On maximal cliques in Paley graphs of square order

Sergey Goryainov
(Hebei Normal University)
based on joint works with
Shamil Asgarli, Vladislav Kabanov, Huiqiu Lin Alexander Masley, Leonid Shalaginov, Alexandr Valyuzhenich, Chi Hoi Yip

Southern University of Science and Technology
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Outline

- Paley graphs
- Maximum cliques in Paley graphs of square order (Erdös-Ko-Rado theorem for Paley graphs of square order)
- Second largest known maximal cliques in Paley graphs of square order (a conjecture on maximal cliques in Paley graphs of square order as an analogue of the Hilton-Milner theorem)
- The weight-distribution bound
- Generalised Paley graphs of square order
- Partial similar results on maximal cliques in generalised Paley graphs of square order

1. Paley graphs

Paley graph $P(q)$

We consider finite undirected graphs without loops and mulptiple edges.

Let q be an odd prime power, $q \equiv 1(4)$.

The Paley graph of order q (denoted by $P(q)$) is a graph defined as follows:

- the vertex set is the finite field $\mathbb{F}_q$;
- two vertices γ_1, γ_2 are adjacent iff $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}_q^*$.

Since -1 is a square in $\mathbb{F}_q^*$ iff $q \equiv 1(4)$, the graph $P(q)$ is undirected.

Maximum and maximal cliques in $P(q)$

A clique (resp. coclique) is a set of pairwise adjacent (resp. non-adjacent) vertices.

Problem 1

What are maximum cliques (cocliques) in $P(q)$?

Since $P(q)$ is self-complementary, the studying cliques and the studying cocliques in $P(q)$ are equivalent.

Since $P(q)$ is strongly regular, we can apply Delsarte-Hoffman bound to $P(q)$. It says that a clique (coclique) in $P(q)$ has at most $\sqrt{q}$ vertices.

Problem 1 is unsolved in general.

2. Maximum cliques in Paley graphs of square order

Delsarte-Hoffman bound

For the clique number $\omega(\Gamma)$ of a distance-regular graph Γ (in particular, of a strongly regular graph), the Delsarte-Hoffman bound holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where $\theta_{\min}$ is the smallest eigenvalue of Γ.

A clique in a distance-regular graph whose size lies on the Delsarte-Hoffman bound is called a Delsarte clique.

The case of Paley graphs of square order q^2

Let q be an odd prime power.

According to the Delsarte-Hoffman bound, a clique in $P(q^2)$ has at most q vertices.

Since every element from $\mathbb{F}_q^*$ is a square in $\mathbb{F}_{q^2}^*$, the subfield $\mathbb{F}_q$ induces a clique of size q in $P(q^2)$, which implies the tightness of the Delsarte-Hoffman bound.

In 1984, Blokhuis classified maximum (Delsarte) cliques in $P(q^2)$ and proved [B84] that such a clique is an affine image of the subfield $\mathbb{F}_q$.

This result can be viewed as the analogue of Erdös-Ko-Rado theorem for Paley graphs of square order (see [GM15]).

[B84] A. Blokhuis, *On subsets of $GF(q^2)$ with square differences*, Indag. Math. **46** (1984) 369–372.

[GM15] C. D. Godsil, K. Meagher, *Erdös-Ko-Rado Theorems: Algebraic Approaches*, Cambridge University Press (2015).

EKR properties of Peisert-type graphs

Given any graph Γ for which we can describe its canonical cliques (that is, typically cliques with large size and simple structure), we can ask whether Γ has any of the following three related Erdős-Ko-Rado (EKR) properties:

- EKR property: the clique number of Γ equals the size of canonical cliques.
- EKR-module property: the characteristic vector of each maximum clique in Γ is a $\mathbb{Q}$-linear combination of characteristic vectors of canonical cliques in Γ.
- strict-EKR property: each maximum clique in Γ is a canonical clique.

EKR properties of Peisert-type graphs

EKR-properties of Peisert-type graphs (a family of Cayley graphs over finite fields that includes Paley graphs of square order) were independently studied in [AGLY22] and [LT22].

