Cayley Neumaier graphs with a spread given by the cosets of a subgroup

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based on joint work in progress with
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Definitions

We consider undirected graphs without loops and multiple edges.

A k-regular graph on v vertices is called edge-regular with parameters (v, k, λ) if every pair of adjacent vertices has λ common neighbours.

An edge-regular graph with parameters (v, k, λ) is called strongly regular with parameters (v, k, λ, μ) if every pair of distinct non-adjacent vertices has μ common neighbours.

A clique in a regular graph is called m-regular if every vertex that doesn't belong to the clique is adjacent to precisely m vertices from the clique. For an m-regular clique, the number m is called the nexus.

A question by Neumaier

For the clique number $\omega(\Gamma)$ of a strongly regular graph Γ, the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of Γ.

A clique in a strongly regular graph is regular if and only if it has $1 - \frac{k}{\theta_{\min}}$ vertices; such a clique is called a Delsarte clique.

In 1981, Neumaier proved [1] that an edge-regular graph which is vertex-transitive, edge-transitive, and has a regular clique is strongly regular.

Neumaier then asked: "**Is it true that every edge-regular graph with a regular clique is strongly regular?**"

[1] A. Neumaier, *Regular Cliques in graphs and Special $1\frac{1}{2}$-designs*, Finite Geometries and Designs, London Mathematical Society Lecture Note Series, 245–259 (1981).

Neumaier graphs

A non-complete edge-regular graph with parameters (v, k, λ) containing an m-regular s-clique is said to be a Neumaier graph with parameters $(v, k, \lambda; m, s)$.

It follows that if a Neumaier graph with parameters $(v, k, \lambda; m, s)$ has an m-regular clique of size s, then all cliques of size s in this graph are m-regular.

Thus, the notion of a Neumaier graph is a generalisation of the notion of a strongly regular graph with a Delsarte clique.

For a Neumaier graph with parameters $(v, k, \lambda; m, s)$, the number m is called the nexus of this graph.

A Neumaier graph that is not strongly regular is said to be a strictly Neumaier graph.

For a Neumaier graph, a spread is a partition of the vertex set into regular cliques.

Outline

- Strictly Neumaier graphs with nexus 1
  - A first construction by Greaves & Koolen;
  - Another construction by Greaves & Koolen;
  - A generalisation of Greaves & Koolen's constructions and application of the Wang-Qiu-Hu switching to it;
  - An infinite class of strictly Neumaier graphs based on the general construction
- Strictly Neumaier graphs with nexus greater than 1
  - Determination of the smallest strictly Neumaier graph and a construction of strictly Neumaier graphs with 2^i-regular cliques, for every positive integer i;
- Cayley Neumaier graphs with a spread given by the cosets of a subgroup
  - Necessary and sufficient conditions;
  - Algorithm for enumeration and numerical results.

The first construction of strictly Neumaier graphs

In [2], Greaves and Koolen constructed an infinite family of strictly Neumaier graphs with 1-regular cliques.

For positive integers ℓ, m and an odd prime power q, consider the group $G_{\ell,m,q} := \mathbb{Z}_\ell \oplus \mathbb{Z}_2^m \oplus \mathbb{F}_q$. Put

$$S_0 := \{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m, (x, y) \neq (0, 0)\}$$

Let $\pi : \mathbb{Z}_2^m \setminus \{0\} \to \{0, \ldots, 2m - 2\}$ be a bijection and ρ be a primitive element of $\mathbb{F}_q$.

For each $y \in \mathbb{Z}_2^m \setminus \{0\}$, define

$$S_{y,\pi} := \{(0, y, \rho^j) \mid \pi(y) \equiv j \ (mod\ 2^m - 1)\}$$

Consider the parametrised Cayley graph $Cay(G_{\ell,m,q}, S(\pi))$, where

$$S(\pi) := S_0 \cup \bigcup_{y \in \mathbb{Z}_2^m \setminus \{0\}} S_{y,\pi}$$

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

The first construction of strictly Neumaier graphs

Let $q = 2nr + 1$ for some positive integer r. For each $i \in \{0, \dots, n-1\}$, define the cyclotomic class

$$C_q^n(i) := \{\rho^{nj+i} \mid j \in 0, \dots, 2r-1\}.$$

For $a, b \in \{0, \dots, n-1\}$, define the cyclotomic number

$$c_q^n(a, b) := |C_q^n(a) + 1 \cap C_q^n(b)|.$$

Put $c := c_q^n(a, b)$ and $\ell := (1 + c)/2$.

Theorem ([2, Theorem 3.6, Corollary 4.4])

Let $q \equiv 1 \pmod 6$, c be odd and $\pi : \mathbb{Z}_2^2 \setminus \{0\} \to \{0, 1, 2\}$ be a bijection. Then $Cay(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters $(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Notes on the first construction

- Set $q := 7^a$, where $a \not\equiv 0 \ (mod\ 3)$. Then $Cay(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters

  $$(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell).$$

  In particular, if $a = 1$, then we have a strictly Neumaier graph with parameters $(28, 9, 2; 1, 4)$. This graph is the smallest example from [2].
- $Cay(G_{\ell,2,q}, S(\pi))$ has a spread of size q given by the cosets of the subgroup $\{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m\}$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Four strictly Neumaier graphs on 24 vertices

Gavrilyuk and Goryainov then searched for examples in a collection of known Cayley-Deza graphs [3] and found four more strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$.

