{(2)}\delta_{ab} e^b \tag{5.86}$$

where $R^{(2)}$ is the scalar curvature of the Σ space. The other three terms are very similar, and we will only work out the first one,

$$e_m \Omega^m{}_a = -4\frac{D_b D_a f_4}{f_4} e^b + 4\frac{D_b f_4}{f_4}\omega^b{}_a \tag{5.87}$$

To continue, we rewrite (5.16) as

$$\omega^b{}_a =e^b \frac{1}{\rho} D_a\rho - e_a \frac{1}{\rho} D^b\rho \tag{5.88}$$

Using this we find

$$e_m \Omega^m{}_a = -4\frac{D_b D_a f_4}{f_4} e^b + 4\frac{D_b f_4}{f_4}\left(e^b \frac{1}{\rho} D_a \rho - e_a \frac{1}{\rho} D^b \rho\right) a \tag{5.89}$$

$$= -4\frac{D_b D_a f_4}{f_4} e^b + 4\left(\frac{D_b f_4}{f_4}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_4}{f_4}\frac{1}{\rho} D^c \rho\right) e^b \tag{5.90}$$

The terms with sphere indices are analogously found to be

$$e_{i_1} \Omega^{i_1}{}_a = -2\frac{D_b D_a f_1}{f_1} e^b + 2\left(\frac{D_b f_1}{f_1}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_1}{f_1}\frac{1}{\rho} D^c \rho\right) e^b \tag{5.91}$$

$$e_{i_2} \Omega^{i_2}{}_a = -2\frac{D_b D_a f_2}{f_2} e^b + 2\left(\frac{D_b f_2}{f_2}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_2}{f_2}\frac{1}{\rho} D^c \rho\right) e^b \tag{5.92}$$

From (5.86), (5.90), (5.91) and (5.92) we can easily read of R_{ab}. Combining this with T_{ab}, we find that the Einstein equation in the Σ components is given by

$$\begin{aligned} 0 = &-4\frac{D_b D_a f_4}{f_4} - 2\frac{D_b D_a f_1}{f_1} - \frac{D_b D_a f_2}{f_2} + R^{(2)}\delta_{ab} \\ &+ 4\left(\frac{D_b f_4}{f_4}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_4}{f_4}\frac{1}{\rho} D^c \rho\right) + 2\left(\frac{D_b f_1}{f_1}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_1}{f_1}\frac{1}{\rho} D^c \rho\right) \\ &+ 2\left(\frac{D_b f_2}{f_2}\frac{1}{\rho} D_a \rho - \delta_{ab}\frac{D_c f_2}{f_2}\frac{1}{\rho} D^c \rho\right) - 2D_a\phi D_b\phi - 4\delta_{ab} f_c f^c + 8 f_a f_b \\ &+ \frac{1}{8}\delta_{ab}(g_c g^c + h_c h^c) - \frac{1}{2} g_a g_b - \frac{1}{2} h_a h_b \end{aligned} \tag{5.93}$$

By now the reader will not be surprised that this equation is also found in [16], implying that this equation is also automatically satisfied for any solution of the form stated in section 4.4.

Thus our final conclusion of this chapter is that we can take a more general metric Ansatz, by replacing AdS_4 by A_4 where A_4 satisfies the same Einstein equations as AdS_4, and this will still be a solution to the type IIB supergravity equations of motion shown in (4.12), (4.13) and (4.14). We will use this fact to introduce a black hole in AdS_4 space-time, allowing us to study the dual conformal field theory at finite temperature.

6 Holographic renormalization

The usual gravitational action for a $n+1$ dimensional space M with a cosmological constant is given by the Einstein-Hilbert action,

$$S = -\int_M d^{n+1}x\sqrt{G}(R-2\Lambda) \tag{6.1}$$

However, if the spacetime under consideration has a boundary, we should add the Gibbons-Hawking surface term in order to maintain a well defined variational principle,[22]

$$S_{surf} = -2\int_{\partial M} K|\gamma|^{1/2}d^n x \tag{6.2}$$

where K is the extrinsic curvature and satisfies

$$\sqrt{\gamma}K = \mathcal{L}_n\sqrt{\gamma} \tag{6.3}$$

AdS space-time has a boundary. Thus this surface term should be added to the Einstein-Hilbert action.

It is easy to see that these integrals will be divergent when calculated on an AdS space background. For instance, consider the bulk term of the gravitational action, (6.1). The Einstein equation that can be deduced from this equation is

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}(R-2\Lambda) = 0 \tag{6.4}$$

Contracting this equation with the metric and using the fact that an $n+1$ dimensional AdS metric has the cosmological constant

$$\Lambda = -\frac{n(n-1)}{2L^2} \tag{6.5}$$

gives the Ricci scalar,

$$R = -\frac{(n+1)n}{L^2} \tag{6.6}$$

If we plug this into (6.1) we find that the action is proportional to the volume of space-time,

$$S_{bulk} = \frac{2n}{L^2} \int d^{n+1}x\sqrt{G} \tag{6.7}$$

which leads to a divergent action due to the non-compactness of AdS space.

Why is this of importance to us? As stated before, we wish to calculate the free energy of the supergravity solutions, which is $F = -lnZ_S$, where Z_S is the supergravity partition function. We remember from chapter 2 that the partition function is defined as the path integral

$$Z_S = \int \mathcal{D}[fields]e^{-S[fields]} \tag{6.8}$$

This is hard to calculate, therefore we made the saddle point approximation. In this approximation, the partition function is given by the exponential of the action evaluated at the solution of the equations of motion,

$$Z_S \approx e^{-S} \tag{6.9}$$

and thus

$$F = -\ln Z_S = S \tag{6.10}$$

If the action diverges, it is clear that this leads to a divergent free energy. To resolve this problem, we regularize the action in an appropriate way, by introducing a universal counterterm which only depends on the induced metric on the boundary. This procedure is called holographic renormalization, and is introduced in the next section, following a treatment presented in [23].

This holographic renormalization is dual to the renormalization performed in CFT on curved backgrounds. It is verified through examples that holographic renormalization gives the same results as the dual renormalized CFT calculations.[24]

6.1 Counterterms

The metric of an $n+1$ dimensional space which is asymptotically AdS with radius L can be written near its boundary at $u = 0$ as

$$ds^2 = L^2 \left(\frac{du^2}{u^2} + \frac{1}{u^2} g_{ij}(u^2, x)dx^i dx^j \right) \tag{6.11}$$

due to a theory by Fefferman and Graham, where

$$g_{ij}(u^2, x) = g_{ij}^{(0)}(x) + u^2 g_{ij}^{(2)}(x) + \cdots + u^n \left[g^{(n)}(x) + \ln(u^2) h^{(n)}(x) \right] + \cdots \tag{6.12}$$

The first term in the expansion is the boundary metric, i.e. the metric of the CFT. The higher order terms can be solved recursively in terms of $g_{ij}^{(0)}$ by plugging in (6.12) into Einstein's equation.

