Project 03

APSC 607 Fall 2017

Seth Goodman

November 15, 2017

1 Introduction

This project explores different numerical methods of solving two well-posed initial value problems (IVPs), y', at discrete points within a range $[a\ b]$ for a given initial value $y(a)$ as defined in Equation 1 and described in Burden and Faires (2010). The methods that will be tested are Euler's method (i.e., first order Taylor method), fourth order Runge-Kutta (i.e., essentially fourth order Taylor method), and an implicit trapezoidal method (trapezoidal with Newton Iteration). The behavior and characteristics of these methods will be reviewed and their effectiveness evaluated based on the true solution to the IVP.

$$y' = \frac{dy}{dt} = f(t, y), \qquad a \leq t \leq b, \qquad y(a) = \alpha \tag{1}$$

All computations are performed using MATLAB using the code accompanying this report. Section 2 will present the methods used in MATLAB to explore functions. Section 3 contains the results and related outputs for each function, and Section 4 includes discussion and conclusion. All figures and tables found in this report are available in the output subdirectory of the accompanying zip file. Additionally, all code and figures found in the zip file can be accessed via GitHub[1]

[1]`https://github.com/sgoodm/apsc607/tree/master/project_03`

2 Methods

The two unique functions which will be explored in this project, Functions **A** and **B**, are defined by Equations 2 and 3, respectively.

$$y' = -9y \tag{2}$$

$$y' = 20(y - t^2) + 2t \tag{3}$$

The associated initial values for Functions A and B, are defined by Equations 4 and 5.

$$y(0) = e \tag{4}$$

$$y(0) = \frac{1}{3} \tag{5}$$

Both IVPs will be examined over the range $0 \leq t \leq 1$, using a baseline step size of $h = 0.1$.

To validate and compare each method for solving the IVPs, the true solution is required. The true solution is generated using the Symbolic Toolkit in MATLAB as seen in the example below for Function A.

```
syms y(t)
ode = diff(y,t) == -9*y;
cond = y(0) == exp(1);
sol = dsolve(ode, cond);
```

Which produces the functions $y(t)$ for A and B seen in 6 and 7.

$$y(t) = e^{1-9t} \tag{6}$$

$$y(t) = \frac{1}{3}e^{-20t} + t^2 \tag{7}$$

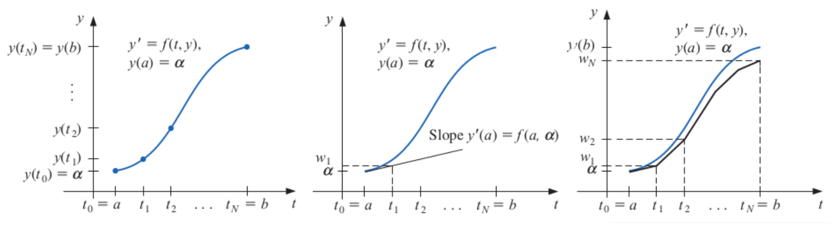
The true values for Functions A and B are compared to the results from solving the IVP using Euler's method, the fourth order Runge-Kutta method, and the implicit trapezoidal method with Newtonian iteration. In addition to the baseline step size, additional values of h will be tested to explore the impact of step size when dealing with stiff equations.

Stiff IVP equations, as described in Burden and Faires (2010), are characterized by having a solution taking the form of e^{-ct}. This portion of the solution, known as the *transient solution*, will decay to zero as t increases. However, the derivatives of the transient solution take the form $e^n e^{-ct}$, which does not decay as quickly as, and can grow very large for small values of t. This is inherently problematic when solving IVPs using methods based on Taylor's Theorem or methods with similar error terms, as the derivatives define the error terms. Given that error terms in these methods are evaluated at values between zero and t, rather than t itself, makes it extremely likely for the derivative of the transient solution to result in large values (Burden and Faires, 2010). Testing

IVPs A and B over a range of increasing step sizes, between 0.01 and 0.1, will illustrate the role of step size in the stability of stiff equations. Additionally, for each function a considerably larger step size will be tested, at which the error due to the transient solution is substantially larger than the *steady-state* component of the solution.

