Introduction to Categories

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Some References:

1. Mac Lane, Saunders. Categories for the Working Mathematician.
2. Baez, John. Categories, Quantization, and Much More.
3. Osborne, M. Scott. Basic Homological Algebra. Chapters 1 and 2

This talk is not going to be on algebraic geometry. It is gonna be on some abstract nonsense I wish I knew before I started learning algebraic geometry.

Please feel free to ask questions.

Also, I will try to be as concrete as possible and I will mention quantum computing at least once in this talk.

1 First, Some Sets

The notion of a *set* is not always sufficient. For instance, one cannot have a *set of all sets*, due to Russell's paradox.

1.1 Russell's Paradox

Let S be the set of all sets. Then

$$A = \{X \in S : S \notin X\} \tag{1}$$

is also a valid set. Then notice that for any set X,

$$X \in A \iff X \notin A. \tag{2}$$

Taking $X = A$, we get

$$A \in A \iff A \notin A, \tag{3}$$

which is a contradiction.

Idea. Define something more general than a set.

1.2 von Neumann-Bernays-Gödel Set Theory

To subvert such issues, we need a better theory. In this section, we restrict ourselves to what are called "material set theories," which means that we think of elements of sets as existing independently of the set.

Von Neumann-Bernays-Gödel set theory (or NBG) is an extension of ZFC that includes *proper classes* (defined below).

The objects of NBG are *classes*. If a class A is contained in another class B, then A is called a *set*. A class that is not a set is called a *proper class*.

For the axioms of NBG, see the nLab entry for NBG.

2 Categories Now

> *If we try to generalize the heck out of the concept of a group, keeping associativity as a sacred property, we get the notion of a category.*
>
> —John Baez

Now that we can talk about all sets at once (using the notion of class), let us define the *category of sets*—which I will denote by Sets. The *objects* of this *category* are sets and the *morphisms* between objects are all the functions between these sets. Now let us define categories more generally. (I promise, this will get less crazy real soon.)

Definition 1. A category C consists of a class of *objects* $\operatorname{obj} C$ and a set of *morphisms*. Every morphism has a *source* and a *target*. There is a function Mor which maps each pair A, B of objects to a set $\operatorname{Mor}_C(A, B)$ of morphisms from A to B. We also have a pairing function called *composition*

$$\circ : \operatorname{Mor}(B, C) \times \operatorname{Mor}(A, B) \longrightarrow \operatorname{Mor}(A, C) \tag{4}$$

$$(g, f) \mapsto gf \tag{5}$$

which is *associative*; that is, if $f \in \operatorname{Mor}(C, D)$, $g \in \operatorname{Mor}(B, C)$ and $h \in \operatorname{Mor}(A, B)$, then $(fg)h = f(gh)$. Finally, each set $\operatorname{Mor}(A, A)$ has a distinguished element i_A called *identity* which satisfies the unit laws; that is, for every $f \in \operatorname{Mor}(A, B)$

$$f = f i_A = i_B f. \tag{6}$$

The important part about categories is that they are associative. We like associativity and we care about it!

Note. Morphisms are sometimes called *arrows*.

Note. What we call Mor is sometimes called Hom for a *homomorphism*. But this is bad notation and we will not use it here.

	Objects	Morphisms
Grp	groups	homomorphisms
Vect	vector spaces	linear maps
Top	topological spaces	continuous functions
Diff	smooth manifolds	smooth maps
Ring	rings	ring homomorphisms
Hilb	Hilbert spaces	bounded operators

Table 1: A Few Concrete Examples of Categories

In all the above examples, the morphisms are a special function, that is not always the case. For example, a group is a category where we have one object and all the morphisms have inverses (such morphisms are called *isomorphisms*.) These morphisms correspond to the elements of the group. (This way of looking at groups—as you shall see—turns out to be very useful. We will return to this when we talk about category representations which are gross generalizations of group representations. These super abstract objects actually turn out to be super useful in quantum gravity!)

Abstract Example 2. Given a category $\mathcal{C}$, define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ as the category with the same objects but with the arrows reversed. Precisely, $\mathrm{obj}\,\mathcal{C}^{\mathrm{op}} = \mathrm{obj}\,\mathcal{C}$ and $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Mor}_{\mathcal{C}}(B, A)$. Also, the composition is reversed, in $\mathcal{C}^{\mathrm{op}}$

$$f \circ g = gf. \tag{7}$$

There is something that all the concrete examples in Table 2 have but the other examples does not. The concrete examples all look like a category of "sets with extra structure." This intuition is formalized using the notion of a concrete category. Don't worry looking at the size of this definition. We will see a simpler definition once we learn about functors.

Definition 3. A category $\mathcal{C}$ is called a *concrete category* if it comes equipped with a function σ from $\mathrm{obj}\,\mathcal{C}$ to sets such that:

1. For any $A \in \mathrm{obj}\,\mathcal{C}$, $\sigma(A)$ is a set. This is called the *underlying set* of A.

2. $\mathrm{Mor}(A, B)$ consists of functions from $\sigma(A)$ to $\sigma(B)$, that is, any $f \in \mathrm{Mor}(A, B)$ is a function from $\sigma(A)$ to $\sigma(B)$.

3. Categorical composition is functional composition.

4. i_A is the identity map on the set $\sigma(A)$

The above definition says that a concrete category is kinda like the category of sets. We will formalize this next time using functors.

3 The Relation To Quantum Computing I Promised

This is just a teaser, for the real deal see the nLab page finite quantum mechanics in terms of dagger-compact categories and for the ones with even more free time

> John C. Baez. Quantum Quandaries: A Category-Theoretic Perspective.

Recall HILB, the category of Hilbert spaces, where the objects are Hilbert spaces and the morphisms are bounded linear operators. This already looks a lot like finite-dimensional bounded-energy quantum theory. But let us take a step back.

Lemme start with a question Bob Coecke asked and answers very well in Kindergarten Quantum Mechanics, what do we really need to do quantum computing? When we apply a channel to a state we get another state; that is,

$$\rho \xrightarrow{\Phi} \sigma, \tag{8}$$

and we can compose such channels

$$\rho \xrightarrow{\Phi} \sigma \xrightarrow{\Psi} \xi. \tag{9}$$

We also have an identity map. This looks a lot like a category. But this is not sufficient. We also want to be able to tensor things together, remember, this is important! That is, if we have states ρ and σ we want to be able to create $\rho \otimes \sigma$. For this, we need what we in the biz call a monoidal category.

I will end here. Next time we will talk about functors and how category theory comes into play in algebraic geometry.