We use this metric to compute the action with a cut-off at $u = \epsilon$,

$$S_\epsilon = -\int_{M_\epsilon} d^{n+1}x\sqrt{G}\left(R + \frac{n(n-1)}{L^2}\right) - 2\int_{\partial M_\epsilon} d^n x K|\gamma|^{1/2} \tag{6.13}$$

The bulk part can easily be verified to be equal to

$$S_{\epsilon\ bulk} = 2nL^{n-1}\int d^n x \int_\epsilon \frac{du}{u^{n+1}}\sqrt{g} \tag{6.14}$$

To calculate the surface action we need the normal vector to the boundary at $u = \epsilon$ and the induced metric on the boundary. The normal vector is

$$n^u = -\frac{u}{L} \tag{6.15}$$

where there is a minus sign because the normal vector points towards $u = 0$, and the induced metric is

$$\gamma_{ij}dx^i dx^j = \frac{L^2}{u^2}g_{ij}(u^2, x)dx^i dx^j \tag{6.16}$$

$$\Rightarrow \sqrt{\gamma} = \left(\frac{L}{u}\right)^n \sqrt{g} \tag{6.17}$$

This implies that the intrinsic curvature of the boundary is

$$\sqrt{\gamma}K = \mathcal{L}_n\sqrt{\gamma} \tag{6.18}$$

$$= -\frac{u}{L}\partial_u\left[\left(\frac{L}{u}\right)^n\sqrt{g}\right] \tag{6.19}$$

$$= \frac{nL^{n-1}}{u^n}\left(1 - \frac{1}{n}u\partial_u\right)\sqrt{g} \tag{6.20}$$

From this we see that the surface action is equal to

$$S_{\epsilon\ surf} = -\frac{2nL^{n-1}}{\epsilon^N}\int d^n x\left(1 - \frac{1}{n}u\partial_u\right)\sqrt{g}|_{u=\epsilon} \tag{6.21}$$

Thus the total action becomes

$$S_\epsilon = 2nL^{n-1}\int d^n x\int_\epsilon \frac{du}{u^{n+1}}\sqrt{g} - \frac{2nL^{n-1}}{\epsilon^N}\int d^n x\left(1 - \frac{1}{n}u\partial_u\right)\sqrt{g}|_{u=\epsilon} \tag{6.22}$$

We can rewrite this in terms of the boundary metric and a power series in ϵ [25],

$$S_\epsilon = L^{n-1}\int d^n x\sqrt{g^{(0)}}\left(\epsilon^{-n}a_{(0)} + \epsilon^{-n+2}a_{(2)} + \cdots - 2\ln(\epsilon)a_{(n)}\right) + \mathcal{O}(\epsilon^0) \tag{6.23}$$

where the logarithmic term is only present if n is even.

Because we will work with $n = 3$ in this thesis, it will only be necessary the calculate the terms $a_{(0)}$ and $a_{(2)}$. To do this we first calculate the determinant of the metric

$$\det g = \det g^{(0)}\left(1 + u^2 Tr\left(g^{(0)-1}g^{(2)}\right) + \cdots\right) \tag{6.24}$$

From this we see that

$$\sqrt{g} = \sqrt{g^{(0)}} \left(1 + \frac{u^2}{2} Tr\left(g^{(0)-1} g^{(2)}\right) + \cdots\right) \tag{6.25}$$

$$\left(1 - \frac{1}{n} u \partial_u\right) \sqrt{g} = \sqrt{g^{(0)}} \left(1 + \frac{n-2}{2n} u^2 Tr\left(g^{(0)-1} g^{(2)}\right) + \cdots\right) \tag{6.26}$$

Plugging this into (6.22) the action becomes

$$S_\epsilon = L^{n-1} \int d^n x \sqrt{g^{(0)}} \left(\frac{2(1-n)}{\epsilon^n} - \frac{n^2 - 5n + 4}{(n-2)\epsilon^{n-2}} Tr\left(g^{(0)-1} g^{(2)}\right) + \cdots\right) \tag{6.27}$$

and we find that

$$a_{(0)} = 2(1-n) \tag{6.28}$$

$$a_{(2)} = -\frac{(n-4)(n-1)}{n-2} Tr\left(g^{(0)-1} g^{(2)}\right) \tag{6.29}$$

Now we define the counterterm as minus the divergent part of action S_ϵ,

$$S_{ct} = L^{n-1} \int d^n x \sqrt{g^{(0)}} \left(\frac{2(n-1)}{\epsilon^n} + \frac{(n-4)(n-1)}{(n-2)\epsilon^{n-2}} Tr\left(g^{(0)-1} g^{(2)}\right) + \cdots\right) \tag{6.30}$$

In this form the counterterm is not particularly useful, since we need the metric in the form of (6.11). We want to rewrite this in term of the induced boundary metric γ. Due to (6.17) and (6.25) we already have an expression for $\sqrt{g^{(0)}}$ in function of $\sqrt{\gamma}$,

$$\sqrt{g^{(0)}} = \left(\frac{\epsilon}{L}\right)^n \left(1 - \frac{\epsilon^2}{2} Tr\left(g^{(0)-1} g^{(2)}\right) + \mathcal{O}(\epsilon^4)\right) \sqrt{\gamma} \tag{6.31}$$

To go further, we need an explicit expression for $g^{(2)}$. As previously mentioned, this can be found by plugging in (6.12) into the Einstein equation. Doing this, it can be shown that [26]

$$g^{(2)}_{ij} = -\frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R g^{(0)}_{ij}\right) \tag{6.32}$$

with R and R_{ij} the Ricci scalar and tensor of $g^{(0)}_{ij}$. With this equation we can verify that

$$Tr\left(g^{(0)-1} g^{(2)}\right) = -\frac{1}{n-2} g^{(0)ij} \left(R_{ij} - \frac{1}{2(n-1)} R g^{(0)}_{ij}\right) \tag{6.33}$$

$$= -\frac{1}{2(n-1)} R \tag{6.34}$$

$$= -\frac{L^2}{2(n-1)\epsilon^2} R(\gamma) + \cdots \tag{6.35}$$

Thus finally, the counterterm is given by

$$S_{ct} = 2 \int d^n x \sqrt{\gamma} \left(\frac{n-1}{L} + \frac{L}{2(n-2)} R(\gamma) + \cdots\right) \tag{6.36}$$

This equation is valid for $n = 3$, for higher dimensions extra counterterms are needed.

In the following chapter we will need the volume of AdS_4 spaces with different boundaries, corresponding to field theories living on different space-times, and the volume of AdS_4 with a black hole, corresponding to finite temperature field theories, in order to calculate the free energy of the supergravity dual theory. Therefore we will now present the calculation of the volume of these space-times.

6.2 Volume calculations

6.2.1 S^3 boundary

In chapter 3 we studied the free energy of $T[SU(N)]$ on a S^3 metric. Since we are interested in calculating the free energy in the supergravity dual theory, we need the volume of AdS_4 with S^3 boundary. This can be described by the metric

$$ds^2 = \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_3^2 \tag{6.37}$$

where $d\Omega_3^2$ is the metric for the unit n-sphere. The bulk action can be calculated using (6.7). In this case we have that

$$\sqrt{G} = \frac{r^3}{\sqrt{1 + \frac{r^2}{L^2}}} \sqrt{g_{S^3}} \tag{6.38}$$

Thus the bulk action is equal to

$$S_{bulk} = \frac{6\text{Vol}(S^3)}{L} \int_0^r d\rho \frac{\rho^3}{\sqrt{L^2 + \rho^2}} \tag{6.39}$$

To calculate the surface action we need the normal vector

$$n = \sqrt{1 + \frac{r^2}{L^2}} \frac{\partial}{\partial r} \tag{6.40}$$

and the induced metric

$$\gamma = r^2 g_{S^3} \tag{6.41}$$

Calculating the intrinsic curvature is now straightforward,

$$\mathcal{L}_n \sqrt{\gamma} = 3r^2 \sqrt{1 + \frac{r^2}{L^2}} \sqrt{g_{S^3}} \tag{6.42}$$

Plugging this into the surface action (6.2) we find

$$S_{surf} = -6r^2 \sqrt{1 + \frac{r^2}{L^2}} Vol(S^3) \tag{6.43}$$

The only thing left to calculate is the counterterm. To calculate this, we need the scalar curvature of the induced metric. Since this is just a three dimensional sphere with radius r^2, it is known that

$$R[\gamma] = \frac{6}{r^2} \tag{6.44}$$

Plugging this into the counterterm (6.36) results in

$$S_{ct} = \frac{4r^3 \mathrm{Vol}(S^3)}{L}\left(1 + \frac{3L^2}{2r^2}\right) \tag{6.45}$$