The remainder of this section will provide details on the implementation of and methods for each of the three approaches to solving IVPs (Euler's, fourth order Runge-Kutta, and the implicit trapezoidal with Newtonian iteration).

2.1 Euler's (first order Taylor)



y
$y(t_N) = y(b)$
$y' = f(t, y)$, $y(a) = \alpha$
$y(t_2)$
$y(t_1)$
$y(t_0) = \alpha$
$t_0 = a$ t_1 t_2 . . . $t_N = b$ t
Slope $y'(a) = f(a, \alpha)$
w_1
α
$y(b)$
w_N
w_2

Figure 1: Example function (left), single step of Euler's method (center), multiple steps of Euler's method (right). (Burden and Faires, 2010)

Euler's method is a first order implementation of higher order Taylor's method. Using the initial condition of the function at a given time step t, the slope at that point, and the step size, the value of the function at *t+1* can be estimated with second order error (Equation 8). This process is illustrated in Figure 1.

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i) \tag{8}$$

Given $w_i \approx y(t_i)$ when the error term is removed, Euler's method can be represented by Equation 9.

$$w_{i+1} = w_i + hf(t_i, w_i) \tag{9}$$

While Euler's method only requires the first derivative provided with the IVP to solve, using higher order implementations of Taylor's method require additional derivatives. The cost and complexity of having to calculate each subsequent derivative make Taylor's method impractical for use in most applications. The equation for Taylor's method of order n can be seen in Equation 10.

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)} + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i) \tag{10}$$

2.2 Runge-Kutta Order Four

The Runga-Kutta method provided the higher order error term achievable using Taylor's method without the need for additional derivatives in the calculation. This is possible by utilizing the slope of the function at sub steps within a given step size to better approximate the value of the function at the subsequent time step.

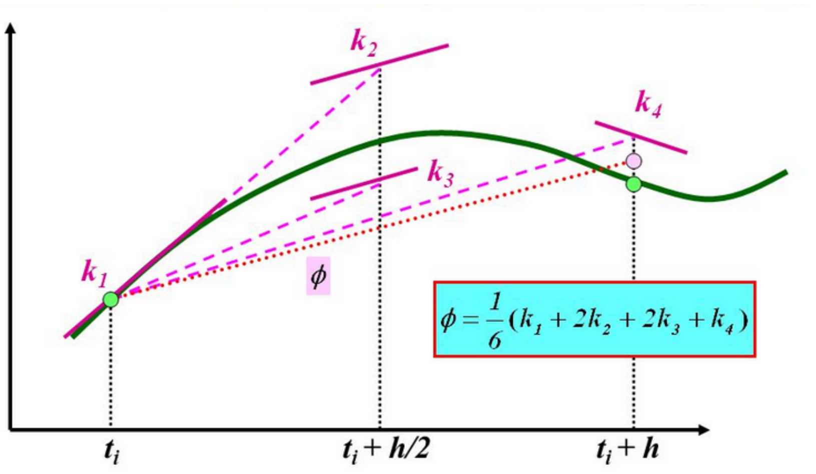


Figure 2: Runge-Kutta fourth order illustration. (?)

As seen in the equations below, four k values are generated which generate pseudo-approximations of the value at t_{i+1} based on different sub steps and pseudo-starting conditions. The first k value is based on the slope and value at t_i, the other k values use either $t_i + 0.5h$ or t_{i+1} along with the value of w_i adjusted by the previously generate k values. The final estimate for w_{i+1} combines these to generate a higher order approximation similar to Taylor's method without requiring additional derivatives to be calculated. An illustration of this process depicting the slopes associated with each k value, as well as the final slope, can be seen in Figure 2.

$$k_1 = hf((t_i, w_i), \tag{11}$$

$$k_2 = hf((t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1), \tag{12}$$

$$k_3 = hf((t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2), \tag{13}$$

$$k_4 = hf((t_{i+1}, w_i + k_3), \tag{14}$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{15}$$

2.3 Implicit Trapezoidal (Trapezoidal with Newtonian Iteration)

3 Results

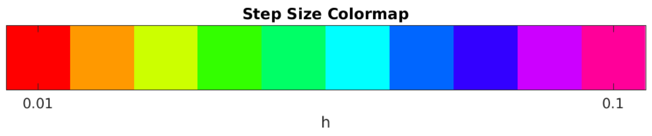


Figure 3: Colormap of h val

This section will review the results of the different integration rules for Functions A and B. An overview of these results can be seen in Table 1, which shows the the number of subintervals required to reach each tolerance specified, as well as the final error achieved. Figure **??** contains the true plots of both function, for reference.