In [4], Greaves and Koolen found 'another' infinite family of strictly Neumaier graphs, which contains one of the four graphs on 24 vertices.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

Antipodal distance-regular graphs

A graph Γ of diameter d is called distance-regular if, for any two vertices $x, y \in V(\Gamma)$, the number of vertices at distance i from x and distance j from y depends only on i, j, and the distance from x to y. It is clear that distance regular graphs are edge-regular.

A distance-regular graph Γ of diameter d is called a-antipodal if the relation of being at distance d or distance 0 is an equivalence relation on the vertices of Γ with equivalence classes of size a.

The second construction of strictly Neumaier graphs

Let Γ be an a-antipodal distance-regular graph of diameter 3 with edge-regular parameters (v, k, λ) such that a is a proper divisor of $\lambda + 2$.

Put $t = \frac{\lambda+2}{a}$ and take t disjoint copies $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ of Γ.

For every antipodal class H in Γ, take the corresponding antipodal classes $H^{(1)}, \ldots, H^{(t)}$ in $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$, respectively, and connect any two vertices from $H^{(1)} \cup \ldots \cup H^{(t)}$ to form a 1-regular clique of size at.

Denote by $F_t(\Gamma)$ the resulting graph.

Theorem ([4])

The graph $F_t(\Gamma)$ is a strictly Neumaier graph having parameters $(tv, k + at - 1, \lambda; 1, at)$ and containing a spread.

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

Notes on the second construction

- In particular, if Γ is the icosahedron, then $a = 2$, $\lambda = 2$, $t = 2$ and $F_2(\Gamma)$ is one of the four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ found in [3].
- The other three graphs can be obtained in a similar way by choosing an appropriate matching of the antipodal classes in the two copies of the icosahedrons.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

Perfect codes in graphs

Let Γ be a simple undirected graph and $e \geq 1$ be an integer.

The ball with radius e and centre $u \in V(\Gamma)$ is the set of vertices of Γ with distance at most e to u in Γ.

A subset C of $V(\Gamma)$ is called a perfect e-code in Γ if the balls with radius e and centres in C form a partition of $V(\Gamma)$.

In particular, a perfect 1-code is a subset of vertices C such that every vertex not in C is adjacent to a unique element of C.

A generalisation of the two constructions

Let $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ be edge-regular graphs with parameters (v, k, λ), and such that each $\Gamma^{(i)}$ has a partition of its vertices into perfect 1-codes of size a, where a is a proper divisor of $\lambda + 2$. Further, we define $t = (\lambda + 2)/a$.

For any $\ell \in \{1, \ldots, t\}$, let $H_1^{(\ell)}, \ldots, H_{v/a}^{(\ell)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(\ell)}$.

If $t \geq 2$, we also take a $(t-1)$-tuple of permutations from $\mathrm{Sym}(\{1, \ldots, v/a\})$, denoted by $\Pi = (\pi_2, \ldots, \pi_t)$.

Using these graphs and the tuple Π, we define the graph $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ as follows.

1. Take the disjoint union of the graphs $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$.
2. For any $i \in \{1, \ldots, v/a\}$, add an edge between any two distinct vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \ldots \cup H_{\pi_t(i)}^{(t)}$ (which forms a 1-regular clique of size at).

A generalisation of the two constructions

Theorem ([5])

The following statements hold.

1. $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *has (a spread of)* 1*-regular cliques, each of size* $\lambda + 2$*;*
2. $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *is an edge-regular graph with parameters* $(vt, k + \lambda + 1, \lambda)$.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. `https://arxiv.org/abs/2109.13884`

A generalisation of the two constructions

Remark

We note that a converse to the theorem above is also true. Let $\Gamma \in NG(v, k, \lambda; 1, s)$ *have a spread of* 1*-regular cliques. The graph* Γ° *created by removing edges from the cliques of this spread in* Γ *is also edge-regular by Soicher [6, Theorem 6.1]. It also follows that connected components of* Γ° *can be partitioned into perfect* 1*-codes, and the parameters have the same restrictions as the conditions of the statement of the theorem above.*

[6] L. H. Soicher, *On cliques in edge-regular graphs*, *Journal of Algebra*, 421, 260–267 (2015). https://doi.org/10.1016/j.jalgebra.2014.08.028

A corollary

In the case for which $t = 1$ can occur, the construction can result in a strictly Neumaier graph. However, this is not necessarily true in all cases when $t = 1$. The following Corollary shows that for $t \geq 2$, our construction always results in a strictly Neumaier graph.