[AGLY22] S. Asgarli, S. Goryainov, H. Lin, C. H. Yip, *The EKR-module property of pseudo-Paley graphs of square order*, January 2022, `https://arxiv.org/abs/2201.03100`, accepted to the Electronic Journal of Combinatorics

[LT22] Cai Heng Li, Venkata Raghu Tej, On the EKR Module property. `https://arxiv.org/abs/2207.05947`

3. Second largest known maximal cliques in Paley graphs of square order

Second largest known maximal cliques in $P(q^2)$

Problem 2

What are maximal but not maximum cliques in $P(q^2)$?

Given an odd prime power q, put $r(q) := \begin{cases} 1, & q \equiv 1(4); \\ 3, & q \equiv 3(4). \end{cases}$

In 1996, Baker et al. found [2] maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$ for any odd prime power q. Let us say that these cliques are of Type I.

In 2018, Goryainov et al. found [3] one more family of maximal cliques in $P(q^2)$ with the same size $\frac{q+r(q)}{2}$. Let us say that these cliques are of Type II.

[BEHW96] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

[GKSV18] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Computations on maximal cliques of size $\frac{q+r(q)}{2}$ in $P(q^2)$

q	3	5	7	9	11	13	17	19	23
Clique size	3	3	5	5	7	7	9	11	13
#Orbits	1	1	1	3	3	4	9	4	4

q	25	27	29	31	37	41	43	47	49
Clique size	13	15	15	17	19	21	23	25	25
#Orbits	2	2	2	2	2	2	2	2	2

q	53	59	61	67	71	73	79	81	83
Clique size	27	31	31	35	37	37	41	41	43
#Orbits	2	2	2	2	2	2	2	2	2

Conjecture 1

For $q \geq 25$, the graph $P(q^2)$ contains exactly two non-equivalent cliques of size $\frac{q+r(q)}{2}$.

Finite field $\mathbb{F}_{q^2}$

Fix a non-square $d \in \mathbb{F}_q^*$.

Consider the polynomial $f(t) = t^2 - d \in \mathbb{F}_q[t]$.

Then

$$\mathbb{F}_{q^2} = \{x + y\alpha \mid x, y \in \mathbb{F}_q\},$$

where α is a root of $f(t)$.

Let β be a primitive element of $\mathbb{F}_{q^2}$.

Note that the elements from $\mathbb{F}_q^* = \langle \beta^{q+1} \rangle$ are squares in $\mathbb{F}_{q^2}^*$ because $q + 1$ is even.

Affine plane $\mathrm{AG}(2, q)$

Let $V(2, q)$ be a 2-dimensional vector space over $\mathbb{F}_q$.

Consider the affine plane $\mathrm{AG}(2, q)$ whose

- points are vectors of $V(2, q)$;
- lines are all cosets of 1-dimensional subspaces in $V(2, q)$;
- incidence relation is natural (whether a vector belongs to a coset).

Since $\mathbb{F}_{q^2}$ can be viewed as a 2-dimensional vector space over $\mathbb{F}_q$, the points of $\mathrm{AG}(2, q)$ can be matched with the elements of $\mathbb{F}_{q^2}$ as follows:

$$(x, y) \leftrightarrow x + y\alpha.$$

Quadratic and non-quadratic lines in $\mathrm{AG}(2, q)$

Given a line ℓ in $\mathrm{AG}(2, q)$, there exist elements $x_1 + y_1\alpha$ and $x_2 + y_2\alpha$ such that

$$\ell = \{x_1 + y_1\alpha + c(x_2 + y_2\alpha) \mid c \in \mathbb{F}_q\}.$$

The line ℓ is called quadratic (reps. non-quadratic) if $x_2 + y_2\alpha$ is a square (resp. non-square) in $\mathbb{F}_{q^2}^*$.

- The subfield $\mathbb{F}_q$ is a quadratic line.
- There are precisely $q + 1$ lines through a point: $\frac{q+1}{2}$ quadratic and $\frac{q+1}{2}$ non-quadratic lines.

$P(q^2)$ as a graph on points of the affine plane $\mathrm{AG}(2, q)$

For any distinct $\gamma_1, \gamma_2 \in \mathbb{F}_{q^2}$, the difference $\gamma_1 - \gamma_2$ is a square in $\mathbb{F}_{q^2}^*$ (equivalently, $\gamma_1 \sim \gamma_2$ in $P(q^2)$) iff the line connecting γ_1 and γ_2 is quadratic.

