Project 03

APSC 607 Fall 2017

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1 Introduction

This project explores different numerical methods of solving two well-posed initial value problems (IVPs), y', at discrete points within a range $[a\ b]$ for a given initial value $y(a)$ as defined in Equation 1 and described in Burden and Faires (2010). The methods that will be tested are Euler's method (i.e., first order Taylor method), fourth order Runge-Kutta (i.e., essentially fourth order Taylor method), and an implicit trapezoidal method (trapezoidal with Newton Iteration). The behavior and characteristics of these methods will be reviewed and their effectiveness evaluated based on the true solution to the IVP.

$$y' = \frac{dy}{dt} = f(t, y), \qquad a \leq t \leq b, \qquad y(a) = \alpha \tag{1}$$

All computations are performed using MATLAB using the code accompanying this report. Section 2 will present the methods used in MATLAB to explore functions. Section 3 contains the results and related outputs for each function, and Section 4 includes discussion and conclusion. All figures and tables found in this report are available in the output subdirectory of the accompanying zip file. Additionally, all code and figures found in the zip file can be accessed via GitHub[1]

[1] `https://github.com/sgoodm/apsc607/tree/master/project_03`

2 Methods

The two unique functions which will be explored in this project, Functions **A** and **B**, are defined by Equations 2 and 3, respectively.

$$y' = -9y \quad (2) \qquad\qquad y' = 20(y - t^2) + 2t \quad (3)$$

The associated initial values for Functions A and B, are defined by Equations 4 and 5.

$$y(0) = e \quad (4) \qquad\qquad y(0) = \frac{1}{3} \quad (5)$$

Both IVPs will be examined over the range $0 \leq t \leq 1$, using a baseline step size of $h = 0.1$.

To validate and compare each method for solving the IVPs, the true solution is required. The true solution is generated using the Symbolic Toolkit in MATLAB as seen in the example below for Function A.

```
syms y(t)
ode = diff(y,t) == -9*y;
cond = y(0) == exp(1);
sol = dsolve(ode, cond);
```

Which produces the functions $y(t)$ for A and B seen in 6 and 7.

$$y(t) = e^{1-9t} \quad (6) \qquad\qquad y(t) = \frac{1}{3}e^{-20t} + t^2 \quad (7)$$

The true values for Functions A and B are compared to the results from solving the IVP using Euler's method, the fourth order Runge-Kutta method, and the implicit trapezoidal with Newtonian iteration method.

The remainder of this section will provide details on the three approaches for solving IVPs (Euler's, fourth order Runge-Kutta, and the implicit trapezoidal with Newtonian iteration).

2.1 Euler's (first order Taylor)

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2.2 Runge-Kutta Order Four

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2.3 Implicit Trapezoidal (Trapezoidal with Newtonian Iteration)

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3 Results

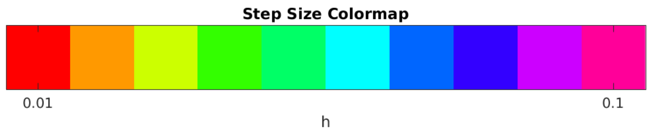

Figure 1: Colormap of h val

This section will review the results of the different integration rules for Functions A and B. An overview of these results can be seen in Table 1, which shows the the number of subintervals required to reach each tolerance specified, as well as the final error achieved. Figure **??** contains the true plots of both function, for reference.

Function	**Method**	**N:** 10^{-4}	**N:** 10^{-8}	**Minimum Error**
a	trapezoidal	768	65536	4.26887e-09
a	midpoint	1024	98304	8.53747e-09
a	simpsons	42	322	1.30384e-12
a	adaptive	24	278	-
b	trapezoidal	12	1082	1.29289e-09
b	midpoint	22	1522	2.58233e-09
b	simpsons	2	22	2.22044e-16
b	adaptive	2	16	-