Corollary ([5])

Let $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ be non-complete edge-regular graphs with parameters (v, k, λ) and let $\Pi = (\pi_2, \dots, \pi_t)$ be a $(t-1)$-tuple of permutations from $Sym(\{1, \dots, v/a\})$.
Further, suppose that each graph $\Gamma^{(\ell)}$ has a partition of its vertices into perfect 1-codes $H_1^{(\ell)}, \dots, H_{v/a}^{(\ell)}$, each of size a, where a is a proper divisor of $\lambda + 2$ and $t = (\lambda + 2)/a$.
If $t \geq 2$, then $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ is a strictly Neumaier graph.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. `https://arxiv.org/abs/2109.13884`

Notes on the generalisation

- Non-isomorphic Taylor graphs with the same parameters give many new examples in the case $t \geq 2$.
- The four strictly Neumaier graphs on 24 vertices from [3] are given by a pair of icosahedrons, and the only difference between them is the choice of the permutation that matches the antipodal classes.
- The generalised construction covers both constructions from [2] and [4] (the cases $t = 1$ and $t \geq 2$, respectively).
- For $t = 1$ we can construct three new strictly Neumaier graphs: with parameters $(28, 9, 2; 1, 4)$, $(40, 12, 2; 1, 4)$ and $(65, 16, 3; 1, 5)$; eight graphs with parameters $(78, 17, 4; 1, 6)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).
[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Sibirskie Èlektronnye Matematicheskie Izvestiya, 11, 268–310 (2014).
[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Dis. Math., 342, Issue 10, (2019) 2818–2820.

Notes on the generalisation

- In [7], Corollary was independently proved and an infinite class of strictly Neumaier graphs based on the general construction was obtained.

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.
https://doi.org/10.1016/j.jcta.2022.105684

Examples from infinite edge-regular lattices (see [5])

Eisenstein integers are the complex numbers of the form $\mathbb{Z}[\omega] = \{b + c\omega : b, c \in \mathbb{Z}\}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. They form a ring with respect to usual addition and multiplication.

The norm mapping $N : \mathbb{Z}[\omega] \mapsto \mathbb{N} \cup \{0\}$ is defined as follows. For an Eisenstein integer $b + c\omega$, $N(b + c\omega) = b^2 + c^2 - bc$ holds. The norm mapping N is known to be multiplicative.

It is well-known that $\mathbb{Z}[\omega]$ forms an Euclidean domain (in particular, a principal ideal domain).

The units of $\mathbb{Z}[\omega]$ are $\{\pm 1, \pm\omega, \pm\omega^2\}$. The natural geometrical interpretation of Eisenstein integers is the 6-regular triangular grid in the complex plane.

If it does not lead to a contradiction, we use the same notation $\mathbb{Z}[\omega]$ for the triangular grid.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. https://arxiv.org/abs/2109.13884

Examples from infinite edge-regular lattices

The grid $\mathbb{Z}[\omega]$ has exactly six elements of norm 7; these are $\{\pm(1+3\omega), \pm(3+2\omega), \pm(2-\omega)\}$. Consider the ideal I generated by an element of norm 7 (say, by the element $2-\omega$). The elements of I form a perfect 1-code in the triangular grid. Note that I is an additive subgroup of index 7 in $\mathbb{Z}[\omega]$; we denote it by I^+. The seven cosets $\mathbb{Z}[\omega]/I^+$ give a partition of the triangular grid into seven perfect 1-codes. Take the following two additive subgroups of $\mathbb{Z}[\omega]$:

$$T_1 := \{2(-2+\omega)x + 14y \mid x, y \in \mathbb{Z}\},$$

$$T_2 := \{(5+\omega)x + 28y \mid x, y \in \mathbb{Z}\}.$$

Since $-2+\omega$, 7 and $5+\omega$ are divisible by $2-\omega$, a generator of I, we have that T_1 and T_2 are subgroups in I^+. Note that there exists a block of four balls of radius 1 such that the additive shifts of the block by the elements of T_1 and T_2 give two tessellations of $\mathbb{Z}[\omega]$.

Examples from infinite edge-regular lattices

Consider the quotient groups

$$G_1 := \mathbb{Z}[\omega]/T_1 \text{ and } G_2 := \mathbb{Z}[\omega]/T_2,$$

where $G_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$ and $G_2 \cong \mathbb{Z}_{28}$. Define two Cayley graphs

$$\Delta_1 := \mathrm{Cay}(G_1, \{\pm(1+T_1), \pm(\omega+T_1), \pm(\omega^2+T_1)\}),$$

$$\Delta_2 := \mathrm{Cay}(G_2, \{\pm(1+T_2), \pm(\omega+T_2), \pm(\omega^2+T_2)\}).$$

Note that Δ_1 and Δ_2 can be interpreted as quotient graphs of the triangular grid by T_1 and T_2, respectively. Each of the graphs Δ_1 and Δ_2 is edge-regular with parameters $(28, 6, 2)$ and admits a partition into perfect 1-codes of size $a = 4$; these partitions are given by the original partition of the triangular grid into perfect 1-codes. We then apply the general construction, which gives two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$. The graph obtained from Δ_1 is isomorphic to the smallest graph from the first Greaves & Koolen's construction, and the graph obtained from Δ_2 is new.