Adding the bulk action, surface action and counterterm and letting $r \to \infty$ we find

$$S = \frac{2\mathrm{Vol}(S^3)}{L}\left(3L^3 \int_0^{r/L} du \frac{u^3}{\sqrt{1+u^2}} - 3r^2\sqrt{r^2+L^2} + 2r^3\left(1 + \frac{3L^2}{2r^2}\right)\right) \tag{6.46}$$

$$= 4\mathrm{Vol}(S^3)L^2 = 8\pi^2 L^2 \tag{6.47}$$

We remember that we have now shown that for the metric (6.37) the action (6.1) is equal to $8\pi^2 L^2$. However, what we are interested in in the following chapter is not the action, but rather the space-time volume,

$$V = \int_M d^{n+1}x\sqrt{G} \tag{6.48}$$

Comparing with (6.1) and plugging in the values for the Ricci scalar and the cosmological constant, it is easy to derive that

$$V = \frac{4}{3}\pi^2 L^4 \tag{6.49}$$

6.2.2 $S^1 \times S^2$ boundary

Another boundary we will consider is the $S^1 \times S^2$ boundary. The AdS_4 metric with this boundary is given by

$$ds^2 = \left(1 + \frac{r^2}{L^2}\right) d\tau^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_2^2 \tag{6.50}$$

$$= h(r)d\tau^2 + h^{-1}(r)dr^2 + r^2 d\Omega_2^2 \tag{6.51}$$

where

$$h(r) = 1 + \frac{r^2}{L^2} \tag{6.52}$$

The bulk action is easily verified to be

$$S_{bulk} = \frac{2r^3\beta Vol(S^2)}{L^2} \tag{6.53}$$

where β is the length of time circle S^1.

The induced metric is in this case

$$\gamma = h(r)d\tau^2 + r^2 d\Omega_2^2 \tag{6.54}$$

and the normal unit vector is

$$n = \sqrt{h(r)}\frac{\partial}{\partial r} \tag{6.55}$$

which implies that

$$\begin{aligned}\mathcal{L}_n\sqrt{\gamma} =& \mathcal{L}_n\left(\sqrt{h(r)}r^2\right) && (6.56)\\ =& r^2\left(\frac{2}{r}h(r) + \frac{1}{2}\frac{dh(r)}{dr}\right) && (6.57)\end{aligned}$$

Thus the surface action is

$$S_{surf} = -2\text{Vol}(S^2)\beta r^2\left(\frac{2}{r}h(r) + \frac{1}{2}\frac{dh(r)}{dr}\right) \tag{6.58}$$

We calculate the scalar curvature of the induced metric,

$$R[\gamma] = \frac{2}{r^2} \tag{6.59}$$

and plug this into (6.36). This gives

$$S_{ct} = 2\text{Vol}(S^2)\beta r^2\sqrt{h(r)}\left(\frac{2}{L} + \frac{L}{r^2}\right) \tag{6.60}$$

Adding all contributions and letting $r \to \infty$, we find the renormalized action to be

$$S = 0 \tag{6.61}$$

and thus also

$$V = 0 \tag{6.62}$$

6.2.3 $S^1 \times S^2$ boundary with black hole

We can generalize the function $h(r)$ we used in the previous case. If we let

$$h(r) = 1 + \frac{r^2}{L^2} - \frac{r_0}{r} \tag{6.63}$$

where r_0 is a constant, this describes a space-time with the same boundary as the previous space-time, but with a black hole.

Since all the calculations are already performed, we immediately know the renormalized action,

$$S = 2Vol(S^2)\beta\left(\frac{1}{L^2}r^3 - r^2\left(\frac{2}{r}h(r) + \frac{1}{2}\frac{dh(r)}{dr}\right) + \sqrt{h(r)}r^2\left(\frac{2}{L} + \frac{L}{r^2}\right)\right) \tag{6.64}$$

Letting $r \to \infty$, we find that

$$S = 4\pi\beta r_0 \tag{6.65}$$

But this is not the complete picture. Because $h(r_+) = 0$ for some $r_+ > 0$, we must restrict the space-time to the region $r > r_+$. It can be shown that the metric is smooth and complete if and only if[27]

$$\beta = \frac{4\pi L^2 r_+}{3r_+^2 + L^2} \tag{6.66}$$

This means that we get an extra contribution to the action from this boundary at finite $r = r_+$. Solving $h(r_+) = 0$ to find $r_+(r_0, L)$, we find that

$$r_+ = \frac{-24^{\frac{1}{3}}L^2 + 2^{\frac{1}{3}}\left(9r_0L^2 + \sqrt{81r_0^2L^4 + 12L^6}\right)^{\frac{2}{3}}}{6^{\frac{2}{3}}\left(9r_0L^2 + \sqrt{81r_0^2L^4 + 12L^6}\right)^{\frac{1}{3}}} \tag{6.67}$$

The extra contribution to the action coming from this boundary is just

$$S_{r_+} = -8\pi\frac{r_+^3}{L^2}\beta \tag{6.68}$$

Thus the total action is

$$S = 4\pi\beta\left(r_0 - 2\frac{r_+^3}{L^2}\right) \tag{6.69}$$

And the volume is found to be

$$V = \frac{2}{3}\pi\beta\left(r_0 - 2\frac{r_+^3}{L^2}\right) \tag{6.70}$$

which is due to (6.66) and (6.67) only a function of r_0 and L.

7 Free energy

As mentioned before, we want to compare the free energy of $T[SU(N)]$ superconformal field theory, which was quoted in chapter 3, with the free energy of its supergravity dual theory. We showed that the free energy on the supergravity side is equal to the action evaluated using the solutions to the equations of motion. In chapter 4 we found the solutions to the equations of motion of the dual supergravity theory in terms of two real harmonic functions h_1 and h_2. We repeat them here for the reader's convenience,

$$h_1 = -\frac{\alpha' N}{4} \ln\left(\tanh\left(\frac{i\pi}{4} - \frac{z-\delta}{2}\right)\right) + c.c. \tag{7.1}$$

$$h_2 = -\frac{\alpha' N}{4} \ln\left(\tanh\left(\frac{z+\delta}{2}\right)\right) + c.c. \tag{7.2}$$

with $\delta = -\frac{1}{2}\ln\left(\tan\left(\frac{\pi}{2N}\right)\right)$.

This corresponds to a configuration of N D5-branes at the position $z = i\frac{\pi}{2} - \frac{1}{2}\ln\left(\tan\left(\frac{\pi}{2N}\right)\right)$ and one stack of N NS5-branes at $z = \frac{1}{2}\ln\left(\tan\left(\frac{\pi}{2N}\right)\right)$ with N D3-branes stretched between them. For large N we have $\delta \approx \frac{1}{2}\ln\left(\frac{2N}{\pi}\right)$. Since $z = x + iy$ is the complex coordinate describing the Σ space, this allows us to divide the Σ space in three different regions. Region (I) with $x \leq -\delta$, region (II) with $-\delta < x < \delta$ and region (III) with $x \geq \delta$. It will be convenient to rescale x to δx. Thus region (II) is now $-1 < x < 1$.

In the limit of $N \to \infty$ region (II) will have the largest contribution to the action. Therefore we will look for a large N approximation for (7.1) and (7.2) and for the fields in this region. This is done in section 7.1. In section 7.2 we will then explicitly calculate the action in region (II) using this large N approximation, and compare the result with the $T[SU(N)]$ result, and make predictions for the free energy of $T[SU(N)]$ on different space-times.