The automorphism group of $P(q^2)$

The automorphism group of $P(q^2)$ acts arc-transitively, and the following equality

$$\operatorname{Aut}\big(P(q^2)\big) = \big\{\, v \mapsto av^{\gamma} + b \mid a \in S,\ b \in \mathbb{F}_{q^2},\ \gamma \in \operatorname{Gal}(\mathbb{F}_{q^2}) \,\big\}$$

holds, where S is the set of square elements in $\mathbb{F}_{q^2}^*$.

The group $\operatorname{Aut}\big(P(q^2)\big)$ preserves the sets of quadratic and non-quadratic lines.

The group $\operatorname{Aut}\big(P(q^2)\big)$ has a subgroup that stabilises the quadratic line $\mathbb{F}_q$ and acts faithfully on the set of points that do not belong to $\mathbb{F}_q$; this subgroup is given by the affine transformations $x \mapsto ax + b$, where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$.

Geometric structure of maximal cliques of Type I

Take an element $\gamma \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Since $\mathbb{F}_q$ is a quadratic line, the line through γ that is parallel to $\mathbb{F}_q$, is quadratic too.

The other $\frac{q-1}{2}$ quadratic lines through γ intersect $\mathbb{F}_q$ in $\frac{q-1}{2}$ points; denote this set of $\frac{q-1}{2}$ intersection points by X_γ.

For the conjugate element $\overline{\gamma}$, the equality $X_{\overline{\gamma}} = X_\gamma$ holds.

If $q \equiv 1(4)$, each of the sets $\{\gamma\} \cup X_\gamma$ and $\{\overline{\gamma}\} \cup X_\gamma$ induce a maximal clique of size $\frac{q+1}{2}$.

If $q \equiv 3(4)$, the set $\{\gamma, \overline{\gamma}\} \cup X_\gamma$ induces a maximal clique of size $\frac{q+3}{2}$.

[BEHW96] R. D. Baker, G. L. Ebert, J. Hemmeter, A. J. Woldar, *Maximal cliques in the Paley graph of square order*, J. Statist. Plann. Inference **56** (1996) 33–38.

The subgroup Q of order $q+1$ in $\mathbb{F}^*_{q^2}$

Put

$$\omega := \beta^{q-1}, Q := \langle \omega \rangle,$$

$$Q_0 := \langle \omega^2 \rangle, Q_1 := \omega \langle \omega^2 \rangle.$$

- Q is a subgroup of order $q+1$ in $\mathbb{F}^*_{q^2}$
- Q is the kernel of the norm mapping $N : \mathbb{F}^*_{q^2} \mapsto \mathbb{F}^*_q$; given an element $\gamma = x + y\alpha \in \mathbb{F}^*_{q^2}$,

$$N(\gamma) := \gamma^{q+1} = \gamma\gamma^q = \gamma\overline{\gamma} = x^2 - y^2 d$$

- Q forms an oval in $\mathrm{AG}(2,q)$ (that is a set of $q+1$ points with no three on a line)
- Q is included to the neighbourhood of 0
- If $q \equiv 1(4)$, then Q induces the complete bipartite graph with parts Q_0 and Q_1
- If $q \equiv 3(4)$, then Q induces a pair of disjoint cliques Q_0 and Q_1

Geometric structure of maximal cliques of Type II

If $q \equiv 1(4)$, each of the sets Q_0 and Q_1 induces a maximal coclique of size $\frac{q+1}{2}$ in $P(q^2)$ (a maximal clique of size $\frac{q+1}{2}$ in $\overline{P(q^2)}$).

If $q \equiv 3(4)$, each of the sets $\{0\} \cup Q_0$ and $\{0\} \cup Q_1$ induces a maximal clique of size $\frac{q+3}{2}$ in $P(q^2)$.

[GKSV18] S. V. Goryainov, V. V. Kabanov, L. V. Shalaginov, A. A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications, **52**, (2018) 361–369.

