CHILE*

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Abstract

We introduce CHILE, an asymmetric information asset-pricing framework with general utilities and payoffs. It features a large economy (LE) with continuous-and-heterogeneous information (CHI). We apply the framework to ask how wealth inequality affects market quality. Holding the quality of private information fixed, making the rich richer and the poor poorer harms information efficiency but improves liquidity. So does making the rich more informed and the poor less informed while holding wealth fixed. With endogenous information, the above effects are reinforced. Overall, widening wealth inequality is a double-edged sword for market quality, increasing liquidity but harming information efficiency.

Keywords: inefficient markets, information aggregation, rational expectations with non-CARA preferences, wealth effects, competition

JEL Codes: D01, D53, D82, E19, G12, G14.

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1 Introduction

Market participants come in different sizes. Financial institutions differ in terms of the total value of assets they manage, while households and individuals differ in terms of wealth they invest in financial markets. As a growing literature establishes, these differences—hereinafter, "wealth inequality"—affect asset prices crucially, thereby also returns, risk premia, and risk-free rates (see Panageas, 2020, for a recent survey). Given that prices are but one outcome of financial trade, one takeaway from this literature is that wealth inequality may also matter for other outcomes, such as liquidity and informational efficiency. Collectively known as "market quality," these outcomes are vital for many economic decisions, including not only those faced by investors but also by policymakers gauging how well markets function. This raises the following question: How does wealth inequality affect market quality?

To answer this question, we need a theory of financial trade in which wealth inequality is an economic primitive, so that we can study changes in wealth inequality through comparative statics. For that, we need a model in which wealth matters, to begin with. Such models, however, are hard to come by; most models of inefficient markets are built on the assumption that investors have Constant Absolute Risk Aversion (CARA) preferences, and thus lack wealth effects.

Aiming to address these issues, we introduce a tractable framework of asymmetric information with preferences that are both general and heterogeneous. We do not restrict attention to Normal payoffs, sidestepping the widely-known limited empirical appeal of standard models, opting instead for general return distributions. We begin by extending the notion of competitive equilibrium in Hellwig (1980) to non-CARA utility functions. Using an information structure that we explain in more detail below, we develop a large economy in which prices are partially revealing, traders are points in an interval, trader characteristics (risk aversion, signal precision, and so on) are arbitrary functions over the interval, and the only source of noise in the economy comes from traders' private signals. Despite the generality of our primitives, our end product is tractable, with all equilibrium objects in closed form.

Turning to our question of how wealth inequality affects market quality, we use our model to study the effect of inequality reductions, representing them by changing the population of traders à la "Robin Hood". More specifically, in the context of individual investors, a Robin-Hood variation changes the distribution of wealth across traders, making the rich less rich and the poor less poor, without necessarily affecting aggregate wealth (for institutional investors, the changes are over fund size). Focusing on decreasing-absolute-risk-aversion (DARA) preferences, a plausibly realistic assumption, we show that reducing inequality makes prices more informative. Widening inequality has the opposite effect.

To see why narrowing inequality improves information efficiency, it helps to first develop a baseline for our intuition. We begin by pointing out that prices reveal the weighted average of all private signals, with weights proportional to the trading intensities of the agents the signals belong to.[1] We then ask the following: is there an "informationally ideal" signal-weighting scheme, however hypothetical, such that the corresponding price reveals maximum information? If so, we intuitively expect that signals of better quality get larger weights. As we show, a weighting scheme confirming this intuition does exist, weighing signals in proportion to their precision alone.

In contrast, the equilibrium weighting scheme is more involved. Accounting for wealth effects, the equilibrium scheme scales signals by risk tolerance on top of precision, placing more weight on the signals of more risk-tolerant traders. As risk tolerance increases in wealth under DARA, the signals of richer traders receive excessive weight, implying that the information content of prices is distorted due to wealth inequality. By transferring wealth from the rich to the poor, a Robin-Hood variation corrects this distortion, moving the equilibrium weighting scheme towards the ideal, making prices more informative.

The mechanism outlined above is empirically relevant at different economic timescales. For long timescales, our result can help interpret the long-term trend towards higher institutionalization, widening wealth inequality, and growing concentration of the asset management industry.[2] For short timescales, such as in high-frequency trading, our model suggests that changes in ef-

[1]As we discuss below, the trading intensity of an agent is defined as the sensitivity of his demand to changes in private information.

[2]The empirical literature offers recent evidence on these trends. See Ben-David, Franzoni, Moussawi, and Sedunov (2021) on the increasing concentration of the asset-management industry and institutionalization, and Saez and Zucman (2016) on widening wealth inequality.

ficiency are influenced by the size diversity of market participants trading at a given moment. For instance, prices become less informative when the active traders are predominately large.

We have, so far, assumed that our agents are endowed with signals and precisions. Given that, as Grossman and Stiglitz (1980) point out, our understanding of price efficiency rests critically on what we assume about the cost of information, one may wonder if our argument continues to hold if information is costly and endogenous. In fact, since people with different wealth may acquire different amounts of information, wealth variations turn on a new channel by changing signal precisions, one that requires more finessed comparative statics.

We isolate this new channel by studying what happens if we change precisions without changing wealth. A Robin Hood variation on precision—decreasing the precision of larger agents and increasing that of smaller ones—again improves information efficiency. When we reduce the precision of larger agents, they end up trading less aggressively, their signals receive smaller weights, thereby pushing the price towards the informational ideal. The opposite happens on the other end of the population, but with the same end effect.

This result has a surprising corollary: prices can become less informative even if everyone receives weakly more information. More concretely, increasing the precision of sufficiently large traders without changing that of others exacerbates the distortion discussed above, because it pushes the signal-weighting scheme further away from the informational ideal. Coming off as an information-aggregation paradox, this property is, in fact, a hallmark of imperfect aggregation with heterogeneous wealth effects. In short, more can be less, because more badly-aggregated information is less information.

Returning to whether wealth effects change if information is endogenous, we revisit our comparative statics in an extension where traders acquire information in the spirit of Verrecchia (1982), with information costs convex in precision (see Appendix B). We show that large traders acquire more information than smaller ones. This is an intuitive result: due to trading more aggressively than others, large traders have stronger incentives to acquire more information because they make more use of it. Consequently, the overall response to a Robin Hood variation in wealth combines two effects. One from wealth changes alone, improving efficiency directly, and the other from precision changes in response to wealth changes, improving efficiency indirectly by amplifying the direct effect.

Turning to other aspects of market quality, we ask how wealth inequality affects liquidity. We show that narrowing wealth inequality induces two conflicting effects on the willingness of agents to provide liquidity, both of which are knock-on effects of prices becoming more efficient, as discussed above. On the one hand, as efficiency improves, each trader puts more weight on commonly-observed price information, aligning his expectations closer to those of others. This effect decreases the agents' willingness to trade, reducing liquidity. On the other hand, as efficiency improves, prices deviate less from fundamentals and are thus de facto less volatile, reducing the risk that traders must absorb when they trade. This effect increases liquidity. As both effects are active in our model, wealth inequality affects liquidity non-monotonically. Nevertheless, we can separate out the risk component if we scale liquidity by return volatility, obtaining a globally monotone comparative static: as wealth inequality decreases, so does risk-adjusted liquidity.

Our framework has tractability rarely seen beyond the case with homogeneous agents and CARA preferences.[3] What enables it in our case is the way we model information. In contrast to the traditional methodology for large markets (Hellwig, 1980, and consequent literature), we do not assume that traders have signals of finite precision, because that would imply that as the number of traders becomes large, so does the total amount of information.[4] What is more, with signals of finite precision, traders would make finite speculative trades, with the unfortunate consequence that aggregate demand would explode for large numbers of traders.

We instead use an assumption similar to that in Section 9 of Kyle (1989), whereby a finite amount of information is distributed among all traders. By definition, then, the total amount of information remains finite regardless of the number of traders. Moreover, as individual de-

[3]Within asymmetric-information asset pricing, papers that go beyond the CARA-Normal framework (imposing, however, other assumptions for tractability) include Peress (2004), Peress (2014), Breon-Drish (2015), Malamud (2015) and Chabakauri, Yuan, and Zachariadis (2022). Allowing for non-Normal distributions, Breon-Drish (2015) and Chabakauri et al. (2022) require CARA, and thus do not incorporate wealth effects. Malamud (2015) requires complete markets. Peress (2004) and Peress (2014) achieve tractability by requiring the risk of assets to be small. See Section 11 for details.

[4]As the total amount of information is the precision of the sufficient statistic of private signals, it equals the sum of signal precisions held by all traders. Thus, in economies where the precision of each signal is finite (as in "neither infinite nor infinitesimal," i.e., neither infinitely large nor infinitely small), the total amount of information becomes infinite with an infinite number of traders.

mands are based on signals with precision inversely related to the size of the economy, aggregate demand is finite, even for infinitely many traders. As we show, we can formally treat information structures with the aforementioned properties as diffusion processes running through a heterogeneous continuum, giving us structures we call continuous-and-heterogeneous information (CHI). Recognizing that our model also requires a large economy (LE), we adopt the name CHILE for the class of models described in this paper.

Working with CHILE yields several advantages. First, it unlocks the full arsenal of stochastic calculus, improving tractability similarly to when one switches from discrete-time to continuous-time formulations. Second, the noise in traders' signals aggregates to a finite random variable, an Ito integral, instead of washing out as in Hellwig (1980). This type of aggregation yields prices that are noisy, ensuring that our competitive equilibrium is well-defined even without external noise.[5] Third, unlike with a discrete economy, working with a continuous economy implies that our main primitives are functions. Consequently, we can carry out comparative statics through the calculus of variations, opening up questions that are otherwise hard to address.

2 Setup

We define a two-period economy on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with trading at $t = 1$ and consumption at $t = 2$.[6] There are two assets, one risky and one risk-free. As we use the risk-free asset as a numeraire, we normalize its gross return to 1. The risky asset pays off $V(v)$ in the second period, where $v \sim N(0, \tau_v^{-1})$ is the *fundamental*, and $V(v)$, a weakly increasing function of v, is the *payoff function*.[7]

The risky asset trades at price P, determined in the first period. Allowing for both price-

[5] The absence of noise traders is beneficial for our research question, as we would otherwise need to take a stand on what happens to noise traders when the wealth distribution changes.

[6] In Appendix B, we study information acquisition, for which we add a period at $t = 0$.

[7] The normalization $\mathbb{E}[v] = 0$ is made without loss of generality, as any mean can be incorporated into the general form of $V(\cdot)$.

inelastic and price-elastic supply components, we write the total supply of the risky asset as

$$\Theta(P) = \bar{\theta} + \theta(P),$$

where $\bar{\theta}$ is a constant, and $\theta(P)$ is a continuous function.

We consider a *large economy* (LE), meaning that the population of our agents is a continuum. A particular agent $a \in [0, 1)$ starts with wealth $W_0(a)$, and trades the risky asset in the first period, learning about the fundamental by observing prices and a private signal. He consumes all his post-trade wealth W in the second period, obtaining utility $u(W, a)$.[8] We assume that $u(W, a)$ is increasing, strictly concave, and thrice continuously differentiable over W in an open neighborhood around $W_0(a)$.

Our economy features *continuous-and-heterogeneous information* (CHI), an information structure in which the *cumulative signal up to agent a* is the Itô integral

$$s(a) = \int_0^a \left(v\, db + \frac{1}{\sqrt{t(b)}} dB(b) \right) = v\, a + \int_0^a \frac{1}{\sqrt{t(b)}} dB(b). \tag{1a}$$

Here, $B(b)$ is a standard Brownian Motion on $[0, 1)$, independent of v. The *signal of agent a*, who lives in segment $[a, a + da)$, is the differential form of (1a),

$$ds(a) = v\, da + \frac{1}{\sqrt{t(a)}} dB(a). \tag{1b}$$

We stress that individual a does not observe $s(a)$ but only $ds(a)$, which can be thought of as an "infinitesimal slice" of $s(a)$.

To represent our CHI structure formally, we define the information set of agents in segment $[b, c)$ as the σ-algebra generated by the increments of the cumulative signal over $[b, c)$, $\mathcal{F}_{b,c} = \sigma\left(\{s(z) - s(b)\}_{b \le z < c}\right)$, and we assume that for any b and c, $0 \le b < c < 1$, the total information available to agents within $[b, c)$ is $\mathcal{F}_{b,c}$. That is, $\mathcal{F}_{b,c}$ is the information set we obtain if we combine the information owned by everyone in $[b, c)$. Under this definition, the information

[8] The agent location a is an index that tracks agents' individual characteristics, entering the utility function as a parameter. We thus refer to the agent and to his location interchangeably.

available in the entire economy is $\mathcal{F}_{0,1}$, denoted more compactly hereafter as $\mathcal{F}_1$.

For an arbitrary information set $\mathcal{F}$, we call the posterior gain in estimating v by observing $\mathcal{F}$, $\mathrm{Var}[v|\mathcal{F}]^{-1} - \mathrm{Var}[v]^{-1}$, the *cumulative precision of* $\mathcal{F}$. For the CHI process in (1), standard results in filtering theory imply that $\mathcal{F}_{0,a}$, the information available to agents in $[0, a)$, has cumulative precision $\int_0^a t(b)db$.[9] For signals of individual agents, we note that $ds(a)$ contributes $t(a)da$ to cumulative precision—we thus refer to the marginal change of cumulative precision of $\mathcal{F}_{0,a}$ at a, which is $\frac{d}{da}\left(\int_0^a t(b)db\right) = t(a)$, as the *precision of* $ds(a)$, or, if the context is clear, as *a's precision*.

To summarize, our agents are heterogeneous along three dimensions: initial wealth $W_0(a)$, preferences $u(W, a)$, and precisions $t(a)$. The profiles $W_0(a)$, $u^{(l)}(W_0(a), a)$, $l = 0$, 1, 2, 3, and $t(a)$ are arbitrary functions of a, continuous almost everywhere over $a \in [0, 1)$.

Remark 1. Beyond assumptions on preferences and payoffs, there are two further distinctions between our economy and the economies in Hellwig (1980) and subsequent literature (hereinafter, "traditional large economies"). First, in our economy the total amount of information is $\int_0^1 t(a)da$, a finite quantity by construction. In traditional large economies, each of an infinite number of agents has a signal of finite precision, implying that the total amount of information is infinite. Consequently, with signals of finite precision, the infinitely many agents make finite speculative trades, implying that aggregate demand blows up. Second, as we explain in detail below, we do not use noise traders, which traditional economies not only require to prevent fully-revealing prices, but they also scale in proportion to the number of agents, effectively assuming that noise variance is infinite, too.[10]

Remark 2. Our agents can be interpreted either as individual traders or as fund managers. In the latter case, $W(a)$ represents the value of the assets managed by fund a, with a certain proportion of $W(a)$ counting as wages for the manager of the fund. To lighten notation, we subsume all coefficients of proportionality in parameter a of the utility profile.

[9] See, e.g., Liptser and Shiryaev (2001), Theorem 10.1.

[10] See Hellwig (1980), Section 5, eq. (B.2).

3 Cumulative demand

There are two equilibrium objects of primary interest with all other objects derived from them: a cumulative demand function, $X_*(a, P)$, and a market-clearing price function, $\mathbf{P}_*$. We denote their non-equilibrium counterparts by dropping the asterisk. We discuss the former object here, leaving the latter for the next section. The equilibrium concept is introduced right after, in Section 5.

The *cumulative demand* is a function $X(a, P)$ such that for any b and c, $0 \leq b < c < 1$, and for every $P \in \mathbb{R}$, the total number of shares of the risky asset that agents in $[b, c)$ are willing to buy at price P is $X(c, P) - X(b, P)$.[11] Normalizing $X(0, P) = 0$, we have that for any $a \in [0, 1)$, the total demand of agents in $[0, a)$ is $X(a, P)$, and $X(1, P)$ is the *aggregate demand*.[12] We focus on equilibria that obey the following restrictions:

Assumption 1. *For every $P \in \mathbb{R}$, the equilibrium cumulative demand $X_*(\cdot, P)$ satisfies*

(i) For any b and c, $0 \leq b < c < 1$, $X_*(c, P) - X_*(b, P)$ is $\mathcal{F}_{b,c}$-measurable.

(ii) For any b and c, $0 \leq b < c < 1$, $\mathbb{E}\left[(X_*(c, P) - X_*(b, P))^2\right] < \infty$.

(iii) $\mathbb{E}[X_*(a, P)]$ is differentiable over $a \in [0, 1)$.

The first restriction above captures the intuition that the demand of traders in any given interval cannot depend on the information they do not have. The other two restrictions are technical, ensuring that demand is "well-behaved." As the restrictions of Assumption 1 apply only to equilibrium demand, they do not constrain our agent's strategy space. In fact, they allow us to represent demand in the following manner.

Lemma 1 (Representation Lemma)**.** *Suppose that Assumption 1 holds. Then there exist deterministic functions $\beta(a, P) : [0, 1) \times \mathbb{R} \to \mathbb{R}$ and $\delta(a, P) : [0, 1) \times \mathbb{R} \to \mathbb{R}$ such that for any $b, c \in [0, 1) : b < c$, $X_*(c, P) - X_*(b, P) = \int_b^c \delta(a, P)da + \int_b^c \beta(a, P)ds(a)$.*

[11] We assume that demand is well-defined for every $P \in \mathbb{R}$, although for some special cases we technically have $\text{supp}(\mathbf{P}_*) \subset \mathbb{R}$—for example, when $V(x) = \exp(x)$ the equilibrium is log-linear, and $\text{supp}(\mathbf{P}_*) = \mathbb{R}_+$. This poses no loss of generality, as we can always extend demand to $P \in \mathbb{R}$ by letting $X(a, P) = 0$ for $P \notin \text{supp}(\mathbf{P}_*)$ without contradicting neither market clearing nor optimization.

[12] We use the term *aggregate* (resp., *cumulative*) for quantities aggregated over the whole (resp., a subset of) population.

Lemma 1 implies that we can write the cumulative demand of agents in $[0, a)$ in differential form as

$$dX_*(a, P) = X_*(a + da, P) - X_*(a, P) = \delta(a, P)da + \beta(a, P)ds(a), \tag{2}$$

which is nothing other the individual demand of agent a. In what follows, we refer to $\beta(a, P)$, the sensitivity of a's demand to his private signal, as his *trading intensity*.

There are two main takeaways from (2). First, like the cumulative signal, cumulative demand is also an Itô integral. Second, individual demands are linear in individual signals, a property that, as we discuss in the next section, makes learning from prices and market clearing tractable. Whereas previous literature achieves this property through particular combinations of preferences with payoff distributions—typically CARA-Normal—in our setting it holds generally.[13] See Section 10.2 for an intuitive discussion of why this property emerges.

