CS 170 Midterm 2

Write in the following boxes clearly and then double check.

Name :

SID :

Exam Room : ◯ Dwinelle 145 ◯ Hearst Field Annex A1 ◯ VLSB 2050 ◯ VLSB 2040 ◯ Evans 10 ◯ Other (Specify):

Name of student to your left :

Name of student to your right :

- **The exam will last 110 minutes.**
- The exam has 8 questions with a total of 100 points. You may be eligible to receive partial credit for your proof even if your algorithm is only partially correct or inefficient.
- Only your writings inside the answer boxes will be graded. **Anything outside the boxes will not be graded.** The last page is provided to you as a blank scratch page.
- Answer all questions. Read them carefully first. Not all parts of a problem are weighted equally.
- Be precise and concise.
- The problems may **not** necessarily follow the order of increasing difficulty.
- The points assigned to each problem are by no means an indication of the problem's difficulty.
- The boxes assigned to each problem are by no means an indication of the problem's difficulty.
- Unless the problem states otherwise, you should assume constant time arithmetic on real numbers. Unless the problem states otherwise, you should assume that graphs are simple.
- If you use any algorithm from lecture and textbook as a blackbox, you can rely on the correctness and time/space complexity of the quoted algorithm. If you modify an algorithm from textbook or lecture, you must explain the modifications precisely and clearly, and if asked for a proof of correctness, give one from scratch or give a modified version of the textbook proof of correctness.
- Unless the problem states otherwise, assume the subparts of each question are **independent**.
- Please write your SID on the top of each page.
- Good luck!

1 True or False (12 points)

True or False (3 points each). Include a short justification (1-2 sentences).

1. The optimal value of a maximizing LP problem can be greater than a feasible value of its dual LP problem.

   ◯ True ◯ False

2. Consider a maximum S-T flow in a directed graph with vertices V and edges E. If we add 1 to the capacity of each edge, then the value of the maximum flow increases by $|E|$.

   ◯ True ◯ False

3. Consider a maximum S-T flow in a directed graph with vertices V and edges E. If we multiply the capacity of each edge by 2, then the value of the maximum flow is also multiplied by 2.

   ◯ True ◯ False

4. The optimal multiplicative weights algorithm always guarantees that the total loss is less than or equal to the expert with the highest loss.

   ◯ True ◯ False

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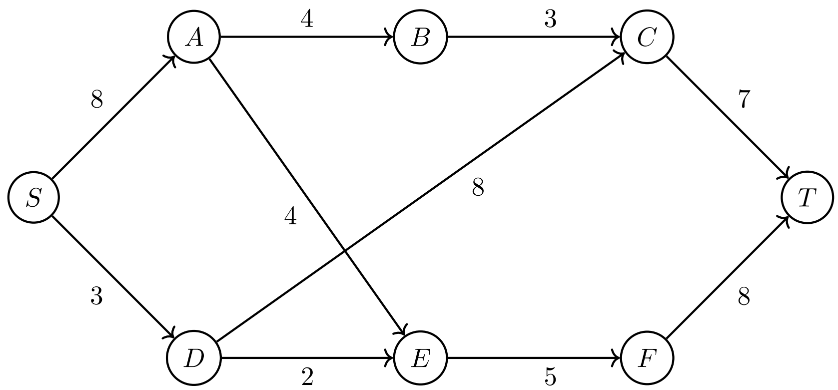
2 Maximize the Flow (8 points)

We wish to compute the maximum flow of the below graph from S to T using the Ford Fulkerson algorithm.



1. (4 points) Let's say that the first path from S to T that we find is $S \to A \to B \to C \to T$.

   (a) What is the maximum flow we can send along this path?

   (b) List the following capacities of edges in the residual graph after sending the maximum possible flow along this path.

$S \to A$

$A \to S$

$A \to B$

$B \to A$

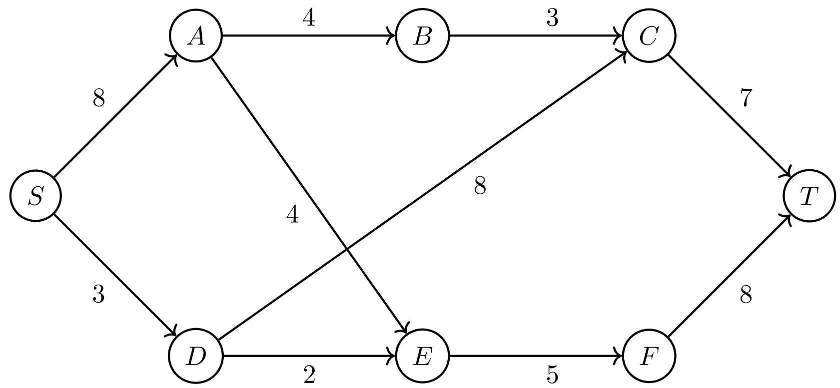
$B \to C$

$C \to B$

$C \to T$

$T \to C$

The graph has been redrawn below for your convenience.



