PI-eigenfunctions of the Star graphs

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Outline

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   - A result by Jucys on eigenvectors of J_n
   - Permutation module M^λ and Specht module S^λ
   - An embedding of the permutation module M^λ into $\mathbb{C}[Sym_n]$
   - Standard basis of the Specht module and eigenvectors of J_n given by a polytabloid
2. Our results
   - A family of eigenfunctions of the Star graph S_n called PI-eigenfunctions
   - A connection between eigenfunctions given by a polytabloid and PI-eigenfunctions

Cayley graph and the neighbourhood of a vertex

Let G be a group. For a non-empty inverse-closed identity-free subset S in G, define the Cayley graph $Cay(G, S)$ with the vertex set G and two vertices x, y being adjacent whenever there is an element $s \in S$ such that $y = sx$ holds.

Given a vertex x in $Cay(G, S)$, the equality

$$N(x) = Sx$$

holds, where $N(x)$ is the neighbourhood of x.

Group algebra $\mathbb{F}[G]$ and the action by multiplication

Take a field $\mathbb{F}$, a group G and the group algebra $\mathbb{F}[G]$.

For a subset S in G, consider the element $\overline{S} \in \mathbb{F}[G]$ given by

$$\overline{S} = \sum_{s \in S} s.$$

Left multiplication of elements from $\mathbb{F}[G]$ by $\overline{S}$ is a linear transformation of $\mathbb{F}[G]$.

The transformation matrix of this linear transformation coincides with the adjacency matrix of $Cay(G, S)$, which gives a bridge between spectral properties of Cayley graphs and the representation theory.

Eigenfunction of a graph

Let $\Gamma = (V, E)$ be a regular graph.

A function $f : V \longrightarrow \mathbb{R}$ is called an eigenfunction of the graph Γ corresponding to an eigenvalue θ if $f \not\equiv 0$ and for any vertex x the local condition

$$\theta \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds, where $N(x)$ is the set of neighbours of the vertex x.

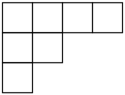
Setting

We put

- $G := Sym_n$
- $\mathbb{F} := \mathbb{C}$
- $S := \{(i\ n) \mid i = 1, \ldots, n-1\}$
- The graph $S_n := Cay(G, S)$ is called the Star graph
- The element $J_n := (1\ n) + \ldots + (n-1\ n)$ from $\mathbb{C}[Sym_n]$ is called the Jucys-Murphy element

Partitions and Ferrers diagrams

Take the partition $\lambda = (4, 2, 1)$ of the number $n = 7$. Then the corresponding Ferrers diagram is as follows:



A tableau t is standard if the rows and columns of t are increasing sequences.

Standard Young tableaux of shape $\lambda = (4,2,1)$

$c(n) = 3$

$$\begin{ytableau} 1&2&6&7\\ 3&5\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&2&5&7\\ 3&6\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&2&6&7\\ 3&4\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&7\\ 3&6\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&2&5&7\\ 3&4\\ 6 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&7\\ 3&5\\ 6 \end{ytableau}\,,$$

$$\begin{ytableau} 1&3&6&7\\ 2&5\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&3&5&7\\ 2&6\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&3&6&7\\ 2&4\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&7\\ 2&6\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&3&5&7\\ 2&4\\ 6 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&7\\ 2&5\\ 6 \end{ytableau}\,,$$

$c(n) = 0$

$$\begin{ytableau} 1&2&5&6\\ 3&7\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&6\\ 3&7\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&5\\ 3&7\\ 6 \end{ytableau}\,,\ \begin{ytableau} 1&3&5&6\\ 2&7\\ 4 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&6\\ 2&7\\ 5 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&5\\ 2&7\\ 6 \end{ytableau}\,,$$

$c(n) = -2$

$$\begin{ytableau} 1&2&5&6\\ 3&4\\ 7 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&6\\ 3&5\\ 7 \end{ytableau}\,,\ \begin{ytableau} 1&2&4&5\\ 3&6\\ 7 \end{ytableau}\,,\ \begin{ytableau} 1&3&5&6\\ 2&4\\ 7 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&6\\ 2&5\\ 7 \end{ytableau}\,,\ \begin{ytableau} 1&3&4&5\\ 2&6\\ 7 \end{ytableau}\,.$$

Decomposition of the regular representation of Sym_n

The regular representation of Sym_n is decomposed into irreducible submodules as follows

$$\mathbb{C}[Sym_n] = \bigoplus_{\lambda \in \mathcal{P}(n)} m_\lambda V_\lambda,$$

where $\mathcal{P}(n)$ the set of partitions of n and $m_\lambda = \dim V_\lambda$.

