

Cayley Neumaier graphs with a spread given by the cosets of a subgroup

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Definitions

We consider undirected graphs without loops and multiple edges.

A k -regular graph on v vertices is called **edge-regular** with parameters (v, k, λ) if every pair of adjacent vertices has λ common neighbours.

An edge-regular graph with parameters (v, k, λ) is called **strongly regular** with parameters (v, k, λ, μ) if every pair of distinct non-adjacent vertices has μ common neighbours.

A clique in a regular graph is called **m -regular** if every vertex that doesn't belong to the clique is adjacent to precisely m vertices from the clique. For an m -regular clique, the number m is called the **nexus**.

A question by Neumaier

For the clique number $\omega(\Gamma)$ of a strongly regular graph Γ , the **Delsarte-Hoffman bound** holds:

$$\omega(\Gamma) \leq 1 - \frac{k}{\theta_{\min}},$$

where θ_{\min} is the smallest eigenvalue of Γ .

A clique in a strongly regular graph is regular if and only if it has $1 - \frac{k}{\theta_{\min}}$ vertices; such a clique is called a **Delsarte clique**.

In 1981, Neumaier proved [1] that an edge-regular graph which is vertex-transitive, edge-transitive, and has a regular clique is strongly regular.

Neumaier then asked: **“Is it true that every edge-regular graph with a regular clique is strongly regular?”**

[1] A. Neumaier, *Regular Cliques in graphs and Special $1\frac{1}{2}$ -designs*, Finite Geometries and Designs, London Mathematical Society Lecture Note Series, 245–259 (1981).

Neumaier graphs

A non-complete edge-regular graph with parameters (v, k, λ) containing an m -regular s -clique is said to be a **Neumaier graph** with parameters $(v, k, \lambda; m, s)$.

It follows that if a Neumaier graph with parameters $(v, k, \lambda; m, s)$ has an m -regular clique of size s , then all cliques of size s in this graph are m -regular.

Thus, the notion of a Neumaier graph is a generalisation of the notion of a strongly regular graph with a Delsarte clique.

For a Neumaier graph with parameters $(v, k, \lambda; m, s)$, the number m is called the **nexus** of this graph.

A Neumaier graph that is not strongly regular is said to be a **strictly Neumaier graph**.

For a Neumaier graph, a **spread** is a partition of the vertex set into regular cliques.

Outline

- ▶ Strictly Neumaier graphs with nexus 1
 - ▶ A first construction by Greaves & Koolen;
 - ▶ Another construction by Greaves & Koolen;
 - ▶ A generalisation of Greaves & Koolen's constructions and application of the Wang-Qiu-Hu switching to it;
 - ▶ An infinite class of strictly Neumaier graphs based on the general construction
- ▶ Strictly Neumaier graphs with nexus greater than 1
 - ▶ Determination of the smallest strictly Neumaier graph and a construction of strictly Neumaier graphs with 2^i -regular cliques, for every positive integer i ;
- ▶ Cayley Neumaier graphs with a spread given by the cosets of a subgroup
 - ▶ Necessary and sufficient conditions;
 - ▶ Algorithm for enumeration and numerical results.

The first construction of strictly Neumaier graphs

In [2], Greaves and Koolen constructed an infinite family of strictly Neumaier graphs with 1-regular cliques.

For positive integers ℓ , m and an odd prime power q , consider the group $G_{\ell,m,q} := \mathbb{Z}_\ell \oplus \mathbb{Z}_2^m \oplus \mathbb{F}_q$. Put

$$S_0 := \{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m, (x, y) \neq (0, 0)\}$$

Let $\pi : \mathbb{Z}_2^m \setminus \{0\} \rightarrow \{0, \dots, 2^m - 2\}$ be a bijection and ρ be a primitive element of \mathbb{F}_q .

For each $y \in \mathbb{Z}_2^m \setminus \{0\}$, define

$$S_{y,\pi} := \{(0, y, \rho^j) \mid \pi(y) \equiv j \pmod{2^m - 1}\}$$

Consider the parametrised Cayley graph $\text{Cay}(G_{\ell,m,q}, S(\pi))$, where

$$S(\pi) := S_0 \cup \bigcup_{y \in \mathbb{Z}_2^m \setminus \{0\}} S_{y,\pi}$$

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

The first construction of strictly Neumaier graphs

Let $q = 2nr + 1$ for some positive integer r . For each $i \in \{0, \dots, n-1\}$, define the **cyclotomic class**

$$C_q^n(i) := \{\rho^{nj+i} \mid j \in 0, \dots, 2r-1\}.$$

For $a, b \in \{0, \dots, n-1\}$, define the **cyclotomic number**

$$c_q^n(a, b) := |C_q^n(a) + 1 \cap C_q^n(b)|.$$

Put $c := c_q^n(a, b)$ and $\ell := (1 + c)/2$.

Theorem ([2, Theorem 3.6, Corollary 4.4])

Let $q \equiv 1 \pmod{6}$, c be odd and $\pi : \mathbb{Z}_2^2 \setminus \{0\} \rightarrow \{0, 1, 2\}$ be a bijection. Then $\text{Cay}(G_{\ell, 2, q}, S(\pi))$ is a strictly Neumaier graph with parameters $(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Notes on the first construction

- ▶ Set $q := 7^a$, where $a \not\equiv 0 \pmod{3}$. Then $\text{Cay}(G_{\ell,2,q}, S(\pi))$ is a strictly Neumaier graph with parameters

$$(4\ell q, 4\ell - 2 + q, 4\ell - 2; 1, 4\ell).$$

In particular, if $a = 1$, then we have a strictly Neumaier graph with parameters $(28, 9, 2; 1, 4)$. This graph is the smallest example from [2].