Consider the mapping $\varphi : \mathbb{F}_{q^2} \mapsto \mathbb{F}_{q^2}$ defined by the rule:

$$\varphi(\gamma) := \begin{cases} \frac{\gamma+1}{\gamma-1} & \text{if } \gamma \neq 1, \\ 1 & \text{if } \gamma = 1. \end{cases}$$

Proposition 1 ([GMS22])

For any $\gamma = x + y\alpha \in Q$, $\gamma \neq 1$, the equality $\varphi(\gamma) = \frac{y}{x-1}\alpha$ holds.

It means that φ maps $Q \setminus \{1\}$ to the line $\{c\alpha \mid c \in \mathbb{F}_q\}$.

Proposition 2 ([GMS22])

For any $\gamma = x + y\alpha \in Q$, $\gamma \neq 1$, the equality $\varphi(\gamma^2) = \frac{x}{yd}\alpha$ holds.

Theorem 3 ([GMS22])

If $q \equiv 1(4)$, then $\varphi(Q_0)$ is a maximal coclique of size $\frac{q+1}{2}$ and of Type I;
if $q \equiv 3(4)$, then $\varphi(Q_0 \cup \{0\})$ is a maximal clique of size $\frac{q+3}{2}$ and of Type I.

[GMS22] S. Goryainov, A. Masley, L. V. Shalaginov, *On a correspondence between maximal cliques in Paley graphs of square order*, Discrete Math. **345** (2022), no. 6, 112853.

More on the correspondence between maximal cliques in Paley graphs of square order

Let q_1 be an odd prime power, $q_1 \equiv 1(4)$. Let $\Gamma_{q_1}(0)$ and $\Gamma_{q_1}(1)$ denote the subgraphs of the Paley graph $P(q_1)$ induced by the set of nonzero squares S_{q_1} and and the set $1 + S_{q^2}$, respectively. Obviously, $\Gamma_{q_1}(0) \simeq \Gamma_{q_1}(1)$.

In [B00], for a Paley graph $P(q_1)$, it was considered the subgroup H in $\mathrm{Aut}(\Gamma_{q_1}(0))$ generated by the maps $\gamma \mapsto a\gamma$ where a is a nonzero square, the field automorphisms, and the map $\gamma \mapsto \gamma^{-1}$. In [MK05], it was shown that $\mathrm{Aut}(\Gamma_{q_1}(0)) = H$.

Let $q_1 = q^2$, where q is an odd prime power, and let $S_{q^2} := (\mathbb{F}_{q^2}^*)^2$.

[B00] A. E. Brouwer, *Locally Paley graphs*, Designs, Codes and Cryptography **21** (2000), 69–76.

[MK05] M. Muzychuk, I. Kovács, *A solution of a problem of A. E. Brouwer*, Designs, Codes and Cryptography **34**, 249–264 (2005).

More on the correspondence between maximal cliques in Paley graphs of square order

In [GMS22], the mapping

$$\varphi(\gamma) := \begin{cases} \frac{\gamma+1}{\gamma-1} & \text{if } \gamma \neq 1, \\ 1 & \text{if } \gamma = 1; \end{cases}.$$

was shown to be a correspondence between two known constructions of maximal cliques in $P(q^2)$ of Type II and Type I.

Proposition 3 ([B22])

The mapping φ restricted to $V(\Gamma_{q^2}(1))$ is an automorphism of $\Gamma_{q^2}(1)$.

[GMS22] S. Goryainov, A. Masley, L. V. Shalaginov, *On a correspondence between maximal cliques in Paley graphs of square order*, Discrete Math. **345** (2022), no. 6, 112853.

[B22] A. E. Brouwer, private communication.

More on the correspondence between maximal cliques in Paley graphs of square order

Consider the mapping φ' obtained from φ by shifting the preimage and image by 1. For this, consider the mapping $\gamma \mapsto \frac{\gamma+1}{\gamma-1}$ and change the variable by $\gamma = \gamma' + 1$.

Then we have $\varphi : \gamma' + 1 \mapsto \frac{\gamma'+2}{\gamma'}$ and $\varphi' : \gamma' \mapsto \frac{2}{\gamma'}$, for any $\gamma' \neq 0$.

Thus, the automorphism φ of $\Gamma_{q^2}(1)$ corresponds to the automorphism φ' of $\Gamma_{q^2}(0)$.

[B22] A. E. Brouwer, private communication.

