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A NOTE ON THE FIRST NONZERO EIGENVALUE OF THE LAPLACIAN ACTING ON P-FORMS

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Introduction. The relations between the eigenvalues of the Laplace operator on functions and the geometry of a Riemannian manifold has been very much studied. For example, the first nonzero eigenvalue $\lambda_1(M)$ of a compact Riemannian manifold M is small if , roughly speaking , M looks like a dumb-bell, which means that Cheeger's isoperimetric constant h(M) of M is small where

$$(0.1) \qquad h(M) = \inf\left[\mathrm{vol}\,\partial\Omega/\mathrm{vol}\,\Omega\ ;\ \mathrm{vol}\,\Omega \ \leq \frac{\mathrm{Vol}\ M}{2}\right]$$

This comes from the following inequalities ([BU1] , [BU2] , [CR])

$$(0.2) \qquad \frac{1}{4}h^2(M) \leq \lambda_1(M) \leq C_1(\delta h(M) + h^2(M))$$

where C_1 is a constant depending on the dimension of M and δ is the lower bound of the Ricci curvature of M.

The left hand side inequality in (0.2) is known as Cheeger's inequality.

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In this paper, we are interested in the first nonzero eigenvalue of the Laplacian acting on p-differential forms of a compact Riemannian manifold.

In the first part (&1,2) , we show that, in contrast with the case of functions, Cheeger's constant h does not permit to control the first nonzero eigenvalue of the p-spectrum. We give examples of compact Riemannian manifolds of any dimension $m \geq 3$ which have arbitrarily small first nonzero eigenvalue of the p-spectrum $1 \leq p \leq m-1$, while Cheeger's constant h stays bounded away from zero.

All our examples have bounded sectional curvatures, some of them have bounded diameter and some others have their diameter tending to infinity.

The examples with bounded diameter (Theorem 0.1) provide a bounded number of arbitrarily small eigenvalues and the examples with diameter going to infinity (Theorems 0.2 , 0.3) provide as many arbitrarily small eigenvalues as we want.

In second part of the paper (&3) , we prove that the property for M to have metrics with uniformly bounded sectional curvature and diameter and arbitrarily small first non zero eigenvalue of the Laplacian acting on p-forms , $p \in \{1,....,m-1\}$, implies that these metrics have arbitrarily small injectivity radius and this fact has topological implications on M (Theorem 0.4 , remark 2 and Corollary 0.6)

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Notations and Results

Let M be a compact Riemannian manifold of dimension $m \geq 3$.

The Laplacian acting on differential p-forms is defined by $\Delta = d\delta + \delta d$, where δ is the formal adjoint of the exterior differential d with respect to the scalar product on p-forms

$$<\alpha, \beta> = \int_M \alpha \wedge *\beta$$ where $*$ is the Hodge operator.

The spectrum of Δ acting on p-forms will be denoted by

$\lambda_0(p,M) = 0 < \lambda_1(p,M) < \lambda_2(p,M) < \ldots$ where each eigenvalue $\lambda_k(p,M)$ has to be considered with its multiplicity m_k.

By standard Hodge theory, the multiplicity m_0 of $\lambda_0(p,M)$ is the dimension of the space $H_p(M)$ of harmonic p-forms of M and then is equal to the p-th Betti number of M.

Theorem 0.1. Let $m \geq 3$ and $0 < p < m$ be integers. There exists a family $M_t = (M,g_t)$, $0 < t \leq 1$, of compact Riemannian manifolds of dimension m having bounded sectional curvatures and diameters such that :

(i) $h(M_t) \geq c > 0$, c a constant independent of t.

(ii) $\lambda_1(p,M_t) \to 0$ as t goes to zero.

Theorem 0.2. Let $m \geq 3$, $0 < p < m$ be integers. There exists a family $M_t = (M,g_t)$, $0 < t \leq 1$, of compact Riemannian manifolds of dimension m having bounded sectional curvatures such that :

(i) $h(M_t) \geq c > 0$, c a constant independent of t.

(ii) For an arbitrary $\varepsilon > 0$, the dimension of the direct sum of the eigenspaces corresponding to eigenvalues $\lambda_k(p,M_t)$ less than ε tends to infinity as t goes to zero.

Theorem 0.3. There exists a sequence M_i of hyperbolic manifolds of dimension 3 with bounded volumes such that $\lambda_1(p,M_i) \to 0$ as i goes to infinity, p = 1,2.

Let $\mathcal{N}(m,a,d,V)$ be the set of m-dimensional Riemannian manifolds with bounded sectional curvatures $|K_M| \leq a$, diameter $d_M \leq d$ and volume $\mathrm{vol}M \geq V$.

Theorem 0.4. There exists a constant $C(m,p,a,d,V) > 0$ such that $\lambda_1(p,M) \geq C(m,p,a,d,V)$ for every $M \in \mathcal{N}(m,a,d,V)$, $p \in 1,\ldots,m-1$.

