Chemical Engineering

Analysis of pipe flow of viscoelastic (polymeric) fluid

Introduction

In order to successfully reproduce the results of the paper we initially focused on research articles which had successfully implemented the Chebyshev collocation method. As mentioned in our previous progress report, these papers helped us develop a thorough understanding of the following concepts:

1. Differentiation matrix to be used in the pseudo spectral method
2. Generation of collocation points
3. QZ Algorithm methods for solving an eigen value problem
4. Neutral stability curves to be used to determine the critical renolds number
5. Applying the appropriate set of boundary conditions and transformations

Therefore, after reproducing a number of papers we felt confident enough to approach the article and problem provided by you. The following summarize our results and findings as **we successfully reproduced the results available in the article**.

Problem Formulation and Code used

As with our previous problems, we reduced the system of equations to an eigen value problem represented as two matrices.

The equations mentioned in the article are as follows:

$$\left(\mathrm{D}+\tfrac{1}{r}\right)\tilde{v}_r+\mathrm{i}k\tilde{v}_z=0, \tag{2.8}$$

$$G\tilde{v}_r=-\mathrm{D}\tilde{p}+\left[\left(\mathrm{D}+\tfrac{1}{r}\right)\tilde{\tau}_{rr}+\mathrm{i}k\tilde{\tau}_{rz}-\tfrac{\tilde{\tau}_{\theta\theta}}{r}\right]+\frac{\beta}{Re}\mathrm{L}\tilde{v}_r, \tag{2.9}$$

$$G\tilde{v}_z+U'\tilde{v}_r=-\mathrm{i}k\tilde{p}+\left[\left(\mathrm{D}+\tfrac{1}{r}\right)\tilde{\tau}_{rz}+\mathrm{i}k\tilde{\tau}_{zz}\right]+\frac{\beta}{Re}\left(\mathrm{L}+\tfrac{1}{r^2}\right)\tilde{v}_z, \tag{2.10}$$

$$H\tilde{\tau}_{rr}=2\frac{(1-\beta)}{Re}\left(\mathrm{D}+W\mathrm{i}kU'\right)\tilde{v}_r, \tag{2.11}$$

$$\begin{aligned}H\tilde{\tau}_{rz}-WU'\tilde{\tau}_{rr}=\frac{(1-\beta)}{Re}[\{&\mathrm{i}k-W\left(U''-U'\mathrm{D}-2\mathrm{i}kWU'^2\right)\}\tilde{v}_r\\&+\left(\mathrm{D}+W\mathrm{i}kU'\right)\tilde{v}_z],\end{aligned} \tag{2.12}$$

$$H\tilde{\tau}_{\theta\theta}=2\frac{(1-\beta)}{Re}\frac{\tilde{v}_r}{r}, \tag{2.13}$$

$$H\tilde{\tau}_{zz}-2WU'\tilde{\tau}_{rz}=2\frac{(1-\beta)}{Re}[-2W^2U'U''\tilde{v}_r+\{\mathrm{i}k+WU'\left(\mathrm{D}+2W\mathrm{i}kU'\right)\}\tilde{v}_z], \tag{2.14}$$

where $G=\mathrm{i}k(U-c)$, $H=1+WG$ and $\mathrm{L}=\left(\mathrm{D}^2+\frac{\mathrm{D}}{r}-\frac{1}{r^2}-k^2\right)$.

NOTE: The solvent to solution viscosity ratio is denoted by $\beta = \eta_s/\eta$

It is important to note that in order to use the qz eigen value solver, we had to re-write the equations as follows:

$$\boxed{G = ikU} \qquad \boxed{H = 1 + WikU}$$

Variables: $\tilde{v}_r, \tilde{v}_z, \tilde{p}$, $\tilde{\tau}_{rr}, \tilde{\tau}_{rz}, \tilde{\tau}_{\theta\theta}, \tilde{\tau}_{zz}$

$$L = D^2 + \frac{D}{r} - \frac{1}{r^2} - k^2$$

Equation 2.8

$$\left(D + \frac{1}{r}\right)\tilde{v}_r + ik\tilde{v}_z = 0 \quad (-1)$$

Equation 2.9

$$\left(\frac{\beta L}{Re} - ikU\right)\tilde{v}_r - D\tilde{p} + \left[\left(D + \frac{1}{r}\right)\tilde{\tau}_{rz} + ik\tilde{\tau}_{zz} - \frac{\tilde{\tau}_{\theta\theta}}{r}\right] = ikc\tilde{v}_r$$