More infinite edge-regular lattices

In the following, we consider countably infinite graphs with vertices consisting of elements of the vector space $\mathbb{R}^n$, for some integer $n \geq 3$. The elements of $\mathbb{R}^n$ are called (n-dimensional) vectors, and we identify the elements with their coordinates with respect to the standard basis of $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$. For a set $A \subseteq \mathbb{R}$, the vector x is an A-vector if the value of all of its entries lie in A. The weight of x is the number of its non-zero entries.

Let $n \geq 3$ be a positive integer and let m be an even positive integer. Let $S^{(1)}_{n,m}$ denote the set of all n-dimensional $\{1, -1, 0\}$-vectors of weight m whose sum of coordinates is zero. Let $S^{(2)}_{n,m}$ denote the set of all n-dimensional $\{1, -1, 0\}$-vectors of weight m. Let $G^{(1)}_{n,m}$ and $G^{(2)}_{n,m}$ be the groups generated by $S^{(1)}_{n,m}$ and $S^{(2)}_{n,m}$ respectively.

More infinite edge-regular lattices

Proposition

For any positive even integer m and any integer n such that $n \geq m+1$, the following statements hold.

1. $G^{(1)}_{n,m}$ *is equal to* $G^{(1)}_{n,2}$*, which consists of all* n*-dimensional vectors with integer coordinates such that the sum of coordinates is equal to* 0*.*
2. $G^{(2)}_{n,m}$ *is equal to* $G^{(2)}_{n,2}$*, which consists of all* n*-dimensional vectors with integer coordinates such that the sum of coordinates is even.*

More infinite edge-regular lattices

From now on, we let

$$G_n^{(1)} := G_{n,2}^{(1)},$$

$$G_n^{(2)} := G_{n,2}^{(2)},$$

and define graphs

$$\Gamma_{n,m}^{(1)} := Cay(G_n^{(1)}, S_{n,m}^{(1)})$$

and

$$\Gamma_{n,m}^{(2)} := Cay(G_n^{(2)}, S_{n,m}^{(2)}).$$

A graph Γ with infinitely many vertices is edge-regular with parameters (k, λ) if it is k-regular and each pair of adjacent vertices have exactly λ common neighbours.

In the following, we show that $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$ are infinite edge-regular graphs, and give the parameters of these graphs in terms of binomial coefficients.

More infinite edge-regular lattices

Proposition

For any positive even integer m and any integer n such that $n \geq m+1$, the following statements hold.

1. *The graph $\Gamma^{(1)}_{n,m}$ is an induced subgraph in $\Gamma^{(2)}_{n,m}$.*
2. *$\Gamma^{(1)}_{n,m}$ is an infinite edge-regular graph with parameters (k_1, λ_1), such that*

$$k_1 = \binom{n}{m}\binom{m}{\frac{m}{2}},\ \lambda_1 = \sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i}\binom{\frac{m}{2}}{\frac{m}{2}-i}\binom{n-m}{\frac{m}{2}-i}\binom{n-\frac{3m}{2}+i}{i}.$$

3. *$\Gamma^{(2)}_{n,m}$ is an infinite edge-regular graph with parameters (k_1, λ_1), such that*

$$k_2 = 2^m\binom{n}{m},\ \ \lambda_2 = \binom{m}{\frac{m}{2}}\binom{n-m}{\frac{m}{2}}2^{\frac{m}{2}}.$$

Comments

Remark

Note that if $n < \frac{3m}{2}$, *then* $\lambda_1 = 0$ *and* $\lambda_2 = 0$. *Otherwise,* $\lambda_1 > 0$ *and* $\lambda_2 > 0$.

Remark

The generating sets $S_{n,2}^{(1)}$ *and* $S_{n,2}^{(2)}$ *are known as root systems* A_{n-1} *and* D_n.

A_2 root lattice is isomorphic to the 6-regular triangular grid.

The root lattices A_3 and D_3 are both isomorphic to the tetrahedral-octahedral honeycomb.

More examples from infinite edge-regular lattices

In the following tables, we present the number of cases for which we find strictly Neumaier graphs using the graphs $\Gamma^{(1)}_{n,m}$ and $\Gamma^{(2)}_{n,m}$, respectively.

The first column of the tables give the corresponding value of n.

The second column gives the Neumaier graph parameters of the graphs we find through the construction. The last column gives the number of pairwise non-isomorphic strictly Neumaier graphs we find from the construction.

More examples from infinite edge-regular lattices

n	parameters of SNG	#
3	$(28, 9, 2; 1, 4)$	2
4	$(78, 17, 4; 1, 6)$	≥ 8
5	$(168, 27, 6; 1, 8)$	≥ 12
6	$(310, 39, 8; 1, 10)$	≥ 1

Table: Number of strictly Neumaier graphs from quotients of $\Gamma^{(1)}_{n,2}$.

n	parameters of SNG	#
3	$(78, 17, 4; 1, 6)$	≥ 8
4	$(250, 33, 8; 1, 10)$	≥ 16

Table: Number of strictly Neumaier graphs from quotients of $\Gamma^{(2)}_{n,2}$.

A problem

However, we have not been able to find more examples of perfect codes and quotients of $\Gamma^{(1)}_{n,2}$ and $\Gamma^{(2)}_{n,2}$ that lead to strictly Neumaier graphs. Therefore, we ask the following.

