CHILE[*]

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Abstract

We introduce an asymmetric-information asset-pricing framework with general utilities and payoffs, called a "Continuous Heterogeneous Information Large Economy" (CHILE). With log-normal payoffs, the CHILE equilibrium is log-linear and in closed form. We endogenize how wealth is distributed across traders by equating its pre- with its post-trade distribution. With CRRA utilities, such "trade-invariant" distributions obey power laws. We also document a spiral between wealth inequality and price efficiency: widening inequality lowers efficiency, which in turn widens inequality even more. Policies aiming to improve efficiency must be designed carefully, because neglecting this spiral can have unanticipated effects on both inequality and efficiency.

Keywords: inefficient markets, information aggregation, rational expectations with non-CARA preferences, wealth effects

JEL Codes: D01, D53, D82, E19, G12, G14.

1 Introduction

How do risk preferences affect market inefficiency? This straightforward question, fundamental as it may be, is also deceptively difficult to answer. As it is widely recognized, standard approaches lack tractability, making the treatment of market inefficiency with general preferences look like an unattainable goal. To gain enough tractability, most of the literature on asymmetric-information asset pricing has been adopting traders with constant absolute risk aversion (CARA) utility functions, and asset payoffs with Normal distributions—a framework known as the CARA-Normal "workhorse." However, while this framework has shown its mettle on many important questions, there are at least two modest research goals that it cannot help us with: incorporating wealth effects, and pushing our understanding beyond linear prices.[1]

Aiming to address these issues, we introduce a tractable framework with heterogeneous traders and general preferences. Our setup begins with a large economy modeled as a continuum of traders indexed by $a \in [0, 1]$, with trader characteristics (risk aversion, signal precision, and so on) represented as arbitrary continuous functions of a. While this framework turns out to be flexible enough for general probability distributions—see our Appendix C for details—if we restrict attention to log-normally distributed assets we get a log-linear equilibrium where all quantities are in closed-form.

This kind of tractability is rarely seen in models with fully-heterogeneous agents and/or non-CARA utilities. What enables it in ours is the way we model information. In contrast to the traditional methodology for large markets (Hellwig, 1980, and consequent literature) we do not assume that traders have signals of finite precision, because that would imply that as the number of traders becomes large, so does the total amount of information.[2] What is more, with signals of finite precision, traders make finite speculative trades, with the unfortunate

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[1] Notable papers that go beyond the CARA-Normal framework (while still requiring other assumptions for tractability) include Peress (2004), Peress (2014), Breon-Drish (2015), Malamud (2015) and Chabakauri, Yuan, and Zachariadis (2022). While allowing for non-Gaussian distribution, both Breon-Drish (2015) and Chabakauri et al. (2022) require CARA utility and thus do not incorporate wealth effects. Malamud (2015) requires complete markets. Peress (2004) and Peress (2014) achieve tractability by requiring the risk of assets to be small. See Section 6 for more details.

[2] As the total amount of information is the precision of the sufficient statistic of private signals, it equals the sum of signal precisions held by all traders. Thus, in economies where the precision of each signal is finite (as in "neither infinite nor infinitesimal," i.e. neither infinitely large nor infinitely small) the total amount of information grows directly in the number of traders.

consequence that aggregate demand would explode for large numbers of traders.

We instead use an assumption similar to Section 9 in Kyle (1989), whereby the total amount of information is distributed among all traders. By definition, then, the total amount of information is always finite, irrespective of the number of traders. In addition, as traders base their demands on signals with precision inversely related to the size of the economy, aggregate demand remains finite even when the number of traders becomes infinite.

A key contribution of our paper is to extend the above information structure from Kyle (1989), turning it into a general formalism for economies with continuums of small heterogeneous signals. As it turns out, the right formalism uses a type of stochastic calculus, where the dimension that typically represents time is "transposed" to represent traders.[3]

To fix ideas, we offer the following example. Suppose we represent a continuum of agents as a unit interval. Using a to denote one of the agents, let us imagine that a lives on a segment of size da, and that he observes the signal

$$ds(a) = v\, da + dB(a),$$

where v is the fundamental value of a traded asset, and where $dB(a)$ is an increment of a Brownian Motion. The aggregate sum of all signals—a sufficient statistic for all private signals— is an Itô integral, a key property that gives us access to the full arsenal of stochastic calculus. But perhaps even more importantly, this type of aggregation also allows us to think of the noise component of the sufficient statistic as consisting of a large number of small idiosyncratic shocks, resembling what Black (1986) calls "noise in the sense of a large number of small events."

Our equilibrium has two main properties that are hard to analyze without our framework. First, the distribution of wealth across different traders affects important equilibrium quantities, such as price efficiency. Using the case of homogeneous precisions and Constant Relative Risk Aversion (CRRA) utilities as an illuminating benchmark, we show that price informativeness is a decreasing function of the coefficient of variation of wealth, the ratio of the cross-sectional standard deviation of wealth over its cross-sectional mean. In short, economies with greater

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[3]See Gârleanu, Panageas, and Yu (2015) for a similar trick applied to firms rather than traders.

wealth inequality have less informative prices.

The intuition is best conveyed by noting that prices reflect the weighted average signal of all traders, with traders' weights proportional to their trading intensities. In the absence of wealth effects, a population of homogeneous traders (same risk aversion, same signal precision) would generate a price that conveys information through an equally-weighted average of trader signals—a weighting scheme which also happens to generate the most accurate possible posteriors for all traders. With wealth effects, however, trading intensities increase in wealth. As a result, the price no longer conveys an equally-weighted average of signals, even if all traders have identical risk aversions and signal precisions. Instead, the price function emphasizes the signals of richer traders more than those of poorer traders, for reasons unrelated to signal accuracy, effectively creating an economy of "wealth-based discrimination for signals." In terms of price efficiency, this discrimination does not benefit anyone. Quite the opposite, by emphasizing the information of richer traders more, wealth inequality distorts the information content of prices away from the purely precision-based weights, thereby lowering price informativeness.

Second, because price efficiency affects speculative profits traders can make, it also affects how wealth is reallocated during trade, changing wealth inequality. Moreover, as we have just seen above, the relationship between price efficiency and wealth inequality also runs the other way around, giving us two parts of the same loop: wealth inequality determines price efficiency, which then determines wealth inequality. An interesting question therefore arises: how both price efficiency and wealth inequality are determined in equilibrium? To endogenize the wealth distribution, we require it to be "trade-invariant" i.e., such that the distribution of wealth across traders does not change as they trade.[4] This way, we obtain two distinct equilibrium objects: in addition to the informational efficiency of prices we typically see in the literature (a scalar), we must now obtain the probability distribution of wealth (a function). While solving for price efficiency works as usual, solving for the wealth distribution requires a tool from stochastic calculus known as the Kolmogorov Forward Equation (KFE). Our results indicate that with Constant-Relative-Risk-Aversion (CRRA) utility, the KFE is a familiar one: it has the same

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[4]This is indeed possible, even though the wealth of each individual trader *does* change. All we need is that, as agents trade, their new wealth values, viewed as a random variable, are drawn from the same distribution as their old wealth values.

form as equations in random-growth models (for a review, see Gabaix, 2009). An immediate implication is that if preferences are CRRA, the trade-invariant distribution of wealth obeys a power law.

Finally, we also study how wealth inequality and price efficiency interact in the overall equilibrium, where both are endogenous. We uncover the following inequality-inefficiency spiral. Greater wealth inequality lowers price efficiency. Lower price efficiency, in turn, makes more room for speculative profits, which disproportionately benefits those traders who act more aggressively—the wealthier traders. This lop-sided response widens inequality relative to exogenous wealth distributions. As a result, policies aiming to improve price efficiency must be designed carefully because neglecting how inequality feeds back into efficiency can backfire. For example, while increasing transparency—i.e. making it easier to acquire private information—has a positive direct effect, it also has a negative indirect effect, boosting the trading benefits that wealthier, more aggressive traders can enjoy from their improved information. Widening inequality, this indirect effect can trump the direct one, decreasing price efficiency despite transparency improvements.

2 Setup

Our economy unfolds over two time periods, $t = 1$ and $t = 2$. It consists of a continuum of agents of total mass one, described in more detail below. There are two assets: one risk-free, and one risky, with the returns of both assets realized at $t = 2$. The risk-free asset has perfectly elastic supply and gross return normalized to 1, while the risky asset has liquidation value

$$V(v), \ v \sim N\left(0, \tau_v^{-1}\right).$$

Our framework is tractable for a general function $V(\cdot)$. Nevertheless we focus on utility-driven (rather than distribution-driven) effects in the main text, assuming that the risky payoff is log-normal, so that

$$V(v) = \exp(v).$$

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We explore general functions $V(\cdot)$ in Appendix C.

We model our agents as points in the interval $[0, 1)$. We use two types of agent, *rational traders* and *noise traders*. We adopt the convention that noise traders live in $\in [0, \nu)$ and rational traders live in $\in [\nu, 1)$, implying that the overall fraction of noise traders is ν.

Our agents trade at $t = 1$, determining the price P of the risky asset.[5] Agent a has initial wealth $W_0(a)$, and he trades a total amount of x *dollars* (i.e. price per share times number of shares). His realized utility is

$$u(W, a),$$

where his wealth at time $t = 2$ is

$$W = W_0(a) + x(R - 1),$$

and where

$$R = V/P$$

is the return of the risky asset in gross terms.[6] We do not impose any restrictions on the above other than that the expected utility $E[u(W, a)]$ of agent a is thrice differentiable with respect to x in a small open neighbourhood around $x = 0$.

The last primitive of our model is a standard Brownian Motion $B(\cdot)$ on the interval $[0, 1)$, running through the "cross-section of traders", that is, through the interval where the agents are located. We will make heavy use of $B(\cdot)$, as it will provide the price noise in our economy.

In what follows, we micro-found the key quantities of our continuous equilibrium as limits of corresponding quantities of an "underlying" discrete economy built on traditional economic primitives. We begin by describing our limit economy in loose terms in our next section, providing our readers with "user-friendly heuristics" involving Brownian differentials. We tighten up our exposition in the sequel (section 2.2), describing the discrete economy and showing how we take the limit.

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[5]Adding information acquisition at $t = 0$ to our framework is straightforward, and we do so in Appendix D.

[6]As we see here, each agent is heterogeneous in two dimensions, initial wealth $W_0(a)$, and preferences $u(\cdot, a)$. We discuss our third and final dimension of heterogeneity, signal precision, in the next section.

2.1 A Heuristic Presentation of the Continuous Economy

A given agent $a \in [0, 1)$ has initial wealth $W_0(a)$ and utility function over final wealth $u(\cdot, a)$. He lives in segment $[a, a + da)$, with his location determining his type as discussed above.

A rational trader a observes the signal

$$ds(a) = v \, da + \frac{1}{\sqrt{t(a)}} dB(a), \, a \in [\nu, 1), \tag{1}$$

where $B(\cdot)$ is the Brownian motion introduced earlier. Thus, conditional on v, trader a's signal is normally distributed with precision

$$t(a)da.$$

To see why, we can (loosely) argue that $ds(a)$ has the same informational content as a signal written in the "truth plus noise" form $v + dB(a)/\left(\sqrt{t(a)}da\right)$. As the precision of a signal is the reciprocal of the variance of the noise, we can think of the precision of (1) as $t(a)(da)^2/\text{Var}\,(dB(a)) = t(a)da$. To simplify exposition from here on, we will abuse terminology, and refer to $t(a)$ as the *precision of trader a*.

An agent a, $a \in [0, \nu)$ believes he is observing the process in (1), but his actual signal is pure noise. We stress that even though noise trader a is wrong about his signal, he has correct beliefs about everything else in the economy, most importantly about the equilibrium price function. Conforming to the notion of Fisher-Black noise traders (see Black, 1986), we model his "signal" as

$$ds(a) = \frac{1}{\sqrt{t(a)}} dB(a), \, a \in [0, \nu). \tag{2}$$

In the interest of brevity, we refer to the $t(a)$ of this noise trader as *precision* if the context allows it, while we refer to it as *perceived precision* if the context requires clarity.[7]

Finally, if we think of stochastic processes as evolving over the agent interval $[0, 1)$, we can see that the "aggregate-signal" process, $s(a) = \int_0^a ds(b)$, is a continuous martingale conditional on v. We refer to economies with continuous-martingale aggregate signals and heterogeneous

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[7]As a modeling choice, our assumption in (2) imposes discipline on the degree of irrationality of the noise traders. It can be relaxed with little loss of general tractability, albeit at the expense of losing our closed form.

traders as economies with *continuous-and-heterogeneous* (CH) information, and economies with an infinite number of traders as *large economies* (LE). Combining these two acronyms, we label the class of models discussed in this paper as "continuous-and-heterogeneous information in a large economy" (CHILE).

2.1.1 Key Insight: Linear Aggregation

Our continuous economy is tractable due to one key property: the aggregate demand is a linear function of the traders' signals. We obtain this result heuristically next, providing some intuition for the limit we later derive in detail.

