APSC 608

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$\boxed{HW \# 4}$

Coleman : 4.1 #1, 2 | 4.2 #1, 2 | 6.1 #2a | 6.4 #1a

Farlow : 12 #1, 3 | 13 #2, 3

$X'' + \lambda X = 0$

$T' + 2\lambda T = 0$

$X(0) = X(L) = 0$

C 4.1 #1 $\quad u_t = 2u_{xx} \qquad u(0,t) = u(L,t) = 0$

A) $L = \pi, \quad u(x,0) = 20$

$$\lambda_n = \frac{(2n-1)^2 \cancel{\pi^2}}{4\cancel{\pi^2}}, \quad X_n(x) = \sin\left(\frac{(2n-1)\cancel{\pi} x}{2\cancel{\pi}}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-2(2n-1)^2}{4}t} \sin\left(\frac{(2n-1)x}{2}\right)$$

$$\cancel{u(x,0) = 20 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)x}{2}\right)}$$

$$c_n = \frac{2}{\pi}\int_0^{\pi} 20 \cdot \sin\left(\frac{(2n-1)x}{2}\right) dx$$

$$c_n = \frac{2}{\pi} \cdot \left[-\frac{40(\cancel{\sin(\pi n)}^{0} - 1)}{(2n-1)}\right]$$

$$c_n = 80 / \pi(2n-1)$$

$$u(x,t) = \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-\frac{(2n-1)^2}{2}t} \sin\left(\frac{(2n-1)x}{2}\right)$$

$$\lambda_n = \left[\frac{(2n-1)\pi}{2L}\right]^2$$

$$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

$$T_n(t) = e^{-2\lambda_n t}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x)$$

$$c_n = \frac{2}{L}\int_0^L u(x,0)\, X_n(x)\, dx$$

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(cont.)

4.1 #1 B) $L = 1, \quad u(x,0) = x$

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{2} \qquad X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

$$T_n(t) = e^{-\frac{\cancel{2}(2n-1)^2\pi^2}{\cancel{2}} t}$$

$$C_n = \frac{2}{1} \int_0^1 x \cdot \sin\left(\frac{(2n-1)\pi x}{2}\right) dx = 2 \cdot \left[-\frac{2\left[\cancel{\pi(2n-1)\sin(\pi n)} + 2\cos(\pi n)\right]}{\pi^2 (2n-1)^2} \right]$$

$$C_n = \frac{-8}{\pi^2} \cdot \frac{(-1)^n}{(2n-1)^2}$$

$$u(x,t) = -\frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cdot e^{-\frac{(2n-1)^2\pi^2 t}{\cancel{2}}} \sin\left(\frac{(2n-1)x\pi}{2}\right)$$

4.1 #1C) $L = 2, \quad u(x,0) = \begin{cases} 20 & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \leq 2 \end{cases}$

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4} \qquad X_n(x) = \sin\left(\frac{(2n-1)\pi x}{4}\right) \qquad T_n(t) = e^{-\frac{2(2n-1)^2\pi^2 t}{4}}$$

$$C_n = \frac{\cancel{2}}{\cancel{2}} \int_0^2 20 \sin\left(\frac{(2n-1)\pi x}{4}\right) dx = -\frac{80\left(\cancel{\sin(\pi n)} - 1\right)}{\pi(2n-1)} = \frac{80}{\pi} \cdot \frac{1}{(2n-1)}$$

$$u(x,t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-\frac{(2n-1)^2\pi^2 t}{2}} \sin\left(\frac{(2n-1)\pi x}{4}\right)$$

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4.1 #2 $\quad u_t = 4u_{xx}$

A) $L = \pi$, $u(x,0) = x^2$, $u(0,t) = u(\pi,t) = 0$

$X'' + \lambda X = 0 \qquad \lambda = \left(\frac{n\pi}{L}\right)^2$

$T' + 4\lambda T = 0 \qquad X(x) = \cancel{\sin}\ \cos\left(\frac{n\pi x}{L}\right) \qquad T(t) = e^{-4\lambda t}$

