APPLIED PROBABILITY FOR MATHEMATICAL FINANCE REPORT: DELTA-GAMMA HEDGING

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Abstract. This project analytically investigates the dynamic Delta and Gamma hedging strategies by building an instantaneously risk-free portfolio trading in a call option, the stock, and the bank account and short the put option written on this asset in discrete time with Black-Scholes model. Under each hedging strategies, move-based and time-based methods are processed consecutively by initializing and rebalancing delta and gamma, position of assets, and book value to obtain the portfolio's profit and loss. Hedging decision is made by analysing a series of plot with tuned parameters.

1. Introduction

Delta hedging is an options trading strategy that aims to hedge the directional risk associated with price movements in the underlying asset by establishing offsetting long and short positions in the same underlying. This approach can isolate volatility changes of either a single option holding or an entire portfolio of holdings. However, one of the drawbacks of delta hedging is the necessity of constantly watching and adjusting positions involved, which can also incur trading costs. To avoid this problem, Delta-Gamma hedging strategy is introduced as it is an options strategy that combines both delta and gamma hedges to mitigate the risk of changes in the underlying asset and in delta itself.

To better understand how Delta-Gamma hedging strategy works in quantitative finance, we can begin with a model of the economy.

Theoretically, a dynamic hedging strategy initializes with the following model setup:

(1) A source of uncertainty given by the process $X = (X_t)_{t\geq 0}$. We assume that X satisfies the SDE:

$$\frac{dX_t}{X_t} = \mu^X(t, X_t)dt + \sigma^X(t, X_t)dW_t$$

where $W = (W_t)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.

(2) A bank account, $B = (B_t)_{t\geq 0}$, satisfying:

$$\frac{dB_t}{B_t} = r(t, X_t)dt$$

where $r = (r_t)_{t\geq 0} = (r(t, X_t))_{t\geq 0}$ is the short rate process. Notice that this is an instantaneously risk-free asset, as it is not driven by a Brownian motion.

(3) A traded claim written on the source of uncertainty whose price process is $f = (f_t)_{t\geq 0}$, and satisfies:

$$\frac{df_t}{f_t} = \mu^f(t, X_t)dt + \sigma^f(t, X_t)dW_t$$

With this model, we are enabled to find the dynamics of a second option, whose price process can be denoted by $g = (g_t)_{0\leq t\leq T}$ and pays $G(X_T)$ at time T. In general form the SDE the g_t satisfies is:

$$\frac{dg_t}{g_t} = \mu^g(t, X_t)dt + \sigma^g(t, X_t)dW_t$$

With this primary model setup, the following quantitative methodology could be introduced.

2. Methodology

In this study, we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}, \mathbb{Q})$ satisfying the usual conditions where T is a fixed time. $\mathbb{P}$ represents the real-world statistical or empirical probability measure, while $\mathbb{Q}$ represents the risk-neutral probability measure. We consider a market with a risky asset (stock) and a bank account. The asset pays no dividend and we assume that the interest rate is zero, without loss of generality. $X(t,\omega)$ represents the stock price per share at time $t \in [0,T]$. Since we are concerned about discrete time hedging, times are equally spaced $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_n = T$, and set time step $\Delta t = t_i - t_{i-1}$. Asset transaction cost c_A per share and option transaction cost c_O per unit are introduced.

Theorem 2.1 (Feynman-Kac Theorem)
The solution to the partial differential equation

$$\begin{cases} \partial_t h + \alpha(t,x)\cdot\partial_x h + \frac{1}{2}b(t,x)^2\cdot\partial_{xx}h = & c(t,x)\cdot h(t,x) \\ & h(T,x) = H(x) \end{cases} \tag{1}$$

admits a stochastic representation given by:

$$h(t,x) = \mathbb{E}_{t,x}^{\mathbb{P}^*}\left[H(X_T)\cdot e^{-\int_t^T c(s,X_s)ds}\right]$$

where $\mathbb{E}_{t,x}\big[\,\cdot\,\big] = \mathbb{E}_{t,x}\big[\,\cdot\,|X_t = x\big]$ *and the process* $X = (X_t)_{t\geq 0}$ *satisfy the SDE*

$$dX_t = a(t,X_t)dt + b(t,X_t)dW_t^{\mathbb{P}^*}$$

where $W^{\mathbb{P}^*} = (W_{t\geq 0}^{\mathbb{P}^*})$ *is a standard Brownian motion under* $\mathbb{P}^*$

Feynman-Kac Theorem established a strong linkage between partial differential equation and stochastic process that was applied to solve the generalized Black-Scholes PDE by converting the PDE to a computation-friendly conditional expectation when the price process of a claim is considered stochastic. The solution assures the availability of Black-Scholes model in this study.

Definition 2.2 (Self-Financing Portfolio)
A self-financing portfolio is a portfolio that satisfies

$$dV_t = \sum_{i=1}^{n} \alpha_t^{(i)} dA_t^{(i)}$$

The self-financing property of a portfolio in the definition of hedging strategy with respect to a European option is required to assure no immediate cash inflows/outflows until the maturity date. In this study, two self-financing portfolios, in delta hedging and delta-gamma hedging, were replicated for the written European put to match the cashflows schedule respectively.

Theorem 2.3 (Generalized Black-Scholes PDE)
Assume that a source of uncertainty given by Itô process $X = (X_t)_{t\geq 0}$ *satisfy the SDE*

$$\frac{dX_t}{X_t} = \mu^X(t,X_t)dt + \sigma^X(t,X_t)dW_t$$

where $\mu^X : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *and* $\sigma^X : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+/\{0\}$ *are functions and* $W = (W_t)_{t\geq 0}$ *is a* $\mathbb{P}$ *- Brownian motion. Further assume that a claim is written on* X *that pays* $G(X_T)$ *at time* T *whose price process,* $g = (g_t)_{0\leq t\leq T}$*, is Markovian in* X*, i.e. there exists a function* $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *such that* $g_t = g(t, X_t)$*. Then the function* g *satisfies the* ***generalized Black-Scholes PDE***

$$\begin{cases} (\partial_t + \mathcal{L})_t^X g(t,x) = & r(t,x)\cdot g(t,x) \\ & g(T,x) = G(x) \end{cases} \tag{2}$$

where $\mathcal{L}_t^X = \big(\mu^X(t,x) - \lambda(t,x)\cdot\sigma^X(t,x)\big)x\cdot\partial_x + \frac{1}{2}(\sigma^X(t,x))^2x^2\cdot\partial_{xx}$, conditioned on $X_t = x$.

Generalized Black-Scholes PDE shows the generasity because it works for many derivatives beside European options. The key to distinguish is to set the derivative payoff functions. Based on the *Feynman-Kac Theorem*, the generalized Black-Scholes PDE, which is derived by equating the process of drift and volatility and the source of uncertainty of asset, can be solved and represented as a stochastic process that is given by

$$g_t = \mathbb{E}_t^{\mathbb{Q}}\big[G(X_T)\cdot e^{-\int_t^T(s,X_s)ds}\big] \tag{3}$$

where X satisfies

$$\frac{dX_t}{X_t} = (\mu_t^X - \lambda_t\sigma_t^X)dt + \sigma_t^X dW_t^*$$

and where W^* is a $\mathbb{P}^*$-Brownian motion. In our study, this PDE provides a boundary condition for the function of g is known as the payoff function, namely that $g(T,x) = G(x)$, implies the claim's price obtained through the price process of claim written on underlying asset is equivalent to some function simulated asset price at the maturity date that is constrianted.

Model 2.4 (Black-Scholes Model)

2.4.1 Model

The introduction describes the initialized set-ups of economy models with stochastic process in a general sense. In this study, we choose to use Black-Scholes model, so defining the parameters is important. The Black-Scholes model requires constant drift and volatility generated from constant parameter geometric Brownian motions in the $\mathbb{P}$-dynamics of risky asset ,

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t^{\mathbb{P}}$$

and requires a constant interest rate given as

$$\frac{dB_t}{B_t} = rdt$$

. In delta hedging, we only use a risky asset, bank account as a portfolio to replicate the written put, while we add a call option to the portfolio in delta-gamma hedging. Thus, we specify the generalized Black-Scholes PDE by setting this call option to satisfy the Black-Scholes PDE(*i.e.* set a boundary condition to the call option)

$$\begin{cases} (\partial_t g(t,X) + rX\cdot\partial_X g(t,X) + \frac{1}{2}\sigma^2X^2\partial_{XX}g(t,X) = & r\cdot g(t,X) \\ & g(T,X) = G(x) \end{cases} \tag{4}$$

, with here replacing the term $\mu_t^X - \lambda_t^X\cdot\sigma_t^X$ to the constant risk-free rate r.

According to *Feynman-Kac Theorem*, the price process of this call option, which is the solution of BS PDE, can be represented as a easily computed expectation form with constant risk-free rate given as

$$g(t,X) = e^{-r(T-t)}\cdot\mathbb{E}_{t,X}^{\mathbb{Q}}\big[G(S_T)\big]$$

.

With these special setting, we can now use Black-Schole model to compute the prices s of options, positions of asset, and values of deltas and gammas shown in the following results part.