Consider the demand of agent a with infinitesimal mass da. This agent lives in segment $[a; a + da)$ and observes the signal $ds(a)$, so for a price p we can (loosely) think of his demand function as

$$X(p, ds; da, a).$$

Applying the usual 'box calculus' heuristics $da^2 = 0$, $da \cdot dB = 0$, and $dB^2 = da$ (see, e.g., Steele, 2001), we can Taylor-expand the function X in two dimensions—signal and mass—around the no-signal-no-mass point, while ignoring terms of order da^2 and higher. We get

$$\begin{aligned}
X(p, ds; da, a) &= X(p, 0; 0, a) + X_a(p, 0; 0, a)da + X_s(p, 0; 0, a)ds(a) + \frac{1}{2}X_{ss}(p, 0; 0, a)ds(a)^2 \\
&= X(p, 0; 0, a) + X_a(p, 0; 0, a)da + X_s(p, 0; 0, a)ds(a) + \frac{1}{2}X_{ss}(p, 0; 0, a)\frac{da}{t(a)}. \quad (3)
\end{aligned}$$

Crucially, the quadratic term $ds(a)^2$ simplifies to $da/t(a)$, implying that the only source of stochastic non-linearity becomes non-stochastic in the limit.

As we show more rigorously below, $X(p, 0; 0, a)$, the demand of a small trader located at a in the limit of zero information, is zero in equilibrium—if it was not, aggregate demand would sum up infinitely many terms of positive size, and it would explode. Finally, the aggregate demand of all traders in segment $[0, y)$ is a stochastic process, given as the aggregation of the

last three terms in (3):

$$\int_0^y X_s(p,0,0,a)ds(a) + \int_0^y \left(\frac{1}{2t_a} X_{ss}(p,0,0,a) + X_m(p,0,0,a) \right) da.$$

2.2 The Continuous Economy as a Limiting Discrete Economy

Here we describe a finite discrete economy, composed of n agents, which becomes a CHILE as n goes to infinity. The agents of this economy are located in disjoint neighbouring segments, forming a partition of the interval $[0,1)$. A particular agent i, $i = 1,2,\ldots,n$, lives in segment $[a^i, a^i + m^i)$, where m_i denotes his size (or *mass*, a positive and finite number), with $a^i = \sum_{j<i} m_j$ and $\sum_i m_i = 1$. The wealth of agent i is $W(a^i)$ and his utility function is $u(\cdot, a^i)$. Noting that i's index and his location a^i are one-to-one, we adopt the convention that if $a^i \in [0,\nu)$, then i is a noise trader, whereas if $a^i \in [\nu, 1)$ then he is a rational trader.

A noise trader a^i observes the signal

$$\Delta s^i = \frac{1}{\sqrt{t(a^i)}} \left(B(a^i + m) - B(a^i) \right), \tag{4}$$

but he acts as if his signal is

$$\Delta s^i = v \cdot m + \frac{1}{\sqrt{t(a^i)}} \left(B(a^i + m) - B(a^i) \right). \tag{5}$$

A rational trader a^i, on the other hand, observes the signal in (5) and behaves in an optimal manner, as described below.

We use X to denote the agents' demand function. As our agents are uniquely identified by their segments, the demand of the agent in $[a, a+m)$ is completely determined by the price p, his realized signal Δs, his size m, and his location a. We thus write

$$X(p, \Delta s; m, a)$$

for this agent's demand, where we note that his location a also captures the agent's heteroge-

neous characteristics, such as wealth, utility, and precision.

2.2.1 The Aggregation Lemma

As we highlight with the following lemma, the key reason for the tractability of our analysis is that as traders become smaller in size, the aggregate demand in our discrete economy above converges to a stochastic integral. This type of aggregation allows us to think in terms of functions, derivatives, and continuous stochastic processes, instead of having to deal with the intractability associated with pointwise sequences of discrete random variables.

Lemma 1. (Aggregation lemma) *Consider a continuous function $X(p, s; m, a) : \mathbb{R} \times \mathbb{R} \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ with continuous partial derivatives X_s, X_{ss} and X_m. Take a partition $[0, y) = \cup_{i=1}^{n}[a^i, a^i + m^i)$, and let $m = \min_i m_i$, $\Delta s^i = s(a^{i+1}) - s(a^i)$, and $\Delta X^i = X(p, \Delta s^i, m, a^i) - X(p, 0, 0, a^i)$. Finally, suppose that there exists an $M > 0$ such that $|\sum_{i=1}^{n} X^i| < M$ for any n. Then, for any partition such that $m \rightarrow 0$ as $n \rightarrow \infty$, we have that*

$$\sum_{i=0}^{n-1} \Delta X^i \rightarrow \qquad (6)$$
$$\int_0^y X_s(p, 0, 0, a)ds(a) + \int_0^y \left(\frac{1}{2t_a} X_{ss}(p, 0, 0, a) + X_m(p, 0, 0, a) \right) da.$$

In what follows we will write that

$$\Delta X \xrightarrow{agg} dX = X_s(p, 0, 0, a)ds(a) + \left(\frac{1}{2t(a)} X_{ss}(p, 0, 0, a) + X_m(p, 0, 0, a) \right) da$$

whenever (6) holds. The idea behind the notation is that whenever $\Delta X \xrightarrow{agg} dX$ the *aggregation* of ΔX and dX is the same in the limit.

The above lemma underlies the tractability of our analysis. Assuming that the demand of each trader becomes 0 as his size goes to 0 (otherwise aggregate demand explodes), we have $\Delta X(a) = X(a)$. Denoting $\beta(p, a) = X_s(p, 0, 0, a)$ and $\aleph(p, a) = \frac{1}{2t(a)} X_{ss}(p, 0, 0, a) +$

$X_m(p, 0, 0, a)$ the aggregation lemma tells us that

$$X(a) \xrightarrow{agg} \beta(p, a)ds(a) + \aleph(p, a)da.$$

This further implies that for any payoff function $V(\cdot)$, our economy would feature a generalized linear equilibrium: given the price, one can always compute the sufficient statistic $\int_0^1 \beta(p, a)ds(a)$, which, as a linear combination of signals of rational traders and noise traders, is conditionally normally distributed. This conditional normality simplifies inferences significantly, overcoming a common obstacle to extending asymmetric information models beyond the CARA-normal framework. Conveniently, our log-normal setup comes with an additional simplification, stemming from that $\beta(p, a)$ does not depend on p, and that $\aleph(p, a) = \alpha(a) - \gamma(a)p$ is affine in p.

Remark 1. The condition $|\sum_{i=1}^n X^i| < M$ is not restrictive, since the aggregate demand is bounded by market-clearing.

2.3 Definition of equilibrium

We define an equilibrium as follows.

1. We consider log-linear equilibria where, with $p = \log P$, the dollar demand of traders in the interval $[a, a + da)$ is

$$dx(a) = \alpha(a) + \beta(a)ds_a - \gamma(a) \, p \, da. \tag{7}$$

   In rigorous terms, the above means that the aggregate demand of traders in some segment $\mathcal{A}$ is $\int_{\mathcal{A}} dx_a$.

2. We interpret $dx(a)$ as the demand of a trader that observes the infinitesimal signal ds, with precision $t(a)da$. We define $dx(a)$ as the limit of the demand function $x(\Delta s(a), p, m, a^i)$ of a trader that observes a finite signal $\Delta s(a) = s(a + m) - s(a)$ with precision $t(a)m$. We position the trader a in the interval $[a; a + m)$, and we move all traders $b > a$ by $+m$

and put them in the interval $[a + m; 1 + m)$. We use $\mathcal{A}^{-a}$ and $\mathcal{A}_r^{-a}$ to denote the set of all traders, and rational traders, excluding a

$$\mathcal{A}^{-a} = [0, 1 + m) \setminus [a, a + m) \text{ and } \mathcal{A}_r^{-a} = [\nu, 1 + m) \setminus [a, a + m).$$

We define $x(\Delta s(a), p, m, a)$ as the best response to all other traders' demands, where all other traders form a continuum. When computing the best response, each trader assumes away the market impact. By aggregation lemma, the demand of other traders in the continuous limit can be written as $d\hat{x}(a) = \hat{\beta}(p, a, m)ds(a) + \hat{\aleph}(p, a, m)da$. We require that the demand of other traders converges to its' limit postulated by (7): $\lim_{m \to 0} \hat{\beta}(p, a, m) = \beta(a)$, and $\lim_{m \to 0} \hat{\aleph}(p, a, m) = \alpha(a) - \gamma(a)p$. We also require that the demand of the trader of interest converges to the one postulated by (7): $x(\cdot, a) \xrightarrow{agg} dx(a)$. More formally:

(a) The demand $x(\Delta s(a), p, m, a)$ solves

$$x(p, \Delta s(a); m, a) = \arg \max_x \{E\left[u\left(W_0(a) + x\left(R - 1\right), a\right) | p, \Delta s(a)\right]\} \qquad (8)$$

Where all traders act as if they get $\Delta s(a)$ given by (5) but noise traders $(a \in [0, \nu))$ actually get (4). The p must clear the market:

$$\int_{\mathcal{A}^{-a}} (\hat{\beta}(a, p; m)ds(a) + \hat{\aleph}(a, p, m)da) = 0.$$

Note that each trader neglects his influence on market clearing and assumes that the price function is generated by all other traders in the economy.

(b) $\lim_{m \to 0} \hat{\beta}(p, a, m) = \beta(a)$, and $\lim_{m \to 0} \hat{\aleph}(p, a, m) = \alpha(a) - \gamma(a)p$

(c) $x(p, \Delta s(a); m, a) \xrightarrow{agg} dx_a = \alpha(a) + \beta(a)ds(a) - \gamma(a) \cdot pda$, which means that for a partition with $m \to 0$ we have,

$$\sum_{i=0}^{n-1} x(p, \Delta s(a); m^i, a) \xrightarrow[n \to \infty]{} \alpha(a) + \beta(a)ds_a - \gamma(a) \cdot pda.$$

Remark 2 (Notation). We will often use "hat" to denote an equilibrium quantity X in the discrete economy with the size of the agent m by $\hat{X}(m)$. Its limit in the continuous economy is

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denoted without hat, $X = \hat{X}(0)$. An example is $\hat{\alpha}(a; 0) = \alpha(a)$ above.

Remark 3. In our equilibrium definition, each trader assumes he has no impact on the price, both in terms of the price level and the informational content of the price. Both of these assumptions constitute a small mistake. Given a trade $dx(a)$, the price changes by λdx, where λ is finite. Similarly, $\text{cov}(ds, p)$ is of order da. These small mistakes aggregate and do not wash away. Because of that, the equilibrium is well-defined even when the mass of noise traders is zero. We also note that it would be hard for traders to prove themselves wrong, as it would be hard to statistically distinguish between $\text{cov}(ds, p)$ of order da and $\text{cov}(ds, p)$ equal to zero.[8]

3 Equilibrium characterisation

3.1 Inference

From the perspective of each agent in the discrete economy, the price function satisfies

$$\int_{\mathcal{A}^{-a}} \left(\hat{\beta}(b, p, m)ds(b) + \hat{\aleph}(p, b, m)db \right) = 0, \tag{9}$$

Substituting (1) and (2) into (9), integrating, and rearranging gives

$$s_p \equiv v + \frac{\int_{\mathcal{A}^{-a}} \hat{\beta}(a; m)/\sqrt{t(a)}dB(a)}{\int_{\mathcal{A}_r^{-a}} \hat{\beta}(a; m)da} = \frac{\int_{\mathcal{A}_r^{-a}} \hat{\aleph}(p, a, m)}{\int_{\mathcal{A}^{-a}} \hat{\beta}(p, a, m)}. \tag{10}$$

We note that the form "truth plus noise" shows up again, this time for the signal s_p defined as the expression on the left-hand side of (10), with truth being v and the noise being $\left(\int_{\mathcal{A}^{-a}} \hat{\beta}(a; m)/\sqrt{t(a)}dB(a) \right) / \left(\int_{\mathcal{A}_r^{-a}} \hat{\beta}(a; m)da \right)$. This noise, in particular, is normally distributed with mean zero and variance

$$\frac{\int_0^1 \hat{\beta}(a; m)^2/t(a)da}{\left(\int_\nu^1 \hat{\beta}(a; m)da \right)^2},$$

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[8]In ongoing work, we also explore a setting where traders behave consistently with competitive REE: they ignore their impact on prices, but they account correctly for the correlation between prices and signals. Our results still hold in this setting, except that the equilibrium wealth distribution is no longer a power law.

a result that follows from standard properties of stochastic integration.[9] We summarize the information content of prices in the following lemma.

Lemma 2. *The information content of prices is the unbiased signal s_p defined in (10). Given the price p, the realization of this signal can be computed as shown on the right-hand side of (10), while its precision can be written as*

$$\hat{\tau}_p(m) = \frac{\left(\int_{\mathcal{A}_r^{-a}} \hat{\beta}(a;m)da\right)^2}{\int_{\mathcal{A}^{-a}} \frac{\hat{\beta}(a;m)^2}{t(a)}da},$$

Moreover,

$$\lim_{m \to 0} \hat{\tau}_p(m) \equiv \tau_p = \frac{\left(\int_{\nu}^{1} \beta(a)da\right)^2}{\int_{0}^{1} \frac{\beta(a)^2}{t(a)}da}.$$

We now turn to deriving the first-order condition (FOC), by first writing the choice problem of trader a as

$$\max_{x} E\left[u\left(W_0(a) + x(R-1), a\right)|\Delta s(a), s_p\right], \tag{11}$$

in a finite economy, and then obtaining the equilibrium conditions for a large economy by clearing the market, combining it with Bayesian inferences, and taking the size of each trader down to zero.