$X(0) = X(L) = 0 \qquad T(t) = e^{-4\left(\frac{n^2\pi^2}{L^2}\right)t}$

$\lambda = \frac{n^2\pi^2}{\pi^2} = n^2 \qquad c_n = \frac{2}{L}\int_0^L u(x,0)\,X(x)\,dx$

$X(t) = \cancel{\sin}\ \cos(nx) \qquad c_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cancel{\sin}\ \cos(nx)\,dx$

$T(t) = e^{-4n^2t}$

$$c_n = \frac{2}{\pi}\left[\cancel{\frac{2\pi n \sin(\pi n) + (2 - \pi^2 n^2)\cos(\pi n) - 2}{n^3}}\right] \qquad \sin(\pi n) = 0,\ \cos(\pi n) = (-1)^n$$

EVEN $\qquad$ ODD

$$\cancel{\frac{2}{\pi}\cdot\frac{-\pi^2 n^2}{n^3}} \qquad \cancel{\frac{2}{\pi}\cdot\frac{\pi^2 n^2 - 4}{n^3}}$$

$$\cancel{-\frac{2\pi}{n}}$$

$$c_n = \frac{2}{\cancel{\pi}}\left[\frac{\cancel{(\pi^2 n^2 - 2)\sin(\pi x)} + 2\cancel{\pi x}\cos(\pi n)}{\cancel{n^3}\ n^2}\right] \qquad \sin(\pi x) = 0,\ \cos(\pi n) = (-1)^n$$

$$c_n = 4\cdot\frac{(-1)^n}{n^2}$$

$$u(x,t) = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}e^{-4n^2t}\cos(nx)$$

4.1#2 B) $L = 1, \quad u(x,0) = 10$

$$\lambda = n^2\pi^2$$

$$X(t) = \overset{\sin}{\cancel{\cos}}(n\pi x)$$

$$T(t) = e^{-4n^2\pi^2 t}$$

$$c_n = 2\int_0^1 10 \cdot \overset{\sin}{\cancel{\cos}}(n\pi x)\,dx$$

$$c_n = 2\left[10 \cdot \frac{1-\cos\ \cancel{10\sin}\,(\pi n)}{\pi n}\right] = \frac{20}{\pi} \cdot \frac{1-\overset{(-1)^n}{\cancel{\cos(\pi n)}}}{n}$$

$$u(x,t) = \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1^n)}{n} e^{-4n^2\pi^2 t} \sin(n\pi x)$$

4.1#2 C) $L = 2, \quad u(x,0) = \begin{cases} 10 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$

$$\lambda = \frac{n^2\pi^2}{4}$$

$$c_n = \cancel{\frac{2}{2}}\int_0^2 10\ \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{2}\right) dx$$

$$X(x) = \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{2}\right)$$

$$c_n = \frac{20}{\pi} \cdot \frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}$$

$$T(t) = e^{-4\frac{n^2\pi^2}{4}t} = e^{-n^2\pi^2 t}$$

$$u(x,t) = \frac{20}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\right) e^{-n^2\pi^2 t} \sin\left(\frac{n\pi x}{2}\right)$$

(c4.2 #1) $u_{tt} = 5u_{xx}$, $u(0,t) = u(L,t) = 0$

$a^2 = 5$, $a = \sqrt{5}$ $\qquad X'' + \lambda X = 0 \qquad T'' + a^2\lambda T = 0$

$X(0) = X(L) = 0$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n = \sin\left(\frac{n\pi x}{L}\right)$

$T_n = c_1 \cdot \cos\left(\sqrt{a^2\lambda}\, t\right) + c_2 \sin\left(\sqrt{a^2\lambda}\, t\right)$