7.1 Large N approximation

7.1.1 h_1 and h_2 approximation

As stated above, h_1 is given by (7.1). We start by rewriting the hyperbolic tangent,

$$\begin{aligned}
\tanh\left(\frac{i\pi}{4}-\frac{z-\delta}{2}\right) &= \frac{1+ie^{iy}e^{\delta(x-1)}}{1-ie^{iy}e^{\delta(x-1)}} && (7.3)\\
&\approx 1+2ie^{iy}e^{\delta(x-1)} \qquad \text{if } x<1 && (7.4)\\
&= 1-2(\sin y-i\cos y)e^{\delta(x-1)} \qquad \text{if } x<1 && (7.5)
\end{aligned}$$

This allows us to rewrite the logarithm

$$\ln\left(1-2(\sin y-i\cos y)e^{\delta(x-1)}\right)\approx -2\sin ye^{\delta(x-1)}+2i\cos ye^{\delta(x-1)} \qquad \text{if } x<1 \tag{7.6}$$

Plugging this into (7.1) gives

$$\begin{aligned}
h_1 &\approx \alpha' N\sin ye^{\delta(x-1)} \qquad \text{if } x<1 && (7.7)\\
&= \alpha' N\sin y\left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} \qquad \text{if } x<1 && (7.8)
\end{aligned}$$

To approximate h_2 we observe that

$$\begin{aligned}
\tanh\left(\frac{z+\delta}{2}\right) &= \frac{1-e^{-iy}e^{-\delta(x+1)}}{1+e^{-iy}e^{-\delta(x+1)}} && (7.9)\\
&\approx 1-2e^{-iy}e^{-\delta(x+1)} \qquad \text{if } x>-1 && (7.10)\\
\Rightarrow \ln\left(1-2e^{-iy}e^{-\delta(x+1)}\right) &\approx -2(\cos y-i\sin y)e^{-\delta(x+1)} \qquad \text{if } x>-1 && (7.11)
\end{aligned}$$

Together with (7.2) this shows that

$$\begin{aligned}
h_2 &\approx \alpha' N\cos ye^{-\delta(x+1)} \qquad \text{if } x>-1 && (7.12)\\
&= \alpha' N\cos y\left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}} \qquad \text{if } x>-1 && (7.13)
\end{aligned}$$

7.1.2 Metric factors approximation

In chapter 5 we proposed an Ansatz for the metric, which was of the form

$$ds^2 = f_4^2 ds^2_{A_4} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + 4\rho^2 dzd\bar{z} \tag{7.14}$$

If we want to evaluate the action we will need the large N approximation for the factors f_4^2, f_1^2, f_2^2 and ρ^2.

In chapter 4 we expressed all the metric factors in function of the real harmonic function h_1 and h_2. To do this, we used some auxiliary quantities, the definition of which we repeat here.

$$W = \partial h_1 \bar{\partial} h_2 + \partial \bar{h}_1 \partial h_2 \tag{7.15}$$
$$N_1 = 2h_1 h_2 |\partial h_1|^2 - h_1^2 W \tag{7.16}$$
$$N_2 = 2h_1 h_2 |\partial h_2|^2 - h_2^2 W \tag{7.17}$$

Using these expressions, the metric factors are

$$f_4^8 = 16 \frac{N_1 N_2}{W^2} \tag{7.18}$$
$$\rho^8 = \frac{N_1 N_2 W^2}{h_1^4 h_2^4} \tag{7.19}$$
$$f_1^8 = 16 h_1^8 \frac{N_2 W^2}{N_1^3} \tag{7.20}$$
$$f_2^8 = 16 h_2^8 \frac{N_1 W^2}{N_2^3} \tag{7.21}$$

Using (7.8) and (7.13) we write

$$\partial h_1 = \frac{1}{2}\left(\frac{\partial}{\partial \delta x} - i\frac{\partial}{\partial y}\right) h_1 \tag{7.22}$$
$$= \frac{1}{2}\left(\alpha' \sin y N \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} - i\alpha' \cos y N \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}}\right) \tag{7.23}$$

where we remembered that in the h_1 approximation the x coordinate was scaled to δx. The other terms necessary for calculating W are

$$\partial h_2 = \frac{1}{2}\left(\alpha' \cos y N \left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}} + i\alpha' \sin y N \left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}}\right) \tag{7.24}$$
$$\bar{\partial} h_1 = \frac{1}{2}\left(\alpha' \sin y N \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} + i\alpha' \cos y N \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}}\right) \tag{7.25}$$
$$\bar{\partial} h_2 = \frac{1}{2}\left(-\alpha' \cos y N \left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}} - i\alpha' \sin y N \left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}}\right) \tag{7.26}$$

Plugging in (7.23), (7.24), (7.25) and (7.26) into (7.15) leads us to

$$W = -(\alpha' N)^2 \sin y \cos y \frac{\pi}{2N} \tag{7.27}$$

With this expression for W we calculate the approximations for N_1 and N_2,

$$N_1 = \left(\frac{2N}{\pi}\right)^{x-2} (N\alpha')^4 \left(\frac{1}{2}\cos y \sin y (1 + 2\sin^2 y)\right) \tag{7.28}$$
$$N_2 = \left(\frac{2N}{\pi}\right)^{-(x+2)} (N\alpha')^4 \left(\frac{1}{2}\cos y \sin y (1 + 2\cos^2 y)\right) \tag{7.29}$$

We now have enough information to calculate the large N approximations of the metric factors. It is a straightforward calculation to show that

$$f_4^8 = \alpha'^4\pi^2N^2(2-\cos(2y))(2+\cos(2y)) \tag{7.30}$$

$$f_1^8 = 16\alpha'^4\pi^2N^2\frac{2+\cos(2y)}{(2-\cos(2y))^3}\sin^8(y) \tag{7.31}$$

$$f_2^8 = 16\alpha'^4\pi^2N^2\frac{2-\cos(2y)}{(2+\cos(2y))^3}\cos^8(y) \tag{7.32}$$

$$\rho^8 = \frac{1}{16}\alpha'^4\pi^2N^2(2-\cos(2y))(2+\cos(2y)) \tag{7.33}$$

7.1.3 Fields approximation

To evaluate the action we also need large N approximations for the fields. Again, we will repeat the explicit equation expressing these quantities in terms of h_1 and h_2 as given in chapter 4.

$$e^{4\phi} = \frac{N_2}{N_1} \tag{7.34}$$

$$H_{(3)} = \omega^{45}\wedge db_1 \tag{7.35}$$

$$F_{(3)} = \omega^{67}\wedge db_2 \tag{7.36}$$

$$\hat{F}_{(5)} = -4f_4^4\omega^{0123}\wedge\mathcal{F} + 4f_1^2f_2^2\omega^{45}\wedge\omega^{67}\wedge(\star_2\mathcal{F}) \tag{7.37}$$

where ω^{45}, ω^{67} and ω^{0123} are the volume forms of the unit-radius spheres S_1^2, S_2^2 and of A_4, and

$$b_1 = 2ih_1\frac{h_1h_2(\partial h_1\bar{\partial}h_2 - \bar{\partial}h_1\partial h_2)}{N_1} + 2h_2^D \tag{7.38}$$

$$b_2 = 2ih_2\frac{h_1h_2(\partial h_1\bar{\partial}h_2 - \bar{\partial}h_1\partial h_2)}{N_2} - 2h_1^D \tag{7.39}$$

where h_1^D and h_2^D were defined by

$$h_1 = -i(\mathcal{A}_1 - \bar{\mathcal{A}}_1) \to h_1^D = \mathcal{A}_1 + \bar{\mathcal{A}}_1 \tag{7.40}$$

$$h_2 = \mathcal{A}_2 + \bar{\mathcal{A}}_2 \to h_2^D = i(\mathcal{A}_2 - \bar{\mathcal{A}}_2) \tag{7.41}$$