Lemma 3. *There exists a unique best response for trader i. It solves (11) and satisfies the FOC*

$$E\left[u'\left(W_0(a) + x(R-1), a\right)(R-1)|\Delta s(a), s_p\right] = 0,$$

which has a unique solution.

The conditional distribution of v given $\mathcal{F}^a = \{\Delta s(a), s_p\}$ is $N\left(E^a[v](\Delta s(a), m), 1/\hat{\tau}^a(m)\right)$, where $\hat{\tau}^a(m)$ can be computed in closed form for any m, but the solution is particularly simple for $m = 0$ (continuous economy limit), which is the only thing we need going forward.

$$\tau = \hat{\tau}^a(0) = \text{Var}[v|p]^{-1} = \tau_v + \tau_p.$$

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[9]For a deterministic function f, the stochastic integral $\int_0^1 f(a)dB(a)$ is normally distributed with mean zero and variance $\int_0^1 f(a)^2 da$.

Note that $\tau(0)$ is the same for all agents, hence we do not use superscript i.

Similarly,

$$E^a(\Delta s(a), m) \equiv E[v | \Delta s(a), p] = \hat{c}_{\bar{v}}(m)\bar{v} + \hat{c}_s(m)\Delta s(a) + \hat{c}_p(m)s_p,$$

where s_p is given by the right-hand side of (10). Again, even though they can be calculated in closed form for any m, we will only need the value of the coefficients $\hat{c}_{\bar{v}}(m)$, $\hat{c}_s(m)$ and $\hat{c}_p(m)$ at 0 :

$$c_{\bar{v}} = \frac{\tau_v}{\tau}, \; c_s = \frac{t(a)}{\tau}, \text{ and } c_p = \frac{\tau_p}{\tau}.$$

Given the information $\mathcal{F}(a) = \{\Delta s(a), p\}$, the random variable $z = \frac{v - E[v | \Delta s(a), p]}{\sqrt{\text{Var}[v | \Delta s(a), p]}}$ is standard normal. Then, we can write

$$v \equiv v^a(p, \Delta s(a), m, z) = E[v | \Delta s(a), p] + z\sqrt{\text{Var}[v | \Delta s(a), p]}, \text{ where } z|\mathcal{F}(a) \sim N(0, 1).$$

Substituting v^a instead of v to (3) yields

$$\int \left(\begin{array}{c} u'\left(W_0(a) + (\exp\left(v^a(p, \Delta s(a), m, z) - p\right) - 1)x, a\right) \\ \cdot (\exp\left(v^a(p, \Delta s(a), m, z) - p\right) - 1) \end{array} \right) d\mathcal{N}(z) = 0, \qquad (12)$$

where $\mathcal{N}(z)$ denotes standard normal CDF. Substitution of v^a instead of v is convenient as it makes explicit the dependence of conditional distribution of v on $\Delta s(a)$ and m. Eq. (12) defines implicitly the demand

$$X(p, \Delta s(a), m, a).$$

In order to proceed we make several observations summarized in the following lemma.

Lemma 4. *The equilibrium exists only if $X(p, 0, 0, a)$ (the limit of $X(p, \Delta s(a), m, a)$, defined by (12) as $m \to 0$) is zero. This implies that*

$$E[R|P] = 1, \qquad (13)$$

$$\frac{\tau_p}{\tau} \frac{\int_0^1 \gamma(a)da}{\int_\nu^1 \beta(a)da} = 1, \tag{14}$$

and

$$\exp\left(\frac{\tau_v}{\tau}\bar{v} - \frac{\tau_p}{\tau}\frac{\int_0^1 \alpha(a)da}{\int_0^1 \gamma(a)da} + \frac{1}{2\tau}\right) = 1.$$

We comment here on the condition (13). In our limiting continuous economy the market is efficient. This makes sense: if it is not, then each trader would make a finite speculative trade and the aggregate demand would explode. What's more, in our economy the risk premium is zero. This is because we assume that the asset is in zero supply. One can introduce risk premium by endowing the agents not only with cash $W_0(a)$ but also with risky asset endowments.[10]

3.2 The Three Greeks of CHILE

According to the Aggregation Lemma, we can compute the demand coefficients $\alpha(a)$, $\beta(a)$ and $\gamma(a)$ (the three Greeks) as follows. First, consider the demand function $X(p, \Delta s, m, a)$: the demand of agent sitting in the interval $[a, a + m)$. This demand is defined implicitly by (12). Then, demand coefficients can be found by first computing the following two functions of partials of $X(\cdot)$:

$$f^1(p, a) \equiv X_s(p, 0, 0, a), \text{ and } f^2(p, a) \equiv \left(\frac{1}{2t(a)}X_{ss}(p, 0, 0, a) + X_m(p, 0, 0, a)\right).$$

According to the Aggregation Lemma, the coefficient $\beta(a)$ is simply

$$\beta(a) = f^1(p, a).$$

Moreover, for our guess of log-linear equilibrium to be correct, $f^1(p, a)$ shall not depend on p. Similarly, coefficients $\alpha(a)$ and $\gamma(a)$ can be found from

$$\alpha(a) - \gamma(a)p = f^2(p, a).$$

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[10]We explore such generalization in our ongoing work.

If our guess is correct, then $\mathfrak{f}^2(p,a)$ is indeed an affine function of p. Then $\gamma(a)$ (resp., $\alpha(a)$) can be identified as a slope (resp., intercept) of that function. Proceeding as we just described is straightforward and we do so in the Appendix. We summarize equilibrium derivation in the theorem below.

Theorem 1. *There exists a unique equilibrium. The demand coefficients are*

$$\beta(a) = \frac{t(a)/\tau}{\rho(a)\mathrm{Var}[R|p]}, \gamma(a) = \frac{h_\gamma}{\rho(a)\mathrm{Var}[R|p]}, \quad and \;\; \alpha(a) = \frac{h_\alpha}{\rho(a)Var[R|p]} + \kappa(a), \quad (15)$$

with

$$\mathrm{Var}[R|p] = (\exp(\tau^{-1}) - 1)$$

and $\tau = \tau_v + \tau_p$, where τ_p is the equilibrium price informativeness, given as

$$\tau_p = \frac{\left(\int_\nu^1 \frac{t(a)}{\rho(a)}da\right)^2}{\int_0^1 \frac{t(a)}{\rho(a)^2}da}. \quad (16)$$

The coefficient h_γ is given by

$$h_\gamma = \frac{\int_0^1 t(a)/\rho(a)^2 da}{\int_\nu^1 t(a)/\rho(a)da \int_0^1 1/\rho(a)da}$$

and the coefficients h_α and $\kappa(a)$ are given in closed-form in (31) and (32) of the Appendix. Moreover, the equilibrium demand is

$$dx(a) = \frac{E[R-1|ds(a),p]}{\rho(a)Var[R|p]} + \kappa(a)da.$$

where $E[R-1|ds(a),p]$ is the infinitesimal conditional excess return from the perspective of trader a.[11]

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[11] The conditional return $E[R-1|ds(a),p]$ is defined as follows. Let

$$ER_e^a(\Delta s(a),m) = E[R-1|\Delta s(a),p] - 1.$$

be the expected excess return process, and denote its stochastic differential at $m=0$ as

$$E[R-1|ds(a),p].$$

Such notation makes sense, since $dER_e^a = E[R-1|ds(a),p] - E[R-1|p]$ and $E[R-1|p]$ is zero.

Theorem 1 above illustrates the power of our framework. It can deliver a closed-form equilibrium under assumptions that are usually quite difficult to incorporate in REE, such as rich heterogeneity, non-CARA utility, and non-normally distributed payoffs.

Directly from (15) we obtain the following corollary that describes the cross-section of $\beta(a)$ and $\gamma(a)$.

Corollary 1. *Traders with higher risk tolerance provide more liquidity, while traders with higher precision-weighted risk tolerance trade more aggressively. More formally, $\gamma(a) > \gamma(b)$ iff $1/\rho(a) > 1/\rho(b)$ and $\beta(a) > \beta(b)$) iff $t(a)/\rho(a) > t(b)/\rho(b)$.*

The above corollary implies that the following is true. Suppose all trader precisions are homogeneous, $t(a) = \bar{t}$, and that they all have the same utility with decreasing absolute risk aversion. Then wealthier traders will trade more aggressively and will provide more liquidity. The next corollary examines the role of wealth distribution for equilibrium price informativeness.

Proposition 1. *Suppose that $t(a) = \bar{t}$, that all traders have CRRA utility function with relative risk aversion RRA and that the wealth distribution is the same between noise and rational traders. Then price informativeness is given by*

$$\tau_p = \bar{t}\frac{\left(\int_\nu^1 W_0(a)da\right)^2}{\int_0^1 W_0(a)^2 da} = \frac{\bar{t}(1-\nu)^2}{1+\mathrm{CV}^2}, \tag{17}$$

Where CV denotes the coefficient of variation $\mathrm{CV} = \frac{\sqrt{\int W^2 dF(W) - (\int W dF(W))^2}}{\int W dF(W)}$ and $F(\cdot)$ denotes the CDF of cross-sectional wealth distribution. The price informativeness increases in $\bar{t}$ and decreases in ν. A mean-preserving spread[12] in cross-section of wealth $W_0(a)$ decreases price informativeness.

Going forward, we take CV as our main measure of wealth inequality. The proposition above shows that higher wealth inequality is associated with lower information efficiency. To

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[12]Preserving the average wealth of both rational and noise traders. By definition, the change in the cross-section of wealth from $W_0(a)$ to $\check{W}(a)$ is mean preserving if both $\int_0^\nu \left(W_0(a) - \check{W}(a)\right) da = 0$ and $\int_\nu^1 \left(W_0(a) - \check{W}(a)\right) da = 0$ and, $\check{W}(a) - W(a)$ is independent of $W(a)$ and $\check{W}(a)$ is different from $W_0(a)$ on a set of points with positive Lebesgue measure.

understand the intuition, note that the price reflects the weighted average of traders' signals, with the weights proportional to their wealth. Suppose we have two traders with signal precisions of 1, i.e., $s_1 = v + \epsilon_1$ and $s_2 = v + \epsilon_2$, where $\text{var}(\epsilon_i) = 1$. If the price reflects the signals of these traders with equal weights, the precision of the resulting signal $s_p = v + 0.5\epsilon_1 + 0.5\epsilon_2$ is $\text{var}(0.5\epsilon_1 + 0.5\epsilon_2)^{-1} = 4$. On the other hand, if the price reflects the signals with weights 90%-10%, , the precision of the resulting signal $s_p = v + 0.1\epsilon_1 + 0.9\epsilon_2$ is $\text{var}(0.1\epsilon_1 + 0.9\epsilon_2)^{-1} \approx 1.22 < 4$. More unequal wealth distribution will result in more unequal distribution of the signals' price weights, resulting in lower information efficiency.

4   Endogenizing cross-sectional wealth distribution

In this section, we endogenize the cross-sectional distribution of wealth. To this end, we assume that initial wealth is the only dimension of heterogeneity in the model. In particular, we assume that all traders have the same utility, their precision is a function of their wealth $t(a) = t(W(a))$,[13] and that the wealth of both rational and noise traders is drawn from the distribution with PDF $\phi(W)$.[14] We assume that the distribution of wealth has strictly increasing CDF $\Phi(W)$, with the inverse function denoted by $\Phi^{-1}(\cdot)$. We arrange traders on the unit interval so that the wealth of traders of the same type increases in a. The initial wealth of a trader a is then given by $W_0(a) = \Phi^{-1}(a/\nu)$ for noise traders and $W_0(a) = \Phi^{-1}((a - \nu)/(1 - \nu))$ for rational traders. In the following section, we endogenize the function $\Phi(\cdot)$ be requiring it to be *trade-invariant*.

Remark 4 (More dimensions of heterogeneity). Our analysis can be extended to allow for heterogeneity not only in terms of initial wealth but also in terms of other characteristics. Suppose that we have n trader types, with traders of the same type being identical in terms of all characteristics except for initial wealth. Suppose that the proportion of type k traders in the population is m_k. Then we can put type-k traders in the segment $\left[\sum_{j<k} m_j, \sum_{j\le k} m_j\right]$ and let their initial wealth be given by $W_0(a) = \Phi_k^{-1}((a - \sum_{j\le k})/(\sum_{j\le k} m_j - m_k))$, where $\Phi_k(\cdot)$ is the

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[13]This function is endogenized in the Appendix D.

[14]Alternatively and equivalently, we can assume that the traders receive a "shock" of being noise trader after receiving their initial wealth, independently from their wealth.

CDF of the distribution of wealth across type-k traders. The setting described above is one with two types, noise and rational, where the distribution of wealth is type-independent.