Function	**Method**	**N:** 10^{-4}	**N:** 10^{-8}	**Minimum Error**
a	trapezoidal	768	65536	4.26887e-09
a	midpoint	1024	98304	8.53747e-09
a	simpsons	42	322	1.30384e-12
a	adaptive	24	278	-
b	trapezoidal	12	1082	1.29289e-09
b	midpoint	22	1522	2.58233e-09
b	simpsons	2	22	2.22044e-16
b	adaptive	2	16	-

Table 1: Results

3.1 Euler's Results

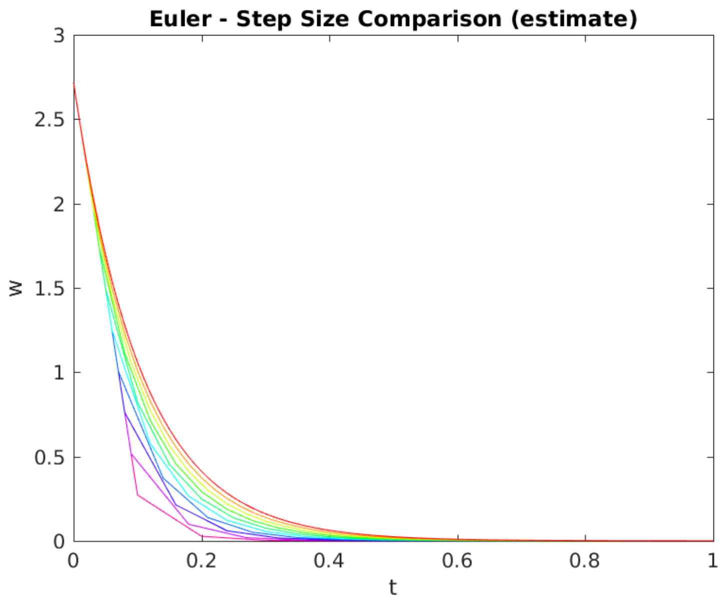


Figure 4: a euler h val

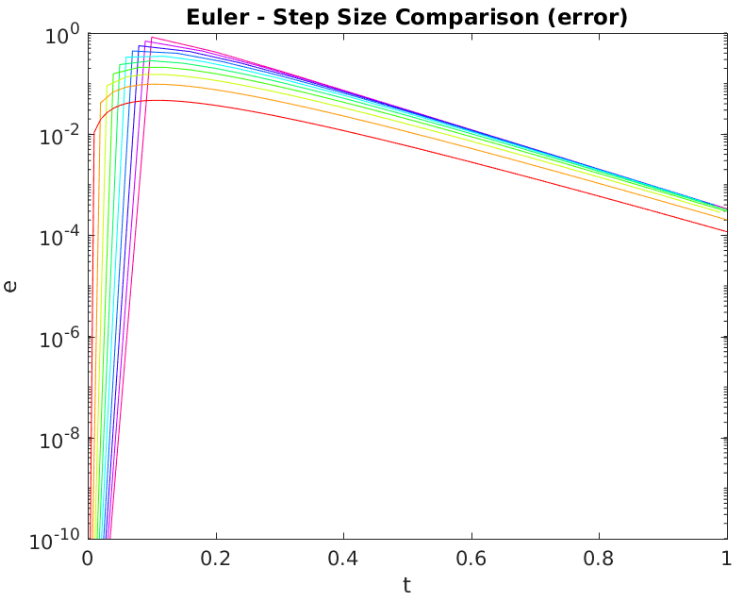


Figure 5: a euler h err

Table 2: My caption

t	true	euler	error1
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	0.27182818	0.83334274
0.2	0.44932896	0.027182818	0.42214615
0.3	0.18268352	0.0027182818	0.17996524
0.4	0.074273578	0.0002718282	0.07400175
0.5	0.030197383	2.7182818E-005	0.030170201
0.6	0.01227734	2.7182818E-006	0.012274622
0.7	0.0049915939	2.7182818E-007	0.0049913221
0.8	0.0020294306	2.7182818E-008	0.0020294035
0.9	0.0008251049	2.7182818E-009	0.0008251022
1	0.0003354626	2.7182818E-010	0.0003354624

Table 3: My caption

t	true	euler	error1
0	0.33333333	0.33333333	0
0.1	0.055111761	-0.33333333	0.38844509
0.2	0.046105213	0.37333333	0.32722812
0.3	0.090826251	-0.25333333	0.34415958
0.4	0.16011182	0.49333333	0.33322151
0.5	0.25001513	-0.093333333	0.34334847
0.6	0.36000205	0.69333333	0.33333129
0.7	0.49000028	0.14666667	0.34333361
0.8	0.64000004	0.97333333	0.3333333
0.9	0.81000001	0.46666667	0.34333334
1	1	1.3333333	0.33333333