Table 1: Results

3.1 Euler's Results

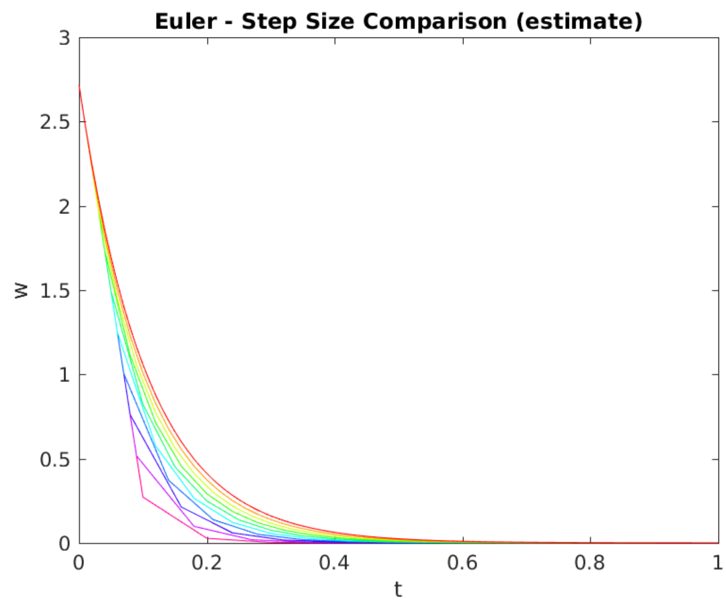

Figure 2: a euler h val

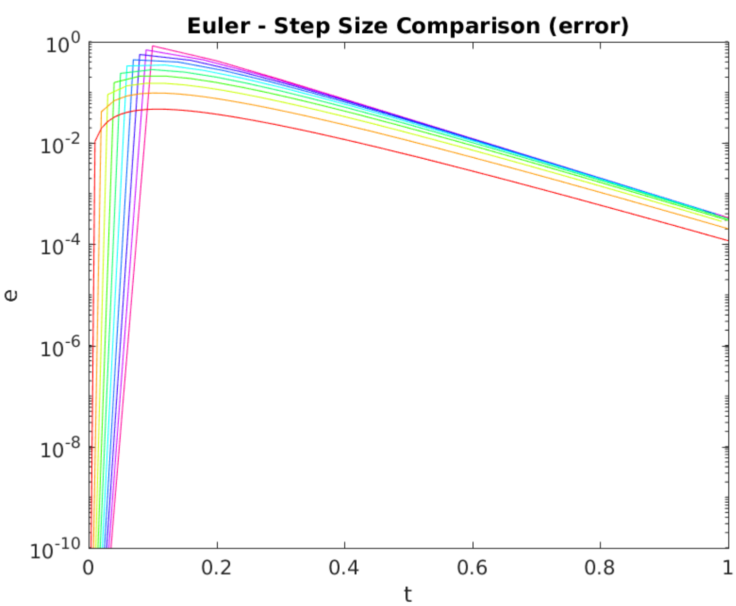

Figure 3: a euler h err

Table 2: My caption

t	true	euler	error1
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	0.27182818	0.83334274
0.2	0.44932896	0.027182818	0.42214615
0.3	0.18268352	0.0027182818	0.17996524
0.4	0.074273578	0.0002718282	0.07400175
0.5	0.030197383	2.7182818E-005	0.030170201
0.6	0.01227734	2.7182818E-006	0.012274622
0.7	0.0049915939	2.7182818E-007	0.0049913221
0.8	0.0020294306	2.7182818E-008	0.0020294035
0.9	0.0008251049	2.7182818E-009	0.0008251022
1	0.0003354626	2.7182818E-010	0.0003354624

Table 3: My caption

t	true	euler	error1
0	0.33333333	0.33333333	0
0.1	0.055111761	-0.33333333	0.38844509
0.2	0.046105213	0.37333333	0.32722812
0.3	0.090826251	-0.25333333	0.34415958
0.4	0.16011182	0.49333333	0.33322151
0.5	0.25001513	-0.093333333	0.34334847
0.6	0.36000205	0.69333333	0.33333129
0.7	0.49000028	0.14666667	0.34333361
0.8	0.64000004	0.97333333	0.3333333
0.9	0.81000001	0.46666667	0.34333334
1	1	1.3333333	0.33333333

