A Lower Bound for the Smallest Eigenvalue of the Laplacian

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Various authors have studied the geometrical and topological significance of the spectrum of the Laplacian Δ^2, on a Riemannian manifold. (The excellent survey article of Berger [2] contains background, references, and open problems.) The purpose of this note is to give a lower bound for the smallest eigenvalue $\lambda > 0$ of Δ^2 applied to functions. The bound is in terms of a certain global geometric invariant, essentially the constant in the isoperimetric inequality. The technique works for compact manifolds of arbitrary dimension with or without boundary.

The author wishes to thank J. Simons for helpful conversations and in particular for suggesting the importance of understanding the following example of E. Calabi. Consider the "dumbbell" manifold homeomorphic to S^2, shown in Fig. 1. The pipe connecting the two halves is to be thought of as having fixed length l and variable radius r.

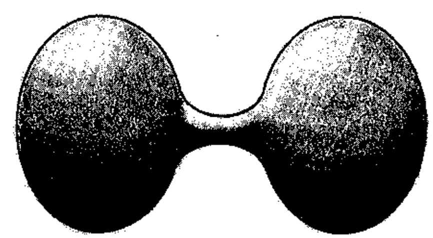

Fig. 1

One sees that $\lambda \to 0$ as $r \to 0$. Calabi's original argument involved consideration of the heat equation, $\dfrac{-\partial T}{\partial t} = \Delta^2 T$.

A somewhat more direct argument is as follows: Let f be a function which is equal to c on the right-hand bulb, $-c$ on the left-hand bulb and changes linearly from c to $-c$ across the pipe. (c chosen so that $\int_M f^2 = 1$.) Then $\int_M f = 0$ and

$$\|\operatorname{grad} f\| \approx \begin{cases} 0 & \text{bulbs,} \\ \dfrac{2c}{l} & \text{pipe.} \end{cases}$$

One has by Stokes' theorem

$$\lambda \leqq \int \Delta^2 f \cdot f = \int \|\operatorname{grad}\|^2 \approx \frac{2c}{l} \cdot 2\pi \cdot r \cdot l. \tag{1}$$

Clearly $\lambda \to 0$ as $r \to 0$. The Calabi example makes it evident that in bounding λ from below, it is not enough to consider just the diameter or volume of M. It also suggests

DEFINITION 1.

(a) Let M be a compact n-dimensional Riemannian manifold, $\partial M = \phi$. Set $h = \inf \dfrac{A(S)}{\min V(M_i)}$, where $A(\ )$ denotes $(n-1)$-dimensional area, $V(\ )$ denotes volume, and the inf is taken over all compact $(n-1)$-dimensional submanifolds S, dividing M into submanifolds with boundary M_1, M_2, with $M = M_1 \cup M_2$, and $\partial M_i = S$.

(b) If $\partial M \neq \phi$, set

$$h = \inf_s \frac{A(S)}{V(M_1)},$$

where we stipulate $S \cap \partial M = \phi$, and there is a submanifold with boundary M_1 such that $S = \partial M_1$. M_1 is necessarily unique.

In the preceding definition ∂M, M_1, M_2, S are not assumed to be connected.

THEOREM. *In the situation just described* $\lambda \geqq \frac{1}{4}h^2$. (*If* $\partial M \neq \phi$ *we assume* $f * df | \partial M = 0$.)

PROOF. If M is not orientable, it will suffice to look at its 2-fold orientable cover. Let f be the eigenfunction corresponding to λ. We make the assumption that f has nondegenerate (and therefore isolated) critical points. If this is not the case we use an obvious approximation argument based on Sard's theorem, which will be left to the reader. First note that for any region R, such that $f * df | \partial R = 0$,

$$\lambda = \frac{\int_R \Delta^2 f \cdot f}{\int_R f^2} = \frac{\int_R \|\operatorname{grad} f\|^2}{\int_R f^2} \tag{2}$$

$$= \frac{\left(\int_R \|\operatorname{grad} f\|^2\right)}{\left(\int_R f^2\right)^2} \cdot \left(\int_R f^2\right) \geqq \frac{\left(\int_R |f| \cdot \|\operatorname{grad} f\|\right)^2}{\left(\int_R f^2\right)^2} \tag{3}$$

$$= \frac{1}{4} \frac{\left(\int_R \|\operatorname{grad} f^2\|\right)^2}{\left(\int_R f^2\right)^2}, \tag{4}$$

where the inequality in (3) is obtained by squaring the Schwarz inequality.