2.4.2 Results

Consider a European option expiring at T with strike price K, cusrrent asset price is X, the option values, relative values of delta and gamma hedging strategy can be computed by Black-Scholes model given as

$$f^{call}(t, X_t) = X\Phi(d_+) - Ke^{-r\tau}\Phi(d_-) \tag{5a}$$

$$\Delta^{call}(t, X_t) = \Phi(d_+) \tag{5b}$$

$$\Gamma^{call}(t, X_t) = \frac{\phi(d_+)}{X\sigma\sqrt{\tau}} \tag{5c}$$

$$f^{put}(t, X_t) = Ke^{-r\tau}\Phi(-d_-) - X\Phi(-d_+) \tag{6a}$$

$$\Delta^{put}(t, X_t) = \Phi(d_+) - 1 \tag{6b}$$

$$\Gamma^{put}(t, X_t) = \Gamma^{call}(t, X_t) = \frac{\phi(d_+)}{X\sigma\sqrt{\tau}} \tag{6c}$$

for cumulative distribution functions $d_\pm = \frac{\log(X/K)+(r\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ of some $\mathbb{P}$-standard Gaussian $\mathcal{Z} \sim \mathcal{N}(0, \sigma^{\mathbb{P}})$ and some $\mathbb{Q}$-standard Gaussian $\mathcal{Z} \sim \mathcal{N}(0, \sigma^{\mathbb{Q}})$, where $\tau = T - t$.

These formulas are used multiple times in the later simulation results section to analyse in many dimensions of result data to determine the optimized hedging strategies under certain conditions.

Model 2.5 (Delta-Hedging Discrete Time)

To apply the Black-Scholes model into determining delta hedging strategies under consideration, let us simply consider $t = k\Delta t$. The risky asset prices were simulated with Brownian Motion as mentioned in the initialization part, amount in bank account is reinvested with risk-free rate r

$$Y_{k\Delta t} = Y_{(k-1)\Delta t} e^{r\Delta t}$$

, and positions of asset is hedged

$$\alpha_{k\Delta t} = \partial_X f(k\Delta t, X_{k\Delta t})$$

to rebalance and locally eliminate risk and cost $(\alpha_{k\Delta t} - \alpha_{(k-1)\Delta t})X_{k\Delta t}$. Notice that no rebalancing plan at the maturity date but simulation generated of asset price and reinvestment of bank account are required. As the claim arrived at the maturity date, compared the profit and loss for the written European put under each hedging strategy and different volatility by liquidating all assets and paying the option payoff

$$PnL = Y_{T-\Delta t} \cdot e^{r\Delta t} + \alpha_{T-\Delta t} \cdot X_T - f(T, X_T)$$

suffices to decide on the optimized delta hedging strategy through this dynamic rebalancing hedging system.

Model 2.6 (Delta-Gamma Hedging Discrete Time)

Compared and based on delta hedging method, delta-gamma-hedging strategy implements the deficiency of merely enabling a small range of asset price change that performs better.

Consider the price process of a contingent claim written on asset X, the second order Taylor expansion

$$\begin{aligned} g(t, X_{t+\Delta t}) &= g(t, X_t + \Delta X_t) \\ &= g(t, X_t) + \partial_x g(t, X_t) \cdot \Delta X_t + \frac{1}{2} \cdot \partial_{xx} g(t, X_t) \cdot (\Delta X_t)^2 + \mathcal{O}^2 \end{aligned} \tag{7}$$

is used to derive the gamma of the claim g written on asset X, which is denoted as $\Gamma^g = \partial_{xx} g(t, X_t)$.

In this study, we choose to apply delta-gamma hedging strategy by holding a bank account, risky asset, X, and a contingent claim(*i.e.* call option more specific), g as a portfolio to firstly event out the response of market movements within a small range of asset price change bringing the net position zero(*i.e.* delta-neutral)

$$\begin{aligned} \partial_X V_t &= \partial_X(\alpha_t X_t + \gamma_t g(t, X_t) - f(t, X_t)) = 0 \\ &= \alpha_t + \gamma_t \cdot \partial_X g(t, X_t) - \partial_X f(t, X_t) = 0 \\ \Rightarrow \qquad & \alpha_t + \gamma \cdot \Delta_t^g - \Delta_t^f = 0 \end{aligned} \tag{8}$$

, with this delta-neutral step is closely followed by removing value change of portfolio due to significant asset price change(*i.e.* gamma-neutral)

$$\partial_{XX} V_t = \gamma_t \cdot \partial_{XX} g(t, X_t) - \partial_{XX} f(t, X_t) = 0$$

. As a result, the gamma and positions on asset of portfolio are derived given as:

$$\gamma_t = \frac{\Gamma_t^f}{\Gamma_t^g} \tag{9}$$

$$\alpha_t = \Delta_t^f - \frac{\Gamma_t^f}{\Gamma_t^g} \cdot \Delta_t^g \tag{10}$$

where $\Gamma_t^g = \partial_{XX} g(t, X_t)$, $\Gamma_t^f = \partial_{XX} f(t, X_t)$

Similar to the delta hedging, the profit and loss for the written European put in both move-based and time-based delta-gamma hedging strategies under different volatility are derived in the form:

$$PnL = Y_{T-\Delta t} \cdot e^{r\Delta t} + \alpha_{T-\Delta t} \cdot X_T + \gamma_{T-\Delta t} \cdot g(T, X_T) - f(T, X_T)$$

that suffices to decide on the optimized delta hedging strategy through this dynamic rebalancing hedging system

Theorem 2.7 (Risk-neutral pricing)
Let g_t be the price at time t of a claim written on a source of uncertainty ,X, with payoff function $G(x)$, where X has $\mathbb{P}$-dynamics given by the SDE

$$\frac{dX_t}{X_t} = \mu_t^X dt + \sigma_t^X dW_t^{\mathbb{P}}$$

where $W^{\mathbb{P}}$ is a $\mathbb{P}$ - Brownian motion. Then g_t is given by

$$g_t = \mathbb{E}_t^{\mathbb{Q}}\big[G(X_T) \cdot e^{-\int_t^T (s, X_s) ds}\big] \tag{11}$$

where the $\mathbb{Q}$ - dynamics of X are given by the SDE

$$\frac{dX_t}{X_t} = (\mu_t^X - \lambda_t \sigma_t^X) dt + \sigma_t^X dW_t^{\mathbb{Q}}$$

Furthermore, if X is a traded asset, then the $\mathbb{Q}$ - dynamics of X are given by

$$\frac{dX_t}{X_t} = r_t dt + \sigma_t^X dW_t^{\mathbb{Q}}$$

Notice that the exponential term can be written in terms of bank account as

$$\frac{B_t}{B_T} = e^{-\int_t^T r(s,X_s)ds} \tag{12}$$

Substitute into equation (11) and simplify

$$\frac{g_t}{B_t} = \mathbb{E}^{\mathbb{P}^*}\big[\frac{g_T}{B_T}|\mathcal{F}_t\big]$$

The simplified results imply that $\mathbb{P}^*$ is actually the risk-neutral probability measure invoked by the bank account as numéraire, which we denote as $\mathbb{Q}$. In this study, we set volatility of risk-neutral measure $\mathbb{Q}$ aligned and distinguished from the volatility of real-world measure $\mathbb{P}$ in both hedging strategies, namely delta hedging and delta-gamma hedging, to compare resulted profit and loss and determine the optimized hedging strategies.

3. Simulation Results

In this section, the simulation results will be displayed and analyzed. The simulation is implemented in discrete time for practical reason, with Black-Scholes model assumed for the underlying asset.

To begin with, consider an asset $S = (S_t)_{t\geq 0}$ follows the Black-Scholes model (Model 2.4), with $S_0 = 100, \sigma = 0.2, \mu = 0.1, r = 0.02$.

Consider that a short position on an 1/4 year put option with strike price $K = 100$ written on the asset is taken. In addition, an at-the-money 1/2 year call option written on the same asset, the asset and the bank account are available for trading in. Assume that the transaction cost for the underlying asset is 0.005\$ per unit, and the transaction cost for options is 0.01\$ per unit. The simulation process will explore the effectiveness of different hedging strategies, namely,

(1) Dynamic delta neutral hedging strategy. (time-based and move-based)

(2) Dynamic delta-gamma neutral hedging strategy. (time-based and move-based)

A self-financing portfolio (Definition 2.2) will be constructed and simulated through time steps. Model 2.5, Model 2.6 will be applied to guarantee the correctness of the simulation. At the end, all statistics will be computed and compared.

The number of simulations is 1000 and the number of time intervals is 10000.

Figure 1 shows a sample path of asset prices with 1000 simulations, where two paths are outlined.

3.1. **Move-based and Time-based Delta Hedging Strategy.**

In delta hedging strategy, only the asset and the bank account will be traded. The goal of delta hedging strategy is to neutralize the delta risk on the portfolio; that is, to ensure that $\partial_S V(t,S) = 0$ by taking $\alpha_t = \Delta_t^{put} := \partial_S f^{put}(t,S)$. There are two ways to develop a delta hedging strategy, namely, time-based hedging strategy and move-based hedging strategy. In time-based hedging process, the risk will be hedged by setting $\alpha_t = \Delta_t^{put}$ at each time step t; whereas in move-based hedging process, the risk will only be hedged when $|\alpha_t - \Delta_t^{put}|$ hits or exceeds some boundaries with a rebalancing-band b. The initial boundary for Δ_t is $(\alpha_0 - b, \alpha_0 + b)$. Then, at each time τ when the portfolio is rebalanced, the boundary will be updated to $(\alpha_\tau - b, \alpha_\tau + b)$. Also, there is a terminal boundary at $(-0.99, -0.01)$, to ensure that the portfolio will be properly rebalanced.

Intuitively, time-based hedging strategies are efficient when mitigating the risk of option, but it may over-balance the portfolio, leading to high total transaction cost. Move-based hedging strategies take transaction costs into consideration, and rebalance only when Δ_t moves out of some specific ranges, in which the risk is tolerated.