Equation 2.10

$$-U'\tilde{v}_r + \left[\frac{\beta}{Re}\left(L + \frac{1}{r^2}\right) - Uik\right]\tilde{v}_z - ik\tilde{p} + \left(D + \frac{1}{r}\right)\tilde{\tau}_{rz} + ik\tilde{\tau}_{zz} \cancel{= ikc\tilde{v}_r} = -ikc\tilde{v}_r$$

Equation 2.11

$$\frac{2(1-\beta)}{Re}(D + WikU')\tilde{v}_r - H\tilde{\tau}_{rr} = -Wikc\tilde{\tau}_{rr}$$

Equation 2.12

$$\frac{(1-\beta)}{Re}\left[\{ik - W(U'' - U'D - 2ikWU')\}\tilde{v}_r + (D + WikU')\tilde{v}_z\right] + WU'\tilde{\tau}_{rr} - H\tilde{\tau}_{rz} = -Wikc\tilde{\tau}_{rz}$$

Equation 2.13

$$\frac{2(1-\beta)}{Re}\frac{\tilde{v}_r}{r} - H\tilde{\tau}_{\theta\theta} = -Wikc\tilde{\tau}_{\theta\theta}$$

Equation 2.14

$$\frac{2(1-\beta)}{Re}\left[-2W^2U'U''\tilde{v}_r + \{ik - WU'(D + 2WikU'^2)\}\tilde{v}_z\right] + 2WU'\tilde{\tau}_{rz} - H\tilde{\tau}_{zz} = -Wikc\tilde{\tau}_{zz}$$

Having formulated the equations in the required form, we were able to write the equations in the form of two matrices to be given as input to the qz eigen value solver.(Matrix A and B in the file stabicy2.m)

It is also important to note that the boundary conditions used by us to solve the problem are NOT the same as that given in the article. The article suggests that the no slip boundary condition is valid only at r = 0. At r = 1 it uses the boundary conditions corresponding to regularity of axisymmetric disturbances in the vicinity of the centerline due to which $v_r = 0$ and v_z = finite.

However, in order to obtain the correct results, we applied boundary conditions similar to those given in the following article:

Lessen, M. (1968). *Stability of Pipe Poiseuille Flow. Physics of Fluids, 11(7), 1404.* doi:10.1063/1.1692122

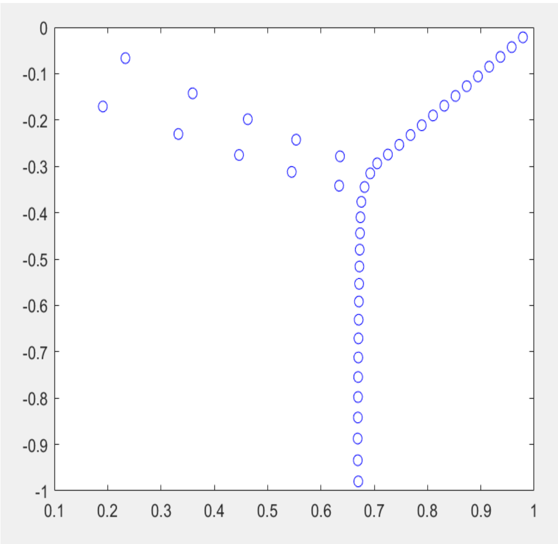
The article suggests that at both r=1 and r=0, v_z=0 and v_r=0. This is because no fluid velocity or pressure may be unbounded or discontinuous at r=0. Further, at r=1, the article states that the fluid and tube wall velocities must be continuous.

Another interesting point to be noted is that the original paper presents two independent formulations to solve the required set of equations in section 2.4. Although our approach is quite similar to the second approach presented, our approach has one distinguishing feature. Instead of taking v_r/r as a dependent variable(as done in the article) we have taken v_r as our variable. Keeping the other variables as mentioned in the paper, we were then able to solve the homogenous eigen value problem using the standard spectral collocation numerical scheme based on Chebyshev polynomials

Once we successfully identified our boundary conditions and equations, we were able to use the qz algorithm to successfully obtain the required eigen values. The function refineceigen.m then works to filter out only the useful eigenvalues as specified in the code. These “refined” eigen values are then fed to the plot function created. The plot function separates the real and imaginary parts enabling us to obtain the required graphs.