$u(x,t) = X_n T_n =$

A) $L = \pi$, $u(x,0) = 3\sin(2x) \qquad u_t(x,0) = \sin(x) - 7\sin(4x)$

$\lambda_n = n^2 \qquad X_n = \sin(nx) \qquad T_n = c_1 \cdot \cos(\sqrt{5}\cdot nt) + c_2 \sin(\sqrt{5}\cdot nt)$

$u_n(x,t) = X_n T_n = \sin(nx)\left[c_n \cos(\sqrt{5}\cdot nt) + d_n \sin(\sqrt{5}\, nt)\right]$

$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[c_n \cos(\sqrt{5}\, nt) + d_n \sin(\sqrt{5}\, nt)\right]$

(Deriv. for t)

$u_t(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[-\sqrt{5}n\, c_n \cdot \sin(\sqrt{5}\, nt) + \sqrt{5}n\, d_n \cos(\sqrt{5}\, nt)\right]$

Use ICs:

$u(x,0) = 3\sin(2x) = \sum_{n=1}^{\infty} \sin(nx)\left[c_n \underbrace{\cos(\sqrt{5}\cdot n \cdot 0)}_{1} + \cancel{d_n \sin(\sqrt{5}\, n 0)}^{0}\right]$

$u(x,0) = 3\sin(2x) = \sum_{n=1}^{\infty} c_n \sin(nx)$

$u_t(x,0) = \sin(x) - 7\sin(4x) = \sum_{n=1}^{\infty} \sin(nx)\left[\cancel{-\sqrt{5}\, n\, c_n \sin(\sqrt{5}\, n 0)}^{0} + \sqrt{5}n\, d_n \underbrace{\cos(\sqrt{5}\, n0)}_{1}\right]$

$u_t(x,0) = \sin(x) - 7\sin(4x) = \sum_{n=1}^{\infty} \sin(nx)\, \sqrt{5}\, n\, d_n$

(c4.2 #1a)

For $u(x,0) = 3\sin(2x) \Rightarrow c_n \sin(nx)$ $n=2$, $c_n = c_2 = 3$

For $u_t(x,0) = \sin(x) - 7\sin(4x) \Rightarrow \sin(nx)\sqrt{5}\, n\, d_n$

$n=1$

$\sqrt{5}\cdot n\cdot d_n = 1$

$d_1 = 1/\sqrt{5}$

$n=4$

$\sqrt{5}\cdot n\cdot d_n = -7$

$d_4 = -7/4\sqrt{5}$

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\cdot\left[c_n\cdot\cos(\sqrt{5}\cdot n\cdot t) + d_n \sin(\sqrt{5}\cdot n\cdot t)\right]$$

$$u(x,t) = \sin(2x)\cdot 3\cdot\cos(\sqrt{5}\cdot 2\cdot t) + \sin(x)\frac{1}{\sqrt{5}}\sin(\sqrt{5}\,t) + \sin(4x)\left(\frac{-7}{4\sqrt{5}}\right)\sin(\sqrt{5}\cdot 4\cdot t)$$

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(c4.2 #1)

B) $L = 4$ $\quad u(x,0) = \begin{cases} 4 & 0 \le x \le 2 \\ 4-x & 2 \le x \le 4 \end{cases}$ $\quad u_t(x,0) = 0$

$$\lambda_n = \frac{n^2\pi^2}{16} \qquad X_n = \sin\left(\frac{n\pi x}{4}\right)$$

$$T_n = c_1 \cos\left(\sqrt{5}\cdot\frac{n\pi}{4}t\right) + d_n \sin\left(\sqrt{5}\cdot\frac{n\pi}{4}t\right)$$