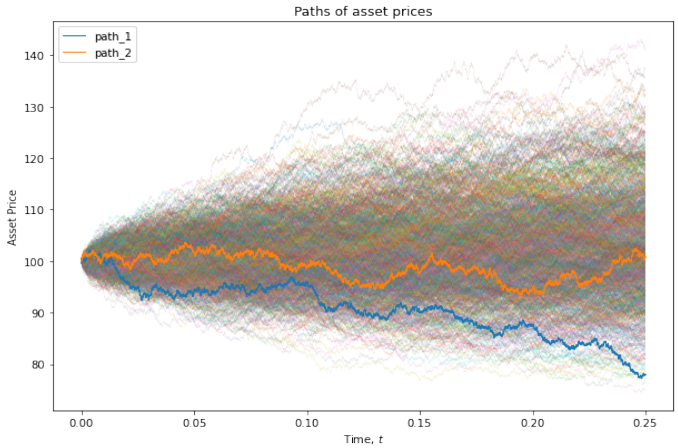
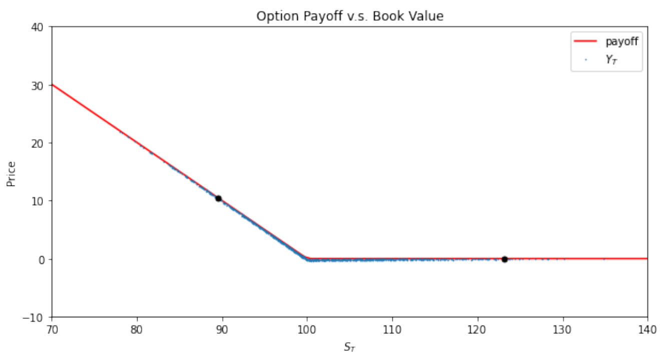
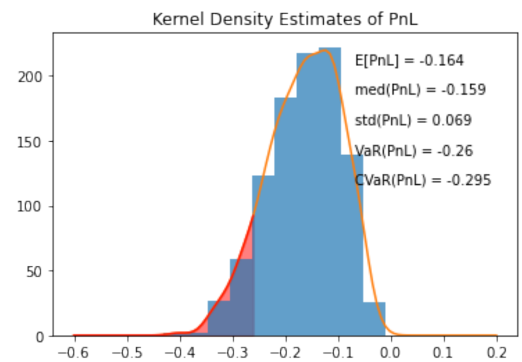


FIGURE 1. Sample Asset Paths



(a) Payoff and Simulated Book Value, at time T



(b) Profit and Loss

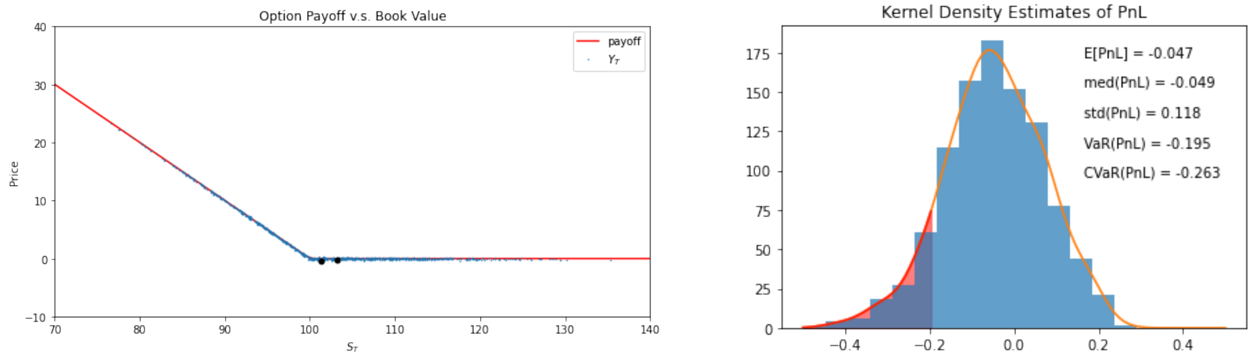
FIGURE 2. Results of Time-based Delta Hedge

3.1.1. *Time-based Results.*

Figure 2 shows the density function of profit and loss, side by side with comparison between option payoff and simulated book value at T. Readers may notice that the strategy yields negative profit in nearly all cases. The expected return is -0.164, with standard deviation 0.069. The value at risk and conditional value at risk (i.e. -0.26 and -0.295, respectively) indicates that the hedging strategy performs well to mitigate risk. But the overall effectiveness of the strategy is not optimal due to negative expected return. One reasonable explanation is that time-based hedging strategy transacts too frequently, incurring high transaction costs. However, this topic will be investigated in detail a bit further later. For now, let us move on to the results from move-based results.

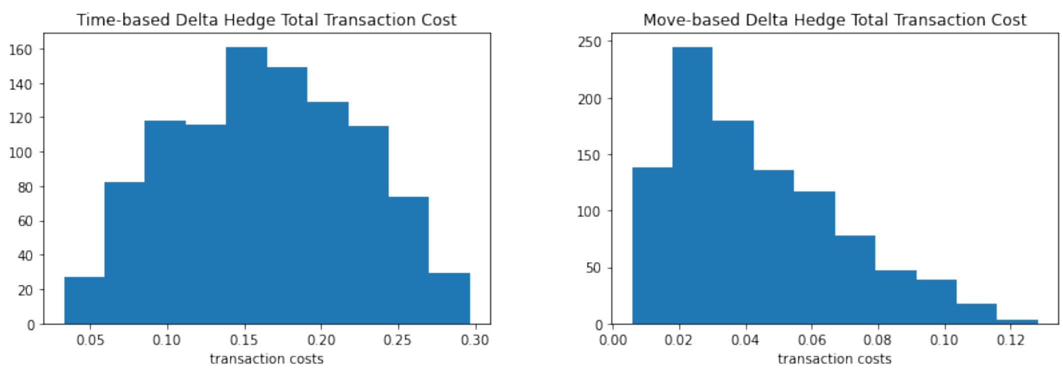
3.1.2. *Move-based Results.*

In move-based hedging strategy, band = 0.05 is used. Figure 3 shows the density function of profit and loss, along with comparison between option payoff and simulated book values at T. Compared to the time-based strategy, one can note that terminal book values scattered more closely to option payoff, with some points beyond the payoff line. The expected profit is -0.047, with standard deviation 0.118. The 90% value at risk and conditional value at risk are -0.195 and -0.263, respectively. Notably, the yields of move-based strategy has higher volatility compared to the time-based strategy, implying less frequent



(a) Payoff and Simulated Book Value, at time T (b) Profit and Loss

FIGURE 3. Results of Move-based Delta Hedge



(a) Time-based Strategy (b) Move-based Strategy

FIGURE 4. Total Transaction Costs

trading and potentially higher risk. However, all other statistics is significantly improved, suggesting that move-based hedging strategy performs better than time-based hedging strategy (i.e. more profitable and better risk minimization).

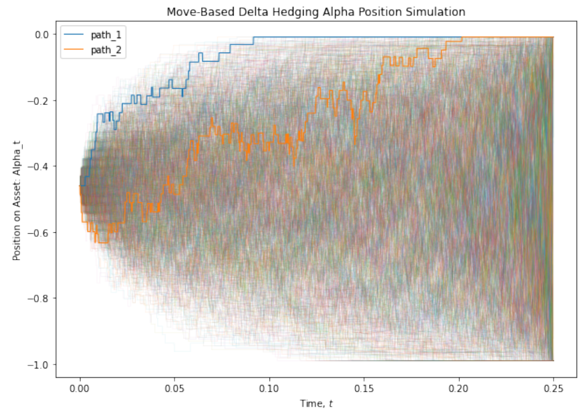
3.1.3. *Comparison and Comments.*

It is interesting to see that, although move-based hedging strategy shows to have higher volatility, it is still more effective at risk minimization, implied by its value at risk and conditional value at risk. The intuition behind the observation is that move-based hedging strategy balances between minimizing risks and minimizing transaction costs. Figure 4 displays the histogram of total transaction costs in time-based strategy and move-based strategy.

Recall that in move-based delta hedging strategy, the portfolio will only be rebalanced when the difference between Δ_t and α_t hits the band. It means that while the delta risk $(\alpha_t - \partial_S f_{put}(t, S_t))$ will always be contained in certain range bounded by our selection of band, the total number of tradings will be massively reduced. Figure 5 shows the paths of simulated α_t (position of the underlying asset), with two paths outlined. Note that α_t appears to have remarkably less "jumps" in move-based strategy than in time-based strategy (where α_t jumps at every time step), indicating less transactions.

In conclusion, move-based delta hedging strategy has higher expected return, VaR (value at risk), cVaR(conditional value at risk) compared to time-based delta hedging strategy. It means that with

(a) Time-based Strategy



(b) Move-based Strategy

FIGURE 5. Simulated Paths for Position on Asset

move-based hedging strategy, investors are expected to see higher profits and less risks, due to the existence of transaction costs on the asset.

3.2. **Move-based and Time-based Gamma Hedging Strategy.**

In a delta-gamma hedging strategy or gamma hedging strategy, the asset, the bank account and the call option will all be traded to minimize risks. The goal of delta-gamma hedging strategy is to neutralize both delta risk and gamma risk on the portfolio. In other words, given a self-financing portfolio with value $V(t,S) = \alpha_t S + Y + \gamma_t g(t,S) - f(t,S)$ (where α_t represents position on asset S, γ_t represents position on call option with price $g(t,S)$, $f(t,S)$ represents the value of put option sold), solve for α_t and γ_t that satisfies $\partial_S V(t,S) = 0$ and $\partial_{SS} V(t,S) = 0$. The mathematical reasoning behind delta-gamma hedging strategy is given in Model 2.7, and hence it will not be repeated here.

Similarly as delta hedging strategy, delta-gamma hedging can also be performed under both time-based and move-based strategies. However, in move-based delta-gamma hedging strategy, the band will not restrict the difference between Δ_t^{put} and α_t, given that the formula for delta-gamma neutral α_t is different. Instead, at each time step t, a set of delta-gamma neutral (α_t, γ_t) will be proposed, and the portfolio will only be rebalanced (i.e. accepting (α_t, γ_t)) if $\alpha_t \notin (\alpha_\tau - b, \alpha_\tau + b)$, where τ is the last time of rebalancing or 0.

