

Math Problem Set # 5 *

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Exercises from **Chapter 8** at “Humpherys, Jeffrey and Tyler J. Jarvis, Foundations of Applied Mathematics, Volume II: Algorithm Design and Optimization, forthcoming. SIAM, Philadelphia, PA.”

Exercise 8.3

$$\begin{aligned} &\text{maximize} && 4b + 3j \\ &\text{subject to} && 15b + 10j \leq 1800 \\ &&& 2b + 2j \leq 300 \\ &&& j \leq 200 \\ &&& b, j \geq 0 \end{aligned}$$

Exercise 8.4

$$\begin{aligned} &\text{maximize} && 2x_{AB} + 5x_{AD} + 5x_{BC} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 2x_{CF} + 4x_{DE} + 3x_{EF} \\ &\text{subject to} && x_{AB} + x_{AD} = 10 \\ &&& x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\ &&& x_{CF} - x_{BC} = -2 \\ &&& x_{DE} - x_{AD} - x_{BD} = -3 \\ &&& x_{EF} - x_{BE} - x_{DE} = 4 \\ &&& -x_{BF} - x_{CF} - x_{EF} = -10 \\ &&& 0 \leq x_{AB}, x_{AD}, x_{BC}, x_{BD}, x_{BE}, x_{BF}, x_{CF}, x_{DE}, x_{EF} \leq 6 \end{aligned}$$

Exercise 8.5

(i)

$$\begin{aligned} &\text{maximize} && 3x_1 + x_2 \\ &\text{subject to} && x_1 + 3x_2 + w_1 = 15 \\ &&& 2x_1 + 3x_2 + w_2 = 18 \\ &&& x_1 - x_2 + w_3 = 4 \\ &&& x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

*I thank Yung-Hsu Tsui for his valuable comments.

ζ	=		$3x_1$	+	x_2	
w_1	=	15	-	x_1	-	$3x_2$
w_2	=	18	-	$2x_1$	-	$3x_2$
w_3	=	4	-	x_1	+	x_2
ζ	=	12	+	$4x_2$	-	$3w_3$
w_1	=	11	-	$4x_2$	+	w_3
w_2	=	10	-	$5x_2$	+	$2w_3$
x_1	=	4	+	x_2	-	w_3
ζ	=	20	-	$\frac{4}{5}w_2$	-	$\frac{7}{5}w_3$
w_1	=	3	+	$\frac{4}{5}w_2$	-	$\frac{3}{5}w_3$
x_2	=	2	-	$\frac{1}{5}w_2$	+	$\frac{2}{5}w_3$
x_1	=	6	-	$\frac{1}{5}w_2$	-	$\frac{3}{5}w_3$

Optimizer: (6, 2)

Optimum value: 20

(ii)

maximize $4x + 6y$ subject to $-x + 3x_2 + w_1 = 11$ $x + y + w_2 = 27$ $2x + 5y + w_3 = 90$ $x, y, w_1, w_2, w_3 \geq 0$

ζ	=		$4x$	+	$6y$	
w_1	=	11	+	x	-	y
w_2	=	27	-	x	-	y
w_3	=	90	-	$2x$	-	$5y$
ζ	=	66	+	$10x$	-	$6w_1$
y	=	11	+	x	-	w_1
w_2	=	16	-	$2x$	+	w_1
w_3	=	35	-	$7x$	+	$5w_1$
ζ	=	116	+	$\frac{8}{7}w_1$	-	$\frac{10}{7}w_3$
y	=	16	-	$\frac{2}{7}w_1$	-	$\frac{1}{7}w_3$
w_2	=	6	-	$\frac{3}{7}w_1$	+	$\frac{2}{7}w_3$
x	=	5	+	$\frac{5}{7}w_1$	-	$\frac{1}{7}w_3$
ζ	=	132	-	$\frac{8}{3}w_2$	-	$\frac{2}{7}w_3$
y	=	12	+	$\frac{2}{3}w_2$	-	$\frac{1}{3}w_3$
w_1	=	14	-	$\frac{7}{3}w_2$	+	$\frac{2}{3}w_3$
x	=	15	-	$\frac{5}{3}w_2$	+	$\frac{1}{3}w_3$

Optimizer: (15, 12)