3.2 Runge-Kutta Results

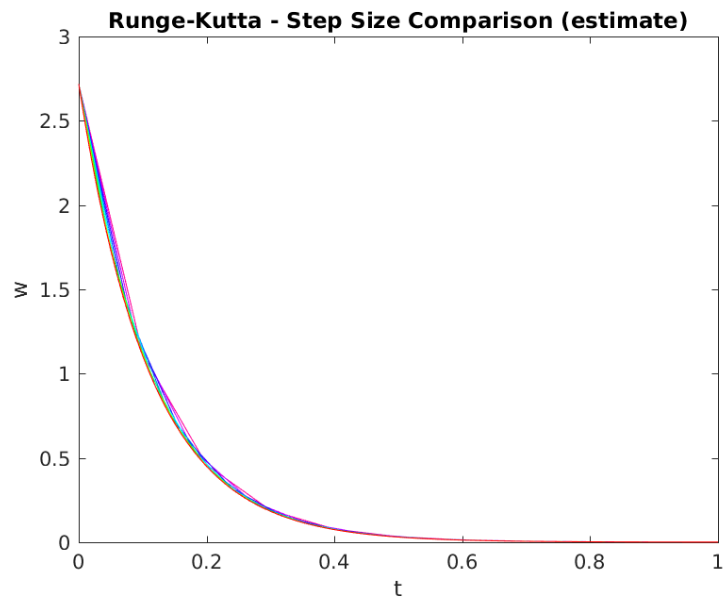

Figure 4: a rk h val

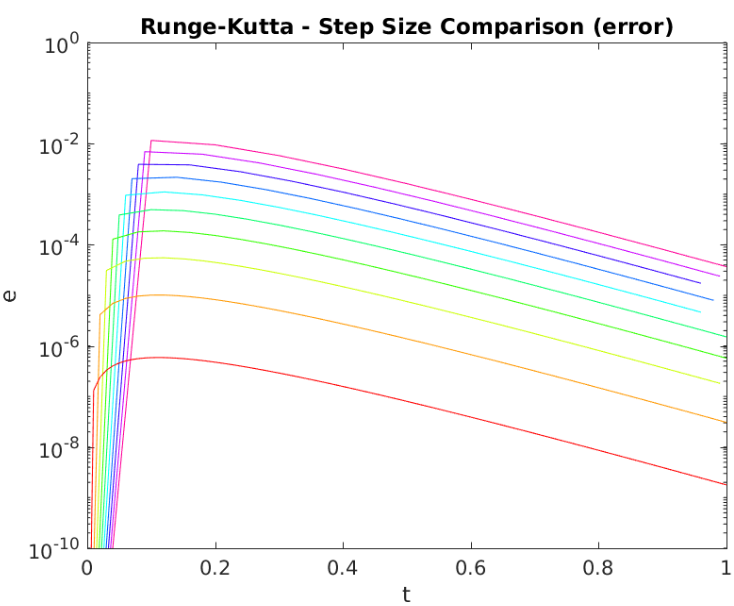

Figure 5: a rk h err

Table 4: My caption

t	true	value	error
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	1.1167721	0.011601193
0.2	0.44932896	0.45881186	0.0094828979
0.3	0.18268352	0.18849712	0.0058135943
0.4	0.074273578	0.077441685	0.0031681067
0.5	0.030197383	0.031815948	0.0016185648
0.6	0.01227734	0.013071185	0.0007938447
0.7	0.0049915939	0.0053701328	0.0003785389
0.8	0.0020294306	0.0022062519	0.0001768213
0.9	0.0008251049	0.000906411	8.1306108E-005
1	0.0003354626	0.0003723876	0.000036925