3.2.1. *Time-based Results.*

Figure 6 shows the general results generated from time-based gamma hedging simulations. Remarkably, the simulated book value scattered way below payoff curve around $S_T = 100$. The PnL plot also shows that time-based delta-gamma hedging leads to negative returns. The expected profit is -1.295 with standard deviation = 0.815, and the value at risk and conditional value at risk are -2.41 and -3.153, respectively.

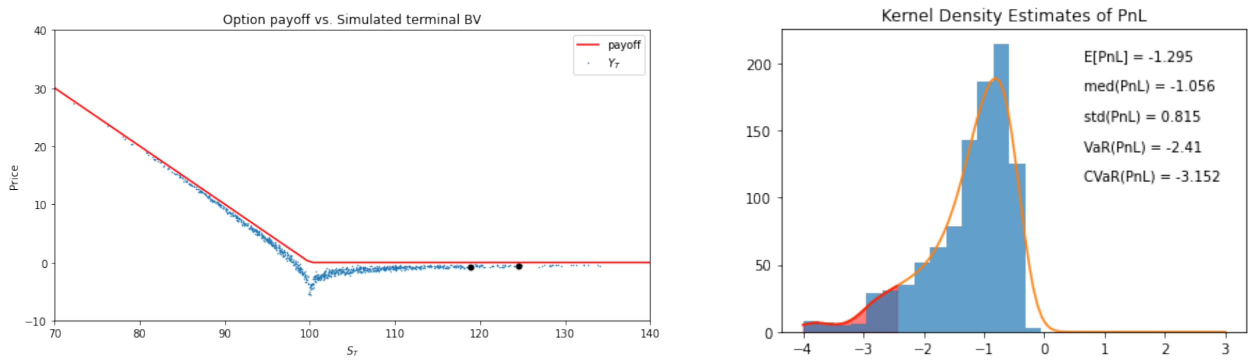
From a mathematical perspective of view, result 2.4.2 suggests that $\Gamma_t^{put} = \phi(d+)/S\sigma\sqrt{\tau}$. If the asset price at time T have $S_T = K$, then S_t would be close to $S_T = K$ for t sufficiently close to T. Then,

$$d_+ = \frac{log(S/K) + (r + 1/2\sigma^2)\tau}{\sigma\sqrt{\tau}} \to 0 \text{ as } \tau \to 0 \text{ where } \tau = T - t$$

.

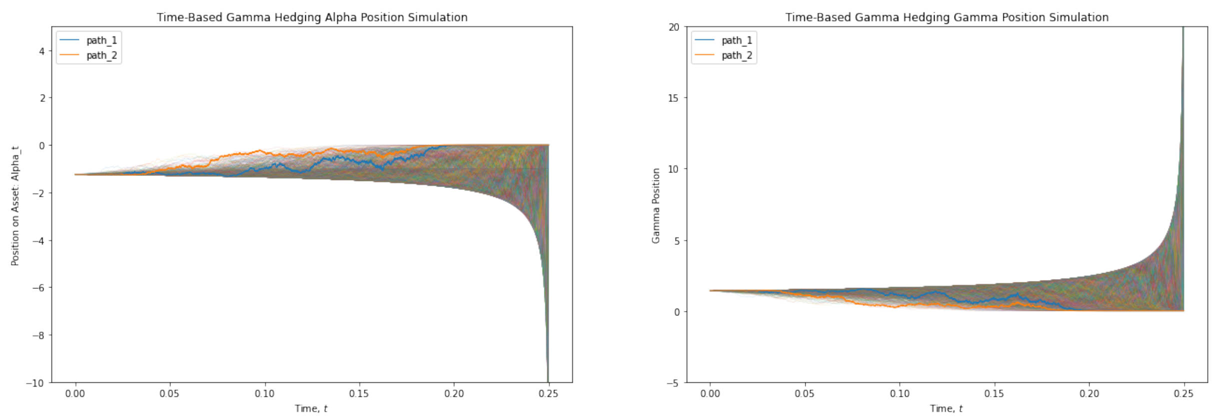
Then $\Gamma_t = \frac{\phi(d+)}{S\sigma\sqrt{\tau}} \to \infty$ as $\tau \to 0$.

It implies, if the asset price at terminal time T, S_T is near to $K = 100$, then for t close to T we would have a large Γ_t^{put}, leading to large $\gamma = \Gamma_t^{put}/\Gamma_t^{call}$. (Note that Γ_t^{call} endows a different equation since



(a) Payoff and Simulated Book Value, at time T (b) Profit and Loss

FIGURE 6. Results of Time-based Delta-Gamma Hedge



(a) Asset position simulation (b) Option position simulation

FIGURE 7. Simulated α_t, γ_t

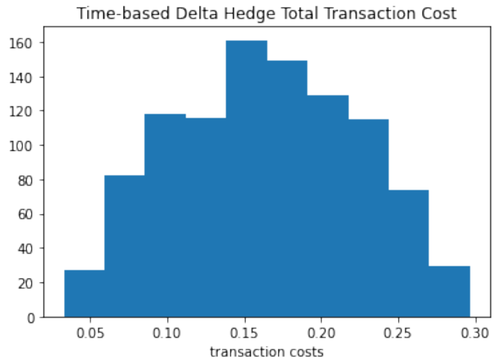
$T_{call} = 0.5$) Hence α_t would also be a large negative value. In spite of the large amount of transaction costs required to reach such (γ_t, α_t), the transaction cost for liquidating them will also be large. Readers should also note that same theory also applies to move-based delta-gamma hedging, but only with a smaller scale. Figure 7 shows the practical phenomenon induced by this theory.

Figure 8 compares the total transaction cost in time-based delta hedging and the total transaction cost in time-based delta-gamma hedging. It is worth noting that delta-gamma hedging can cost ten times more transaction fees than delta hedging.

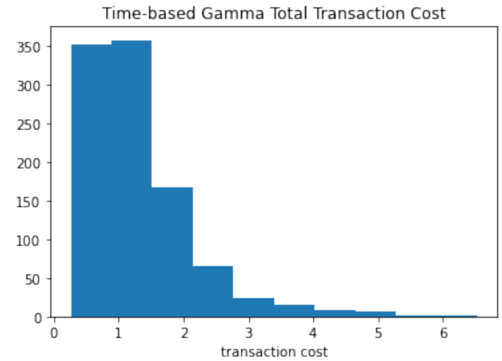
3.2.2. *Move-based Strategy.*

Recall that move-based delta-gamma hedging strategy put boundaries on α_t instead of Δ_t, figure 9 shows the general result of a move-based delta-gamma hedging strategy simulation.

The expected return is -0.722; standard deviation is 0.747; value at risk and conditional value at risk are -1.64 and -2.499, respectively. It is not surprised to observe that the expected return, value at risk and conditional value at risk statistics are improved in move-based delta-gamma hedging, since move-based strategies can mitigate the negative returns by reducing transaction costs. However, as opposed to the observation from delta hedging strategies, the standard deviation of return in move-based gamma hedging is slightly less than that in time-based gamma hedging. The intuitive reason is that move-based

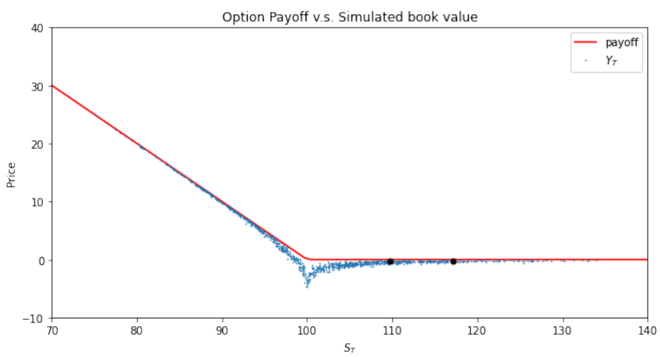


(a) Time-based Delta

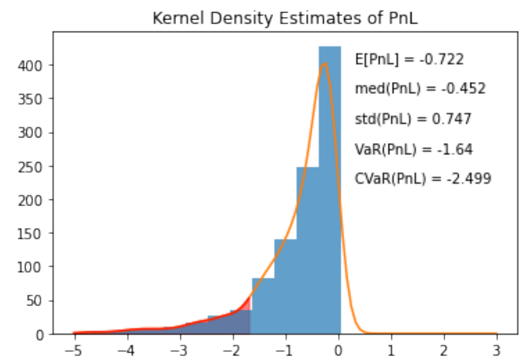


(b) Time-based Gamma

FIGURE 8. Total Transaction Costs

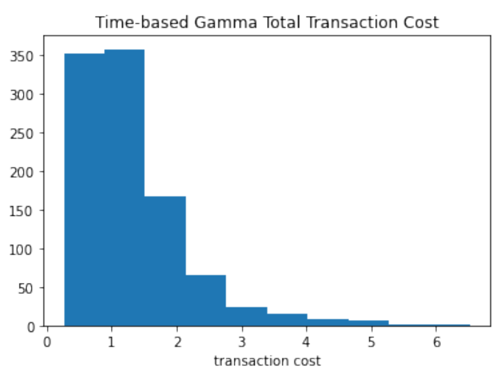


(a) Payoff and Simulated Book Value, at time T

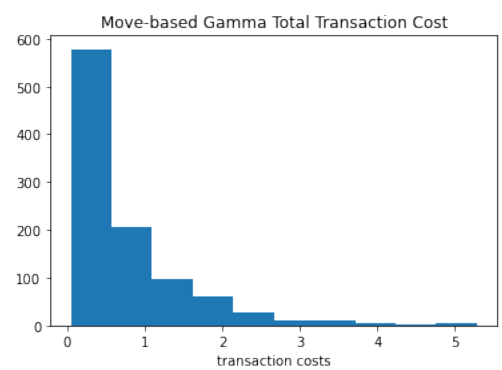


(b) Profit and Loss

FIGURE 9. Results of Move-based Delta-Gamma Hedge



(a) Time-based Gamma



(b) Move-based Gamma

FIGURE 10. Total Transaction Costs, Time-based v.s. Move-based Gamma

strategies reduce transaction cost to a large extent, as shown in Figure 10. Hence some extreme negative returns are pulled back to normal level.

