2018-02-06

(Fluids)

② The "inviscid" Burger's eq: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$
is also known as the nonlinear, first order wave equation.
Governs propagation of nonlinear waves in the 1D case

③ Burger's eq. ("viscous") $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \overset{\text{"nu"}}{\nu}\frac{\partial^2 u}{\partial x^2}$
This is eq ② with diffusion added. Very similar to eq of fluid flow and used as a single nonlinear model for numerical experiments

④ Poisson's eq. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ If $f(x,y)=0$ this is Laplace's eq

This Elliptic equation (no time deriv) governs temp dist in a solid w/ heat sources defined by $f(x,y)$. Only steady state (not temporal). This eq also determines the electric field in a region w/ a charge density $f(x,y)$

- Transforms and Convolution

– Transforms General Philosophy:
1) use transform to simplify problem
2) solve simpler problem
3) transform solution back to get sol. to orig. prob

differentiation prob → $\boxed{\text{integral transform}}$ → multiplication problem

* Example use: PDE in n vars is transformed into ODE in (n-1) vars

General Notation: $F(s) = \int_A^B \underbrace{k(s,t)}_{\text{transformation kernel: maps } f(t) \text{ into } F(s)} f(t)\, d(t)$

* always some kernel and some function, but kernel and limits vary from transform to transform

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- Transforms exist as <u>transform pairs</u>

1
$$\mathcal{F}_s[f] = F(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin(wt)\,dt$$ Fourier sine transform

$$\mathcal{F}_s^{-1}[F] = f(t) = \int_0^\infty F(w)\sin(wt)\,dw$$ inverse sine transform

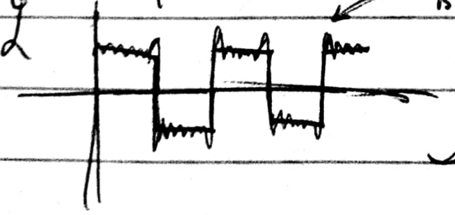
2
$$\mathcal{F}[f] = F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx$$ Fourier transform

$$\mathcal{F}^{-1}[F) = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(w)\,e^{iwx}\,dw$$ inv. Fourier trans.

3
$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$ Laplace transform

$$\mathcal{L}^{-1}[F] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{st}\,ds$$ inv Laplace trans

$\mathcal{L}$ square wave



overshoot is called Gibbs' phenomenon

transformed to sines and cosines

not perfect so...

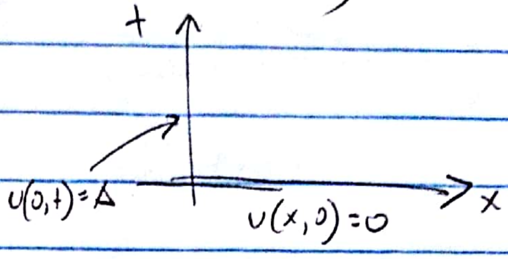
2018-02-08

# Infinite Diffusion Prob (in semi infinite medium)

PDE $u_t = \alpha^2 u_{xx}$ $\quad 0 < x < \infty$, $0 < t < \infty$

BCs $u(0,t) = A$ $\quad 0 < t < \infty$

IC $u(x,0) = 0$ $\quad 0 \le x \le \infty$




① transform each side of PDE

$$\mathcal{F}_s[u_t] = \alpha^2 \mathcal{F}_s[u_{xx}] \qquad \mathcal{F}_s[u_t] = \frac{2}{\pi}\int_0^\infty u_t(x,t)\sin(wx)\,dx$$
$$= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int_0^\infty u(x,t)\sin(wx)\,dx\right]$$
$$= \frac{d}{dt}\mathcal{F}_s[u] = \frac{d}{dt}\bar{U}(t)$$

$$\mathcal{F}_s[u_{xx}] = \frac{2}{\pi} w\, u(0,t) - w^2 \mathcal{F}_s[u]$$
$$= \frac{2}{\pi} w\, u(0,t) - w^2 \bar{U}(t)$$
$$= \frac{2Aw}{\pi} - w^2 \bar{U}(t)$$

sub into PDE...

$$\frac{d\bar{U}}{dt} = \alpha^2\left[-w^2\bar{U}(t) + \frac{2Aw}{\pi}\right]$$

Now transform IC...

$$\mathcal{F}_s[u(x,0)] = \bar{U}(0) = 0$$

$\Rightarrow$ ODE

$$\frac{d\bar{U}}{dt} + w^2\alpha^2\bar{U} = \frac{2Aw\alpha^2}{\pi} \qquad \text{IC} \quad \bar{U}(0) = 0$$

Solve ODE (integrating factor approach)...

$$\Rightarrow \bar{U}(t) = \frac{2A}{\pi w}\left(1 - e^{-w^2\alpha^2 t}\right)$$

Now we inverse transform (lookup in Tables) *hardest part of this process

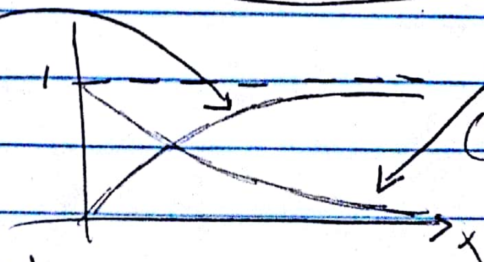
$$u(x,t) = A \cdot \operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

complementary error function

Note: this is DIFFUSION prob, so erfc, which goes down, makes sense

error func

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

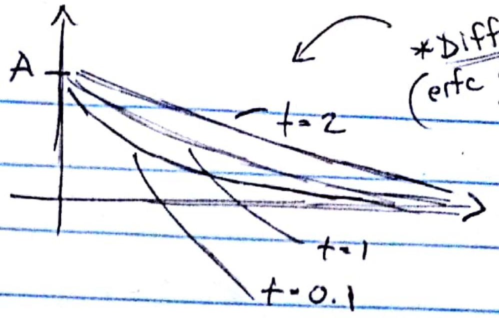



$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$
$(0 < x < \infty)$

Note: $\operatorname{erfc}(x) + \operatorname{erf}(x) = 1$

→

... solution looks like...




*Diffusion...
(erfc gets drawn out w/ t)

(NOT THE SAME)

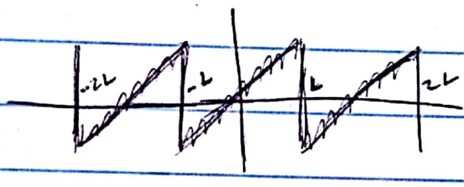
- <u>Fourier series and transform</u>

periodic functions in $(-\infty, \infty)$ of funcs on finite intervals can be approximated by infinite series of sines and cosines (this is call decomposition)

For sawtooth func:

$$f(x) = x \qquad -L < x < L$$

$$f(x+2L) = f(x) \quad \text{(periodicity)}$$




Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

with Fourier coefs :

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\left(\frac{2L}{n\pi}\right)(-1^n) \qquad n = 1, 2, \ldots$$

⇒ approx to sawtooth func :

$$f(x) = \frac{2L}{\pi} \left[ \sin\left(\frac{\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) - \ldots \right]$$

(sign alternates)

<u>* harmonics</u> : increasing multiples of fundamental frequencys

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Fourier Transform:

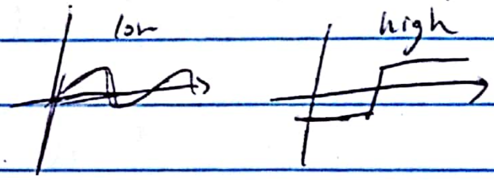
Fourier series $\xrightarrow{\text{non-periodic}}$ Fourier integral representation

$$f(x) = \int_0^\infty a(\xi) = \cos(\xi x)\, d\xi + \int_0^\infty b(\xi) \sin(\xi x)\, d\xi$$

with $a(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\xi x)\, dx$

$$b(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\xi x)\, dx$$

sharp edges $\rightarrow$ high frequency




Fourier Transform

$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

Useful Properties:

① Fourier transform pair

ex $\quad e^{-|x|} \xrightarrow{\mathcal{F}} \sqrt{\frac{2}{\pi}} \left( \frac{1}{1+\xi^2} \right) \xrightarrow{\mathcal{F}^{-1}} e^{-|x|}$

② Linear Transformation

$$\mathcal{F}[af + bg] = a\,\mathcal{F}[f] + b\,\mathcal{F}[g]$$

so... $\quad \mathcal{F}\left[\frac{1}{x^2+1} + 3e^{-x^2}\right] = \mathcal{F}\left[\frac{1}{x^2+1}\right] + \mathcal{F}\left[3e^{-x^2}\right]$

③ Transformation of Partial Derivatives

For $u = u(x,t)$ and F.T. of x-var

Derivatives of $x$

$$\mathcal{F}[u_x] = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_x(x,t) e^{-i\xi x}\, dx = i\xi\,\mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_{xx}(x,t) e^{-i\xi x}\, dx = -\xi^2\,\mathcal{F}[u]$$

Deriv. of $t$

$$\mathcal{F}[u_t] = \cdots\ u_t\ \cdots dx = \partial/\partial t\ \mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \cdots = \partial^2/\partial t^2\ \mathcal{F}[u]$$

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④ Convolution property

since $\mathcal{F}[f(x)g(x)] \neq \mathcal{F}[f]\,\mathcal{F}[g]$

Is there a operation that does satisfy this?

yes → convolution

... convolution of $f$ with $g$ = $f * g$

and $\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g]$

Proof:
$$\mathcal{F}[f*g] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi\right] e^{-iwx}\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,e^{-iwx}\,dx\,d\xi$$

Let $p = x-\xi \Rightarrow x = p+\xi$
$$= \int_{-\infty}^{\infty} f(\xi)\left[\int_{-\infty}^{\infty} g(p)\,e^{-iw(p+\xi)}\,dp\right] d\xi$$

since $e^{-iw(p+\xi)} = e^{-iwp}\,e^{-iw\xi}$
$$\Rightarrow \mathcal{F}[f*g] = \left[\int_{-\infty}^{\infty} f(\xi)\,e^{-iw\xi}\,d\xi\right]\left[\int_{-\infty}^{\infty} g(p)\,e^{-iwp}\,dp\right]$$
$$= \mathcal{F}[f]\,\mathcal{F}[g] \quad \text{Q.E.D.}$$

*** CONVOLUTION IN TIME = MULTIPLICATION IN FREQUENCY

$$u(t) \longrightarrow \boxed{\text{LTI}} \xrightarrow{} y(t) \qquad \text{with } h(t) \text{ above LTI}$$

$\underline{y = h * u}$ CONVOLUTION METHOD TO SOLVE

other method..

$$\left.\begin{aligned} u(t) &\xrightarrow{FT} \bar{U}(w) \\ h(t) &\xrightarrow[FT]{} \bar{H}(w) \end{aligned}\right\} \bar{Y}(w) = \bar{H}(w)\,\bar{U}(w) \xrightarrow{FT^{-1}} y(t)$$

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## Solution of a Cauchy Problem

Heat flow in an infinite rod w/ initial temp. dist,

PDE $\quad u_t = \alpha^2 u_{xx} \quad \begin{matrix} -\infty < x < \infty \\ 0 < t < \infty \end{matrix}$ $\quad$ (No BC because infinite rod)

IC $\quad u(x,0) = \phi \quad -\infty < x < \infty$

① Transform (F.T. in x): $\quad \mathcal{F}[u_t] = \alpha^2 \mathcal{F}[u_{xx}]$

$$\mathcal{F}[u(x,0)] = \mathcal{F}[\phi(x)]$$

ODE
$$\Rightarrow \frac{d\bar{U}(t)}{dt} = -\alpha^2 \xi^2 \bar{U}(t)$$

$$\bar{U}(0) = \bar{\Phi}(\xi) \longleftarrow \text{IC}$$

② Solve ODE: $\quad \bar{U}(t) = \bar{\Phi}(\xi) e^{-\alpha^2 \xi^2 t} \longleftarrow$ Solution to 1st order ODE, always exponential, and the IC is always the coefficient

③ Inverse Transform:

$$u(x,t) = \mathcal{F}^{-1}[\bar{U}(\xi,t)] = \mathcal{F}^{-1}[\bar{\Phi}(\xi) e^{-\alpha^2 \xi^2 t}]$$

• can use conv. property here...

CONVOLUTION

$$u(x,t) = \mathcal{F}^{-1}[\phi(\xi)] * \mathcal{F}^{-1}[e^{-\alpha^2 \xi^2 t}]$$

HE WILL EMAIL EXAMPLE STEPS FOR THIS

$\downarrow \qquad \downarrow$ FROM TABLES (w/ ADJUSTMENTS)

$$\phi(x) * \frac{1}{\alpha\sqrt{2t}} e^{-(x-\xi)^2/4\alpha^2 t} \, d\xi$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/4\alpha^2 t} \, d\xi \qquad *$$

this is impulse response of system

this is input to system, initial cond

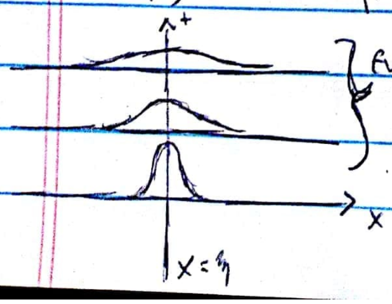
since $\quad f*g = \mathcal{F}^{-1}[\mathcal{F}[f]\mathcal{F}[g]]$

Let's look at solution...

A) $\phi(x)$ is initial temp (input)

B) $G(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} e^{-(x-\xi)^2/4\alpha^2 t}$ known as Green's func. (impulse resp function)

$G(x,t)$ is the temp. resp. to an initial temp impulse at $x = \xi$




Eventually damps out

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## Laplace Transform

$$\mathcal{L}[f] = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

$$\mathcal{L}^{-1}[F] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds$$

- Advantage over F.T. : damping factor $e^{-st}$
  FT limited to $L_2$ functions ($L_2$ = have limit : $\int |f^2|\,dt < M$) ← $L_2$ : square integrable, Hilbert space
  ↳ So LT can be used on wider ~~array~~ class of functions
- since the LT operates on funcs from $[0,\infty)$, it is applied mostly to time variable

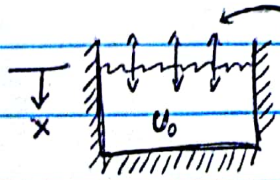
## Rules of Transformation (on time variable)

① $$\mathcal{L}[U_t] = s\bar{U}(x,s) - U(x,0)$$
$$\mathcal{L}[U_{tt}] = s^2\bar{U}(x,s) - sU(x,0) - U_t(x,0)$$
$$\mathcal{L}[U_x] = \frac{d\bar{U}(x,s)}{dx}$$
$$\mathcal{L}[U_{xx}] = \frac{d^2\bar{U}(x,s)}{dx^2}$$

② Finite convolution:
$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

and
$$\mathcal{L}[f*g] = \mathcal{L}[f]\,\mathcal{L}[g]$$
or
$$\mathcal{L}^{-1}\left[\mathcal{L}[f]\,\mathcal{L}[g]\right] = f*g$$

## Example : Heat conduction in a semi infinite medium




Heat flow $\begin{cases} U(0,t) > 0 \to \text{out} \\ U(0,t) < 0 \to \text{in} \end{cases}$

- Deep container, insulated at sides
- Air temp above liquid is zero
- Find temp dist as $f(t)$

→

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**PDE** $u_t = u_{xx}$ $\quad 0 < x < \infty \quad 0 < t < \infty$

**BC** $u_x(0,t) - u(0,t) = 0 \quad 0 < t < \infty$

**IC** $u(x,0) = u_0 \quad 0 < x < \infty$

① Transform time variable...