- ▶ $\text{Cay}(G_{\ell,2,q}, S(\pi))$ has a spread of size q given by the cosets of the subgroup $\{(x, y, 0) \mid x \in \mathbb{Z}_\ell, y \in \mathbb{Z}_2^m\}$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

Four strictly Neumaier graphs on 24 vertices

Gavrilyuk and Goryainov then searched for examples in a collection of known Cayley-Deza graphs [3] and found four more strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$.

In [4], Greaves and Koolen found ‘another’ infinite family of strictly Neumaier graphs, which contains one of the four graphs on 24 vertices.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

Antipodal distance-regular graphs

A graph Γ of diameter d is called **distance-regular** if, for any two vertices $x, y \in V(\Gamma)$, the number of vertices at distance i from x and distance j from y depends only on i, j , and the distance from x to y . It is clear that distance regular graphs are edge-regular.

A distance-regular graph Γ of diameter d is called **a -antipodal** if the relation of being at distance d or distance 0 is an equivalence relation on the vertices of Γ with equivalence classes of size a .

The second construction of strictly Neumaier graphs

Let Γ be an a -antipodal distance-regular graph of diameter 3 with edge-regular parameters (v, k, λ) such that a is a proper divisor of $\lambda + 2$.

Put $t = \frac{\lambda+2}{a}$ and take t disjoint copies $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ of Γ .

For every antipodal class H in Γ , take the corresponding antipodal classes $H^{(1)}, \dots, H^{(t)}$ in $\Gamma^{(1)}, \dots, \Gamma^{(t)}$, respectively, and connect any two vertices from $H^{(1)} \cup \dots \cup H^{(t)}$ to form a 1-regular clique of size at .

Denote by $F_t(\Gamma)$ the resulting graph.

Theorem ([4])

The graph $F_t(\Gamma)$ is a strictly Neumaier graph having parameters $(tv, k + at - 1, \lambda; 1, at)$ and containing a spread.

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Discrete Mathematics, 342, Issue 10, (2019) 2818–2820.

Notes on the second construction

- ▶ In particular, if Γ is the icosahedron, then $a = 2$, $\lambda = 2$, $t = 2$ and $F_2(\Gamma)$ is one of the four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ found in [3].
- ▶ The other three graphs can be obtained in a similar way by choosing an appropriate matching of the antipodal classes in the two copies of the icosahedrons.

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Siberian Electronic Mathematical Reports, 11, 268–310 (2014) (in Russian).

Perfect codes in graphs

Let Γ be a simple undirected graph and $e \geq 1$ be an integer.

The **ball** with radius e and centre $u \in V(\Gamma)$ is the set of vertices of Γ with distance at most e to u in Γ .

A subset C of $V(\Gamma)$ is called a **perfect e -code** in Γ if the balls with radius e and centres in C form a partition of $V(\Gamma)$.

In particular, a perfect 1-code is a subset of vertices C such that every vertex not in C is adjacent to a unique element of C .

A generalisation of the two constructions

Let $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ be edge-regular graphs with parameters (v, k, λ) , and such that each $\Gamma^{(i)}$ has a partition of its vertices into perfect 1-codes of size a , where a is a proper divisor of $\lambda + 2$. Further, we define $t = (\lambda + 2)/a$.

For any $\ell \in \{1, \dots, t\}$, let $H_1^{(\ell)}, \dots, H_{v/a}^{(\ell)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(\ell)}$.

If $t \geq 2$, we also take a $(t - 1)$ -tuple of permutations from $\text{Sym}(\{1, \dots, v/a\})$, denoted by $\Pi = (\pi_2, \dots, \pi_t)$.

Using these graphs and the tuple Π , we define the graph $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ as follows.

1. Take the disjoint union of the graphs $\Gamma^{(1)}, \dots, \Gamma^{(t)}$.
2. For any $i \in \{1, \dots, v/a\}$, add an edge between any two distinct vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \dots \cup H_{\pi_t(i)}^{(t)}$ (which forms a 1-regular clique of size at).

A generalisation of the two constructions

Theorem ([5])

The following statements hold.

1. $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ has (a spread of) 1-regular cliques, each of size $\lambda + 2$;
2. $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ is an edge-regular graph with parameters $(vt, k + \lambda + 1, \lambda)$.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. <https://arxiv.org/abs/2109.13884>

A generalisation of the two constructions

Remark

We note that a converse to the theorem above is also true. Let $\Gamma \in NG(v, k, \lambda; 1, s)$ have a spread of 1-regular cliques. The graph Γ° created by removing edges from the cliques of this spread in Γ is also edge-regular by Soicher [6, Theorem 6.1]. It also follows that connected components of Γ° can be partitioned into perfect 1-codes, and the parameters have the same restrictions as the conditions of the statement of the theorem above.

[6] L. H. Soicher, *On cliques in edge-regular graphs*, *Journal of Algebra*, 421, 260–267 (2015). <https://doi.org/10.1016/j.jalgebra.2014.08.028>

A corollary

In the case for which $t = 1$ can occur, the construction can result in a strictly Neumaier graph. However, this is not necessarily true in all cases when $t = 1$. The following Corollary shows that for $t \geq 2$, our construction always results in a strictly Neumaier graph.