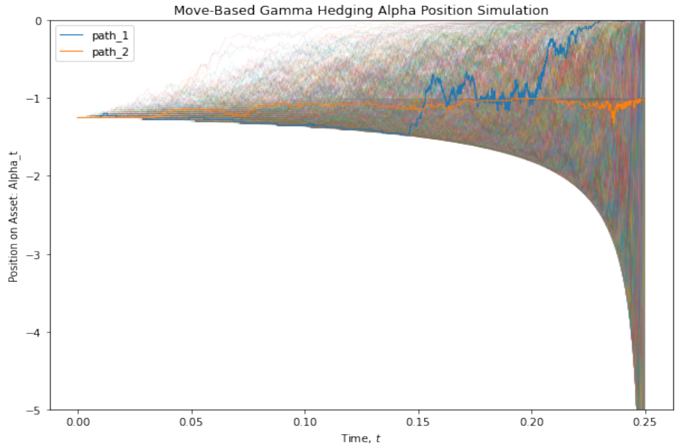


FIGURE 11. Simulated Position on Asset, move-based gamma hedging

3.2.3. *Comparison and Comments.*

Compared to delta hedging, gamma hedging strategies are under-performing in general due to heavy transactions in both option and asset. Figure 11 reveals that even in move-based gamma hedging strategy, α_t is reacting more sensitively at t nearing the maturity time, T.

One may argue that by enlarge the band can effectively restrict the amount of transactions. The study will investigate the impact of band selection in greater details in later sections. However, the nature of sensitive α_t, especially when $t \to T$, is independent from the selection of band. Investors may also consider to reduce a delta-gamma hedging strategy to a delta hedging strategy when certain conditions are met (eg. when t is close to T). This topic is out of the scope of this study, hence will not be explored in further details.

3.3. **Different Volatility.**

By far, the real-world probability measure is assumed to be the risk-neutral probability measure. While Theorem 2.7 (Risk-neutral price) suggests that we should use the assumed probability measure (i.e. risk-neutral probability measure) to correctly price an option, the real-world probability which determines the price of the underlying asset may be different. In this section, the study will explore how the hedging strategies behave if the real-world probability measure is different from risk-neutral probability measure.

Assume the real-world volatility of underlying asset is $\sigma_P = 0.15$; whereas the risk-neutral volatility is $\sigma_Q = 0.2$. Then, while the pricing function for options written on the underlying asset stays the same, the actual sample path of prices for the asset will change. Figure 12 shows a simulated sample path of asset prices with real-world volatility $\sigma_P = 0.15$.

Readers may notice that sample asset paths generated with $\sigma = 0.15$ real-world volatility are denser compared to figure 1, in which paths generated with $\sigma = 0.2$ are shown. Recall that under Black-Scholes model, the asset S_t endows $S_{t+\Delta t} = S_t e^{(\mu - 1/2\sigma^2)\Delta t + \sigma(W_{t+\Delta t} - W_t)}$, where W_t is a Brownian motion under $\mathbb{P}$ measure. It implies that the asset $(S_t)_{t\geq 0}$ would shift at a smaller scale with respect to time interval Δt, if we choose real-world volatility $\sigma_P = 0.15 < 0.2 = \sigma_Q$. Then, the most important question is: how does it change our hedging strategy? The study will investigate how it will alter the hedging strategy and how it will impact the hedging strategy returns. Let us begin with a delta hedging strategy.

3.3.1. *Delta Hedging Strategy.*

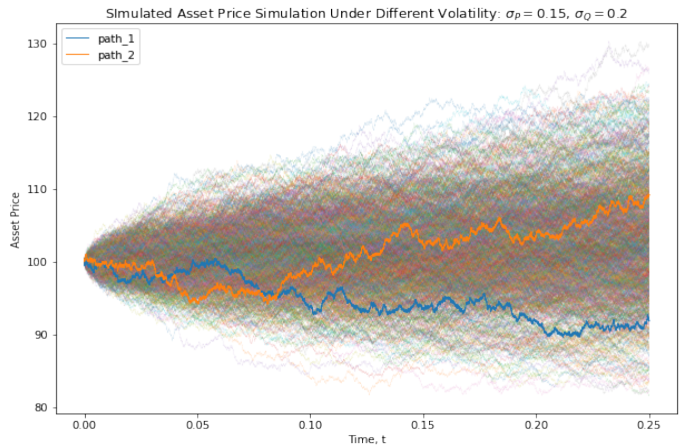
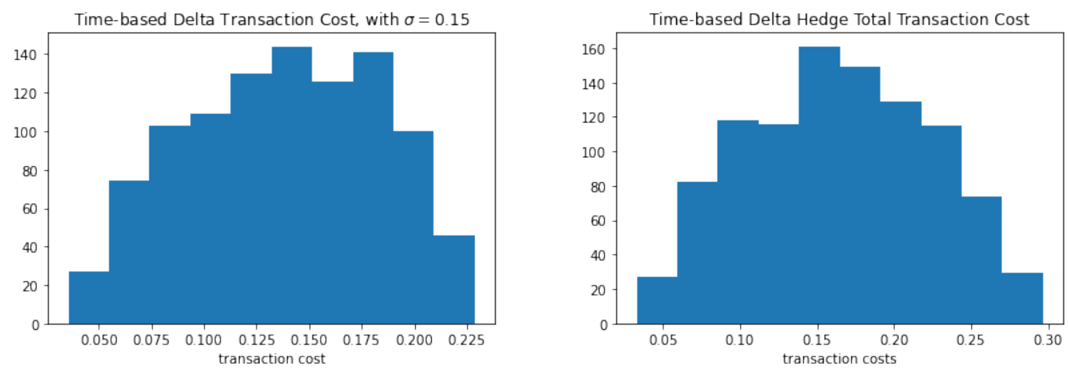


FIGURE 12. Sample Asset Paths



(a) Time-based Delta $\sigma = 0.15$

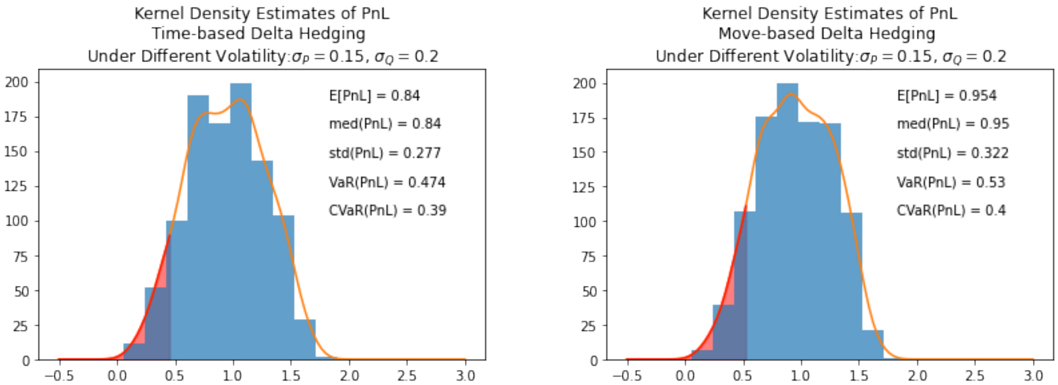
(b) Time-based Delta $\sigma = 0.2$

FIGURE 13. Total Transaction Costs, $\sigma_P = 0.15$ vs. $\sigma_P = 0.2$

Consider condition of zero delta risk, $\alpha_t = \Delta_t^{put}$. By result 2.4.2 we can expand the equation $\Delta_t = \Phi(d+) - 1$, where $d+ = \frac{log(S/K)+(r+1/2\sigma^2)\tau}{\sigma\sqrt{\tau}} \propto log(S)$ by taking all other variables constant. Also note that the function $\Phi()$ is a monotonically increasing function since it is a cumulative distribution function. It directly implies that $\Phi(d+)$ will change in smaller scales when S takes smaller steps at each time interval. In other worlds, the absolution change in α_t in each time steps is expected to be lower, leading to lower transaction costs. Figure 13 compared the simulated total transaction cost between time-based delta hedging with different real world volatility.

Again consider the value of portfolio $V = Y + \alpha S - f^{put}$. Intuitively, delta hedging strategies use the position on S to hedge the risk imposed by the option $-f^{put}$. Although the real-world volatility is altered, all options are still priced with risk-neutral volatility. Then at $t = 0$, all things should remain unchanged. It means that with real-world volatility is 15%, the hedging strategy would rebalance the portfolio at smaller scale, while Y still accumulated with the unchanged risk-free rate $r = 0.02$. Hence it is expected to see higher returns on the hedging strategy. Figure 14 shows the statistics and estimated density function of profit, using real-world $\sigma = 0.15$.

Notably, both time-based delta strategy and move-based delta strategy shows significant improvement in expected return and value at risk. It is not surprised to see that move-based strategy still outperforms



(a) Time-based Delta PnL (b) Move-based Delta PnL

FIGURE 14. PnL and statistics at $\sigma = 0.15$

time-based strategy, meaning that adding band to reduce the intensity of trading will effectively improve the performance, with either $\sigma_P = \sigma_Q = 0.2$ or $\sigma_P = 0.15$.