4 Market clearing and price inference

Turning to market clearing, we require that the equilibrium price $\mathbf{P}_*$ is $\mathcal{F}_1$-measurable, as the price cannot reflect more information than what is available in the entire economy. Due to (2), when agents see the price realisation $\mathbf{P}_* = P$, they extract the sufficient statistic

$$s_p = \int_0^1 \omega(a, P)ds(a) = v + \int_0^1 \omega(a, P)\frac{dB(a)}{\sqrt{t(a)}}. \tag{3a}$$

Thus, the price reveals a weighted-average signal, made up by weighing the signal of each agent a by

$$\omega(a, P) = \frac{\beta(a, P)}{\int_0^1 \beta(a, P)da},$$

[13] The class of REE with linear demand functions has been recently enlarged by extending Normal payoffs to the exponential family. See Breon-Drish (2015) for a single-asset case and Chabakauri et al. (2022) for the case of many assets.

namely, in proportion to a's trading intensity, $\beta(a,P)$.[14] The agents compute s_p from the market-clearing condition, as follows:

$$\int_0^1 dX_*(a,P)da = \Theta(P) \Rightarrow$$
$$s_p = \frac{\Theta(P) - \int_0^1 \delta(a,P)da}{\int_0^1 \beta(a,P)da} \equiv h(P). \tag{3b}$$

Next, we assume the following applies to the function $h(P)$ in equilibrium.

Assumption 2. *The function $h(P)$ defined in (3b) is strictly monotone.*

Under Assumption 2, the equilibrium price $\mathbf{P}_*$ and the sufficient statistic s_p have the same informational content. The sufficient statistic s_p in (3a) has a familiar "truth plus noise" form. Moreover, both the truth, v, and the noise, represented by the Itô integral in (3a), are normally distributed. This allows one to use the familiar Bayes rule with normal random variables, resulting in the characterization below.[15]

Lemma 2 (Price Inference)**.** *Suppose that Assumptions 1 and 2 hold. Then the conditional distribution of v given $\mathbf{P}_* = P$ is*

$$\mathcal{N}\left(\frac{\tau_p}{\tau}s_p, \frac{1}{\tau}\right).$$

The sufficient statistic s_p can be computed as $s_p = h(P)$, with $h(P)$ as in (3b). Here $\tau = \mathrm{Var}[v|P]^{-1} = \tau_p + \tau_v$, and τ_p is the precision of s_p, given as

$$\tau_p = Var[v|s_p]^{-1} - \mathrm{Var}[v]^{-1} = \left(\int_0^1 \frac{\omega(a,P)^2}{t(a)}da\right)^{-1}. \tag{4}$$

Remark 3. Our model does not require noise traders. Traditional large economies use noise traders to ensure that prices are not fully revealing—the noise entering prices through the aggregate signal is not enough, as it washes out in aggregation due to the exact Law of Large Numbers. In our economy, however, the aggregate signal does not reveal the fundamental, despite aggregating a large number of private signals. Instead, similarly to Avdis (2018), the

[14] Thus, our economy features generalized linear equilibria, as in Breon-Drish (2015), Glebkin, Gondhi, and Kuong (2021), and others.

[15] See Section 10.3 for a heuristic derivation and an intuitive discussion of Lemma 2.

aggregate noise entering prices is an Itô integral, namely, a random variable with positive and finite variance.[16]

5 Notion of competitive equilibrium

We maintain Assumptions 1 and 2 from here on. Our equilibrium concept is as follows.

Definition 1. *A* ***competitive equilibrium*** *is a cumulative demand function $X_*(a, P)$ and an $\mathcal{F}_1$-measurable price function $\mathbf{P}_*$ such that*

(i) $\mathbf{P}_*$ *clears the market, i.e.* $X_*(1, \mathbf{P}_*) = \Theta(\mathbf{P}_*)$, *and*

(ii) $X_*(a, P)$ *is optimal given* $\mathbf{P}_* = P$.

Definition 1 follows standard conventions in the literature, describing equilibrium as a juncture of market clearing and demand optimality. Nevertheless, we have not yet explained what optimality means for cumulative demand. To do so, we first note that in an economy with finitely many agents—we call such economies "discrete"—optimality of cumulative demand means that individual demands are optimal.

In our large economy, defining equilibrium using optimality for individual demands is tricky, as each individual demand is infinitesimal. As, however, cumulative demand is not, and optimality is well-defined in discrete economies, we can extend the notion of optimality for cumulative demand to large economies by continuity.

We proceed in exactly that manner below. We first define a sequence of discrete economies that are "neighbors" of the continuous economy introduced above, in the sense that the discrete economies converge to our continuous one. We then define optimal cumulative demand in the discrete economies. After that, we revisit demand optimality in our continuous economy, defining it as the limit of optimal discrete demands. As we explain below, our notion of

[16] Our price noise also ensures there is trade in our large economy in a way that, unlike Avdis (2018), is consistent with the notion of perfect competition in Hellwig (1980). See Remark 7 and our literature review for details.

optimality contains a crucial assumption: agents behave competitively, disregarding their influence on prices completely, in a way that sidesteps what Hellwig (1980) calls the "schizophrenia problem"—see section 6.3 for details.

6 The large economy as a limit of discrete economies

In this section, the terms "continuous" or "large" refer to the economy introduced in Section 2, while the terms "discrete" and "finite" refer to economies along a sequence, which, as mentioned above, converge to the one in Section 2, allowing us to extend optimality of demand to the continuum.

6.1 Technical conditions on the primitives of the large economy

Our next assumption, which we maintain everywhere below, ensures that certain equilibrium objects are well-defined.

Assumption 3. *The following hold for the candidate price function* $\mathbf{P}$*, the payoff function* $V(v)$*, and the agent primitives.*

(i) $\mathbf{P}$ *is an* $\mathcal{F}_1$*-measurable continuous random variable. Its joint density with* v*,* $f_{v,\mathbf{P}}(v, P)$*, is such that for any fixed* P*, there exist* $A > 0$ *and* $k > 0$ *(that may depend on* P*), such that* $f_{v,\mathbf{P}}(v, P) < A \exp(-kv^2)$*.*

(ii) $V(v)$ *satisfies* $\lim_{v \pm \infty} \frac{\ln|V(v)|}{v^2} \leq 0$*.*

(iii) The agent primitives jointly satisfy the regularity conditions

$$\int_0^1 \frac{1}{\rho(a)} da < \infty, \quad \int_0^1 \frac{t(a)}{\rho(a)} da < \infty, \quad \int_0^1 \frac{t(a)}{\rho^2(a)} da < \infty, \quad \int_0^1 \frac{t(a)\pi(a)}{\rho^2(a)} da < \infty,$$

where

$$\rho(a) = -\frac{u''(W_0(a))}{u'(W_0(a))}, \quad \pi(a) = -\frac{u'''(W_0(a))}{u''(W_0(a))}. \tag{5}$$

The first two conditions refer to the risky asset. The first one requires that prices and fundamentals have a joint density dominated by a Gaussian function, to which the second one adds something similar, but for the shape of the payoff function. The last condition is a certain type of joint integrability for the agent primitives.[17]

6.2 Primitives of the discrete economies

The agents of the nth discrete economy live in n subintervals of size $\mu = 1/n$ that form a uniform partition of $[0, 1)$. A particular agent i, $i = 1, \ldots, n$, lives in segment $[a_i, a_{i+1})$, with $a_i = (i-1)\mu$. His initial wealth is $W_0(a_i)$, his precision is $t(a_i)$, and his utility function over terminal wealth W is $u(W; a_i)$. Thus, discrete primitives are obtained by "sampling" those in the continuous economy at the leftmost points of agent subintervals. His private signal is

$$\Delta s_i = v\,\Delta a_i + \frac{1}{\sqrt{t\,(a_i)}}\Delta B(a_i), \tag{6}$$

with $\Delta a_i = a_{i+1} - a_i = \mu$ and $\Delta B(a_i) = B(a_i + \mu) - B(a_i) \sim \mathcal{N}(0, \Delta a_i)$. Note that if we replace Δ with d, (6) becomes (1b).

Remark 4. Partitioning the agent interval uniformly and using the same profile of preferences in the discrete economy as in the continuous economy are intended to simplify exposition. Our results hold more generally, as long as the norm of the partition vanishes as $n \to \infty$, and the discrete preference profile converges to the continuous one. See Section C.4.1 for details.

6.3 Optimal competitive demands

A competitive economy is widely understood as one consisting of agents that have no market impact whatsoever. Existing work on noisy Rational Expectations Equilibria (REE) recognizes this idea explicitly, using the term "price taking" for how competitive behavior is modeled. And yet, as Hellwig (1980) points out, competition in REE is not entirely consistent with price taking: even though traders ignore their influence on the price level, they do account for

[17] Condition (i) holds in equilibrium by Lemma 2, (ii) holds if $V(v)$ has bounded support, while (iii) is easily met if risk aversion, prudence, and precision are continuous functions over $a \in [0, 1]$.

their influence on the informational content of the price (a behavior that Hellwig (1980) dubs "schizophrenic").

Our approach differs from traditional REE in that our agents fulfill both aspects of price taking. The first aspect, that agents should not be able to influence price levels, is a standard property of any competitive equilibrium. The second aspect, that agents should not be able to influence price information, is the one that Hellwig finds problematic, identifying the crux as agents taking "account of the covariance between 'noise' in their own information and 'noise' in the price" (Hellwig, 1980, p. 478). Neither problem arises in our large market because, as (3) shows, changing $\beta(a, P)$ or $\delta(a, P)$ for a single agent a has no effect on the price.

The above notwithstanding, to extend demand optimality from discrete markets to continuous ones, we must first clarify the notion of competition among agents in our discrete economies. We conduct the same exercise as Hellwig (1980), passing from finite economies to the large limit. We fulfill the first aspect of competitive behavior by treating prices as fixed in the demand-choice problem of our agents. For the second aspect, we take a page right out of Hellwig's critique: our agents assume that the noise in their private information is independent of the noise in the price.

Formally, for each discrete economy, each agent assumes that his signal is independent of the price conditional on the fundamental.[18] More specifically, agent a, who in the nth discrete economy lives in subinterval $a \in [a_i, a_i + \mu)$, believes that the market-clearing price is the realization of an $\mathcal{F}_1$-measurable continuous random variable $\mathbf{P}_i^n$, which we call a *price-function conjecture*. These conjectures indexed over a, $\mathbf{P}^n(a) = \mathbf{P}_i^n$, for all $a \in [a_i, a_i + \mu)$ and each i, constitute a *profile of price-function conjectures*. We note that the profile $\mathbf{P}^n(a)$ can differ from the actual price function $\mathbf{P}$ for finite n, but it must converge to $\mathbf{P}$ for every a as n becomes large, guaranteeing that the price conjectures are correct in the limit. We summarize our assumptions on price conjectures below.

Assumption 4. *Agent i in the nth discrete economy*

(i) believes that the market-clearing prices are realizations of an $\mathcal{F}_1$-measurable continuous

[18]The price and the signal cannot be independent unconditionally as they both reflect the fundamental v. Conditioning on v isolates the noise in both.

random variable $\mathbf{P}_i^n$ that has joint density $g(v, P, \mu)$ with v.

(ii) assumes that conditional on v, $\mathbf{P}_i^n$ and Δs_i are independent, and

(iii) makes the right price conjecture in the large economy, that is, $\mathbf{P}^n(a) \xrightarrow[n\to\infty]{\mathbb{P}} \mathbf{P}$, for all $a \in [0, 1)$.

We refer to the joint density of fundamentals and prices, $g(v, P, \mu)$, as the *price belief*. We make this belief symmetric across agents, a natural assumption given that it applies to quantities common to all agents.[19] Highlighting that we have defined the price belief using $\mu \equiv 1/n$ as an index for the sequence of economies instead of n, we also impose the following restrictions on it.[20]

Assumption 5. *The density $g(v, p, \mu)$ is continuously differentiable in μ for every $v \in \mathbb{R}$ and $P \in \mathbb{R}$. Moreover, for any fixed P, there exist constants A_1, $A_2 > 0$ and k_1, $k_2 > 0$ which may depend on P, such that $g(v, P, \mu) < A_1 \exp(-k_1 v^2)$ and $g_\mu(v, P, \mu) < A_2 \exp(-k_2 v^2)$, for all $\mu \in (0, 1)$.*

With the price belief as above, our next Lemma characterizes the conditional density of fundamentals given the signal and price.

Lemma 3. *Suppose Assumptions 4 and 5 hold. The conditional density of v given $\mathbf{P}_i^n = P$ and $\Delta s_i = s$ can then be written as*

$$f_{v|\Delta s_i, \mathbf{P}_i^n}(v, s, P) = \frac{\exp\left(t(a_i)(sv - \frac{\mu}{2}v^2)\right) g(v, P, \mu)}{\int_{\mathbb{R}} \exp\left(t(a_i)(sy - \frac{\mu}{2}y^2)\right) g(y, P, \mu) dy}. \tag{7}$$

Lemma 3 demonstrates the conditional density of fundamentals given signals and price is completely determined by the price belief. As the conditional distribution of fundamentals in (7) is the only distribution we need to describe demand choice, we are now able to define demand optimality in the discrete economies.

[19]Symmetry over price beliefs guarantees a unique solution for $\delta(a, P)$. Our main conclusions do not change without this symmetry, as our measures of market quality depend only on $\beta(a, P)$, which is uniquely determined even without symmetric price beliefs.

[20]Working with $\mu \in (0, 1)$, which is one-to-one with n, is more convenient because it has a bounded range.

Definition 2. *Given the realizations $\mathbf{P}_i^n = P$ and $\Delta s_i = s$, the demand of agent i in the nth discrete economy is **optimal** if it solves*[21]

$$x_i^*(s, P) = \arg\max_x \int_{\mathbb{R}} u\Big(W_0(a_i) + x\left(V(v) - P\right), a_i\Big) f_{v|\Delta s_i, \mathbf{P}_i^n}(v, s, P) dv. \tag{8}$$

Equation (8) defines a standard problem of maximizing expected utility, with fundamentals distributed as in Lemma 3. We maintain Assumptions 4 and 5 from here on.

Our last task in this section is to define optimal cumulative demand in the large economy as a limit of the same object in finite economies. To do so, we first establish if and when such limit exists. Hereafter, we denote $\mathbb{E}[\cdot|\mathbf{P} = P]$ as $\mathbb{E}[\cdot|P]$.

Lemma 4. *For a fixed realization $\mathbf{P} = P$, let*

$$X(a, P) = \operatorname*{plim}_{n\to\infty} \sum_{i:a_i<a} x_i^*(\Delta s_i, P) \tag{9}$$

with $x_i^*(\cdot)$ as in (8). This limit exists if and only if $\mathbf{P}$ satisfies

$$\mathbb{E}[V(v)|P] = P. \tag{10}$$

If (10) holds, $X(a, P)$ satisfies Assumption 1 and there exist deterministic functions $\beta(a, P)$ and $\delta(a, P)$ such that

$$X(a, P) = \int_0^a \beta(b, P) ds(b) + \int_0^a \delta(b, P) db.$$

As the above result establishes, optimal cumulative demand is well-defined if and only if the market is weak-form efficient as in (10). In intuitive terms, the large market cannot accommodate non-infinitesimal trades from infinitely many agents: if, for example, $\mathbb{E}[V(v)|P] > P$ holds, all agents would buy finite amounts, making the cumulative demand explode. Consequently, price functions that violate (10) are not viable candidates for equilibrium, and thus we can focus on those that do satisfy (10).

[21] Given that preferences are strictly concave, the maximum in (8) is unique.

Definition 3. *The limiting cumulative demand $X(a, P)$ in (9) is **optimal** given $\mathbf{P} = P$ for those price functions $\mathbf{P}$ which satisfy (10).*

To summarize, Definition 3 not only describes the notion of demand optimality in the large economy, but it also establishes that this notion is well-defined. Namely, an optimal cumulative demand exists, and it satisfies all necessary equilibrium restrictions, such as Assumption 1 and Lemma 1.

Remark 5 (On risk premia). The weak-form efficiency of Equation (10) implies that the risk premium is zero in our economy. This type of efficiency makes sense within our context: our agents start without endowments in the risky asset, they trade infinitesimal amounts, and they end up holding infinitesimal amounts of risk post-trade. We abstract away from how inequality affects the risk premium, as such effects are relatively better understood in existing work (see Panageas, 2020, for a survey). Instead, we focus on the less explored effects of inequality on market quality.[22]

Remark 6 (CHILE vs REE). As highlighted in Hellwig (1980), another concern with traditional models of competition under asymmetric information lies within their version of noiseless equilibrium: the information conveyed by prices is unaffected by preferences (see also Grossman, 1976). As Hellwig points out, this is somewhat counter-intuitive: "One would expect that the weight with which agent i 's information I_i affects the equilibrium price should depend on the strength of agent i 's reaction to this information, which in turn should depend on his preferences. Presumably, it should make a difference whether the news of an increase in a firm's profits is passed to somebody who is almost risk-neutral and responds by buying a large number of shares or whether this piece of news is passed to a risk averter who hardly responds at all." As we discuss below, the informational content of prices *does* depend on preferences in our equilibrium, and in a way that aligns with the intuition in the quote.

Remark 7 (No no-trade). Our equilibrium involves trade despite lacking external noise—a property that may seem surprising given well-known results (Milgrom and Stokey, 1982; Tirole, 1982). By conjecturing that the noise in their signals is independent of that in the price, our

[22]In ongoing work, we introduce a positive risk premium in CHILE through endowments in the risky asset. As we show, a certain type of efficiency analogous to (10) emerges, but it holds under the risk-neutral measure (as opposed to under the physical one).

agents treat their signals as incrementally informative relative to the price, in contrast to Grossman (1976)'s agents who treat their signals as dominated by the price. As a result, our agents employ their signals to formulate their demands, and trade ensues.

Nevertheless, we stress that our agents make the right price conjectures in the large limit. Although it may thus appear that trade would disappear in the limit, within the logic of our model the no-trade intuition works differently. As the economy grows, the size of individual demands shrinks to zero, but their number grows, with cumulative demand counterbalancing their shrinking size by summing up more of them. This balance is maintained all the way to the large limit, guaranteeing that aggregate demand converges to a well-defined and non-trivial quantity. Restated colloquially, in our model the no-trade theorem applies to individual—but not cumulative—demand.

7 Competitive equilibrium in CHILE

We present the main theorem of the paper. In Section 10, we provide heuristic derivation for the most important statements of the theorem, while the rigorous proof is in the Appendix.