Table 5: My caption

t	true	value	error
0	0.33333333	0.33333333	0
0.1	0.055111761	0.12277778	0.067666017
0.2	0.046105213	0.079259259	0.033154046
0.3	0.090826251	0.10475309	0.013926836
0.4	0.16011182	0.16658436	0.0064725413
0.5	0.25001513	0.25386145	0.0038463207
0.6	0.36000205	0.36295382	0.0029517699
0.7	0.49000028	0.49265127	0.0026509955
0.8	0.64000004	0.64255042	0.0025503867
0.9	0.81000001	0.81251681	0.002516803
1	1	1.0025056	0.002505602

3.3 Implicit Trapezoidal Results

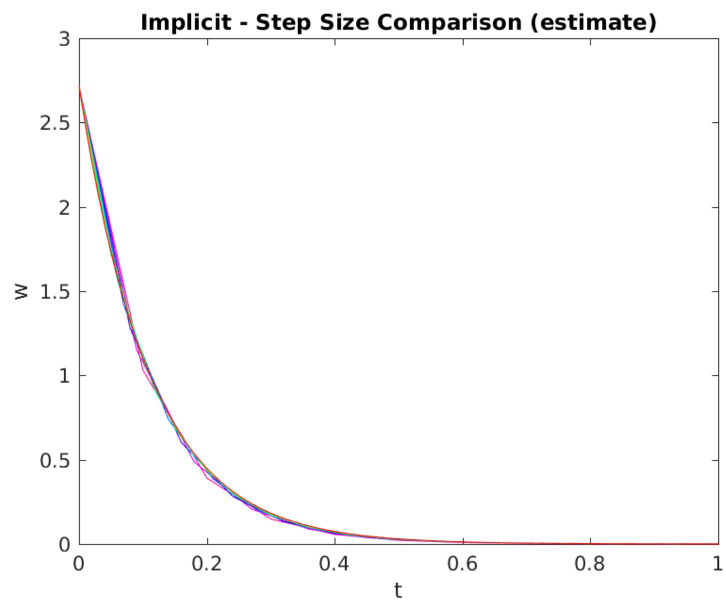

Figure 6: a implicit h val

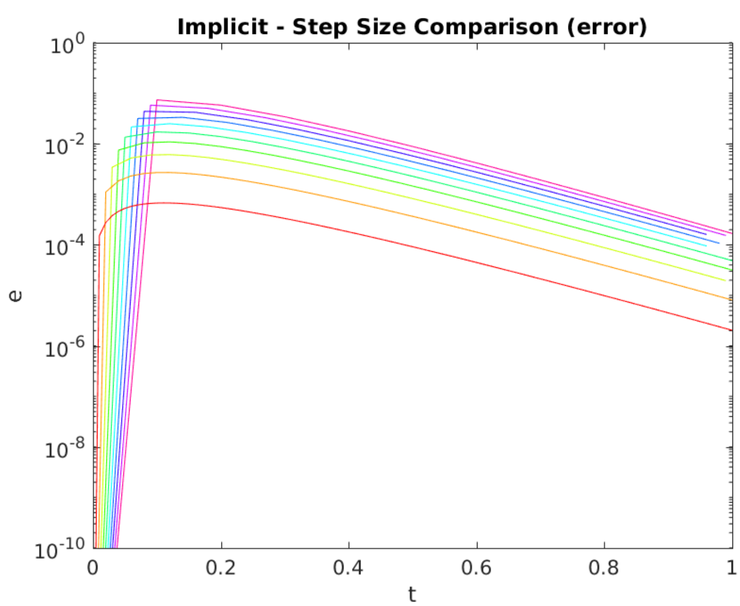

Figure 7: a implicit h err

Table 6: My caption

t	true	value	error
0	2.7182818	2.7182818	4.4408921E-016
0.1	1.1051709	1.0310724	0.0740985
0.2	0.44932896	0.39109643	0.05823253
0.3	0.18268352	0.14834692	0.034336601
0.4	0.074273578	0.056269523	0.018004056
0.5	0.030197383	0.021343612	0.0088537714
0.6	0.01227734	0.0080958528	0.0041814871
0.7	0.0049915939	0.0030708407	0.0019207532
0.8	0.0020294306	0.0011648017	0.000864629
0.9	0.0008251049	0.0004418213	0.0003832836
1	0.0003354626	0.0001675874	0.0001678752