3.3.2. *Delta-Gamma Hedging Strategy.*

In delta-gamma hedging strategy, the derivation of α_t and γ_t becomes different. Recall that in delta hedging, α_t only depends on Δ_t^{put}; however, in delta-gamma hedging strategy, α_t depends on ratios and differences between (Δ, Γ) of two options. With analogy to the discussion in last section, (Δ, Γ) of both options should change in a smaller scale (with respect to time interval Δt) if S_t takes smaller steps. But it remains unclear on how the ratios and differences of the two set of (Δ, Γ) changes. It is worth noting that the option that is less sensitive to change in S_t could be intuitively ignored when analyzing the real-world volatility impact on hedging strategy (i.e. α_t, the position on asset). In another word, if we can identify which option is more sensitive to change in S_t, then we can conclude whether the change in scale of α_t is increased or decreased based on that option, and thus provides a mathematical reasoning in the perspective of change in total transaction cost.

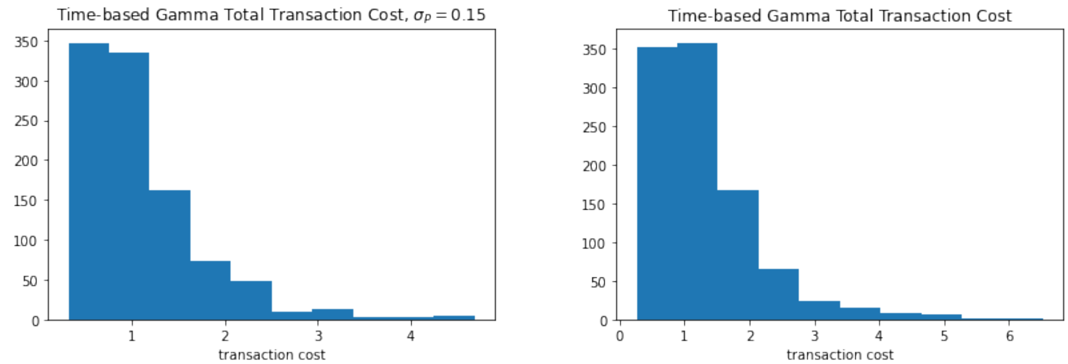
Recall that the scale of change in (Δ, Γ) depends on the scale of change in $d+$. (Note that $\Gamma = \phi(d+)\partial_S(d+) \implies$ smaller change in $d+$ leads to smaller change in Γ). Expanding $d+$ gives $d+ = \frac{log(S/K)+(r+1/2\sigma^2)\tau}{\sigma\sqrt{\tau}}$. Then, one could find that if τ is large, then $log(S/K)$ will have relatively smaller impact on $d+$; that is, $d+$ will be less sensitive to S. Then, it is obvious that since the call option has maturity $T_{call} = 0.5 > 0.25 = T_{put}$. The call option is less sensitive to the change in S.

Hence we could only consider $\Gamma^{put}, \Delta^{put}$. With smaller scale of change in S, both terms will change in smaller scale, resulting smaller changes in α_t. As what is previously discussed, it implies that the intensity of transaction will be reduced if a real-world volatility $\sigma = 0.15$ is used. Figure 15 shows the empirical observation.

Also, figure 16 shows that the delta-gamma hedging strategy yields improved return when using $\sigma_P = 0.15$. However, it appears that the delta-gamma hedging strategy is still no better than delta hedging. Clearly, the issue of over-exaggerated $|\alpha_t|$ and $|\gamma_t|$ at t close to T, S_t close to K would still be a problem, but only it happens with a smaller scale.

3.3.3. *Final Comments.*

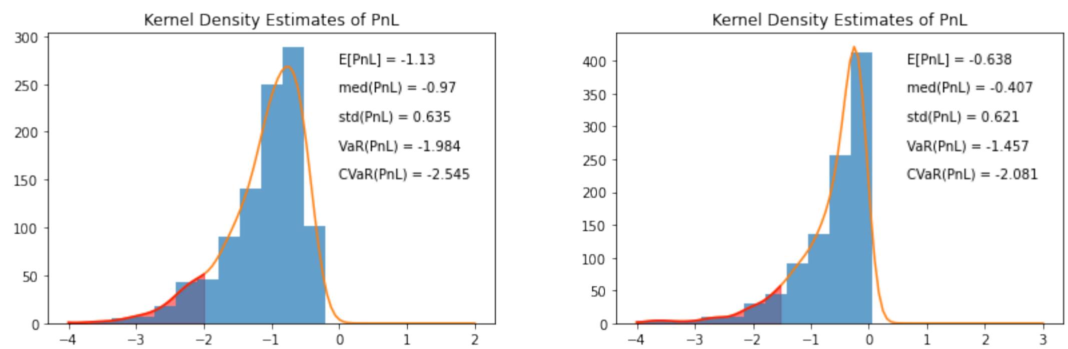
The simulation has shown that if the real-world probability volatility is 0.15 instead of 0.2, then the hedging strategies yield higher return. The mathematical reasoning has already been investigated in the last two sections.



(a) Time-based Gamma $\sigma = 0.15$

(b) Time-based Gamma $\sigma = 0.2$

FIGURE 15. Total Transaction Costs, $\sigma_P = 0.15$ vs. $\sigma_P = 0.2$



(a) Time-based Gamma PnL

(b) Move-based Gamma PnL

FIGURE 16. PnL and statistics at $\sigma = 0.15$

From a financial perspective, both options are priced under the risk-neutral probability measure. It implies that if the real-world volatility is lower, then the options will be over-priced. If a short position on the option is taken, then the investor will gain more cash-flow than the real value of the option. Hence, it follows that with a proper hedging strategy that can eliminate the risk from the option, the investor will be gaining profit in this portfolio.

Likewise, if the real-world volatility is in fact higher than risk-neutral volatility, then the option will be under-priced, leading to a potential loss in the portfolio. In either case, the construction of a solid hedging strategy is the key. The simulation already shows that move-based strategies are performing better than time-based strategies in general. It is now time to investigate how one can improve the move-based strategies. Recall that the difference between time-based strategy and move-based strategy is that a move-based strategy uses a band to tolerate the risk. Hence, it is undoubted that the key of a successful move-based hedging strategy is a proper selection of the band. Thus, the study will investigate the role of rebalancing-band in further details.

3.4. **Role of Rebalancing-Band.**

In move-based delta and move-based delta-gamma hedging strategies, there are several questions concerned with respect to the role that rebalancing-bands play: How do different bands affect the output under each strategy? How to valuate a good band and a bad band using metrics? In this section, the study will analyze the outputs as well as explore the answers to these questions.

3.4.1. *Move-based Delta Hedging Strategy.*

In this section, rebalancing-bands are set to 0.01, 0.05, 0.2, 0.5, representing narrow bands, medium bands and wide bands.

First, sample paths of asset prices under each rebalancing-band out of 1000 simulations could be generated. Intuitively, it does not show an obvious pattern of the impact that rebalancing-bands have on asset price.

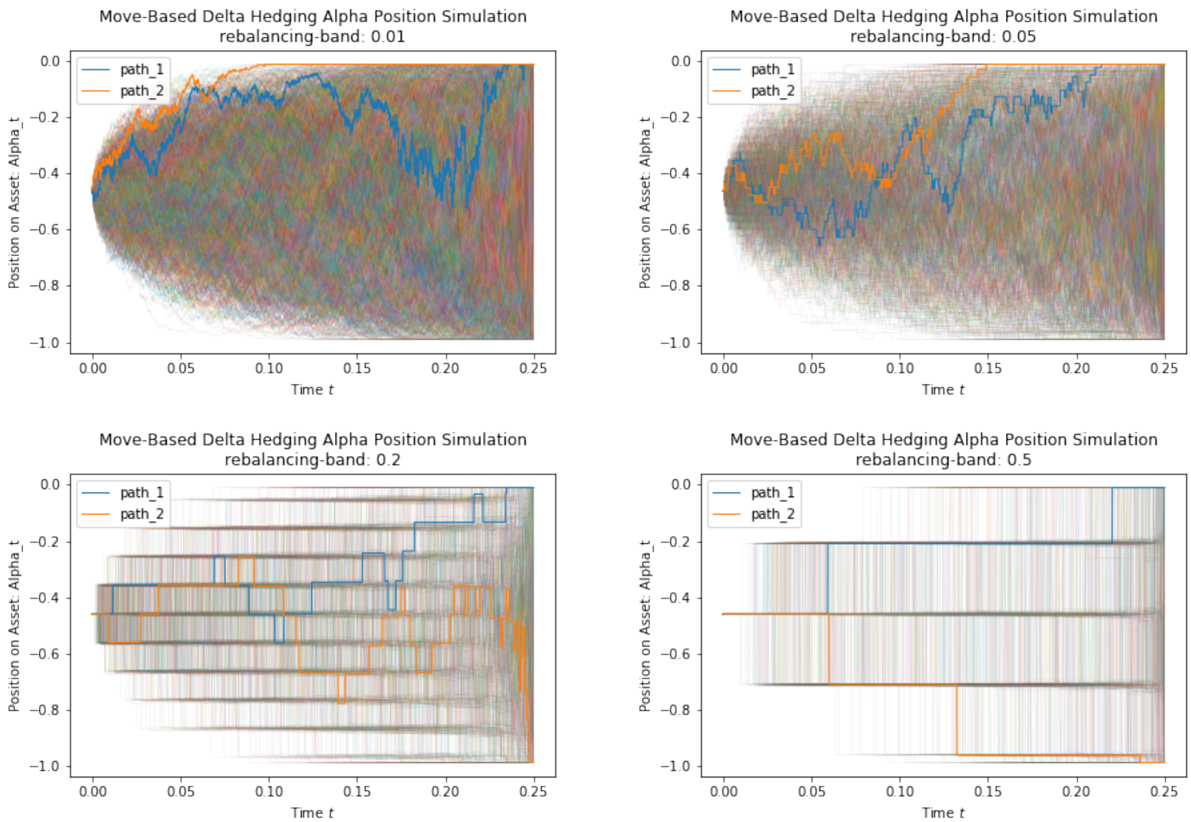


FIGURE 17. Simulated Paths for Position on Asset, move-based delta hedging

The figure of simulated paths for alpha position on asset clearly shows that move-based delta hedging strategy transacts less frequently as the range of band increases and hence the paths fluctuate less.