4.1   Kolmogorov in CHILE

In this section, we consider the following exercise. Consider an economy where a trader a is of size m, whereas the rest form a continuum. Suppose that the initial wealth of a trader a is drawn from the distribution $\hat{\phi}(w; m)$. After the trade, the distribution of wealth will be different and will depend on realized v and p, as well as the distribution the initial wealth drawn from. Denote it $\hat{\phi}_+(w; v, p, m)$. We say that the distribution is *trade-invariant* if ex-ante distribution of post-trade wealth coincides with $\hat{\phi}(w)$, i.e.,

$$E[\hat{\phi}_+(w; v, p, m)] = \hat{\phi}(w; m),$$

The expectation is taken with respect to the realizations of v and p, everywhere in this section. The trade-invariant density in the continuous economy $\phi(w)$ is simply the limit, as $m \to 0$, of the invariant density $\hat{\phi}(w, m)$, i.e., $\phi(w) = \hat{\phi}(w, 0)$.

We proceed below with a heuristic derivation, and we leave the rigorous derivation for the Appendix. First, write the change in wealth due to trade, $dW = W_2 - W_0$ as follows

$$
\begin{aligned}
dW &= (R - 1)dx \\
&= (R - 1)\left((\alpha - \gamma p)\, da + \beta ds(a)\right) \\
&= (R - 1)\left((\alpha - \gamma p + \beta v)\, da + \beta/\sqrt{t(a)}dB(a)\right) \\
&= \mu_W da + \sigma_W dB(a).
\end{aligned}
$$

Here we denoted

$$\mu_W = (R - 1)\left(\alpha(W) - \gamma(W)p + \beta(W)v\right) \text{ and } \sigma_W = \beta(W)/\sqrt{t(W)}.$$

We emphasize that the coefficients α, β and γ depend on W.

Recall that $\hat{\phi}_+(w; v, p, m) = Pr(W \in [w, w + dw])/dw$ is the PDF of time-2 wealth when trader a's size is m. Fix $\hat{\phi}(w) = Pr(W_0 \in [w, w + dw])/dw$, the PDF of the initial wealth, to be the same for all m. Now note that in the limit as the size of agent a becomes zero $(m \to 0)$, agents do not trade and pre- and post-trade distributions are the same. Thus, we have $\hat{\phi}_+(w, 0) = \hat{\phi}(w)$. Then $\hat{\phi}_+(w, m) - \hat{\phi}(w) = \hat{\phi}_+(w, m) - \hat{\phi}_+(w, 0)$. The Kolmogorov Forward Equation (KFE), written at $m = 0$ tells us that

$$\frac{\partial \hat{\phi}_+}{\partial m} = -\frac{\partial}{\partial W}(\mu_W \hat{\phi}_+(W, 0)) + \frac{1}{2}\frac{\partial^2}{\partial W^2}(\sigma_W^2 \hat{\phi}_+(W, 0)).$$

The trade-invariant density is such that $E[\hat{\phi}_+(w, m) - \hat{\phi}(w)] = E[\hat{\phi}_+(w, m) - \hat{\phi}_+(w, 0)] = 0$ which implies that $\frac{\partial E[\hat{\phi}_+(w,0)]}{\partial m} = 0$. Taking expectation over v and p, we obtain the following ODE on trade-invariant density

$$0 = -\frac{\partial}{\partial W}(E[\mu_W]\phi(W)) + \frac{1}{2}\frac{\partial^2}{\partial W^2}(E[\sigma_W^2]\phi(W)). \tag{18}$$

We can compute in close-form

$$E[\mu_W] = (1 - \nu)\beta(W)\tau^{-1}$$

and

$$E\left[\sigma_W^2\right] = \beta^2(W)/t(w)Var[R|p]$$

(see Appendix for the derivation).

Now consider a special case of CRRA utility and $t(W) = const = \bar{t}$. We obtain $\rho(W) = W/RRA$ and so $E[\mu_W] \propto W$, and $E\left[\sigma_W^2\right] \propto W^2$, and the KFE coincides with the KFE describing the stationary density of a Geometric Brownian Motion (GBM). It is well-known that such density does not exist (the GBM density is log-normal, with variance increasing over time). This is well-understood in the literature (e.g. Gabaix (1999)) and so the papers additionally introduce a "stabilizing force": a reflecting lower barrier, exit and reinjection or death (see Gabaix (1999) and Gabaix, Lasry, Lions, and Moll (2016, Appendix D)).

For our context, we use death as a natural stabilizing force. More specifically, we assume

that a "death shock," modeled as a Poisson arrival of intensity δ, lands somewhere in the cross-section of agents. If this shock hits agent a, this agent dies before $t = 2$ and his wealth is replaced with a draw from the distribution $\psi(\cdot)$, satisfying certain regularity conditions. In that case the KFE becomes (see the Proof)

$$0 = -\frac{\partial}{\partial W}(E[\mu_W]\phi(W)) + \frac{1}{2}\frac{\partial^2}{\partial W^2}(E[\sigma_W^2]\phi(W)) - \delta \cdot \phi(W) + \delta \cdot \psi(W). \tag{19}$$

The intuition for the transition from (18) to (19) is as follows. With intensity δ the agent dies and his wealth is taken out, which contributes to "outflow" $-\delta\phi(\cdot)$. His wealth is then replaced with a draw from $\psi(\cdot)$ which contributes to an "inflow" $\delta\psi(\cdot)$.[15]

For the case of CRRA utility and $t(W) = \bar{t}$, the ODE above can be translated to the ODE for the log-wealth, which would have the form identical to the ODE (5) in Gabaix et al. (2016). The results in Gabaix et al. (2016) then imply that $\phi(W)$ has a power-law tail

$$Pr(W > x) \sim Cx^{-\zeta},$$

where

$$\zeta = -\frac{E[\mu_W]/W}{E[\sigma_W^2]/W^2} + \frac{1}{2} + \sqrt{\left(\frac{E[\mu_W]/W}{E[\sigma_W^2]/W^2} - \frac{1}{2}\right)^2 + \frac{2\delta}{E[\sigma_W^2]/W^2}}.$$

Substituting closed-form solutions for $E[\mu_W]$ and $E[\sigma_W^2]$ and simplifying we obtain

$$\zeta = -(1-\nu)RRA + \frac{1}{2} + \sqrt{\left((1-\nu)RRA - \frac{1}{2}\right)^2 + RRA^2\tau^2\left(\exp\left(\tau^{-1}\right) - 1\right)\frac{2\delta}{\bar{t}}}. \tag{20}$$

We summarize in the proposition below.

Proposition 2. *In a general setup, the trade-invariant density solves ODE* (19). *Suppose that (i) agents have CRRA utility with relative risk aversion* RRA, *(ii)* $t(W) = \bar{t}$ *for all* W *and (iii)* $\psi \sim x^{-\zeta_\psi}$, *where* ζ_ψ *is the greater than the right-hand side of* (20). *Then, stationary distribution has a Pareto tail with exponent* ζ *given by* (20). *Moreover, the characteristic function of the*

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[15]It is also important to note that the death shocks do not alter agents' investment policies. They would simply maximize their utilities, conditional on not getting a shock. Given that shocks are independent from v, p and investment policy, such a problem is equivalent to the one considered in the section 3.2.

distribution of $\ln W$ *is given by*

$$CF_{\phi,log}(s) = \frac{2CF_{\psi,log}(s)}{2 - \frac{s\bar{t}(-2\mathrm{i}(1-\nu)RRA+s+\mathrm{i})}{\delta RRA^2(e^{1/\tau}-1)\tau^2}},$$

Where $CF_{\psi,log}(s)$ *is a characteristic function of the distribution of* $\tilde{y} = \ln \tilde{W}$, *where* $\tilde{W}$ *is drawn from a distribution* $\psi(\cdot)$:

$$CF_{\psi,log}(s) = \int e^{is\tilde{y}} e^y \psi\left(e^y\right) dy.$$

Provided it exists, the n-th (noncentral) moment of the wealth distribution is given by

$$M_\phi(n) = \frac{2M_\psi(n)}{2 - \frac{n\bar{t}(1-n-2(1-\nu)RRA)}{\delta RRA^2(e^{1/\tau}-1)\tau^2}},$$

where $M_\psi(n)$ *is the n-th (noncentral) moment of the distribution* ψ.

The proposition above allows computing the moments of the cross-sectional wealth distribution in closed form. According to Proposition 1, the price informativeness τ_p is decreasing in wealth inequality CV. The proposition above allows to compute the CV in closed form and to study the reverse channel. We do so in the following section.

5   How wealth inequality and information efficiency interact?

To study how information efficiency affects wealth inequality, we note that by Theorem 1, the information efficiency τ is a statistic summarizing the actions of all traders in the economy. To come up with his optimal trading strategy, each trader does not need to know the strategies of all other traders in the economy. Knowing τ suffices. The equilibrium can then be viewed in the following way: given traders' beliefs $\hat{\tau}$ about the information efficiency, they choose their optimal trading strategies. The trading strategies then result in the information efficiency τ that is consistent with the beliefs, $\tau = \hat{\tau}$. To study how information efficiency affects wealth inequality, we vary the exogenously postulated beliefs $\hat{\tau}$.

Proposition 3. *For an exogenously assumed wealth distribution with a coefficient of variation CV, the information efficiency τ is given by*

$$\tau\left(\text{CV}\right) = \frac{\bar{t}(1-\nu)^2}{1+\text{CV}^2} + \tau_v.$$

It is decreasing in the coefficient of variation of wealth distribution. Suppose that all traders believe that the price inforativeness is given by τ. Denote $\underline{l} = \min_{x>0} x^2(\exp(1/x)-1) \approx 1.52$. Suppose $1 - \frac{\bar{t}(2(1-\nu)RRA+1)}{\underline{l}\delta RRA^2} > 0$. Then the first and the second moment of cross-sectional wealth distribution exist for all $\tau > 0$ and the coefficient of variation of trade-invariant distribution is given by

$$\text{CV}(\tau) = \sqrt{\frac{M_\psi(2)\left(\delta RRA\left(e^{1/\tau}-1\right)\tau^2 + (\nu-1)\bar{t}\right)^2}{\delta M_\psi(1)^2\left(e^{1/\tau}-1\right)\tau^2\left(\delta RRA^2\left(e^{1/\tau}-1\right)\tau^2 + \bar{t}(2(\nu-1)RRA-1)\right)} - 1}.$$

Moreover, for small enough (large enough) $\hat{\tau}$, the coefficient of variation of trade-invariant wealth distribution is increasing (decreasing) in τ.

Figure 1 illustrates the Proposition. The dashed line represents the curve $\tau(\text{CV})$. Higher levels of inequality correspond to lower levels of information efficiency. As we discussed, this happens because the best way to reflect traders' information in price is via an equally weighted combination of their signals. With more inequality, the weights are unequal and are skewed towards wealthier traders who trade more aggressively. The solid line represents the curve $\text{CV}(\tau)$. There are two ways for information efficiency to affect inequality. First, with higher information efficiency, traders use their private signals less, contributing to lower inequality. Second, higher information efficiency reduces uncertainty, making investors trade more aggressively on their private signals. The second (first) effect dominates for high (low) τ, explaining the hump shape of the curve $\text{CV}(\tau)$.

For high enough levels of τ we get the self-reinforcing inequality-inefficiency relationship depicted in Figure 2. When the market is less informationally efficient, there is more money that can be made via speculation. Since wealthy traders speculate more, they benefit from the informational inefficiency more. This results in greater inequality. The market with more inequality aggregates the information less well, which feeds back into a less informationally
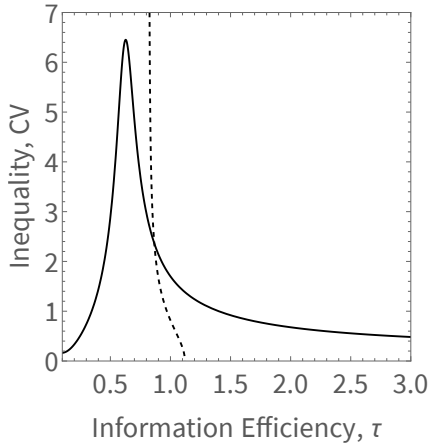
efficient market.

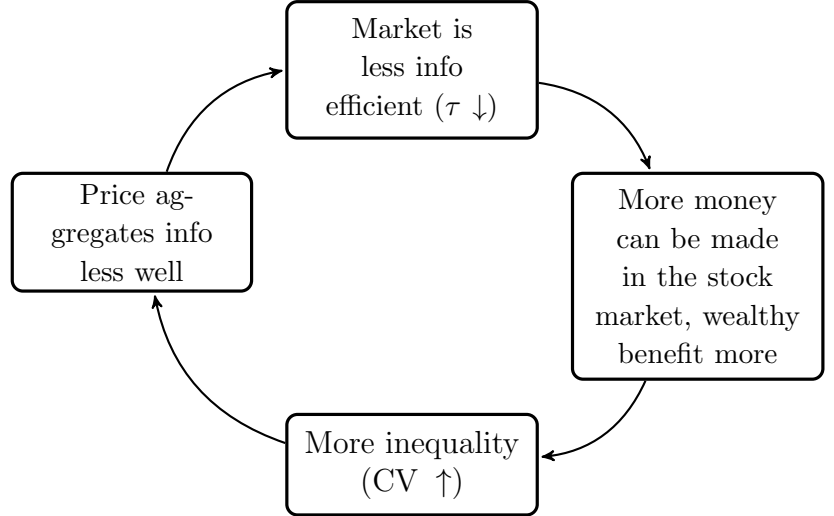Figure 1: Determination of overall equilibrium

Figure 2: Inequality-inefficiency complementarity

5.1 Overall equilibrium

In this section, we look at the comparative statics in the overall equilibrium, where the distribution of wealth is endogenous. We focus on two comparative statics exercises: increasing transparency (i.e., increasing $\bar{t}$) and providing information about firm fundamentals (i.e., increasing τ_v). While the previous literature has extensively studied the effect of such policies on different aspects of market quality, such as information efficiency and liquidity, the effect of such policies on inequality is relatively unexplored. When the wealth distribution is exogenous, Proposition 1 implies that public information disclosure and increasing transparency both increase τ_v. We show that some of these results are overturned once the wealth distribution is endogenized.