We now recall the notion of collapsing of a manifold ([C-G1] , [C-G2] , [PA]).

Definition 0.5. A manifold M is said to be collapsing if there exists a family of metrics $(g_\varepsilon)_{\varepsilon \in]0,1]}$ on M with bounded sectional curvatures such that the injectivity radius of points of (M,g_ε) goes to zero with ε.
If in addition the diameter stays bounded , the last condition can be replaced by the requirement that $\mathrm{Vol}(M,g_\varepsilon)$ goes to zero with ε.

A collapsing with bounded diameter of a manifold M does occur typically when there exists a Riemannian metric g and a locally free isometric torus action on (M,g) ; the family of metrics g_ε obtained by multiplying g by ε^2 in the direction T tangent to the action and leaving g unchanged in the orthogonal directions , i.e. $g_\varepsilon = \varepsilon^2 g_T + g_{T\perp}$, satisfies the required conditions and (M, g_ε) is a collapsing of M with bounded diameter [PA] .

Remark 1. The examples of Theorem 0.1 will be of this type.

Remark 2. In ([C-G1] , [C-G2]) J.Cheeger and M.Gromov defined the notion of F-structure of positive rank on a manifold M , which is a topological property generalizing the torus action and prove the following :

Theorem ([C-G1] , [C-G2]) A manifold M collapses if and only if M carries an F-structure of positive rank.

As a consequence , we deduce the following corollary of Theorem 0.4 :

Let $N(m,a,d)$ the set of Riemannian compact manifold M of dimension m with bounded sectional curvatures $|K_M| \leq a$ and diameter $d_M \leq d$.

Corollary 0.6. There exists $\varepsilon = \varepsilon(m,a,d) > 0$ such that for every $M \in N(m,a,d)$ and $0 < p < m$, if $\lambda_1(p,M) < \varepsilon$ then M carries a F-structure.

Remark 3. The manifolds M_t of Theorem 0.2 are topological tori with a cusp-like metric of constant curvature -1 on a large part of M_t.

Remark 4. The Theorem 0.3 contrasts with a theorem of Schoen [SN] which says that the first nonzero eigenvalue $\lambda_1(0,M)$ of a hyperbolic compact three manifold M of volume V verifies $\lambda_1(0,M) \geq C/V^2$.

Remark 5. In [GA1], [GA2] the author gives an explicit upper bound C'(n,p,r,d,t) of the number N(p,t) of eigenvalues less than t of the Laplacian acting on p-forms on an n-dimensional Riemannian manifold whose Ricci curvature is bounded below by r and whose diameter is bounded above by d (See also [B-G] , [GA3]).

Remark 6 In the case of curvature $K > 0$, Theorem 0.4 is a consequence of [Li] , remark 1 , p.466-67 . In this case , Li's result is better than ours because it gives explicite bounds. But in the case of $K < 0$ the result becomes ineffective for $\lambda_1(p,M)$.

1. Some examples with bounded diameter

Let us consider a compact Riemannian manifold (M,g) , which admit an locally free isometric action of a torus T^n .

The metrics constructed on M by shrinking the initial metric g on M in the direction tangent to the orbits of T^n define a collapsing of M onto the orbit space $X = M/T^n$ ([C-G1] , [GR2] appendix 2) .

These collapsing metrics g_t are defined by $g_t = t^2 g' + h$, where g' (resp. h) is the restriction of g to the tangent space (resp. orthogonal space) to the orbits, and keep their sectional curvatures and diameter bounded as t goes to zero.

We will consider the case where the action of T^n on (M,g) has the following properties (Cf. examples 1.2, 1.3).

H_1 : $X = M/T^n$ is a compact Riemannian manifold having possibly finitely many conical singularities along submanifolds of codimension greater than or equal to two and such that the canonical map $\Pi : M \to X$ is a Riemannian submersion outside the inverse image of these singularities.

H_2 : The orbits are embedded totaly geodesic submanifolds of (M,g) diffeomorphic to n-dimensional tori.

H_3 : The L^2-harmonic p-forms of X are bounded.

This last condition is trivialy satisfied as soon as X has no singularity.

Let us denote by $H_p(X)$ the space of L^2-harmonic p-forms on X and by $b_p(M)$ the p-th Betti number of M.

Theorem 1.1. Suppose the torus T^n is acting by isometries on a compact Riemannian manifold (M,g) of dimension n+k with the orbit space X and let us assume that hypothesis H_1 , H_2 and H_3 are satisfied.

If for some p , $0< p \leq k$, $\dim H_p(X) > b_p(M)$, the manifolds $M_t = (M,g_t)$ which collapse along the torus action have their first nonzero eigenvalues $\lambda_1(p,M_t)$ going to zero with t .

Before giving the proof of Theorem1.1 , let us give some examples of situations where it can be applied.