Corollary ([5])

Let $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ be non-complete edge-regular graphs with parameters (v, k, λ) and let $\Pi = (\pi_2, \dots, \pi_t)$ be a $(t - 1)$ -tuple of permutations from $\text{Sym}(\{1, \dots, v/a\})$.

Further, suppose that each graph $\Gamma^{(\ell)}$ has a partition of its vertices into perfect 1-codes $H_1^{(\ell)}, \dots, H_{v/a}^{(\ell)}$, each of size a , where a is a proper divisor of $\lambda + 2$ and $t = (\lambda + 2)/a$.

If $t \geq 2$, then $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ is a strictly Neumaier graph.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. <https://arxiv.org/abs/2109.13884>

Notes on the generalisation

- ▶ Non-isomorphic Taylor graphs with the same parameters give many new examples in the case $t \geq 2$.
- ▶ The four strictly Neumaier graphs on 24 vertices from [3] are given by a pair of icosahedrons, and the only difference between them is the choice of the permutation that matches the antipodal classes.
- ▶ The generalised construction covers both constructions from [2] and [4] (the cases $t = 1$ and $t \geq 2$, respectively).
- ▶ For $t = 1$ we can construct three new strictly Neumaier graphs: with parameters $(28, 9, 2; 1, 4)$, $(40, 12, 2; 1, 4)$ and $(65, 16, 3; 1, 5)$; eight graphs with parameters $(78, 17, 4; 1, 6)$.

[2] G. R. W. Greaves, J. H. Koolen, *Edge-regular graphs with regular cliques*, Europ. J. Combin., 71, 194–201 (2018).

[3] S. V. Goryainov, L. V. Shalaginov, *Cayley-Deza graphs with fewer than 60 vertices*, Sibirskie Èlektronnye Matematicheskie Izvestiya, 11, 268–310 (2014).

[4] G. R. W. Greaves, J. H. Koolen, *Another construction of edge-regular graphs with regular cliques*, Dis. Math., 342, Issue 10, (2019) 2818–2820.

Notes on the generalisation

- In [7], Corollary was independently proved and an infinite class of strictly Neumaier graphs based on the general construction was obtained.

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.
<https://doi.org/10.1016/j.jcta.2022.105684>

Examples from infinite edge-regular lattices (see [5])

Eisenstein integers are the complex numbers of the form $\mathbb{Z}[\omega] = \{b + c\omega : b, c \in \mathbb{Z}\}$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. They form a ring with respect to usual addition and multiplication.

The norm mapping $N : \mathbb{Z}[\omega] \mapsto \mathbb{N} \cup \{0\}$ is defined as follows. For an Eisenstein integer $b + c\omega$, $N(b + c\omega) = b^2 + c^2 - bc$ holds. The norm mapping N is known to be multiplicative.

It is well-known that $\mathbb{Z}[\omega]$ forms an Euclidean domain (in particular, a principal ideal domain).

The units of $\mathbb{Z}[\omega]$ are $\{\pm 1, \pm \omega, \pm \omega^2\}$. The natural geometrical interpretation of Eisenstein integers is the 6-regular triangular grid in the complex plane.

If it does not lead to a contradiction, we use the same notation $\mathbb{Z}[\omega]$ for the triangular grid.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. <https://arxiv.org/abs/2109.13884>

Examples from infinite edge-regular lattices

The grid $\mathbb{Z}[\omega]$ has exactly six elements of norm 7; these are $\{\pm(1 + 3\omega), \pm(3 + 2\omega), \pm(2 - \omega)\}$. Consider the ideal I generated by an element of norm 7 (say, by the element $2 - \omega$). The elements of I form a perfect 1-code in the triangular grid. Note that I is an additive subgroup of index 7 in $\mathbb{Z}[\omega]$; we denote it by I^+ . The seven cosets $\mathbb{Z}[\omega]/I^+$ give a partition of the triangular grid into seven perfect 1-codes. Take the following two additive subgroups of $\mathbb{Z}[\omega]$:

$$T_1 := \{2(-2 + \omega)x + 14y \mid x, y \in \mathbb{Z}\},$$

$$T_2 := \{(5 + \omega)x + 28y \mid x, y \in \mathbb{Z}\}.$$

Since $-2 + \omega$, 7 and $5 + \omega$ are divisible by $2 - \omega$, a generator of I , we have that T_1 and T_2 are subgroups in I^+ . Note that there exists a block of four balls of radius 1 such that the additive shifts of the block by the elements of T_1 and T_2 give two tessellations of $\mathbb{Z}[\omega]$.

Examples from infinite edge-regular lattices

Consider the quotient groups

$$G_1 := \mathbb{Z}[\omega]/T_1 \text{ and } G_2 := \mathbb{Z}[\omega]/T_2,$$

where $G_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$ and $G_2 \cong \mathbb{Z}_{28}$. Define two Cayley graphs

$$\Delta_1 := \text{Cay}(G_1, \{\pm(1 + T_1), \pm(\omega + T_1), \pm(\omega^2 + T_1)\}),$$

$$\Delta_2 := \text{Cay}(G_2, \{\pm(1 + T_2), \pm(\omega + T_2), \pm(\omega^2 + T_2)\}).$$

Note that Δ_1 and Δ_2 can be interpreted as quotient graphs of the triangular grid by T_1 and T_2 , respectively. Each of the graphs Δ_1 and Δ_2 is edge-regular with parameters $(28, 6, 2)$ and admits a partition into perfect 1-codes of size $a = 4$; these partitions are given by the original partition of the triangular grid into perfect 1-codes. We then apply the general construction, which gives two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$. The graph obtained from Δ_1 is isomorphic to the smallest graph from the first Greaves & Koolen's construction, and the graph obtained from Δ_2 is new.