Optimum value: 132

Exercise 8.6

$$\begin{aligned}
&\text{maximize } 4b + 3j \\
&\text{subject to } 15b + 10j + w_1 = 1800 \\
&\quad 2b + 2j + w_2 = 300 \\
&\quad j + w_3 = 200 \\
&\quad b, j, w_1, w_2, w_3 \geq 0
\end{aligned}$$

$$\begin{array}{rcllcl}
\zeta & = & & 4b & + & 3j \\
\hline
w_1 & = & 1800 & - & 15b & - & 10j \\
w_2 & = & 300 & - & 2b & - & 2j \\
w_3 & = & 200 & - & j & & \\
\hline
\zeta & = & 450 & + & b & - & \frac{3}{2}w_2 \\
\hline
w_1 & = & 300 & - & 5b & + & 5w_2 \\
j & = & 150 & - & b & - & \frac{1}{2}w_2 \\
w_3 & = & 50 & + & b & + & \frac{1}{2}w_2 \\
\hline
\zeta & = & 510 & - & \frac{1}{5}w_1 & - & \frac{1}{2}w_2 \\
b & = & 60 & - & \frac{1}{5}w_1 & + & w_2 \\
j & = & 90 & + & \frac{1}{5}w_1 & - & \frac{3}{2}w_2 \\
w_3 & = & 110 & - & \frac{1}{5}w_1 & + & \frac{3}{2}w_2
\end{array}$$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls

Maximal profit: \$510

Exercise 8.7

(i)

$$\begin{aligned}
&\text{maximize } x_1 + 2x_2 \\
&\text{subject to } -4x_1 - 2x_2 + x_3 = -8 \\
&\quad -2x_1 + 3x_2 + x_4 = 6 \\
&\quad x_1 + x_5 = 3 \\
&\quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

Auxiliary problem:

$$\begin{aligned}
&\text{maximize } -x_0 \\
&\text{subject to } -4x_1 - 2x_2 + x_3 - x_0 = -8 \\
&\quad -2x_1 + 3x_2 + x_4 - x_0 = 6 \\
&\quad x_1 + x_5 - x_0 = 3 \\
&\quad x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

ζ	=					-	x_0	
x_3	=	-8	+	$4x_1$	+	$2x_2$	+	x_0
x_4	=	6	+	$2x_1$	-	$3x_2$	+	x_0
x_5	=	3	-	x_1			+	x_0
ζ	=	-8	+	$4x_1$	+	$2x_2$	-	x_3
x_0	=	8	-	$4x_1$	-	$2x_2$	+	x_3
x_4	=	14	-	$2x_1$	-	$5x_2$	+	x_3
x_5	=	11	-	$5x_1$	-	$2x_2$	+	x_3
ζ	=						-	x_0
x_1	=	2	-	$\frac{1}{2}x_2$	+	$\frac{1}{4}x_3$	-	$\frac{1}{4}x_0$
x_4	=	10	-	$4x_2$	+	$\frac{1}{2}x_3$	+	$\frac{1}{2}x_0$
x_5	=	1	+	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$	+	$\frac{5}{4}x_0$
ζ	=	2	+	$\frac{3}{2}x_2$	+	$\frac{1}{4}x_3$		
x_1	=	2	-	$\frac{1}{2}x_2$	+	$\frac{1}{4}x_3$		
x_4	=	10	-	$4x_2$	+	$\frac{1}{2}x_3$		
x_5	=	1	+	$\frac{1}{2}x_2$	-	$\frac{1}{4}x_3$		
ζ	=	3	+	$2x_2$	-	x_5		
x_1	=	3			-	x_5		
x_4	=	12	-	$3x_2$	-	$2x_5$		
x_3	=	4	+	$2x_2$	-	$4x_5$		
ζ	=	11	-	$\frac{2}{3}x_4$	-	$\frac{7}{3}x_5$		
x_1	=	3			-	x_5		
x_2	=	4	-	$\frac{1}{3}x_4$	-	$\frac{2}{3}x_5$		
x_3	=	4	-	$\frac{2}{3}x_4$	-	$\frac{16}{3}x_5$		

Optimal point: (3, 4)