Example 1.2. The Berger's sphere

Let us start with the Hopf fibration given by the S^1-isometric action $e^{i\theta}(z_1,\ldots,z_{n+1}) = (e^{i\theta} z_1,\ldots\ldots,e^{i\theta} z_{n+1})$ on $S^{2n+1} = \{(z_i)_{i=1}^{n+1} \in \mathbb{C}^{n+1}, \sum_{i=1}^{n+1} |z_i|^2=1\}$.

Clearly this action satisfies H_1 and H_2. Here, the orbit space is the complex projective space $\mathbb{P}^n(\mathbb{C})$ whose sectional curvatures lie between 1 and 4. Note that in this case, $X = \mathbb{P}^n(\mathbb{C})$ has no singularities.

Since $b_{2k}(\mathbb{P}^n(\mathbb{C})) = \dim H_{2k}(\mathbb{P}^n(\mathbb{C})) = 1$, $0 \le k \le n$, the Theorem1.1 applies and the collapsing spheres $S_t = (S^{2n+1}, g_t)$ have their first nonzero eigenvalues $\lambda_1(2k, S_t)$ going to zero with t , $k > 0$.

By Hodge duality , we also have $\lambda_1(2k+1, S_t)$ going to zero with t , $0 \le k \le (n-1)$.

The same remains true on the quotient of S^3 by a finite subgroup of S^1 , because S^1 act also by isometries with the same orbit space $X = \mathbb{P}^n(\mathbb{C})$.

Note that S. Gallot and D. Meyer computed $\lambda_1(2k, S_t)$ in [G-M].

Example 1.3 Compacts manifolds modelled on $\widetilde{SL}(2,\mathbb{R})$

Denote the universal cover of $PSL(2,\mathbb{R})$ by $\widetilde{SL}(2,\mathbb{R})$. The unitary tangent bundle of $\mathbb{H}^2$ is naturally identified with $PSL(2,\mathbb{R})$ and the induced metric on $PSL(2,\mathbb{R})$ and therefore on $SL(2,\mathbb{R})$ is left invariant. The group $\widetilde{SL}(2,\mathbb{R})$ is a line bundle over $\mathbb{H}^2$ which is topologically but not metrically trivial. Note that every fiber is a geodesic.

The connected component G of identity in the isometry group of $\widetilde{SL}(2,\mathbb{R})$ is generated by $\widetilde{SL}(2,\mathbb{R})$ (acting on itself by left multiplication) and its center isomorphic to $\mathbb{R}$, acting by translations on the geodesic fibers, cf. [ST]. This is summarized in the following exact sequence .

$$1 \to \mathbb{R} \to G \to PSL(2,\mathbb{R}) \to 1.$$

Consider now a compact quotient $M = \widetilde{SL}(2,\mathbb{R})/\Gamma$ of $\widetilde{SL}(2,\mathbb{R})$ by a discrete subgroup Γ . M inherits an isometric S^1-action which satisfies the properties (H1) and (H2) because of [ST] , Th. 4.15. M is a Seifert fibration whose base space is a compact hyperbolic surface X with possibly finitely many conical points. Therefore, the manifolds $M_t = (M, g_t)$ collapsing on X along the S^1-action on M have their first eigenvalues $\lambda_1(p, M_t)$ going to zero with t if the condition $\dim H_p(X) > b_p(M)$ of Theorem 1.1 can be checked. This is the case for the following two examples.

A) The first is the unit tangent bundle $M_{p,q,r}$ of the hyperbolic sphere $X_{p,q,r}$ with three conical points of angles $2\pi/p$, $2\pi/q$, $2\pi/r$, where (p,q,r) are integers such that $1/p + 1/q + 1/r < 1$.

$X_{p,q,r}$ is the quotient $\mathbb{H}^2/\Sigma(p,q,r)$ where $\Sigma(p,q,r)$ is the subgroup of index 2 of orientation preserving isometries of the triangle group $\Sigma^*(p,q,r)$ generated by the reflections in the edges of the hyperbolic triangle whose interior angles are π/p, π/q, π/r (cf [MI]).

$M_{p,q,r}$ is defined as $\tilde{S}L(2,\mathbb{R})/\Gamma(p,q,r)$ with $\Gamma(p,q,r) = \beta^{-1}(\Sigma(p,q,r)$ where β is the canonical projection $\beta : \tilde{S}L(2,\mathbb{R}) \to SL(2,\mathbb{R})$ and $M_{p,q,r}$ has its first and second Betti numbers equal to zero ([MI]).

Here, we have : $\dim H_2(X_{p,q,r}) = 1$, because the volume element of $X_{p,q,r}$ is a L^2-harmonic 2-form. Moreover, the volume element is bounded.

Therefore Theorem 1.1. applies and the manifolds $M_t = (M_{p,q,r}, g_t)$ collapsing along the S^1-action have their first eigenvalue $\lambda_1(2,M_t)$ going to zero with ε, and so does $\lambda_1(1,M_t)$ because of Hodge duality.

B) The same conclusion holds for the Brieskorn manifolds $N_{p,q,r}$ defined as follows.

