

# Math Problem Set # 3 \*

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Exercises from **Chapter 4** at “Humpherys, Jeffrey, Tyler J. Jarvis, and Emily J. Evans, Foundations of Applied Mathematics, Volume I: Mathematical Analysis, 2017. SIAM, Philadelphia, PA.”

**Problem 2.** Let  $V = \text{span}(\{1, x, x^2\})$  and  $D$  is the derivative operator  $D : V \rightarrow V$  such that  $D[p](x) = p'(x)$ . In the Chapter 3 exercises we showed that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Then  $\det(D - \lambda I) = (-\lambda)^3 = 0$ . Therefore the eigenvalue is  $\lambda = 0$  with algebraic multiplicity 3 and geometric multiplicity of 0 since all the eigenvectors are in the form  $(a, 0, 0)$ .

**Problem 4.** Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) If  $A = A^H$  then

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Then  $\text{tr}(A) = a + d$  and  $\det(A) = ad - b^2$ .  $A$  has real eigenvalues if  $\text{tr}(A)^2 - 4\det(A) \geq 0$  where  $\text{tr}(A)^2 - 4\det(A)$  is the term under the square root in the quadratic formula. We have that  $\text{tr}(A)^2 - 4\det(A) = a^2 + d^2 + 2ad - 4ad + 4b^2 = (a - d)^2 + 4b^2 \geq 0$  so  $A$  has real eigenvalues.

(ii) If  $A = -A^H$  then

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

Then  $\text{tr}(A) = 0$  and  $\det(A) = b^2$ . We have that  $\text{tr}(A)^2 - 4\det(A) = -4b^2 < 0$  so  $A$  has imaginary eigenvalues.

**Problem 6.** Let the matrix  $A$  be upper triangular. Then, since upper triangular matrices are closed under addition and subtraction,  $A - \lambda I$  is also an upper triangular matrix with diagonal entries  $a_{ii} - \lambda$ . The determinant of an upper triangular matrix is the product of the diagonal entries so  $p_A(\lambda) = \det(A - \lambda I) = \prod_{i=1}^n (a_{ii} - \lambda) = 0$ . Therefore the eigenvalues of  $A$  are the values of  $\lambda$  such that  $a_{ii} - \lambda = 0 \forall i$  or equivalently  $a_{ii} = \lambda \forall i$  where the  $a_{ii}$  are the diagonal entries of  $A$ .

**Problem 8.** Let  $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$  and  $V = \text{span}(\{S\})$ .

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- (i)  $S$  is a basis for  $V$  if  $S$  spans  $V$  and is linearly independent. The first part follows from our assumptions. We now show the second. We proved in problem 3.8 that the set  $S$  is orthonormal which implies that they are linearly independent and are hence a basis for  $V$ .
- (ii) The derivatives of the basis are  $\{\cos(x), -\sin(x), 2\cos(2x), -2\sin(2x)\}$  respectively. Then

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

- (iii) The two complementary  $D$ -invariant subspaces of  $V$  are  $\{\sin(x), \cos(x)\}$  and  $\{\sin(2x), \cos(2x)\}$ .

**Problem 13.** To find the diagonalization of  $A$ , we compute its eigenvalues using the characteristic polynomial which is  $p(\lambda) = \lambda^2 - 1.4\lambda + 0.4 = (\lambda - 1)(\lambda - 0.4) = 0$  so  $\lambda = 1, 0.4$ . The associated eigenvectors are respectively

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We therefore have that

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

where  $D = P^{-1}AP$ .

**Problem 15.** Let  $A$  be a semisimple matrix and therefore diagonalizable where  $A = PDP^{-1}$ . Then  $f(A) = f(PDP^{-1}) = a_0I + a_1PDP^{-1} + \dots + a_nPD^nP^{-1} = P(a_0I + a_1D + \dots + a_nD^n)P^{-1} = Pf(D)P^{-1}$ . Then the diagonal elements of  $f(D)$  are equal to  $f(\lambda_i)$ , and since  $f(D)$  is the diagonal matrix that results from diagonalizing  $f(A)$ ,  $f(\lambda_i)$  are the eigenvalues of  $f(A)$ .

**Problem 16.**

- (i) We have that

$$B = \lim_{n \rightarrow \infty} A^n = \lim_{n \rightarrow \infty} PD^nP^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

Similarly,

$$A^k = PD^kP^{-1} = \begin{bmatrix} \frac{2 + .4^k}{3} & \frac{2 - 2(.4)^k}{3} \\ \frac{1 - .4^k}{3} & \frac{1 + 2(.4)^k}{3} \end{bmatrix}$$

and therefore,

$$\|A^k - B\|_1 = \left\| \begin{bmatrix} \frac{.4^k}{3} & -\frac{2(.4)^k}{3} \\ -\frac{.4^k}{3} & \frac{2(.4)^k}{3} \end{bmatrix} \right\| = \frac{2}{3}(.4)^k$$

So if  $\frac{4}{3}(.4)^k < \epsilon$  then  $k > \frac{\ln(\frac{3}{4}\epsilon)}{\ln(.4)}$  and  $\exists B$  such that  $B = \lim_{n \rightarrow \infty} A^n$  with respect to the 1-norm.