Optimal value: 11

(ii)

$$\begin{aligned}
&\text{maximize} && 5x_1 + 2x_2 \\
&\text{subject to} && 5x_1 + 3x_2 + x_3 = 15 \\
&&& 3x_1 + 5x_2 + x_4 = 15 \\
&&& 4x_1 - 3x_2 + x_5 = -12 \\
&&& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

Auxiliary problem:

$$\begin{aligned}
 &\text{maximize} && -x_0 \\
 &\text{subject to} && 5x_1 + 3x_2 + x_3 - x_0 = 15 \\
 &&& 3x_1 + 5x_2 + x_4 - x_0 = 15 \\
 &&& 4x_1 - 3x_2 + x_5 - x_0 = -12 \\
 &&& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

ζ	=						-	x_0
<hr/>								
x_3	=	15	-	$5x_1$	-	$3x_2$	+	x_0
x_4	=	15	-	$3x_1$	-	$5x_2$	+	x_0
x_5	=	-12	-	$4x_1$	+	$3x_2$	+	x_0
<hr/>								
ζ	=	-12	-	$4x_1$	+	$3x_2$	-	x_5
<hr/>								
x_3	=	27	-	x_1	-	$6x_2$	+	x_5
x_4	=	27	+	x_1	-	$8x_2$	+	x_5
x_0	=	12	+	$4x_1$	-	$3x_2$	+	x_5
<hr/>								
ζ	=	$-\frac{15}{8}$	-	$\frac{29}{8}x_1$	-	$\frac{3}{8}x_4$	-	$\frac{5}{8}x_5$
<hr/>								
x_3	=	$\frac{27}{4}$	-	$\frac{7}{4}x_1$	+	$\frac{3}{4}x_4$	+	$\frac{1}{4}x_5$
x_2	=	$\frac{27}{8}$	+	$\frac{1}{8}x_1$	-	$\frac{1}{8}x_4$	+	$\frac{1}{8}x_5$
x_0	=	$\frac{15}{8}$	+	$\frac{29}{8}x_1$	+	$\frac{3}{8}x_4$	+	$\frac{5}{8}x_5$

The original problem has no feasible solutions.

(iii)

$$\begin{aligned}
 &\text{maximize} && -3x_1 + x_2 \\
 &\text{subject to} && x_2 + x_3 = 4 \\
 &&& -2x_1 + 3x_2 + x_4 = 6 \\
 &&& x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

ζ	=			-	$3x_1$	+	x_2
<hr/>							
x_3	=	4			-	x_2	
x_4	=	6	+	$2x_1$	-	$3x_2$	
<hr/>							
ζ	=	2	-	$\frac{7}{3}x_1$	-	$\frac{1}{3}x_4$	
<hr/>							
x_3	=	2	-	$\frac{2}{3}x_1$	+	$\frac{1}{3}x_4$	
x_2	=	2	+	$\frac{2}{3}x_1$	-	$\frac{1}{3}x_4$	

Optimal point: $(0, 2)$

Optimal value: 2

Exercise 8.8

Give an example of a three-dimensional linear problem where the feasible region is closed

and unbounded, but where the objective function still has a unique feasible maximizer.

$$\begin{array}{ll}\text{maximize} & -x - y - z \\ \text{subject to} & x, y, z \geq 0\end{array}$$

Maximizer: $(0, 0, 0)$

Exercise 8.9

Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded and where the objective function has no maximizer.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x, y, z \geq 0\end{array}$$

Exercise 8.10

Give an example of a three-dimensional linear problem where the feasible region is empty.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x + y + z \leq -1 \\ & x, y, z \geq 0\end{array}$$

Exercise 8.11

Give an example of a three-dimensional linear problem where the feasible region is nonempty, closed, and bounded, but $\mathbf{0}$ is not feasible.

$$\begin{array}{ll}\text{maximize} & x + y + z \\ \text{subject to} & x + y + z \geq 1 \\ & x + y + z \leq 3 \\ & x, y, z \geq 0\end{array}$$

Auxiliary problem:

$$\begin{array}{ll}\text{maximize} & -w \\ \text{subject to} & -x - y - z - w \leq -1 \\ & x + y + z - w \leq 3 \\ & x, y, z, w \geq 0\end{array}$$