More infinite edge-regular lattices

In the following, we consider countably infinite graphs with vertices consisting of elements of the vector space \mathbb{R}^n , for some integer $n \geq 3$. The elements of \mathbb{R}^n are called (**n -dimensional**) **vectors**, and we identify the elements with their coordinates with respect to the standard basis of \mathbb{R}^n .

Let $x \in \mathbb{R}^n$. For a set $A \subseteq \mathbb{R}$, the vector x is an **A -vector** if the value of all of its entries lie in A . The **weight** of x is the number of its non-zero entries.

Let $n \geq 3$ be a positive integer and let m be an even positive integer. Let $S_{n,m}^{(1)}$ denote the set of all n -dimensional $\{1, -1, 0\}$ -vectors of weight m whose sum of coordinates is zero. Let $S_{n,m}^{(2)}$ denote the set of all n -dimensional $\{1, -1, 0\}$ -vectors of weight m . Let $G_{n,m}^{(1)}$ and $G_{n,m}^{(2)}$ be the groups generated by $S_{n,m}^{(1)}$ and $S_{n,m}^{(2)}$ respectively.

More infinite edge-regular lattices

Proposition

For any positive even integer m and any integer n such that $n \geq m + 1$, the following statements hold.

- 1. $G_{n,m}^{(1)}$ is equal to $G_{n,2}^{(1)}$, which consists of all n -dimensional vectors with integer coordinates such that the sum of coordinates is equal to 0.*
- 2. $G_{n,m}^{(2)}$ is equal to $G_{n,2}^{(2)}$, which consists of all n -dimensional vectors with integer coordinates such that the sum of coordinates is even.*

More infinite edge-regular lattices

From now on, we let

$$G_n^{(1)} := G_{n,2}^{(1)},$$

$$G_n^{(2)} := G_{n,2}^{(2)},$$

and define graphs

$$\Gamma_{n,m}^{(1)} := \text{Cay}(G_n^{(1)}, S_{n,m}^{(1)})$$

and

$$\Gamma_{n,m}^{(2)} := \text{Cay}(G_n^{(2)}, S_{n,m}^{(2)}).$$

A graph Γ with infinitely many vertices is **edge-regular** with **parameters** (k, λ) if it is k -regular and each pair of adjacent vertices have exactly λ common neighbours.

In the following, we show that $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$ are infinite edge-regular graphs, and give the parameters of these graphs in terms of binomial coefficients.

More infinite edge-regular lattices

Proposition

For any positive even integer m and any integer n such that $n \geq m + 1$, the following statements hold.

- 1. The graph $\Gamma_{n,m}^{(1)}$ is an induced subgraph in $\Gamma_{n,m}^{(2)}$.*
- 2. $\Gamma_{n,m}^{(1)}$ is an infinite edge-regular graph with parameters (k_1, λ_1) , such that*

$$k_1 = \binom{n}{m} \binom{m}{\frac{m}{2}}, \quad \lambda_1 = \sum_{i=0}^{\frac{m}{2}} \binom{\frac{m}{2}}{i} \binom{\frac{m}{2}}{\frac{m}{2} - i} \binom{n-m}{\frac{m}{2} - i} \binom{n - \frac{3m}{2} + i}{i}.$$

- 3. $\Gamma_{n,m}^{(2)}$ is an infinite edge-regular graph with parameters (k_2, λ_2) , such that*

$$k_2 = 2^m \binom{n}{m}, \quad \lambda_2 = \binom{m}{\frac{m}{2}} \binom{n-m}{\frac{m}{2}} 2^{\frac{m}{2}}.$$

Comments

Remark

Note that if $n < \frac{3m}{2}$, then $\lambda_1 = 0$ and $\lambda_2 = 0$. Otherwise, $\lambda_1 > 0$ and $\lambda_2 > 0$.

Remark

The generating sets $S_{n,2}^{(1)}$ and $S_{n,2}^{(2)}$ are known as root systems A_{n-1} and D_n .

A_2 root lattice is isomorphic to the 6-regular triangular grid.

The root lattices A_3 and D_3 are both isomorphic to the tetrahedral-octahedral honeycomb.

More examples from infinite edge-regular lattices

In the following tables, we present the number of cases for which we find strictly Neumaier graphs using the graphs $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$, respectively.

The first column of the tables give the corresponding value of n .

The second column gives the Neumaier graph parameters of the graphs we find through the construction. The last column gives the number of pairwise non-isomorphic strictly Neumaier graphs we find from the construction.

More examples from infinite edge-regular lattices

n	parameters of SNG	#
3	$(28, 9, 2; 1, 4)$	2
4	$(78, 17, 4; 1, 6)$	≥ 8
5	$(168, 27, 6; 1, 8)$	≥ 12
6	$(310, 39, 8; 1, 10)$	≥ 1

Table: Number of strictly Neumaier graphs from quotients of $\Gamma_{n,2}^{(1)}$.

n	parameters of SNG	#
3	$(78, 17, 4; 1, 6)$	≥ 8
4	$(250, 33, 8; 1, 10)$	≥ 16

Table: Number of strictly Neumaier graphs from quotients of $\Gamma_{n,2}^{(2)}$.