$$u_n(x,t) = X_n T_n = \sin\left(\frac{n\pi x}{4}\right)\left[c_n \cos\left(\sqrt{5}\frac{n\pi}{4}t\right) + d_n \sin\left(\sqrt{5}\frac{n\pi}{4}t\right)\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{4}\right)\left[-\sqrt{5}\frac{n\pi}{4} c_n \sin\left(\sqrt{5}\frac{n\pi}{4}t\right) + \sqrt{5}\frac{n\pi}{4} d_n \cos\left(\sqrt{5}\frac{n\pi}{4}t\right)\right]$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{4}\right) = \begin{cases} x & 0:2 \\ 4-x & 2:4 \end{cases}$$

$$c_n = \int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx = \frac{2}{4}\int_0^2 x\sin\left(\frac{n\pi x}{4}\right)dx + \frac{2}{4}\int_2^4 (4-x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$c_n = \left[\frac{\overset{8}{\cancel{16}}\sin\left(\frac{\pi n}{2}\right) - \cancel{8\pi n\cos\left(\frac{\pi n}{2}\right)}}{\cancel{2}(\pi n)^2}\right] + \left[\frac{\cancelto{0}{-16\sin(\pi n)} + \overset{8}{\cancel{16}}\sin\left(\frac{\pi n}{2}\right) + \cancel{\overset{4}{\cancel{8}}\pi n\cos\left(\frac{\pi n}{2}\right)}}{\cancel{2}(\pi n)^2}\right]$$

$$c_n = \frac{16\sin\left(\frac{\pi n}{2}\right)}{\pi^2 n^2}$$

(c4.2 #1B)

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{4}\right) \sqrt{5}\, \frac{n\pi}{4}\, d_n \longrightarrow d_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{4}\right) \cdot \frac{16 \sin\left(\frac{\pi n}{2}\right)}{n^2\pi^2} \cdot \cos\left(\sqrt{5}\, \frac{n\pi}{4}\, t\right)$$

$$u(x,t) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{4}\right) \cdot \frac{1}{n^2} \cdot \sin\left(\frac{\pi n}{2}\right) \cdot \cos\left(\sqrt{5}\, \frac{n\pi}{4}\, t\right)$$

(c4.2 #1C) $L = 2$ $\quad u(x,0) = 0 \quad u_t(x,0) = 3$

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 \qquad X_n = \sin\left(\frac{n\pi x}{2}\right)$$

$$T_n = c_1 \cos\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right) + c_2 \sin\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right)$$

$$u_n(x,t) = X_n T_n = \sin\left(\frac{n\pi x}{2}\right)\left[c_n \cos\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right) + d_n \sin\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right)\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left[-\sqrt{5}\,\frac{n\pi}{2}\,c_n \sin\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right) + \sqrt{5}\,\frac{n\pi}{2}\,d_n \cos\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right)\right]$$

$$u(x,0) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) c_n \Longrightarrow c_n = 0$$

$$u_t(x,0) = 3 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \sqrt{5}\,\frac{n\pi}{2}\,d_n \Longrightarrow \frac{\sqrt{5}\,n\pi x}{2}\,d_n = \int_0^2 3\sin\left(\frac{n\pi x}{2}\right)dx$$

$$\frac{\sqrt{5}\,n\pi x}{2}\,d_n = \frac{6 - 6\cos(\pi n)}{\pi n}$$

$$d_n = \frac{12}{\sqrt{5}} \cdot \frac{1-(-1)^n}{\pi^2 n^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \cdot \frac{12}{\sqrt{5}} \cdot \frac{1-(-1)^n}{\pi^2 n^2} \cdot \sin\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right)$$

$$u(x,t) = \frac{12}{\sqrt{5}\cdot\pi^2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right) \sin\left(\sqrt{5}\,\frac{n\pi}{2}\,t\right) \frac{1-(-1)^n}{n^2}$$

ODD N REPRESENTATION

$$u(x,t) = \frac{24}{\sqrt{5}\cdot\pi^2} \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2}\right) \sin\left(\sqrt{5}\,\frac{(2n-1)\pi}{2}\,t\right) \frac{1}{(2n-1)^2}$$