Exercise 8.12

$$\begin{array}{ll}\text{maximize} & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\ & x_1 + x_7 = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0\end{array}$$

ζ	=		$10x_1$	-	$57x_2$	-	$9x_3$	-	$24x_4$	
x_5	=	-	$0.5x_1$	+	$1.5x_2$	+	$0.5x_3$	-	x_4	
x_6	=	-	$0.5x_1$	+	$5.5x_2$	+	$2.5x_3$	-	$9x_4$	
x_7	=	1	-	x_1						
ζ	=	-	$27x_2$	+	x_3	-	$44x_4$	-	$20x_5$	
x_1	=		$3x_2$	+	x_3	-	$2x_4$	-	$2x_5$	
x_6	=		$4x_2$	+	$2x_3$	-	$8x_4$	+	x_5	
x_7	=	1	-	$3x_2$	-	x_3	+	$2x_4$	+	$2x_5$
ζ	=	1	-	$30x_2$	-	$42x_4$	-	$18x_5$	-	x_7
x_1	=	1							-	x_7
x_6	=	2	-	$2x_2$	-	$4x_4$	+	$5x_5$	-	$2x_7$
x_3	=	1	-	$3x_2$	+	$2x_4$	+	$2x_5$	-	x_7

Optimal point: $(1, 0, 1, 0)$ Optimum value: 1

Exercise 8.15

If $\mathbf{x} \in \mathbb{R}^n$ is feasible for the primal and $\mathbf{y} \in \mathbb{R}^m$ is feasible for the dual, then $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$.

Proof:

Let $\mathbf{x} \in \mathbb{R}^n$ be feasible for the primal and $\mathbf{y} \in \mathbb{R}^m$ be feasible for the dual. Then we know that $A\mathbf{x} \leq \mathbf{b}$ and $A^T \mathbf{y} \leq \mathbf{c}$, so

$$\begin{aligned}
 A\mathbf{x} &\leq \mathbf{b} \\
 \mathbf{x}^T A^T &\leq \mathbf{b}^T \\
 \mathbf{x}^T A^T \mathbf{y} &\leq \mathbf{b}^T \mathbf{y} \\
 \mathbf{x}^T \mathbf{c} &\leq \mathbf{b}^T \mathbf{y} \\
 \mathbf{c}^T \mathbf{x} &\leq \mathbf{b}^T \mathbf{y}
 \end{aligned}$$

Exercise 8.17

For a linear optimization problem in standard form, the dual of the dual optimization problem is again the primal problem.

Proof:

Consider the primal problem:

$$\begin{aligned}
 &\text{maximize } \mathbf{c}^T \mathbf{x} \\
 &\text{subject to } A\mathbf{x} \preceq \mathbf{b} \\
 &\quad \mathbf{x} \succeq \mathbf{0}
 \end{aligned}$$

with the dual problem:

$$\begin{aligned}
 &\text{minimize } \mathbf{b}^T \mathbf{y} \\
 &\text{subject to } A^T \mathbf{y} \succeq \mathbf{c} \\
 &\quad \mathbf{y} \succeq \mathbf{0}
 \end{aligned}$$

The dual in standard form becomes:

$$\begin{aligned} & \text{maximize } (-\mathbf{b}^T)\mathbf{y} \\ & \text{subject to } (-A)^T\mathbf{y} \preceq -\mathbf{c} \\ & \mathbf{y} \succeq \mathbf{0} \end{aligned}$$

and the dual of the dual is:

$$\begin{aligned} & \text{minimize } (-\mathbf{c}^T)\mathbf{x} \\ & \text{subject to } ((-A)^T)^T\mathbf{x} \succeq -\mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{aligned}$$

In standard form, this is:

$$\begin{aligned} & \text{maximize } \mathbf{c}^T\mathbf{x} \\ & \text{subject to } A\mathbf{x} \preceq \mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{aligned}$$

which is the original primal problem.