ODE : $$s\,\bar{U}(x) - u_0 = \frac{d^2\bar{U}(x)}{dx^2} \qquad 0 < x < \infty$$

BC : $$\frac{d\bar{U}(0)}{dx} = \bar{U}(0)$$

② Solve... $$\bar{U}(x) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{u_0}{s}$$

$\Rightarrow c_1 = 0$ (physics, grows forever)

$$\Rightarrow \bar{U}(x) = -u_0\left\{\frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s}+1)}\right\} + \frac{u_0}{s}$$

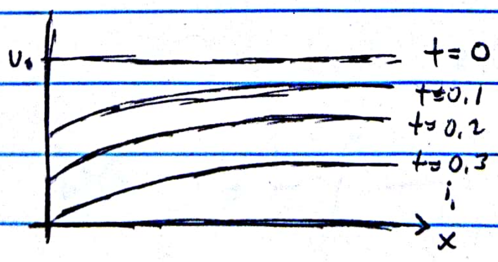
③ Inverse...

$$u(x,t) = \mathcal{L}^{-1}\left[\bar{U}(x,s)\right] \qquad \text{<USING TABLES>}$$

$$\Rightarrow u(x,t) = u_0 - u_0\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)e^{(x+t)}\right]$$

with: $$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\,d\xi$$

Solution looks like

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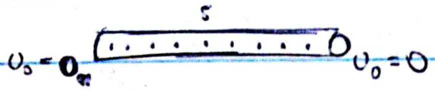
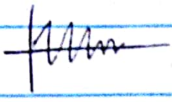
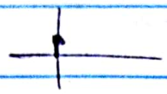
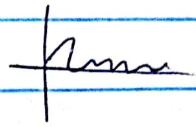


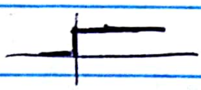
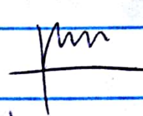

$u_1 = \sin(2\pi f_1(t))$

$u_1$ 

use unit input  to get ~~how~~ h at S 

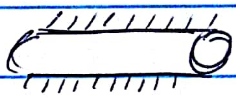
then $u_S = h_S * u_1$

OR

step  $\rightarrow$ LTI $\rightarrow$  S $\quad h = \frac{dS}{dt}$

$$h * u \iff S * \frac{du}{dt}$$

## Heat Flow in Rod (time varying BCs)

$u(t,0) = 0$  $u(1,t) = f(t)$

PDE : $u_t = u_{xx}$ $\quad 0 \le x \le 1 \quad 0 \le t \le \infty$

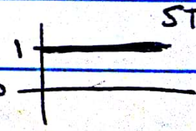
BCs : $u(0,t) = 0 \quad u(1,t) = f(t) \quad 0 < t < \infty$

IC : $u(x,0) = 0 \quad 0 \le x \le \infty$

Find sol. to simple prob. $\rightarrow$ constant temp on boundaries

constant BC : PDE $w_t = w_{xx}$ $\quad$ BCs $w(0,t) = 0$ $\quad$ STEP!

IC $w(x,0) = 0$ $\quad$ $w(1,t) = 1$




Solve and compare both... Laplace $\rightarrow$ $\frac{d^2W}{dx^2} - sW(x) = 0$

$$W(0) = 0, \quad W(1) = \frac{1}{s}$$

solve ODE...

$$W(x,s) = \frac{1}{s}\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$$

inverse...

$$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-1^n}{n} e^{-(n\pi)^2 t} \sin(n\pi x)$$

CONT. $\Delta \longrightarrow$

CONT.
A →

2018-02-13

Now solve harder prob...

Transform: $\frac{d^2\bar{U}}{dx^2} - s\bar{U}(x) = 0 \qquad \bar{U}(0) = 0 \qquad \bar{U}(1) = F(s)$

Solve ODE: $\bar{U}(x,s) = F(s)\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]$

But before taking inverse, multiply and divide by $s$

$$\bar{U}(x,s) = F(s)\left\{s\left[\frac{\sinh(x\sqrt{s})}{\sinh(\sqrt{s})}\right]\right\}$$

and with $\mathcal{L}[w_t] = sw - w(\cancel{x,0})^{0}$

$$\Rightarrow \bar{U}(x,s) = F(s)\,\mathcal{L}[w_t]$$

so that $u(x,t) = \mathcal{L}^{-1}\left\{F(s)\,\mathcal{L}[w_t]\right\} = f(t) * w_t(t)$

$$= \int_0^t f(\tau)\, w_t(x, t-\tau)\, d\tau$$

integrate by parts...

$$= \int_0^t \underbrace{w(x, t-\tau)}_{\substack{\downarrow \\ S,\ \text{step} \\ \text{resp.}}} \underbrace{f'(\tau)}_{\substack{\downarrow \\ \text{derivative} \\ \text{of} \\ \text{input}}} d\tau + \underbrace{f(0)\, w(x,t)}_{IC}$$

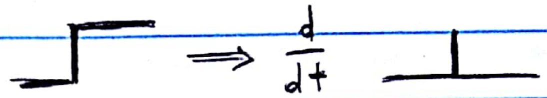
2018-02-15

$$u(x,t) = \int_0^t \underbrace{w(x,t-\tau)}_{\text{S, step response}} \underbrace{f'(\tau)}_{\text{derivative of input}} d\tau + f(0)\underbrace{w(x,t)}_{\text{IC}}$$

→ known as Duhamel's integral (convolution using step response)

$$\underset{\text{impulse response}}{h(t)} = \frac{d}{dt} \underset{\text{step response}}{S(t)}$$

step input



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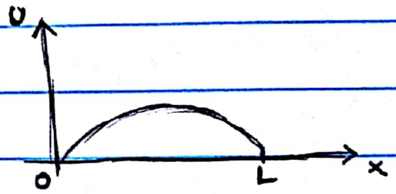
## Hyperbolic PDEs

### Vibrating String Problem




Apply $\vec{F} = ma$ to problem w/ simplifying assumptions...

**Assume:**

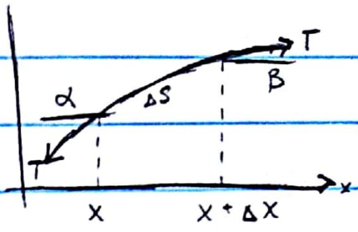
1) String rigidly attached at both endpoints
2) String vibrates in one plane
3) No external forces act on string
4) No damping forces affect string
5) String is homogeneous → linear density ($\rho$) and mass per unit length ($m$) are constant
6) Deflection of string from equilibrium point and slope are small: A) magnitude of tension force $T(x,t)$ in string is constant ⇒ $|\vec{T}(x,t)| = T$
   B) rigid longitudinally → motion in vertical direction only
7) tension force always tangential to string → known as perfectly flexible

→

2018-02-15

$\rightarrow$

To derive eq., look at small segment of string




- Many segment in $\rho \Delta s$, but for $|u| << 1$: $\Delta s \approx \Delta x$ $\Leftrightarrow$ mass $= \rho \Delta x$

- acceleration of segment is $u_{tt}$
- sum of vertical forces: $T \sin\beta - T \sin\alpha$

Combining $(m\vec{a})$

$$\Rightarrow (\rho \Delta x)\, u_{tt} = T(\sin\beta - \sin\alpha)$$

but small slopes $\Rightarrow$ $\sin\alpha \approx \tan\alpha = u(x,t)$

$\sin\beta \approx \tan\beta = u(x+\Delta x, t)$

$$\Rightarrow (\rho \Delta x)\, u_{tt} = T\big(u_x(x+\Delta x, t) - u_x(x,t)\big)$$

$$\rho u_{tt} = T\left[\frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}\right]$$

As $\Delta x \to 0$, $u_{tt} = c^2 u_{xx}$ with $c^2 = T/\rho$ 1-D wave eq.

If we include effect of additional forces,

1. External force, $F(x,t)$
   a) gravity $\Rightarrow F(x,t) = -mg$
   b) impulses along the string at different values of time
   c) 2-D wave eq. (vibrating drum head), sound waves impinging on surface of membrane
2. Frictional force (against string) $\Rightarrow -\beta u_t$
3. Restoring force $\Rightarrow -\gamma u$ (opposite to motion)

putting together $\Rightarrow$ $u_{tt} = c^2 u_{xx} - \beta u_t - \gamma u + F(x,t)$ "Telephone Eq."