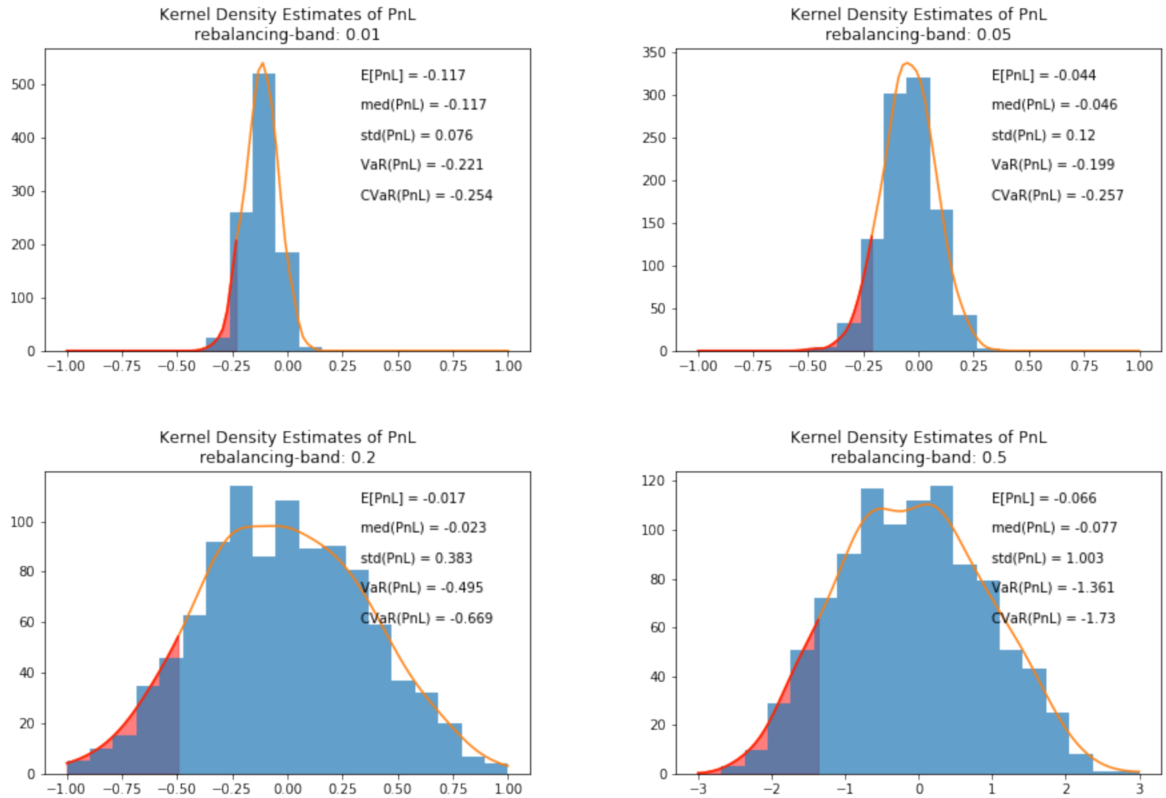


FIGURE 18. KDE Results, move-based delta hedging

The figure of KDE on PnL indicates a strong tendency that when the bands are wider, the standard deviation of PnL increases as well as the absolute values of VaR and CVaR decrease.

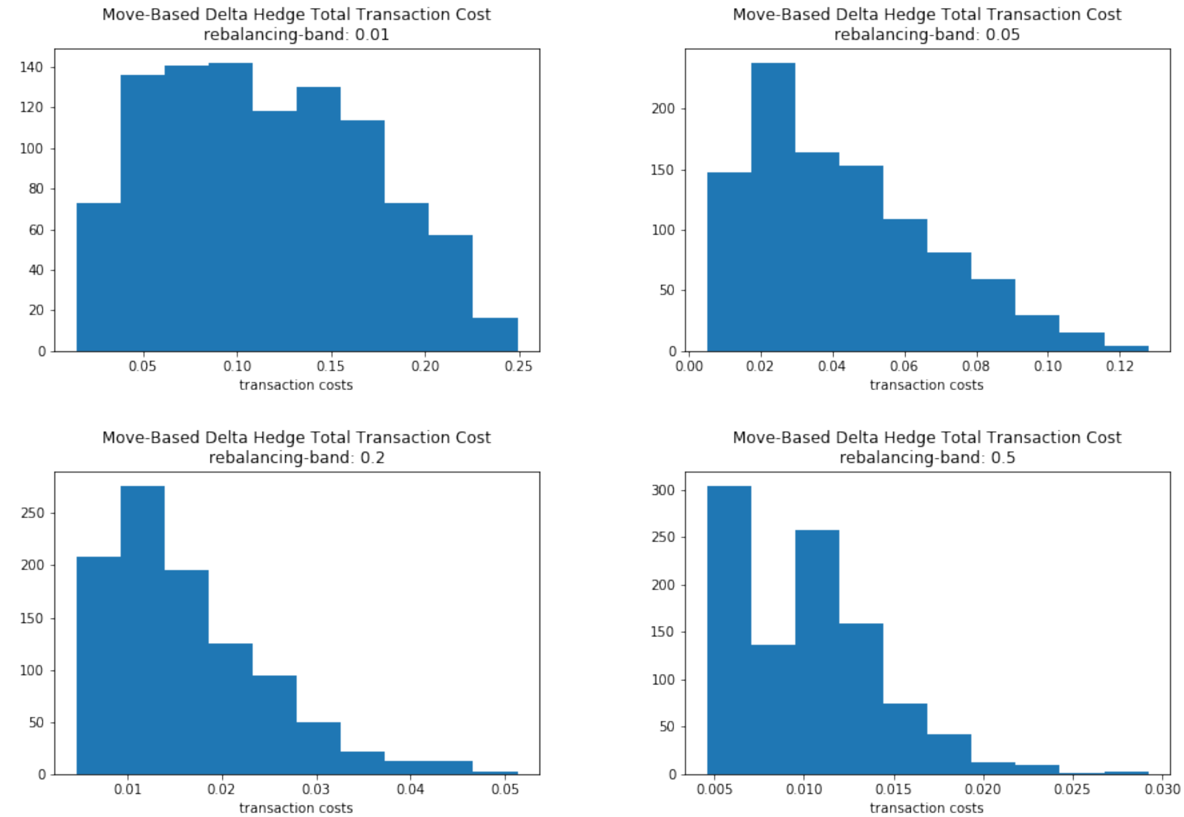


FIGURE 19. Total Transaction Costs, move-based delta hedging

According to the histograms, total transaction costs cut down dramatically as the range of band increases, which is consistent with the finding on alpha position.

Generally in move-based delta hedging, when rebalancing-bands are narrower, the strategy incurs lower volatility but the transaction costs add up due to the model's less tolerance on fluctuation. On the contrary, when rebalancing-bands are wider, the hedging process transacts much less and the corresponding risk increases.

3.4.2. *Move-based Gamma Hedging Strategy.*

In this section, rebalancing-bands are set to 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, representing narrow bands, medium bands and wide bands.

Similarly, for this part we first generate the simulated paths of asset price and find that rebalancing-bands and asset price are independent.

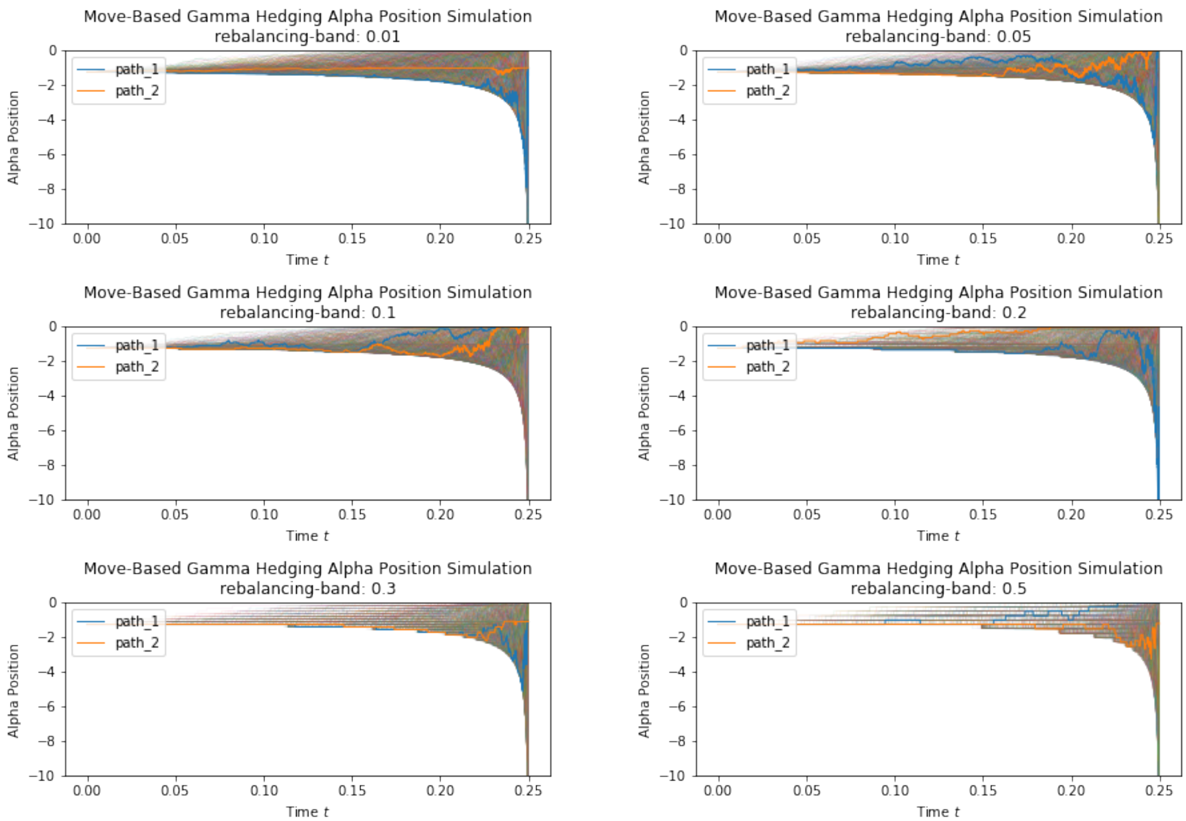


FIGURE 20. Simulated Alpha Position, move-based gamma hedging

It is obvious to conclude from the figure that, under move-based gamma hedging strategy, alpha position changing frequency decreases as the range of rebalancing bands increases, which is consistent with the case in move-based delta hedging.