We now assume that zero is not a critical value of f. (Again if this is not the case the argument undergoes a trivial modification.) Now the submanifold $Z = \{x | f(x) = 0\}$ divides M into n-dimensional submanifolds with boundary $M_1 = \{x | f(x) \geqq 0\}$ and $M_2 = \{x | f(x) \leqq 0\}$. It is in asserting that Z, M_1, and M_2 exist that we are using the information that f is a nonconstant eigenfunction ($\lambda \neq 0$) and hence must take on positive and negative values. Let h, h_1, and h_2 be the constants corresponding to M, M_1, and M_2. Clearly if, say, $V(M_1) \leqq V(M_2)$, then $h_1 \geqq h$. It will then suffice to prove the estimate for the submanifold with boundary M_1 and, moreover, the same argument will work for any manifold with boundary. Now the regions of M_1 lying between the critical levels of f^2 have a natural product structure $L \times I$ given by the level surfaces and their orthogonal trajectories. We introduce product coordinates (x, t) by choosing local coordinates $\{x_i\}$ on some L and setting $t = f^2$. Since dt is orthogonal to dx_i, the volume element dv may be written in coordinates as

$$dV = v_1(t, x)\, dt \times v_2(t, x)\, dx. \tag{5}$$

Since $f^2 = dt$, $v_1(t, x) = \left\|\dfrac{\partial}{\partial t}\right\|$, we have

$$\|\operatorname{grad} f^2\| \cdot v_1(t, x) = \left\langle \operatorname{grad} t, \frac{\partial}{\partial t} \Big/ \left\|\frac{\partial}{\partial t}\right\| \right\rangle \cdot \left\|\frac{\partial}{\partial t}\right\| \tag{6}$$

$$= dt\left(\frac{\partial}{\partial t}\right) = 1. \tag{7}$$

Let $V(t)$ denote the volume of the set $\{x \in M_1 | f^2(x) \geqq t\}$. $V(t)$ is continuous and differentiable.

$$\int_{M_1} \|\operatorname{grad} f^2\| \cdot dv = \int_L \left(\int_0^\infty \|\operatorname{grad} f^2\| \cdot v_1 \cdot v_2 \cdot dt\right) dx. \tag{8}$$

By (7) this is equal to

$$\int_L \left(\int_0^\infty v_2 \cdot dt\right) dx = \int_0^\infty \left(\int_L v_2 \cdot dx\right) dt \tag{9}$$

$$= \int_0^\infty A(L_t)\, dt \geqq h_1 \int_0^\infty V(t) \cdot dt \tag{10}$$

$$= -h_1 \int_0^\infty t \cdot \frac{dV(t)}{dt} \cdot dt. \tag{11}$$

Moreover,

$$V(t) = V(M_1) - \int_0^t \left\{\int_L v_1(x, t) \cdot v_2(x, t) \cdot dx\right\} dt, \tag{12}$$

and $t = f^2$. Thus (11) becomes

$$h_1 \int_0^\infty t\left\{\int_L v_1(t, x)\cdot v_2(t, x)\, dx\right\} dt$$

$$(13) \qquad = h_1 \int_0^\infty \left\{\int_L t\cdot v_1(t, x)\cdot v_2(t, x)\, dx\right\} dt$$

$$(14) \qquad = h_1 \int_{M_1} f^2 \cdot dV.$$

Squaring the inequality (8)–(14) and dividing through by $\left(\int_{M_1} f^2\right)^2$ yields

$$(15) \qquad \frac{\left(\int_M \|\operatorname{grad} f^2\|\right)^2}{\left(\int_{M_1} f^2\right)^2} \geqq h_1^2 \geqq h^2.$$

Combining (15) with (2)–(4) completes the proof.[1]

In dimension 2, it is relatively easy to see that h is always strictly greater than zero. In fact, let $V(M) = V$, and let c be such that a metric ball of radius $r < c$ is always convex. Then, if

$$(16) \qquad \frac{A(S)}{\min V(M_i)} \leqq \frac{c}{V},$$

it follows that each component of S must lie in a convex ball. On such balls the metric g satisfies $k\cdot E \geqq g \geqq \frac{1}{k} E$ where E is the Euclidean metric in normal coördinates. Hence $h > 0$ is implied by the usual isoperimetric inequality in the plane. Now, according to a theorem of the author (see [3]), c may be estimated from below by knowing a bound on the absolute value of the sectional curvature s_M, an upper bound for the diameter $d(M)$, and a lower bound for the volume. Once this is done, it is elementary that k may be estimated from $|s_M|$. This yields

COROLLARY. *If* $\dim M = 2$, $\partial M = \phi$, *then given* δ *there exists* ε *such that if* $\frac{1}{V} + d(M) + |s_M| < \delta$, *then* $\lambda > \varepsilon$.

In case $\dim M > 2$ the situation is not so elementary but 6.1 and 6.2 of [4], or [5], will still imply that $h > 0$.[2] Actually the results of [4] and [5] show the existence of an integral current T whose boundary in S, such that $A(S)$ divided by the mass of the current is always bounded away from zero independent of S. However, since T is of top dimension, it is known that T

[1] The equality $\int_M \|\operatorname{grad} f^2\|\, dV = \int_0^\infty A(L_t)\, dt$ is actually a special case of the "co-area formula" (see [6]).

[2] Thanks are due to F. Almgren for supplying these references.

may be taken to be either M_1 or M_2. If $\partial M \neq \phi$, the fact that $h > 0$ may also be deduced from Theorem 1 of [1] without too much difficulty.

It would be of interest to generalize the argument given here to Δ^2 acting on k-forms. Singer has pointed out that this would give a new proof of the fact that the dimension of the space of harmonic forms is independent of the metric and that the techniques might be applicable to other situations. To date, we have not been able to accomplish this except in the case $\dim M = 2$. The essential point here is that for any eigenvalue of Δ^2 on 1-forms one can find an eigenform of the form df, where f is an eigenfunction corresponding to λ. This observation is probably of little help if $n > 2$.

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