(ii) With respect to the  $\infty$ -norm,

$$\|A^k - B\|_\infty = (.4)^k$$

Then if  $(.4)^k < \epsilon$  then  $k > \frac{\ln(\epsilon)}{\ln(.4)}$ . With respect to the Frobenious norm,

$$\|A^k - B\|_F = \sqrt{\frac{10}{9}}(.4)^k$$

Then if  $\sqrt{\frac{10}{9}}(.4)^k < \epsilon$  then  $k > \frac{\ln(\epsilon/\sqrt{\frac{10}{9}})}{\ln(.4)}$ . The answer of B therefore does not depend on choice of norm.

(iii) By theorem 4.3.12, we have that since A is semi-simple, that the eigenvalues of  $f(A) = 3I + 5A + A^3$  are the values  $f(\lambda_i)$  and therefore the eigenvalues of  $f(A)$  are  $f(0.4) = 3 + 5 * 0.4 + 0.4^3 = 5.064$  and  $f(1) = 3 + 5 + 1 = 9$ .

**Problem 18.** Let  $\lambda$  be an eigenvalue of  $A$ . Then it is also an eigenvalue of  $A^T$  because  $A$  and  $A^T$  have the same characteristic polynomial. Then  $\exists x$  such that  $A^T x = \lambda x$  implying that  $(A^T x)^T = (\lambda x)^T$  and therefore  $x^T A = \lambda x^T$ .

**Problem 20.** Let  $A$  be Hermitian and orthonormally similar to  $B$ . Then  $B = PAP^H = PA^H P^H = (PAP^H)^H = B^H$  so  $B$  is also Hermitian.

**Problem 24.** The denominator of the Rayleigh quotient is always real, so whether  $\rho(x)$  is real or imaginary is determined solely by the numerator. Let  $A$  be Hermitian. Then  $\langle x, Ax \rangle = x^H Ax = x^H A^H x = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle}$ . Since  $\langle x, Ax \rangle = \overline{\langle x, Ax \rangle}$  the numerator is real and the Rayleigh quotient takes on real values. Now let  $A$  be Skew-Hermitian. Then  $\langle x, Ax \rangle = x^H Ax = -x^H A^H x = -\langle Ax, x \rangle = -\overline{\langle x, Ax \rangle}$ . Since  $\langle x, Ax \rangle = -\overline{\langle x, Ax \rangle}$  the numerator is imaginary and the Rayleigh quotient takes on imaginary values.

**Problem 25.**

(i) If  $A \in M_n(\mathbb{C})$  is normal, then it has orthonormal eigenvectors  $[x_1, \dots, x_n]$ . Note that  $\langle x_j, x_j \rangle = x_j^H x_j = 1$  and  $\langle x_i, x_j \rangle = x_i^H x_j = 0$  for  $i \neq j$ . Then  $(x_1 x_1^H + \dots + x_n x_n^H) x_j = x_j x_j^H x_j = I x_j$ . Then  $I = x_1 x_1^H + \dots + x_n x_n^H$ .

- (ii) Consider  $(\lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H)x_j = \lambda x_j x_j^H x_j = \lambda x_j = Ax_j$ . Then  $A = \lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H$ .

**Problem 27.** Let  $A$  be positive definite and let  $x \neq 0$ . Then  $\langle x, Ax \rangle = x^H Ax > 0$ . Let  $x = e_i$  where  $e_i$  is vector  $i$  of the standard basis. Then  $e_i^H A e_i > 0$  is entry  $(i, i)$  on the diagonal of  $A$  and is real and positive.

**Problem 28.** Let  $A, B$  be positive semi-definite, then the diagonal has all non-negative elements so the sum of the diagonal elements is non-negative. Therefore  $0 \leq \text{tr}(A)$  and therefore  $0 \leq \text{tr}(AB)$ . Then using the Frobenius norm and Cauchy-Schwarz,  $\|AB\| \leq \|A\| \|B\|$  and  $\sqrt{\|AB\|} \leq \sqrt{\|A\|} \sqrt{\|B\|}$  and finally  $\text{tr}(AB) \leq \text{tr}(A)\text{tr}(B)$ .

**Problem 31.**

- (i) Let  $A = U\Sigma V^H$ ,  $y = V^H x$ , and  $\sigma_1$  be the largest singular value of  $A$ . Then

$$\begin{aligned} \|A\|_2 &= \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|U\Sigma V^H x\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|\Sigma V^H x\|_2}{\|x\|_2} = \\ &= \sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|V y\|_2} = \sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|y\|_2} = \sup_{\|y\|=1} \|\Sigma y\|_2 = \sigma_1 \end{aligned}$$

- (ii) Let  $A = U\Sigma V^H$  and  $\sigma_n$  be the smallest singular value of  $A$ . Then  $A^{-1} = V\Sigma^{-1}U^H$ . The diagonal elements of  $\Sigma^{-1}$  are  $\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_n}$ . Since  $\sigma_n$  is the smallest singular value of  $A$ ,  $\frac{1}{\sigma_n}$  is the largest singular value of  $A^{-1}$  and by part (i) of this problem  $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$ .