Finally $f_4^4\mathcal{F} = dj_1$ and

$$j_1 = 3\mathcal{C} + 3\bar{\mathcal{C}} - 3\mathcal{D} + i\frac{h_1h_2}{W}(\partial h_1\bar{\partial}h_2 - \bar{\partial}h_1\partial h_2) \tag{7.42}$$

$$\partial\mathcal{C} = \mathcal{A}_1\partial\mathcal{A}_2 - \mathcal{A}_2\partial\mathcal{A}_1 \tag{7.43}$$

$$\mathcal{D} = \bar{\mathcal{A}}_1\mathcal{A}_2 + \mathcal{A}_1\bar{\mathcal{A}}_2 \tag{7.44}$$

The large N approximation for the dilaton is easily obtained from (7.28) and (7.29),

$$e^{4\phi} = \left(\frac{2N}{\pi}\right)^{-2x}\frac{2+\cos(2y)}{2-\cos(2y)} \tag{7.45}$$

Next we want to calculate b_1 and b_2. The only extra ingredient we need is the dual harmonic functions h_1^D and h_2^D. We observe that

$$h_1 = \alpha' N \sin y \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} \tag{7.46}$$

$$= \frac{\alpha' N}{2}(-i)\left(e^{z-\delta} - e^{\bar{z}-\delta}\right) \tag{7.47}$$

$$\Rightarrow h_1^D = \frac{\alpha' N}{2}\left(e^{z-\delta} + e^{\bar{z}-\delta}\right) \tag{7.48}$$

$$= \alpha' N \left(\frac{2N}{\pi}\right)^{\frac{x-1}{2}} \cos y \tag{7.49}$$

An analogous calculation shows that

$$h_2^D = \alpha' \left(\frac{2N}{\pi}\right)^{-\frac{x+1}{2}} \sin y \tag{7.50}$$

Putting everything together leads to

$$b_1 = 4\alpha'\pi \left(\frac{2N}{\pi}\right)^{\frac{1-x}{2}} \frac{\sin^3 y}{2 - \cos(2y)} \tag{7.51}$$

$$b_2 = -4\alpha'\pi \left(\frac{2N}{\pi}\right)^{\frac{1+x}{2}} \frac{\cos^3 y}{2 + \cos(2y)} \tag{7.52}$$

We now have all we need to calculate the fields $H_{(3)}$ and $F_{(3)}$.

The last field we need to calculate is $\hat{F}_{(5)}$. The only unknown in (7.37) is $\mathcal{F}$. Thus we calculate $\mathcal{C}$, $\mathcal{D}$ and j_1.

$$\partial\mathcal{C} = -\frac{(\alpha' N)^2}{2}\frac{\pi}{2N} \tag{7.53}$$

$$\Rightarrow \mathcal{C} = -\frac{(\alpha' N)^2}{2}\frac{\pi}{2N}z \tag{7.54}$$

$$\mathcal{D} = \frac{(\alpha' N)^2}{2}\frac{\pi}{2N}\cos(2y) \tag{7.55}$$

$$\Rightarrow j_1 = \frac{1}{2}(\alpha' N)^2 \frac{\pi}{2N}\left(-6\delta x - 2\cos(2y)\right) \tag{7.56}$$

Using these large N approximations for the metric factors and the fields, we will evaluate the type IIB supergravity action in the next section.

7.2 Action evaluation

The action which we want to evaluate to leading order in N is

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g}\Big[R - 2\partial_M\phi\partial^M\phi - \frac{1}{2}e^{4\phi}\partial_M\chi\partial^M\chi - \frac{1}{2}e^{-2\phi}|H_{(3)}|^2 \\ & - \frac{1}{2}|F_{(3)} - \chi H_{(3)}|^2 - \frac{1}{4}|\hat{F}_{(5)}|^2\Big] - \frac{1}{4\kappa_{10}^2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \end{aligned} \tag{7.57}$$

As mentioned previously in chapter 4, any solution can be transformed into a solution with vanishing axion, i.e. $\chi = 0$. Setting the axion to zero in the the action, and performing an integration by parts on the Chern-Simons term, $\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}$, lets us write the action as

$$S_{IIB} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{g}\left[R - 2\partial_M\phi\partial^M\phi - \frac{1}{2}e^{-2\phi}|H_{(3)}|^2 - \frac{1}{2}e^{2\phi}|F_{(3)}|^2 - \frac{1}{4}|\hat{F}_{(5)}|^2\right] \quad (7.58)$$

$$- \frac{1}{4\kappa_{10}^2} \int F_{(5)} \wedge H_{(3)} \wedge C_{(2)} \quad (7.59)$$

where the ten dimensional coupling constant κ_{10}^2 is related to the string length parameter α' by $2\kappa_{10}^2 = (2\pi)^7(\alpha')^4$

We will evaluate this action term by term. The first term is the integral over the kinetic term of the dilaton,

$$\frac{1}{(2\pi)^7\alpha'^4} \int d^{10}x\sqrt{g}(-2)\partial_M\phi\partial^M\phi \quad (7.60)$$

The derivatives can be rewritten as

$$\partial_M\phi\partial_N\phi g^{MN} = \frac{1}{4\rho^2}\left(\frac{1}{\delta^2}\partial_x\phi\partial_x\phi + \partial_y\phi\partial_y\phi\right) \quad (7.61)$$

$$= \frac{1}{4\rho^2}\left(\frac{1}{4\delta^2}\ln^2\left(\frac{2N}{\pi}\right) + \frac{16\sin^2(2y)}{(\cos(4y) - 17)^2}\right) \quad (7.62)$$

Plugging this into Eq. 7.60, simplifying and evaluating all integrals results in

$$\frac{1}{(2\pi)^7\alpha'^4} \int d^{10}x\sqrt{g}(-2)\partial_M\phi\partial^M\phi = \mathrm{Vol}(A_4)\frac{9(-2+\sqrt{3})}{32\pi^2}N^2\ln\left(\frac{2N}{\pi}\right) \quad (7.63)$$

where $\mathrm{Vol}(A_4) = \int d^4x\sqrt{g_{A_4}}$.

Next up is the $|H_{(3)}|^2$ integral,

$$\frac{1}{(2\pi)^7\alpha'^4} \int d^{10}x\sqrt{g}\left(-\frac{1}{2}\right)\left(e^{-2\phi}|H_{(3)}|^2\right) \quad (7.64)$$

where

$$|H_{(3)}|^2 = \frac{1}{4\rho^2 f_2^4}\partial_\alpha b_1\partial_\beta b_2 g^{\alpha\beta} \quad (7.65)$$

Thus the $|H_{(3)}|^2$ integral is

$$\frac{1}{(2\pi)^7\alpha'^4} \int d^{10}x\frac{f_4^4 f_1^2}{f_2^2}\sqrt{g_{A_4}}\sqrt{g_{S_1^2}}\sqrt{g_{S_2^2}}\left(-\frac{1}{2}\right)e^{-2\phi}\left((\partial_x b_1)^2 + (\partial_y b_1)^2\right) \quad (7.66)$$

$$= \mathrm{Vol}(A_4)\frac{(-8+3\sqrt{3})N^2}{16\pi^2}\ln\left(\frac{2N}{\pi}\right) \quad (7.67)$$

The calculation for the $|F_{(3)}|^2$ is completely analogous, and we only state the result,

$$\frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{g}\left(-\frac{1}{2}\right)e^{2\phi}|F_{(3)}|^2 = \mathrm{Vol}(A_4)\frac{(-8+3\sqrt{3})N^2}{16\pi^2}\ln\left(\frac{2N}{\pi}\right) \tag{7.68}$$