Results

In this section we have attached figure 2-5 as obtained by us using the codes submitted.



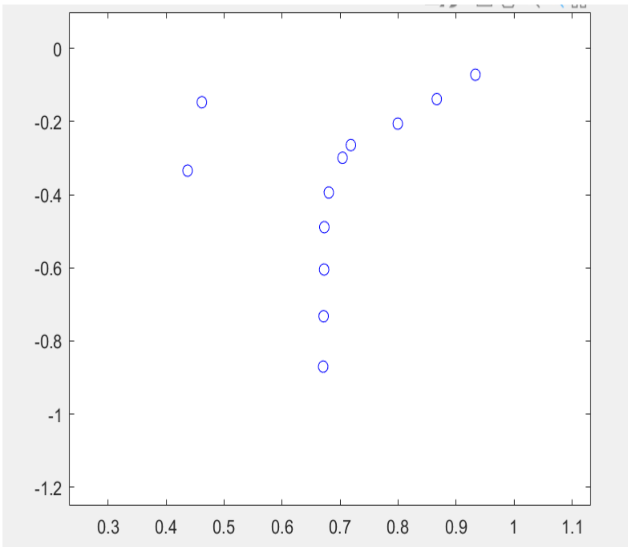
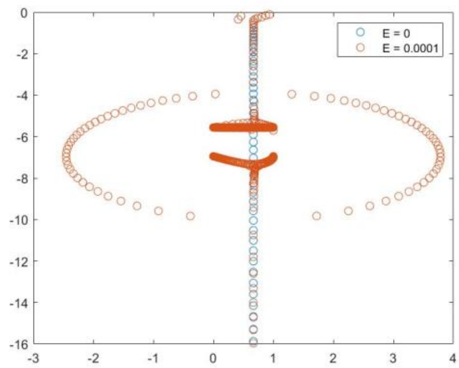
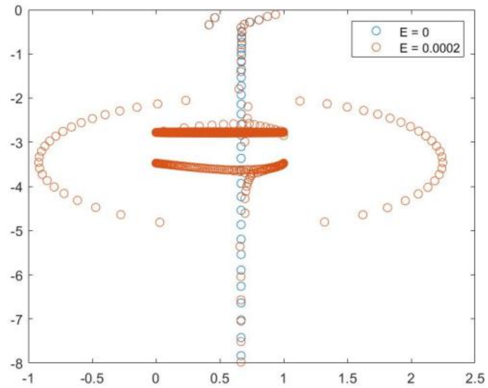


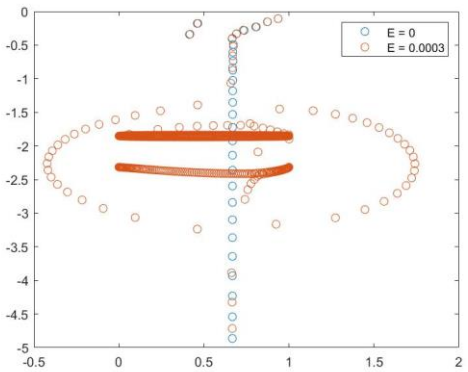
Figure 2: The ‘Y’-shaped eigenspectrum for Newtonian pipe flow subjected to axisymmetric disturbances for k = 3, and for Re = 6000 and 600.