A conjecture on maximal cliques in Paley graphs of square order

Conjecture 2

Each second largest maximal clique in $P(q^2)$, where $q \geq 25$, is equivalent to a clique of Type I or a clique of Type II.

In view of the correspondence φ, the cliques of Type I and of Type II (with a removed vertex) are equivalent as cliques in the local subgraph.

This means we can talk about a unique type of cliques in Conjecture 2.

Thus, a proof of Conjecture 2 would be an analogue of the well-known Hilton-Milner theorem on the largest intersecting families of r-element sets that are maximal but not maximum [Y22].

[Y22] C. H. Yip, private communication.

4. The weight-distribution bound

Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a k-regular graph on n vertices and λ be an eigenvalue of its adjacency matrix A. Let $u = (u_1, \ldots, u_n)^t$ be an eigenvector of A corresponding to λ. Then u defines a function $f_u : V \mapsto \mathbb{R}$, which is called a λ-eigenfunction of Γ.

For an eigenfunction f_u of Γ, the *support* is the set

$$Supp(f_u) := \{x \in V \mid f_u(x) \neq 0\}.$$

MS-problem

The following problem was first formulated in [VK15] (see also [SV21] for the motivation and details).

Problem 4 (MS-problem)

Given a graph Γ *and its eigenvalue* λ*, find the minimum cardinality of the support of a* λ*-eigenfunction of* Γ*.*

A λ-eigenfunction having the minimum cardinality of support is called optimal.

Problem 5 (Strong MS-problem)

Given a graph Γ *and its eigenvalue* λ*, characterise optimal* λ*-eigenfunctions of* Γ*.*

[VK15] K. V. Vorobev, D. S. Krotov, *Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph*, Journal of Applied and Industrial Mathematics 9(1) (2015) 141–146.

[SV21] E. Sotnikova, A. Valyuzhenich, *Minimum supports of eigenfunctions of graphs: a survey*, Art Discrete Appl. Math. 4 (2021), no. 2, Paper No. 2.09, 34 pp.

A survey on Problem 5

Recently, Problem 5 was solved for several classes of graphs:

- all eigenvalues of Hamming graphs $H(n, q)$ when $q = 2$ or $q > 4$ and some eigenvalues of $H(n, q)$ when $q = 3, 4$;
- all eigenvalues of Johnson graphs (asymptotically);
- the smallest eigenvalue of Hamming, Johnson and Grassmann graphs;
- the largest non-principal eigenvalue of a Star graph S_n, $n \geq 8$;
- the largest non-principal eigenvalue of Doob graphs.

A survey on Problem 4

Excepting the results from the previous slide, Problem 4 was solved for several more classes of graphs:

- both non-principal eigenvalues of Paley graphs of square order;
- strongly regular bilinear forms graphs over a prime field.

Weight-distribution bound

Let Γ be a distance-regular graph of diameter $D(\Gamma)$ with intersection array $(b_0, b_1, \ldots, b_{D(\Gamma)-1}; c_1, c_2, \ldots, c_{D(\Gamma)})$ and nonprincipal eigenvalue λ.

Theorem 6 (Weight-distribution bound, [KMP16, Corollary 1])

A λ-eigenfunction f of Γ has at least $\sum_{i=0}^{D(G)} |W_i|$ nonzeros, where

$$W_0 = 1,$$

$$W_1 = \lambda$$

and

$$W_i = \frac{(\lambda - a_{i-1})W_{i-1} - b_{i-2}W_{i-2}}{c_i}.$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Known results when the weight-distribution bound is tight

- the eigenvalue -1 of the Boolean Hamming graph of an odd dimension and the minimum eigenvalue of an arbitrary Hamming graph;
- both non-principal eigenvalues of Paley graphs of square order;
- the minimum eigenvalue of Johnson graphs;
- the minimum eigenvalue of Grassmann graphs;
- the minimum eigenvalue of strongly regular bilinear forms graphs over a prime field.