(Note: Linear, no cross products of u)

$\rightarrow$

2018-02-15

**Comments:**

1. wave eq. also describes the longitudinal and torsional vibrations of rod: $u_{tt} = k u_{xx}$, $k \equiv$ Young's modulus (elasticity value for material)
2. For the case of a string w/ variable density $\rho(x)$
$$u_{tt} = \partial/\partial x \left[ c^2(x) u_x \right]$$
3. Wave eq. requires two ICs for unique solution (b.c. it is higher order)
$$u(x,0) = f(x) \quad \text{initial position}$$
$$u_t(x,0) = g(x) \quad \text{initial velocity}$$
4. Electric current and voltage along a wire

$$i_{xx} = CL\, i_{tt} + (CR + GL) i_t + GRi$$
$$v_{xx} = CL\, v_{tt} + (CR + GL) v_t + GRv$$

$C =$ capacitance per unit length
$G =$ leakage conductance per unit length
$R =$ resistance per unit length
$L =$ self inductance per unit length

"Transmission Line Equations"

- 2nd order hyperbolic, but become parabolic if C or L are zero
- And if $G = R = 0$, $i_{tt} = \alpha^2 i_{xx}$ and $v_{tt} = \alpha^2 v_{xx}$ $\left(\alpha^2 = \frac{1}{CL}\right)$

## D'Alembert's Solution of Wave Eq.

PDE $\quad u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty, \; 0 < t < \infty$

ICs $\quad u(x,0) = f(x) \quad -\infty < x < \infty$
$\quad\quad u_t(x,0) = g(x)$

- Solve the 1-D wave eq in free space:
- Could solve by FT or LT...

but we will look at method of canonical coordinates instead

**Step 1** Replace $(x,t)$ with new canonical coords. $(\xi, \eta)$
$$\xi = x + ct, \quad \eta = x - ct$$
$$\Rightarrow u_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_t = c(u_\xi - u_\eta) \qquad u_{tt} = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$\Rightarrow$ new PDE $\quad u_{\xi\eta} = 0$

2018-02-15

Step ② Solve new PDE $\quad u_\eta(\xi,\eta) = \phi(\eta)$ and $u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$

Step ③ Transform back to original coordinates $\quad \xi = x + ct, \quad \eta = x - ct$

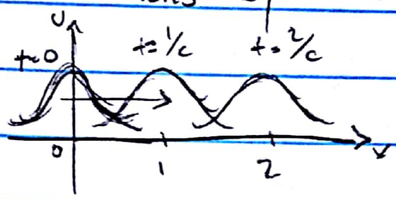
$$\Longrightarrow u(x,t) = \phi(x - ct) + \psi(x + ct)$$

General solution of wave eq.

- Represents the sum of any two moving waves, moving in opposite directions w/ velocity $c$.

ex:

$$u(x,t) = e^{-(x-ct)^2}$$




(propagation)

Step ④ Substitute into ICs, $\quad u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$$\Longrightarrow \phi(x) + \psi(x) = f(x)$$

$$-c\,\phi'(x) + c\,\psi'(x) = g(x)$$

Integrate 2nd eq, $\quad -c\,\phi(x) + c\,\psi(x) = \int_{x_0}^{x} g(\xi)\,d\xi + K$

Solve for each,

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi$$

$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\,d\xi$$

Finally,

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

D'Alembert's Solution (from ~1750)

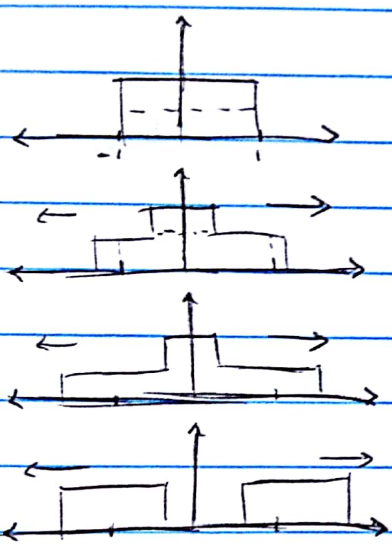
2018-02-15

Example - Motion of Square Wave:

IC: $u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$

$u_t(x,0) = 0$

Pulled Apart



If initial velocity defined

$u(x,0) = 0$

$u_t(x,0) = \sin(x)$

$$\Rightarrow u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(\eta)\, d\eta$$

$$= \frac{1}{2c} \left[ \cos(x+ct) - \cos(x-ct) \right]$$

→

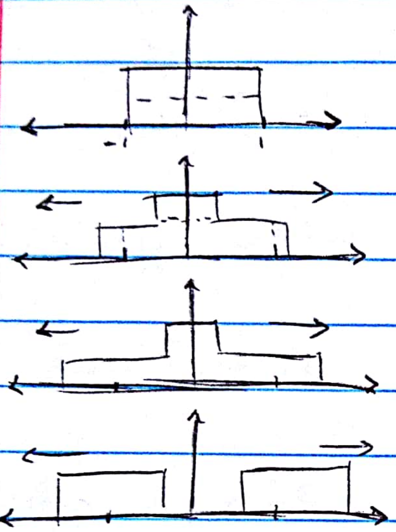
2018-02-18

Example – Motion of Square Wave:

IC: $u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$

$u_t(x,0) = 0$

Pulled Apart



If initial velocity defined

$u(x,0) = 0$

$u_t(x,0) = \sin(x)$

$$\Rightarrow u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(\eta)\,d\eta$$

$$= \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

2018-02-20

1-D Problem in Infinite Domain (Another D'Alembert example)

PDE: $u_{tt} - u_{xx} = p(x,t)$ $\quad -\infty < x < \infty$, $t \geq 0$

ICs: $u(x,0) = f(x)$ $\quad u_t(x,0) = g(x)$

with $p = 0$ (homogeneous) represents,

- perturbation velocity (or surface height) for small amplitude water waves
- perturbation velocity or density for small amp. disturbance in a 1-D compressible gas
- lateral or axial displacement of a vibrating string in tension in the limit of small amp. oscillation

→

2018-02-20

→ • <u>Fundamental</u> <u>Solution</u>:

$$\underline{\text{PDE}}: \quad u_{tt} - u_{xx} = \delta(x)\delta(t)$$

$$\underline{\text{ICs}}: \quad u(x,0) = u_t(x,0) = 0$$

Consider equivalent homogeneous equation...

$$\underline{\text{PDE}} \quad u_{tt} - u_{xx} = 0 \qquad \underline{\text{ICs}} \quad \begin{array}{l} u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{array}$$

Can apply D'Alembert's solution... $u(x,t) = \phi(x+t) + \psi(x-t)$

– Apply first IC: $\phi(x) + \psi(x) = 0 \Rightarrow \psi = -\phi$

$$u(x,t) = \phi(x+t) - \phi(x-t)$$

– Apply second IC: $u_t(x,0) = \delta(x)$

$$\phi'(x) - (-\phi'(x)) = \delta(x)$$

$$2\phi'(x) = \delta(x)$$

Integrate. Recall that, $\int \delta(x)\,d(x) = H(x) \equiv$ Heaviside Function (step function)

so...

$$\phi(x) = \frac{1}{2} H(x)$$

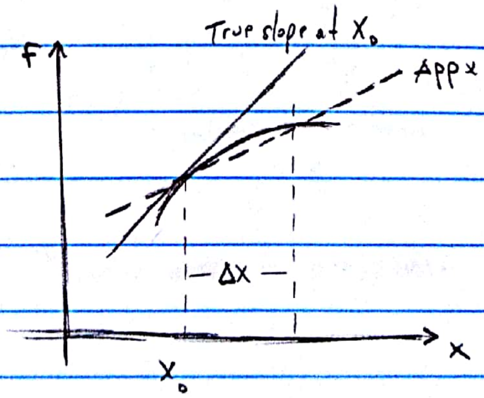
$$\Rightarrow u(x,t) = \frac{1}{2}\left[H(x+t) - H(x-t)\right] \quad \text{Green's Function}$$

2018-02-20

# Numerical Methods for PDEs

- Can be used to solve linear OR nonlinear PDEs (D'Alembert's is linear only)
- Basic idea is to model derivatives by finite differences
- Entire solution field must be discretized
- Usually need a large # of mesh points
- Leads to very large system of equations ($\rightarrow$ matrices linear algebra)
- Computational Fluid dynamics (CFD) - very important technology
- Solution process includes:
  - grid/mesh generation
  - "Flow field" discretization algorithms (doesn't have to be fluids/flow - stress, strain, etc.)
  - efficient solution of large system of eq.s
  - massive data storage and transmission technology methods
  - computational flow visualization (Tecplot) ex. plot software

Many ways to obtain finite difference representation of derivative.