A problem

However, we have not been able to find more examples of perfect codes and quotients of $\Gamma_{n,2}^{(1)}$ and $\Gamma_{n,2}^{(2)}$ that lead to strictly Neumaier graphs. Therefore, we ask the following.

Problem

What strictly Neumaier graphs can be obtained from quotients of infinite edge-regular graphs $\Gamma_{n,m}^{(1)}$ and $\Gamma_{n,m}^{(2)}$?

Examples from infinite edge-regular lattices

We can also use two infinite edge-regular graphs to get a new infinite edge-regular graph by taking the Cartesian product of the graphs.

Proposition

Let Γ_1 and Γ_2 be two infinite edge-regular graphs with parameters (k_1, λ) and (k_2, λ) , respectively. Then the Cartesian product of Γ_1 and Γ_2 is an edge-regular graph with parameters $(k_1 + k_2, \lambda)$.

An example from Cartesian products of infinite edge-regular lattices

Consider the Cartesian product of two 6-regular triangular grids; the resulting infinite graph is edge-regular with parameters $(12, 2)$. This graph has a subgroup perfect 1-code, and there exists an edge-regular quotient graph with parameters $(52, 12, 2)$. We then apply the general construction to this graph, which gives a strictly Neumaier graph having parameters $(52, 15, 2; 1, 4)$ and isomorphic to the second largest graph from the first Greaves & Koolen's construction. As we have seen this example using Cartesian products, we ask the following.

Problem

What strictly Neumaier graphs can be obtained from quotients of Cartesian products of infinite edge-regular graphs?

Spectrum of a graph

The **spectrum** of a graph Γ is the multiset of eigenvalues of the adjacency matrix of Γ .

Two graphs are **cospectral** if they have the same spectra.

The following switching (Wang-Qu-Hu switching), which produces cospectral graphs, was discovered in [8], applied in [9] to obtain new strongly regular graphs, and discussed in [10].

[8] W. Wang, L. Qiu, Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p , Linear Algebra and its Applications, Volume 563, 15 (2019), 154–177.

[9] F. Ihringer, A. Munemasa, New strongly regular graphs from finite geometries via switching, Linear Algebra and its Applications Volume 580, (2019), 464–474. <https://doi.org/10.1016/j.laa.2019.07.014>

[10] L. Qiu, Y. Ji, W. Wang, On a theorem of Godsil and McKay concerning the construction of cospectral graphs, Linear Algebra and its Applications, Volume 603 (2020), 265–274.

<https://doi.org/10.1016/j.laa.2020.05.025>

Wang-Qu-Hu switching

Lemma (WQH-switching)

Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that the induced subgraphs on C_1, C_2 , and $C_1 \cup C_2$ are regular, where the degrees in the induced subgraphs on C_1 and C_2 are the same. Suppose that all $x \in D$ satisfy one of the following:

- 1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, or*
- 2. $|\Gamma(x) \cap (C_1 \cup C_2)| \in \{|C_1|, |C_2|\}$.*

Construct a graph Γ' from Γ by modifying the edges between $C_1 \cup C_2$ and D as follows:

$$|\Gamma'(x) \cap (C_1 \cup C_2)| = \begin{cases} |C_1|, & \text{if } |\Gamma(x) \cap (C_1 \cup C_2)| = |C_2|; \\ |C_2|, & \text{if } |\Gamma(x) \cap (C_1 \cup C_2)| = |C_1|; \\ |\Gamma(x) \cap (C_1 \cup C_2)|, & \text{otherwise,} \end{cases}$$

for all $x \in D$. Then Γ' is cospectral with Γ .

WQH-switching for the general construction

Proposition ([5])

Let $t \geq 2$ and $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ be a strictly Neumaier graph obtained from the general construction. Then for any non-empty subset $I \subseteq \{1, \dots, t\}$ containing 1 and distinct $i, j \in \{1, \dots, v/a\}$, the partition

$$C_1 := \bigcup_{\ell \in I} H_i^{(\ell)}, \quad C_2 := \bigcup_{\ell \in I} H_j^{(\ell)},$$

$$D := V(F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})) \setminus (C_1 \cup C_2)$$

satisfies the conditions of WQH-switching.

Moreover, we have the equality

$(F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)}))' = F_{\Pi'}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$, where

$$(\Pi')_r = \begin{cases} \pi_r & \text{if } r \in I \\ (i \ j) \circ \pi_r & \text{if } r \notin I \end{cases}$$

Corollaries

Corollary ([5])

For any Π, Π' , $(t-1)$ -tuples of elements of $\text{Sym}(\{1, \dots, v/a\})$, the graphs $F_{\Pi}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ and $F_{\Pi'}(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ are cospectral.

It would be interesting to investigate how many pairwise non-isomorphic graphs can be constructed using our construction. In doing so, we may find a prolific construction of cospectral strictly Neumaier graphs. Although this has not been investigated in detail, we have already observed several pairwise non-isomorphic graphs with relatively small order.

Corollary ([5])

The four strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$ obtained from two copies of icosahedron are cospectral.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, A. D. Mednykh, *A general construction of strictly Neumaier graphs and a related switching*, September 2021. <https://arxiv.org/abs/2109.13884>

The smallest strictly Neumaier graph

In [11], Evans, G. and Panasenko found the smallest strictly Neumaier graph, which is a Cayley graph, has parameters $(16, 9, 4; 2, 4)$ and contains a spread given by the cosets of a subgroup.