Tightness of the weight-distribution bound for the smallest eigenvalue of a DRG

It was shown in [KMP16] that, for the smallest eigenvalue of a distance-regular graph Γ with a Delsarte clique, such that every edge is included in a constant number of Delsarte cliques, the tightness of the weight-distribution bound is equivalent to the existence of an isometric bipartite distance-regular induced subgraph $T_0 \cup T_1$, where T_0 and T_1 are parts, such that an optimal eigenfunction, up to multiplication by a non-zero constant, has the following form:

$$f(x) = \begin{cases} 1, & \text{if } x \in T_0; \\ -1, & \text{if } x \in T_1; \\ 0, & \text{otherwise.} \end{cases}$$

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

The weight-distribution bound for a non-principal eigenvalue of an SRG

If Γ is a strongly regular graph with non-principal eigenvalues θ, τ, where $\tau < 0 < \theta$, then the following holds.

Theorem 7 ([KMP16], Weight-distribution bound for SRG))

(1) *A τ-eigenfunction f of Γ has at least (-2τ) nonzeros;*
(2) *A θ-eigenfunction f of Γ has at least $2(\theta + 1)$ nonzeros.*

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for the negative eigenvalue of a special SRG with a Delsarte clique

Let Γ be a strongly regular graph with a Delsarte clique such that every edge is included in a constant number of Delsarte cliques. Let τ be the negative eigenvalue of Γ and $\overline{\theta}$ be the positive non-principal eigenvalue of the complement $\overline{\Gamma}$. (Note that $\theta = -1 - \tau$ holds for all strongly regular graphs)

Let f be a τ-eigenfunction of Γ (a $\overline{\theta}$-eigenfunction of $\overline{\Gamma}$).

Theorem 8 (Follows from [KMP16])

(1) $|Supp(f)|$ *meets WDB if and only if there exists an induced complete bipartite subgraph with parts* T_0, T_1 *of size* $-\tau$*;*
(2) $|Supp(f)|$ *meets WDB if and only if there exists an induced disjoint union of two cliques* T_0, T_1 *of size* $\theta + 1$.

[KMP16] D. S. Krotov, I. Yu. Mogilnykh, V. N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics 339 (3) (2016) 1150–1157.

Tightness of the weight-distribution bound for Paley graphs of square order

In [GKSV18], for Paley graphs $P(q^2)$, we showed the tightness of the weight-distribution bound for both non-principal eigenvalues, which are $\tau = \frac{-1-q}{2}$ and $\theta = \frac{-1+q}{2}$.

Let β be a primitive element in $\mathbb{F}_{q^2}$. Put $\omega := \beta^{q-1}$. Then $Q = \langle \omega \rangle$ is the subgroup of order $q+1$ in $\mathbb{F}^*_{q^2}$.

Facts about Q:

- Q is an oval in the corresponding affine plane;
- Q is the kernel of the norm mapping $N : \mathbb{F}^*_{q^2} \mapsto \mathbb{F}^*_q$, which means that $Q = \{\gamma \in \mathbb{F}^*_{q^2} \mid \gamma^{q+1} = 1\}$, or, equivalently, $Q = \{x + y\alpha \mid x, y \in \mathbb{F}_q, x^2 - y^2 d = 1\}$, where d is a non-square in $\mathbb{F}^*_q$ and $\alpha^2 = d$.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

Tightness of the weight-distribution bound for Paley graphs of square order

Let $Q_0 = \langle \omega^2 \rangle$ and $Q_1 = \omega Q_0$.

Facts about Q:

- if $q \equiv 1(4)$, then $Q = Q_0 \cup Q_1$ induces a complete bipartite graph with parts Q_0 and Q_1;
- if $q \equiv 3(4)$, then $Q = Q_0 \cup Q_1$ induces a pair of disjoint cliques Q_0 and Q_1.

Corollary 9 ([GKSV18, Theorem 2])

The weight-distribution bound is tight for both non-principal eigenvalues of Paley graphs of square order.

[GKSV18] S. Goryainov, V. Kabanov, L. Shalaginov, A. Valyuzhenich, *On eigenfunctions and maximal cliques of Paley graphs of square order*, Finite Fields and Their Applications 52 (2018) 361–369.

