

Problem 1. For the Brock and Mirman model, find the value of A in the policy function by guessing a policy function $K_{t+1} = Ae^{z_t}K_t^\alpha$ and plugging it into the Euler equation. We therefore have then that

$$\begin{aligned}\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} &= \beta E_t \left\{ \frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}} \right\} \\ \frac{1}{(1-A)e^{z_t}K_t^\alpha} &= \beta E_t \left\{ \frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{(1-A)e^{z_{t+1}}K_{t+1}^\alpha} \right\} \\ \frac{1}{e^{z_t}K_t^\alpha} &= \beta \frac{\alpha}{K_{t+1}} \\ \frac{1}{e^{z_t}K_t^\alpha} &= \beta \frac{\alpha}{Ae^{z_t}K_t^\alpha} \\ 1 &= \frac{\alpha\beta}{A} \\ A &= \alpha\beta\end{aligned}$$

Problem 2. Given following functional forms,

$$\begin{aligned}u(c_t, \ell_t) &= \ln c_t + a \ln (1 - \ell_t) \\ F(K_t, L_t, z_t) &= e^{z_t}K_t^\alpha L_t^{1-\alpha}\end{aligned}$$

there are 7 characterizing equations:

$$\begin{aligned}c_t &= (1 - \tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\ \frac{1}{c_t} &= \beta E_t \left\{ \frac{1}{c_{t+1}} ((r_{t+1} - \delta)(1 - \tau) + 1) \right\} \\ \frac{a}{1 - l_t} &= \frac{w_t}{c_t} (1 - \tau) \\ r_t &= \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \\ w_t &= (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} \\ \tau(w_t l_t + (r_t - \delta)k_t) &= T_t \\ z_t &= (1 - p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)\end{aligned}$$

Problem 3. Using the new given functional forms,

$$\begin{aligned}u(c_t, \ell_t) &= \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \ln (1 - \ell_t) \\ F(K_t, L_t, z_t) &= e^{z_t}K_t^\alpha L_t^{1-\alpha}\end{aligned}$$

we can again solve for the 7 characterizing equations:

$$c_t = (1 - \tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1}$$

$$\begin{aligned}
\frac{1}{c_t^{-\gamma}} &= \beta E_t \left\{ \frac{1}{c_{t+1}^{-\gamma}} ((r_{t+1} - \delta)(1 - \tau) + 1) \right\} \\
\frac{a}{1 - l_t} &= \frac{w_t}{c_t^{-\gamma}} (1 - \tau) \\
r_t &= \alpha e^{zt} K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1 - \alpha) e^{zt} K_t^{\alpha} L_t^{-\alpha} \\
\tau(w_t l_t + (r_t - \delta)k_t) &= T_t \\
z_t &= (1 - p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}$$

Problem 4. Using the new given function forms,

$$\begin{aligned}
u(c_t, \ell_t) &= \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi} \\
F(K_t, L_t, z_t) &= e^{zt} [\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta}]^{\frac{1}{\eta}}
\end{aligned}$$

we can again solve for the 7 characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{-\gamma}} &= \beta E_t \left\{ \frac{1}{c_{t+1}^{-\gamma}} ((r_{t+1} - \delta)(1 - \tau) + 1) \right\} \\
\frac{a}{(1 - l_t)^{-\xi}} &= \frac{w_t}{c_t^{-\gamma}} (1 - \tau) \\
r_t &= \alpha e^{zt} K_t^{\eta-1} (\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta})^{\frac{1-\eta}{\eta}} \\
w_t &= (1 - \alpha) e^{zt} L_t^{\eta-1} (\alpha K_t^{\eta} + (1 - \alpha) L_t^{\eta})^{\frac{1-\eta}{\eta}} \\
\tau(w_t l_t + (r_t - \delta)k_t) &= T_t \\
z_t &= (1 - p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}$$

Problem 5. Using the new given functional forms.