Theorem 1. *There exists a unique equilibrium. The equilibrium price function has the representation* $\mathbf{P}_* = \mathcal{P}(s_p)$, *where*

$$s_p = \int_0^1 \omega(a) ds(a) = v + \int_0^1 \frac{\omega(a)}{\sqrt{t(a)}} dB(a) \tag{11}$$

is the equilibrium sufficient statistic and $\omega(a)$ *is a weighting function given by*

$$\omega(a) = \frac{t(a)}{\rho(a)} \left(\int_0^1 \frac{t(b)}{\rho(b)} db \right)^{-1}. \tag{12}$$

The function $\mathcal{P}(x)$ *is given by*

$$\mathcal{P}(x) = \int V\left(\frac{\tau_p}{\tau} x + \frac{z}{\sqrt{\tau}} \right) d\Phi(z). \tag{13}$$

Here $\Phi(z)$ *denotes the standard normal cumulative distribution function (cdf). Consequently,* *the price function is completely determined by $V(\cdot)$ and two other quantities, the precision of s_p, given by*

$$\tau_p = \frac{\left(\int_0^1 \frac{t(a)}{\rho(a)} da\right)^2}{\int_0^1 \frac{t(a)}{\rho(a)^2} da}, \tag{14}$$

and the posterior precision of v, given by $\tau = \mathbb{V}ar(v|P)^{-1} = \tau_v + \tau_p$.

The equilibrium cumulative demand function has the representation $dX(a) = \beta(a,P)ds(a) + \delta(a,P)da$, where

$$\beta(a,P) = \frac{t(a)}{\rho(a)} \frac{\tau^{-1}\,\mathbb{E}[V'(v)|P]}{\mathrm{Var}\,[V(v)|P]}, \text{ and}$$

$$\delta(a,P) = \frac{\beta(a,p)^2}{2t(a)}\pi(a)\frac{\mathrm{Sk}[V(v)|P]}{\mathrm{Var}[V(v)|P]} - \beta(a,p)\frac{\mathbb{E}[v(V(v)-P)^2|P]}{\mathrm{Var}[V(v)|P]} + \frac{\psi(P)}{\rho(a)\mathrm{Var}[V(v)|P]}. \tag{15}$$

Here $\rho(a)$ and $\pi(a)$ denote the absolute risk aversion and prudence coefficients, defined in (5). The conditional moments of $V(v)$ and the function $\psi(P)$ are given in the closed form in Appendix C.4.

The theorem above highlights the notable tractability of CHILE. All equilibrium objects are available in closed form despite rich heterogeneity and the generality of preferences. We will make use of this tractability in Section 9, where we examine the effects of changes in wealth distribution on market quality. We now discuss the main features of our equilibrium.

Trading intensity $\beta(a,P)$. The trading intensity has an intuitive structure. First, $\beta(a,P)$ is proportional to $t(a)/\rho(a)$: More informed traders (those with higher $t(a)$) and more risk-tolerant ones (those with higher $1/\rho(a)$) trade more aggressively. Second, $\beta(a,P)$ is inversely proportional to $\mathrm{Var}(V(v)|P)$: Higher uncertainty makes all traders scale down their trading intensities. Third, $\beta(a,P)$ is proportional to $\mathbb{E}[V'(v)|P]$: Trading intensities are higher for assets with payoffs that are more sensitive to changes in fundamentals.[23]

The key results of our paper operate through *wealth effects*. With non-CARA utility, the risk tolerance $1/\rho(a)$ depends on the initial wealth $W_0(a)$. In a realistic case of decreasing absolute risk aversion, risk tolerance increases with wealth. In this scenario, Theorem 1 suggests that

[23]The last property is intuitive because higher $\mathbb{E}[V'(v)|P]$ implies that news about fundamentals is more payoff-relevant: signals about fundamental v tell more about payoffs $V(\cdot)$.

wealthier investors trade more aggressively, and their signals receive greater weight in the price.

Price function. The price reflects the weighted average of traders' signals $ds(a)$. Moreover, the weights are proportional to $t(a)/\rho(a)$: the signals of more informed and more risk-tolerant traders have greater weights. This is intuitive as such traders trade more aggressively.

We also note that the closed-form expression (13) for the price function is a restatement of the efficiency condition (10). Indeed, Lemma 2 implies that conditional distribution of v given P is Normal with a mean of $\tau_p/\tau \cdot s_p$ and a variance of $1/\tau$. Thus, $z = \sqrt{\tau}(v - \tau_p/\tau \cdot s_p)$ has standard normal distribution. Therefore, one can write $v = \tau_p/\tau \cdot s_p + z/\sqrt{\tau}$. Substituting this into the efficiency condition yields $\mathbb{E}[V(v) \mid P] = \mathbb{E}[V(\tau_p/\tau \cdot s_p + z/\sqrt{\tau})] = \int V(\tau_p/\tau \cdot s_p + z/\sqrt{\tau})\, d\Phi(z)$. The same change of variable is used to obtain the closed-form expression in (52) for other conditional moments of $V(v)$.

The coefficients $\delta(a, P)$. The first two terms in (15) indicate that our equilibrium is influenced by the higher moments of the payoff (skewness, $\mathrm{Sk}(V(v)|P)$), as well as by the higher derivatives of the utility function (prudence, $\pi(a)$). This contrasts with the approach of Peress (2004), where these effects do not play a role. Peress (2004) employs a "small risk" approximation (assuming the variance of the fundamental is small), rendering higher-order risk negligible in his model. In contrast, we use a "small information" approximation, where the risk faced by each trader remains substantial even in the limit.[24]

The last term in (15) represents the demand of a trader with no private signal, referred to as "uninformed demand" hereafter. To see this, note that if we set $t(a) = 0$, the term $\psi(P)/(\rho(a)\mathrm{Var}[V(v)|P])$ becomes the only non-zero component in the coefficients $\beta(a, P)$ and $\delta(a, P)$. To better understand this term, let us first discuss the risk premium in our economy.

[24] Other papers that use a local second-order approximation to the utility function, such as Samuelson (1970), Campbell and Viceira (2002), Farboodi, Singal, Veldkamp, and Venkateswaran (2022b), and Mihet (2022), also do not account for higher-order effects. However, our results demonstrate that a second-order approximation is not always sufficient. Under small information asymptotics employed here, a third-order approximation to utility is necessary. Even under small-risk asymptotics, the second-order approximation may not be valid. For example, in the setting of Peress (2004), it is valid when traders learn about the mean of the fundamental, but it fails when they learn about the payoff itself (see Peress (2011)).

Risk premium. As previously noted, our efficiency condition (10) implies the absence of a risk premium in our economy. However, a more accurate description of the risk premium in CHILE is

$$\mathbb{E}[V(v) - P \mid P] = \psi(P)\, da.$$

This expression means that the risk premium in the discrete economy with an average trader mass of $\mu = da$ is $\psi(P)\, da$.[25]

With this understanding, the uninformed demand dX^u takes on a familiar mean-variance form:

$$dX^u = \frac{\psi(P)}{\rho(a)\mathrm{Var}[V(v) \mid P]}\, da = \frac{\mathbb{E}[V(v) - P \mid P]}{\rho(a)\mathrm{Var}[V(v) \mid P]}.$$

8 Market quality

Our measure of information efficiency is based on how much prices reduce uncertainty about the fundamental and is defined as

$$\mathcal{I} = 1 - \frac{\mathbb{V}\mathrm{ar}(v|P)}{\mathbb{V}\mathrm{ar}(v)}.$$

This measure is common in both theoretical (e.g., Rostek and Weretka, 2012) and empirical work (e.g., Dessaint, Foucault, and Fresard, 2024; Dávila and Parlatore, 2023). In practice, $\mathcal{I}$ corresponds to the R^2 of forecasting fundamentals by prices, with the fundamental v typically proxied by earnings in empirical work.

Our measure of liquidity is based on how much supply shocks can move prices

$$\mathcal{L} = -\left(\left.\frac{\partial \mathcal{P}(s_p, \bar{\theta})}{\partial \bar{\theta}}\right|_{\mathcal{P}=P} \right)^{-1} \mathbb{V}\mathrm{ar}(V|P). \tag{16}$$

The first term in (16) captures price sensitivity to unexpected price-inelastic supply shocks, a standard measure of liquidity in the theoretical literature (e.g. Vayanos and Wang, 2013). As our equilibrium is generally non-linear, this term depends on the realized price signal s_p,

[25]Rigorously, this means that $\lim_{n\to\infty} \int (V(v) - P) g(v, P, \mu)/\mu = \psi(P)$, as we formally show in the Appendix.

a stochastic quantity with few (if any) empirical counterparts. As we show next, scaling by $\mathbb{V}\mathrm{ar}(R|P)$ in (16) allows us to obtain the risk-adjusted liquidity measure that does not depend on the aggregate signal, and we thus adopt it as our primary notion of liquidity.

Proposition 1. *The equilibrium expressions for information efficiency $\mathcal{I}$ and liquidity $\mathcal{L}$ are*

$$\mathcal{I} = \left(1 + \tau_v \frac{\int_0^1 \frac{t(a)}{\rho(a)^2} da}{\left(\int_0^1 \frac{t(a)}{\rho(a)} da\right)^2}\right)^{-1}, \quad \mathcal{L} = \frac{\int_0^1 \frac{t(a)}{\rho(a)^2} da}{\int_0^1 \frac{t(a)}{\rho(a)} da}.$$

As we see above, information efficiency and liquidity are in closed form. What is more, they are parsimoniously characterized by just three primitives of the economy, two profiles and one parameter: risk tolerances $1/\rho(a)$, precisions $t(a)$, and prior uncertainty $1/\tau_v$. This enables tractable comparative statics that we turn to next.

9 Comparative statics

We now explore how market quality is affected by inequality, starting with inequality in wealth, followed by inequality in precision. In more concrete terms, we consider an equilibrium object $\mathcal{O}$—our placeholder notation for either information efficiency $\mathcal{I}$ or liquidity $\mathcal{L}$—and we change the primitives $W_0(a)$ and $t(a)$ around those associated with the original equilibrium. That is, the task at hand for this section is to carry out comparative statics with respect to functions. The right tool for this job is the Gateaux derivative.

Definition 4. *The* ***Gateaux derivative*** *an equilibrium object $\mathcal{O}$ with respect to a parameter $h(a)$ in the direction $h^{\Delta}(a)$ is*

$$\mathcal{O}'\left(h(a)\right)\left[h^{\Delta}(a)\right] = \lim_{\varepsilon \to 0} \frac{\mathcal{O}\left(h(a) + \varepsilon h^{\Delta}(a)\right) - \mathcal{O}\left(h(a)\right)}{\varepsilon}.$$

To facilitate exposition, we use the following conventions and types of notation in our definition above. The *parameter* (namely, a function) with respect to which we differentiate appears inside round brackets, while the *direction* (another function) along which we perturb the parameter appears inside square brackets. Our notation also distinguishes the parameter from the direction by indicating the direction with a Δ superscript.[26]

Of particular interest are directions that correspond to reduced inequality. To this end, we refer to a variation of model parameters that makes poor agents better off (either richer or more informed) and rich agents worse off (either poorer or less informed) as a *Robin Hood variation*. We then index our agents in the same order as their wealth, implying that $W_0(a)$ *increases* in a, an assumption maintained hereafter without loss of generality. The poor and rich are defined with respect to thresholds $\underline{a}$ and $\bar{a}$, with the poor lying below $\underline{a}$ and the rich above $\bar{a}$.

Definition 5. *A* ***Robin-Hood variation*** *of parameter $h(a)$ is a direction $h^{\Delta}(a)$, bounded over $a \in [0, 1)$, and two associated thresholds $\underline{a} < \bar{a}$ such that*

(i) $h^{\Delta}(a) \geq 0$ *for* $a < \underline{a}$*,*

(ii) $h^{\Delta}(a) \leq 0$ *for* $a > \bar{a}$*, and*

(iii) $h^{\Delta}(a)$ *is not zero for some set of indices a with positive Lebesgue measure.*

Returning to our questions on wealth inequality, we can arrive at answers by considering how Robin-Hood variations in wealth affect market quality. As, however, wealth effects may also enter through information acquisition, we must do so with care, isolating different possible effects. We thus first take Robin-Hood variations with respect to wealth keeping precisions fixed, then with respect to precisions keeping wealth fixed, then allowing for both effects by letting precisions depend on wealth.

As we see below, this sequence of comparative statics not only uncovers useful intuition but also reveals some surprising results. Before we proceed, however, we establish an important benchmark for the discussion of later subsections.

[26] Once we fix a direction, computing a Gateaux derivative can be done with standard calculus. One differentiates $\mathcal{O}\left(h(a) + \varepsilon h^{\Delta}(a)\right)$ with respect to the scalar ε, evaluating the result at $\varepsilon = 0$.

9.1 Ideal information aggregation in CHILE

We begin by pointing out that prices reveal the weighted average of all private signals—see equation (11). Revisiting how information is aggregated in our economy, we ask: is there an "informationally ideal" version of the weighting function $\omega(a)$, in the sense that the corresponding price would reveal maximum information to the agents?

We answer this question for a hypothetical weighting function $\omega^H(a)$, using it to define the notion of an *aggregate signal* as

$$s\left[\omega^H(a)\right] = \int_0^1 \omega^H(b)ds(b), \qquad \omega^H(a) \geq 0, \qquad \int_0^1 \omega^H(a)da = 1.$$

Writing $s_p = s[\omega(a)]$ then shows that the equilibrium-price weights $\omega(a)$ are but one possibility for $\omega^H(a)$, with the more general $\omega^H(a)$ yielding an aggregate signal with precision

$$\mathbb{V}\mathrm{ar}\big(v \big| s\left[\omega^H(a)\right]\big)^{-1} - \mathbb{V}\mathrm{ar}(v)^{-1}.$$

The informationally-ideal weighting function maximizes this precision, and is characterized in the following result.

Lemma 5 (Ideal information aggregation)**.** *No aggregate signal can exceed the cumulative precision of the entire economy, that is,*

$$\mathbb{V}ar\big(v \big| s\left[\omega^H(a)\right]\big)^{-1} - \mathbb{V}ar(v)^{-1} \leq \int_0^1 t(a)da \tag{17}$$

for any weighting function $\omega^H(a)$. The weighting function that maximizes signal precision satisfies (17) as an equality, and is given by

$$\omega^I(a) = \frac{t(a)}{\int_0^1 t(a)da}. \tag{18}$$

As we can see in (18), if we were to design an aggregate signal with maximizing informativeness as our only goal, we should be weighing each signal in proportion to its precision. Comparing (18) with (12) now highlights the key to understanding information inefficiency:

the informationally-ideal weights are proportional to precisions, $\omega^I \propto t(a)$, whereas the price weights ω are distorted by risk tolerances, $\omega \propto t(a)/\rho(a)$. With DARA, it is the wealthier agents that can tolerate more risk. Consequently, DARA preferences yield equilibria where prices overweigh the signals of the rich, underweighing those of the poor.

9.2 Wealth inequality

Using the result above as an illuminating benchmark, we examine the effects of transferring wealth from the rich to the poor on market quality holding precisions fixed. To proceed, we need the following technical conditions.

Assumption 6. *The following hold for the profiles of wealth, $W_0(a)$, absolute risk tolerances, $1/\rho(a)$, and relative risk aversions, $\rho(a)/W_0(a)$.*

1. *The cross-sectional cdf of wealth is continuous, strictly increasing, and has support $[0, \infty)$;*
2. *The cross-sectional cdf of absolute risk tolerances $1/\rho(a)$ is continuous and strictly increasing;*
3. *There exists constants $\underline{\eta}$ and $\overline{\eta}$ such that $0 < \underline{\eta} \le \rho(a)/W_0(a) \le \bar{\eta} < \infty$.*[27]

As in earlier parts of the paper, we maintain the above assumption, from its statement onwards. We next state the main result of this section.

Proposition 2. *Suppose that agent preferences are DARA. Then, there exist thresholds $0 < a_1^W \le a_2^W < 1$, such that for any Robin Hood variation $W_0^\Delta(a)$ with $\underline{a} \le a_1^W \le a_2^W \le \bar{a}$ we have*

$$\mathcal{I}'(W_0(a))[W_0^\Delta(a)] > 0 \text{ and } \mathcal{I}'(W_0(a))[-W_0^\Delta(a)] < 0; \tag{19}$$

$$\mathcal{L}'(W_0(a))[W_0^\Delta(a)] < 0 \text{ and } \mathcal{L}'(W_0(a))[-W_0^\Delta(a)] > 0. \tag{20}$$

Reducing inequality by transferring wealth from the sufficiently rich to the sufficiently poor leads to higher informational efficiency. The intuition behind this is rooted in the distortion

[27]The cross-sectional cdf of wealth (resp., absolute risk tolerances) is defined as $F_{W_0}(x) = \Lambda(a : W_0(a) \le x)$ (resp., $F_{1/\rho}(x) = \Lambda(a : 1/\rho(a) \le x)$. Here $\Lambda(\cdot)$ denotes the Lebesgue measure.

highlighted in the previous section: the price overweights the signals of the rich and underweights the signals of the poor. By redistributing wealth from the wealthy to the poor, this distortion is corrected, thereby improving informational efficiency.

We now turn to the liquidity result (20). There are two channels at play. First, a reduction in inequality is associated with increased information efficiency. This means each individual trader faces more adverse selection, making them less willing to provide liquidity. Second, there is the uncertainty reduction channel: higher information efficiency leads to less uncertainty about fundamentals, potentially decreasing $\mathrm{Var}[V(v)|P]$ and making traders more willing to provide liquidity. By examining the risk-adjusted measure $\mathcal{L}$, we isolate the first effect. As a result, a reduction in inequality negatively impacts $\mathcal{L}$.

Beyond providing comparative statics, our results connect with two sets of empirical observations. First, regarding secular trends, it is known that while stocks have become more liquid since the beginning of the 20th century (e.g., Chordia, Roll, and Subrahmanyam, 2001), the informativeness of the average US stock has deteriorated (Farboodi, Matray, Veldkamp, and Venkateswaran, 2022a). This is puzzling given the improved data availability in modern markets.[28] Our model can explain both trends by appealing to the growing wealth inequality among individual investors (Saez and Zucman, 2016), or to the unequal distribution of assets under management for institutional investors (Ben-David et al., 2021).

Second, applied at a higher frequency, our results suggest that market quality responds to temporal changes in the size distribution of market participants. Relative to a market composed of traders with similar sizes, a market composed of traders with disparate sizes has noisier prices and higher liquidity. Given that noisier, more liquid markets constitute better trading environments for all investors irrespective of size, if large institutions ("whales") and small institutions or individuals ("small fry") were able to choose who to trade with, they would prefer those least like them.[29] Correspondingly, the trading volume clustering in McInish and Wood (1992) can be interpreted as an outcome of "coordination" among traders' choices of

[28]There is also evidence that the trend in price informativeness is not uniform: Bai, Philippon, and Savov (2016) find that the informativeness of S&P 500 firms has increased over time, while Farboodi et al. (2022a) show that the negative average trend is driven by small firms.