$N_{p,q,r} = \tilde{S}L(2,\mathbb{R})/[\Gamma(p,q,r), \Gamma(p,q,r)]$ (where $[\Gamma(p,q,r), \Gamma(p,q,r)]$ is the commutator subgroup of $\Gamma(p,q,r)$) which is a normal covering of $M_{p,q,r}$ of degree $d = |pqr - pq - pr - qr|$ cf [MI] .

Namely the 1^{th} and 2^{nd} Betti numbers of $N_{p,q,r}$ are zero cf [MI], [BR] and the base surface $S_{p,q,r}$ possesses and L^2-harmonic 2-form (the volume element).

<u>Remark</u> : It is possible to deduce a similar example from [GH].

<u>Proof of Theorem 1.1.</u>

Denote by $\Omega_p(M)$ the space of C^∞ p-forms on M and $H_p(M)$ the space of harmonic p-forms on (M,g). The first nonzero eigenvalue $\lambda_1(p,M)$ of the Laplacian acting on p-forms of (M,g) has the following min-max characterisation :

(1.1) $\lambda_1(p,M) = \inf\ [R(\Omega)\ ;\ \Omega \in \Omega_p(M)\ ,\ \Omega \perp H_p(M)\ ]$ where $R(\Omega)$ is the Rayleigh

quotient of Ω defined by $$R(\Omega) = \frac{\int_M |d\Omega|^2 + \int_M |\delta\Omega|^2}{\int_M |\Omega|^2}.$$

In particular, we have the following

Lemma 1.5. Let E be a subspace of $\Omega_p(M)$ such that

(i) $\dim E > \dim H_p(M)$, (ii) $\sup\,[R(\Omega)\,,\Omega \in E] < t$. Then $\lambda_1(p,M) \leq t$.

The Theorem 1.1 will then come from an appropriate choice of a subspace E of $\Omega_p(M)$.

The idea is to use the pull-back $\Pi^*(H_p(X))$ of $H_p(X)$ as a test space E. However, because these forms are not defined above the conical points, we begin by modifying the elements of $H_p(X)$ around the conical points.

For this we will use the next lemma.

Lemma 1.6. Let N a differentiable submanifold of codimension greater than or equal to 2. Then there exist a family $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$ of functions C^∞ such that : for sufficiently small ε , f_ε is equal to 1 outside a 2ε-tubular neighbourhood of N and $\| 1- f_\varepsilon \|_{H^1}$ goes to 0 as ε goes to 0 where $\|\bullet\|_{H^1}$ is the H^1-norm.

Proof [CS] , prop.1.3.1.

The test space will be now $\{\Pi^*(f_\varepsilon\,\Omega) \mid \Omega \in H_p(X)\}$. To estimate $R_t(\Pi^*(f_\varepsilon\,\Omega))$, where R_t denotes the Rayleigh quotient on M_t , we have to do some calculations.

I. The calculation of $d(\Pi^*(f_\varepsilon\,\Omega))$

(1.2) $d(\Pi^*(f_\varepsilon\,\Omega)) = \Pi^*(d(f_\varepsilon\,\Omega)) = \Pi^*(df_\varepsilon \wedge \Omega)$ because Ω is harmonic on X.

II. The calculation of $*\Pi^*(f_\varepsilon\,\Omega)$

We introduce the local orthonormal moving frame for the metric g_t

$(\frac{1}{t}x_1;\dots,\frac{1}{t}x_n, y_1,\dots,y_k)$ where $(x_1,\dots,x_n)$ is orthonormal for g and tangent to the orbits of T^n and $y_1,\dots,y_k$ are orthogonal to the orbits.

Let us denote by ω the n-form dual of $\frac{1}{t}x_1 \wedge \dots \wedge \frac{1}{t}x_n$ with respect to g_t.

We have : (1.3) $*\Pi^*(f_\varepsilon\,\Omega) = (-1)^\gamma\,\omega \wedge \Pi^*(*f_\varepsilon\,\Omega)$, γ integer.

proof If $\xi_1,\ldots,\xi_{n+k-p} \in \{\frac{1}{t}x_1,\ldots,\frac{1}{t}x_n,\ldots,y_1,\ldots,y_k\}$

$*\Pi^*(f_\varepsilon\,\Omega)(\xi_1,\ldots,\xi_{n+k-p}) = (-1)^\alpha\,\Pi^*(f_\varepsilon\,\Omega)(\eta_1,\ldots,\eta_p)$

where $\eta_1,\ldots,\eta_p$ is the complement of $\xi_1,\ldots,\xi_{n+k-p}$ in $\{\frac{1}{t}x_1,\ldots,\frac{1}{t}x_n,y_1,\ldots,y_k\}$ and $(-1)^\alpha$ the corresponding signature so, $*\Pi^*(f_\varepsilon\,\Omega)(\xi_1,\ldots,\xi_{n+k-p}) =$

$(-1)^\alpha(f_\varepsilon\,\Omega)(\Pi^*(\eta_1),\ldots,\Pi^*(\eta_p))$.

thus, if $\{\frac{1}{t}x_1,\ldots,\frac{1}{t}x_n\} \notin \{\xi_1,\ldots,\xi_{n+k-p}\}$ the value is 0.

