Midterm 2

- **The exam has 4 questions, is worth 100 points, and will last 120 minutes.**
- **Each question is worth 25 points.** Not all parts of a problem are weighted equally.
- Read the instructions and the questions carefully first.
- Begin each problem on a new page.
- Be precise and concise.
- The problems may **not** necessarily follow the order of increasing difficulty.
- If you use any algorithm from the textbook as a black box, you can rely on the correctness of the quoted algorithm. If you modify an algorithm from the textbook, you must explain the modifications precisely and clearly, and if asked for a proof of correctness, give one from scratch or give a modified version of the textbook proof of correctness.
- Good luck!

1 Smorgasbord Short Answer

(a) Consider a Huffman code for the character-frequency pairs $(A, \frac{1}{2}), (B, \frac{1}{8}), (C, \frac{1}{8}), (D, \frac{1}{8}), (E, \frac{1}{16}), (F, \frac{1}{16})$. Draw a Huffman tree (mid-exam clarification: removed "break ties alphabetically").

(b) Show that the following Horn formula is not satisfiable:

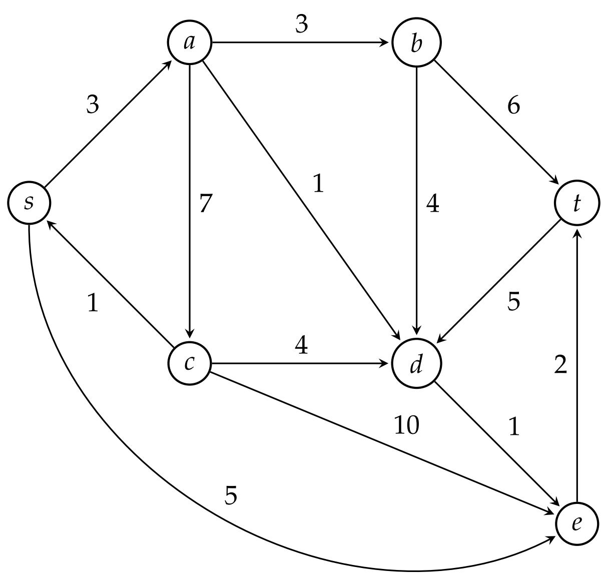
$$\Rightarrow w, \Rightarrow x, (v \wedge w \wedge x) \Rightarrow y, (w \wedge x) \Rightarrow z, (w \wedge x \wedge y) \Rightarrow z, (\bar{y}), (\bar{v} \vee \bar{x}), (\bar{w} \vee \bar{x} \vee \bar{z}).$$

(c) Suppose G has an MST T with edge weights 1, 4, 4, and 8. Argue why G cannot have another MST T' with edge weights 2, 3, 6, and 6.

(d) Draw a graph of four nodes A, B, C, D with **unique** edge weights such that it has a cycle whose minimum edge is *not* part of the MST of the graph.

(e) Consider the knapsack problem with repetition (as discussed in lecture) with a weight bound of W. Suppose the weight bound is increased to W^2. Explain why this increases the runtime of the DP algorithm by an exponential factor.

(f) Suppose you are given the graph G as below. Argue that the maximum $s - t$ flow in this graph is 5.



2 Fruits

Jack Sparrow is shopping for apples, bananas, and cantaloupes at Raider Moe's, in preparation for his upcoming voyage. Jack doesn't want to get scurvy, so he needs to buy fruit containing at least 4500mg of vitamin C in total. Moreover, Jack's ship can carry at most 100 pounds of fruit. Each fruit has different properties, given in the table below:

Fruit	Weight per fruit	Vitamin C per fruit	Calories per fruit	Cost per fruit
Apple	0.4 pounds	8 mg	100 calories	4 copper pieces
Banana	0.3 pounds	10 mg	100 calories	5 copper pieces
Cantaloupe	2 pounds	100 mg	300 calories	50 copper pieces

Jack wants to maximize the total number of calories in his purchased fruit.

Note: You may assume that Jack can purchase fractional quantities of fruit.

(a) Write down (but do not solve) a linear program that Jack can solve to obtain the optimal quantities of apples/bananas/cantaloupes he should buy. (Ignore the cost per fruit for this part.)

(b) Argue that the optimal number of apples Jack buys is 0. (Again, ignore the cost per fruit in this part.)

(c) What is the minimum number of copper pieces Jack needs to spend on fruit to meet the vitamin C and weight constraints? Show that Jack cannot spend less than your minimum and show that the minimum is achievable. (Ignore the calories per fruit in this part.)

Hint: Compare the cost (in copper pieces) with the amount of vitamin C for each fruit.

3 Wells

There are n cities numbered $1, 2, \ldots, n$, each of which needs water. To obtain water, a city must either drill its own well or be connected via a sequence of pipes to a city which has its own well. It costs w_i for city i to drill its own well and costs $p_{ij} = p_{ji}$ to construct a pipe between city i and city j. A set of drilling locations and pipe constructions is a *viable plan* if all cities have access to water after all drilling/piping is complete.

- *For full credit:* Design an algorithm to find the minimum-cost viable plan.
- *For half credit:* Design an algorithm to find the minimum-cost viable plan **which involves only one well drilling**.

Clearly indicate whether you are giving a full credit algorithm or half credit algorithm. If you attempt both without any indication, we will only grade the first attempt shown on Gradescope. For the option you choose:

(a) Describe your algorithm.

(b) Prove that your algorithm is correct.

(c) Provide a runtime analysis for your algorithm (in terms of n).

Note: Credit for the proof of correctness and the runtime analysis will only be awarded for correct algorithms that run in asymptotically optimal time.

4 Mittens

You are running Toasty Digits, a company that produces mittens. To make sure that your company can meet demands, you are planning out the production for the coming n months. The following information is given to you:

- **Demand:** In month i, the demand will be $d_i \geq 0$ (i.e. you sell exactly d_i pairs of mittens in month i). The total demand for all n months is given by $D = \sum_{i=1}^{n} d_i$.
- **Production:** Your full-time knitting staff can produce at most m pairs of mittens per month. You can hire additional knitters who will produce as many additional mittens as you need, but at a cost of c dollars per pair (whereas you do not pay anything per pair for your full-time staff).
- **Storage:** If, at the end of a month, you have any leftover mittens, you have to store them at a cost. In particular, if you have k pairs of mittens left, you pay $h_k \geq 0$ dollars. You can store at most D pairs of mittens.

Provide a dynamic programming algorithm for computing the minimum total cost required to meet all demand. Your solution should include the following:

(a) A description of your subproblems and how you will use them to compute the final answer.

Hint: Let one of the subproblem parameters be the number of leftover mittens.

(b) A recurrence relation for your subproblems and the relevant base cases.

(c) A justification for your recurrence relation and your base cases.

(d) The order in which to solve the subproblems.

(e) The runtime of solving all subproblems and computing the final answer.