Proposition 4. *Increasing τ_v always leads to an increase in τ. When equilibrium τ is small (large) enough, increasing τ_v leads to an increase (decrease) in* CV*. Increasing $\bar{t}$ has an ambiguous effect on both τ and* CV*.*

Providing public disclosure (increasing τ_v) has an ambiguous effect on inequality. When the uncertainty reduction effect dominates (low levels of equilibrium τ), increasing τ_v leads to a reduction in uncertainty faced by the traders so that they trade more aggressively on their
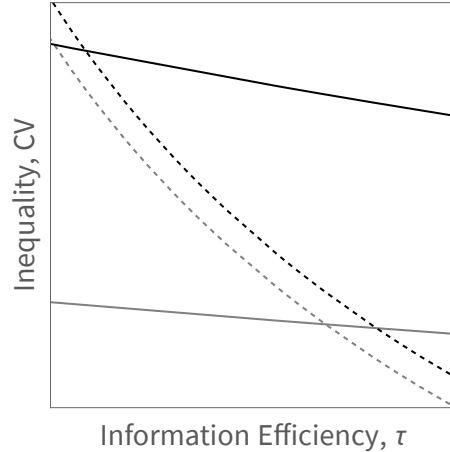
25

Figure 3: An increase in $\bar{t}$. Intersection of grey lines correspond to the equilibrium with lower $\bar{t}$.

private information. This leads to an increase in inequality. For high levels of τ increasing τ_v implies a reduction in potential profits that can be made via speculation and reduction in inequality. The effects of changes in transparency on both inequality and information efficiency are ambiguous. Figure 3 provides an example of when increasing $\bar{t}$ *reduces* information efficiency. There are two effects. When transparency improves, for the same level of inequality CV, the information efficiency increases: the dashed curve shifts to the right. However, it also makes investors trade more aggressively on their private information, inequality: the solid curve moves up. As Figure 2 highlights, the increase in inequality feeds back into lower equilibrium price efficiency.

6 Literature review

There are several branches of literature that our paper is related to. First, there is a literature on REE models that go beyond the CARA-Normal framework. Breon-Drish (2015) extends the CARA-Normal framework beyond normality in a single asset setup. Albagli, Hellwig, and Tsyvinski (2021) consider a setup with general distribution and risk-neutral traders subject to position limits. Chabakauri et al. (2022) further extends Breon-Drish (2015) by allowing for multiple assets. All of these papers assume CARA utility and so abstract away from wealth

effects that are central to our paper.[16]

Malamud (2015) considers an REE model with a continuum of assets. Central to the tractability of his framework is the assumption of market completeness.[17] In contrast, we have one asset and continuum of states of the world. Hence the market is incomplete in our setup. One of the central results in Malamud (2015, Theorem 2.1) is that with non-CARA utility the equilibrium is fully revealing. In contrast, in our setup, there is no full revelation for any utility function.

Peress (2004) was the first (to our knowledge) to study wealth effects in a noisy REE model. Similarly to our paper, his model features log-normally distributed payoffs and non-CARA utilities.[18] The key difference is that Peress (2004) relies on "small risk" approximation, where the riskiness of the risky asset is small. In the limit, the variance of risky asset return is zero, making such approach not suitable for quantitative work (it will be hard to match variance). Our approximation is essentially "small information". In contrast to Peress (2004), in our model the asset stays risky even in the limit. Additionally: (i) in our model the equilibrium quantities are affected by absolute risk aversion and absolute prudence, whereas in Peress (2004) only risk-aversion plays a role and (ii) conditional skewness plays a role in our model, but not in Peress (2004). Peress (2014) embeds the REE model of Peress (2004) into a growth model. An interesting direction for future research is to embed our model of financial market into a growth model.

Second, our paper is related to the literature on power laws in economics and finance. See Gabaix (1999), Gabaix et al. (2016) and, for a review (Gabaix (2009) and Gabaix et al. (2016)). As in most of the papers in that literature, the power law emerges via a random growth mechanism. To the best of our knowledge we are the first to make the heterogeneity of information a part of the random growth mechanism and thus to relate the tail exponent in the economy to information efficiency.

Third, our paper is also related to the literature on mean-field games. See Lasry and Lions

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[16]Here we consider risk-neutral preferences as a special case of CARA with risk-aversion equal to zero.

[17]Related, DeMarzo and Skiadas (1998) and DeMarzo and Skiadas (1999) analyze REE models, where the market is *quasi-complete.*

[18]The wealth effect can arise even in CARA model, because the wealth may affect the tightness of financial constraints, as in Glebkin, Gondhi, and Kuong (2021).

(2007), Achdou, Han, Lasry, Lions, and Moll (2022) and, for a review, Guéant, Lasry, and Lions (2011). A typical mean-field game is described by HJB equation determining the dynamics of optimal policy of each agent type and KFE describing the dynamics of the distribution of agent types over time. Our model is static. Yet the tools we use are similar. In our model the KFE describes the trade-invariant distribution that is analogous to a stationary distribution in a dynamic game. Moreover, as Achdou et al. (2022) note: " *The name (Mean Field Games) comes from an analogy to the continuum limit taken in "Mean Field theory" which approximates large systems of interacting particles by assuming that these interact only with the statistical mean of other particles.*" This analogy holds in our model. The effect of other traders on a trader of interest in our economy is summarized by several statistics of cross-sectional distribution of traders' characteristics. These statistics can be viewed as a "mean field" that influences each trader's equilibrium behaviour. As in Mean Field theory other traders do not affect a trader of interest directly, but only through their (infinitesimal) contribution to the mean field.

Fourth, on a technical side, our paper is also related to literature that uses stochastic calculus tools outside the domain of continuous time finance and economics. Examples include Malamud (2015) who models the noise in a continuum of assets as a cross-sectional stochastic process; Gârleanu et al. (2015) who use Brownian bridge to represent the dividends for firms located on a circle; and Glebkin et al. (2021) who use stochastic calculus techniques to derive a marginal value of information in a static model.[19] The most closely related paper is Avdis (2018), that introduces a model with continuous heterogeneous information, albeit with CARA preferences and, as a result, without wealth effects.

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[19]There is also a related econometric literature on the unit roots. A good example is Phillips (1987).

7 Conclusion

Thank you for reaching this part of the paper. At some point we will add a conclusion here.

Appendices

A   Summary of notation

Notation	Explanation
General mathematical notation	
$dB(a)$	An increment of the Brownian motion
a, b	Index of agents in the continuous limit
i, j	Index of agents in the discrete economy
n	Number of agents in the discrete economy
$m = 1/n$	Mass of an agent in the discrete economy
$E^i[\cdot]$	Conditional expectation, given agent i's information set
$E_y[\cdot]$	Expectation taken with respect to r.v. y
$\mathcal{N}(\cdot)$	Standard normal CDF
Model variables	

General note. Subscripts indicate partial derivatives.

$dx(a) = \alpha(a)da + \beta(a)ds_a - \gamma(a)pda$	Demand of trader in $[a, a + da]$
$\rho(a) = -u''(a)/u'(a)$	Absolute risk aversion of trader a
$\pi(a) = -u'''(a)/u''(a)$	Absolute prudence of trader a
Π	Profit

B   Derivation of demand coefficients

The Lemma 7 in the appendix implies that $\mathfrak{f}^1(p,a)$ and $\mathfrak{f}^2(p,a)$ can be computed as follows. Denote $F(\Delta s^i(m),m)$ the left-hand side of the first order condition (12), where one substitutes the demand of an agent $X(\cdot)$ instead of x.

$$
F(\Delta s^i(m),m) = \int \left(\begin{array}{c} U'\left(W_0^i + (\exp\left(v^i(p,\Delta s^i,m,z) - p\right) - 1\right) X(\cdot)) \\ \cdot \left(\exp\left(v^i(p,\Delta s^i,m,z) - p\right) - 1\right) \end{array} \right) d\mathcal{N}(z)
$$

Note, that $F(\Delta s^i(m),m)$ is an Ito process, driven by the process

$$
ds^i = vda + \frac{1}{\sqrt{t(a)}} dB_a.
$$

We then apply Ito's lemma to the process F at $m = 0$ and equalize both drift and diffusion to zero.

To proceed we note that $X(p,\Delta s^i,m,a)$ and $v^i(p,\Delta s^i,m,z)$ are also Ito processes and by Ito's lemma their differentials at $m = 0$ can be written as

$$
dX = \mathcal{D}X da + \sigma_X ds(a) \text{ and } dv^i = \mathcal{D}v^i \, da + \sigma_{v,i} ds(a),
$$

where

$$
\mathcal{D}X = X_{ss}(p,0,0,a) + X_m(p,0,0,a),
$$
$$
\mathcal{D}v^i = v^i_{ss}(p,0,0,z) + v^i_m(p,0,0,z),
$$
$$
\sigma_X = X_s(p,0,0,a),
$$
$$
\sigma_{v,i} = v^i_s(p,0,0,z).
$$

The functions $\mathfrak{f}^1(p,a)$ and $\mathfrak{f}^2(p,a)$ are identified as

$$
\mathfrak{f}^1 = \sigma_X \text{ and } \mathfrak{f}^2 = \mathcal{D}X.
$$

The computation of equilibrium can be done in several steps. First, we determine the equilibrium coefficients $\beta(a)$. Once these are known, we can compute all other coefficients in closed-form.

B.1 Finding $\beta(a)$

Computing the diffusion term of $F(\cdot)$ at $m = 0$ and equalizing it to zero yields

$$\sigma_{v,i} u'(W_0(a)) \int \left(\exp \left(v^i(p,0,0,z) - p \right) \right) d\mathcal{N}(z) + \tag{21}$$
$$\sigma_X u''(W_0(a)) \int \left(\exp \left(v^i(p,0,0,z) - p \right) - 1 \right)^2 d\mathcal{N}(z) = 0$$

Now we note that

$$\int \exp \left(v^i(p,0,0,z) - p \right) d\mathcal{N}(z) = E[R|p] = 1,$$

where the last equality follows from (13). It then follows that

$$\int \left(\exp \left(v^i(p,0,0,z) - p \right) - 1 \right)^2 d\mathcal{N}(z) = Var[R|p] = (\exp \left(\tau(0)^{-1} \right) - 1),$$

where the last equality computes $Var[R|p]$ using the fact that $R|p$ is lognormally distributed.[20]

Denoting absolute risk aversion of agent a

$$\rho(a) \equiv -\frac{u''(W_0)}{u'(W_0)},$$

and noting that $\sigma_{v,i} = c_s(0) = \frac{t(a)}{\tau}$ we express $\beta(a) = \sigma_X$ from (21) as follows

$$\beta(a) = \sigma_X = \frac{t(a)/\tau}{\rho(a)\text{Var}[R|p]} = \frac{t(a)}{\rho(a)\tau(\exp \left(\tau^{-1} \right) - 1)}.$$

---

[20]The calculation is as follows. We know that $v|p \sim N(\mu_{v|p}, \sigma_{v|p})$. Moreover, $E[R|p] = E[\exp(v - p)|p] = \exp(\mu_{v|p} + \sigma^2_{v|p}/2) = 1$. Here the second transition uses the standard formula for mean of lognormal distribution and third transition uses (13). Then, $\text{Var}[R|P] = (\exp(\sigma^2_{v|p}) - 1)\exp(2\mu_{v|p} + \sigma^2_{v|p}) = (\exp(\sigma^2_{v|p}) - 1)^2$.

In accordance with our guess σ_X does not depend on p.

The following proposition then summarizes how the solution $\beta(a)$ is found.