Exercise 8.18

Primal:

$$\begin{aligned} & \text{maximize } x_1 + x_2 \\ & \text{subject to } 2x_1 + x_2 + w_1 = 3 \\ & \quad \quad \quad x_1 + 3x_2 + w_2 = 5 \\ & \quad \quad \quad 2x_1 + 3x_2 + w_3 = 4 \\ & \quad \quad \quad x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

ζ	$=$		x_1	$+$	x_2	
w_1	$=$	3	$-$	$2x_1$	$-$	x_2
x_2	$=$	5	$-$	x_1	$-$	$3x_2$
x_5	$=$	4	$-$	$2x_1$	$-$	$3x_2$
<hr style="border-top: 3px double #000;"/>						
ζ	$=$	$\frac{3}{2}$	$+$	$\frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
x_1	$=$	$\frac{3}{2}$	$-$	$\frac{1}{2}x_2$	$-$	$\frac{1}{2}w_1$
w_2	$=$	$\frac{7}{2}$	$-$	$\frac{5}{2}x_2$	$+$	$\frac{1}{2}w_1$
w_3	$=$	1	$-$	$2x_2$	$+$	w_1
<hr style="border-top: 3px double #000;"/>						
ζ	$=$	$\frac{7}{4}$	$-$	$\frac{1}{4}w_1$	$-$	$\frac{1}{4}w_1$
x_1	$=$	$\frac{5}{4}$	$-$	$\frac{3}{4}w_1$	$+$	$\frac{1}{4}w_3$
w_2	$=$	$\frac{9}{4}$	$-$	$\frac{3}{4}w_1$	$+$	$\frac{5}{4}w_3$
x_2	$=$	$\frac{1}{2}$	$+$	$\frac{1}{2}w_1$	$-$	$\frac{1}{2}w_3$

Optimal point: $(\frac{5}{4}, \frac{1}{2})$

Optimum value: $\frac{7}{4}$

Dual:

$$\begin{aligned}
 &\text{minimize} && 3y_1 + 5y_2 + 4y_3 \\
 &\text{subject to} && 2y_1 + y_2 + 2y_3 \geq 1 \\
 &&& y_1 + 3y_2 + 3y_3 \geq 1 \\
 &&& y_1 + 3y_2 + 3y_3 \geq 1 \\
 &&& y_1, y_2, y_3 \geq 0
 \end{aligned}$$

Dual in standard form:

$$\begin{aligned}
 &\text{maximize} && -3y_1 - 5y_2 - 4y_3 \\
 &\text{subject to} && -2y_1 - y_2 - 2y_3 + v_1 - v_0 = -1 \\
 &&& -y_1 - 3y_2 - 3y_3 + v_2 - v_0 = -1 \\
 &&& y_1, y_2, y_3, v_1, v_2 \geq 0
 \end{aligned}$$

ζ	=											-	v_0
v_1	=	-1	+	$2y_1$	+	y_2	+	$2y_3$	+				v_0
v_2	=	-1	+	y_1	+	$3y_2$	+	$3y_3$	+				v_0
ζ	=	-1	+	$2y_1$	+	y_2	+	$2y_3$	-				v_1
v_0	=	1	-	$2y_1$	-	y_2	-	$2y_3$	+				v_1
v_2	=			-	y_1	+	$2y_2$	+	y_3	+			v_1
ζ	=											-	v_0
y_2	=	1	-	$2y_1$	-	$2y_3$	+	v_1	-			v_0	
v_2	=	2	-	$5y_1$	-	$3y_3$	+	$3v_1$	-			$2v_0$	
ζ	=	-2	+	y_1	-	$3y_2$	-	$2v_1$					
y_3	=	$\frac{1}{2}$	-	y_1	-	$\frac{1}{2}y_2$	+	$\frac{1}{2}v_1$					
v_2	=	$\frac{1}{2}$	-	$2y_1$	+	$\frac{3}{2}y_2$	+	$\frac{3}{2}v_1$					
ζ	=	$-\frac{7}{4}$	-	$\frac{3}{2}y_2$		-	$\frac{5}{4}v_1$	-	$\frac{1}{2}v_2$				
y_3	=	$\frac{1}{4}$	-	$2\frac{3}{2}y_2$		-	$\frac{1}{4}v_1$	+	$\frac{1}{2}v_2$				
y_1	=	$\frac{1}{4}$	+	$\frac{3}{2}y_2$		+	$\frac{3}{4}v_1$	-	$\frac{1}{2}v_2$				

Optimal point: $(\frac{1}{4}, 0, \frac{1}{4})$

Optimum value: $\frac{7}{4}$