We can suppose $\xi_i = x_i\frac{1}{t}$ $\quad i = 1,\ldots,n$. We obtain

$*\Pi^*(f_\varepsilon\,\Omega)(\frac{1}{t}x_1,\ldots,\frac{1}{t}x_n,y_{j_1},\ldots,y_{j_{k-p}}) = (-1)^\beta\,\Pi^*(f_\varepsilon\,\Omega)(y_{i_1},\ldots,y_{i_p})$

$= (-1)^\gamma\,\Pi^*(*f_\varepsilon\,\Omega)(y_{j_1},\ldots,y_{j_{k-p}})$ where $\{i_1\ldots i_p\} \cup \{j_1\ldots j_{k-p}\} = \{1,\ldots,k\}$.

Hence $*\Pi^*(f_\varepsilon\,\Omega) = (-1)^\gamma\,\omega \wedge \Pi^*(*f_\varepsilon\,\Omega)$ for an appropriate γ.

III The calculation of $d*\Pi^*(f_\varepsilon\,\Omega)$

Because of (1.3), we have

(1.4) $d*\Pi^*(f_\varepsilon\,\Omega) = (-1)^\gamma\,[d\,\omega \wedge \Pi^*(*f_\varepsilon\,\Omega) + \omega \wedge \Pi^*(d*f_\varepsilon\,\Omega)]$, with

(1.5) $d(*f_\varepsilon\,\Omega) = d(f_\varepsilon *\Omega) = d\,f_\varepsilon \wedge *\,\Omega$ since $\delta\,\Omega = 0$.

IV An estimate of $|d\omega|^2$

Lemma 1.7 $|d\omega|^2 = 0(t^2)$ as t goes to zero.

Proof. We use the formula

$d\omega(u_o,\ldots u_n) = \frac{1}{n+1}\sum_{0\le i\le j\le n}(-1)^{i+j}\,\omega([u_i,u_j],u_o,\ldots\hat{u}_i,\ldots\hat{u}_j,\ldots,u_n)$ where $\{u_o,\ldots u_n\} \subset \{\frac{x_1}{t},\ldots,\frac{x_n}{t},y_1,\ldots,y_k\}$.

Observe first that $[y_i,y_j] = \sum_{\ell} c_{ij}^{\ell} x_{\ell} + y$ where c_{ij}^{ℓ} are coefficients independent of t and y is perpendicular to $(\frac{x_1}{t},\dots,\frac{x_n}{t})$. Therefore $[y_i,y_j] = t \sum_{\ell} c_{ij}^{\ell} \frac{1}{t} x_{\ell} + y$

Because of this and the fact that the orbits of T^n are totaly geodesic (cf.H_2) one sees that

$$d\omega = t \sum_{i<j,\ell} c_{ij}^{\ell} (y_i \wedge y_j \wedge \frac{x_1}{t} \wedge \dots \wedge \frac{\hat{x_i}}{t} \wedge \dots \wedge \frac{\hat{x_j}}{t} \wedge \dots \wedge \frac{x_n}{t})^{\#}$$

where $(u_o \wedge \dots \wedge u_n)^{\#}$ is the dual form of $(u_o \wedge \dots \wedge u_n)$.

Hence $|d\omega|^2 = 0(t^2)$.

By Lemma 1.6 , 1.7 and (1.2), (1.4), (1.5) and H_3 , for every $t \in (0,1)$, there exists $\varepsilon(t)$ such that $\| \Pi^*(d * f_{\varepsilon(t)} \wedge \Omega) \|^2 = 0(t^2) \| \Pi^*(f_{\varepsilon(t)} \Omega)\|^2$ as $t \to 0$ and $\| \Pi^*(df_{\varepsilon(t)} \wedge \Omega) \| = 0(t^2) \| \Pi^*(f_{\varepsilon(t)} \Omega)\|^2$ for every $\Omega \in H_p(X)$ as $t \to 0$.

It follows that $R_t(\Pi^*(f_{\varepsilon(t)} \Omega)) \leq 0(t^2)$ for every $\Omega \in H_p(X)$ as $t \to 0$.

This finish the proof of Theorem 1.1.

1.2. Proof of theorem 0.1

We use example 1.2 of collapsing $M_t = S_t$. The spheres M_t have bounded sectional curvature and diameter, therefore Cheeger's constant $h(S_t)$ stays bounded away from zero, by a theorem of Gallot ([GA1] ,[B-B-G]). The same conclusion hold for $M_t = S_t \times S^1$ and the Theorem 0.1 is proved.