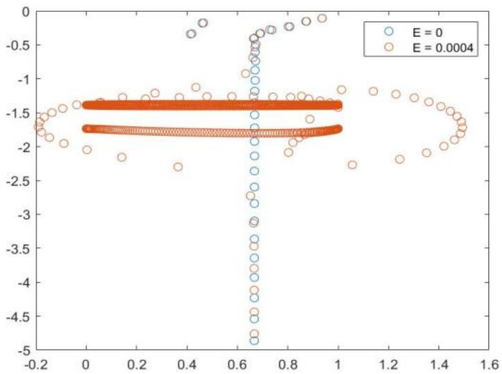
(a) E = 10^{-4}



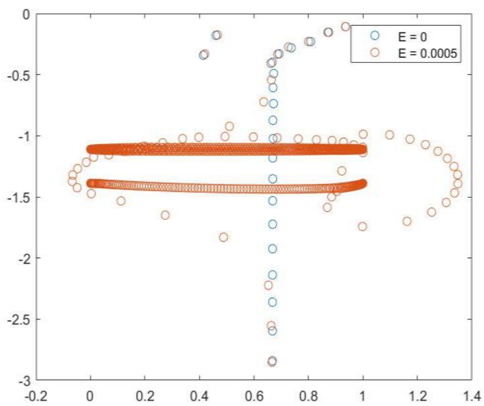
(b) E = 2 x 10^{-4}



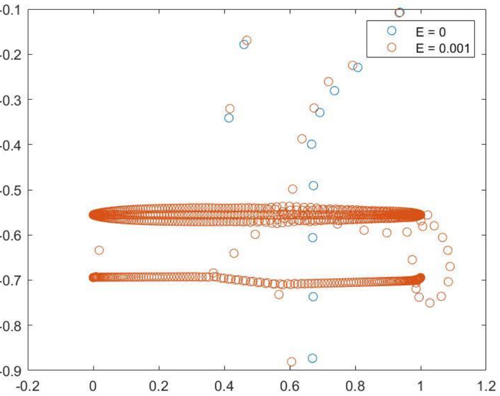
(c) E = 3 x 10^{-4}



(d) E = 4 x 10^{-4}

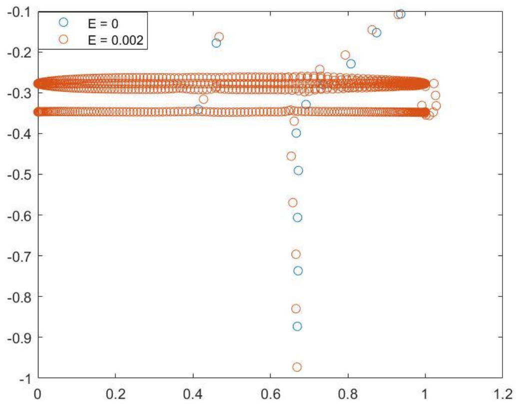


(e) E = 5 x10^{-4}

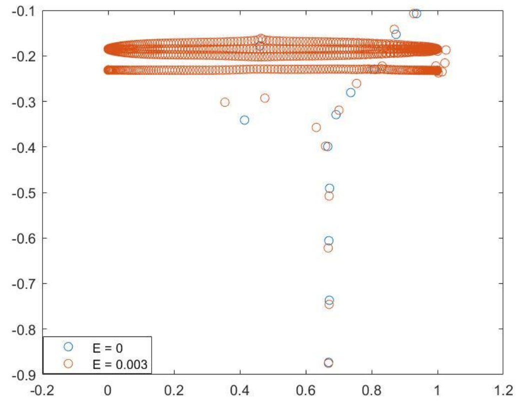


(f) E = 10^{-3}

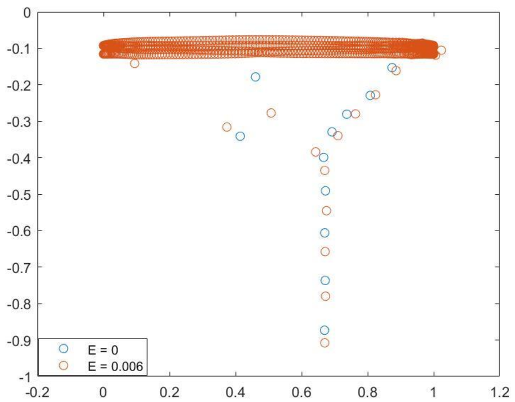
Figure 3: Eigenspectra for pipe flow of an Oldroyd-B fluid at β = 0.8,Re = 600 and k = 3, and for different E in the range 5×10−4–10−3. The eigenspectra are obtained for N = 200



