Q7:

Consider $f(x) = \ln(x) - (x-1)$

$$f'(x) = \frac{1}{x} - 1.$$

For $x \in (0,1]$, $f'(x) > 0 \Rightarrow f(x)$ is increasing.

$x \in (1, \infty)$ $f'(x) < 0 \Rightarrow f(x)$ is decreasing

Function attains local maxima

at $f'(x) = 0 \Rightarrow x = 1.$

The value at $x=1 \Rightarrow 0.$

So, $\ln x \leq x-1$ is true $\forall x > 0$

and equal only when $x=1.$

Q8: a. $KL(p,q) = \sum_{i=1}^{k} p_i \log\left(\frac{p_i}{q_i}\right)$.

Consider $-KL$. $\Rightarrow \sum_{i=1}^{p} p_i \log \frac{q_i}{p_i}$

$$\leq \sum p_i \left(\frac{q_i}{p_i} - 1\right)$$

$$\leq \sum q_i - \sum p_i$$

$$= 0.$$

$$\Rightarrow \quad -KL \leq 0 \Rightarrow \boxed{KL \geq 0}$$

b. From prev question (Q7) inequation is equal only when $x=1$.

so $p_i = q_i \ \forall i$.

c. Consider 2 prob dist $p = (0.3, 0.7)$
$q = (0.4, 0.6)$

$KL(p,q) = 0.3 \ln(3/4) + 0.7 \ln(7/6) = 0.021$

$KL(q,p) = 0.4 \ln(4/3) + 0.6 \ln(6/7) = 0.022$

Q9. A function f is convex if

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

$\forall \alpha \in [0$

Given fun, $L(\omega) = -\log\left(\frac{e^{z_y}}{\sum_k e^{z_k}}\right)$

$$= -\log\left(\frac{e^{\omega^y x}}{\sum_k e^{\omega^k x}}\right)$$, where ω^y is y^{th} row in

RHS $= \alpha L(\omega_1) + (1-\alpha) L(\omega_2)$

$$= -\alpha \log\left(\frac{e^{\omega_1^y x}}{\sum e^{\omega_1^k x}}\right) + (1-\alpha)\left(-\log\left(\frac{e^{\omega_2^y x}}{\sum e^{\omega_k^y x}}\right)\right.$$

$$= -\log\left(\frac{e^{\alpha \omega_1^y x + (1-\alpha)\omega_2^y x}}{\left(\sum e^{\omega_1^k x}\right)^{\alpha}\left(\sum e^{\omega_2^k x}\right)^{(1-\alpha)}}\right)$$

LHS $= L(\alpha \omega_1 + (1-\alpha)\omega_2)$

$$= -\log\left(\frac{e^{\alpha w_1^y x + (1-\alpha) w_2^y x}}{\sum e^{\alpha w_1^k x + (1-\alpha) w_2^k x}} \right).$$

Let $e^{w_1^k x} = a_k$, $e^{w_2^k x} = b_k$.

we can rewrite denominators as.

$$RHS_{den} = \left(\sum_k a_k\right)^{\alpha} \left(\sum_k b_k\right)^{1-\alpha}.$$

$$LHS_{den} = \sum_k a_k^{\alpha} b_k^{1-\alpha}.$$

when $\alpha \in (0,1)$, we know.

$a_k, b_k > 0$

$$\left(\sum a_k\right)^{\alpha} \left(\sum b_k\right)^{1-\alpha} \geq \sum a_k^{\alpha} b_k^{1-\alpha}.$$

As numerators are same in LHS & RHS.

$$-\log\left(\frac{c}{RHS_{den}}\right) \geq -\log\left(\frac{c}{LHS_{den}}\right),$$

when $RHS_{den} \geq LHS_{den}$

Hence LHS $\leq$ RHS