Problem

What strictly Neumaier graphs can be obtained from quotients of infinite edge-regular graphs $\Gamma^{(1)}_{n,m}$ *and* $\Gamma^{(2)}_{n,m}$*?*

Examples from infinite edge-regular lattices

We can also use two infinite edge-regular graphs to get a new infinite edge-regular graph by taking the Cartesian product of the graphs.

Proposition

Let Γ_1 *and* Γ_2 *be two infinite edge-regular graphs with parameters* (k_1, λ) *and* (k_2, λ)*, respectively. Then the Cartesian product of* Γ_1 *and* Γ_2 *is an edge-regular graph with parameters* $(k_1 + k_2, \lambda)$*.*

An example from Cartesian products of infinite edge-regular lattices

Consider the Cartesian product of two 6-regular triangular grids; the resulting infinite graph is edge-regular with parameters (12,2). This graph has a subgroup perfect 1-code, and there exists an edge-regular quotient graph with parameters $(52, 12, 2)$. We then apply the general construction to this graph, which gives a strictly Neumaier graph having parameters $(52, 15, 2; 1, 4)$ and isomorphic to the second largest graph from the first Greaves & Koolen's construction. As we have seen this example using Cartesian products, we ask the following.

Problem

What strictly Neumaier graphs can be obtained from quotients of Cartesian products of infinite edge-regular graphs?

Spectrum of a graph

The spectrum of a graph Γ is the multiset of eigenvalues of the adjacency matrix of Γ.

Two graphs are cospectral if they have the same spectra.

The following switching (Wang-Qu-Hu switching), which produces cospectral graphs, was discovered in [8], applied in [9] to obtain new strongly regular graphs, and discussed in [10].

[8] W. Wang, L. Qiu, Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p, Linear Algebra and its Applications, Volume 563, 15 (2019), 154–177.

[9] F. Ihringer, A. Munemasa, New strongly regular graphs from finite geometries via switching, Linear Algebra and its Applications Volume 580, (2019), 464–474. https://doi.org/10.1016/j.laa.2019.07.014

[10] L. Qiu, Y. Ji, W. Wang, On a theorem of Godsil and McKay concerning the construction of cospectral graphs, Linear Algebra and its Applications, Volume 603 (2020), 265–274. https://doi.org/10.1016/j.laa.2020.05.025

Wang-Qu-Hu switching

Lemma (WQH-switching)

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that the induced subgraphs on C_1, C_2, and $C_1 \cup C_2$ are regular, where the degrees in the induced subgraphs on C_1 and C_2 are the same. Suppose that all $x \in D$ satisfy one of the following:

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, *or*
2. $\Gamma(x) \cap (C_1 \cup C_2)| \in \{C_1, C_2\}$.

Construct a graph Γ' from Γ by modifying the edges between $C_1 \cup C_2$ and D as follows:

$$\Gamma'(x) \cap (C_1 \cup C_2) = \begin{cases} C_1, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_2; \\ C_2, & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_1; \\ \Gamma(x) \cap (C_1 \cup C_2), & \text{otherwise}, \end{cases}$$

for all $x \in D$. Then Γ' is cospectral with Γ.

WQH-switching for the general construction

Proposition ([5])

Let $t \geq 2$ and $F_{\Pi}(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ be a strictly Neumaier graph obtained from the general construction. Then for any non-empty subset $I \subseteq \{1, \ldots, t\}$ containing 1 and distinct $i, j \in \{1, \ldots, v/a\}$, the partition

$$C_1 := \bigcup_{\ell \in I} H_i^{(\ell)}, \quad C_2 := \bigcup_{\ell \in I} H_j^{(\ell)},$$

$$D := V(F_{\Pi}(\Gamma^{(1)}, \ldots, \Gamma^{(t)})) \setminus (C_1 \cup C_2)$$

satisfies the conditions of WQH-switching.
Moreover, we have the equality $(F_{\Pi}(\Gamma^{(1)}, \ldots, \Gamma^{(t)}))' = F_{\Pi'}(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$, where

$$(\Pi')_r = \begin{cases} \pi_r & \text{if } r \in I \\ (i \ j) \circ \pi_r & \text{if } r \notin I \end{cases}$$

Corollaries

Corollary ([5])

For any Π, Π', $(t-1)$*-tuples of elements of* $Sym(\{1, \ldots, v/a\})$, *the graphs* $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *and* $F_{\Pi'}(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ *are cospectral.*

It would be interesting to investigate how many pairwise non-isomorphic graphs can be constructed using our construction. In doing so, we may find a prolific construction of cospectral strictly Neumaier graphs. Although this has not been investigated in detail, we have already observed several pairwise non-isomorphic graphs with relatively small order.

Corollary ([5])

The four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ *obtained from two copies of icosahedron are cospectral.*

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. https://arxiv.org/abs/2109.13884

The smallest strictly Neumaier graph

In [11], Evans, G. and Panasenko found the smallest strictly Neumaier graph, which is a Cayley graph, has parameters $(16, 9, 4; 2, 4)$ and contains a spread given by the cosets of a subgroup.