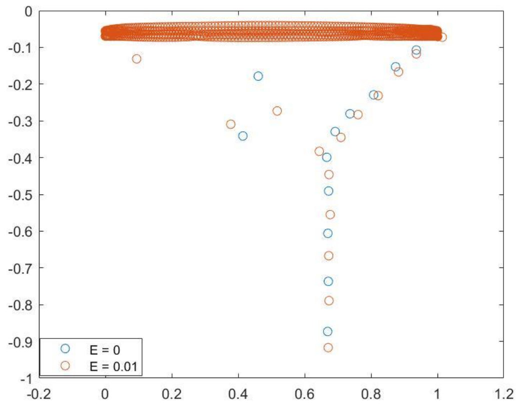
(a) E = 0.002



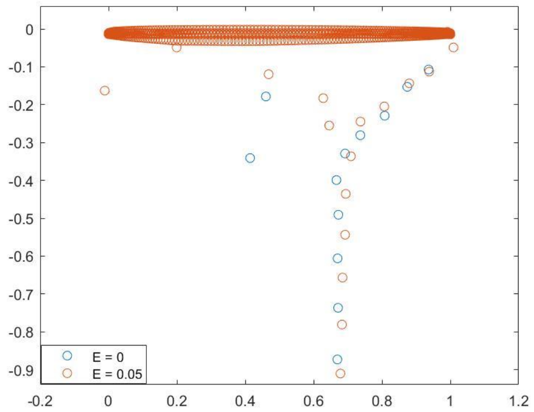
(b) E = 0.003



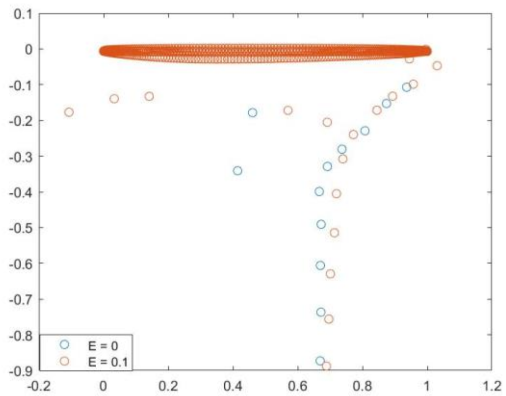
(c) E = 0.006



(d) E = 0.01

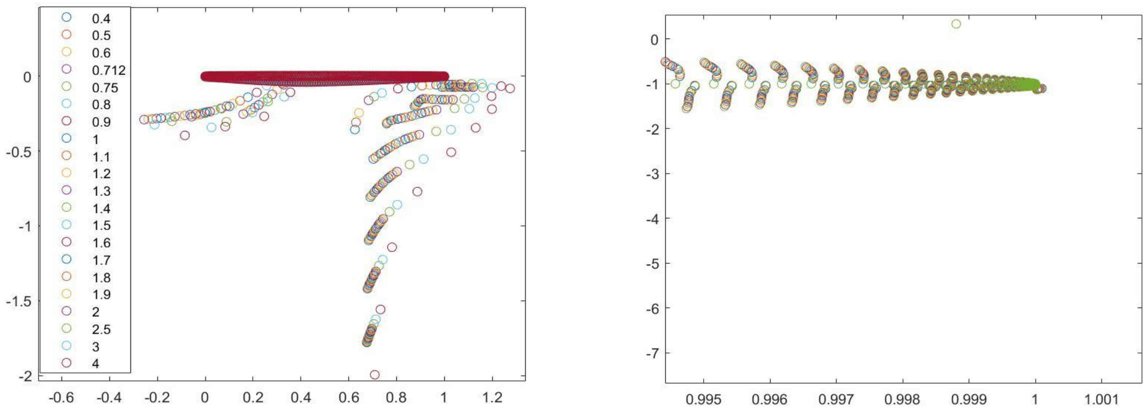


(e) E = 0.05



(f) E = 0.1

Figure 4: Eigenspectra for the Oldroyd-B fluid for different E in the range 0.002–0.1, at β = 0.8, Re = 600, N = 200, and k = 3



(a) Unfiltered spectra (b) Magnified at $c_r = 1$

Figure 5: (a) Eigenspectra for different values of E for β = 0.96,Re = 500,k = 1; (b) Enlarged version of region in panel (a) near the unstable center mode. The scaled growth rate kWci fixes the vertical location of both the CS

The obtained figures show striking similarities with that presented in the original article. Hence the results of the paper have been successfully reproduced