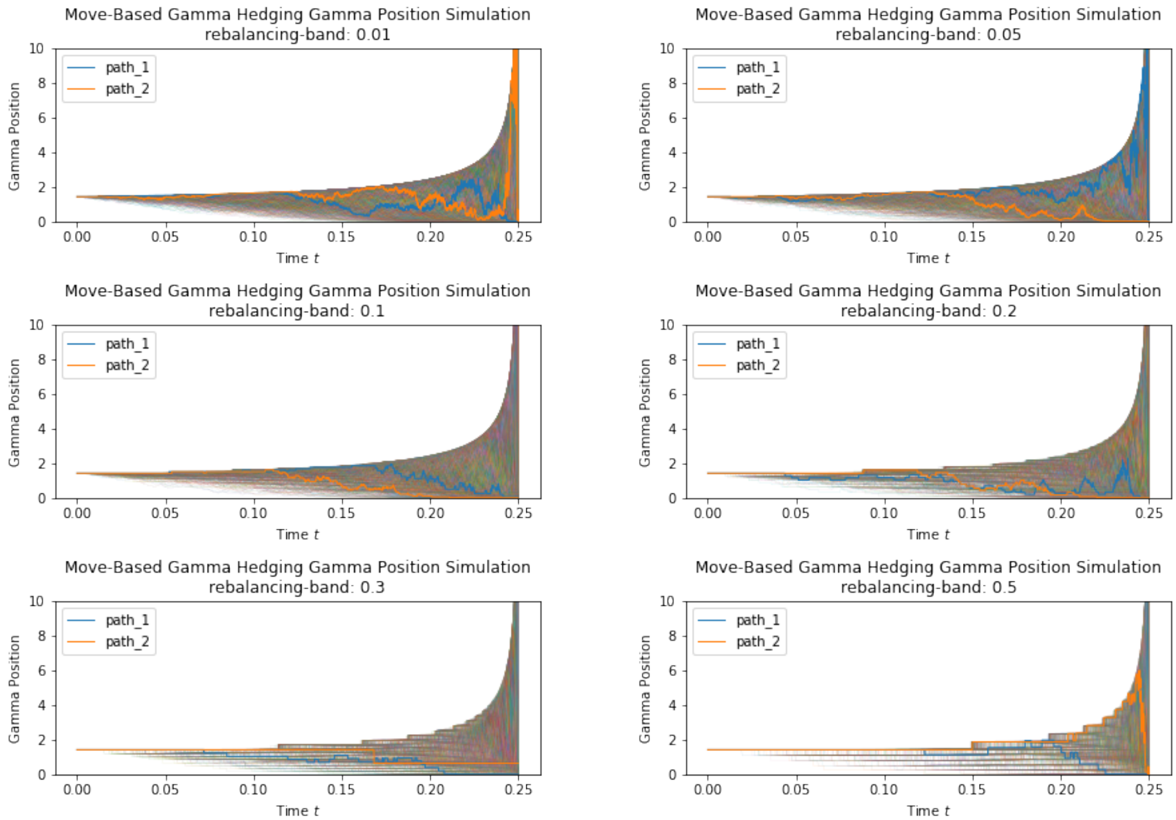
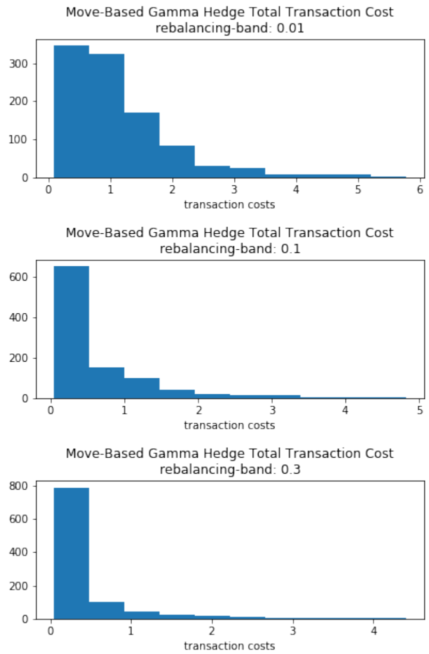
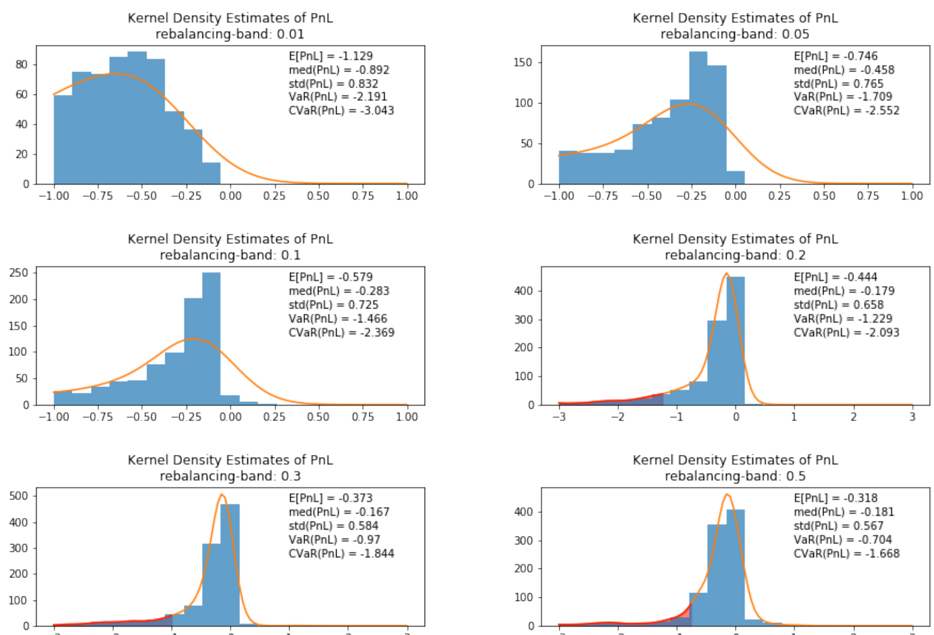


FIGURE 21. Simulated Gamma Position, move-based gamma hedging

The figure of simulated gamma position indicates that different settings of rebalancing-bands are also correlated with gamma position updating frequency. As the range of bands is larger, less frequently gamma position changes. It is also critical to notice that the dramatic increase at maturity is correspondent with the sharp decrease in alpha position curve.



(a) Total Transaction Costs, move-based gamma hedging



(b) KDE of PnL, move-based gamma hedging

FIGURE 22. Results of Move-based Gamma Hedge

One intriguing finding of the results of move-based gamma hedge is that the absolute values of VaR and CVaR all decrease as the range of bands increases. Intuitively, it implies that with a less frequently rebalancing strategy, investors will take less risk. To some extent, the total transaction costs can account for the situation, which offset the benefit that a rebalancing strategy can bring.

4. Comments and Conclusions

This study investigated and evaluated the effectiveness of delta hedging strategy and delta-gamma hedging strategy. Firstly, Move-based strategy works more effectively than time-based strategy in both delta hedging and delta-gamma hedging because move-based strategy allows volatility to incur within a specific boundary which reduces total transaction costs to a large extent. Second, delta hedging strategy behaves better than delta-gamma hedging strategy as a consequence of alpha position's and gamma position's dramatic changes near the maturity time under delta-gamma hedging strategy. In addition, we also explored how different probability measures function under both strategies. It is not surprising to discover that all types of hedging strategies perform better with a lower volatility setting. Last but not the least, rebalancing bands play an important role in both delta hedging and delta-gamma hedging strategies. Result shows the hedging strategy becomes less effective in move-based delta hedging as band increases, since the density of PnL flattens with higher band. However, it is intriguing to see that the delta-gamma hedging becomes more effective as band increases up to 0.5. Indeed, in delta hedging strategy, α_t only takes on $(0, -1)$; whereas in delta-gamma hedging strategy, α_t takes on all negative values, and shifts at a much greater scale near maturity. It is worth debating if only a simple band should be applied to delta-gamma hedging strategy, since α_t shifts at different scales as t approaches T. But due to the scope of this study, the topic will not be investigated with further details.

In conclusion, the simulation suggests that move-based strategies perform better than time-based strategies, and delta hedging strategies perform better than delta-gamma hedging strategies, provided the existence of market imperfection (i.e. transaction costs).

5. References

Jaimungal, S., Al-Aradi, A. (n.d.). *PricingTheoryNotes* [PDF].