Table 7: My caption

t	true	value	error
0	0.33333333	0.33333333	0
0.1	0.055111761	0.01	0.045111761
0.2	0.046105213	0.04	0.006105213
0.3	0.090826251	0.09	0.0008262507
0.4	0.16011182	0.16	0.0001118209
0.5	0.25001513	0.25	1.513331E-005
0.6	0.36000205	0.36	2.0480708E-006
0.7	0.49000028	0.49	2.7717624E-007
0.8	0.64000004	0.64	3.7511725E-008
0.9	0.81000001	0.81	5.0766599E-009
1	1	1	6.870513E-010

4 Discussion and Conclusions

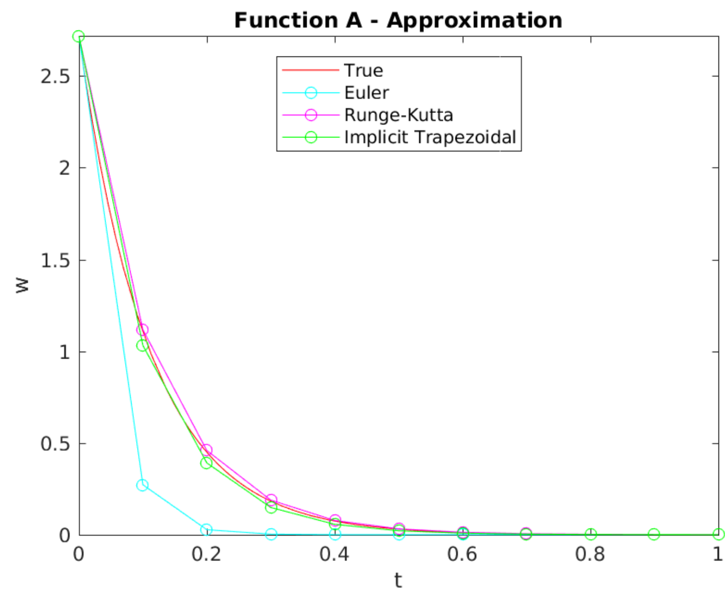

Figure 8: a compare val

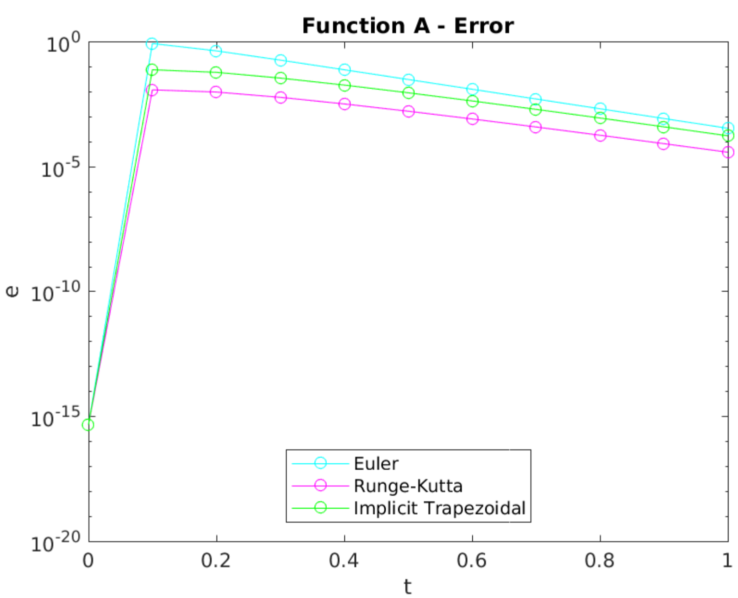

Figure 9: a compare err

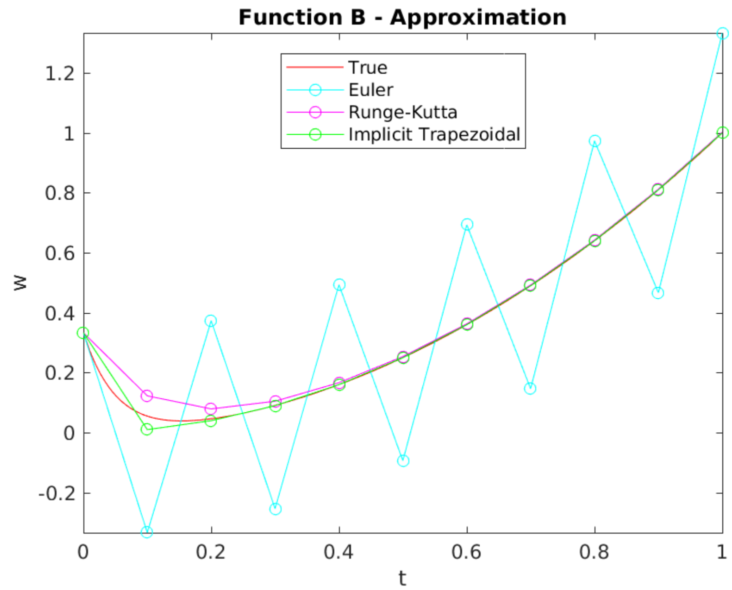

Figure 10: b compare val

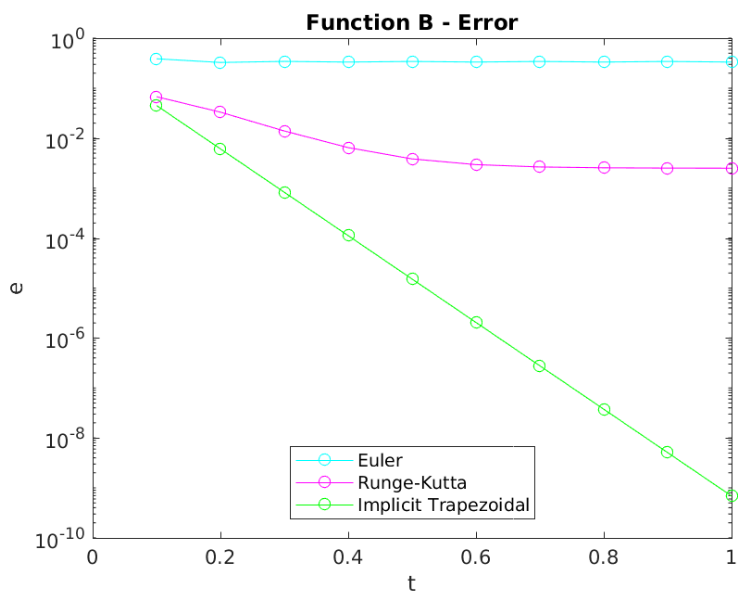

Figure 11: b compare err

unstable

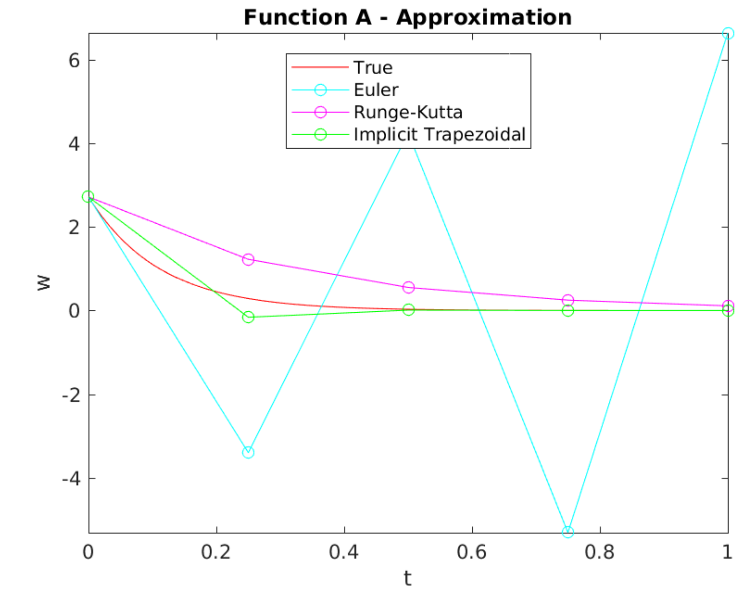