[29]Our model can be extended to incorporate two trading periods (or two trading venues), allowing investors to choose when (or where) to trade. We leave such extensions for future work.

when—and thus against whom—to trade: small fry attract whales, and whales attract small fry.

9.3 Information inequality

Here, we examine the effects of changing the distribution of information across agents on market quality holding the wealth profile fixed.

Proposition 3. *Suppose that traders have DARA utilities. Then, there exist thresholds* $0 < a_1^t \leq a_2^t < 1$*, such that for any Robin Hood variation* $t^{\Delta}(a)$ *with* $\underline{a} \leq a_1^t \leq a_2^t \leq \bar{a}$

$$\mathcal{I}'(t(a))[t^{\Delta}(a)] > 0 \text{ and } \mathcal{I}'(t(a))[-t^{\Delta}(a)] < 0; \tag{21}$$
$$\mathcal{L}'(t(a))[t^{\Delta}(a)] < 0 \text{ and } \mathcal{L}'(t(a))[-t^{\Delta}(a)] > 0.$$

Making the rich less informed and the poor more informed improves information efficiency but reduces liquidity. We discuss the intuition for the information result, from which the liquidity result follows. Recall the key distortion highlighted in Section 9.1: the trading intensities of the rich are too large, while those of the poor are too small. To correct this inefficiency, one needs to increase the trading intensities of the poor relative to the rich. This can be achieved by making the poor more informed and the rich less informed.

Note that the definition of a Robin Hood variation allows for weak inequalities. Therefore, the proposition above applies to a variation that decreases the precisions of the rich while leaving other precisions unchanged.

Corollary 1 (An information-aggregation paradox)**.** *Suppose that traders have DARA utilities. Then, there exists a threshold* a_2^t *such that for any* $h^{\Delta}(a) \neq 0$ *such that* $h^{\Delta}(a) \geq 0$ *for* $a > a_2^t$*, and* $h^{\Delta}(a) = 0$ *otherwise,* $\mathcal{I}'(t(a))[t^{\Delta}(a)] > 0$ *and* $\mathcal{I}'(t(a))[-t^{\Delta}(a)] < 0$.

Weakly increasing the information of all traders can hurt information efficiency, while weakly decreasing it can have the opposite effect. Banerjee, Davis, and Gondhi (2018), Dugast and Foucault (2017), and Glebkin and Kuong (2023) also showed that improving the quality of private information can reduce information efficiency. In their studies, the mechanism is that giving more information to some traders invites more noise from another source. In our paper, we focus on a pure information aggregation channel: the only noise comes from the traders' signals themselves. Yet, reducing the noise in some traders' signals can harm information efficiency. This occurs because better information is aggregated less effectively: increasing the precisions of the rich makes them trade more aggressively, causing their signals to be even more overweighted in price, exacerbating the existing distortion.

The mentioned papers also do not feature wealth effects, so the effect there is not specific to large traders (i.e., traders with large wealth). An implication of our result is that making large investors differentially more informed could harm information efficiency. One of the changes introduced in the MiFID II regulation was to unbundle investment research from trading costs. Simplifying, before MiFID, everyone who traded obtained information. After that, only whoever is willing to pay gets it. Since large traders have a higher value of information (see Section B), such regulation makes large trades differentially more informed, potentially negatively affecting price informativeness.

Remark 8 (Why quality of aggregation dominates quantity of information). The result of Corollary 1 arises from the balance of two effects: on the one hand, making the rich less informed decreases the overall quantity of information available in the economy; on the other hand, the information is aggregated more effectively. Why does the second effect dominate? To understand this, consider the contribution of trader a to price informativeness τ_p. We can express τ_p as $\tau_p = I_1^2/I_2$, where $I_1 = \int_0^1 t(a)/\rho(a)da$ and $I_2 = \int_0^1 t(a)/\rho(a)^2 da$. Price informativeness is related to the signal-to-noise ratio. The "signal" in this ratio is related to I_1, and trader a's contribution to it, $t(a)/\rho(a)da$, is proportional to their risk tolerance, $1/\rho(a)$. Their contribution to the "noise", captured in I_2, is $t(a)/\rho(a)^2$, which is proportional to the square of their risk tolerance, $1/\rho(a)^2$. For sufficiently wealthy traders, their contribution to the noise always dominates their contribution to the signal, resulting in an overall negative contribution to price informativeness.

9.4 Wealth inequality redux: the role of information acquisition

Do the results in Proposition 2 hold when the precisions are endogenous (i.e., can depend on $W_0(a)$)? Consider the information efficiency result. The mechanism there is that the signals of the rich are overweighted, while the signals of the poor are underweighted. With endogenous information acquisition, one can expect the rich to acquire higher quality signals: they trade more aggressively and thus have more use for their information, making them value it more.[30] Is overweighting the signals of better quality a bad idea?

The transfer of wealth from rich to poor has two effects. The indirect one of decreasing the precisions of the rich and increasing that of the poor (via the dependence of precisions of wealth) is captured by Proposition 3. Proposition 2 captures the direct one. Since both of them work in the same direction, the results (19)–(20) are reinforced in the presence of information acquisition. We summarise this in the proposition below.

Proposition 4. *Suppose that Assumption 6 holds. Suppose that traders have DARA utilities. Suppose that precisions are a function of wealth $t(W_0(a), a)$ and are increasing in $W_0(a)$. Then, for any Robin Hood variation $W_0^{\Delta}(a)$ with $\underline{a} \leq \min\{a_1^t, a_1^W\} \leq \max\{a_2^t, a_2^W\} \leq \bar{a}$, results (19)–(20) hold.*

The result of Proposition 4 is perhaps surprising. Since the rich acquire more information than the poor, a Robin Hood variation could lead to a smaller overall amount of information being produced in the economy. Why does the information efficiency improve? Similar to our information aggregation paradox (Corollary 1), the effect of better aggregation of information dominates: Less information, but aggregated better results in more information efficiency (see Remark 8).

[30] In Section B, we consider the information acquisition problem and show that agent a's precision $t(a)$ is indeed an increasing function of his wealth $W_0(a)$.

10 Heuristic derivation of equilibrium

In this section, we sketch out the derivation of key equilibrium objects, applying familiar heuristics that have become available in CHILE due to its setup.[31] To lighten the notation, we drop the dependence of the payoff function on v.

10.1 Key stochastic calculus heuristics

We begin by recalling some properties of Brownian Motion and its increments. For $a \neq b \in [0, 1)$, we have

$$dB(a) \perp\!\!\!\perp dB(b) \qquad dB(a), dB(b) \sim \mathcal{N}(0, da),$$

allowing us to interpret the signals in (1) as a continuum of pointwise mutually-independent Normal random variables.

In what follows, we employ second-order Taylor expansions ignoring terms of orders higher than da, following a method commonly referred to as the "box calculus" (Steele, 2001, Chapter 8.4):

$$da \cdot da = 0, \quad dB(a) \cdot da = 0, \quad dB(a) \cdot dB(a) = da; \quad dB(a) \cdot dB(b) = 0. \tag{22}$$

10.2 Information continuity linearizes demand

As our Representation Lemma demonstrates, the demands *must* be linear in signals (see (2)). Suppose, by contradiction, that demands are non-linear in signals—for example, assume that (2) contained a squared-signal term. Denoting the coefficient of the quadratic signal as $\zeta(\cdot)$ and denoting the new version of the remaining quantities by adding tildes on top, we have

$$\begin{aligned} dX(a) &= \tilde{\beta}(P, a)ds(a) + \tilde{\delta}(P, a)da + \zeta(P, a)ds^2(a) \\ &= \tilde{\beta}(P, a)ds(a) + \left[\tilde{\delta}(a, P) + \frac{\zeta^2(a, P)}{t(a)}\right]da, \end{aligned} \tag{23}$$

[31] See, e.g., Cochrane (2009), Appendix A.3, for an application of these heuristics over time, instead of over agents.

where the second equality follows by the box-calculus heuristic. What is more, by comparing (23) to (2) we can see that the demand function in (2) can subsume the extra term through its da term without affecting the signal term, giving us a contradiction. To summarize, the functional form in (2) already accounts for squared signals.[32]

10.3 Price inference

Equation (3) implies that information in the price is summarized by:

$$s_p = v + \int_0^1 \omega(a, P) \frac{dB(a)}{\sqrt{t(a)}}.$$

This signal has a familiar "truth plus normally distributed noise" structure. Once the precision τ_p is known, the inference from prices, summarized in Lemma 2, follows from Bayes's rule for normal random variables. Here, we derive expression (4) for τ_p.

Note that the "noise" term $\int_0^1 \omega(a, P)/\sqrt{t(a)}\, dB(a)$ is normally distributed with mean 0 and variance $\int_0^1 \omega(a, P)^2/t(a)\, da$. This follows from the basic properties of stochastic integrals but can also be derived heuristically as outlined at the beginning of this section. Indeed, the noise $\int_0^1 \omega(a, P)/\sqrt{t(a)}\, dB(a)$ accumulates terms $\omega(a, P)/\sqrt{t(a)}\, dB(a)$, each normally distributed with mean zero and variance $\omega(a, P)^2/t(a)\, da$. As an accumulation of normal mean-zero independent random variables, the integral is normally distributed with a mean of zero and a variance that sums the variances of the individual terms, i.e.,

$$\mathrm{Var}\left[\int_0^1 \omega(a, P) \frac{dB(a)}{\sqrt{t(a)}}\right] = \int_0^1 \omega(a, P)^2/t(a)\, da.$$

This variance is the reciprocal of τ_p.

[32] Restricting attention to quadratic adjustments is without loss of generality, as any sufficiently differentiable non-linear function can be expanded into a polynomial of $ds(a)$, with terms of order three and above becoming zero due to (22).

10.4 A market-efficiency result

In the discrete economy, trader a's optimal demand satisfies the first-order condition

$$\mathbb{E}[u'(W_0(a) + (V - P)x)(V - P)|\Delta s(a), P] = 0. \tag{24}$$

We note that demands must go to zero as the number of agents goes to infinity—if that was not the case, aggregate demand would sum up infinitely many terms of finite size, and it would explode. Consequently, $x = 0$ must solve (24) in the large economy limit. Further noting that the information set of $(\Delta s(a), P)$ collapses to that of just P in the limit, setting $x = 0$ in (24) yields

$$\mathbb{E}[u'(W_0(a))(V - P)|P] = 0 \iff P = \mathbb{E}[V|P], \tag{25}$$

implying that under perfect competition, prices in CHILE are weak-form efficient.

10.5 Deriving trading intensity

As we have seen above, market quality is completely determined by the trading-intensity $\beta(a, P)$, on which we now focus. We Taylor-expand (24) over x up to the second order, and then we replace x by $dX(a)$ and $\Delta s(a)$ by $ds(a)$, obtaining

$$\begin{aligned} 0 = u'(W_0(a))\,\mathbb{E}[(V-P)|ds(a),P] \\ + u''(W_0(a))\,\mathbb{E}\big[(V-P)^2\big|ds(a),P\big]\,dX(a) \\ + \frac{1}{2}u'''(W_0(a))\,\mathbb{E}\big[(V-P)^3\big|ds(a),P\big]\,dX(a)^2. \end{aligned} \tag{26}$$

To solve for trading intensity, we plug (2) into (26) and compare the result with (2). We then solve for $\beta(a, P)$ by matching $ds(a)$ coefficients (for $\delta(a, P)$, we can match da terms).

After some simplifications explained in detail in Appendix C.10, we obtain

$$0 \;=\; u'(W_0(a))\,\mathbb{E}[(V-P)|ds(a),P] \;+\; u''(W_0(a))\,\mathbb{E}\big[(V-P)^2\big|P\big]\,\beta(a,P)ds(a) \;+\; \cdots, \tag{27}$$

where we have omitted terms that do not contain the signal, as they do not influence our subsequent calculations. Such "non-ds" terms are denoted by "$\cdots$" in the remainder of the subsection.

Next, note that by the market efficiency condition (25) we have $P = E[V|P]$ and so $\mathbb{E}[(V-P)^2|P] = \mathrm{Var}[V|P]$. Note also that we can replace $\rho(a) = -u''(W_0(a)/u'(W_0(a))$, where $\rho(a)$ is the risk-aversion coefficient of agent a. This simplifies the above even further, eventually yielding

$$\beta(a,P)ds(a) = \frac{\mathbb{E}[(V-P)|ds(a),P]}{\rho(a)\,\mathbb{V}\mathrm{ar}(V|P)} + \cdots \tag{28}$$

Juxtaposing (28) with (2) brings forth a striking property: even though we use general preferences, the demand function depends on the signal as if preferences were mean-variance. We can think of this property as the converse of linearizing demand—by applying the box calculus on the expanded first-order condition, all higher-order signal terms drop out, and thus the only way for signals to influence demand is through the first two terms in (26).[33]

To pin down $\beta(a,P)$ we need to separate out the $ds(a)$ term on the right-hand side of (28). To this end, we compute the $ds(a)$ term in the conditional expectation $\mathbb{E}[(V-P)|ds(a),P]$. Note that for small $ds(a)$, this conditional expectation is linear in $ds(a)$ and so can be computed using a familiar linear regression formula[34]

$$\mathbb{E}[(V-P)|ds(a),P] = \frac{\mathbb{C}\mathrm{ov}(V(v),ds(a)|P)}{\mathbb{V}\mathrm{ar}(ds(a)|P)}ds(a) + \cdots = \frac{\mathbb{C}\mathrm{ov}(v,ds(a)|P)}{\mathbb{V}\mathrm{ar}(ds(a)|P)}\,\mathbb{E}[V'(v)|P]ds(a) + \cdots \tag{29}$$

Here, the second equality follows by Stein's Lemma. Moreover, up to terms of order higher than da, we have

$$\mathbb{C}\mathrm{ov}(v,ds(a)|P) = \mathbb{C}\mathrm{ov}(v,v\,da|P) = da\,\mathbb{V}\mathrm{ar}(v|P) = da/\tau.$$

[33]In equilibrium, preferences cannot be substituted by a mean-variance tradeoff locally approximating the utility. While higher-order moments of returns and higher-order marginal preferences (i.e. prudence) may not affect $\beta(a,P)$, they still affect $\delta(a,P)$.

[34]More precisely, the linearity of expectation allows to substitute it with the Best Linear Predictor formula (see, e.g., Goldberger (1991), Chapter 5.4.) that is written following the first equality in (29).

Where the last equality uses $\tau = 1/\mathbb{V}\text{ar}(v|P)$. Similarly,

$$\mathbb{V}\text{ar}(ds(a)|P) = \mathbb{V}\text{ar}(dB(a)/\sqrt{t(a)}|P) = da/t(a).$$

Combining everything together, (28) becomes

$$\beta(a,P)ds(a) = \frac{t(a)}{\tau}\frac{\mathbb{E}[V'(v)|P]}{\rho(a)\,\mathbb{V}\text{ar}(V|P)}ds(a) + \cdots \iff \beta(a,P) = \frac{t(a)}{\rho(a)}\frac{\mathbb{E}[V'(v)|P]}{\tau\,\mathbb{V}\text{ar}(V|P)}.$$

11 Literature review

There are several branches of literature that our paper is related to. First, there is literature on REE models that go beyond the CARA-Normal framework.[35] Breon-Drish (2015) extends the CARA-Normal framework beyond normality in a single asset setup. Chabakauri et al. (2022) further extends Breon-Drish (2015) by allowing for multiple assets. Albagli, Hellwig, and Tsyvinski (2021) consider a setup with general distribution and risk-neutral traders subject to position limits. All of these papers assume CARA utility and so abstract away from wealth effects that are central to our paper.[36]

Malamud (2015) considers an REE model with a continuum of assets. Central to the tractability of his framework is the assumption of market completeness.[37] In contrast, we have one asset and continuum of states of the world. Hence the market is incomplete in our setup. One of the central results in Malamud (2015, Theorem 2.1) is that with non-CARA utility, the equilibrium is fully revealing.[38] In contrast, in our incomplete market setup, there is no full revelation for any utility function, thanks to the aggregate price noise.

[35] CARA-Normal framework is also used to model markets where traders have market power (see Rostek and Yoon (2020) for a review). Glebkin, Malamud, and Teguia (2023a) and Glebkin, Malamud, and Teguia (2023b) allow for, respectively, non-normal payoffs in a setup with CARA traders and non-normal payoffs in a setup that also allows for non-CARA utilities. These papers abstract from information frictions that are central to this paper.

[36] Here, we consider risk-neutral preferences as a special case of CARA with risk aversion equal to zero.

[37] Relatedly, DeMarzo and Skiadas (1998) and DeMarzo and Skiadas (1999) analyze REE models, where the market is *quasi-complete.*

[38] See also Chabakauri (2024) who shows that the results about full revelation with non-CARA utility are more nuanced.

Peress (2004) was the first (to our knowledge) to study wealth effects in noisy REE.[39] His model features log-normally distributed payoffs and non-CARA utilities. The key difference is that Peress (2004) relies on a "small risk" approximation, where the riskiness of the asset is small. In the limit, the variance of risky asset return is zero, making such an approach not suitable for quantitative work (it will be hard to match variance). Our approximation is essentially "small information." In contrast to Peress (2004), in our model, the asset stays risky even in the limit. Additionally: (i) in our model, the equilibrium quantities are affected by absolute risk aversion and absolute prudence, whereas in Peress (2004) only risk-aversion plays a role, (ii) conditional skewness plays a role in our model, but not in Peress (2004), and (iii) our model allows for the general distribution of asset payoffs.

Second, our paper is related to asset pricing literature studying the implications of heterogeneity in preferences and wealth for asset prices. Examples include Dumas (1989), Gârleanu and Panageas (2015) and Gomez et al. (2016), and Panageas (2020) for a review. This literature focuses on the implications of wealth heterogeneity on risk premia and risk-free rates but abstracts away from informational frictions and does not derive implications for market quality that are central to this paper.

Third, our paper is also related to the literature on mean-field games. See Lasry and Lions (2007), Achdou, Han, Lasry, Lions, and Moll (2022) and, for a review, Guéant, Lasry, and Lions (2011). As Achdou et al. (2022) note: "The name (Mean Field Games) comes from an analogy to the continuum limit taken in 'Mean Field theory' which approximates large systems of interacting particles by assuming that these interact only with the statistical mean of other particles." This analogy holds in our model. The effect of other traders on a trader of interest in our economy is summarized by several statistics of the cross-sectional distribution of traders' characteristics. These statistics can be viewed as a "mean field" that influences each trader's equilibrium behavior. As in Mean Field theory other traders do not affect a trader of interest directly, but only through their (infinitesimal) contribution to the mean field.