It can be constructed by switching edges in the affine polar graph $VO^+(4, 2)$, which is isomorphic to the complement to (4×4)-lattice.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Strictly Neumaier graphs with 2^i-regular cliques

We then generalised the smallest strictly Neumaier graph and, for every positive integer i, by switching in certain affine polar graphs, found a strictly Neumaier graph with 4^{i+1} vertices containing a 2^i-regular clique and having parameters of these affine polar graphs as edge-regular graphs.

However, the graphs for $i \geq 2$ were not vertex-transitive, and it was an open question whether there exists a vertex-transitive strictly Neumaier graph with nexus greater than 1 except the smallest strictly Neumaier graph. In this project we solve this question in positive.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Problem on strictly Neumaier graphs

In general, we are interested in the following problem.

Problem

For which positive integers m does there exist a strictly Neumaier graph with an m-regular clique?

Remark

All previously known strictly Neumaier graphs had regular cliques with nexus equal to a power of 2. The only known strictly Neumaier graphs having regular cliques with nexus greater than 1 were found in [11].

Motivated by the fact that many known examples of strictly Neumaier graphs are Cayley graphs with a spread given by the cosets of a subgroup, we decided to have a general look at Cayley Neumaier graphs with a spread given by the cosets of a subgroup.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Equitable partitions

Let Γ be a k-regular graph with the vertex set $V(\Gamma)$.

Let $\Pi := (V_1, \ldots, V_t)$ be a partition of $V(\Gamma)$ into t parts (t-partition).

The partition Π is said to be an equitable t-partition if for any $i, j \in \{1, \ldots, t\}$ there is a constant p_{ij} such that any vertex from the part V_i is adjacent to precisely p_{ij} vertices from the part V_j.

The square matrix $P_\Pi := (p_{ij})_{i,j=1}^{t}$ is called the quotient matrix of the equitable t-partition Π.

Subgroup equitable 2-partitions in Cayley graphs

Lemma (Evans, G., Zhao, 2022+)

Let G be a finite group, $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Further, let $S \subseteq G, 1 \notin S$.
Then $\{H, G \setminus H\}$ is an equitable 2-partition in $\Gamma = \mathrm{Cay}(G, S)$ if and only if

1. *$S \cap H$ is closed under inversion.*
2. *There exists subsets $T_2, T_3, \ldots, T_n \subseteq S$, such that*
   2.1 $|T_i| = m$
   2.2 *T_i consists of representatives of the coset Hg_i*
   2.3 *$T = \bigcup T_i$ is closed under inversion.*

In this case, the induced subgraph $\Gamma[H]$ is $|S \cap H|$-regular with nexus m, and the induced subgraph $\Gamma[G \setminus H]$ is $(|S| - m)$-regular with nexus $|S \setminus H|$. In other words, we have quotient matrix

$$A(\Gamma/H) = \begin{pmatrix} |S \cap H| & |S \setminus H| \\ m & |S| - m \end{pmatrix}$$

Coset equitable partitions in Cayley graphs

Corollary (Evans, G., Zhao, 2022+)

Suppose $\Gamma = \mathrm{Cay}(G, S)$ is a Cayley graph with group equitable 2-partition corresponding to $H < G, [G : H] = n$. Then Γ has an equitable n-partition X, with the parts the cosets of the group H, and quotient matrix

$$A(\Gamma/X) = \begin{pmatrix} |S \cap H| & m & \cdots & m \\ m & |S \cap H| & \cdots & \vdots \\ \vdots & \cdots & |S \cap H| & m \\ m & \cdots & m & |S \cap H| \end{pmatrix}$$

Edge-regularity of special Cayley graphs with coset equitable partition

Lemma (Evans, G., Zhao, 2022+)

Let G be a group, $H < G$ and $S \subseteq G, 1 \notin S$, where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets $T_2, \ldots, T_n$ and $T = \bigcup T_i$. Furthermore, assume $H^* \cap S = \emptyset$.
Then Γ is edge-regular with parameters (v, k, λ) if and only if for all $g \in T$, the condition

$$|Tg \cap T| = \lambda$$

holds.

Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Lemma (Evans, G., Zhao, 2022+)

Let G be a group, $H < G$ and $S \subseteq G, 1 \notin S$ where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets $T_2, \ldots, T_n$ and $T = \bigcup T_i$. Furthermore, assume $H^* \subseteq S$, so $S = H^* \cup T$. Then $\Gamma = \mathrm{Cay}(G, S)$ is an edge-regular (and thus Neumaier) graph with parameters (v, k, λ) if and only if

1. *For all $h \in H^*$, $|Th \cap T| = \lambda - |H| + 2$*
2. *For all $g \in T$, $|Tg \cap T| = \lambda - 2m + 2$*

Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Corollary (Evans, G., Zhao, 2022+)

Such a Neumaier graph $\Gamma = \mathrm{Cay}(G, S)$ *has parameters*

$$\begin{aligned} v &= ns \\ k &= s - 1 + (n-1)m \\ \lambda &= s - 2 + \frac{(n-1)m(m-1)}{(s-1)} \end{aligned}$$

where $s = |H|$.