(c4.2 #2) $u_{tt} = 4u_{xx}$, $L = \pi$, $u_x(0,t) = u_x(\pi,t) = 0$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2 \qquad X_n = \cos\left(\frac{n\pi x}{L}\right) = \cos(nx) \qquad a = 2$$

$$T_n = c_1 \cos(2nt) + c_2 \sin(2nt)$$

$$u_n(x,t) = X_n T_n = \cos(nx) \cdot \left[c_n \cos(2nt) + d_n \sin(2nt)\right]$$

$$u_0(x,t) = X_0 T_0 = c_0 + d_0 t \longrightarrow u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)$$

$$u_t(x,t) = d_0 + \sum_{n=1}^{\infty} \cos(nx)\left[-2nc_n \sin(2nt) + 2nd_n \cos(2nt)\right]$$

A) $u(x,0) = 4\cos(3x) \longrightarrow = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$

$$u_t(x,0) = 6\cos(2x) - \cos(5x)$$

$$= d_0 + \sum_{n=1}^{\infty} \cos(nx) \cdot 2n \cdot d_n \cdot ~~(\ldots)~~$$

$n = 3 \longrightarrow c_n = 4$

$n = 2 \longrightarrow 2nd_n = 6$, $d_n = 6/4 = 3/2$

$n = 5 \longrightarrow 2nd_n = -1$, $d_n = -\frac{1}{10}$

$$u(x,t) = \cos(3x) \cdot 4 \cdot \cos(6t) + \cos(2x) \cdot \frac{3}{2} \cdot \sin(4t) + \cos(5x) \cdot \left(-\frac{1}{10}\right) \cdot \sin(10t)$$

(c4.2 # 2B) $u(x,0) = \sin(x)$, $u_t(x,0) = 0$

$$\sin(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) \qquad 0 = d_0 + \sum_{n=1}^{\infty} \cos(nx)\, 2n\, d_n$$

$$d_n = 0$$

$$c_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{2}{\pi} \cdot \frac{1+\cos(\pi n)}{(1-n^2)}$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \longrightarrow \cancel{2}\cdot\frac{1}{\cancel{2}L}\int_0^{L} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx = \frac{1}{\pi}\cdot 2$$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} \cos(nx)\, c_n \cos(2nt)$$

$$u(x,t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \cos(nx)\left[\frac{2}{\pi}\cdot\frac{1+\overset{(-1)^n}{\cos(\pi n)}}{(1-n^2)}\right]\cos(2nt)$$

$$u(x,t) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1+(-1)^n}{(1-n^2)} \cos(nx)\cos(2nt)$$

(c4.2 + 2c) $u(x,0)=1$, $u_t(x,0)=x$

$$1 = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx)$$

$$x = d_0 + \sum_{n=1}^{\infty} d_n \cdot 2 \cdot n \cdot \cos(nx)$$

$$c_n = \cancel{2} \cdot \frac{2}{\cancel{2}\pi} \int_0^{\pi} 1 \cdot \cos(nx)\,dx$$

$$c_n = \frac{2}{\pi} \cdot \frac{\cancel{\sin(\pi n)}^{\,0}}{n} = 0$$

$$d_n = \frac{1}{\cancel{2}n} \cdot \frac{\cancel{2}}{\pi} \int_0^{\pi} x\cos(nx)\,dx$$

$$d_n = \frac{1}{n\pi} \cdot \left[\frac{\cancel{\pi n \sin(\pi n)}^{\,0} + \cos(\pi n) - 1}{n^2} \right]$$

$$d_n = \frac{1}{\pi} \cdot \frac{\cos(\pi n) - 1}{n^3}$$

$$c_0 = \frac{1}{\pi} \int_0^{\pi} 1\,dx = \frac{1}{\pi} \cdot \pi = 1$$

$$d_0 = \frac{1}{\pi} \int_0^{\pi} x\,dx = \frac{1}{\pi} \cdot \frac{\pi^2}{2} = \frac{\pi}{2}$$

$$u(x,t) = \left(1 + \frac{\pi}{2} t\right) + \sum_{n=1}^{\infty} \cos(nx) \left[\frac{1}{\pi} \cdot \frac{\overset{(-1)^n}{\cos(\pi n)} - 1}{n^3} \right] \sin(2nt)$$

$$u(x,t) = \left(1 + \frac{\pi}{2} t\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^3} \cos(nx) \sin(2nt)$$