Use Taylor Series...

$$f(x_0 + \Delta x) = f(x_0) + \Delta x \left.\frac{df}{dx}\right|_{x_0} + \frac{(\Delta x)^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x_0} + \frac{(\Delta x)^3}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x_0} + \ldots$$

$\rightarrow$

$\longrightarrow$

2018-02-20

Solve for $\left.\frac{df}{dx}\right|_{x_0}$ ...

forward diff

$$\left.df/dx\right|_{x_0} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - \Delta x\left(\frac{1}{2}\right)\left.\frac{d^2f}{dx^2}\right|_{x_0} - \dots$$

or $$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} + \underbrace{\Theta(\Delta x)}_{\text{Truncation Error}}$$

- Truncation error means that the error of the approximation vanishes as $\Delta x$ goes to zero.
- When T.E. is $\Theta(\Delta x) \rightarrow 1^{st}$ order accurate
  - $\Rightarrow$ this form known as a $1^{st}$ order, one-sided, forward difference, approximation
- If we expand using info prior to (behind) point of interest,

$$f(x_0-\Delta x) = f(x_0) - \Delta x\left.\frac{df}{dx}\right|_{x_0} + \frac{(\Delta x)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x_0} - \frac{(\Delta x)^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x_0} + \dots$$

backwards diff

or $$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0)-f(x_0-\Delta x)}{\Delta x} + \Theta(\Delta x)$$

$\longrightarrow$ $1^{st}$ order accurate, one-sided, backward difference

- If TE can be reduced, we can use larger steps or less grid points (Faster to calculate)
- If we subtract forward and backward Taylor Series

$$f(x_0+\Delta x) - f(x_0-\Delta x)$$

$\longrightarrow$

$$f(x_0+\Delta x) - f(x_0-\Delta x) = 2\Delta x \left.\frac{df}{dx}\right|_{x_0} + \frac{(\Delta x)^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x_0} + \dots$$

- The $\Theta(\Delta x)$ terms cancel when subtracted
- Solving for $\left.\frac{df}{dx}\right|_{x_0}$ ...

$$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2(\Delta x)} + \Theta(\Delta x)^2$$

2nd order central difference formula

- Can generate finite difference (f.d.) approx. to the 2nd derivative.
- Adding Taylor's series for forward and backward expansions (odd terms cancel out)

$$f(x_0+\Delta) + f(x_0-\Delta x) = 2f(x_0) + (\Delta x)^2\left.\frac{d^2f}{dx^2}\right|_{x_0} + \Theta(\Delta x)^2$$

or...

$$\left.\frac{d^2f}{dx^2}\right|_{x_0} = \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{(\Delta x)^2} + \Theta(\Delta x)^2$$

- for partial derivatives, using $(i,j)$ notation..

$$\frac{\partial f}{\partial x} = \frac{f_{i+1,j} - f_{i,j}}{\Delta x} + \Theta(\Delta x)$$ 1st order forward diff

$$\frac{\partial f}{\partial x} = \frac{f_{i,j} - f_{i-1,j}}{\Delta x} + \Theta(\Delta x)$$ 1st order backwards diff

$$\frac{\partial f}{\partial x} = \frac{f_{i+1,j} - f_{i-1,j}}{2(\Delta x)} + \Theta(\Delta x)^2$$ 2nd order central diff.

→

2018-02-20

→

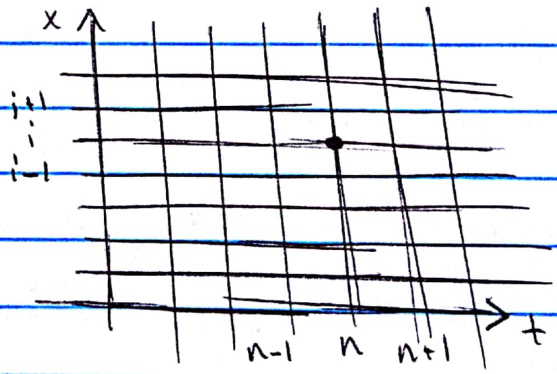
... and the $2^{nd}$ deriv. is $\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta x)^2} + \Theta(\Delta x)^2$

* IMPORTANT

1. Physical features of PDE should so reflected in the numerical approach
2. So, the type of PDE can determine the numerical method to use

- Apply to heat equation... $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

  - discretized using a forward difference in time and a central diff. in space




Written as...

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + TE$$

$$TE = \Theta[\Delta t, (\Delta x)^2]$$

This is discretization.

This is an explicit scheme (depends on inputs that have already happend on RHS, not future states)

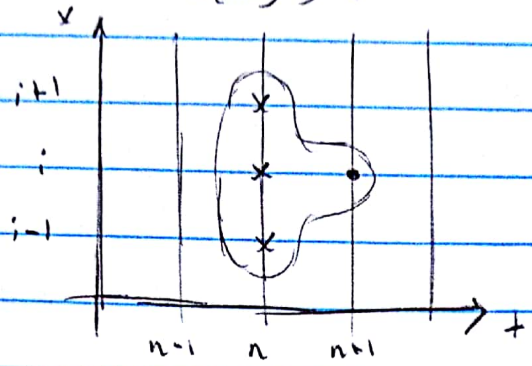
because solution at time n is known, so only one unknown at time n+1 (on LHS)

→

2018-02-20

or...

$$u_i^{n+1} = u_i^n + \alpha\left(\frac{\Delta t}{(\Delta x)^2}\right)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$




"Stencil" for this method

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2018-02-22

**EXPLICIT**

- Algebra is simple
- Stability requires very small step sizes
- Easy to implement in parallel computations

~ Consider alternate scheme...

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

- Need to solve simultaneously for all values at $n+1$

$\Rightarrow$ matrix (linear algebra)

- is an implicit method
- Solution of a system of eqs. required at each step
- Stability reqs. allow larger step size
- Harder to parallelize

Define:

$$\lambda = \alpha\frac{\Delta t}{(\Delta x)^2}$$

Can rewrite eq. as

$$-\lambda u_{i-1}^{n+1} + (1+2\lambda)u_i^{n+1} - \lambda u_{i+1}^{n+1} = u_i^n$$

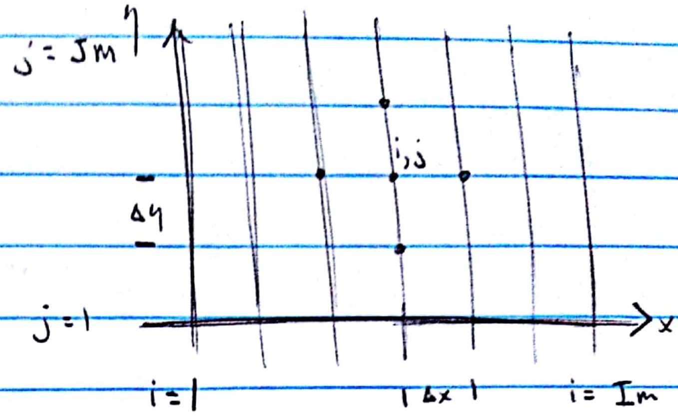
In matrix form... $n = 1, 2, \ldots, N$

$$\begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & (1+2\lambda) & -\lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & (1+2\lambda) \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \\ u_N^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n \\ u_N^n \end{bmatrix}$$

Tridiagonal form

2018-02-22

## Nomenclature / Terminology




1) **Consistency** - a finite difference (f.d.) approximation of a PDE is consistent if the f.d. eq approaches the PDE as the grid size approaches zero.