Fourth, on a technical side, our paper is also related to literature that uses stochastic calculus tools outside the domain of continuous time finance and economics. Examples include Malamud

[39] The wealth effect can arise even in CARA model, because the wealth may affect the tightness of financial constraints, as in Glebkin et al. (2021).

(2015) who models the noise in a continuum of assets as a cross-sectional stochastic process; Gârleanu, Panageas, and Yu (2015) who use Brownian bridge to represent the dividends for firms located on a circle; and Glebkin et al. (2021) who use stochastic calculus techniques to derive a marginal value of information in a static model.[40] Finally, the most closely related paper is Avdis (2018), which introduces a model with continuous heterogeneous information, albeit with CARA preferences and, as a result, without wealth effects.

12 Conclusion

We introduce a new asymmetric-information asset-pricing framework called "Continuous-and-Heterogeneous Information in a Large Economy" (CHILE). In this economy, we study perfect competition with rich agent heterogeneity, arbitrary preferences, and general payoff distributions. A unique equilibrium features all quantities in closed form. Leveraging the tractability of our model and its ability to work with wealth effects, we show how changes in the distribution of wealth affect different aspects of market quality: information efficiency and liquidity.

There are many potentially fruitful extensions. While we focus on a competitive CHILE equilibrium in this paper, in ongoing and preliminary work we show that the strategic equilibrium has a different limit. Moreover, even though we model markets as uniform-price auctions, our techniques can also find applications in discriminatory-price auctions. Finally, our framework can help study several interesting environments, such as those with dynamic trading, feedback effects, and endogenous growth.

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[40]There is also a related econometric literature on the unit roots. A good example is Phillips (1987).

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Appendices

A Summary of notation

Notation	Explanation
	General mathematical notation
$X = \text{plim}_{n\to\infty} X_n$ or $X_n \xrightarrow[n\to\infty]{\mathbb{P}} X$	X_n converges to X in probability
$dB(a)$	An increment of the Brownian motion
$u^{(l)}(\cdot)$	l-th derivative of $u(\cdot)$
a, b	Index of agents in the continuous economy
i, j	Index of agents in the discrete economy
n	Number of agents in the discrete economy
m_i	Mass of an agent in the discrete economy
$\mu = 1/n$	Average mass of an agent in the discrete economy
$\text{Sk}[Y] = \frac{\mathbb{E}[(Y-E[Y])^3]}{\text{Var}[Y]^{3/2}}$	Skewness of r.v. Y

Notation	Explanation
$\Phi(\cdot)$	Standard normal cdf
$\mathcal{O}'\left(h(a)\right)\left[h^{\Delta}(a)\right]$	Gateaux derivative of an equilibrium object $\mathcal{O}$ with respect to a parameter $h(a)$ in the direction $h^{\Delta}(a)$
	Model variables
$v \sim N(0, \tau_v^{-1})$	Fundamental, with ex-ante precision τ_v
$V(v)$	Payoff function
$\Theta(P) = \bar{\theta} + \theta(P)$	Supply of risky asset
$ds(a) = v\, da + \frac{1}{\sqrt{t(a)}} dB(a)$	Trader a's signal, with precision $t(a)$
$\mathcal{F}_{b,c} = \sigma\left(\{s(z) - s(b)\}_{b \leq z < c}\right)$	Information available to agents living in $[b, c)$
$\mathcal{F}_1 = \mathcal{F}_{0,1}$	Information in the entire economy
$dX(a) = \beta(a, P)ds(a) + \delta(a, P)da$	Demand of trader a; $\beta(a, P)$ is a's trading intensity
$\rho(a) = -u''(a)/u'(a)$	Absolute risk aversion of trader a
$\pi(a) = -u'''(a)/u''(a)$	Absolute prudence of trader a
$s\left[\omega^H(a)\right] = \int_0^1 \omega^H(b)ds(b)$	Aggregate signal; $\omega^H(a)$ is a weighting function
$s_p = s[\omega(a)] = \int_0^1 \omega(a)ds(a)$	Equilibrium sufficient statistic; $\omega(a)$ is equilibrium weighting function
$\mathcal{P}(s_p)$	Equilibrium price function
$h(\cdot)$	The inverse of $\mathcal{P}(\cdot)$

B Information acquisition

Information acquisition happens at $t = 0$. The precisions $t(a)$ are endogenized by requiring them to be optimal, given an information acquisition cost. We define the notion of optimality of the profile of precisions $t(a)$ similarly to how we defined the optimality of demands: precision profile $t(a)$ is optimal in CHILE if it is a limit of optimal precisions in the discrete economies. We make this definition precise after we describe the information acquisition in the discrete economy. We assume that agents are uncertain about the distribution of wealth $W_0(a)$ (but they know their wealth).[41] This distribution becomes known before the start of the trade at $t = 1$.

The new primitive in CHILE is the information acquisition cost $c(t, a)$. We assume that the cost of acquiring an infinitesimal signal $ds(a) = vda + 1/\sqrt{t}dB(a)$ for a trader a is $c(t; a)da$, where $c(\cdot)$ is continuous, strictly increasing and convex function of t. Thus, the cost of acquiring a finite signal

$$\Delta s = \int_{a_i}^{a_i+m_i} vda + \int_{a_i}^{a_i+m_i} \frac{1}{\sqrt{t(a)}}dB(a) \text{ is } \int_{a_i}^{a_i+m_i} c(t(a), a_i)da.$$

A finite signal can be split into a collection of infinitesimal ones, with associated costs. Using Jensen's inequality, one can show that it is not optimal to split a finite signal into infinitesimal ones of varying precisions. If a trader wants to get information of precision tm_i, he should acquire a signal

$$\Delta s = \int_{a_i}^{a_i+m_i} vda + \frac{1}{\sqrt{t}}dB(a) = vm_i + \frac{B(a_i + m_i) - B(a_i)}{\sqrt{t}} \text{ at a cost } c(t; a_i)m_i. \tag{30}$$

Thus, without loss of generality, we restrict finite signals in the discrete economy to be of the form (30).

Denote $\mathcal{U}_i(W_0^i, t_i, m_i)$ the maximum utility in (8) for a given precision t_i and initial wealth W_0^i.[42] Define $c(t_i, a_i)m_i$ as a (monetary) cost of acquiring signal Δs_i. We require that precisions

[41]Done this way, the information choice of agent a will only depend on his wealth, but not the wealth of others, which helps to simplify some of the analysis.

[42]Here we follow the setup of the main paper, where the profile of preferences in the discrete and continuous economy are the same. This approach can be generalized to allow them to differ for finite n but coincide in the $n \to \infty$ limit. Such an alternative approach could allow to lift the technical restrictions from the primitives of CHILE and "shift" those restrictions to the primitives of discrete economies instead.

t_i are *optimal*, i.e.

$$t_i = \arg\max_t \left\{ \int \mathcal{U}_i(W_0^i - c(t, a_i)m_i, t, m_i)g(v, P, \mu)dvdP \right\}. \tag{31}$$

Here $g(\cdot)$ denotes the PDF of the joint distribution of $\mathbf{P}_i^n$ and Δs_i. We denote $t^n(a)$ the profile of optimal precisions in the discrete economy, $t^n(a) = t_i$, for all $a \in [a_i, a_i + m_i)$, where t_i solves (31).

We introduce some technical restrictions on $c(t; a)$.

Assumption 7. *The information acquisition cost $c(t; a)$ is such that there exists M_t such that for any i, $\hat{t}_i > M_t$ is not optimal.*

The additional technical restriction allows us to make a choice set in (31) compact. Thus, without loss of generality, we assume $t \in [0, M_t]$ everywhere in the sequel. We define the notion of optimal profile of precisions $t(a)$ by continuity, analogously to Section 6.

Definition 6. *A profile $t(a)$ is **optimal** if, for every a, the optimal precision in discrete economy $t^n(a)$ converges to $t(a)$ as $n \to \infty$.*

Finally, we define the information acquisition equilibrium at $t = 0$.

Definition 7. *A profile $t(a)$ is an information acquisition **equilibrium** if for every a, $t(a)$ is optimal.*

With these definitions at hand, we are ready to state this section's main result in the subsection below.

B.1 Information acquisition: heuristic derivation

It is illuminating to start with the heuristic derivation. Consider a change in trader a's time-2 realized utility due to trade

$$
\begin{aligned}
d\mathcal{U}_{t=2}(dX(a);a) &= u(W_0(a) - c(t(a))da + dX(a)(V(v) - P);a) - u(W_0(a);a) \\
&= u'(W_0(a))\left(dX(a)(V(v) - P) - c(t(a))da + \frac{\rho(a)}{2}dX(a)^2(V(v) - P)^2\right)
\end{aligned}
$$

In the second line, we Taylor expanded the $u(\cdot)$, up to terms of order da. Due to the heuristics of Section 10, we have $dX(a)^2 = \beta(a,P)^2/t(a)da$. Now substitute $P = \mathbf{P}(a)$ (trader a's conjecture about the market-clearing price) and take the expectation:

$$
\frac{\mathbb{E}[d\mathcal{U}_{t=2}(dX(a);a)]}{u'(W_0(a);a)} = \mathbb{E}[dX(a)(V(v)-\mathbf{P}(a)] - c(t(a))da + \frac{\rho(a)}{2t(a)}\mathbb{E}[\beta(a,\mathbf{P}(a))^2(V(v)-\mathbf{P}(a))^2]da. \tag{32}
$$

Now simplify

$$
\begin{aligned}
\mathbb{E}[dX(a)(V(v) - \mathbf{P}(a))] &= da\,\mathbb{E}[\beta(a,\mathbf{P}(a))v(V(v) - \mathbf{P}(a))] \\
&\quad + \mathbb{E}[dB(a)\beta(a,\mathbf{P}(a))(V(v) - \mathbf{P}(a))] \\
&\quad + da\,\mathbb{E}[\delta(a,\mathbf{P}(a))(V(v) - \mathbf{P}(a))] \\
&= da\,\mathbb{E}[\beta(a,\mathbf{P}(a))v(V(v) - \mathbf{P}(a))].
\end{aligned} \tag{33}
$$

In the first transition, we substituted $dX(a) = \beta(a,P)ds(a) + \delta(a,P)da$. To get (33), we substituted $\mathbb{E}[dB(a)\beta(a,\mathbf{P}(a))(V(v) - \mathbf{P}(a))] = 0$ ($dB(a)$ is independent of $V(v)$ and $\mathbf{P}(a)$ as the traders are price takers and assume the noise in their signals in independent from that in the price) and $\mathbb{E}[\delta(a,\mathbf{P}(a))(V(v) - \mathbf{P}(a))] = 0$ (market efficiency condition).

To proceed further, we divide (32) by da and pass to $da \to 0$ limit. In that limit, $\mathbf{P}(a)$ becomes the market clearing price $\mathbf{P}_*$, which for ease of notation we simply denote P going forward. We can further simplify (33) by noting that $\mathbb{E}[v(V(v) - P)|P] = \beta(a,P)Var[V(v)|P]\rho(a)/t(a)$

(see (49)). Then, after substituting $\mathbb{E}[(V(v)-P)^2|P] = \text{Var}[V(v)|P]$, and $\beta(a,P) = t(a)/\rho(a)\beta_P(P)$ we can finally obtain

$$\frac{\mathbb{E}[d\mathcal{U}_{t=2}(dX(a);a)]}{u'(W_0(a);a)da} = \frac{t(a)}{2\rho(a)}\,\mathbb{E}[\beta_P(P)^2\text{Var}[V(v)|P]] - c(t(a)) \tag{34}$$

The optimal precision maximizes (34) with respect to $t(a)$. We summarize in the Proposition below. The rigorous proof is in the Appendix C.11.

Proposition 5. *The optimal precision choice for a trader a solves*

$$c'(t,a) = \frac{1}{2\rho(a)}\,\mathbb{E}[\beta_P(P)^2\text{Var}[V(v)|P]]. \tag{35}$$

Here $\beta_P(P) = \frac{E[v(V(v)-P)|P]}{\text{Var}[V(v)|P]}$. When $V(v) = \exp(v)$, the optimal precision solves

$$c'(t,a) = \frac{1}{2\rho(a)}\,\mathbb{E}\left[\frac{1}{\tau^2\left(\exp\left(\tau^{-1}\right)-1\right)}\right]. \tag{36}$$

Provided that the equilibrium exists, the equilibrium precision for trader a is an increasing function of his wealth $W_0(a)$ under DARA preferences.

Note that the expectations in (35) and (36) are over prices P, distribution of aggregate signal paths $\{s(a)\}_a$, and potential wealth distributions.

C Online appendix: derivations and proofs

C.1 The Representation Lemma

Proof of Lemma 1.

For ease of notation, we denote $X(a, P)$ simply as X_a. We fix some b and c, $0 \leq b < c < 1$. Consider a filtration $\mathbb{F}_{b,c} = \{\mathcal{F}_{b,z}\}_{z \in [b,c)}$. Denote

$$\mu_b(z) = E[v|\mathcal{F}_{b,z}].$$

Note that

$$B^s(z) = \int_b^z \sqrt{t(a)}\, (ds(a) - \mu_b(a)da)$$

is a Brownian motion with respect to $\mathbb{F}_{b,c}$ (Liptser and Shiryaev (2001), Theorem. 8.1).

Consider a martingale

$$Y_z = E[X_c - X_b|\mathcal{F}_{b,z}].$$

By Martingale Representation Theorem (Cohen and Elliott (2015), Theorem 14.5.1), there exists an $\mathbb{F}_{b,c}$-adapted process $\zeta(a, P)$, denoted by $\zeta(a)$ hereafter, such that Y_z can be written as

$$Y_z = Y_0 + \int_b^c \zeta(a)\sqrt{t(a)}\, (ds(a) - \mu_b(a)da)\,.$$

Letting $z \to c$ and substituting the definition of Y we obtain

$$Y_c = X_c - X_b = E[X_c - X_b] + \int_b^c \zeta(a)\sqrt{t(a)}\, (ds(a) - \mu_b(a)da)\,.$$

Lemma 6 (to follow) shows that $\int_b^c \zeta(a)\sqrt{t(a)}\mu_b(a)da = \int_b^c \hat{\zeta}(a)ds(a)$ for some $\mathbb{F}_{b,c}$ - adapted process $\hat{\zeta}(a)$, explicitly given in the Lemma.[43]

[43] The process $\hat{\zeta}(a)$ may depend on P, but we drop the P argument for compactness of notation, as we do

Thus, we have

$$Y_c = X_c - X_b = E[X_c - X_b] + \int_b^c \beta(a, P) ds(a),$$

where $\beta(a, P) = \zeta(a)\sqrt{t(a)} - \hat{\zeta}(a)$.

It remains to prove that $\beta(a, P)$ must be deterministic. First, we argue that the process $\beta(a, P)$ is the same for any chosen b and c. Denote $\beta_{b,c}(a)$ the process $\beta(a, P)$ constructed for some fixed b and c (again, we drop the argument P for compactness of notation). Then, for any $0 \leq b < c < 1$, we can write $X_1 - X_0 = X_1 - X_c + X_c - X_b + X_b - X_0 = E[X_1 - X_0] + \int_0^1 \beta_{0,1}(z)ds(z) = E[X_1 - X_0] + \int_0^b \beta_{0,b}(z)ds(z) + \int_b^c \beta_{b,c}(z)ds(z) + \int_c^1 \beta_{c,1}(z)ds(z)$. Then, we must have $\int_0^b (\beta_{0,1}(z) - \beta_{0,b}(z))ds(z) + \int_b^c (\beta_{0,1}(z) - \beta_{b,c}(z))ds(z) + \int_c^1 (\beta_{0,1}(z) - \beta_{c,1}(z))ds(z) = 0$, for any b and c. This is only possible when $\beta_{b,c}(z) = \beta_{0,1}(z)$ for any $z \in [b, c)$.

Second, note that $\mathcal{F}_{b,c}$-measurable $\beta(z, P)$ is deterministic at $z = b$. Since b is arbitrary, $\beta(z, P)$ is deterministic for any $z \in [0, 1)$.

Finally, note that since $E[X_c - X_b]$ is differentiable, Leibniz's rule implies that $X(c) - X(b) = \int_b^c \delta(a, P)da$, with $\delta(a, P)$ being the derivative of $E[X(a, P)]$ with respect to a ■

Lemma 6. *We have* $\int_b^c \zeta(a)\sqrt{t(a)}\mu_b(a)da = \int_b^c \hat{\zeta}(a)ds(a)$, *where* $\hat{\zeta}(a)$ *is given by* (39).

Proof. Denote $\tau_b(z) = Var[v|\mathcal{F}_{b,z}]^{-1}$. By Kalman-Bucy filtering equations (Liptser and Shiryaev (2001), Theorem 10.1), we have

$$d\mu_b(z) = \frac{t(z)}{\tau_b(z)} \left(ds(z) - \mu_b dz\right) \tag{37}$$

$$\frac{d}{dz}\left(\tau_b(z)\right) = t(z).$$

Moreover, the SDE (37) can be solved explicitly, as follows:

$$\mu_b(z) = \frac{\int_b^z t(a)ds(a)}{\tau_v + \int_b^z t(a)da}. \tag{38}$$

with the $\zeta(a)$.

Let, for this proof only, $h(a) = \zeta(a)/(1 + t(a)/\tau_b(a))$ and write

$$\begin{aligned}
\int_b^c h(a)\sqrt{t(a)}\mu_b(a)da &= \mu_b(c)\int_b^c h(a)\sqrt{t(a)}da - \int_b^c h(a)\sqrt{t(a)}d\mu_b \\
&= \mu_b(c)\int_b^c h(a)\sqrt{t(a)}da - \int_b^c h(a)\sqrt{t(a)}\frac{t(a)}{\tau_b(a)}\left(ds(a) - \mu_b(a)da\right)
\end{aligned}$$

We integrate by parts to obtain the first transition and then substitute $d\mu_b$ with (37).

Rearranging, we obtain

$$\begin{aligned}
\int_b^c h(a)\sqrt{t(a)}\left(1 + \frac{t(a)}{\tau_b(a)}\right)\mu_b(a)da &= \mu_b(c)\int_b^c h(a)\sqrt{t(a)}da - \int_b^c h(a)\sqrt{t(a)}\frac{t(a)}{\tau_b(a)}ds(a) \\
&= \int_b^c \left(\frac{t(a)\int_b^c h(a)\sqrt{t(a)}da}{\tau_v + \int_b^c t(a)da} - h(a)\sqrt{t(a)}\frac{t(a)}{\tau_b(a)}\right)ds(a).
\end{aligned}$$

Here, in the second line, we substituted (38). Now note that, by construction, $h(a)(1 + t(a)/\tau_b(a)) = \zeta(a)$ and so the statement of the lemma holds with

$$\hat{\zeta}(a) = \frac{t(a)\int_b^c h(a)\sqrt{t(a)}da}{\tau_v + \int_b^c t(a)da} - h(a)\sqrt{t(a)}\frac{t(a)}{\tau_b(a)}. \tag{39}$$

■

C.2 Proof of Lemma 2

Proof of Lemma 2. By the properties of the Ito integral, $\int_0^1 \omega(a)/\sqrt{(}t(a))dB(a)$ is distributed normally with a mean of zero. By Ito isometry, the variance of the integral is given by $\int_0^1 \frac{w(a)^2}{t(a)}da$. The statements of the lemma then follow from the standard results on Bayes rule with normal random variables. ■

C.3 Proof of Lemma 3

Proof of Lemma 3.