- (iii) Let  $A = U\Sigma V^H$ . Then  $A^T = V\Sigma^T U^T$  and  $A^H = V\Sigma^H U^H$ . Since the singular values of  $A$  are all positive real numbers, and  $\Sigma$  is a diagonal matrix composed of the singular values, it is equal to both its transpose and its Hermitian. Therefore by part (i) of this problem  $\|A\|_2^2 = \|A^T\|_2^2 = \|A^H\|_2^2 = \sigma_1^2$ . Now consider  $A^H A = V\Sigma^H U^H U \Sigma V^H = V\Sigma^H \Sigma V^H = V\Sigma^2 V^H$ . Diagonal matrices are closed under multiplication so  $\Sigma^2$  is a diagonal matrix with  $\sigma_i^2$  on the diagonal. Then by part (i) of this problem  $\|A^H A\|_2 = \sigma_1^2$ .

- (iv)  $\|UAV\|_2^2 = \|(UAV)^H UAV\|_2 = \|V^H A^H AV\|_2 = \|A^H AVV^H\|_2 = \|A^H A\|_2 = \|A\|_2^2$  where the first and last equality follow from part (iii) and the equalities in the middle follow from the properties of norm.

**Problem 32.**

- (i)  $\|UAV\|_F = \sqrt{\text{tr}(V^H A^H U^H UAV)} = \sqrt{\text{tr}(V^H A^H AV)} = \sqrt{\text{tr}(A^H AVV^H)} = \sqrt{\text{tr}(A^H A)} = \|A\|_F$

- (ii) Let  $A = U\Sigma V^H$ . Then  $U \in M_m(\mathbb{F})$  and  $V^H \in M_n(\mathbb{F})$ . Then by the part (i) of this problem,  $\|A\|_F = \|U\Sigma V^H\|_F = \sqrt{\text{tr}(\Sigma^H \Sigma)} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$

**Problem 33.** From problem 31 part (i),  $\|A\|_2 = \sigma_1$  where  $\sigma_1$  is the largest singular value of  $A$ . Then

$$\sup_{\|x\|=1, \|y\|=1} |y^H Ax| \leq \sup_{\|x\|=1, \|y\|=1} \|y\|_2 \|\Sigma x\|_2 = \sup_{\|x\|=1} \|\Sigma x\|_2 \leq \sigma_1$$

where the last inequality follows from problem 31 part (i). However, if we define  $x$  and  $y$  as the standard eigenvector  $e_1$ , then  $\sup_{\|x\|=1, \|y\|=1} |y^H Ax| \geq |y^H Ax| = \sigma_1$ . Therefore  $\sup_{\|x\|=1, \|y\|=1} |y^H Ax| = \sigma_1 = \|A\|_2$ .

**Problem 36.** Let  $A = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$ . Then  $\det(A) = -18 \neq 0$  and  $\lambda = -3, 6$  and  $\sigma = 6.1088, 2.9466$ . More generally, if a matrix is positive semi-definite, then the eigenvalues are equal to singular values. Making an entry in the matrix negative or change an entry so that the matrix is no longer symmetric, would result in the eigenvalues and singular values that are not equal to one another.

**Problem 38.** Let  $A = U_1 \Sigma_1 V_1^H$ . Then  $A^\dagger = V_1 \Sigma_1^{-1} U_1^H$ .

- (i)  $AA^\dagger A = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = U_1 \Sigma_1 \Sigma_1^{-1} \Sigma_1 V_1^H = U_1 \Sigma_1 V_1^H = A$
- (ii)  $A^\dagger AA^\dagger = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} \Sigma_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} U_1^H = A^\dagger$
- (iii)  $(AA^\dagger)^H = (U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H)^H = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = AA^\dagger$
- (iv)  $(A^\dagger A)^H = (V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H)^H = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = A^\dagger A$ .
- (v) Let  $P = AA^\dagger$ . Then,  $P^2 = (AA^\dagger)^2 = AA^\dagger AA^\dagger = AA^\dagger = P$ . Furthermore,  $AA^\dagger x = A(A^\dagger x) \in R(A)$  so  $P$  is a projection onto  $R(A)$ . Lastly,  $(AA^\dagger)^H = AA^\dagger$  so  $P$  is an orthogonal projection.
- (vi) Let  $P = A^\dagger A$ . Then,  $P^2 = (A^\dagger A)^2 = A^\dagger AA^\dagger A = A^\dagger A = P$ . Furthermore,  $A^\dagger Ax = A^\dagger(Ax) \in R(A^H)$  so  $P$  is a projection onto  $R(A^H)$ . Lastly,  $(A^\dagger A)^H = A^\dagger A$  so  $P$  is an orthogonal projection.