The smallest strictly Neumaier graph as a Cayley graph

- $G = C_2 \times C_8$
  $H = \langle (1,2) \rangle$,
  $T_2 = \{(0,1),(1,7)\}$,
  $T_3 = \{(0,2),(0,6)\}$,
  $T_4 = \{(0,7),(1,1)\}$.
- $G = D_{16} = \langle a, b | a^8 = b^2 = 1, bab = a^{-1} \rangle$,
  $H = \langle a^2 \rangle, \{1, a^4, a^2 b, a^6 b\}$,
  $S = \{a, a^2, a^4, a^6, a^7, ab, a^2 b, a^6 b, a^7 b\}$.
- $G = C_2 \times D_8 = \langle 1 \rangle \times \langle a, b | a^4 = b^2 = baba = 1 \rangle$,
  $H = \langle (0,a) \rangle, \langle (1,a) \rangle$,
  $S = \{(0,a),(0,a^2),(0,a^3),(0,a^2 b),(0,a^3 b),(1,a),(1,a^3),$
  $(1,a^2 b),(1,a^3 b)\}$

In all cases, $\Gamma = \mathrm{Cay}(G, H^* \cup T)$ is the smallest strictly Neumaier graph, with a spread given by the cosets of H and parameters $(16,9,4;2,4)$.

The four graphs from the icosahedron as Cayley graphs

1. $G = S_4, H = \langle(1,3,2,4)\rangle, T_2 = \{(1,2,3)\}, T_3 = \{(1,3,2)\}, T_4 = \{(1,4,2)\}, T_5 = \{(1,3)(2,4)\}, T_6 = \{(1,2,4)\}$
2. $G = S_4, H = \langle(1,2),(3,4)\rangle, T_2 = \{(1,3,4)\}, T_3 = \{(1,4)(2,3),\}, T_4 = \{(2,4,3),\}, T_5 = \{(1,4,3),\}, T_6 = \{(2,3,4)\}$
3. $G = C_2 \times A_4, H = \langle(1,(1,2)(3,4)),(1,(1,3)(2,4))\rangle, T_2 = \{(0,(1,2,4)),\}, T_3 = \{(0,(1,3,4)),\}, T_4 = \{(0,(1,3)(2,4))\}, T_5 = \{(0,(1,4,2))\}, T_6 = \{(0,(1,4,3))\}$
4. $G = C_2 \times A_4, H = \langle(0,(1,4)(2,3)),(1,())\rangle, T_2 = \{(0,(1,2,3))\}, T_3 = \{(0,(2,3,4))\}, T_4 = \{(0,(2,4,3))\}, T_5 = \{(0,(1,3,2))\}, T_6 = \{(0,(1,2)(3,4))\}$

Then for each case, $\Gamma = \mathrm{Cay}(G, H^* \cup T)$ is a strictly Neumaier graph with with a spread given by the cosets of H and parameters $(24, 8, 2; 1, 4)$.

Two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$

1. $G = C_{28} = C_4 \times C_7, H = \langle (1,0) \rangle, T_2 = \{(1,1)\}, T_3 = \{(3,6)\}, T_4 = \{(1,3)\}, T_5 = \{(3,4)\}, T_6 = \{(0,2)\}, T_7 = \{(0,5)\}$
2. $G = C_{28} = C_4 \times C_7, H = \langle (1,0) \rangle, T_2 = \{(3,1)\}, T_3 = \{(1,6)\}, T_4 = \{(3,3)\}, T_5 = \{(1,4)\}, T_6 = \{(0,2)\}, T_7 = \{(0,5)\}$
3. $G = C_2 \times C_{14} = C_2 \times C_2 \times C_7, H = \langle (1,0,0), (0,1,0) \rangle, T_2 = \{(1,0,1)\}, T_3 = \{(1,0,4)\}, T_4 = \{(0,1,2)\}, T_5 = \{(0,1,5)\}, T_6 = \{(1,1,3)\}, T_7 = \{(1,1,6)\}$

Then for the cases 1,2 we find that the graphs $\mathrm{Cay}(G, H^* \cup T)$ are isomorphic with different generating sets.

In all cases above, $\Gamma = \mathrm{Cayley}(G, H^* \cup T)$ is a strictly Neumaier graph with with a spread given by the cosets of H and parameters $(28, 9, 2; 1, 4)$

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Lemma (Evans, G., Zhao, 2022+)

Let G be a finite group and $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Then, for any $i \in \{2, \ldots, n\}$,

$$\mathrm{Stab}_{\mathrm{Aut}\, G}(Hg_i) < \mathrm{Stab}_{\mathrm{Aut}\, G}(H)$$

holds, where the stabilisers are setwise.

Corollary (Evans, G., Zhao, 2022+)

Let G be a finite group and $H < G$ with cosets $Hg_1, \ldots, Hg_n$, with $g_1 = 1$. Then, for any $i \in \{2, \ldots, n\}$, each automorphism $\varphi \in \mathrm{Stab}_{\mathrm{Aut}\, G}(Hg_i)$ preserves the partition

$$\{Hg_2, \ldots, Hg_{i-1}, Hg_{i+1}, \ldots, Hg_n\}.$$

Moreover, if $g_j = g_i g$, then $\varphi(Hg_j) = Hg_i\varphi(g)$.