Proposition 5. *The coefficients $\beta(a)$ are given by*

$$\beta(a) = \frac{t(a)/\tau}{\rho(a)\mathrm{Var}[R|p]}.$$

where

$$\mathrm{Var}[R|p] = (\exp(\tau^{-1}) - 1)$$

and $\tau = \tau_v + \tau_p$. The price informativeness τ_p is given by

$$\tau_p = \frac{\left(\int_\nu^1 \frac{t(a)}{\rho(a)} da\right)^2}{\int_0^1 \frac{t(a)}{\rho(a)^2} da}. \tag{22}$$

B.2   Finding $\gamma(a)$

Computing the drift term of $dF(\cdot)$ at $m = 0$ and equalizing it to zero yields:

$$\int \mathcal{D}v^i \exp\left(v^i(p, 0, 0, z) - p\right) d\mathcal{N}(z) - \tag{23}$$

$$\mathcal{D}X\rho(a) \int \left(\exp\left(v^i(p, 0, 0, z) - p\right) - 1\right)^2 d\mathcal{N}(z) + \tag{24}$$

$$\frac{\pi(a)\rho(a)\sigma_X^2}{2t(a)} \int (\exp\left(v^i(p, 0, 0, z) - p\right) - 1)^3 d\mathcal{N}(z) - \tag{25}$$

$$\frac{2\rho(a)\sigma_{v,i}\sigma_X}{t(a)} \int \exp\left(v^i(p, 0, 0, z) - p\right) \left(\exp\left(v^i(p, 0, 0, z) - p\right) - 1\right) d\mathcal{N}(z) + \tag{26}$$

$$\frac{\sigma_{v,i}^2 \int \exp\left(v^i(p, 0, 0, z) - p\right) d\mathcal{N}(z)}{2t(a)} = 0, \tag{27}$$

where we have denoted

$$\pi(a) = -\frac{u'''(W_0(a))}{u''(W_0(a))}$$

The absolute prudence of agent a. While the above equation is big, only the first two lines depend on p. Denoting

$$\hat{h}(m) \equiv c_p(m) \frac{\int_0^1 \gamma(a,m)da}{\int_\nu^1 \hat{\beta}(a,m)da},$$

and letting "$\cdots$" represent the terms that do not depend on p we write

$$\mathcal{D}v^i = \hat{h}(m)'p + \cdots$$

Noting that $\mathcal{D}X = -\gamma(a)p + \cdots$ equation (23)-(27) then becomes

$$\hat{h}(0)'p + \gamma(a)\rho(a)Var[R|p]p + \cdots = 0$$

From which we find

$$\gamma(a) = -\frac{\hat{h}(0)'}{\rho(a)\mathrm{Var}[R|p]}. \tag{28}$$

Given that our equilibrium concept only requires the functions $\hat{\alpha}(a,m)$, $\hat{\beta}(a,m)$ and $\hat{\gamma}(a,m)$ to be equal to $\alpha(a)$, $\beta(a)$ and $\gamma(a)$ and does not put explicit restrictions on the derivatives of these functions with respect to m, one might be concerned about equilibrium indeterminacy, as the aforementioned conditions put no restrictions on $\hat{h}(m)'$. However, the observations of Lemma 4 allow to pin down $\hat{h}(m)'$ as follows. The key is the condition (14).

Note that τ_p, τ and $\int_\nu^1 \beta(a)da$ are determined once $\beta(a)$ is solved for. Then

$$\hat{h}(0)' \equiv -h_\gamma = -\frac{\tau \int_\nu^1 \beta(a)da \mathrm{Var}[R|p]}{\tau_p \int_0^1 1/\rho(a)da} = -\frac{\int_0^1 t(a)/\rho(a)^2 da}{\int_\nu^1 t(a)/\rho(a)da \int_0^1 1/\rho(a)da}. \tag{29}$$

We summarize in the proposition below.

Proposition 6. *The coefficients $\gamma(a)$ are given by (28), where $\hat{h}(0)'$ is given by (29).*

B.3 Finding $\alpha(a)$

We turn to finding $\alpha(a)$. We work our way through (23)-(27) line by line. We start with (23). Denoting "$\cdots$" the terms proportional to p (which are already accounted for) we write

$$\mathcal{D}v^i = c_{\bar{v}}(0)'\bar{v} + (c_p(m)\bar{\alpha}(m)\bar{\beta}(m)))' - \frac{1}{2}\tau^{-3/2}\hat{\tau}(0)'z + \cdots$$

While the next term can be computed in closed-form, it is only important for us to note that it does not depend on a and denote it by h_α.

$$\int \mathcal{D}v^i \exp\left(v^i(p,0,0,z) - p\right) d\mathcal{N}(z) = h_\alpha.$$

We turn to (24). Sice $\mathcal{D}X = \alpha(a) + \cdots$, we have

$$\mathcal{D}X\rho(a) \int \left(\exp\left(v^i(p,0,0,z) - p\right) - 1\right)^2 d\mathcal{N}(z) = \alpha(a)\rho(a)Var[R|p] + \cdots$$

We turn to (25). Substituting $\sigma_X = \beta(z)$, and $\int (\exp(v^i(p,0,0,z) - p) - 1)^3 d\mathcal{N}(z) = \text{Sk}[v|p]\text{Var}[R|p]^{3/2}$ we get

$$\frac{\pi(a)\rho(a)\sigma_X^2}{2t(a)} \int (\exp\left(v^i(p,0,0,z) - p\right) - 1)^3 d\mathcal{N}(z) = \frac{\pi(a)\rho(a)\beta(a)^2}{2t(a)}\text{Var}[R|p]^{3/2}\text{Sk}[R|p],$$

where the skewness can be computed in closed-form

$$\text{Sk}[R|p] = \left(\exp\left(\tau^{-1}\right) + 2\right)\sqrt{\exp\left(\tau^{-1}\right) - 1}.$$

We proceed to (26). Here we substitute $\sigma_{v,i} = t(a)/\tau$ and

$$\int \exp\left(v^i(p,0,0,z) - p\right)\left(\exp\left(v^i(p,0,0,z) - p\right) - 1\right) d\mathcal{N}(z) = \text{Var}[R|p]$$

and get

$$\frac{2\rho(a)\sigma_{v,i}\sigma_X}{t(a)} \int \exp\left(v^i(p,0,0,z) - p\right)\left(\exp\left(v^i(p,0,0,z) - p\right) - 1\right)d\mathcal{N}(z) =$$

$$\frac{2\rho(a)\beta(a)}{\tau}Var[R|p]$$

Finally, for (27) we get

$$\frac{\sigma_{v,i}^2 \int \exp\left(v^i(p,0,0,z) - p\right)d\mathcal{N}(z)}{2t(a)} = \frac{1t(a)}{2\tau^2}$$

Combining it all together we get

$$h_\alpha - \alpha(a)\rho(a)Var[R|p]$$

$$+\frac{\pi(a)\rho(a)\beta(a)^2}{2t(a)}\text{Var}[R|p]^{3/2}\text{Sk}[R|p] - \frac{2\rho(a)\beta(a)}{\tau}Var[R|p] + \frac{1t(a)}{2\tau^2} = 0$$

After mild simplification we get

$$\alpha(a) = \frac{h_\alpha}{\rho(a)Var[R|p]} + \frac{\pi(a)\beta(a)^2}{2t(a)}\text{Var}[R|p]^{1/2}\text{Sk}[R|p] - \frac{3\beta(a)}{2\tau}. \tag{30}$$

As before, the discipline on h_α comes from the Lemma 4 and it is found as a unique solution to

$$\int_0^1 \alpha(a)da = h_\alpha \int_0^1 \frac{da}{\rho(a)Var[R|p]} + \text{Var}[R|p]^{1/2}\text{Sk}[R|p] \int_0^1 \frac{\pi(a)\beta(a)^2}{2t(a)}da - \frac{3}{2\tau}\int_0^1 \beta(a)da. \tag{31}$$

We summarize.

Proposition 7. *The coefficients $\alpha(a)$ are given by (30), where h_α is the unique solution to (31).*

B.4 An alternative representation of demand

Define the expected excess return process

$$ER_e^i(\Delta s^i, m) = E[R - 1 | \Delta s^i, p].$$

Denote its stochastic differential at $m = 0$

$$E[R - 1 | ds^i, p].$$

Such notation makes sense, since $dER_e^i = E[R - 1 | ds^i, p] - E[R - 1 | p]$ and the term $E[R - 1 | p]$ is zero. Applying Ito's lemma it's easy to compute

$$E[R - 1 | ds^i, p] = 1 c_s(0) ds^i + 1 \hat{h}(0)' p dm + h_\alpha da.$$

Then the following proposition follows immediately.

Proposition 8. *The equilibrium demand can be written as*

$$dX_a = \frac{E[R - 1 | ds^i, p]}{\rho(a) Var[R|p]} + \kappa(a) da,$$

where
$$\kappa(a) = \frac{\pi(a)\beta(a)^2}{2t(a)} Var[R|p]^{1/2} Sk[R|p] - \frac{3\beta(a)}{2\tau}. \tag{32}$$

C General distribution of asset payoffs

In this section, we assume the general function $V(v)$: our only assumption is that it is weakly increasing for all $v \in \mathbb{R}$. For notational simplicity, we work with the case of no noise traders, $\nu = 0$. With general distribution, working in terms of dollar demands does not yield any significant advantages. We thus formulate equilibrium in terms of unit demands (number of stocks purchased/sold during trade at $t = 1$), which we denote by y (in the discrete economy) and dy in the continuous economy. According to the aggregation lemma, the demand of a

trader in $[a, a + da)$ is given by

$$dy(a) = \beta(a)da + \aleph(a, P)da.$$

We conjecture that $\beta(a, P)$ is of the form

$$\beta(a, p) = \beta_a(a)\beta_p(P). \tag{33}$$

The informational content of the price is then summarized by

$$s_p = \frac{\int_0^1 \beta_a(a)ds(a)}{\int_0^1 \beta_a(a)da} = v + \int_0^1 \frac{\beta_a(a)}{\sqrt{t(a)}}dB(a).$$

The inference from price is identical to that in the main part of the paper, and is summarized in the Lemma 2.

The first-order condition in the discrete economy is

$$\int U'(W_0^a + (V(\hat{v}^a(P, \Delta s^a, m, z)) - P) y) (V(\cdot) - P) d\mathcal{N}(z) = 0 \tag{34}$$

Where we denoted

$$\hat{v}^a = E[v|\Delta s^a, p] + z\sqrt{\text{Var}[v|\Delta s^a, p]}$$

It must be true that as $m \to 0$ the y satisfying the FOC above goes to 0. Then it follows that the price function must be given by

$$\int V \left(\frac{\tau_v}{\tau}\bar{v} + \frac{\tau_p}{\tau}s_p + \frac{z}{\sqrt{\tau}} \right) d\mathcal{N}(z) = P.$$

We summarize in the lemma below.

Lemma 5. *Provided that conjecture* (33) *holds, the equilibrium price function is given by* $P(s_p)$,

where $s_p = v + \int_0^1 \frac{\beta_a(a)}{\sqrt{t(a)}} dB(a)$ and $P(x)$ is a strictly increasing function defined implicitly by

$$\int V\left(\frac{\tau_v}{\tau}\bar{v} + \frac{\tau_p}{\tau}x + \frac{z}{\sqrt{\tau}}\right) d\mathcal{N}(z) = P.$$

Moreover, $\tau = \tau_v + \tau_p$ and τ_p is given by $\tau_p = \frac{\left(\int_0^1 \beta(a)da\right)^2}{\int_0^1 \frac{\beta(a)^2}{t(a)} da}$. Let $h(P)$ be the inverse of $P(\cdot)$. For a function $f(v, P)$ we have

$$E[f(v, P)|P] = \int f\left(\frac{\tau_v}{\tau}\bar{v} + \frac{\tau_p}{\tau}h(P) + \frac{z}{\sqrt{\tau}}, P\right) d\mathcal{N}(z). \tag{35}$$

C.1 Finding the coefficients β and $\aleph$.

Once the price function and the informational content of price is known, the derivation of demand coefficients follows that in the main part of the paper. We differentiate the FOC (34) implicitly and calculate the limit as $m \to 0$. The economics are very similar to that in the case of the main model, with a bit algebra is a bit more convoluted. The result is summarized in the Theorem below.

Theorem 2. *There exists a unique equilibrium. The price informativeness is given by*

$$\tau_p = \frac{\left(\int_0^1 \frac{t(a)}{\rho(a)}da\right)^2}{\int_0^1 \frac{t(a)}{\rho(a)^2} da}.$$

Let $\tau = \tau_v + \tau_p$. The coefficient $\beta(\cdot)$ is given by $\beta(a, P) = \beta_a(a)\beta_p(P)$, where

$$\beta_a(a) = \frac{t(a)}{\rho(a)} \text{ and } \beta_p(P) = \frac{E[V'(v)|P]}{\tau \text{Var}[V(v)|P]}.$$

The functions $E[V'(v)|P]$ and $Var[V(v)|P] = E\left[(V(v) - P)^2 |P\right]$ can be computed in closed form using (35).

The coefficient $\aleph(\cdot)$ is given by

$$\aleph(a, P) = \frac{1}{2\rho(a)t(a)\tau^2 \mathrm{Var}[V(v)|P]} \begin{pmatrix} 2h_{\aleph}(P)E[V'(v)|P]t(a) + E[V''(v)|P]t(a)^2 + \\ +\beta(a,P)\rho(a)\tau(4E[V'(v)\,(V(v)-P)\,|P]t(a) \\ -\beta(a,P)\pi(a)\mathrm{skew}(V(v)|P)\tau) \end{pmatrix},$$

where $h_{\aleph}(P)$ is the unique solution to

$$\int \aleph(a, P)da = h(P)\int \beta(a, P)da,$$

and $E[V'(v)\,(V(v)-P)\,|P]$ and $\mathrm{skew}[V(v)|P] = E\left[(V(v)-P)^3\,|P\right]$ can be computed in closed form using (35).

Proof of Theorem 2.