(c 6.1 #2a) Laplace transform to solve ~~of~~ heat problem:

$$u_t = u_{xx} \qquad x>0,\ t>0$$

$$u(x,0) = 10e^{-x}$$

$$u(0,t) = 0 \qquad \lim_{x\to\infty} u(x,t) = 0$$

$$s\bar{U}(x,s) = \bar{U}_{xx}(x,s) + 10e^{-x}$$

ODE $\cancel{\bar{U}(x,s) = c_1(s)e^{x\sqrt{s}} + c_2(s)e^{-x\sqrt{s}}}$

$$\cancel{\bar{U}(0,s) = \mathcal{L}\{u(0,t)\} = \mathcal{L}\{0\} = 0 = c_2(s)}$$

$$\bar{U}(x) = \cancel{c_1 e^{x\sqrt{s}}} + c_2 e^{-x\sqrt{s}} + \frac{10e^{-x}}{s} \qquad c_1 = 0$$

$$\cancel{\bar{U}(x) = -\frac{10e^{-x}}{s}\left(\frac{e^{-x\sqrt{s}}}{s(\sqrt{s}+1)}\right) + \frac{10e^{-x}}{s}}$$

$$\cancel{u(x,t) = 10e^{-x} - 10e^{-x}\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)e^{(x+t)}\right]}$$

$$c_2 \cancel{e^{-x\sqrt{0}}}^{\,1} + \frac{10e^{-x}}{s} = 0 \longrightarrow c_2 = -\frac{10e^{-x}}{s}$$

$$\bar{U}(x) = -\frac{10e^{-x}}{s}\cdot e^{-x\sqrt{s}} + \frac{10e^{-x}}{s}$$

$$\bar{U}(x) = \frac{-10e^{-x^2\sqrt{s}}}{s} + \frac{10e^{-x}}{s}$$

$$u(x,t) = -10\operatorname{erfc}\left(\frac{x^2}{2\sqrt{t}}\right) + 10e^{-x}$$

(c6.4 #1a) $u_t = k^2 u_{xx} \quad -\infty < x < \infty, \quad t > 0$

$$u(x,0) = \begin{cases} T_1 & x < 0 \\ T_2 & x > 0 \end{cases}$$

Not sure where to start for this one

Coleman pg 243

$$\bar{U}_t(\alpha, t) = -\alpha^2 k^2 \bar{U}(\alpha, t) \qquad U(\alpha, t) = e^{-\alpha^2 k^2 t} G(\alpha)$$

$$U(\alpha, 0) = F(\alpha) = \mathcal{F}[f(x)]$$

$$U(\alpha, 0) = e^0 G(\alpha)$$

$$U(\alpha, t) = e^{-\alpha^2 k^2 t} F(\alpha)$$

$$\mathcal{F}^{-1}\left[e^{-\alpha^2 k^2 t}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 k^2 t} e^{i\alpha x}\, d\alpha$$

Known Transform:

$$\mathcal{F}^{-1}\left[e^{-\alpha^2/4}\right] = \sqrt{2} \cdot e^{-x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2/4} e^{i\alpha x}\, d\alpha$$