A result by Jucys

Theorem (Jucys, 1974)

Let $\lambda \in \mathcal{P}(n)$. Then there exists a basis $\{v_t\}$ of the irreducible module V_λ, indexed by standard Young tableaux t of shape λ such that for all $i \in \{2, \dots, n\}$, the equality

$$J_i v_t = c_t(i) v_t$$

holds.

If $i = n$, the theorem says that there exists a basis of an irreducible module V_λ consisting of eigenvectors of J_n. Moreover, the number of eigenvectors corresponding to the same eigenvalue is given by the number of standard Young tableaux of the shape λ with the same value $c(n)$.

$\lambda = (4, 2, 1)$

The irreducible module V_λ has dimension 24.

The basis contains

- 12 eigenvectors of J_7 with eigenvalue 3,
- 6 eigenvectors with eigenvalue 0,
- 6 eigenvectors with eigenvalue -2.

Corollary

For any $n \geq 4$, the spectrum of the Star graph S_n consists of integers $-(n-1), \ldots, n-1$, and the multiplicity of an eigenvalue θ is given by the formula

$$mul(\theta) = \sum_{\lambda \in \mathcal{P}(n)} m_\lambda \cdot C_\lambda(k),$$

where m_λ is the number of standard Young tableaux of shape λ and $C_\lambda(k)$ is the number of standard Young tableaux of shape λ with $c(n) = k$.

Permutation module M^λ

Let λ be a shape with n cells.

For a tableau t of shape λ, the λ-tabloid $\{t\}$ is the set of all tableaux of shape λ that can be obtained from t by permutations of elements in rows.

Let $M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\}$ be the permutation module corresponding to λ, where $\{t_1\}, \ldots, \{t_k\}$ is a complete list of λ-tabloids.

Further, we consider the action of the group algebra $\mathbb{C}[Sym_n]$ on M^λ.

Polytabloids

Let t be a tableau of shape λ.

Let C_t be the column stabilizer of C_t.

Put

$$\mathbf{e}_t := \sum_{\pi \in C_t} sgn(\pi)\{\pi(t)\}.$$

The element $\mathbf{e}_t \in M^\lambda$ is called the polytabloid given by the tableau t.

Specht module S^λ and its standard basis

Given a partition λ, the corresponding Specht module S^λ, is the submodule of M^λ spanned by all polytabloids $\mathbf{e}_t$, where t is of shape λ.

A polytabloid $\mathbf{e}_t$ is standard if the tableau t is standard.

The set of standard polytabloids

$$\{\mathbf{e}_t \mid t \text{ is a standard tableau of shape } \lambda\}$$

forms a basis of the Specht module S^λ.

An embedding ϕ of M^λ into $\mathbb{C}[Sym_n]$

Let id_λ be the standard tableau of shape λ whose rows consist of the consecutive elements.

Let T_λ be the set of all tableaux of shape λ. For any tableau $t \in T_\lambda$, denote by τ_t the permutation defined be the equation

$$\tau_t(t) = id_\lambda, \tag{1}$$

where τ_t acts on t by replacing the values of the cells of t.

Let us define a linear mapping $\phi : M^\lambda \to \mathbb{C}[\mathrm{Sym}_n]$. Since the set of all λ-tabloids is a basis for M_λ, it is enough to define images for λ-tabloids. For any λ-tabloid $\{t\}$, where $t \in T_\lambda$, we put $\phi(\{t\}) = \sum\limits_{t' \in \{t\}} \tau_{t'}$.

An embedding ϕ of M^λ into $\mathbb{C}[Sym_n]$

Lemma (1)

For any polytabloid $\mathbf{e}_t$*, the equality* $\phi(J_n(\mathbf{e}_t)) = J_n(\phi(\mathbf{e}_t))$ *holds.*

Lemma (2)

Let $\mathbf{v} \in S^\lambda$ *be an eigenfunction of the operator* $J_n : M^\lambda \to M^\lambda$ *with eigenvalue* θ*. Then* $\phi(\mathbf{v})$ *is an eigenfunction of the operator* $J_n : \mathbb{C}[\mathrm{Sym}_n] \to \mathbb{C}[\mathrm{Sym}_n]$ *with eigenvalue* θ*.*

An eigenvector of J_n given by a polytabloid $\mathbf{e}_t$

Let $\lambda \in \mathcal{P}(n)$ be a partition $(\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $s \geqslant 2$, $\lambda_1 > \lambda_2$ and $\lambda_i \geqslant \lambda_{i+1}$ for any $i \in \{2, \ldots, s-1\}$. Put $m = \lambda_2 + \ldots + \lambda_s$. In this setting m is the number of cells in all rows of λ but the first.