It can be constructed by switching edges in the affine polar graph $VO^+(4, 2)$, which is isomorphic to the complement to (4×4) -lattice.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Strictly Neumaier graphs with 2^i -regular cliques

We then generalised the smallest strictly Neumaier graph and, for every positive integer i , by switching in certain affine polar graphs, found a strictly Neumaier graph with 4^{i+1} vertices containing a 2^i -regular clique and having parameters of these affine polar graphs as edge-regular graphs.

However, the graphs for $i \geq 2$ were not vertex-transitive, and it was an open question whether there exists a vertex-transitive strictly Neumaier graph with nexus greater than 1 except the smallest strictly Neumaier graph. In this project we solve this question in positive.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Problem on strictly Neumaier graphs

In general, we are interested in the following problem.

Problem

For which positive integers m does there exist a strictly Neumaier graph with an m -regular clique?

Remark

All previously known strictly Neumaier graphs had regular cliques with nexus equal to a power of 2. The only known strictly Neumaier graphs having regular cliques with nexus greater than 1 were found in [11].

Motivated by the fact that many known examples of strictly Neumaier graphs are Cayley graphs with a spread given by the cosets of a subgroup, we decided to have a general look at Cayley Neumaier graphs with a spread given by the cosets of a subgroup.

[11] R. J. Evans, S. V. Goryainov, D. I. Panasenko, The smallest strictly Neumaier graph and its generalisations, The Electronic Journal of Combinatorics, 26(2) (2019), #P2.29.

Equitable partitions

Let Γ be a k -regular graph with the vertex set $V(\Gamma)$.

Let $\Pi := (V_1, \dots, V_t)$ be a partition of $V(\Gamma)$ into t parts (t -partition).

The partition Π is said to be an **equitable** t -partition if for any $i, j \in \{1, \dots, t\}$ there is a constant p_{ij} such that any vertex from the part V_i is adjacent to precisely p_{ij} vertices from the part V_j .

The square matrix $P_\Pi := (p_{ij})_{i,j=1}^t$ is called the **quotient matrix** of the equitable t -partition Π .

Subgroup equitable 2-partitions in Cayley graphs

Lemma (Evans, G., Zhao, 2022+)

Let G be a finite group, $H < G$ with cosets Hg_1, \dots, Hg_n , with $g_1 = 1$. Further, let $S \subseteq G, 1 \notin S$.

Then $\{H, G \setminus H\}$ is an equitable 2-partition in $\Gamma = \text{Cay}(G, S)$ if and only if

- 1. $S \cap H$ is closed under inversion.*
- 2. There exists subsets $T_2, T_3, \dots, T_n \subseteq S$, such that*

2.1 $|T_i| = m$

2.2 T_i consists of representatives of the coset Hg_i

2.3 $T = \bigcup T_i$ is closed under inversion.

In this case, the induced subgraph $\Gamma[H]$ is $|S \cap H|$ -regular with nexus m , and the induced subgraph $\Gamma[G \setminus H]$ is $(|S| - m)$ -regular with nexus $|S \setminus H|$. In other words, we have quotient matrix

$$A(\Gamma/H) = \begin{pmatrix} |S \cap H| & |S \setminus H| \\ m & |S| - m \end{pmatrix}$$

Coset equitable partitions in Cayley graphs

Corollary (Evans, G., Zhao, 2022+)

Suppose $\Gamma = \text{Cay}(G, S)$ is a Cayley graph with group equitable 2-partition corresponding to $H < G$, $[G : H] = n$. Then Γ has an equitable n -partition X , with the parts the cosets of the group H , and quotient matrix

$$A(\Gamma/X) = \begin{pmatrix} |S \cap H| & m & \cdots & m \\ m & |S \cap H| & \cdots & \vdots \\ \vdots & \cdots & |S \cap H| & m \\ m & \cdots & m & |S \cap H| \end{pmatrix}$$

Edge-regularity of special Cayley graphs with coset equitable partition

Lemma (Evans, G., Zhao, 2022+)

Let G be a group, $H < G$ and $S \subseteq G, 1 \notin S$, where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets T_2, \dots, T_n and $T = \bigcup T_i$. Furthermore, assume $H^ \cap S = \emptyset$.*

Then Γ is edge-regular with parameters (v, k, λ) if and only if for all $g \in T$, the condition

$$|Tg \cap T| = \lambda$$

holds.

Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Lemma (Evans, G., Zhao, 2022+)

Let G be a group, $H < G$ and $S \subseteq G, 1 \notin S$ where $\{H, G \setminus H\}$ is an equitable 2-partition with corresponding sets T_2, \dots, T_n and $T = \bigcup T_i$. Furthermore, assume $H^ \subseteq S$, so $S = H^* \cup T$.*

Then $\Gamma = \text{Cay}(G, S)$ is an edge-regular (and thus Neumaier) graph with parameters (v, k, λ) if and only if

- 1. For all $h \in H^*$, $|Th \cap T| = \lambda - |H| + 2$*
- 2. For all $g \in T$, $|Tg \cap T| = \lambda - 2m + 2$*

Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Corollary (Evans, G., Zhao, 2022+)

Such a Neumaier graph $\Gamma = \text{Cay}(G, S)$ has parameters

$$v = ns$$

$$k = s - 1 + (n - 1)m$$

$$\lambda = s - 2 + \frac{(n - 1)m(m - 1)}{(s - 1)}$$

where $s = |H|$.