3.2 Runge-Kutta Results

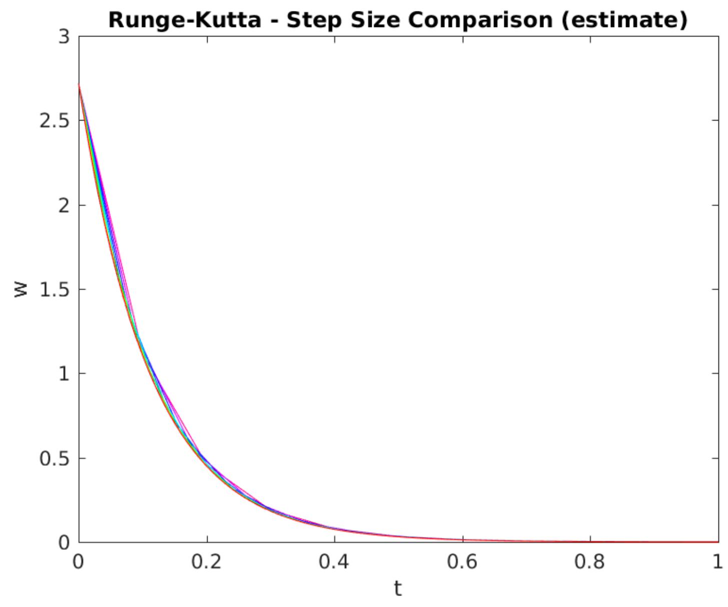


Figure 6: a rk h val

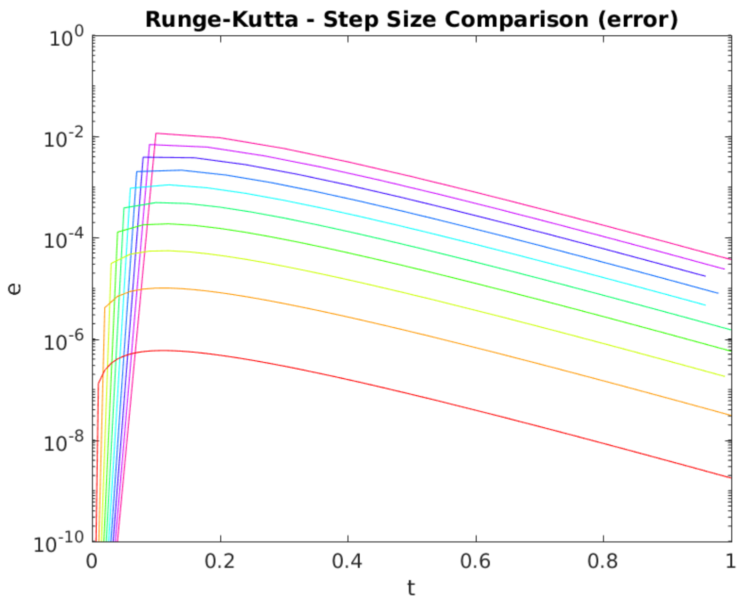


Figure 7: a rk h err

Table 4: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	1.1167721	0.011601193
0.2	0.44932896	0.45881186	0.0094828979
0.3	0.18268352	0.18849712	0.0058135943
0.4	0.074273578	0.077441685	0.0031681067
0.5	0.030197383	0.031815948	0.0016185648
0.6	0.01227734	0.013071185	0.0007938447
0.7	0.0049915939	0.0053701328	0.0003785389
0.8	0.0020294306	0.0022062519	0.0001768213
0.9	0.0008251049	0.000906411	8.1306108E-005
1	0.0003354626	0.0003723876	0.000036925

Table 5: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.1	0.055111761	0.12277778	0.067666017
0.2	0.046105213	0.079259259	0.033154046
0.3	0.090826251	0.10475309	0.013926836
0.4	0.16011182	0.16658436	0.0064725413
0.5	0.25001513	0.25386145	0.0038463207
0.6	0.36000205	0.36295382	0.0029517699
0.7	0.49000028	0.49265127	0.0026509955
0.8	0.64000004	0.64255042	0.0025503867
0.9	0.81000001	0.81251681	0.002516803
1	1	1.0025056	0.002505602