Figure 12: a compare val

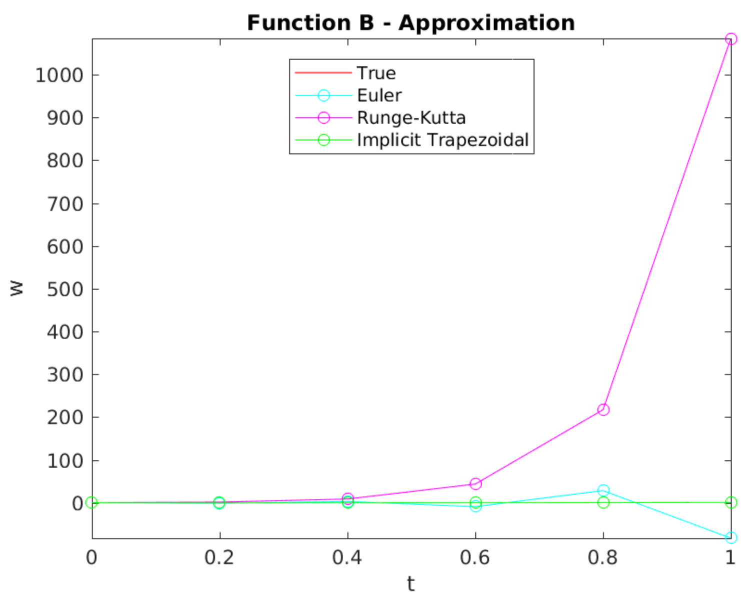

Figure 13: a compare err

Table 8: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	-3.3978523	3.6843571
0.5	0.030197383	4.2473154	4.217118
0.75	0.0031827808	-5.3091442	5.312327
1	0.0003354626	6.6364302	6.6360948

Table 9: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.2	0.046105213	-1	1.0461052
0.4	0.16011182	3.24	3.0798882
0.6	0.36000205	-8.92	9.280002
0.8	0.64000004	28.44	27.8
1	1	-82.44	83.44

Table 10: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	1.225085	0.93858023
0.5	0.030197383	0.55212572	0.52192834
0.75	0.0031827808	0.248834	0.24565122
1	0.0003354626	0.1121454	0.11180994

Table 11: My caption

t	**true**	**value**	**error**
0	0.33333333	0.33333333	0
0.2	0.046105213	1.76	1.7138948
0.4	0.16011182	8.8133333	8.6532215
0.6	0.36000205	43.68	43.319998
0.8	0.64000004	217.29333	216.65333
1	1	1084.32	1083.32

Table 12: My caption

t	**true**	**value**	**error**
0	2.7182818	2.7182818	4.4408921E-016
0.25	0.2865048	-0.15989893	0.44640373
0.5	0.030197383	0.0094058195	0.020791564
0.75	0.0031827808	-0.0005532835	0.0037360643
1	0.0003354626	3.2546088E-005	0.0003029165

Table 13: My caption

t	true	value	error
0	0.33333333	0.33333333	0
0.2	0.046105213	-0.071111111	0.11721632
0.4	0.16011182	0.19703704	0.036925216
0.6	0.36000205	0.34765432	0.012347727
0.8	0.64000004	0.64411523	0.0041151888
1	1	0.99862826	0.0013717428

References

R. Burden and J Faires. *Numerical Analysis*. Brooks/Cole, 9th edition, 2010.