2. (2 points) Compute the value of the maximum flow of the graph.

3. (2 points) In the minimum S-T cut of the graph, which vertices are in S's side of the cut? (List them in alphabetical order).

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3 Liam's Linear Program (8 points)

Consider the following LP.

$$\begin{aligned} \max\ & 2x + y \\ \text{subject to } & x - y \geq -4 \\ & 6 - x \geq y \\ & x - 4y \leq 1 \\ & x, y \geq 0 \end{aligned}$$

We have provided a blank graph on the next page if you need it, but **we will not grade anything written on the graph**.

1. (4 points) What is the optimal x, y and objective? Answer in the boxes below.

   x^*: ☐ , y^*: ☐

   Optimal objective value: ☐

2. (4 points) Now complete the dual of this LP in its canonical form.

   ○ max ○ min $4a + 6b + c$

   subject to

   ☐ $a +$ ☐ $b +$ ☐ $c \geq 2$

   ☐ $a +$ ☐ $b +$ ☐ $c \geq 1$

   $a, b, c \geq 0$

   Optimal objective value of the dual: ☐

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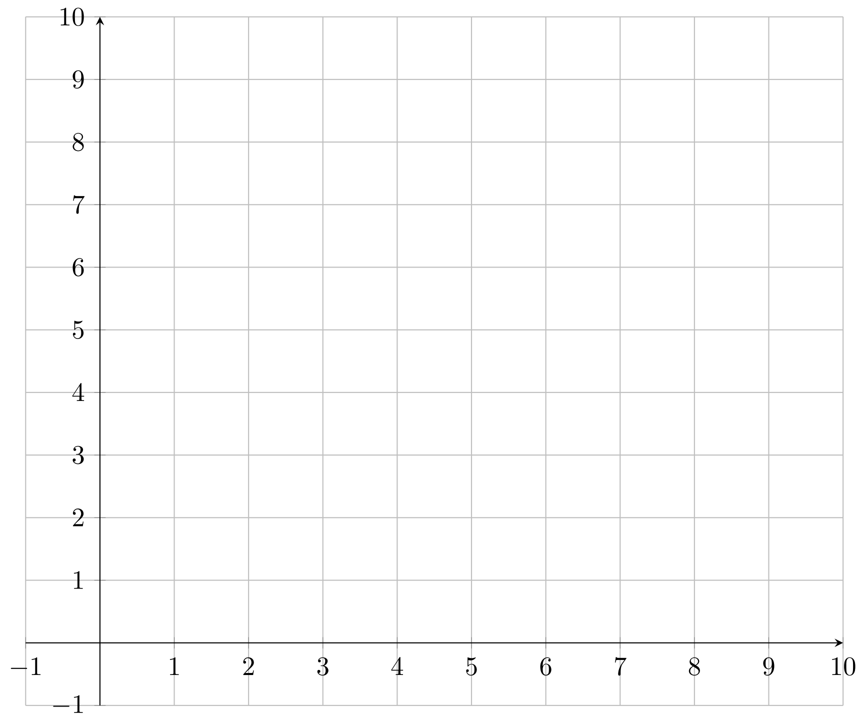
As a reminder, you may use this graph for scratch work, but it will not be graded.



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4 Pokemon Battle 0 (10 points)

Jonathan has 3 pokemon and Ajit has 2 pokemon. They will each choose just 1 of their pokemon. If Jonathan chooses his i-th pokemon and Ajit chooses his j-th pokemon, then Jonathan gets a score of s_{ij} (could be negative). Jonathan would like to maximize his score and Ajit would like to minimize Jonathan's score. Given all of the values of s_{ij}, please write out the LP formulation for both Ajit and Jonathan. Use a and b for the probabilities for Ajit's strategy and x, y, and z for the probabilities for Jonathan's strategy. **You do not have to solve the LPs**. In the table below, Jonathan is the row player while Ajit is the column player.

$$S_{ij} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \\ -2 & 7 \end{bmatrix}$$

Ajit:

Jonathan:

5 Pokemon Battle 1 (16 points)

Jonathan and Ajit each have n pokemon. They are playing a game that takes n rounds. Each round, they each choose 1 of their remaining pokemon (one that hasn't been chosen before). If Jonathan chooses his i-th pokemon and Ajit chooses his j-th pokemon, then Jonathan gets a score of s_{ij} (could be negative). However, Jonathan knows that Ajit will just play his pokemon in order (i.e. play pokemon 1 in round 1, pokemon 2 in round 2, etc.). Given this information, please describe a dynamic programming algorithm to help Jonathan determine the order to play his pokemon to maximize his total score. Your algorithm should run in time $O(n2^n)$.