The next part of the integral, the $|\hat{F}_{(5)}|^2$ term, is a bit trickier. The problem is that $|F_{(5)}|^2 = 0$. This is because after the derivation of the field equations from the action, we imposed an extra constraint on the $\hat{F}_{(5)}$ field, namely that it is self-dual, or

$$\hat{F}_5 = \star\hat{F}_5 \tag{7.69}$$

From this we can derive that

$$|\hat{F}_{(5)}|^2 = \hat{F}_{(5)\mu_1\mu_2\mu_3\mu_4\mu_5}\hat{F}_{(5)}{}^{\mu_1\mu_2\mu_3\mu_4\mu_5} \tag{7.70}$$

$$= \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\nu_1\nu_2\nu_3\nu_4\nu_5}\hat{F}_{(5)}{}^{\nu_1\nu_2\nu_3\nu_4\nu_5}F_{(5)}{}^{\mu_1\mu_2\mu_3\mu_4\mu_5} \tag{7.71}$$

$$= -\epsilon_{\nu_1\nu_2\nu_3\nu_4\nu_5\mu_1\mu_2\mu_3\mu_4\mu_5}\hat{F}_{(5)}{}^{\nu_1\nu_2\nu_3\nu_4\nu_5}F_{(5)}{}^{\mu_1\mu_2\mu_3\mu_4\mu_5} \tag{7.72}$$

$$= -|\hat{F}_{(5)}|^2 \tag{7.73}$$

However, we still want a contribution from this term to the action, so we need a prescription on how to do this. Our prescription is to reduce $\hat{F}_{(5)}$ to its components on the A_4 space and double the contribution from $|\hat{F}_{(5)}|^2$ and from the Chern-Simons term. This prescription will be made a bit more plausible in the next section.

The prescription now tells us that

$$\hat{F}_{(5)} = -\frac{\alpha'^2\pi N}{\rho f_4^4}\left(-3e^{01238} + 4\sin y\cos y e^{01239}\right) \tag{7.74}$$

$$\Rightarrow |\hat{F}_{(5)}|^2 = -\frac{\alpha'^4\pi^2N^2}{\rho^2 f_4^8}\left(9 + 16\sin^2 y\cos^2 y\right) \tag{7.75}$$

Furthermore, the $|\hat{F}_{(5)}|^2$ integral becomes

$$\frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{g}\left(-\frac{1}{2}\right)|\hat{F}_{(5)}|^2 = \mathrm{Vol}(A_4)\frac{-2+3\sqrt{3}}{16\pi^2} \tag{7.76}$$

To calculate the Chern-Simons integral, the only extra bit of information we need is that

$$C_{(2)} = b_2\omega^{67} = \frac{b_2}{f_2^2}e^{67} \tag{7.77}$$

and that

$$\hat{F}_{(5)}\wedge H_{(3)}\wedge C_{(2)} = \frac{b_2}{f_2^2}\left(F_{(5)01238}H_{(3)459} - F_{(5)01239}H_{(3)458}\right)e^{0123456789} \tag{7.78}$$

$$= \frac{b_2}{f_2^2}\left(F_{(5)01238}H_{(3)459} - F_{(5)01239}H_{(3)458}\right)\sqrt{g}d^{10}x \tag{7.79}$$

Thus the contribution of the Chern-Simons term is (we need to remember to double the term found in the action because of our prescription)

$$-\frac{1}{(2\pi)^7\alpha'^4}\int \hat{F}_{(5)}\wedge H_{(3)}\wedge C_{(2)} = \mathrm{Vol}(A_4)\frac{8-3\sqrt{3}}{8\pi^2} \tag{7.80}$$

The last term in the action is the R term. Using the Einstein equation of motion Eq. 4.14 we can write this term as a combination of the previous terms. Contracting with the metric we find

$$R = 2P_\mu\bar{P}^\nu + \frac{1}{96}|\hat{F}_{(5)}|^2 + g^{\mu\nu}\left(\frac{1}{8}\left(G_\mu{}^{\rho\sigma}\bar{G}_{\nu\rho\sigma} + \bar{G}_\mu{}^{\rho\sigma}G_{\nu\rho\sigma}\right) - \frac{1}{48}g_{\mu\nu}G^{\rho\sigma\kappa}\bar{G}_{\rho\sigma\kappa}\right) \tag{7.81}$$

Because $|\hat{F}_{(5)}|^2 = 0$ this cancels from the equation and we get

$$R = 2P_\mu\bar{P}^\mu + 3!\left(\frac{1}{4}G^{\nu\rho\sigma}\bar{G}_{\nu\rho\sigma} - \frac{10}{48}G^{\rho\sigma\kappa}\bar{G}_{\rho\sigma\kappa}\right) \tag{7.82}$$

$$= 2P_\mu\bar{P}^\mu + \frac{1}{4}G^{\nu\rho\sigma}\bar{G}_{\nu\rho\sigma} \tag{7.83}$$

To calculate the contribution of this term to the action, we first need to express this in terms of the fields we know. From chapter 5 we remember that

$$P_\mu = \partial_\mu\phi \tag{7.84}$$

It follows immediately from the definition of G in (4.5) that

$$G^{\mu\nu\rho}G_{\mu\nu\rho} = e^{-2\phi}|H_{(3)}|^2 + e^{2\phi}|F_{(3)}|^2 \tag{7.85}$$

Now we can calculate the R term of the action, and it turns out it is equal to

$$\frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{g}R = \mathrm{Vol}(A_4)\frac{34-15\sqrt{3}}{32\pi^2}N^2\ln\left(\frac{2N}{\pi}\right) \tag{7.86}$$

Collecting the results from (7.63), (7.67), (7.68), (7.76), (7.80) and (7.86), we find that the total action is

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g}\left[R - 2\partial_M\phi\partial^M\phi - \frac{1}{2}e^{-2\phi}|H_{(3)}|^2 - \frac{1}{2}e^{2\phi}|F_{(3)}|^2 - \frac{1}{4}|\hat{F}_{(5)}|^2\right] \tag{7.87}$$

$$-\frac{1}{4\kappa_{10}^2}\int F_{(5)}\wedge H_{(3)}\wedge C_{(2)} \tag{7.88}$$

$$= \mathrm{Vol}(A_4)\frac{3}{8\pi^2}N^2\log N + \mathcal{O}(N^2) \tag{7.89}$$

We see that the free energy is proportional to the volume of the A_4 space-time. In section 6.2 we calculated the volume of different space-times, which we can now simply plug into (7.89) in order to get the free energy of the supergravity theory on that particular space-time.

The first volume we consider is the volume of AdS_4 space-time with an S^3 boundary. The volume was found in (6.49), giving a free energy

$$F_{S^3} = \frac{1}{2} N^2 \log N + \mathcal{O}(N^2) \tag{7.90}$$

Since we obtained the free energy for $T[SU(N)]$ on S^3 in (3.21), we can compare the two. We see that both expressions are exactly the same to highest order in N, as predicted by the AdS/CFT correspondence. Thus we have explicitly checked that the AdS/CFT correspondence works in this case, which is one of the main results of this thesis.

Now that we have checked the AdS/CFT correspondence, we can use it to make new predictions. Since we have calculated the volume for AdS_4 with an $S^1 \times S^2$ boundary in (6.62), we can now predict the free energy of $T[SU(N)]$ on $S^1 \times S^2$ to be

$$F_{S^1 \times S^2} = 0 + \mathcal{O}(N^2) \tag{7.91}$$

We also added a black hole to this metric, which allows us to calculate the free energy of $T[SU(N)]$ on $S^1 \times S^2$ at a finite temperature. The volume was found in (6.70). Our solutions used an AdS space-time with unit radius, therefore we have to put $L = 1$. Plugging in the volume we find,

$$F_{S^1 \times S^2 \text{black hole}} = \frac{\beta}{4\pi} \left(r_0 - 2r_+^3 \right) N^2 \log N + \mathcal{O}(N^2) \tag{7.92}$$

Since $L = 1$, we know from chapter 6 that the prefactor $\frac{\beta}{4\pi} \left(r_0 - 2r_+^3 \right)$ can be written as a function of β alone. In figure 7.2 we show the volume of the Schwarzschild AdS space-time with $S^1 \times S^2$ boundary as a function of β. For every β there are two different volumes. This is so because for every temperature we find two black hole solutions of different r_0. The physically relevant solution is of course the solution with the lowest free energy. It is clear from figure 7.2 that for $\beta = \beta_0$ the volume becomes negative, which will correspond to negative free energy. Thus at this point a phase transition occurs, since the normal AdS space-time will no longer have the lowest free energy. It was calculated that $\beta_0 = \pi$.