The smallest strictly Neumaier graph as a Cayley graph

- ▶ $G = C_2 \times C_8$
 $H = \langle (1, 2) \rangle$,
 $T_2 = \{(0, 1), (1, 7)\}$,
 $T_3 = \{(0, 2), (0, 6)\}$,
 $T_4 = \{(0, 7), (1, 1)\}$.
- ▶ $G = D_{16} = \langle a, b | a^8 = b^2 = 1, bab = a^{-1} \rangle$,
 $H = \langle a^2 \rangle, \{1, a^4, a^2b, a^6b\}$,
 $S = \{a, a^2, a^4, a^6, a^7, ab, a^2b, a^6b, a^7b\}$.
- ▶ $G = C_2 \times D_8 = \langle 1 \rangle \times \langle a, b | a^4 = b^2 = baba = 1 \rangle$,
 $H = \langle (0, a) \rangle, \langle (1, a) \rangle$,
 $S = \{(0, a), (0, a^2), (0, a^3), (0, a^2b), (0, a^3b), (1, a), (1, a^3), (1, a^2b), (1, a^3b)\}$

In all cases, $\Gamma = \text{Cay}(G, H^* \cup T)$ is the smallest strictly Neumaier graph, with a spread given by the cosets of H and parameters $(16, 9, 4; 2, 4)$.

The four graphs from the icosahedron as Cayley graphs

1. $G = S_4, H = \langle (1, 3, 2, 4) \rangle, T_2 = \{(1, 2, 3)\}, T_3 = \{(1, 3, 2)\}, T_4 = \{(1, 4, 2)\}, T_5 = \{(1, 3)(2, 4)\}, T_6 = \{(1, 2, 4)\}$
2. $G = S_4, H = \langle (1, 2), (3, 4) \rangle, T_2 = \{(1, 3, 4)\}, T_3 = \{(1, 4)(2, 3)\}, T_4 = \{(2, 4, 3)\}, T_5 = \{(1, 4, 3)\}, T_6 = \{(2, 3, 4)\}$
3. $G = C_2 \times A_4, H = \langle (1, (1, 2)(3, 4)), (1, (1, 3)(2, 4)) \rangle, T_2 = \{(0, (1, 2, 4))\}, T_3 = \{(0, (1, 3, 4))\}, T_4 = \{(0, (1, 3)(2, 4))\}, T_5 = \{(0, (1, 4, 2))\}, T_6 = \{(0, (1, 4, 3))\}$
4. $G = C_2 \times A_4, H = \langle (0, (1, 4)(2, 3)), (1, ()) \rangle, T_2 = \{(0, (1, 2, 3))\}, T_3 = \{(0, (2, 3, 4))\}, T_4 = \{(0, (2, 4, 3))\}, T_5 = \{(0, (1, 3, 2))\}, T_6 = \{(0, (1, 2)(3, 4))\}$

Then for each case, $\Gamma = \text{Cay}(G, H^* \cup T)$ is a strictly Neumaier graph with a spread given by the cosets of H and parameters $(24, 8, 2; 1, 4)$.

Two strictly Neumaier graphs with parameters (28, 9, 2; 1, 4)

1. $G = C_{28} = C_4 \times C_7, H = \langle (1, 0) \rangle, T_2 = \{(1, 1)\}, T_3 = \{(3, 6)\}, T_4 = \{(1, 3)\}, T_5 = \{(3, 4)\}, T_6 = \{(0, 2)\}, T_7 = \{(0, 5)\}$
2. $G = C_{28} = C_4 \times C_7, H = \langle (1, 0) \rangle, T_2 = \{(3, 1)\}, T_3 = \{(1, 6)\}, T_4 = \{(3, 3)\}, T_5 = \{(1, 4)\}, T_6 = \{(0, 2)\}, T_7 = \{(0, 5)\}$
3. $G = C_2 \times C_{14} = C_2 \times C_2 \times C_7, H = \langle (1, 0, 0), (0, 1, 0) \rangle, T_2 = \{(1, 0, 1)\}, T_3 = \{(1, 0, 4)\}, T_4 = \{(0, 1, 2)\}, T_5 = \{(0, 1, 5)\}, T_6 = \{(1, 1, 3)\}, T_7 = \{(1, 1, 6)\}$

Then for the cases 1,2 we find that the graphs $\text{Cay}(G, H^* \cup T)$ are isomorphic with different generating sets.

In all cases above, $\Gamma = \text{Cay}(G, H^* \cup T)$ is a strictly Neumaier graph with with a spread given by the cosets of H and parameters (28, 9, 2; 1, 4)

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Lemma (Evans, G., Zhao, 2022+)

Let G be a finite group and $H < G$ with cosets Hg_1, \dots, Hg_n , with $g_1 = 1$. Then, for any $i \in \{2, \dots, n\}$,

$$\text{Stab}_{\text{Aut } G}(Hg_i) < \text{Stab}_{\text{Aut } G}(H)$$

holds, where the stabilisers are setwise.

Corollary (Evans, G., Zhao, 2022+)

Let G be a finite group and $H < G$ with cosets Hg_1, \dots, Hg_n , with $g_1 = 1$. Then, for any $i \in \{2, \dots, n\}$, each automorphism $\varphi \in \text{Stab}_{\text{Aut } G}(Hg_i)$ preserves the partition

$$\{Hg_2, \dots, Hg_{i-1}, Hg_{i+1}, \dots, Hg_n\}.$$

Moreover, if $g_j = g_i g$, then $\varphi(Hg_j) = Hg_i \varphi(g)$.