Denote $F(\Delta s(m), m)$ the left-hand side of the first order condition (34)

$$F(\Delta s^a(m), m) = \int \begin{pmatrix} U'\left(W_0^i + (V\left(\hat{v}^a(p, \Delta s^a, m, z)\right) - P)\,y(\cdot)\right) \\ \cdot\left(V\left(\hat{v}^a(p, \Delta s^a, m, z)\right) - P\right) \end{pmatrix} d\mathcal{N}(z)$$

The function $F(\cdot)$ can be seen as an increment of an Ito process $\tilde{F}(\Delta\tilde{s}(l), l)$ between $l = m$ and $l = 0$. The process $\tilde{F}(\cdot)$ is in turn driven by

$$d\tilde{s}(l) = vdl + \frac{1}{\sqrt{t(a)}}dB(l)$$

with $\Delta\tilde{s}(l) = \tilde{s}(l) - \tilde{s}(0)$, and

$$d\hat{v}^a(l) = d\left(\frac{\tau_v}{\hat{\tau}}\bar{v} + \frac{t(a)}{\hat{\tau}(l)}\Delta\tilde{s}(l) + \frac{\hat{\tau}_p(l)}{\hat{\tau}(l)}\hat{s}_p(l) + \frac{z}{\sqrt{\hat{\tau}(l)}}\right).$$

$$dy = \hat{\beta}(a, P, m)ds(l) + \hat{\aleph}(a, P, m)da.$$

We note that can be computed as

$$\hat{h}(P, l) = \frac{\int_0^1 \hat{\aleph}(a, P, l)da}{\int_0^1 \hat{\beta}(a, P, l)da}.$$

We then compute the diffusion coefficient of dF and equalize it to zero. This gives the solution for $\beta(\cdot)$. Setting the drift of dF to zero gives the solution for $\aleph(\cdot)$. The function $h_\aleph(P)$ is a linear combination of the derivatives of $\hat{h}(P, m)$ and $\hat{\tau}(m)$ with respect to m at $m = 0$. We pin down $h_\aleph(P)$ (the limit of $\hat{h}(P, m)$ as $m \to 0$) by requiring the price function in the $m \to 0$ limit to coincide with the function $P(\cdot)$ described in Lemma 5. $\blacksquare$

D Information acquisition

We start with the following lemma.

Lemma 6. *Consider a continuously differentiable function $f(t, m)$ such that $f(t, 0)$ does not depend on t. Consider $t(m) \in \arg\max_t f(t, m)$. Suppose that $t(m)$ is single-valued and bounded for small enough m. Then, $t(0) \in \arg\max_t f_m(t, 0)$.*

Our approach to endogenizing $t(a)$ is as follows. Denote $\mathcal{U}^i(W_0^i, t^i, m)$ the maximum in (8) for a given precision t^i and initial wealth W_0^i. Define $\mathcal{C}^i(t^i, m)$ as a (monetary) cost of acquiring signal Δs_i. Then we define $t^i(m)$ as

$$t^i(m) = \arg\max_t \{\mathcal{U}^i(W_0^i - \mathcal{C}^i(t^i, m), t^i, m)\}$$

and

$$t(a) = \lim_{m \to 0} t^{i=a}(m).$$

Assumption 1. $\mathcal{C}^i(t, m) = c(t)m + o(m)$, *where $c(t)$ is a convex function.*

Using Lemma 6, one can derive that the precision $t(a)$ solves the following problem:

$$t(a) \in \arg\max_t \left\{ \left. \frac{\partial \mathcal{U}^a(W_0^i - \mathcal{C}^i(t^i, m), t^i, m), t, m)}{\partial m} \right|_{m=0} c(t) \right\}.$$

40

The following proposition solves for optimal precision choice.

Proposition 9. *The optimal precision choice solves*

$$c'(t) = \frac{1}{2\rho(a)\tau(\exp{(\tau^{-1})} - 1)}.$$

E An OLG economy

The time is discrete, $t \in \{1, 2, \ldots \infty\}$. The economy is populated by households who live for two periods. The young households can either be *locals*, in which case they receive their initial wealth from their parent households, or they can be *immigrants*, in which case their initial wealth is drawn from the distribution $\psi(\cdot)$. The households can vanish without kids with intensity δ. These vanished households are replaced by immigrants. The wealth of vanished households is distributed equally across the whole economy. Thus, the economy always has a unit mass of young households.

The households optimize the family wealth by investing in the risk-free asset with a return 1 and a risky asset. We assume that the risky asset is in unit supply. The risky asset is a claim to the next period's dividend generated by a Lucas tree. The dividends V_t are i.i.d. with $V_t = \exp(v_t)$, and $v_t \sim N(0, \tau_v^{-1})$. The goal of each household is to choose the investments to maximize the bequest utility $u(\cdot, a)$. We, therefore, abstract away from the consumption-investment tradeoff. In the economy described here, families manage the wealth inherited from their parents, which is not used for consumption (the households manage "family offices").

The evolution of wealth distribution across households is given by

$$f(w, l) = \left(f(w, 0) + \int_0^m \left(-\frac{\partial}{\partial w}(\mu_W(l, w)f(w, l)) + \frac{1}{2}\frac{\partial^2}{\partial W^2}(\sigma_W^2(w, l)f(w, l)) \right) dl \right) (1 - Pr(\text{no kids})) +$$
$$+ Pr(\text{no kids})\psi(w)$$

The wealth of the family changes due to investments made in the assets (the term in the square brackets), provided that the family has offsprings. If there are no offsprings, the wealth is seized and redistributed. Given the large size of the economy, such redistribution does not

result in a sizeable increase in other families' wealth. When one family vanishes it is replaced by immigrants (the last term).

The analysis of the OLG economy described here is the same as the analysis of the static economy presented in the main paper. The trade-invariant distribution of the main paper corresponds to the (stochastic) steady-state distribution here.

F Proofs

F.1 Proof of Lemma 1

Proof of Lemma 1. Fix p. We drop the argument p in what follows. That is, $X(\Delta s^i; m, a^i)$ denotes $X(p, \Delta s^i; m, a^i)$.

By Ito's lemma we have

$$
X(\Delta s^i, m, a^i) - X(0, 0, a^i) =
$$
$$
= \int_{a^i}^{a^{i+1}} X_s(s(a) - s(a^i); a - a^i, a^i) ds_a +
$$
$$
+ \int_{a^i}^{a^{i+1}} \left(\frac{1}{2t_a} X_{ss}(p, s(a) - s(a^i); a - a^i, a^i) + X_m(p, s(a) - s(a^i); a - a^i, a^i) \right) da.
$$

Denote the i-th segement $\mathcal{A}^i = [a^i, a^{i+1})$ and $\hat{a}^i = \sum_{i=1}^{n-1} a^i 1(a \in \mathcal{A}^i)$ ($\hat{a}^i$ gives a^i whenever a falls in the i-th segment $\mathcal{A}^i$). With such notation we can write

$$
\sum_{i=0}^{n-1} \left(X(\Delta s^i; m, a^i) - X(0; 0, a^i) \right) =
$$
$$
= \int_0^y X_s(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) ds_a +
$$
$$
+ \int_0^y \left(\frac{1}{2t_a} X_{ss}(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) + X_d(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) \right) da.
$$

Denote $LI = \int_0^y \left(\frac{1}{2t_a} X_{ss}(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) + X_d(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) \right) da$ and $SI = \int_0^y X_s(s(a) - s(\hat{a}^i); a - \hat{a}^i, \hat{a}^i) ds_a$. That $LI \to \int \left(\frac{1}{2t_a} X_{ss}(p, 0; 0, a) + X_d(p, 0; 0, a) \right) da$ follows

from the Continuous Mapping Theorem.[21] We can also apply Continuous Mapping Theorem to the stochastic integral, since in our case it is bounded (since $\sum_{i=1}^{n} X^i$ is bounded). Thus, we get

$$\sum_{i=0}^{n-1} \left(X(p, \Delta s^i; m, a^i) - X(p, 0; 0, a^i) \right) \xrightarrow[n \to \infty]{}$$
$$\int X_s(p, 0; 0, a) ds_a + \int \left(\frac{1}{2t_a} X_{ss}(p, 0; 0, a) + X_d(p, 0; 0, a) \right) da.$$

∎

F.2 Proof of Lemma 3

Proof of Lemma 3. We start with FOC (we use shortcut $E^i[\cdot] = E[\cdot | \Delta s^i, s_p]$)

$$E^i \left[U' \left(W_0 1 + (R-1) x \right) ((R-1)) \right] = 0.$$

The second derivative of the objective function is

$$E^i \left[U'' (\cdot) (R-1)^2 \right] < 0$$

Hence, the objective function is strictly concave. Hence, the FOC is both necessary and sufficient. ∎

F.3 Proof of Lemma 4

Proof of Lemma 4. Denote, for this proof only $\bar{\gamma}(m) = \int_0^1 \gamma(a, m) da$, $\bar{\beta}(m) = \int_\nu^1 \beta(a, m) da$ and $\bar{\alpha}(m) = \int_0^1 \alpha(a, m) da$.

Indeed, if the limit is not zero, then the aggregate demand is infinite. Eq. (13) follows by

---

[21]Here we use the fact that Lebesgue integral is a continuous mapping from $C[0, 1]$ to $C[0, 1]$, the space of functions continuous on $[0, 1]$.

substituting $x = 0$ to (12) and noting that $\mathcal{F}_i = p$ in the limit when $m = 0$. Note that

$$E[R|p] = \exp\left(E[v|p] - p + \frac{1}{2\tau}\right) = 1 \text{ as } m \to 0.$$

We must have that $\partial/\partial p$ of LHS is zero. It's only possible if

$$\partial/\partial p E^i[v] = 1 \implies \frac{\tau_p^i(0)\,\bar{\gamma}(0)}{\tau^i(0)\,\bar{\beta}(0)} = 1.$$

It then follows that $E[v|p] - p + \frac{1}{2\tau} = \frac{\tau_v}{\tau}\bar{v} - \frac{\tau_p}{\tau}\frac{\bar{\alpha}}{\bar{\gamma}} + \frac{1}{2\tau}$ and the last statement follows.

■

F.4 Proof of Corollary 1

Proof of Corollary 1.   Eq.(17) follows from (22) by substituting $\rho(a) = RRA/W(a)$ and $t(a) = \bar{\tau}$. From there, the only part of the corollary that requires a proof is a statement about mean preserving spread.

By definition, the change in the cross-section of wealth from $W_0(a)$ to $\check{W}(a)$ is mean preserving if

$$\int_0^\nu \left(W_0(a) - \check{W}(a)\right) da = 0 \text{ and } \int_\nu^1 \left(W_0(a) - \check{W}(a)\right) da = 0$$

and $\check{W}(a)$ is different from $W_0(a)$ on a set of points with positive Lebesgue measure. Then, the only "affected" term is $\int_0^1 W_0(a)^2 da$. We have

$$
\begin{aligned}
\int_0^1 \check{W}(a)^2 da &= \int_0^1 \left(W_0(a) + \check{W}_0(a) - W_0(a)\right)^2 da \\
&= \int_0^1 W_0(a)^2 da + \int_0^1 \left(\check{W}_0(a) - W_0(a)\right)^2 da + 2\int_0^1 W_0(a) da \int_0^1 \left(\check{W}_0(a) - W_0(a)\right) da \\
&= \int_0^1 W_0(a)^2 da + \int_0^1 \left(\check{W}_0(a) - W_0(a)\right)^2 da \\
&> \int_0^1 W_0(a)^2 da
\end{aligned}
$$

■

F.5 Proof of Proposition 2

Proof of Proposition 2.

Consider a process $\Delta \tilde{s}(l) = s(a + l) - s^i(a)$. The signal that the discrete trader a receives is $\Delta \tilde{s}(m)$. Similarly, the time-2 optimal wealth of the trader with a signal $\Delta \tilde{s}(l)$ is $\tilde{W}(l)$. Then the optimal time-2 wealth of the discrete trader is $\Delta \tilde{W}(m)$.

An optimal wealth of a discrete trader of size l with signal $\Delta \tilde{s}(l) = s(a + l) - s(a)$ must satisfy the FOC

$$\int u'(W)\left(\exp(v^a(l) - p) - 1\right) d\mathcal{N}(z) = 0.$$

Where,

$$v^a(l)(p, \Delta \tilde{s}(l), l, z) = E[v|\Delta \tilde{s}(l), p] + z\sqrt{\mathrm{Var}[v|\Delta \tilde{s}(l), p]}.$$

Thus, the optimal wealth can be viewed as an Ito process, driven by the process $ds(a)$. Applying Ito's lemma to the FOC we can write

$$dW = \mu_W(\Delta \tilde{s}(l), l; W, v, p)da + \sigma_W(\Delta \tilde{s}(l), l; W, v, p)dB_a.$$

From FOC, we can express $\Delta \tilde{s}(l)$ as a function of optimal wealth, and write the dirft and diffusion coefficients as functions of W, l, v, p only. In what follows we'll only write W and l as arguments of μ_W and σ_W.