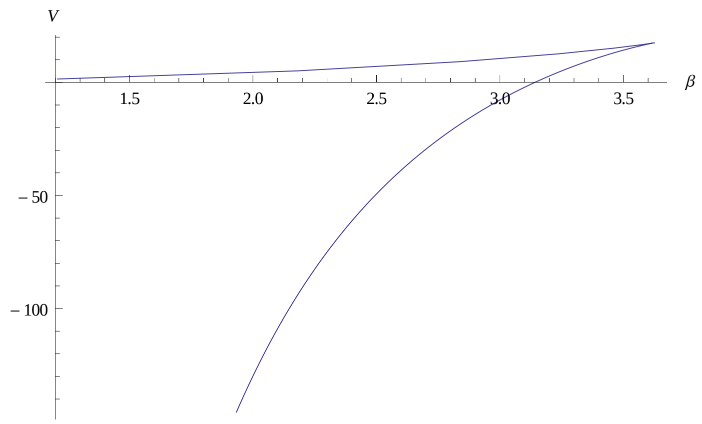

Figure 7.1: The volume of the Schwarzschild AdS space-time with boundary $S^1 \times S^2$ as a function of the inverse temperature β

7.3 $\hat{F}_{(5)}$ prescription

When we were calculating the free energy (7.89), we used an ad hoc prescription to calculate the $\hat{F}_{(5)}$ term of the action. In this section we will make an argument in favour of this prescription. It can be proven to be true, but due to time constrictions we were not able to include the full proof in the thesis.

We start with a simplified type IIB action, given by

$$S = \int dx^{10}\sqrt{g}\left(R + |F_{(5)}|^2\right) \tag{7.93}$$

from which we derive the equation of motion

$$R_{\mu\nu} - \frac{1}{2}(R + |F_{(5)}|^2)g_{\mu\nu} - \frac{1}{4!}F_{(5)\mu\alpha_1\alpha_2\alpha_3\alpha_4}F_{(5)\nu}{}^{\alpha_1\alpha_2\alpha_3\alpha_4} = 0 \tag{7.94}$$

We want a solution where the metric is $AdS_5 \times S_5$ with radius L and $F_{(5)}$ should be self-dual. Therefore we make an Ansatz for $F_{(5)}$,

$$F_{(5)} = -Ae^{01234} + Ae^{56789} \tag{7.95}$$

from which it follows that

$$\frac{1}{4!}F_{(5)\mu\alpha_1\alpha_2\alpha_3\alpha_4}F_{(5)\nu}{}^{\alpha_1\alpha_2\alpha_3\alpha_4} = -g_{\alpha\beta}A^2 + g_{\gamma\delta}A^2 \tag{7.96}$$

where α and β are space-time coordinates in the AdS_5 part of the total space-time and γ and δ are in the S_5 part. Using the fact that the space-time should be $AdS_5 \times S_5$ with radius L, we find that

$$R_{\mu\nu} = -\frac{4}{L^2}g_{\alpha\beta} + \frac{4}{L^2}g_{\gamma\delta} \tag{7.97}$$

where we have the same conventions on α, β, γ and δ. Using these expressions (7.96) and (7.97), and plugging them into (7.94) we find a relation between A and L,

$$A = \frac{2}{L} \tag{7.98}$$

and thus

$$F_{(5)} = -\frac{2}{L}e^{01234} + \frac{2}{L}e^{56789} \tag{7.99}$$

Now we can use this to compute the action, using the prescription of course.

$$\int dx^{10}\sqrt{g}\left(R + |F_{(5)}|^2\right) = \int d^{10}x\sqrt{g}(-2A^2) \tag{7.100}$$

since $R = 0$, and using the prescription we have $F_{(5)} = Ae^{01234} \Rightarrow |F_{(5)}|^2 = -A^2$, and we had to double this contribution. Thus we find that

$$S = -\,2A^2\mathrm{Vol}(S^5)\mathrm{Vol}(AdS_5) \tag{7.101}$$

$$= -\,\frac{8}{L^2}\mathrm{Vol}(S^5)\mathrm{Vol}(AdS_5) \tag{7.102}$$

Alternatively, we could have reduced the ten dimensional integral to a five dimensional one

$$S_{5D} = B \int d^5x \sqrt{g}\,(R - \Lambda) \tag{7.103}$$

where B is a normalization constant, which should be equal to $\mathrm{Vol}(S^5)$ since we "integrated out" the S^5 part of the space-time. Demanding that AdS_5 with the same radius L as before remains a solution will dictate the value of the cosmological constant to be

$$\Lambda = -\frac{12}{L^2} \tag{7.104}$$

We can calculate this action without using the prescription, and find it to be

$$S_{5D} = -\frac{8}{L^2}\mathrm{Vol}(S^5)\mathrm{Vol}(AdS_5) \tag{7.105}$$

which is exactly the same as the ten dimensional result using the prescription.

As mentioned in the beginning of the section, this does not explain why the prescription should work in section 7.2, but it can be checked in the same way we did here that it is correct.

8 Conclusion

The AdS/CFT correspondence, is the conjectured duality between a field theory and a superstring theory, meaning that both theories describe the same physics. Usually appropriate limits are taken, such that we can approximate string theory by supergravity. In this thesis we have studied an example of the AdS/CFT correspondence, between a family of field theories called $T^{\rho}_{\hat{\rho}}[SU(N)]$ and a set of solutions to the equations of motion of type IIB supergravity.

On the field theory side of the duality, we sketched the most important features of $T^{\rho}_{\hat{\rho}}[SU(N)]$ and how the theory is found by brane constructions in type IIB string theory. One of the reasons this family of field theories is interesting, is because the partition function can be calculated exactly at any value of the coupling constant. From this the free energy can be derived, which is done for a special case called $T[SU(N)]$.

On the supergravity side of the duality, we discussed a family of supergravity solutions dual to the $T^{\rho}_{\hat{\rho}}[SU(N)]$ field theories, and thus also of the $T[SU(N)]$ field theory. Using this supergravity solution, we were also able to calculate the free energy on the supergravity side of the duality. One of the implications of the AdS/CFT correspondence is that the free energy should be the same on both sides of the duality. We were able to verify this prediction, which was the first part of the original work for this thesis and a result which cannot yet be found in the literature.

In most checks of the AdS/CFT correspondence in the literature, a quantity is computed using perturbation theory on both sides of the duality. It is then argued that the quantity will not change when it is considered in the strongly coupled regime, for example due to protection by supersymmetry. This allows the comparison of the quantity. In our check of the correspondence no such argument was needed, since the free energy is known at any coupling strength. There are not many examples of this kind of check in the literature.

The second part of the original work was to show that the original supergravity solution is extendable, allowing black holes in its geometry. Introducing a black hole into the supergravity solution corresponds to putting the field theory at finite temperature. Using the AdS/CFT correspondence and the extended supergravity solution we were able to predict the free energy of the field theory $T[SU(N)]$ at finite temperature. A phase transition is found in this strongly coupled field theory. It would be interesting to find real condensed matter systems which are described by this theory.

Future research may include extending the supergravity solution even more, by extending the metric Ansatz to include more general space-times. It would also be interesting to study transport properties of this theory, like the viscosity to entropy density example discussed in the introduction. Another question deserving attention is wether a superconducting phase transition is possible in this system. To answer this we could check if it is possible to introduce a charged black hole into the supergravity solution and if we are able to generate non-zero expectation values for complex scalar fields.