$\vdots$ (Rest of Steps pg 243 Coleman)

$$u(x,t) = \frac{1}{2k\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-\xi)\, e^{-\xi^2/4k^2 t}\, d\xi$$

or $$u(x,t) = \frac{1}{2k\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4k^2 t} f(\xi)\, d\xi$$

$$u(x,t) = \frac{1}{2k\sqrt{\pi t}} \int_{-\infty}^{0} T_1 e^{-(x-\xi)^2/4k^2 t}\, d\xi + \frac{1}{2k\sqrt{\pi t}} \int_{0}^{\infty} T_2 e^{-(x-\xi)^2/4k^2 t}\, d\xi$$

$$= \frac{1}{2k\cancel{\sqrt{\pi t}}} \cdot (-\cancel{\sqrt{\pi}}\, T_1 k\sqrt{t}) \left(\operatorname{erf}\left(\frac{x}{2k\sqrt{t}}\right) - 1\right) + \frac{1}{2k\cancel{\sqrt{\pi t}}} \cdot T_2 k\cancel{\sqrt{\pi t}} \left(\operatorname{erf}\left(\frac{x}{2k\sqrt{t}}\right) + 1\right)$$

$$= \frac{T_1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{2k\sqrt{t}}\right)\right) + \frac{T_2}{2}\left(1 + \operatorname{erf}\left(\frac{x}{2k\sqrt{t}}\right)\right)$$

$$u(x,t) = \left[\frac{T_1}{2} \operatorname{erfc}\left(\frac{x}{2k\sqrt{t}}\right)\right] + \left[\frac{T_2}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2k\sqrt{t}}\right)\right]\right]$$

(F12 #1) Fourier Transform

$$F(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & 0 < x \end{cases}$$

(Right sided exponential decay)

$$\mathcal{F}\left[e^{-at} u(t)\right] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{-1}{a + j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a + j\omega}$$

$$= \frac{1}{1 + j\omega}$$

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(F12 #3) Solve IVP w/ FT

$u_t = \alpha u_{xx} \qquad -\infty < x < \infty$

$u(x,0) = e^{-x^2} \qquad -\infty < x < \infty$

$$u(x,t) = \frac{1}{\sqrt{4\alpha^2+1}} \cdot e^{-x^2/(4\alpha^2 t+1)}$$

(F13 #2) Solve IVP with LT

$u_t = \alpha^2 u_{xx} \quad -\infty < x < \infty \quad 0 < t < \infty$

$u(x,0) = \sin x \quad -\infty < x < \infty$
$= u_0$

* Cannot find example for $-\infty < x < \infty$

$$sU(x) - u_0 = \frac{d^2U}{dx^2}(x)$$

$$sU(x) - \sin(x) = \frac{d^2U}{dx^2}(x)$$

$$u(x,t) = e^{-\alpha^2 t} \cdot \sin(x)$$

(FIS #3) Solve w/ LT (Give physical interpretation)

$u_t = u_{xx} \quad 0 < x < \infty \quad 0 < t < \infty$

$u(0,t) = \sin t \quad 0 < t < \infty$

$u(x,0) = 0 \quad 0 \leq x < \infty$

$$s\bar{U}(x,s) = \bar{U}_{xx}(x,s) + 0$$

$$\bar{U}(x) = \cancel{c_1 e^{x\sqrt{s}}} + c_2 e^{-x\sqrt{s}} + 0 \rightarrow c_1 = 0$$

$$c_2 \cancel{e^{-x\sqrt{0}}}^{1} + 0 = \frac{1}{s^2+1} \rightarrow c_2 = \frac{1}{s^2+1}$$

$$\bar{U}(x) = \frac{1}{s^2+1} \cdot e^{-x\sqrt{s}}$$

Transform from Coleman pg 218

$$u(x,t) = \sin(t) * \frac{1}{\sqrt{\pi t}} e^{-x^2/4t}$$

Transform from Internet

$$u(x,t) = \sin(t) * \frac{x e^{-x^2/4t}}{2\sqrt{\pi t^3}}$$

Still different?

$$u(x,t) = \sin(t) * \frac{\alpha x}{2\sqrt{\pi t^3}} \cdot e^{-\frac{\alpha^2 x}{4t}}$$