2) **Stability** - a numerical scheme is said to be stable if any error introduced in the f.d. eq. does not grow with the solution of f.d. equation

3) **Convergence** - A f.d. scheme is convergent if a solution of a f.d. eq. approaches that of the PDE as the grid size approaches zero (or temporal size)

4) Lax's Equivalence Theorem - For a FDE (finite difference eq) which approximates a well-posed, linear, initial value problem, the necessary & sufficient condition for convergence is that the FDE must be stable and consistent.

Example ⟶

2018-02-22

# Dirichlet Problem (Laplace's Eq)

Elliptic (no time)
↳ Steady State
∴ No ICs

PDE $\quad U_{xx} + U_{yy} = 0 \qquad 0 < x < 1, \; 0 < y < 1$

BCs $\quad U = 0$ on top, sides of square

$U(x,0) = \sin(\pi x) \qquad 0 \le x \le 1$

Using central diff. formula...

$$\nabla^2 U = \frac{1}{h^2}\left(U_{i,j+1} - 2U_{i,j} + U_{i,j-1}\right)$$

$$+ \frac{1}{k^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right) = 0$$

If $h = k$:

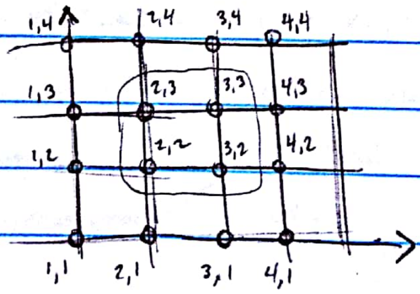
$$\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}\right) = 0$$

OR

$$U_{i,j} = \frac{1}{4}\left(U_{i-1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right)$$

AVG OF FOUR NEIGHBORING GRID POINTS *

Write eqs. for 4 grid pts:




Edges are known by boundary conditions

Interior are unknown

For (2,2):

$$-4U_{22} + U_{32} + U_{12} + U_{23} + U_{21} = 0$$

$$-4U_{22} + U_{32} + 0 + U_{23} + \sin(\pi/3) = 0$$

(3,2) $\quad -4U_{32} + U_{42} + U_{22} + U_{33} + U_{31} = 0$

$$-4U_{32} + 0 + U_{22} + U_{33} + \sin\left(\frac{2\pi}{3}\right) = 0$$

(2,3) $\quad -4U_{23} + U_{33} + U_{13} + U_{24} + U_{22} = 0$

$$-4U_{23} + U_{33} + 0 + 0 + U_{22} = 0$$

(3,3) $\quad -4U_{33} + U_{43} + U_{23} + U_{34} + U_{32} = 0$

$$-4U_{23} + 0 + U_{23} + 0 + U_{32} = 0$$

2017-02-22

So...

$$-4U_{22} + U_{23} + U_{32} + \sin(\pi/3) = 0$$
$$-4U_{23} + U_{33} + U_{22} = 0$$
$$-4U_{32} + U_{22} + U_{33} + \sin(2\pi/3) = 0$$
$$-4U_{33} + U_{23} + U_{32} = 0$$

or

$$\underbrace{\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_{22} \\ U_{23} \\ U_{32} \\ U_{33} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -\sin(\frac{\pi}{3}) \\ 0 \\ -\sin(\frac{2\pi}{3}) \\ 0 \end{bmatrix}}_{b}$$

Condition Number of Matrix: measure of "goodness" or condition of a matrix

MATLAB> cond(A)

cond(A) = 1 is best possible, larger values = ill-conditioned, eventually becoming singular (bad) (determinant = 0)

best linear independence

loses linear independence

Solve via Implicit Method (Heat Eq)

PDE: $U_t = U_{xx}$ $\quad 0<x<1$, $0<t<\infty$

BCs: $\begin{cases} U(0,t) = 0 \\ U(1,t) = 0 \end{cases}$ $\quad 0<t<\infty$

ICs: $U(x,0) = 1 \quad 0 \le x \le 1$

$\lambda$ can be between [0,1] ⇒ weighted avg of central diff approx.

So,

$$U_t(x,t) = \frac{1}{k}\left(U(x,t+k) - U(x,t)\right)$$

$$U_{xx}(x,t) = \frac{\lambda}{h^2}\left[U(x+h,t+k) - 2U(x,t+k) + U(x-h,t+k)\right] + \frac{(1-\lambda)}{h^2}\left[U(x+h,t) - 2U(x,t) + U(x-h,t)\right]$$

when $\lambda = 0.5$, standard avg

$\lambda > 0.5$, more emphasis on $t+k$

$\lambda = 0$ is standard explicit method

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2/27/2018. Tuesday.

Combining

$$\frac{1}{k}(U_{i+1,j} - U_{i,j}) = \frac{\lambda}{k^2}(U_{i+1,j+1} - 2U_{i+1,j} + U_{i+1,j-1}) + \frac{(1-\lambda)}{k^2}(U_{i,j+1} - 2U_{i,j} + U_{i,j+1})$$

BCs: $\begin{cases} U_{ij} = 0 \\ U_{i,n} = 0 \end{cases}$ $i = 1, 2 \cdots m$

IC. $U_{ij} = 1$ $j = 2, 3 \ldots n-1$

or. $-\lambda r_{i+1,j+1} + (1+2r\lambda)U_{i+1,j} - \lambda r U_{i+1,j-1}$

$= r(1-\lambda)U_{i,j+1} + [1-2r(1-\lambda)]U_{ij} + r(1-\lambda)U_{i,j-1}$

with $r = \frac{k}{h^2}$

Solve as follows:

① select $\lambda \Rightarrow \lambda = 0.5$ ⇒ Crank-Nicolson method

② select $h = \Delta x = 0.2$ and $k = \Delta t = 0.08$ $r = \frac{k}{h^2} = 2$

⇒ six grid points in (four interior ~~grid~~ grid points)

$$\Rightarrow -U_{21} + 3U_{22} - U_{23} = U_{11} - U_{12} + U_{13} = 1$$

$$-U_{22} + 3U_{23} - U_{24} = U_{12} - U_{13} + U_{14} = 1$$

$$-U_{23} + 3U_{24} - U_{25} = U_{13} - U_{14} + U_{15} = 1$$

$$-U_{24} + 3U_{25} - U_{26} = U_{14} - U_{15} + U_{16} = 1$$

$$\Downarrow \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} U_{22} \\ U_{23} \\ U_{24} \\ U_{25} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

orthogonal = linear independent (Gram Schmidt ⇒ turn matrix to orthogonal

go: is not gonna be singular

tridiagonal

* inverse (~~of~~ orthogonal) = transpose (matrix)

Other explicit methods:

FTCS (forward in time, central in space)

$$u_t = \alpha u_{xx}$$

$$u_i^{n+1} = u_i^n + \frac{\alpha(\Delta t)}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \quad [(\Delta t)(\Delta x)^2]$$

errors

Stable for $\frac{\alpha(\Delta t)}{(\Delta x)^2} \le \frac{1}{2}$

Richardson method

Central differences for both time and space (CTCS)

$$\frac{u_i^{n+1} - u_i^{n-1}}{2(\Delta t)} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, \quad [(\Delta t)^2, (\Delta x)^2]$$

unconditionally unstable

DuFort-Frankle method

central difference in time and central difference in space, with an averaged $u_i^n$

$$\frac{u_i^{n+1} - u_i^{n-1}}{2(\Delta t)} = \alpha \frac{u_{i+1}^n - 2\left(\frac{u_i^{n+1} + u_i^{n-1}}{2}\right) + u_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow u_i^{n+1} = u_i^{n-1} + \frac{2\alpha(\Delta t)}{(\Delta x)^2}\left[u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n\right]$$

can solve explicitly

$$\left[1 + \frac{2\alpha(\Delta t)}{(\Delta x)^2}\right]u_i^{n+1} = \left[1 - 2\frac{\alpha(\Delta t)}{(\Delta x)^2}\right]u_i^{n-1} + \frac{2\alpha(\Delta t)}{(\Delta x)^2}\left[u_{i+1}^n + u_{i-1}^n\right.$$

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$\left[(\Delta t)^2, (\Delta x)^2, \left(\frac{\Delta t}{\Delta x}\right)^2\right]$ unconditionally stable

(does not mean unconditionally accurate. define n &

## Numerical methods for hypobolic ~~##~~ equations.

methods of characteristics: valuable but difficult to use for 3-D and nonlinear problems.