2. Examples with unbounded diameter.

Proof of theorem 0.2. : As the Hodge operator $*$ is an isometry from $\Omega_p(M)$ to $\Omega_{n-p}(M)$, it suffices to prove the theorem for $p < \frac{n}{2}$.
M_t is the n-torus T^n with the metric g_t given by

$$ds^2 = dx_1^2 + \dots\dots\dots\dots + dx_{n-2}^2 + a_t^2(x_1) (dy_1^2 + dy_2^2)$$

$-t < x_1 < t$, $-1 < x_2,\ldots\ldots,x_{n-2},y_1,y_2 < 1$.

with a_t defined as follows.

$$a_t(r) = \begin{cases} \dfrac{\operatorname{ch} r}{\operatorname{ch} t} & \text{if } |r| \leq t-1 \\ \dfrac{\operatorname{ch} r}{\operatorname{ch} t}\left[1+\psi(1-t-r)\left(\dfrac{\operatorname{ch} t}{\operatorname{ch} r}-1\right)\right] & \text{if } -t < r < -t+1 \\ \dfrac{\operatorname{ch} r}{\operatorname{ch} t}\left[1+\psi(r+1-t)\left(\dfrac{\operatorname{ch} t}{\operatorname{ch} r}-1\right)\right] & \text{if } t-1 < r < t \end{cases},$$

where

$$\psi(r) = \frac{\int_0^r \gamma(s)\, ds}{\int_0^{\frac{1}{2}} \gamma(s)\, ds}, \qquad \gamma(r) = \eta(r)\,\eta\left(\frac{1}{2}-r\right),$$

$$\eta(r) = \exp\frac{-1}{r^2} \text{ if } r > 0 \quad , \ 0 \text{ elsewhere.}$$

Proof of (ii) . Observe first that the space of harmonic p-forms $H_p(M_t)$ has finite dimension $N = \dfrac{n!}{(n-p)!p!}$ independent of t.
So, by the minimax principle, it suffices to exhibit , for each N , a subspace F_t of $\Omega_p(M_t)$ with $\dim F_t = N$ such that $R(\omega) < \varepsilon(t)$ for every $\omega \in F_t$, with $\varepsilon(t) \to 0$ as $t \to \infty$. F_t will consist of elements of the following type

$\omega_f = f(x_1)\, dx_2 \wedge \ldots\ldots\ldots \wedge dx_p \wedge dy_1$ if $p > 1$, $\omega_f = f(x_1)\, dy_1$ if $p=1$

First, we calculate the Rayleigh quotient of such elements.

LEMMA 2.1 $$R(\omega_f) = \frac{\int_{-t}^{t} f'^2(r)\, dr}{\int_{-t}^{t} f^2(r)\, dr}$$

Proof: By standard calculations

$\omega_f \wedge * \omega_f = f^2(x_1)\, dx_1 \wedge \ldots\ldots\ldots\ldots \wedge dx_{n-2} \wedge dy_1 \wedge dy_2$

$d\omega_f \wedge * d\omega_f = f'^2(x_1)\, dx_1 \wedge \ldots\ldots\ldots\ldots \wedge dx_{n-2} \wedge dy_1 \wedge dy_2$

$\delta\omega_f = 0$.

and the lemma follows .

We separate the interval $[-t\ ,\ t]$ into $N+1$ disjoints intervals of equal length $\frac{2t}{N+1}$, denoted by $[a_i\ ,\ a_{i+1}]$ $i = 0\ ,1\ , \ldots\ldots\ldots\ , N$ and define test functions f_i as follows :

On the interval $[a_i\ ,\ a_{i+1}]$, f_i is the eigenfunction associated to the first eigenvalue $(\frac{N+1}{t}\pi)^2$ of the one dimensional Dirichlet problem , and $f_i(r) = 0$ if $r \notin [a_i\ ,\ a_{i+1}]$.
Let F_t be the space spanned by $\omega_{f_0}, \ldots\ldots\ldots, \omega_{f_N}$. As $R(\omega_{f_i}) = (\frac{N+1}{t}\pi)^2$ and as the ω_{f_i} are clearly orthogonal, the same is true for every element of F_t , and (ii) is proved.

<u>Proof of (i).</u> Let $f \in C^\infty(M_t)$ such that $\int_{M_t} f = 0$. We are going to show that:

(2.1) $R(f) \geq C' > 0$, C' constant not dependent on t.

Assuming (2.1) the minimax principle shows that $\lambda_1(M_t) \geq C'$. Moreover, the Ricci curvature of M_t is bounded below by a constant not depending on t (because we may write the metric
$ds^2 = (dx_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots + dx_{n-2}^2) + (dx_1^2 + a_t^2(x_1)\,(dy_1^2 + dy_2^2))$
and it is easy to compute that the two terms have bounded curvature.) thus (0.2) shows that $h(M_t) \geq C$, which achieves the proof of (ii).