The conditional density of v given $\mathbf{P}_i^n$ and Δs_i can be written as

$$f_{v|\Delta s_i,\mathbf{P}_i^n}(v,s,P) = \frac{f_{\Delta s_i|v}(v,s)g(v,P,\mu)}{\int_{\mathbb{R}} f_{\Delta s_i|v}(u,s)g(u,P,\mu)du} \tag{40}$$
$$= c(s,P,\mu)\exp((sv - mv^2/2)t(a_i))g(v,P,\mu).$$

In the first line, we write in the numerator the joint density $f_{v,\Delta s_i,\mathbf{P}_i^n}(\cdot) = f_{\Delta s_i|v,\mathbf{P}_i^n}(\cdot)f_{v|\Delta s_i,\mathbf{P}_i^n}(\cdot) = f_{\Delta s_i|v,\mathbf{P}_i^n}(s,v,P)g(v,P,\mu)$ and then use the conditional independence of Δs_i and $\mathbf{P}_i^n$ given v to substitute $f_{\Delta s_i|v,\mathbf{P}_i^n}(\cdot) = f_{\Delta s_i|v}(\cdot)$. In the second line, we substitute the density $f_{\Delta s_i|v}(\cdot)$. It follows from (6), that this density is Gaussian, with mean mv and variance $m/t(a_i)$. We then collect all terms that do not depend on v into a function $c(s,P)$ (this subsumes the denominator of the fraction in (40)). The function $1/\int_{\mathbb{R}} \exp((sv - mv^2/2)t(a_i))g(v,P,\mu)dv$ is finite since Assumption 5 implies that the integrand $\exp((sv - mv^2/2)t(a_i))g(v,P,\mu)$ is dominated by a function $A\exp(-kv^2)\exp((sv - mv^2/2)t(a_i))$, which is integrable. ■

C.4 Proof of Lemma 4 and Theorem 1

The structure of this section is as follows. First, we introduce the most general technical restrictions on the sequences of discrete economies. Second, we prove the aggregation lemma that is central to proofs here. Finally, we prove Lemma 4 and Theorem 1 with the proof split into parts, such as deriving the efficiency condition, deriving $\beta(a,P)$, etc. Subsections up to C.4.4 prove Lemma 4, whereas the rest of the sections prove Theorem 1.

In what follows, we use circumflex to denote objects in the discrete economy.

C.4.1 Technical conditions on the sequence of discrete economies

Here, we describe the most general sequence of discrete economies. The agents of the n-th discrete economy are located in n disjoint neighboring segments, forming a partition of the interval $[0,1)$. The size of subinterval i of the partition is m_i. A particular agent i, $i = 1,\ldots,n$, lives in segment $[a_i, a_{i+1})$, with $a_i = \sum_{j<i} m_j$. His initial wealth is $\hat{W}_0(a_i)$, his precision is $\hat{t}(a_i)$, and his utility function over terminal wealth W is $\hat{u}(W; a_i)$. His private signal is given by (6). We denote $m = \max_i m_i$ and we consider any partition such that $m \to 0$ as $n \to \infty$. We impose the following restrictions on the primitives of the discrete economies.

Assumption 8. *We have*

- *For every $a \in [0, 1)$, $\hat{u}(W, a)$ is thrice continuously differentiable in W in the neighborhood of $W = W_0(a)$.*
- *For $l = 1, 2, 3$, $\hat{u}^{(l)}(W_0(a), a)$ converges uniformly to $u^{(l)}(W_0(a), a)$, a.e., over $a \in [0, 1)$.*
- *For any fixed P, for small enough ϵ, and for all n there exist $M(v)$ such that $|\hat{u}^{(l)}(\hat{W}_0(a) + \epsilon(V(v) - P))| < M(v)$ for $l = 1, 2, 3$ and $M(v)$ is such that $\lim_{v\to\infty} \frac{\ln M(v)}{v^2} \leq 0$.*
- *$\hat{t}(a)$ converges uniformly to $t(a)$ and $\hat{W}(a)$ converges uniformly to $W(a)$, a.e., over $a \in [0, 1)$*

C.4.2 The Aggregation Lemma

Lemma 7. ***(Aggregation lemma)*** *Consider a sequence of processes $d\hat{s}^n(a) = v\, da + \frac{1}{\sqrt{\hat{t}^n(a)}} dB(a)$. Consider a sequence of functions $\hat{x}^n(P, s, m, a) : \mathbb{R} \times \mathbb{R} \times [0, 1] \times [0, 1] \to \mathbb{R}$ with partial derivatives $\hat{x}^n_s(P, s, m, a)$, $\hat{x}^n_{ss}(P, s, m, a)$ and $\hat{x}^n_m(P, s; m, a)$ that are continuous functions of s and m in the neighbourhood of $s = m = 0$ for every $P \in \mathbb{R}$ and $a \in [0, 1)$. Suppose that $1/\hat{t}^n(a)$, $\hat{x}^n_s(P, 0, 0, a)$, $\hat{x}^n_s(P, 0, 0, a)$, $\hat{x}^n_{ss}(P, 0, 0, a)$ and $\hat{x}^n_m(P, 0, 0, a)$ are continuous functions of a, a.e., and converge uniformly, a.e., over $a \in [0, y]$ for every $P \in \mathbb{R}$ and $0 < y < 1$. Denote the respective limits by $1/t(a)$, $x_s(P, a)$, $x_s(P, a)$, $x_{ss}(P, a)$ and $x_m(P, a)$. Let*

$$\beta(P, a) = x_s(P, a) \text{ and } \delta(P, a) = \frac{1}{2t(a)} x_{ss}(P, a) + x_m(P, a).$$

For a fixed $y \in (0,1)$ take a partition $[0,y] = \cup_{i=1}^{n}[a_i, a_i + m_i]$, and let $\bar{m} = \max_i m_i$, $\Delta s_i = vm_i + \frac{1}{\sqrt{\hat{t}(a_i)}}(B(a_i+m_i) - B(a_i))$. Assume that $\int_0^y \beta(a,P)^2/t(a)da < \infty$, $\int_0^y \beta(a,P)da < \infty$, and $\int_0^y \delta(a,P)da < \infty$ for every $P \in \mathbb{R}$. For any partition sequence such that $\bar{m} \to 0$ as $n \to \infty$, we have that

$$\sum_{i:\, a_i<y} (\hat{x}^n(P, \Delta s_i, m_i, a_i) - \hat{x}^n(P,0,0,a_i)) \xrightarrow[n\to\infty]{p} \int_0^y \beta(P,a)ds(a) + \int_0^y \delta(P,a)da.$$

Proof of Lemma 7. Fix P and let $\hat{x}(\Delta\hat{s}_i, m_i, a_i)$ denote $\hat{x}_n(P, \Delta\hat{s}_i, m_i, a_i)$. Thus, we don't indicate the dependence of $\hat{x}_n(\cdot)$ on P and n explicitly. The sequence $\hat{x}(\cdot)$ can be distinguished from its' limit $x(\cdot)$ by the presence of circumflex in the former. By Itô's Lemma,

$$\begin{aligned}\hat{x}(\Delta\hat{s}_i, m_i, a_i) - \hat{x}(0,0,a_i) &= \int_{a_i}^{a_{i+1}} \hat{x}_s\Big(\hat{s}(a) - \hat{s}(a_i), a - a_i, a_i\Big)\left(v\,da + dB(a)/\sqrt{\hat{t}^n(a_i)}\right) \\ &\quad + \int_{a_i}^{a_{i+1}} \left[\hat{x}_m\Big(\hat{s}(a) - \hat{s}(a_i), a - a_i, a_i\Big) + \frac{1}{2\hat{t}(a_i)}\hat{x}_{ss}\Big(\hat{s}(a) - \hat{s}(a_i,n), a - a_i, a_i\Big)\right] da.\end{aligned}$$

For the n-th element of a partition sequence, denote $\hat{a}_n(a) = \sum_{i=1}^{n-1} a_i \mathbb{1}_{[a_i,a_{i+1})}(a)$, so that for any $a \in [0,1)$, $\hat{a}_n(a)$ equals the left point of the segment $[a_i, a_{i+1})$ that a belongs to. Similarly, denote $\check{a}_n(a) = \sum_{i=1}^{n-1} a_{i+1}\mathbb{1}_{[a_{i-1},a_{i+1})}(a)$, so that for any $a \in [0,1)$, $\check{a}_n(a)$ equals the right point of the segment $[a_i, a_{i+1})$ that a belongs to. With this notation, we can write

$$\begin{aligned}&\sum_{i:\, a_i<y} \hat{x}(\Delta\hat{s}_i, m_i, a_i) - \hat{x}(0,0,a_i) \\ &\quad = \int_0^{\check{a}_n(y)} \hat{x}_s\Big(s(a) - s(\hat{a}_n(a)), a - \hat{a}_n(a), \hat{a}_n(a)\Big)\left(v\,da + dB(a)/\sqrt{\hat{t}^n(\hat{a}_n(a))}\right) \\ &\qquad + \int_0^{\check{a}_n(y)} \hat{x}_m\Big(s(a) - s(\hat{a}_n(a)), a - \hat{a}_n(a), \hat{a}_n(a)\Big)da \\ &\qquad + \int_0^{\check{a}_n(y)} \frac{1}{2\hat{t}^n(\hat{a}_n(a))}\hat{x}_{ss}\Big(s(a) - s(\hat{a}_n(a)), a - \hat{a}_n(a), \hat{a}_n(a)\Big)da. \qquad (41)\end{aligned}$$

The proof is concluded by passing to the limit in the integrals above. Note that as $n \to \infty$,

$\hat{a}_n(a) \rightrightarrows a$ and $\check{a}_n(a) \rightrightarrows a$ over $a \in [0,1)$.[44] Then, in the limit, the partials of $\hat{x}(\cdot)$ are substituted by the respective partials of $x(\cdot)$, and the terms $\check{a}_n(y)$, $s(a) - s(\hat{a}_n(a))$, and $a - \hat{a}_n(a)$ are respectively substituted by y, 0, and 0. We obtain

$$\sum_{i:\, a_i<y} \left(\hat{x}(\Delta\hat{s}_i, m_i, a_i) - \hat{x}(0,0,a_i)\right) \xrightarrow[n\to\infty]{p} \int_0^y x_s(P,a)ds(a) + \int_0^y \left[x_m(P,a) + \frac{1}{2t(a)}x_{ss}(P,a)\right] da.$$

The remainder of the proof justifies passing to the limit in (41). Note that since the points of discontinuity of $1/\hat{t}^n(a)$, $\hat{x}^n_s(P,0,0,a)$, $\hat{x}^n_s(P,0,0,a)$, $\hat{x}^n_{ss}(P,0,0,a)$ and $\hat{x}^n_m(P,0,0,a)$ have a Lebesgue measure zero, we can assume, without loss of generality that these functions are continuous over $a \in [0,y]$. Similarly, we can assume that these functions converge uniformly over $a \in [0,y]$. For the Lebesgue integrals in (41), passing to the limit is justified by the Uniform Convergence Theorem. The hypotheses of the UCT hold since Lemma 8 (to follow) implies that the sequence of integrands converges uniformly to $x_s(P,a)v + x_m(P,a) + \frac{1}{2t(a)}x_{ss}(P,a)$. For the stochastic integral, the passage to the limit follows from Theorem IV.2.12 in Revuz and Yor (2013).[45] The hypotheses of the Theorem hold, because the integrands $\hat{y}^n(\cdot) = \hat{x}_s(s(a) - s(\hat{a}_n(a)), a - \hat{a}_n(a), \hat{a}_n(a))/\sqrt{\hat{t}^n(\hat{a}^n(a))}$ are bounded (by the Extreme Value Theorem) and Lemma 8 implies that the sequence $\hat{y}^n(\cdot)$ converges uniformly. Then, the sequence $\hat{y}^n(\cdot)$ is uniformly bounded. ■

Lemma 8. *Consider a continuous function $Y(a) : [0,1) \to \mathbb{R}^d$. Consider a sequence of functions $f_n(y,a) : \mathbb{R}^d \times [0,1) \to \mathbb{R}$ such that: (1) $f_n(0,a)$ converges uniformly to $f(0,a)$ over $a \in [0,1)$, (2) $f_n(y,a)$ is continuous in y in the neighborhood of 0 for every $a \in [0,1)$ and for every n, and (3) $f(0,a)$ is continuous in a. Take a partition $[0,y] = \cup_{i=1}^n [a_i, a_i + m_i]$, and let $\bar{m} = \max_i m_i$ and $\hat{a}_n(a) = \sum_{i=1}^{n-1} a_i \mathbb{1}_{[a_i,a_{i+1})}(a)$. For any partition sequence such that $\bar{m} \to 0$ as $n \to \infty$, we have that $f_n\Big(Y(a) - Y(\hat{a}_n(a)), \hat{a}_n(a)\Big)$ converges uniformly to $f\Big(0,a\Big)$*

[44] The notation $\rightrightarrows$ stands for uniform convergence.

[45] We are grateful to Christoph Frei for suggesting this reference.

over $a \in [0, y]$.

Proof. We need to establish that

$$\lim_{n\to\infty} \sup_{a\in[0,y]} \left| f_n\Big(Y(a) - Y(\hat{a}_n(a)), \hat{a}_n(a)\Big) - f\,(0,a) \right| = 0$$

Note that, for a given n,

$$\begin{aligned}
&\sup_{a\in[0,y]} \left| f_n\Big(Y(a) - Y(\hat{a}_n(a)), \hat{a}_n(a)\Big) - f_n\,(0,a) \right| = \\
&\max_i \sup_{a\in[a_i,a_{i+1}]} \left| f_n\Big(Y(a) - Y(\hat{a}_n(a)), \hat{a}_n(a)\Big) - f\,(0,a) \right| = \\
&\max_i \left| f_n\Big(Y(a_i^*) - Y(a_i), a_i\Big) - f\,(0,a_i^*) \right|
\end{aligned}$$

Here $a_i^* \in [a_i, a_{i+1}]$ is maximand in the optimization over $a \in [a_i, a_{i+1}]$ above. The a_i^* exists by Weiestrass's extreme value theorem. Note that since $a_i^* \in [a_i, a_{i+1}]$ and $a_{i+1} - a_i = m_i \to 0$ as $n \to \infty$, we have $a_i^* - a_i \to 0$ as $n \to \infty$. Note further that

$$\begin{aligned}
&\max_i \left| f_n\Big(Y(a_i^*) - Y(a_i), a_i\Big) - f\,(0,a_i^*) \right| \le \\
&\max_i \left| f_n\Big(Y(a_i^*) - Y(a_i), a_i\Big) - f_n\,(0,a_i) \right| + \max_i \left| f_n\Big(0,a_i\Big) - f\,(0,a_i) \right| + \max_i \left| f\Big(0,a_i\Big) - f\,(0,a_i^*) \right|.
\end{aligned} \tag{42}$$

The proof is concluded by noting that for any $\epsilon > 0$ there exists a large enough $N(\epsilon)$ such that for all $n > N(\epsilon)$, each of the three terms in (42) are less then $\epsilon/3$. Indeed, if, on the contrary, there exists an $\epsilon > 0$ such that $\max_i \left| f_n\Big(Y(a_i^*) - Y(a_i), a_i\Big) - f_n\,(0,a_i) \right| > \epsilon/3$ for all large enough n, a contradiction with the hypothesis (2) would occur. Similarly, if $\max_i \left| f_n\Big(0,a_i\Big) - f\,(0,a_i) \right| > \epsilon/3$ a contradiction with the hypothesis (1) would occur. Finaly, if $\max_i \left| f\Big(0,a_i\Big) - f\,(0,a_i^*) \right| > \epsilon/3$, a contradiction with the hypotheis (3) would occur. ■

C.4.3 Optimal demand in a discrete economy

Consider a trader a living in the interval $[a, a+m)$ in the n-th discrete economy. The price belief in that economy is $g(v, P, \mu)$. Denote trader a'a signal realization by s. We denote the optimal demand of the trader a by $x^*(P, s, m, a, \mu)$. (Note that we can index economies by $\mu = 1/n$ instead of n as there is a one-to-one correspondence between the two.) Given the strict concavity of the utility function, the optimality condition (8) holds if and only if $x^*(P, s, m, a, \mu)$ satisfies the first-order condition

$$\int \hat{u}'\left(\hat{W}_0(a) + x^*(P, s, m, a, \mu)(V(v) - P),\ a\right)(V(v) - P) \\ \cdot \exp((sv - mv^2/2)\hat{t}(a))g(v, P, \mu)dv = 0. \quad (43)$$

Note that we dropped the constant $c(s, P, \mu)$ in the conditional density as it does not affect maximization.

For the n-th discrete economy, let $\hat{X}(y)$ be the cumulative demand, defined as

$$\hat{X}(y) = \sum_{i:\, a_i < y} x^*(P, \Delta s_i, m_i, a_i, \mu).$$

We can decompose the cumulative demand as follows

$$\begin{aligned} \hat{X}(y) = \sum_{i:\, a_i < y} x^*(P, \Delta s_i, m_i, a_i, \mu) = \\ \sum_{i:\, a_i < y} (x^*(P, \Delta s_i, m_i, a_i, \mu) - x^*(P, 0, 0, a_i, \mu)) + \\ \sum_{i:\, a_i < y} (x^*(P, 0, 0, a_i, \mu) - x^*(P, 0, 0, a_i, 0)) + \\ \sum_{i:\, a_i < y} x^*(P, 0, 0, a_i, 0). \end{aligned}$$

By Aggregation Lemma, we have

$$\sum_{i:\, a_i<y} \left(x^*(P,\Delta s_i, m_i, a_i, \mu) - x^*(P,0,0,a_i,\mu)\right) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^y x_s^*(P,0,0,a,0)ds(a) + \int_0^y \left(x_m^*(P,0,0,a,0) + \frac{1}{2}x_{ss}^*(P,0,0,a,0)\right) da \quad (44)$$

The hypotheses of the Lemma hold. The fact that $x_s^*(P,s,m,a,\mu)$, $x_{ss}^*(P,s,m,a,\mu)$ and $x_m^*(P,s,m,a,\mu)$ are continuous functions of s and m in the neighborhood of $s=m=0$ follows via the Implicit Function Theorem. We demostrate below that $x_s^*(P,0,0,a,\mu)$, $x_{ss}^*(P,0,0,a,\mu)$ and $x_m^*(P,0,0,a,\mu)$ converge uniformly to, respectively, $x_s^*(P,0,0,a,0)$, $x_{ss}^*(P,0,0,a,0)$ and $x_m^*(P,0,0,a,0)$ over $a\in[0,y]$, using Lemma 12 (to follow).