1st order wave eq: $\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$   $a > 0$

Explicit for solution

Euller's FTFS method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

unconditionally unstable

Euller's FTCS method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_{i+1}^n - u_{i-1}^n}{2(\Delta x)}$$

Stable when $C \le 1$, $C = \frac{a(\Delta t)}{\Delta t} \equiv$ courant number

## Lax method

Using an average value of $U_i^n$ in Euler's FTCS method

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{a(\Delta t)}{2(\Delta x)}(U_{i+1}^n - U_{i-1}^n)$$

method is stable when $C \le 1$

## Mid-point leapfrog? method

$$\frac{U_i^{n+1} - U_i^{n-1}}{2(\Delta t)} = -a\frac{U_{i+1}^n - U_{i-1}^n}{2(\Delta x)} \quad O[(\Delta t)^2, (\Delta x)^2]$$

Stable when $C \le 1$

- Need two sets of initial data, $n$, $n-1$
- Need to use a ~~stable~~ starter solution → affect order of $\square$

Two independant solutions may ~~be~~ develop.

## Lax-Wandroff method

Derived from Taylor's Series.

$$U(x, t+\Delta t) = U(x,t) + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^2$$

or

$$U_i^{n+1} = U_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{(\Delta t)^2}{2!} + O(\Delta t)^3$$

consider for 1-D wave eq.

Taking time derivatives

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = a^2\frac{\partial^2 u}{\partial x^2}$$

substituting into Taylor

$$u_i^{n+1} = u_i^n - \left(-a\frac{\partial u}{\partial x}\right)\Delta t + \frac{(\Delta t)^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2}\right)$$

using central difference for spatial term.

$$u_i^{n+1} = u_i^n - a(\Delta t)\left[\frac{u_{i+1}^n - u_{i-1}^n}{2(\Delta x)}\right] + \frac{1}{2}a^2(\Delta t)^2\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

Stable for $c \leq 1$

$O[(\Delta t)^2, (\Delta x)^2]$

Implicit for solution (need a matrix solution)

Euler's BTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2(\Delta x)}\left[u_{i+1}^{n+1} - u_{i-1}^{n+1}\right] \text{ or}$$

$$\frac{1}{2}c\,u_{i-1}^{n+1} - u_i^{n+1} - \frac{1}{2}c\,u_{i+1}^{n+1} = -u_i^n$$

unconditionally stable ! $O[(\Delta t), (\Delta x)^2]$

$\Rightarrow$ yields a tridiagonal coefficient matrix

## Crank-Nicolson method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \cdot \frac{1}{2}\left[ \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2(\Delta x)} + \frac{u_{i+1}^n - u_{i-1}^n}{2(\Delta x)} \right]$$

$$O[(\Delta t)^2, (\Delta x)^2]$$

$\Rightarrow$ tridiagonal matrix

## Comments

- extending methods to greater dimensions not straightforward.
- Implicit methods for multidimensional could result in very large matrix.
- more complex problem $\Rightarrow$ Alternating Direction ~~Implid~~ Implicit (ADI), Approximate Factorization (AF)
- other issues is nonlinearity
- ~~Multi~~ multi-step methods (split time levels) $\rightarrow$ predictor $\boxed{\phantom{xx}}$ methods

## ~~Note~~ Mac-Cormak method

First eq $\rightarrow$ forward diff

$$\frac{u_i^* - u_i^n}{\Delta t} = -a \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

$$\frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\frac{1}{2}(\Delta t)} = -a \frac{u_i^* - u_{i-1}^*}{\Delta x}$$

the value for $u_i^{n+\frac{1}{2}}$ is replaced by an average

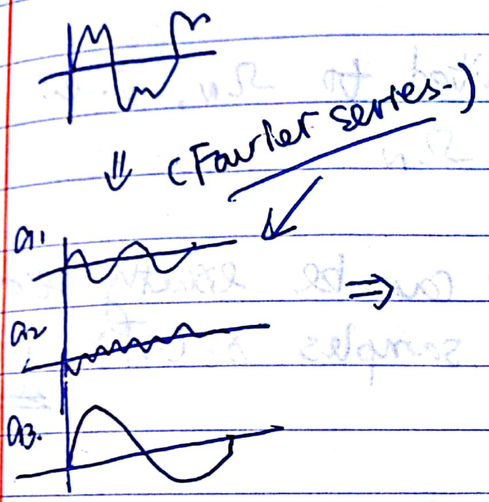
$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_i^{n} + u_i^{*}\right)$$

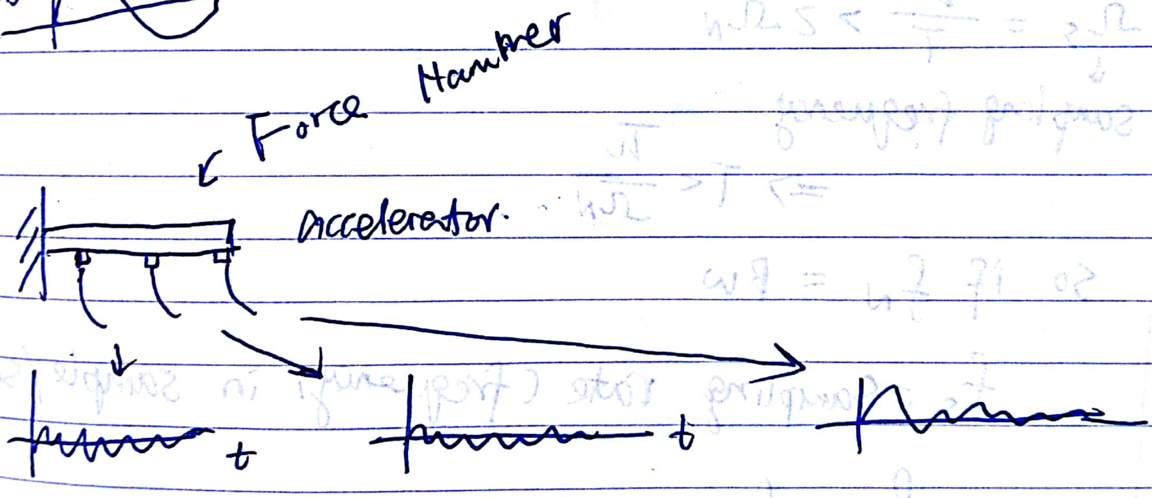
So. predictor step:

$$u_i^{*} = u_i^{n} - \frac{a(\Delta t)}{\Delta t}\left(u_{i+1}^{n} - u_i^{n}\right)$$

corrector step: $u_i^{n+1} = \frac{1}{2}\left[\left(u_i^{n} + u_i^{*}\right) - \frac{a(\Delta t)}{\Delta x}\left(u_i^{*} - u_{i-1}^{*}\right)\right]$

second order accurate: $\frac{a(\Delta t)}{\Delta x} \le 1$

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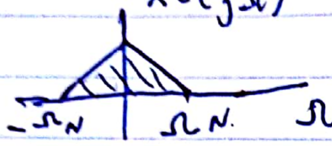
Nyquist Sampling

Let $x_c(t)$ be limited to $\Omega_N$, i.e.