3.3 Implicit Trapezoidal Results

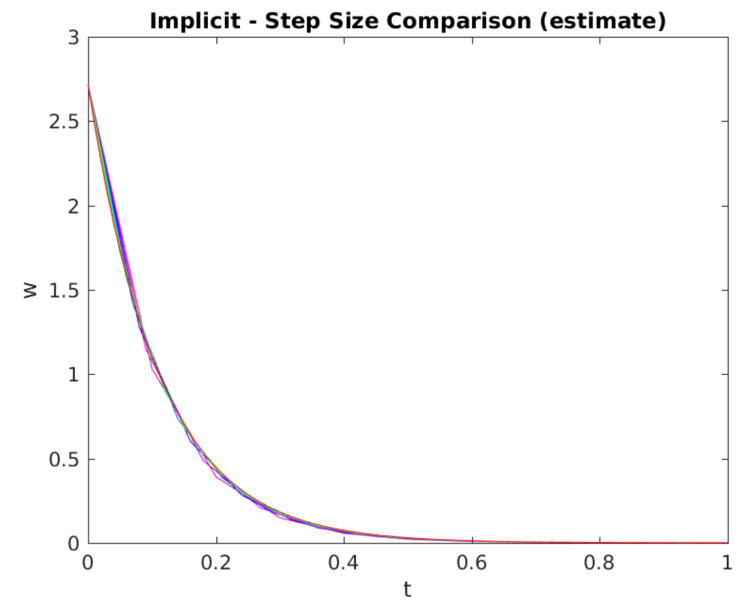


Figure 8: a implicit h val

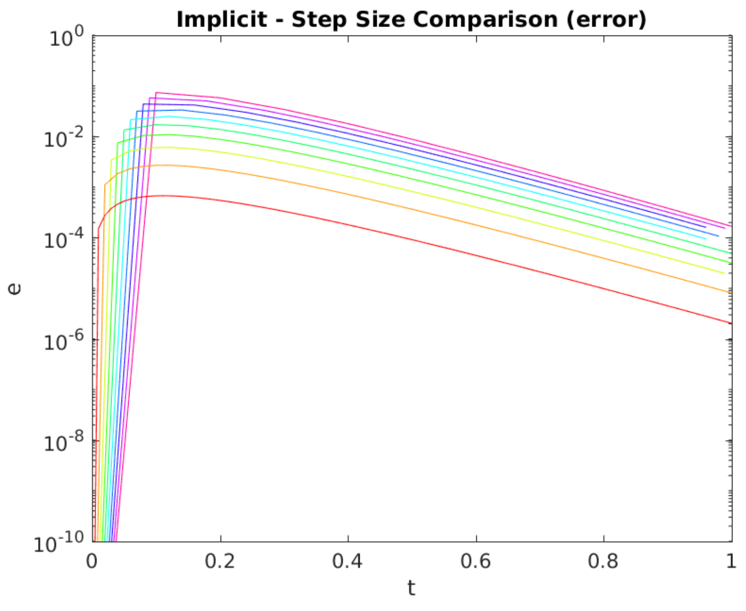


Figure 9: a implicit h err

Table 6: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	1.0310724	0.0740985
0.2	0.44932896	0.39109643	0.05823253
0.3	0.18268352	0.14834692	0.034336601
0.4	0.074273578	0.056269523	0.018004056
0.5	0.030197383	0.021343612	0.0088537714
0.6	0.01227734	0.0080958528	0.0041814871
0.7	0.0049915939	0.0030708407	0.0019207532
0.8	0.0020294306	0.0011648017	0.000864629
0.9	0.0008251049	0.0004418213	0.0003832836
1	0.0003354626	0.0001675874	0.0001678752

Table 7: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.1	0.055111761	0.01	0.045111761
0.2	0.046105213	0.04	0.006105213
0.3	0.090826251	0.09	0.0008262507
0.4	0.16011182	0.16	0.0001118209
0.5	0.25001513	0.25	1.513331E-005
0.6	0.36000205	0.36	2.0480708E-006
0.7	0.49000028	0.49	2.7717624E-007
0.8	0.64000004	0.64	3.7511725E-008
0.9	0.81000001	0.81	5.0766599E-009
1	1	1	6.870513E-010