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

Data:

a feasible tuple of parameters $(v, k, \lambda; m, s)$ of a Neumaier graph;

a finite group G , $|G| = v$;

a subgroup $H < G$, $|H| = s$, with cosets Hg_1, \dots, Hg_n , where $g_1 = 1$;

the automorphism group $\text{Aut } G$;

the stabiliser $\text{Stab}_{\text{Aut } G}(Hg_2)$;

Result:

the list of Cayley Neumaier graphs with a spread given by the cosets of the subgroup H with parameters $(v, k, \lambda; m, s)$ over G (each graph is given as $\text{Cay}(G, H^* \cup T)$)

An algorithm for enumeration of Cayley Neumaier graphs with a spread given by the cosets of a subgroup

```
for non-equivalent (under  $\text{Stab}_{\text{Aut } G}(Hg_2)$ )  $m$ -subsets  $T_2 \subset Hg_2$ 
  for all correct  $m$ -subsets  $T_3 \subset Hg_3$ 
    for all correct  $m$ -subsets  $T_4 \subset Hg_3$ 
      ...
      for all correct  $m$ -subsets  $T_n \subset Hg_n$ 
        if the necessary and sufficient conditions hold then
          save  $\text{Cay}(G, H^* \cup \underbrace{T_2 \cup T_3 \dots \cup T_n}_T)$ ;
        end if;
      end for;
    end for;
  end for;
end for;
```

Comments on the algorithm

- ▶ feasible tuples of parameters of Neumaier graphs can be computed according to necessary conditions on the existence of Neumaier graphs;
- ▶ feasible tuples parameters for strictly Neumaier graphs up to 64 vertices can be found in [7, Table 1];
- ▶ the multiplication tables of the groups and the generators of their automorphism groups can be taken from GAP;
- ▶ representatives of classes of conjugate subgroups of appropriate order can be taken from GAP;
- ▶ having the automorphism group of the group and the representatives of classes of conjugate subgroups, it is possible to compute the list of all non-equivalent subgroups;
- ▶ the isomorphism tests for the resulting graphs can be done in SAGE or MAGMA

[7] A. Abiad, W. Castryck, M. De Boeck, J. H. Koolen, S. Zeijlemaker, *An infinite class of Neumaier graphs and non-existence results*, Journal of Combinatorial Theory, Series A Volume 193, January 2023, 105684.

<https://doi.org/10.1016/j.jcta.2022.105684>

Numerical data

- ▶ all our computational efforts so far are devoted to the following feasible tuples of parameters:

$$(64, 21, 8; 2, 8),$$

$$(64, 28, 12; 3, 8),$$

$$(64, 35, 18; 4, 8),$$

$$(64, 42, 26; 5, 8)$$

$$(64, 49, 36; 6, 8);$$

these are the parameters of the block graphs of orthogonal arrays $\text{OA}(8,3)$, $\text{OA}(8,4)$, $\text{OA}(8,5)$, $\text{OA}(8,6)$ and $\text{OA}(8,7)$, respectively; we are interested in both strongly regular graphs (SRGs) and strictly Neumaier graphs (SNGs);

- ▶ the first and the fourth tuples are complementary (for SRGs) and correspond to parameters of a Latin square graph and its complement, respectively;

Numerical data

- ▶ the second and the third tuples are complementary (for SRGs) and correspond to parameters of prolific Wallis and Wallis2 constructions;
- ▶ the fifth tuple correspond to the parameters of the complement of 8×8 -lattice (there exists a unique strongly regular graph with these parameters);
- ▶ there are exactly 267 groups of order 64;
- ▶ according to the algorithm, the most difficult groups are those that have many involutions; thus, the most difficult group is $C2 \times C2 \times C2 \times C2 \times C2 \times C2$;
- ▶ it is expected to take about 2 months to make computations over $C2 \times C2 \times C2 \times C2 \times C2 \times C2$ (one month has been passed already);
- ▶ in other cases, the time needed to execute the program varies from several minutes to several days;
- ▶ currently, only 7 groups of order 64 (among 267 such groups) are unfinished; they have numbers 250,261,263,264,265,266 and 267 in GAP

Findings so far

- ▶ at least 6 SRGs and no SNGs with parameters $(64, 21, 8; 2, 8)$;
- ▶ at least 40 SRGs and at least 6 SNGs with parameters $(64, 28, 12; 3, 8)$;
- ▶ at least 123 SRGs and at least 138 SNGs with parameters $(64, 35, 18; 4, 8)$;
- ▶ at least 13 SRGs and a unique SNG with parameters $(64, 42, 26; 5, 8)$; some of the complements of the 13 SRGs have no regular cliques and thus cannot be Latin-square graphs; it is known that, for sufficiently large number of vertices, an SRG with parameters of a Latin square graph is a Latin square graph;
- ▶ a unique SRG and no SNGs with parameters $(64, 49, 36; 6, 8)$;
- ▶ note that some of these graphs may be Cayley graphs over more than one group G and may be given by more than one subgroup H for the same group G .

Further problems

- ▶ generalisation of the obtained examples;
- ▶ for the SRGs, we need to make isomorphism tests with known examples; this may be difficult because Wallis and Wallis2 constructions, which give SRGs with parameters $(64, 28, 12; 3, 8)$ and $(64, 35, 18; 4, 8)$, are prolific;
- ▶ investigating another feasible tuples of parameters including open tuples of parameters of SRGs from Brouwer's list (the smallest are $(96, 35, 10; 2, 6)$ and $(96, 60, 38; 9, 16)$);
- ▶ generalisation of the general construction with use of a partition into completely regular codes of radius 1 instead of a partition into perfect 1-codes;

Thank you for your attention!