5. Generalised Paley graphs of square order

Generalised Paley graphs of square order; WDB for the smallest eigenvalue

Let $m > 1$ be a positive integer. Let q be an odd prime power, $q \equiv 1\ (2m)$. The m-Paley graph on $\mathbb{F}_q$, denoted $GP(q, m)$, is the Cayley graph $Cay(\mathbb{F}_q^+, (\mathbb{F}_q^*)^m)$, where $(\mathbb{F}_q^*)^m$ is the set of m-th powers in $\mathbb{F}_q^*$.

We consider the graphs $GP(q^2, m)$, where q is an odd prime power and m divides $q + 1$; these graphs are strongly regular and form a generalisation of Paley graphs of square order (the usual Paley graphs of square order are just 2-Paley graphs of square order).

The eigenvalues of $GP(q^2, m)$ are $\tau = (-\frac{q+1}{m})$ and $\theta = \frac{(m-1)q-1}{m}$.

Given an odd prime power q and an integer $m > 1$ such that m divides $q + 1$, a $(-\frac{q+1}{m})$-eigenfunction of the generalised Paley graph $GP(q^2, m)$ has at least $\frac{2(q+1)}{m}$ non-zeroes.

6. Partial similar results on maximal cliques in generalised Paley graphs of square order

Structure of Q (I)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, ..., $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 4 ([GSY22, Lemma 3.8])

Let q be a prime power and m be an integer such that $m > 1$, m divides $q + 1$. The mapping $\gamma \mapsto \gamma^{q-1}$ is a homomorphism from $\mathbb{F}_{q^2}^*$ to Q. Moreover, an element γ is an m-th power in $\mathbb{F}_{q^2}^*$ if and only if γ^{q-1} is an m-th power in Q.

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (II)

Let us divide Q into m parts

$$Q = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1},$$

where $Q_0 = \langle \omega^m \rangle$, $Q_1 = \omega Q_0$, $\ldots$, $Q_{m-1} = \omega^{m-1} Q_0$.

Proposition 5 ([GSY22, Lemma 3.10])

Let γ be an arbitrary element from Q, $\gamma \neq 1$. Then, for the image of $(\gamma - 1)$ under the action of the homomorphism, the following equality holds:

$$(\gamma - 1)^{q-1} = -\frac{1}{\gamma}.$$

[GSY22] S. Goryainov, L. Shalaginov, C. H. Yip, *On eigenfunctions and maximal cliques of generalised Paley graphs of square order*, arXiv:2203.16081

Structure of Q (III)

The following theorem basically states that each of the sets $Q_0, Q_1 \ldots, Q_{m-1}$ induces either a clique or an independent set, and there are at most two cliques among them.

Moreover, the theorem states that for every independent set Q_{i_1}, there exists uniquely determined independent set Q_{i_2} among $Q_0, Q_1 \ldots, Q_{m-1}$ such that there are all possible edges between Q_{i_1} and Q_{i_2} and there are no edges between Q_{i_1} and $Q \setminus Q_{i_2}$.

Structure of Q (IV)

Theorem 10 ([GSY22, Theorem 4.1])

Given an odd prime power q and an integer $m > 1$, m divides $q+1$, the following statements hold for the subgraph of $GP(q^2, m)$ induced by Q.
(1) If m divides $\frac{q+1}{2}$ and m is odd, then Q_0 is a clique, and $Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \leq i_1 < i_2 \leq m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if $i_1 + i_2 \equiv 0 \pmod{m}$, and there are no such edges if $i_1 + i_2 \not\equiv 0 \pmod{m}$. In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m-1}{2}} \cup Q_{\frac{m+1}{2}}$ induce $\frac{m-1}{2}$ complete bipartite graphs.

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Structure of Q (V)

(2) *If* m *divides* $\frac{q+1}{2}$ *and* m *is even, then* $Q_0, Q_{\frac{m}{2}}$ *are cliques, and* $Q_1, \ldots, Q_{\frac{m}{2}-1}, Q_{\frac{m}{2}+1}, \ldots, Q_{m-1}$ *are independent sets; moreover, for any distinct* i_1, i_2 *such that* $0 \leq i_1 < i_2 \leq m-1$*, there are all possible edges between the sets* Q_{i_1} *and* Q_{i_2} *if*

$$i_1 + i_2 \equiv 0 \ (\mathit{mod}\ m) \ \mathit{and}\ \{i_1, i_2\} \neq \{0, \frac{m}{2}\}$$

and there are no such edges if

$$i_1 + i_2 \not\equiv 0 \ (\mathit{mod}\ m) \ \mathit{or}\ \{i_1, i_2\} = \{0, \frac{m}{2}\}.$$