4 Discussion and Conclusions

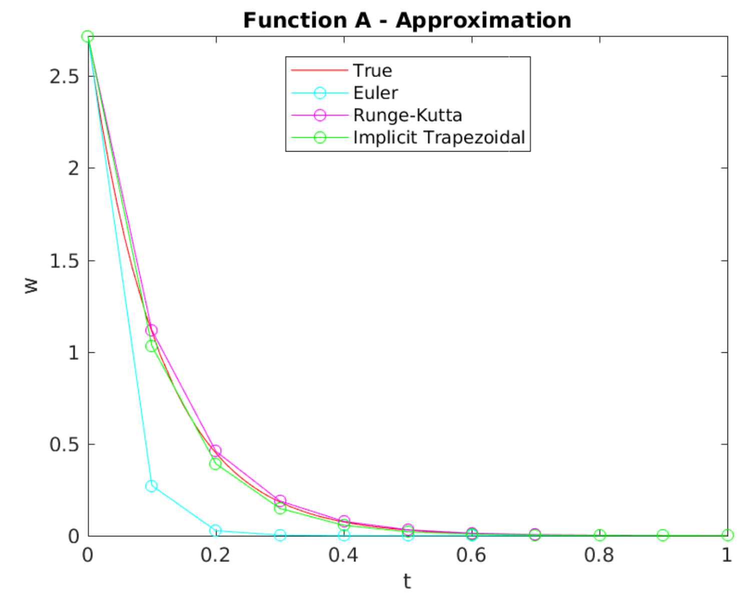


Figure 10: a compare val

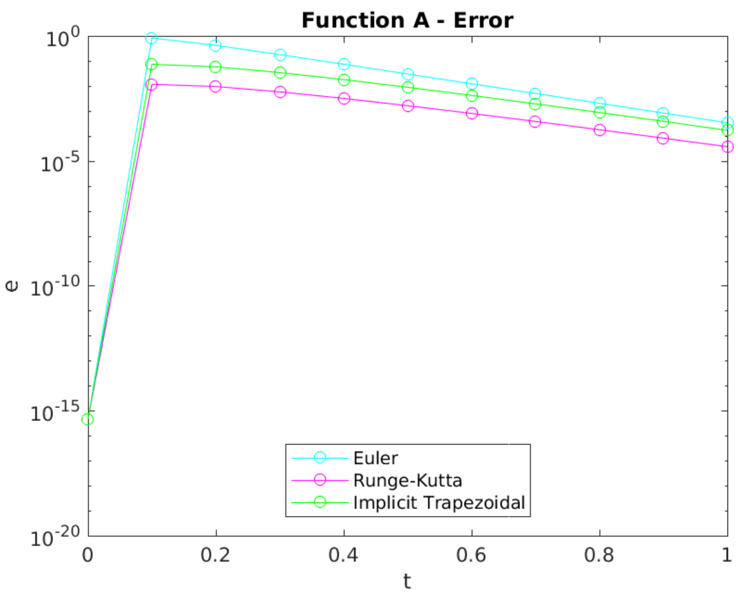


Figure 11: a compare err

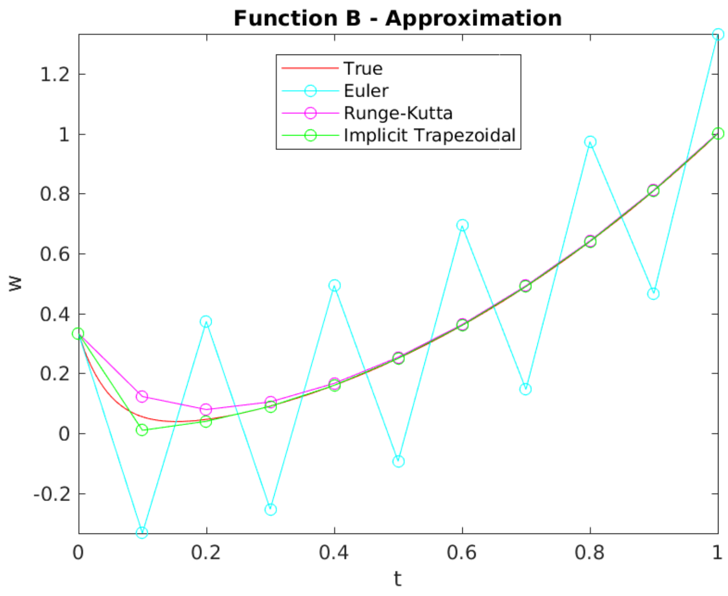


Figure 12: b compare val

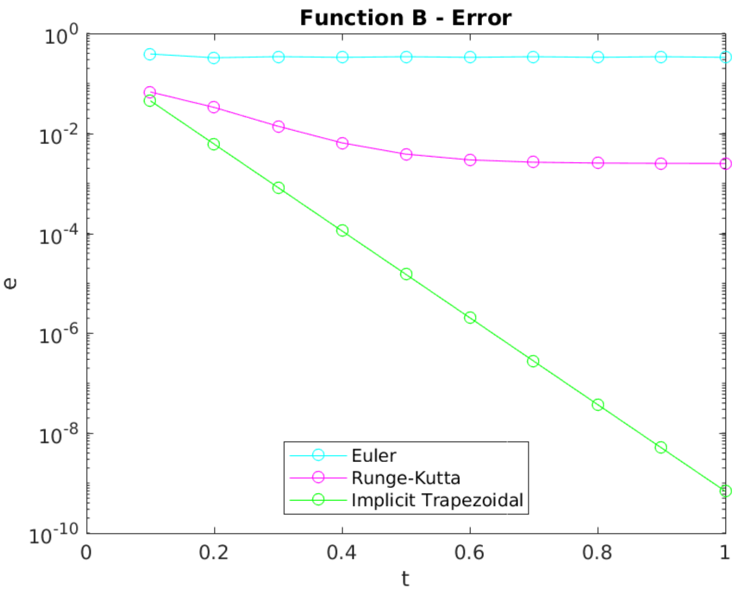


Figure 13: b compare err

unstable

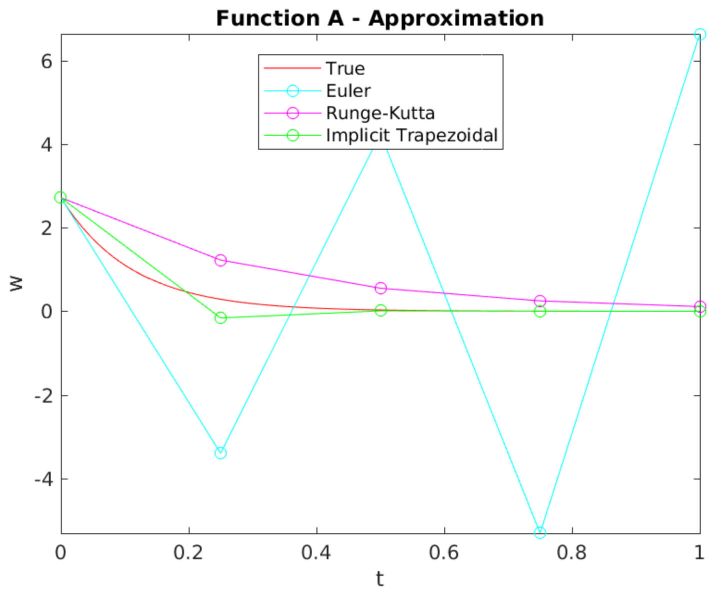


Figure 14: a compare val

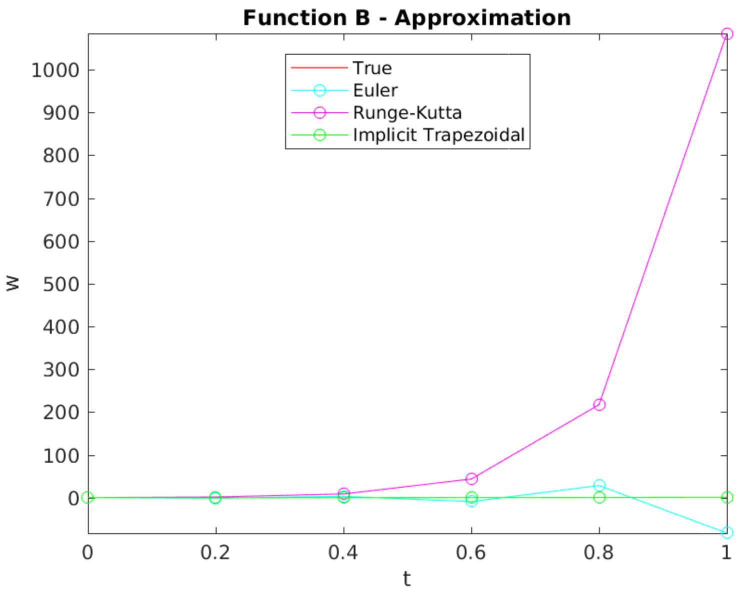


Figure 15: a compare err

Table 8: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	-3.3978523	3.6843571
0.5	0.030197383	4.2473154	4.217118
0.75	0.0031827808	-5.3091442	5.312327