It remains to prove (2.1). Let

$$F(x_1) = \int dx_2 \ldots\ldots dx_{n-2} dy_1 dy_2\ f(x_1, x_2, \ldots\ldots\ldots x_{n-2}, y_1, y_2)$$

Two cases occur:

(A) $F(x_1) \equiv 0$ (B) $F(x_1) \not\equiv 0$.

In case (A) , for every fixed x_1 we have :

$\int dx_2 \ldots\ldots dx_{n-2} dy_1 dy_2 \; f(x_1, x_2, \ldots\ldots x_{n-2}, y_1, y_2) \; = 0$. Therefore in this case $\dfrac{A_t(x_1)}{B_t(x_1)} \geq \pi^2$, where

$$A_t(x_1) = \int dx_2 \ldots\ldots dx_{n-2} dy_1 dy_2 \; a_t^2(x_1) \Big(\sum_{i=1}^{n-2} \big(\frac{\partial f}{\partial x_i}\big)^2 + \frac{1}{a_t^2(x_1)} \sum_{j=1}^{2} \big(\frac{\partial f}{\partial y_j}\big)^2 \Big) \quad ,$$

$$B_t(x_1) = \int dx_2 \ldots\ldots dx_{n-2} dy_1 dy_2 \; a_t^2(x_1) \, f^2(x_1, x_2, \ldots\ldots x_{n-2}, y_1, y_2) \quad ,$$

since $\dfrac{1}{a_t^2(x_1)} > 1$ and [B-G-M] Ch3-B . It follows that

$$R(f) = \frac{\int A_t(x_1) \, dx_1}{\int B_t(x_1) \, dx_1} \geq \pi^2 \quad .$$

In the case (B) we use Cheeger's inequality as $\int_{M_t} F = 0$. However , since F depends only on x_1 , the appropriate version of Cheeger's inequality is

$R(F) \geq \dfrac{h'^2}{4}$ where $h'(M_t) = \inf \big(\dfrac{\mathrm{Vol}\,\partial\Omega}{\mathrm{Vol}\,\Omega} \big)$ where Ω ranges over all open submanifolds of M_t whose boundary is a union of hypersurfaces x_1=constant.

We will show that $h'(M_t) \geq D$, a constant independent of t. By an easy computation we see that Vol $M_t < 2^{n+3}$.
It suffices to consider Ω such that $\partial\Omega = \{x_1=c_1\} \cup \{x_1=c_2\}$ with $-t+1 \leq c_1, c_2 \leq t-1$ .We may assume that $|c_1| \geq |c_2|$.

Writing $\partial\Omega_1 = \{x_1=c_1\}$ and $\partial\Omega_2 = \{x_1=c_2\}$ we have:

$$\frac{\mathrm{Vol}\,\partial\Omega}{\mathrm{Vol}\,\Omega} = \frac{\mathrm{Vol}\,\partial\Omega_1 + \mathrm{Vol}\,\partial\Omega_2}{\mathrm{Vol}\,\Omega} \geq \frac{a_t^2(c_1)}{2\int_0^{|c_1|} a_t^2(u)\,du} \geq \frac{1}{6} \quad .$$

This concludes the proof of Theorem 0.2.

<u>Proof of Theorem 0.3</u> : In order to prove Theorem 0.3, let us recall some facts about hyperbolic three manifolds (cf [TH] , [GR1]).
Any complete hyperbolic three manifold M of finite volume is made of two parts : the thick part $M_{[\mu,\infty[}$ (resp. the thin part $M_{]0,\mu]}$) whose points have injectivity radius greater(resp. smaller) than the Margulis constant μ , where $M_{[\mu,\infty[}$ is connected and $M_{]0,\mu]}$ is a finite disjoint union of tubular neighbourhoods of closed geodesics and of cusps , ie a flat tori cross the half line : $[0,\infty[\times T^2$ with the metric $dr^2 + e^{-2r} ds^2$, where ds^2 is the flat metric of T^2 .
Moreover, any non compact hyperbolic manifold M may be "approximated" by a sequence of compact hyperbolic manifolds M_i with the procedure of closing the cusps of M ([GR1] &5). From a topological viewpoint ,the M_i are constructed by cutting M along a flat torus on each cusp and gluing on it a full torus by a Dehn surgery.
<u>Theorem</u> ([TH] , [GR1]) Let M be a finite volume hyperbolic 3-manifold with p >0 cusps. There exists a sequence $\{M_i\}_{i\in N}$ of compact hyperbolic 3-manifolds with p closed geodesics whose length tends to 0 as i tends to infinity , such that (M_i , m_i) is converging to (M , m) for the pointed Lipschitz distance , where m_i [resp. m] lies in the thick part of M_i [resp. M].