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Data:

a feasible tuple of parameters $(v, k, \lambda; m, s)$ of a Neumaier graph;
a finite group G, $|G| = v$;
a subgroup $H < G$, $|H| = s$, with cosets $Hg_1, \dots, Hg_n$, where $g_1 = 1$;
the automorphism group $\operatorname{Aut} G$;
the stabiliser $\operatorname{Stab}_{\operatorname{Aut} G}(Hg_2)$;

Result:

the list of Cayley Neumaier graphs with a spread given by the cosets of the subgroup H with parameters $(v, k, \lambda; m, s)$ over G (each graph is given as $\operatorname{Cayley}(G, H^* \cup T)$)

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

for non-equivalent (under $\mathrm{Stab}_{\mathrm{Aut}\,G}(Hg_2)$) m-subsets $T_2 \subset Hg_2$
  for all correct m-subsets $T_3 \subset Hg_3$
    for all correct m-subsets $T_4 \subset Hg_3$
      ...
      for all correct m-subsets $T_n \subset Hg_n$
        if the necessary and sufficient conditions hold then
          save $\mathrm{Cay}(G, H^* \cup \underbrace{T_2 \cup T_3 \ldots \cup T_n}_{T})$;
        end if;
      end for;
      ...
    end for;
  end for;
end for;

Comments on the algorithm

- feasible tuples of parameters of Neumaier graphs can be computed according to necessary conditions on the existence of Neumaier graphs;
- feasible tuples parameters for strictly Neumaier graphs up to 64 vertices can be found in [7, Table 1];
- the multiplication tables of the groups and the generators of their automorphism groups can be taken from GAP;
- representatives of classes of conjugate subgroups of appropriate order can be taken from GAP;
- having the automorpism group of the group and the representatives of classes of conjugate subgroups, it is possible to compute the list of all non-equivalent subgroups;
- the isomorphism tests for the resulting graphs can be done in SAGE or MAGMA

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.
https://doi.org/10.1016/j.jcta.2022.105684

Numerical data

- all our computational efforts so far are devoted to the following feasible tuples of parameters:

$$(64, 21, 8; 2, 8),$$

$$(64, 28, 12; 3, 8),$$

$$(64, 35, 18; 4, 8),$$

$$(64, 42, 26; 5, 8)$$

$$(64, 49, 36; 6, 8);$$

  these are the parameters of the block graphs of orthogonal arrays OA(8,3), OA(8,4), OA(8,5) and OA(8,6), respectively; we are interested in both strongly regular graphs (SRGs) and strictly Neumaier graphs (SNGs);
- the first and the fourth tuples are complementary (for SRGs) and correspond to parameters of a Latin square graph and its complement, respectively;

Numerical data

- the second and the third tuples are complementary (for SRGs) and correspond to parameters of prolific Wallis and Wallis2 constructions;
- the fifth tuple correspond to the parameters of the complement of 8×8-lattice (there exists a unique strongly regular graph with these parameters);
- there are exactly 267 groups of order 64;
- according to the algorithm, the most difficult groups are those that have many involutions; thus, the most difficult group is $C2 \times C2 \times C2 \times C2 \times C2 \times C2$;
- it is expected to take about 2 months to make computations over $C2 \times C2 \times C2 \times C2 \times C2 \times C2$ (one month has been passed already);
- in other cases, the time needed to execute the program varies from several minutes to several days;
- currently, only 7 groups of order 64 (among 267 such groups) are unfinished; they have numbers 250,261,263,264,265,266 and 267 in GAP

Findings so far

- at least 6 SRGs and no SNGs with parameters $(64, 21, 8; 2, 8)$;
- at least 40 SRGs and at least 6 SNGs with parameters $(64, 28, 12; 3, 8)$;
- at least 123 SRGs and at least 138 SNGs with parameters $(64, 35, 18; 4, 8)$;
- at least 13 SRGs and a unique SNG with parameters $(64, 42, 26; 5, 8)$; some of the complements of the 13 SRGs have no regular cliques and thus cannot be Latin-square graphs; it is known that, for sufficiently large number of vertices, an SRG with parameters of a Latin square graph is a Latin square graph;
- a unique SRG and no SNGs with parameters $(64, 49, 36; 6, 8)$;
- note that some of these graphs may be Cayley graphs over more than one group G and may be given by more than one subgroup H for the same group G.

Further problems

- generalisation of the obtained examples;
- for the SRGs, we need to make isomorphism tests with known examples; this may be difficult because Wallis and Wallis2 constructions, which give SRGs with parameters $(64, 28, 12; 3, 8)$ and $(64, 35, 18; 4, 8)$, are prolific;
- investigating another feasible tuples of parameters including open tuples of parameters of SRGs from Brouwer's list (the smallest are $(96, 35, 10; 2, 6)$ and $(96, 60, 38; 9, 16)$);
- generalisation of the general construction with use of a partition into completely regular codes of radius 1 instead of a partition into perfect 1-codes;

Thank you for your attention!