$X_c(j\Omega) = 0$ for $\Omega > \Omega_N$




the $x_c(t)$ can be exactly reconstructed from its samples $x_c(nT)$ provided

$\Omega_s = \frac{2\pi}{T} > 2\Omega_N$

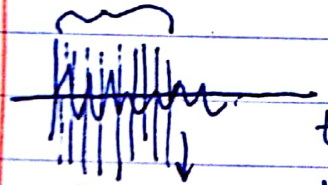
sampling frequency

$\Rightarrow T < \frac{\pi}{\Omega_N}$

so if $f_N = BW$

$f_s$ = Sampling rate (frequency) in sample/sec

$f_s > 2f_N \rightarrow$ Nyquist frequency



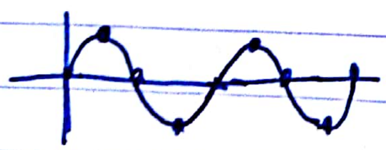

me → $\Delta t = 0.1$ . $\frac{1}{\Delta t}$ = sampling frequency .
step.

If the samples are not enough #, then it can not reflect the real ~~sig~~ ~~signal~~ signal

⇓

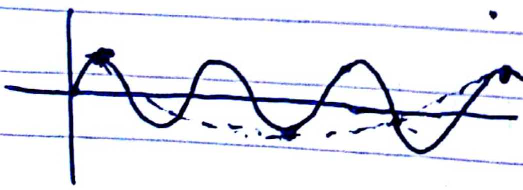
alising

eg:



$f = 1$ Hz.

sample . $f_s = 4$ Hz. ✓



$f_s = \frac{4}{3}$ Hz

sine ~~wave~~ wave @ $f = \frac{1}{3}$ Hz time function → frequency function

Matlab.

fft.

time = (0: 0.1 : 10)

~~y=2sint~~ y = 2 sin(2 * pi * 5 * time)

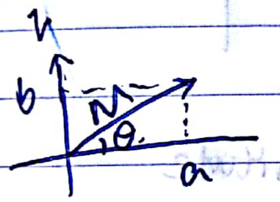
plot (time, y)

yft = fft(y).

↑

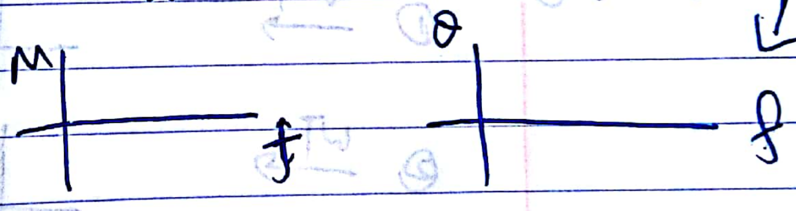
complex function $\int f(t) e^{-iwt} dt . \rightarrow a+ib.$

↑ real



$M = \sqrt{a^2+b^2} \Rightarrow$ abs(yft)

$\theta = \tan^{-1}\frac{b}{a} \Rightarrow$ phase = angle(yft)



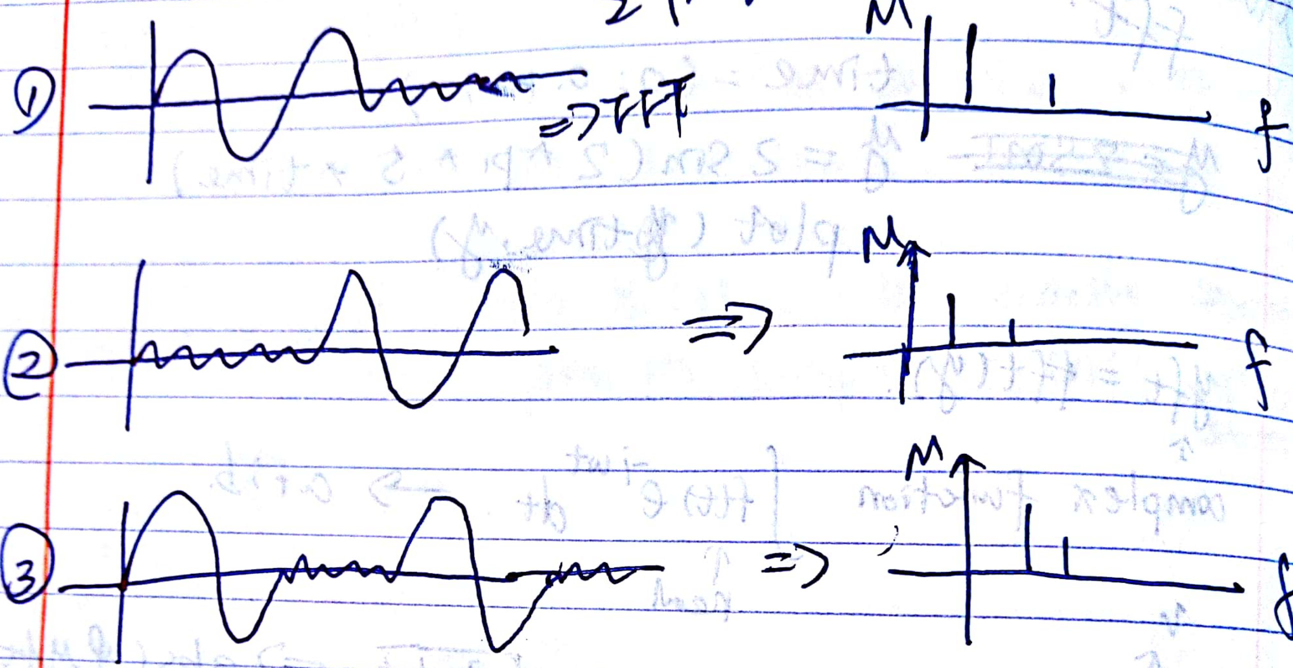
$\Delta f = \frac{1}{N(\Delta t)}$

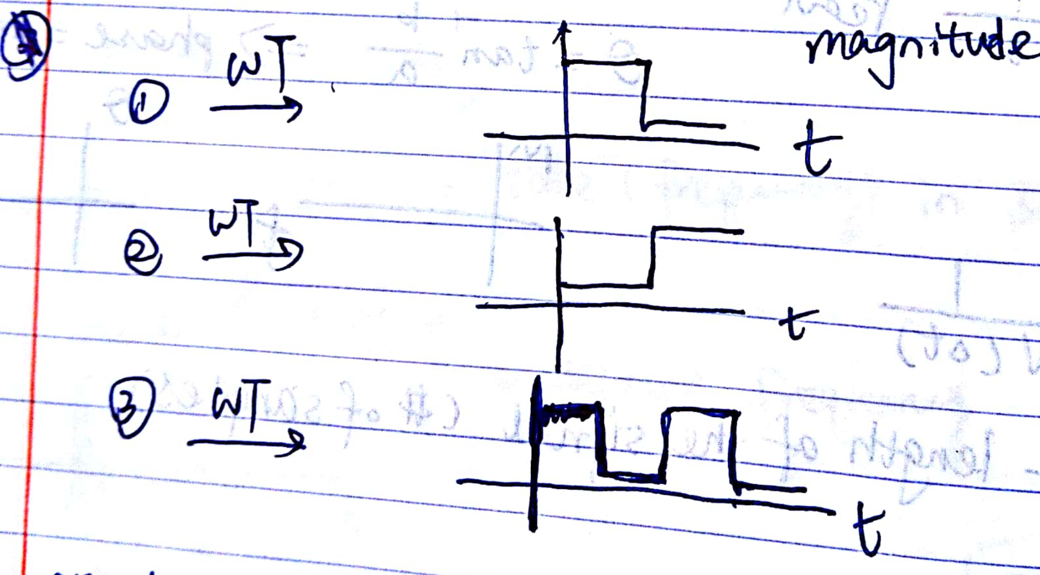
- N - length of the signal (# of samples).

freq = (0: df : fN)

$N_2 = \frac{N}{2}$

2 frequencies (1, 5)




create a signal that compose multiple sine.

$\sin(2\pi f_1 t) + 2\sin(2\pi f_2 t) + 7\sin(\varphi_2 \pi f_2 t)$