In particular, $Q_1 \cup Q_{m-1}, \ldots, Q_{\frac{m}{2}-1} \cup Q_{\frac{m}{2}+1}$ *induce* $(\frac{m}{2} - 1)$ *complete bipartite graphs.*

Structure of Q (VI)

(3) *If m does not divide $\frac{q+1}{2}$, then m is even.*
(3.1) *If $\frac{m}{2}$ is odd, then $Q_0, Q_1, \ldots, Q_{m-1}$ are independent sets; moreover, for any distinct i_1, i_2 such that $0 \le i_1 < i_2 \le m-1$, there are all possible edges between the sets Q_{i_1} and Q_{i_2} if*

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m),$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m).$$

In particular, if $m = 2$, $Q = Q_0 \cup Q_1$ is a complete bipartite graph; if $m \ge 6$,
$Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-2}{4}} \cup Q_{\frac{m+2}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-2}{4}} \cup Q_{\frac{3m+2}{4}}$
induce $\frac{m}{2}$ complete bipartite graphs.

Structure of Q (VII)

(3.2) *If* $\frac{m}{2}$ *is even, then* $Q_{\frac{m}{4}}, Q_{\frac{3m}{4}}$ *are cliques, and* $Q_0, \ldots, Q_{\frac{m}{4}-1}, Q_{\frac{m}{4}+1}, \ldots, Q_{\frac{3m}{4}-1}, Q_{\frac{3m}{4}+1}, \ldots, Q_{m-1}$ *are independent sets; moreover, for any distinct* i_1, i_2 *such that* $0 \leq i_1 < i_2 \leq m-1$*, there are all possible edges between the sets* Q_{i_1} *and* Q_{i_2} *if*

$$i_1 + i_2 \equiv \frac{m}{2} \ (mod\ m) \ and\ \{i_1, i_2\} \neq \{\frac{m}{2}, \frac{3m}{2}\},$$

and there are no such edges if

$$i_1 + i_2 \not\equiv \frac{m}{2} \ (mod\ m) \ or\ \{i_1, i_2\} = \{\frac{m}{2}, \frac{3m}{2}\}.$$

In particular, if $m = 4$*,* $Q_0 \cup Q_2$ *is a complete bipartite graph; if* $m \geq 8$*, then* $Q_0 \cup Q_{\frac{m}{2}}, \ldots, Q_{\frac{m-4}{4}} \cup Q_{\frac{m+4}{4}}, Q_{\frac{m}{2}+1} \cup Q_{m-1}, \ldots, Q_{\frac{3m-4}{4}} \cup Q_{\frac{3m+4}{4}}$ *induce* $\frac{m-2}{2}$ *complete bipartite graphs.*

Structure of Q (VIII) and tightness of WDB for the smallest eigenvalue of $GP(q^2, m)$

Corollary 11 ([GSY22, Corollary 4.2])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then, except for the case $m = 2$ and 2 divides $\frac{q+1}{2}$, there is at least one pair Q_{i_1}, Q_{i_2} among $Q_0, \dots, Q_{m-1}$ such that $Q_{i_1} \cup Q_{i_2}$ induces a complete bipartite subgraph.

Corollary 12 ([GSY22, Theorem 4.3])

Let q be an odd prime power and m be an integer $m \geq 2$, m divides $q+1$. Then the weight-distribution bound is tight for the eigenvalue $(-\frac{q+1}{m})$ of $GP(q^2, m)$.

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Cliques of Type I and II for generalised Paley graphs of square order

Cliques of Type I and II can be similarly defined for generalised Paley graphs of square order.

Moreover, a similar correspondence φ between them can be established; this correspondence is also an automorphism of the subgraph induced by the neighbourhood of 1. This means that the cliques of Type I are maximal if and only if the cliques of Type II maximal.

Problem 13

Investigate the maximality or near-maximality of the cliques of Type I and Type II in generalised Paley graphs of square order.

Partial progress towards the solution of Problem 13 can be found in [GSY22].

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Thank you for your attention!