$$\begin{aligned}
u(c_t) &= \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \\
F(K_t, L_t, z_t) &= K_t^{\alpha} (L_t e^{zt})^{1-\alpha}
\end{aligned}$$

we can again solve for the 6 characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau)(w_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{-\gamma}} &= \beta E_t \left\{ \frac{1}{c_{t+1}^{-\gamma}} ((r_{t+1} - \delta)(1 - \tau) + 1) \right\} \\
r_t &= \alpha (e^{zt})^{(1-\alpha)} K_t^{\alpha-1}
\end{aligned}$$

$$\begin{aligned}
w_t &= (1 - \alpha)(e^{z_t})^{(1-\alpha)} K_t^\alpha \\
\tau(w_t + (r_t - \delta)k_t) &= T_t \\
z_t &= (1 - p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}$$

which in the steady state become:

$$\begin{aligned}
\bar{c} &= (1 - \tau)(\bar{w} + (\bar{r} - \delta)\bar{k}) + \bar{T} \\
\frac{1}{\bar{c}^{-\gamma}} &= \beta \frac{1}{\bar{c}^{-\gamma}} ((\bar{r} - \delta)(1 - \tau) + 1) \\
\bar{r} &= \alpha(e^{\bar{z}})^{1-\alpha} \bar{k}^{\alpha-1} \\
\bar{w} &= (1 - \alpha)(e^{\bar{z}})^{1-\alpha} \bar{k}^\alpha \\
\tau(\bar{w} + (\bar{r} - \delta)\bar{k}) &= \bar{T}
\end{aligned}$$

where $\bar{z} = 0$. Solving algebraically for the steady state values of capital, output, and investment, we have that:

$$\begin{aligned}
\bar{r} &= \frac{\frac{1}{\beta} - 1}{1 - \tau} + \delta = 0.12 \\
\bar{k} &= \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{\alpha-1}} = 7.28 \\
\bar{I} &= \delta \bar{k} = 0.73 \\
\bar{Y} &= \bar{k}^\alpha = 2.21
\end{aligned}$$

For the numerical solution, refer to my DSGE python notebook.

Problem 6. Using the new given functional forms.

$$\begin{aligned}
u(c_t, \ell_t) &= \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi} \\
F(K_t, L_t, z_t) &= K_t^\alpha (L_t e^{z_t})^{1-\alpha}
\end{aligned}$$

we can again solve for the 7 characterizing equations:

$$\begin{aligned}
c_t &= (1 - \tau)(w_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{-\gamma}} &= \beta E_t \left\{ \frac{1}{c_{t+1}^{-\gamma}} ((r_{t+1} - \delta)(1 - \tau) + 1) \right\} \\
\frac{a}{(1 - \ell_t)^{-\xi}} &= \frac{w_t}{c_t^{-\gamma}} (1 - \tau) \\
r_t &= \alpha(e^{z_t})^{1-\alpha} K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t &= (1 - \alpha)(e^{z_t})^{1-\alpha} K_t^\alpha L_t^{-\alpha} \\
\tau(w_t + (r_t - \delta)k_t) &= T_t \\
z_t &= (1 - p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}$$

which in the steady state become:

$$\bar{c} = (1 - \tau)(\bar{w} + (\bar{r} - \delta)\bar{k}) + \bar{T}$$

$$\frac{1}{\bar{c}^{-\gamma}} = \beta \frac{1}{\bar{c}^{-\gamma}} ((\bar{r} - \delta)(1 - \tau) + 1)$$

$$\frac{a}{(1 - \bar{l})^{-\xi}} = \frac{\bar{w}}{\bar{c}^{-\gamma}} (1 - \tau)$$

$$\bar{r} = \alpha(e^{\bar{z}})^{1-\alpha} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}$$

$$\bar{w} = (1 - \alpha)(e^{\bar{z}})^{1-\alpha} \bar{k}^{\alpha} \bar{l}^{-\alpha}$$

$$\tau(\bar{w} + (\bar{r} - \delta)\bar{k}) = \bar{T}$$

where $\bar{z} = 0$. For the numerical solution, refer to my DSGE python notebook.