A Differential geometry

This appendix gives a short overview of differential geometry and the conventions used in this thesis. More complete treatments can be found in [28][29].

The basic units to construct differential forms are coordinate differentials, dx^μ. We define the wedge product, also known as the exterior product, as the antisymmetric tensor product,

$$dx^\mu \wedge dx^\nu = \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right) = -dx^\nu \wedge dx^\mu \tag{A.1}$$

Using this definition we can construct differential p-forms, for $p = 1, \cdots, d$ and d the dimension of space-time.

$$F_{(1)} = F_{(1)\mu} dx^\mu \tag{A.2}$$

$$\omega_{(2)} = \frac{1}{2} F_{(2)\mu\nu} dx^\mu \wedge dx^\nu \tag{A.3}$$

$$\vdots \tag{A.4}$$

$$F_{(p)} = \frac{1}{p!} F_{(p)\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{A.5}$$

We can go from a p-form to a $(p+1)$-form using the exterior derivative,

$$dF_{(p)} = \frac{1}{p!} \partial_\mu F_{(p)\mu_1\cdots\mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{A.6}$$

It is also possible to go from a $(p+1)$-form to a p-form using the interior derivative, but we need a vector V for this. The interior derivative is defined as

$$i_V F_{(p)} = \frac{1}{(p-1)!} V^\mu F_{(p)\mu\mu_1\cdots\mu_{p-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{A.7}$$

In the thesis we also often use the contraction of two p-forms,

$$(F, G) = \frac{1}{p!} F_{(p)}{}^{\mu_1\cdots\mu_p} G_{(p)\mu_1\cdots\mu_p} \tag{A.8}$$

Usually the p-form is contracted with the complex conjugate of the p-form, which we denote by $(F_{(p)}, \bar{F}_{(p)}) = |F|^2_{(5)}$.

Suppose now that we have a metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{A.9}$$

We will introduce the frame fields, also called 'vielbeins', by

$$g_{\mu\nu}(x) = e^a_\mu e^b_\nu \eta_{ab} \tag{A.10}$$

where η_{ab} is the Minkowski metric, $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$. The indices a and b are usually called frame indices, while μ and ν are space-time indices. The frame field is often written as a one-form,

$$e^a = e^a_\mu dx^\mu \tag{A.11}$$

Now we can consider tensors with both space-time and frame indices. In general relativity one usually works with covariant derivatives, to ensure that the result transforms like the indices suggest. Since we now work with two kinds of indices, we introduce the covariant derivative which is covariant in both types of indices,

$$D_\mu V^a_\nu = \nabla_\mu V^a_\nu + \omega_\mu{}^a{}_b V^b_\nu \tag{A.12}$$

where $\omega_\mu{}^a{}_b$ is called the spin connection. The spin connection can be found by the first Cartan structure equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{A.13}$$

In general relativity the covariant derivative has to satisfy $\nabla_\alpha g_{\mu\nu} = 0$. Analogously we require that

$$D_\mu e^a_\nu = 0 \tag{A.14}$$

A very important tensor in general relativity is the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$. Using

$$e^\mu_a = \eta_{ab} g^{\mu\nu} e^b_\nu \tag{A.15}$$

we can define the curvature two-form

$$\Omega^a{}_b = \frac{1}{2} R^\lambda{}_{\nu\kappa\mu} dx^\kappa \wedge dx^\mu e^a_\lambda e^\nu_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d \tag{A.16}$$

which can be calculated using the second Cartan structure equation

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{A.17}$$

Finally, we define the Hodge dual of a p-form as a $(d-p)$-form defined by

$$\star F_{(p)} = \frac{1}{p!(n-p)!} \epsilon_{a_1 \cdots a_p b_1 \cdots b_{n-p}} F_{(p)}{}^{a_1 \cdots a_p} e^{b_1 \cdots b_{n-p}} \tag{A.18}$$

where $e^{b_1 \cdots b_{n-p}} = e^{b_1} \wedge \cdots \wedge e^{b_{n-p}}$.

B Nederlandstalige samenvatting

Quantum velden theorie is een theoretisch model dat de laatste jaren groot succes heeft behaald onder meer in de vorm van het standaardmodel, dat in staat is om drie van de vier fundamentele interacties in de natuur te beschrijven. Het standaardmodel gebruikt zogenaamde ijktheorieën voor deze interacties. In een ijktheorie wordt er op natuurlijke wijze een koppelingsconstante ingevoerd, die aangeeft hoe sterk deeltjes met elkaar interageren. De term koppelingsconstante is echter misleidend: de waarde hangt namelijk af van de energie waarop experimenten worden uitgevoerd. Berekeningen in ijktheorieën gebeuren traditioneel door middel van storingsrekening. Dit houdt in dat verondersteld wordt dat de koppelingsconstante klein is, waardoor een lineaire benadering kan gemaakt worden. Wanneer echter experimenten worden uitgevoerd op energieën waarbij de koppelingsconstante groot is, is er tot nog toe geen standaard manier om theoretische voorspellingen te maken.

In deze thesis bestuderen we sterk gekoppelde veldentheorieën door middel van de Anti-de Sitter/Conformal Field Theory correspondentie, afgekort AdS/CFT correspondentie. Deze correspondentie die geponeerd werd in 1998 door Juan Maldacena[4] stelt dat er een dualiteit is tussen een ijktheorie en een theorie die zwaartekracht bevat. Dit betekent dat beide theorieën fysisch equivalent zijn. Eén van de interessantste gevolgen van die dualiteit is dat het over een sterke/zwakke koppelingsdualiteit gaat. Een sterk gekoppelde ijktheorie betekent een zwak gekoppelde zwaartekrachttheorie en visa versa. Deze dualiteit zal ons toelaten om sterk gekoppelde veldentheorieën te bestuderen door middel van conventionele methoden in de zwaartekrachttheorie.

Hoewel de AdS/CFT correspondentie tot nog toe niet bewezen is, zijn er aanwijzingen dat deze juist is. In de literatuur zijn voorbeelden van uitgerekende grootheden die zowel in de ijktheorie als in de zwaartekrachttheorie overeenkomen. Beide grootheden werden uitgerekend door middel van storingsrekening. Er werd gebruik gemaakt van symmetrie om te argumenteren dat deze waarde niet zal veranderen in het sterk gekoppelde regime. In deze thesis is de AdS/CFT correspondentie geverifieerd zonder gebruik te maken van dergelijke symmetrie argumenten.

In hoofdstuk 3 bespreken we een familie van veldentheorieën, waarvan we een speciaal geval, $T[SU(N)]$ genaamd, bestuderen. Deze theorie is interessant omdat de vrije energie gekend is voor een arbitraire koppelingsconstante, wat ons in staat stelt om een expliciete verificatie van de AdS/CFT correspondentie te verkrijgen.

In hoofdstuk 4 bespreken we de duale zwaartekrachttheorie. Een eerste deel van het originele werk gerapporteerd in deze thesis is het berekenen van de vrije energie van de zwaartekrachttheorie. Dit resulteert in exact dezelfde vrije energie als die van de veldentheorie. Hiermee bevestigen we dus de AdS/CFT correspondentie.

Om de zwaartekrachttheorie te bekomen werd er een Ansatz gemaakt voor de velden en de metriek van de theorie. In hoofdstuk 5 presenteren we het tweede originele deel van dit werk: we tonen aan dat deze Ansatz kan uitgebreid worden, zodat de metriek zwarte gaten kan bevatten. Het invoeren van een zwart gat in de zwaartekrachttheorie komt overeen met de duale veldentheorie op een eindige temperatuur. Gebruik makend van de AdS/CFT correspondentie kunnen we dan de vrije energie van de veldentheorie op eindige temperatuur voorspellen. Dit wordt besproken in hoofdstuk 7. We stellen vast dat er zich een faseovergang voordoet in de sterk gekoppelde veldentheorie.

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