Similarly, by Aggregation Lemma, we have

$$\sum_{i:\, a_i<y} \left(x^*(P,0,0,a_i,\mu) - x^*(P,0,0,a_i,0)\right) \xrightarrow[n\to\infty]{p} \int_0^y x_\mu^*(P,0,0,a,0)da. \quad (45)$$

Similarly to the previous step, the hypotheses of the lemma follow via the Implicit Function Theorem and Lemma 12.

We show below that the limiting integrals in (44) and (45) are finite. Thus, we must have $\lim_{n\to\infty}\sum_{i:a_i<y} x^*(P,0,0,a_i,0)<\infty$ for the limit of $\hat{X}(y)$ to be well-defined. This gives rise to the efficiency condition, as we derive below.

C.4.4 Efficiency condition

We start by characterizing $x^*(P,a)=\lim_{n\to\infty} x^*(P,0,0,a,\mu)$. Substituting $m=s=0$ to (43) and taking the $n\to\infty$ limit we obtain

$$\int u'\left(W_0(a) + x^*(P,a)(V(v)-P),\, a\right)(V(v)-P)\cdot g(v,P,0)dv = 0. \quad (46)$$

To get from (43) to (46), we first passed the limit inside the integral, which is justified by the Dominated Convergence Theorem. The hypotheses of the DCT hold since the Assumption 8 implies that the integrand in (43) admits an integrable majorant of the form $A\exp(-kv^2)$ with some constants A, $k \geq 0$, provided that $|x^*(\cdot)|$ is small enough. We then pass to the limiting functions (i.e., we go from $\hat{u}(\cdot)$ to $u(\cdot)$ and from $g(\cdot, \mu)$ to $g(\cdot, 0)$), which the Uniform Limit Theorem justifies. The $|x^*(\cdot)|$ is indeed small enough and is, in fact, zero in the limit, as implied by the Lemma below.

Lemma 9. *For a fixed $y \in (0,1)$ take a partition $[0, y] = \cup_{i=1}^{n}[a_i, a_i + m_i]$, and let $m = \max_i m_i$. For any partition sequence such that $m \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty} \sum_{i:a_i<y} x^*(P, a_i) < \infty$ if, and only if, $x^*(P, a) = 0$ for any $a \in [0, y]$ and*

$$\int (V(v) - P) \cdot g(v, P, 0)dv = 0. \tag{47}$$

Proof of Lemma 9. Suppose, on the contrary, that $\int (V(v) - P) \cdot g(v, P, 0)dv > 0$. (The case of the opposite inequality is considered analogously and is omitted for brevity.) Then, for (46) to hold, $x^*(P, a_k)$ must be positive for any fixed $a_k < y$. Thus, all terms in the sum $\lim_{n\to\infty} \sum_{i:a_i<y} x^*(P, a_i)$ are positive. By continuity of $x^*(P, a)$ there exist a δ such that for all $a : |a - a_k| < \delta$ we have $x^*(P, a) > \epsilon$, for some $\epsilon > 0$ and some fixed k. We then have $\sum_{i:a_i<y} x^*(P, a_i) \geq \sum_{i:|a_i-a_k|<\delta} x^*(P, a_i)$. Since $\#\{i : |a_i - a_k| < \delta\} \to \infty$ and each $x^*(P, a_i) > \epsilon > 0$ we have that $\sum_{i:|a_i-a_k|<\delta} x^*(P, a_i) \to \infty$. A contradiction. ■

Note that (47) and the efficiency condition (10) are equivalent.

C.4.5 $\beta(a, P)$

By the Aggregation Lemma, $\beta(a, P)$ is given by the limit of $x_s^*(P, 0, 0, a, \mu)$ as $n \to \infty$. This partial can be computed by differentiating (43) implicitly, as follows

$$\lim_{n\to\infty} \frac{\partial}{\partial s} \int \hat{u}' \left(\hat{W}_0(a_i) + x^*(\cdot)(V(v) - P)\right)(V(v) - P) \cdot \exp((sv - mv^2/2)\hat{t}(a))g(v, P, \mu)dv = 0.$$

We then interchange the limit and differentiation with the integral, which, as in the previous steps, is justified by the Dominated Convergence Theorem, and evaluate the resulting expression at $s = m = \mu = 0$.

We get:

$$\int \lim_{n\to\infty} \frac{\partial}{\partial s} \left(\hat{u}' \left(\hat{W}_0(a) + x^*(\cdot)(V(v) - P)\right)(V(v) - P) \exp((sv - mv^2/2)\hat{t}(a))g(v, P, \mu)\right) dv =$$
$$x_s^*(P, 0, 0, a, 0)\, u''(W_0(a))(V(v) - P)^2 g(v, P, 0)+$$
$$u'(W_0(a))(V(v) - P)t(a)\, v\, g(v, P, 0). \quad (48)$$

The terms in the second line of (48) arise from differentiating $x^*(\cdot)$, whereas the terms in (48) arise from differentiating the density. We accounted for the fact that $x^*(P, 0, 0, a, 0)$ is zero (Lemma 9). The transitions $\lim_{n\to\infty} \hat{u}' (W_0(a) + x^*(\cdot)(V(v) - P)) = u'(W_0(a))$—and similarly for $\hat{u}''(\cdot)$—are due to the Uniform Limit Theorem. The convergence $\lim_{n\to\infty} x_s(P, 0, 0, a, \mu) = x_s(P, 0, 0, a, 0)$ is uniform over $a \in [0, y]$ by Lemma 12.

Integrating over v and accounting for the fact that $E[V(v)-P|P] = \int_{\mathbb{R}}(V(v)-P)g(v, P, 0)dv = 0$ (Lemma 9) yields

$$\beta(a, P) = \frac{t(a)}{\rho(a)} \frac{E[v(V(v) - P)|P]}{\text{Var}[V(v)|P]}.$$

We summarize in the following lemma.

Lemma 10. *$\beta(a,P)$ is multiplicatively separable, that is, $\beta(a,P)=\beta_a(a)\beta_P(P)$ where*

$$\beta_a(a)=\frac{t(a)}{\rho(a)} \text{ and } \beta_P(P)=\frac{E[v(V(v)-P)|P]}{\mathrm{Var}[V(v)|P]}. \tag{49}$$

If $v|\mathbf{P}_*$ is distributed normally with variance τ^{-1}, we can write

$$\beta_P(P)=\frac{\tau^{-1}E[V'(v)|P]}{\mathrm{Var}[V(v)|P]}.$$

The last equation of the Lemma follows via integration by parts.[46] In the following part of the proof, we proceed by deriving the equilibrium price function. This derivation is greatly simplified by the multiplicative separability of $\beta(a,P)$.

C.4.6 The price function

Given that $\beta(a,P)$ is of the multiplicatively separable (Lemma 10), we have

$$\beta(a,P)=\beta_a(a)\beta_p(P).$$

The informational content of the price is summarized by

$$s_p=\frac{\int_0^1\beta_a(a)ds(a)}{\int_0^1\beta_a(a)da}=v+\int_0^1\frac{\omega(a)}{\sqrt{t(a)}}dB(a).$$

[46]Indeed:

$$\begin{aligned}
E[v(V(v)-P)|P] &= \int_{\mathbb{R}} v(V(v)-P)\frac{1}{\sqrt{2\pi/\tau}}\exp\left(-\frac{(v-E[v|P])^2\tau}{2}\right)dv \\
&= \int_{\mathbb{R}}(v-E[v|P])\,(V(v)-P)\frac{1}{\sqrt{2\pi/\tau}}\exp\left(-\frac{(v-E[v|P])^2\tau}{2}\right)dv \\
&= -\tau^{-1}\int_{\mathbb{R}}(V(v)-P)d\left[\frac{1}{\sqrt{2\pi/\tau}}\exp\left(-\frac{(v-E[v|P])^2\tau}{2}\right)\right] \\
&= \tau^{-1}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi/\tau}}\exp\left(-\frac{(v-E[v|P])^2\tau}{2}\right)V'(v)dv.
\end{aligned}$$

Here

$$\omega(a) = \frac{\beta_a(a)}{\int_0^1 \beta_a(a)da}.$$

The inference from price is then the same as summarized in the Lemma 2. In particular, the conditional distribution of v given $\mathbf{P}_* = P$ is Normal with mean $\tau_p/\tau\, s_p$ and variance $1/\tau$. This means that v can be written as

$$v = \frac{\tau_p}{\tau} s_p + \frac{1}{\sqrt{\tau}} z.$$

Here z is a standard Normal random variable. The condition $E[V(v) - P|P] = 0$ (Lemma 9) can be then be rewritten

$$\int V\left(\frac{\tau_p}{\tau} s_p + \frac{z}{\sqrt{\tau}}\right) d\Phi(z) = P. \tag{51}$$

This gives the explicit expression for the function $\mathcal{P}(s_p)$. We summarize in the lemma below.

Lemma 11. *The equilibrium price function is given by* $\mathbf{P}_* = \mathcal{P}\left(s_p\right)$*, where* $s_p = v + \int_0^1 \frac{\beta_a(a)}{\sqrt{t(a)}} dB(a)$ *and* $\mathcal{P}(x)$ *is a strictly increasing function defined by*

$$\int V\left(\frac{\tau_p}{\tau} x + \frac{z}{\sqrt{\tau}}\right) d\Phi(z) = \mathcal{P}(x).$$

Moreover, $\tau = \tau_v + \tau_p$ *and* τ_p *is given by* $\tau_p = \frac{\left(\int_0^1 \beta(a)da\right)^2}{\int_0^1 \frac{\beta(a)^2}{t(a)} da}$*. Let* $h(P)$ *be the inverse of* $\mathcal{P}(\cdot)$*. For a function* $f(v, P)$ *we have*

$$E[f(v, P)|P] = \int f\left(\frac{\tau_p}{\tau} h(P) + \frac{z}{\sqrt{\tau}}, P\right) d\Phi(z).$$

The last expression of the lemma follows by the same change of variables that yields (51). It allows us to compute the conditional moments of $V(v)$. In particular, for k-th moment, we have:

$$\mathbb{E}[(V(v) - \mathbb{E}[V(v)|P])^k|P] = \mathbb{E}[(V(v) - P)^k|P] = \int \left(V\left(\frac{\tau_P}{\tau} h(P) + \frac{z}{\sqrt{\tau}}\right) - P\right)^k d\Phi(z). \tag{52}$$

C.4.7 $\delta(a, p)$

By Aggregation Lemma,

$$\delta(a,p) = \lim_{n\to\infty} \frac{1}{2t(a)} x_{ss}(P,0,0,a,\mu) + x_m(P,0,0,a,\mu) + x_\mu(P,0,0,a,\mu).$$

To compute these partials, we follow the same steps as in Section C.4.5: We differentiate the first-order condition in (43) implicitly and then take the $n \to \infty$ limit. We split the calculation into several steps.

Step 1. Calculating $\lim_{n\to\infty} \frac{1}{2t(a)} x_{ss}(P,0,0,a,\mu) + x_m(P,0,0,a,\mu)$.

Differentiating (43) twice with respect to s, dividing by $2t(a)$, adding the derivative of (43) with repect to m (i.e., we compute $\frac{1}{2t(a)}\frac{\partial^2}{\partial s^2}(43) + \frac{\partial}{\partial m}(43)$), passing to the limit we get:

$$\begin{aligned}\frac{1}{2t(a)} u'''(W_0(a)) x_s^*(P,0,0,a,0)^2 \int (V(v) - P)^3 g(v,P,0)dv \\ + u''(W_0(a)) x_s^*(P,0,0,a,0) \int v(V(v) - P)^2 g(v,P,0)dv \\ + u''(W_0(a)) \left(\frac{1}{2t(a)} x_{ss}^*(P,0,0,a,0) + x_m^*(P,0,0,a,0) \right) \int (V(v) - P)^2 g(v,P,0)dv = 0.\end{aligned}$$

The convergence $\lim_{n\to\infty} x_{ss}(P,0,0,a,\mu) = x_s(P,0,0,a,0)$ and $\lim_{n\to\infty} x_m(P,0,0,a,\mu) = x_m(P,0,0,a,0)$ is uniform over $a \in [0,y]$ by Lemma 12.

We then note that $-u'''(W_0(a))/u''(W_0(a)) = \pi(a)$, $x_s^*(P,0,0,a,0) = \beta(a,P)$, $\int (V(v) - P)^3 g(v,P,0)dv = \mathbb{E}[(V(v) - \mathbb{E}[V(v)|P])^3|P] = \text{Sk}[V(v)|P]\text{Var}[V(v)|P]^{3/2}$ and $\int (V(v) - P)^2 g(v,P,0)dv = \text{Var}[V(v)|P]$. With this, we obtain

$$\begin{aligned}\frac{1}{2t(a)} x_{ss}^*(P,0,0,a,0) + x_m^*(P,0,0,a,0) = \\ \frac{\beta(a,p)^2}{2t(a)} \pi(a) \frac{\text{Sk}[V(v)|P]}{\text{Var}[V(v)|P]} - \beta(a,p) \frac{E[v(V(v) - P)^2|P]}{\text{Var}[V(v)|P]}.\end{aligned}$$

Step 2. Calculating $\lim_{n\to\infty} x_\mu(P,0,0,a,\mu)$.

Differentiating (43) with respect to μ and passing to the limit we obtain

$$u'(W_0(a))\int (V(v)-P)g_\mu(v,P,0) + u''(W_0(a))x_\mu(P,0,0,a,0)\int (V(v)-P)^2 g(v,P,0) = 0$$

Now note that $-u''(W_0(a))/u'(W_0(a)) = \rho(a)$, $\int (V(v)-P)^2 g(v,P,0)dv = \mathrm{Var}[V(v)|P]$ and

$$\int (V(v)-P)g_\mu(v,P,0) = \frac{\partial}{\partial\mu}\int (V(v)-P)g(v,P,\mu)dv\bigg|_{\mu=0} = \frac{\partial}{\partial\mu}E[V(v)-P|P] = \psi(P).$$

With this, we obtain

$$x_\mu(P,0,0,a,0) = \frac{\psi(P)}{\rho(a)\mathrm{Var}[V(v)|P]}.$$

Combining everything together, we obtain the following expression for $\delta(a,p)$:

$$\delta(a,P) = \frac{\beta(a,p)^2}{2t(a)}\pi(a)\frac{\mathrm{Sk}[V(v)|P]}{\mathrm{Var}[V(v)|P]} - \beta(a,p)\frac{E[v(V(v)-P)^2|P]}{\mathrm{Var}[V(v)|P]} + \frac{\psi(P)}{\rho(a)\mathrm{Var}[V(v)|P]}.$$

The function $\psi(P)$ can be uniquely pinned down from the market clearing condition

$$\int \delta(a,P)da + h(P)\int \beta(a,P)da = \Theta(p).$$

C.4.8 Existence

Our derivation so far shows that an equilibrium is unique if it exists. We still need to show that a sequence of discrete economies satisfying the technical conditions in Section C.4.1 exists.

To this end, we consider a sequence of economies with the primitives given by $\hat{t}(a) = t(a)$, $\hat{W}(a) = W(a)$ and $\hat{u}''(w;a) = \mathrm{Tr}(u''(w;a), u''(W_0(a)-\epsilon;a), u''(W_0(a)+\epsilon,a))$ for some fixed ϵ, where $\mathrm{Tr}(x,l_1,l_2)$ denotes the function that truncates x to lower limit $\min\{l_1,l_2\}$ and upper limit $\max\{l_1,l_2\}$.

We next consider a uniform partition of the unit interval with $a_i = (i-1)/n$ and $m_i = \mu = 1/n$ for all i. To define the price beliefs, consider

$$s_{p,i}^n \equiv v + k_i \int_{-i} \beta(a)/\sqrt{t(a)}dB(a).$$

where

$$k_i = \sqrt{\frac{\int_0^1 \beta(a)^2/t(a)da}{\int_{-i} \beta(a)^2/t(a)da}}.$$

Here $-i \equiv [0,1) \setminus [a_i, a_i + m_i)$. Note that k_i is chosen such that the distribution on $s_{p,i}^n|v$ is the same for all i. Moreover, we have $k_i \to 1$ as $n \to 0$ and so $s_p = \text{plim}_{n\to\infty} s_{p,i}^n$. We define $\mathbf{P}_i^n = \mathcal{P}^n(s_{p,i}^n)$, where the function $\mathcal{P}^n(\cdot)$ is such that its inverse is $h^n(P) = h(P) + \mu q(P)$, for some function $q(P)$ that will be linked to $\psi(P)$ in equilibrium. One can verify that such a sequence satisfies all technical conditions. In particular, the symmetry of the joint density of v and $\mathbf{P}_i^n$ follows from the symmetry of the conditional distribution of $s_{p,i}^n|v$ and we have $\mathbf{P}_* = \text{plim}_{n\to\infty} \mathbf{P}_i^n$ by Continuous Mapping Theorem.

C.4.9 Auxiliary lemmata

Lemma 12. *Consider a sequence of functions $\hat{x}(P,a)$ and $\hat{q}(P,a)$ that converge, as $n \to \infty$ to 0 and $q(P,a)$, respectively, for all $a \in [0,1)$ and all $P \in \mathbb{R}$. Suppose that for any fixed P, there exist A and k that may depend on P, such that $\hat{q}(v,p) < A\exp(-kv^2)$, for all n. Take a partition $[0,y] = \cup_{i=1}^n [a_i, a_i + m_i]$, and let $m = \max_i m_i$ and $\hat{a}_n(a) = \sum_{i=1}^{n-1} a_i \mathbb{1}_{[a_i,a_{i+1})}(a)$. For any partition sequence such that $m \to 0$ as $n \to \infty$, we have that*

$$\int_{\mathbb{R}} \hat{u}^{(l)} \left(\hat{W}_0(a) + \hat{x}(P,a)(V(v) - P),\, a\right) \hat{q}(v,P)dv$$

converges uniformly to

$$u^{(l)}\left(W_0(a),\, a\right) \int_{\mathbb{R}} q(v,P)dv$$

over $a \in [0, y]$.