For this problem, write your answer in the following 3 part format (you do not need to prove that the recurrence is correct):

(a) Define a function $f(\cdot)$ in words, including how many parameters are and what they mean, and tell us what inputs you feed into f to get the answer to your problem.

(b) Write the "base cases" along with a recurrence relation for f.

(c) Analyze the runtime and space complexity of your final DP algorithm.

6 Migration (16 points)

A group of k PNPenguins are currently in Guinland but have decided to move to Hawaii, using m self-driving sailboats that travel between n islands (Guinland and Hawaii are both islands).

Each island can host unlimited number of penguins, but sailboat i can only have c_i penguins onboard. Each sailboat has a list of destinations d_i which it will travel through each of them repeatedly: for example, $d_3 = [1, 5, 6]$ means sailboat 3 will stop at island 1 on day 0, island 5 on day 1, island 6 on day 2, island 1 on day 3, island 5 on day 4, island 6 on day 5, and so forth. Each sailboat takes 1 day to travel from any island to any other island. Penguins can only embark and disemembark when the sailboat is at an island.

Given m, n, c_i and d_i, your job determine whether or not it is possible to transport all of the penguins from Guinland to Hawaii within 170 days.

Model this problem as a flow network. Specify the vertices, edges, and capacities; show that a maximum flow in your network can be transformed into an optimal solution for the original problem. You do not need to explain how to solve the max-flow instance itself.

7 Alarming Knapsack (17 points)

During a robbery, a burglar finds much more loot than he had expected and has to decide what to take. His bag (or knapsack) will hold a total weight of at most W pounds ($W > 0$). There are n items to pick from, of positive weight $w_1, \ldots, w_n$, and positive dollar value $v_1, \ldots, v_n$. However, if the burglar takes 4 items in a row (i.e. the burglar takes items $k, k+1, k+2$, and $k+3$ for some k), then an alarm will trigger and the burglar will get caught. The burglar can first look at all of the items and then decide which ones to take. What is the dollar value of the most valuable combination of items he can fit into his bag without taking 4 items in a row?

For this problem, write your answer in the following 4 part format:

(a) Define a function $f(\cdot)$ in words, including how many parameters are and what they mean, and tell us what inputs you feed into f to get the answer to your problem.

(b) Write the "base cases" along with a recurrence relation for f.

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(c) Prove that the recurrence correctly solves the problem.

(d) Analyze the runtime and space complexity of your final DP algorithm.

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8 Param's Paper (13 points)

1. (3 points) Suppose we run Multiplicative Weight Updates with 3 experts and $\epsilon = \frac{1}{2}$. The losses are

$$\ell^{(1)} = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}$$
$$\ell^{(2)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}$$

Here, the i-th element of $\ell^{(1)}$ denotes the loss of the i-th expert in the first iteration, and the i-th element of $\ell^{(2)}$ denotes the loss of the i-th expert in the second iteration. As a reminder, we initialize the weights of each expert to 1 and our update rule is

$$w_i^{(t+1)} = w_i^{(t)} \cdot (1 - \epsilon)^{\ell_i^{(t)}}$$

(a) Compute $w_i^{(1)}$ for $i = 1, 2, 3$.

$w_1^{(1)}$ $w_2^{(1)}$ $w_3^{(1)}$

(b) Compute $w_i^{(2)}$ for $i = 1, 2, 3$.

$w_1^{(2)}$ $w_2^{(2)}$ $w_3^{(2)}$

(c) Compute the probability that the 1st expert is chosen on the 3rd iteration.

2. (10 points) Param is playing k games of Rock Paper Scissors. Each game, Param must decide whether to play a rock, paper, or scissors. Thankfully, Param can listen to the advice of N experts, each of whom will tell Param which action to take for each game (it is up to Param to decide which one(s) to listen to).

   Param decides that that he will use a modified version of the MWU algorithm. He initializes all of the weights of each expert to 1. Every turn he chooses the action that maximizes the sum of the weights of the experts voting for that action. After each game, Param will multiply the weights of the experts that got the action wrong by half. Param would like to do as well as the best expert. If the best expert was wrong M times, show that the number of times Param is wrong at most $c \cdot (M + \log_2 N)$ for some constant c, and compute c.

   (a) First, show that if Param is wrong, then the sum of the weights of the experts is reduced from W to at most $\frac{3}{4}W$.

   (b) Second, show that if the best expert makes M mistakes, then the sum of the weights of the experts at the end is at least $\left(\frac{1}{2}\right)^M$.

(c) Finally, show that the number of times Param is wrong is at most $c \cdot (M + \log_2 N)$ for some constant c, and compute c. You may use the fact that

$$\log_a b = \frac{\log_x b}{\log_x a}$$

c:

Blank scratch page.
This page **will not be graded**.