1	0.0003354626	6.6364302	6.6360948

Table 9: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.2	0.046105213	-1	1.0461052
0.4	0.16011182	3.24	3.0798882
0.6	0.36000205	-8.92	9.280002
0.8	0.64000004	28.44	27.8
1	1	-82.44	83.44

Table 10: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	1.225085	0.93858023
0.5	0.030197383	0.55212572	0.52192834
0.75	0.0031827808	0.248834	0.24565122
1	0.0003354626	0.1121454	0.11180994

Table 11: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.2	0.046105213	1.76	1.7138948
0.4	0.16011182	8.8133333	8.6532215
0.6	0.36000205	43.68	43.319998
0.8	0.64000004	217.29333	216.65333
1	1	1084.32	1083.32

Table 12: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	-0.15989893	0.44640373
0.5	0.030197383	0.0094058195	0.020791564
0.75	0.0031827808	-0.0005532835	0.0037360643
1	0.0003354626	3.2546088E-005	0.0003029165

Table 13: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.2	0.046105213	-0.071111111	0.11721632
0.4	0.16011182	0.19703704	0.036925216
0.6	0.36000205	0.34765432	0.012347727
0.8	0.64000004	0.64411523	0.0041151888
1	1	0.99862826	0.0013717428

References

R. Burden and J Faires. *Numerical Analysis.* Brooks/Cole, 9th edition, 2010.