Proof of Lemma 12. Note that, for a given n,

$$\sup_{a\in[0,y]}\left|\int_{\mathbb{R}}\hat{u}^{(l)}\left(\hat{W}_0(a)+\hat{x}(P,a)(V(v)-P),\,a\right)\hat{q}(v,P)dv-\int_{\mathbb{R}}u^{(l)}\left(W_0(a),\,a\right)q(v,P)dv\right|\leq$$
$$\sup_{a\in[0,y]}\int_{\mathbb{R}}\left|\hat{u}^{(l)}\left(\hat{W}_0(a)+\hat{x}(P,a)(V(v)-P),\,a\right)\hat{q}(v,P)-u^{(l)}\left(W_0(a),\,a\right)q(v,P)\right|dv\leq$$
$$\int_{\mathbb{R}}\sup_{a\in[0,y]}\left\{\left|\hat{u}^{(l)}\left(\hat{W}_0(a)+\hat{x}(P,a)(V(v)-P),\,a\right)-u^{(l)}\left(W_0(a),\,a\right)\right|\right\}|\hat{q}(v,P)|dv+$$
$$\int_{\mathbb{R}}\sup_{a\in[0,y]}\left\{\left|u^{(l)}\left(W_0(a),\,a\right)\right|\right\}\left|\hat{q}(v,P)-q(v,P)\right|dv.$$

Now consider the $n\to\infty$ limit of the above. We have

$$\lim_{n\to\infty}\int_{\mathbb{R}}\sup_{a\in[0,y]}\left\{\left|\hat{u}^{(l)}\left(\hat{W}_0(a)+\hat{x}(P,a)(V(v)-P),\,a\right)-u^{(l)}\left(W_0(a),\,a\right)\right|\right\}|\hat{q}(v,P)|dv=$$
$$\int_{\mathbb{R}}\lim_{n\to\infty}\sup_{a\in[0,y]}\left\{\left|\hat{u}^{(l)}\left(\hat{W}_0(a)+\hat{x}(P,a)(V(v)-P),\,a\right)-u^{(l)}\left(W_0(a),\,a\right)\right|\right\}|q(v,P)|dv=0 \quad (53)$$

In the first transition, we passed the limit under the integral sign, which is permitted by the Dominated Convergence Theorem. We then noted that in the last equation above, $\lim_{n\to\infty}\sup_{a\in[0,y]}\{\cdots\}=0$. Here $\cdots$ denote the term in the curly brackets in (53). The fact that the limit is zero can be proved using the the"$\epsilon/3$" argument as in the proof of Lemma 8.

We also have that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\sup_{a\in[0,y]}\left\{\left|u^{(l)}\left(W_0(a),\,a\right)\right|\right\}\left|\hat{q}(v,P)-q(v,P)\right|dv=0.$$

Similarly to previous calculation, the limit can be passed under the integral by DCT, the sup is finite by the Extreme Value Theorem and $\hat{q}(\cdot)\to q(\cdot)$ by the hypothesis of the Lemma. ∎

C.5 Proof of Proposition 1

Proof of Proposition 1. From definition of $\mathcal{I}$ we have $\mathcal{I} = \tau_p/\tau$. Substituting (14) into the last equation and rearranging, we obtain the stated expression for $\mathcal{I}$.

We turn to deriving expression for $\mathcal{L}$. The market clearing price $\mathcal{P}(s_p, \bar{\theta})$ is P that solves

$$s_p = \frac{\bar{\theta} + \theta(P) - \int \delta(a, P)da}{\int \beta(a, P)da}.$$

Then define

$$\begin{aligned}
\text{Liquidity} &\equiv -\left(\frac{\partial}{\partial \bar{\theta}} \mathcal{P}(s_p, \bar{\theta})\right)^{-1} \\
&= -s_p \frac{\partial}{\partial P}\left(\int \beta(a, P)da\right) - \frac{\partial}{\partial P}\left(\int \delta(a, P)da - \theta(P)\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial s_p} \mathcal{P}(s_p, \bar{\theta}) &= \frac{-\int \beta(a, P)da}{s_p \frac{\partial}{\partial P}\left(\int \beta(a, P)da\right) + \frac{\partial}{\partial P}\left(\int \delta(a, P)da - \theta(P)\right)} \\
&= \text{Liquidity}^{-1} \cdot \int \beta(a, P)da.
\end{aligned}$$

From (13) we get that

$$\frac{\partial}{\partial s_p} \mathcal{P}(s_p, \bar{\theta}) = \frac{\tau_p}{\tau} \int V'\left(\frac{\tau_p}{\tau} s_p + \frac{z}{\sqrt{\tau}}\right) d\Phi(z) = \frac{\tau_p}{\tau} E[V'(v)|P].$$

Combining the two preceding equations and substituting the expression for $\beta(a, P)$ we get

$$\frac{\tau_p}{\tau} E[V'(v)|P] = \text{Liquidity}^{-1} \cdot \int \frac{t(a)}{\rho(a)} \frac{\tau^{-1}\, \mathbb{E}[V'(v)|P]}{\text{Var}\,[V(v)|P]} da \implies$$

$$\text{Liquidity} = \frac{1}{\tau_p \text{Var}[V(v)|P]} \int \frac{t(a)}{\rho(a)} da.$$

Multiplying by $\text{Var}[V(v)|P]$ and substituting (14) we get the stated equation for $\mathcal{L}$. ∎

C.6 Proof of Lemma 5

Proof of Lemma 5. First, from Lemma 2 we have $\tau_{agg} \equiv \mathbb{V}\text{ar}\big(v \big| s\left[\omega^H(a)\right]\big)^{-1} - \mathbb{V}\text{ar}(v)^{-1} = 1/\Big(\int_0^1 \omega^H(a)^2/t(a)da\Big)$. Second, we apply the Cauchy-Bunyakovsky-Schwartz inequality

$$\left(\int_0^1 f(a)g(a)da\right)^2 \leq \int_0^1 f(a)^2 da \int_0^1 g(a)^2 da \tag{54}$$

with $f(a) = \omega^H(a)/\sqrt{t(a)}$ and $g(a) = \sqrt{t(a)}$ to obtain $\tau_{agg}(b(a)) \leq \int_0^1 t(a)da$. The equality in (54) is attained if, and only if, $f(a)$ and $g(a)$ are linearly dependent, i.e., when $\omega^H(a)/\sqrt{t(a)} = c\sqrt{t(a)}$. The constant c is pinned down by the condition $\int_0^1 w(a)da = 1$. ■

C.7 Proof of Proposition 2

Proof of Proposition 2. Denote absolute risk tolerance $y(a) \equiv 1/\rho(a)$. Without loss of generality, index traders such that $y(b)$ increases in b. (This is in contrast to index a, which is such that $W_0(a)$ is increasing in a.) We first compute the Gateaux derivatives of $\mathcal{I}$ and $\mathcal{L}$ with respect to $y(b)$. We then show that under DARA utilities and the technical conditions imposed, the signs of the derivatives with respect to $y(b)$ and $W_0(a)$ are the same. We start by proving the following statement.

Step 1. There exist thresholds $0 < b_1^y \leq b_2^y < 1$, such that for for any Robin Hood variation $y^\Delta(b)$ with $\underline{b} \leq b_1^y \leq b_2^y \leq \bar{b}$

$$\begin{aligned} &\mathcal{I}'(y(b))[y^\Delta(b)] > 0 \text{ and } \mathcal{I}'(y(b))[-y^\Delta(b)] < 0; \\ &(\mathcal{L})'(y(b))[y^\Delta(b)] < 0 \text{ and } (\mathcal{L})'(y(b))[-y^\Delta(b)] > 0. \end{aligned} \tag{55}$$

From (14) we obtain

$$\tau_p = \left(\int_0^1 y(b)t(b)db\right)^2 \Big/ \left(\int_0^1 y(b)^2 t(b)db\right). \tag{56}$$

Substituting this expression into $\mathcal{I} = \tau_p/(\tau_p + \tau_v)$ and computing the Gateaux derivative (this entails substituting $y(b)+\epsilon y^{\Delta}(b)$ instead of $y(b)$, differentiating with respect to ϵ, and evaluating the resulting expression at $\epsilon = 0$) yields:

$$\mathcal{I}'(y(b))[y^{\Delta}(b)] = C_{\mathcal{I}} \int_0^1 t(b)y^{\Delta}(b)(I_2 - I_1 y(b))\, db.$$

Here, $C_{\mathcal{I}} > 0$ is positive (we have the closed-form expressions for $C_{\mathcal{I}}$ via parameters of the model, but it is not important here), $I_1 = \int_0^1 t(b)y(b)\, db$ and $I_2 = \int_0^1 t(b)y(b)^2\, db$. Lemma 13 (to follow) implies that there exists a unique b_y^* such that $I_2 - I_1 y(b) \gtrless 0$ iff $b \lesseqgtr b_*^y$. Then, for a $y^{\Delta}(b)$ that is Robin Hood with $\underline{b} < b_*^y < \bar{b}$, $y^{\Delta}(b)(I_2 - I_1 y(b)) > 0$ and the first statement of this step follows by letting $b_1^y = b_2^y = b_*^y$.

One can obtain $\mathcal{L} = \int_0^1 t(b)y(b)db/\tau_p$. Substituting (56) in this equation and computing the Gateaux derivative yields

$$(\mathcal{L})'(y(b))[y^{\Delta}(b)] = C_{\mathcal{L}} \int_0^1 t(b)y^{\Delta}(b)(2I_1 y(b) - I_2)\, db.$$

Here, $C_{\mathcal{L}} > 0$ is positive, and Lemma 13 (to follow) implies that there exists a unique $b_{**}^y > b_*^y$ such that $2I_1 y(b) - I_2 \gtrless 0$ iff $b \gtrless b_{**}^y$. Then, for a $y^{\Delta}(b)$ that is Robin Hood with $\underline{b} < b_{**}^y < \bar{b}$, $y^{\Delta}(b)(2I_1 y(b) - I_2) > 0$ and the first three statements of this step follow by letting $b_1^y = b_*^y$ and $b_2^y = b_{**}^y$.

Step 2. There exist thresholds $0 < a_1^W \leq a_2^W < 1$, such that for for any Robin Hood variation $W_0^{\Delta}(a)$ with $\underline{a} \leq a_1^W \leq a_2^W \leq \bar{a}$ (55) follows.

Let a_2^W be a unique solution to $W_0(a) = \underline{\eta} b_2^y$. Note that for any $a > a_2^W$, we have $y(a) > W_0(a)/\underline{\eta} > b_2^y$. Thus, by the previous step of the proposition, decreasing risk tolerances for traders $a > a_2^W$ leads to improvement in information efficiency and reduction in liquidity. Note also that decreasing wealth for traders $a > a_2^W$ induces their risk tolerances to decrease as well (DARA utilities). Similarly, letting a_1^W be a unique solution to $W_0(a) = \overline{\eta} b_2^y$, we get that for any $a < a_1^W$, $y(a) < b_1^y$ and that increasing wealth for traders $a < a_a^W$ induces their risk tolerances to increase as well (DARA utilities). Then, by the previous step, the statement of the proposition holds. ■

Lemma 13. *Assume* $0 < \underline{\eta} \leq \eta(a) \leq \bar{\eta} < \infty$. *Assume that absolute risk tolerance* $y(a)$ *is a continuous and strictly increasing function of* a. *For any* $c > 0$, *there exists a unique solution* $\hat{a}(c)$ *to* $y(a) = c$, *moreover,* $\hat{a}(c)$ *increases in* c *and* $0 < \hat{a}(c) < 1$ *for any* $0 < c < \infty$.

Proof of Lemma 13. Since $y(a)$ increases in a, at most one solution to $y(a) = c$ exists. Monotonicity also implies $\hat{a}(c)$ increases in c. For risk tolerance $y(a)$ we can write $y(a) = W_0(a)/\eta(a)$. We have $y(a) < W_0(a)/\underline{\eta}$ and so $0 \leq \lim_{a\to 0} y(a) = \inf y(a) \leq \inf\{W_0(a)\}/\underline{\eta} = 0$. Thus, $\lim_{a\to 0} y(a) = 0$. One can show analogously that $\lim_{a\to\infty} y(a) = \infty$. Then, by Intermediate Value Theorem, $\hat{a}(c)$ exists and $0 < \hat{a}(c) < 1$. ■

C.8 Proof of Proposition 3

Proof of Proposition 3. This proof follows the same steps as the proof of Proposition 2. Without loss of generality, index traders such that $y(b)$ increases in b.

Step 1. There exist thresholds $0 < b_1^t \leq b_2^t < 1$, *such that for for any Robin Hood variation* $t^\Delta(b)$ *with* $\underline{b} \leq b_1^t \leq b_2^t \leq \bar{b}$

$$\mathcal{I}'(t(b))[t^\Delta(b)] > 0 \text{ and } \mathcal{I}'(t(b))[-t^\Delta(b)] < 0;$$
$$(\mathcal{L})'(t(b))[t^\Delta(b)] < 0 \text{ and } (\mathcal{L})'(t(b))[-t^\Delta(b)] > 0.$$

Here, the proof is identical to step 1 of Proposition 2, with the difference of expressions for the Gateaux derivatives, which we reproduce below:

$$\mathcal{I}'(t(b))[t^\Delta(b)] = C_\mathcal{I}\int_0^1 y(b)t^\Delta(b)(2I_2 - I_1 y(b))\,db;$$

$$(\mathcal{L})'(t(b))[t^{\Delta}(b)] = C_{\mathcal{L}} \int_0^1 y(b)t^{\Delta}(b)(I_1 y(b) - I_2)\, db;$$

Step 2. There exist thresholds $0 < a_1^W \leq a_2^W < 1$*, such that for for any Robin Hood variation* $W_0^{\Delta}(a)$ *with* $\underline{a} \leq a_1^W \leq a_2^W \leq \bar{a}$ (21) *holds.*

Let a_2^W be a unique solution to $W_0(a) = \underline{\theta} b_2^t$. Note that for any $a > a_2^W$, we have $y(a) > W_0(a)/\underline{\theta} > b_2^t$. Similarly, letting a_1^W be a unique solution to $W_0(a) = \overline{\theta} b_2^t$, we get that for any $a < a_1^W$, $y(a) < b_1^t$. Then, by the previous step, the statement of the proposition holds. ■

C.9 Proof of Proposition 4

Proof of Proposition 4. With endogenous information acquisition and DARA utilities, the precision of information is increasing in wealth. The transfer of wealth from rich to poor has two effects. The indirect one of decreasing the precisions of the rich and increasing that of the poor (via endogenous information acquisition) is captured by the Proposition 3. Proposition 2 captures the direct one. By taking the thresholds $\underline{a} \leq \min\{a_1^t, a_1^W\}$ and $\bar{a} \geq \max\{a_2^t, a_2^W\}$ we make sure that both propositions apply ■

C.10 Completing heuristic derivation in Section 10

Here, we justify the transition from (26) to (27). Note that by the stochastic calculus heuristics (22), $dX(a)^2 = \beta(a,P)^2/t(a)da$ and so $\mathbb{E}[(V-P)^3|ds(a),P]\, dX(a)^2$ does not contain any $ds(a)$ terms. (Indeed, the terms in $\mathbb{E}[(V-P)^3|ds(a),P]$ that contain $ds(a)$ will be zeroed out after multiplying by $dX(a)^2 = \beta(a,P)^2/t(a)da$ and applying (22).)

Similarly, the term $\mathbb{E}[(V_P)^2|ds(a),P]$ in (26) can be replaced by $\mathbb{E}[(V_P)^2|P]$ in (27). The difference $\mathbb{E}[(V_P)^2|ds(a),P] - \mathbb{E}[(V_P)^2|P]$ contains at most a $ds(a)$ term, which will become a da term after being multiplied by $\beta(a,P)ds(a)$ in (27).

C.11 Proof of Proposition 5

Proof of Proposition 5.

Fix a trader i. Given the price P, his realized utility at time $t = 2$ is

$$\mathcal{U}_{i,t=2}\left(\Delta s_i, m_i\right) = u\left(W_0(a_i) + x(\Delta s_i, m_i; a_i)(V(v) - P)\right).$$

The Δs_i is a finite increment of a diffusion process

$$ds(b) = vdb + \frac{1}{\sqrt{t(a_i)}}dB(b)$$

between $b = a_i$ and $b = a_i + m_i$. Similarly, $\mathcal{U}_{i,t=2}\left(\Delta s_i, m_i\right) - \mathcal{U}_{i,t=2}\left(0,0\right)$ can be viewed as a finite increment of a diffusion process driven by $ds(b)$, between $b = a_i$ and $b = a_i + m_i$. By Ito's lemma, we can write

$$\mathcal{U}_{i,t=2}\left(\Delta s_i, m_i\right) - \hat{\mathcal{U}}_{i,t=2}\left(0,0\right) = \int_0^m \mu_u(b)db + \int_0^m \sigma_u(b)dB,$$

where μ_u and σ_u denote the drift (the "db" coefficient) and the diffusion coefficients (the "$dB(b)$" coefficient) of $\mathcal{U}_{i,t=2}$ process.

By Lemma 14 (to follow), the optimal precision solves

$$t(a_i) \in \arg\max_t \left\{ \left.\frac{\partial\, \mathbb{E}[\mathcal{U}_{i,t=2}]}{\partial m_i}\right|_{m_i=0} \right\}. \tag{57}$$

Thus, $t(a_i)$ maximized the expected drift of drift $\hat{\mathcal{U}}_{i,t=2}$ at $m_i = 0$, $\mathbb{E}[\mu_u(0)]$. The expected drift is then computed as in Section B.1, with the first order (necessary and sufficient) condition in (57) reducing to (35). ■

Lemma 14. *Consider a continuously differentiable function $f(t,m)$ such that $f(t,0)$ does not depend on t. Consider $t(m) \in \arg\max_t f(t,m)$. Suppose that $t(m)$ is bounded for small enough*

m. *Then,* $t(0) \in \arg\max_t f_m(t, 0)$.

Proof of Lemma 14. Suppose, on the contrary, that there exists some $\check{t}$ such that $f_m(\check{t}, 0) > f_m(t(0), 0)$. Then, by continuity, there exists $\bar{m}$ such that

$$f_m(\check{t}, m) > f_m(t(m), m) \text{ for } m < \bar{m}. \tag{58}$$

Integrate (58) with respect to m:[47]

$$f(\check{t}, m) - f(\check{t}, 0) > f(t(m), m) - f(t(0), 0).$$

Since $f(t, 0)$ does not depend on t we have $f(t(0), 0) = f(\check{t}, 0)$ and so

$$f(\check{t}, m) > f(t(m), m).$$

We obtained a contradiction with $t(m) \in \arg\max_t f(t, m)$. ■

[47]To integrate the right-hand side we use the Envelope Theorem $\frac{df(t(m),m)}{dm} = f_m(t(m), m)$