While we can compute $\mu_W(l, W)$ and $\sigma_W(l, W)$ for any l, we will only need them at $l = 0$. There it is easily calculated from $dW = (R - 1)dx$, recalling that $dx = \alpha da + \beta ds_a - \gamma pda$

$$\mu_W(0, W) = (R - 1)\left(\alpha(W) - \gamma(W)p + \beta(w)v\right),$$

for rational traders and

$$\mu_W(0, W) = (R - 1)\left(\alpha(W) - \gamma(W)p\right),$$

for noise traders. We also have

$$\sigma_W(0, W) = (R - 1)\beta/\sqrt{t(a)}.$$

By Kolmogorov Forward Equation, the PDF of optimal wealth for the size of an agent equal to l, $f(w, l) = Pr(W(l) \in [w, w + dw])/dw$ o satisfies

$$f(w, l) = \left(f(w, 0) + \int_0^m \left(-\frac{\partial}{\partial w}(\mu_W(l, w)f(w, l)) + \frac{1}{2}\frac{\partial^2}{\partial w^2}(\sigma_W^2(w, l)f(w, l)) \right) dl \right) (1 - Pr(\text{death})) +$$
$$+ Pr(\text{death})\psi(w)$$

The death probability is $\delta m + O(m^2)$. The stationary density is such that $E[f(w, l)] = f(w, 0)$. Taking expectations, dividing by m and taking the limit as $m \to 0$ we obtain

$$0 = -\frac{\partial}{\partial w}(E[\mu_W(0, w)]f(w, 0)) + \frac{1}{2}\frac{\partial^2}{\partial w^2}(E[\sigma_W^2(w, l)]f(w, 0)) + \delta f(w, 0) - \delta\psi(w).$$

Now we compute

$$E[\mu_W(0, w)] = E[E[(R - 1)|p](\alpha(W) - \gamma(W)p)] + (1 - \nu)\beta(w)E[v(R - 1)]$$
$$= (1 - \nu)\beta(w)E[v(R - 1)]$$
$$= (1 - \nu)\beta(w)\tau^{-1}$$

Where the last transition is thanks to Lemma 8.

$$E[\sigma_W^2(w, l)] = \beta^2(w)/t(a)Var[R|p].$$

$$\frac{E[\mu_W(w, 0)]}{E[\sigma_W^2(w, 0)]} = (1 - \nu)\frac{t(a)\tau^{-1}1}{\beta(w)Var[R|p]} = (1 - \nu)\rho(W).$$

The statements about the tail exponent follow from the results in Gabaix et al. (2016).

We now turn to the statements about the characteristic function and the moments. First,

we make the change of variable

$$y = \ln W,$$

and derive the PDF of y, when W is drawn from $\phi(\cdot)$ and $\psi(\cdot)$, as follows:

$$\phi_{\log}(y) = e^y \phi\left(e^y\right),$$

$$\psi_{\log}(y) = e^y \psi\left(e^y\right).$$

Then, the KFE can be rewritten as

$$-\left(c_\mu + \frac{c_\sigma}{2}\right)\frac{\partial \phi_{\log}(y)}{\partial y} + \frac{1}{2}c_\sigma \frac{\partial^2 \phi_{\log}(y)}{\partial y^2} - \delta\phi_{\log}(y) + \delta\psi_{\log}(y) = 0,$$

where we have denoted

$$c_\mu = E[\mu_W]/W = (1-\nu)\beta(W)/W\tau^{-1}, \text{ and}$$

$$c_\sigma = \beta^2(W)\left(\exp(1/\tau) - 1\right)/\left(W^2\bar{t}\right).$$

Taking the Fourier transform of the last ODE and expressing $CF_{\phi,log}(s)$ from the resulting linear equation yields the answer for the characteristic function. The moment generating function for log wealth is $MGF_{\phi,log}(s) = CF_{\phi,log}(-is)$. The n-th moment is $MGF_{\phi,log}(n)$.

∎

F.6   Proof of Proposition 5

Proof of Proposition 5. To get (22) note that

$$\tau_p = \frac{\left(\int_\nu^1 \beta(a)da\right)^2}{\int_0^1 \frac{\beta(a)^2}{t(a)}da}.$$

Then (22) follows by substituting $\beta(a) = \frac{t(a)}{\rho(a)\tau(\exp(\tau^{-1})-1)}$ to integrals $\int_0^1 \beta(a)da$ and $\int_0^1 \frac{\beta(a)^2}{t(a)}da$ and rearranging.

■

F.7   Proof of Lemma 7.

Lemma 7. *Consider an Ito process $dz(a) = \mu(z, a)da + \sigma(z, a)dB(a)$. Consider a function $y = \phi(z(a), a)$ defined implicitly by $F(y, z, a) = 0$. Then, $\sigma_y = \frac{\partial \phi}{\partial z}$ and $\mu_y = \frac{\partial \phi}{\partial a} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial z^2}$ are given by*

$$\sigma_y = -\frac{F_z}{F_y},$$

$$\mu_y = -(F_a + F_z\mu + \frac{1}{2}F_{yy}\sigma_y^2\sigma^2 + F_{yz}\sigma_y\sigma + \frac{1}{2}F_{zz}\sigma^2)/F_y,$$

where μ, σ and partials of ϕ are evaluated at $(z(0), 0)$ and partials of F are evaluated at $(\phi(z(0), 0), 0, 0)$. These expressions can be obtained by applying Ito's lemma to $F(y, z, a)$ at $a = 0$, $dF = \mu_F da + \sigma_F dz$ and equalizing both μ_F and σ_F to zero.

Proof. We apply Ito's lemma at $a = 0$ to the identity

$$F(\phi(z(a), a), z, a) = 0.$$

We have

$$dF = F_y d\phi + F_z dz + F_a da + \frac{1}{2}F_{yy}d\phi^2 + F_{yz}d\phi dz + \frac{1}{2}F_{zz}dz^2.$$

Since $d\phi = \phi_z dz + \phi_a da + \frac{1}{2}\phi_{zz}dz^2 = \sigma_y dz + \mu_y da$ we have

$$\mu_F = F_y\mu_y + F_z\mu + F_a + \frac{1}{2}F_{yy}\sigma_y^2\sigma^2 + F_{yz}\sigma_y\sigma^2 + \frac{1}{2}F_{zz}\sigma^2 = 0$$

$$\sigma_F = F_y\sigma_y + F_z = 0.$$

Solving for μ_y and σ_y from the two equations above yields the stated result. ■

F.8 Proof of Lemma 6

Proof of Lemma 6. Suppose that there exists $\hat{t}$ such that $f_m(t(0), 0) < f_m(\hat{t}, 0)$. Then, by continuity, there exists $\bar{m}$ such that

$$f_m(t(m), m) < f_m(\hat{t}, m) \text{ for } m < \bar{m}.$$

Integrate the above with respect to m:[22]

$$f(t(m), m) - f(t(0), 0) < f(\hat{t}, m) - f(\hat{t}, 0).$$

Since $f(t, 0)$ does not depend on t we have $f(t(0), 0) = f(\hat{t}, 0)$ and so

$$f(t(m), m) < f(\hat{t}, m).$$

A contradiction with $t(m) \in \arg\max_t f(t, m)$. ■

F.9 Proof of Proposition 3

Proof of Proposition 3.

The first part of the proposition is a restatement of the Proposition 1.

For the second part, we use the closed form-solutions for the moments and obtain

$$\text{CV}^2 = \frac{M_\psi(2) \left(\delta RRA \left(e^{1/\tau} - 1\right) \tau^2 + (\nu - 1)\bar{t}\right)^2}{\delta M_\psi(1)^2 \left(e^{1/\tau} - 1\right) \tau^2 \left(\delta RRA^2 \left(e^{1/\tau} - 1\right) \tau^2 + \bar{t}(2(\nu - 1)RRA - 1)\right)} - 1.$$

Condition $1 - \frac{\bar{t}(2(1-\nu)RRA+1)}{\underline{l}\delta RRA^2} > 0$ is sufficient to ensure $M_\phi(2) > 0$ for all $\hat{\tau}$. It can be verified that both moments indeed exist provided that this condition holds. Differentiation of closed-form

---

[22]To integrate the left-hand side we use the Envelope Theorem $\frac{df(t(m),m)}{dm} = f_m(t(m), m)$

expression for CV with respect to τ yields

$$
d\text{CV}^2/d\tau = -\frac{M_\psi(2)\left(e^{1/\tau}(2\tau-1)-2\tau\right)\bar{t}\left(\delta RRA\left(e^{1/\tau}-1\right)\tau^2+(\nu-1)\bar{t}\right)}{\delta M_\psi(1)^2\left(e^{1/\tau}-1\right)^2\tau^4\left(\delta RRA^2\left(e^{1/\tau}-1\right)\tau^2+\bar{t}(2(\nu-1)RRA-1)\right)^2} \cdot
$$
$$
\cdot\left(\delta RRA\left(e^{1/\tau}-1\right)\tau^2+(\nu-1)\bar{t}(2(\nu-1)RRA-1)\right)
$$

which is positive (negative) for small enough (large enough) τ, provided that the condition $1-\frac{\bar{t}(2(1-\nu)RRA+1)}{\underline{l}\delta RRA^2}>0$ holds. ∎

F.10 Proof of Proposition 9

Proof of Proposition 9 .

Denote $\Delta\tilde{s}^i(l)=s(a^{i+1})-s(a^i+l)$. Fix the precision of trader i, $t_i=t$ and assume that $t(a)=t(a;m)$ for $a\notin\mathcal{A}^i$. Consider the relized utility process

$$
\mathcal{U}_r^i(l)=u((W_0^i-\mathcal{C}^i(t^i,l))+(\exp(v-p)-1)\hat{x}^i).
$$

Here $\hat{x}^i=\hat{x}^i(\Delta\tilde{s}^i(l),p,l))$ is the equilibrium allocation to trader i, implicitly defined by

$$
E^i\left[U'\left((W_0^i-\mathcal{C}^i(t^i,l))+(\exp(v-p)-1)x\right)(\exp(v-p)-1)\right]=0.
$$

By Ito's lemma we can write

$$
\mathcal{U}_r^i(m)-\mathcal{U}_r^i(0)=\int_0^m\mu_u(b)db+\int_0^m\sigma_u(b)dB,
$$

where μ_u and σ_u denote the drift and the diffusion coefficients of $\mathcal{U}_r^i(m)$ process. Since $E[\int_0^m\sigma_u(b)dB]=0$ we have that

$$
\frac{\partial\tilde{\mathcal{U}}^i(m)}{\partial m}(t;m)=\frac{\partial E[\mathcal{U}_r^i(m)-\mathcal{U}_r^i(0)]}{\partial m}=E[\mu_u(m)].
$$

Thus,

$$
\frac{\partial\tilde{\mathcal{U}}^i(0)}{\partial m}(t;0)=E[\mu_u(0)].
$$

To compute the drift of $\mathcal{U}_r^i$ at 0, we calculate the diffusion coefficient (the 'dB' coefficient) of $\hat{x}^i$ at 0, $\sigma_x(0)$. From Ito's lemma it follows that $\sigma_x(0) = \frac{1}{\sqrt{t(a)}} \frac{\partial \hat{x}^i}{\partial \Delta \tilde{s}^i(l)}(0, p^{-i}(0), 0)$. Calculations similar to that in section 3 yield that

$$\sigma_x(0) = \beta(a)/\sqrt{t(a)}.$$

Applying Ito's lemma to the process $\tilde{\mathcal{U}}^i(l)$ we get

$$daE[\mu_u(0)]/u'(W_0) = E[d\hat{x}^i(R-1)] - 1c(t^i) - \frac{1}{2}(d\hat{x}^i)^2 \rho(a)E[(R-1)^2].$$

Applying the familiar box calculus we have $(d\hat{x}^i)^2 = \sigma_x^2 da$. Note that $d\hat{x}^i = const \cdot da + \frac{\partial \hat{x}^i}{\partial \Delta \tilde{s}^i(l)}(0, p^{-i}(0), 0)vda + \sigma_x dB_a$. Then, since $E[R-1] = 0$ we have

$$E[d\hat{x}^i(R-1)] = \beta(a)E[v(R-1)] = \beta(a)\tau^{-1}1,$$

where the last transition is due to Lemma 8 (to follow). We also have

$$E[(R-1)^2] = E[E[(R-1)^2|p]] = (\exp(\tau^{-1}) - 1).$$

Combining everything, we get

$$\frac{E[\mu_u(0)]}{1u'(W_0)} = \frac{1}{2}\frac{1t^i/\tau}{\rho(a)\text{Var}[R|p]}\tau^{-1} - c(t^i).$$

Taking first-order condition with respect to t^i yields the stated result.

∎

Lemma 8. $E[v(R-1)] = \tau^{-1}1$.

Proof. Note that by law of iterated expectations we have

$$E[v(R-1)] = E[E[v(R-1)|p]]$$
$$= E[E[v(R-1)|p]].$$

Denoting $\mu_{v|p} = E[v|p]$ and $\tau^{-1} = Var[v|p]$ we have

$$
\begin{aligned}
E\left[v\left(R-1\right)|p\right] &= \frac{\partial}{\partial t}\, E\left[\exp\left(tv - p\right)\right]\big|_{t=1} - \mu_{v|p}1 \\
&= \frac{\partial}{\partial t}\, \exp\left(t\mu_{v|p} + \frac{1}{2}t^2\tau^{-1} - p\right)\bigg|_{t=1} - \mu_{v|p}1 \\
&= \left(\mu_{v|p} + \tau^{-1}\right)\underbrace{\exp\left(\mu_{v|p} + \frac{1}{2}\tau^{-1} - p\right)}_{=E[R|p]=1} - \mu_{v|p}1 \\
&= \tau^{-1}1
\end{aligned}
$$

∎

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