Let t be a standard tableau of shape λ with n placed at its upper right cell.

Lemma (3)

The polytabloid $\mathbf{e}_t$ *is an eigenfunction of* J_n *with eigenvalue* $n - m - 1$.

Main result 1: the family of PI-eigenfunctions of S_n

Let us take a vector

$$P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$$

of $2m$ pairwise different elements from the set $\{1, \ldots, n-1\}$ arranged into m pairs and a vector

$$I_m = (i_1, i_2, \ldots, i_m)$$

of m pairwise different elements from the set $\{1, \ldots, n\}$. Define a function $f_{I_m}^{P_m} : \mathrm{Sym}_n \to \mathbb{R}$. For a permutation $\pi = [\pi_1 \pi_2 \ldots \pi_n] \in \mathrm{Sym}_n$, we put $f_{I_m}^{P_m}(\pi) = 0$, if there exists $t \in \{1, 2, \ldots, m\}$ such that $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$. If for every $t \in \{1, 2, \ldots, m\}$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$, then we define a binary vector $X_\pi = (x_1, x_2, \ldots, x_m)$ as follows:

$$x_t = \begin{cases} 1, & \text{if } \pi_{j_t} = i_t; \\ 0, & \text{if } \pi_{k_t} = i_t. \end{cases}$$

Main result 1: the family of PI-eigenfunctions of S_n

We use the vector X_π to complete the definition of the function $f_{I_m}^{P_m}$:

$$f_{I_m}^{P_m}(\pi) = \begin{cases} 1, & \text{if } X_\pi \text{ contains an even number of 1s;} \\ -1, & \text{if } X_\pi \text{ contains an odd number of 1s;} \\ 0, & \text{there exists } t \text{ such that } \pi_{j_t} \neq i_t \text{ and } \pi_{k_t} \neq i_t. \end{cases}$$

Theorem (1)

For $n \geqslant 3$, the function $f_{I_m}^{P_m}$ is an eigenfunction with eigenvalue $n-m-1$ of the Star graph S_n.

Main result 2: an expression of an eigenfunction given by a polytabloid in PI-eigenfunctions

Let $\mathbf{e}_t$ be an eigenfunction from the Lemma 3, and $n > 2m$ holds.

For any $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, k\}$, denote by $R_t(i)$ and $C_t(j)$ the symmetric groups on the elements of ith row and jth column of the tableau t, respectively. Then we have

$$R_t = R_t(1) \times R_t(2) \times \ldots \times R_t(s),$$

$$C_t = C_t(1) \times C_t(2) \times \ldots \times C_t(k),$$

where R_t and C_t are the row-stabilizer and the column-stabilizer of t, respectively.

Main result 2: an expression of an eigenfunction given by a polytabloid in PI-eigenfunctions

Put

$$CA_t = CA_t(1) \times CA_t(2) \times \ldots \times CA_t(k),$$

where $CA_t(j)$ denotes the subgroup of even permutations in $C_t(j)$.

Theorem (2)

The equality

$$f_{\phi(\mathbf{e}_t)} = \sum_{\sigma \in R_t(2) \times \ldots \times R_t(s)} \sum_{\pi \in CA_{\sigma(t)}} f^{P_\pi}_{(n-m+1,\ldots,n)}$$

holds, where P_π is a vector of m pairs uniquely determined by π.

Questions

1. Can we construct linearly independent embeddings of a Specht module into $\mathbb{C}[Sym_n]$?
2. Can we write down explicitly a basis of an eigenspace of the Star graph?
3. Is the family of PI-eigenfunctions complete? It is true in the case of the largest non-principal eigenvalue $n-2$ of S_n. Moreover, we can find a basis among PI-eigenfunctions.
4. Does there exist an analogue of the PI-eigenfunctions for the other half of positive eigenvalues of the Star graph?

Thank you for your attention!