In particular, the thick part of each M_i is diffeomorphic to the thick part of M and therefore the dimension of $H_p(M_i)$, (p = 1,2) , is less than or equal to the number of generators of the fundamental group of $M_{[\mu,\infty[}$ plus q, and therefore is bounded.
Moreover, the thin parts of the M_i's , which are neighborhoods of small closed geodesics have the following parametrisation in Fermi coordinates : $g(r,t,\theta) = dr^2 + (\varepsilon \mathrm{ch} r)^2 dt^2 + (\mathrm{sh} r)^2 d\theta^2 \quad 0 < r < l_i$ with $l_i \to \infty$ as $i \to \infty$.
It remains to construct examples exactly as in Theorem 0.2.
Furthermore, we have $\mathrm{Vol}M_i \to \mathrm{Vol}M$ ([GR1]) so that $h(M_i) > C > 0$ and the proof is complete.

3. Proof of Theorem 0.4 and corollary 0.6..

Let us consider a compact manifold M with a C^1 metric g. The spectrum of p-forms of (M,g) is the set of eigenvalues of the quadratic form
$q(\omega) = \|d\omega\|^2 + \|\delta\omega\|^2$ with respect to the scalar product

$<\omega_1, \omega_2> = \int_M \omega_1 \wedge * \omega_2$. The domain of q is the set of square-integrable p-forms ω such that $d\omega$ and $\delta\omega$ are also square-integrable.
By standard arguments of analysis [BD] p.52-64. those eigenvalues form an increasing discrete sequence : $\mu_1(p,g) \leq \mu_2(p,g) \leq$ .... tending to $+\infty$ where the k-th eigenvalue has the following definition :

Definition 3.1. $\mu_k(p,g) = \min \max [\ R(\omega) ; \omega \in E\ ]$ where E is a k-dimensional subspace of the space $\Omega_p(M)$ of the C^1 forms on M and $R(\omega)$ is the Rayleigh quotient of ω , i.e. $R(\omega) = \dfrac{q(\omega)}{<\omega,\omega>}$.
It may happen that 0 is an eigenvalue. In this case the space $H_p(M,g) = \{\omega \in \Omega_p(M,g) \ / \ d\omega = 0 , \delta\omega = 0 \}$ is non trivial.

Although the metric is only C^1 , Hodge theory still works and
(3.2) $\dim H_p(M,g) = b_p(M)$ the p-th Betti number ([T]) .

The following lemma is a direct consequence of the min-max characterization (3.1) of eigenvalues :

Lemma 3.3 The function $g \to \mu_k(p,g)$ is continuous on the set of C^1 metrics g on a fixed manifold M for the C^1 topology.

Proof: $R(\omega)$ depends on the metric and the first derivative of the metric.

Let us remark that $g \to \lambda_k(0,g)$ is continuous for the C^0 topology because if f is a function , R(f) depends only on the metric.

Proof of Theorem 0.4. . Suppose the conclusion is not true. Then there exists a sequence of Riemannian manifolds (M_i,g_i) in $\mathcal{N}(m,a,d,V)$ for which the first nonzero eigenvalue $\lambda_1(p,g_i)$ of the p-forms approaches zero when i tends to infinity.
By a theorem of S.Peters , ([PE] Theorem 4.4) , we can assume that each M_i is a fixed manifold M and that there exists a subsequence g_j of metrics and diffeomorphisms φ_j of M such that $h_j = (\varphi_j)^*(g_j)$ converges in the C^1 topology to a C^1 metric g on M.

The p-spectrum of (M,h_j) is then

$0 = \mu_1(p,h_j) = \ldots = \mu_N(p,h_j) < \mu_{N+1}(p,h_j) = \lambda_1(p,h_j) \leq \ldots$ where N equals the p-th Betti number of M. By the lemma 3.3 , we deduce that $\mu_{N+1}(p,g) = 0$ which contradicts (3.2).

Proof of corollary 0.6. Let us recall the critical radius Theorem of Cheeger-Gromov (c.f. [C-G 2] , corollary 0.9 p.0.3) . There exists $c(m,a) > 0$ such that , for any Riemannian manifold M of dimension m and bounded sectional curvature $|K_M| < a$, if the injectivity radius at every points of M is less than $c(m,a)$ then M admits a collapsing and , so , M carries a F-structure. On the other hand , because of the volume comparison Theorem of Bishop , Theorem 0.4 is equivalent to the following :
There exists $c' = c'(m,p,a,d,c) > 0$ such that $\lambda_1(p,M) > c'$ for any manifold M of bounded curvature $|K_M| < a$, diameter $d_M < d$ and with injectivity radius greater than c at one point.
Corollary 0.6 follows , taking $\varepsilon(m,a,d) = \inf\{c'(m,p,a,d,c(m,a)) ,0 < p < m\}$.

BIBLIOGRAPHY

[BR] E.Brieskorn , Beispiele zur Differentialtopologie von Singularitäten, Invent. Math 2 (1966) p.1-14

[B-B-G] Bérard, P.- Besson, G.- Gallot, S., Sur une inégalité isopérimétrique qui généralise celle de Paul Lévy-Gromov, Invent. Math. 80 (1985) 295

[BD] P.Bérard : Spectral geometry : Direct and inverse problems ; Lect.notes in Math. 1207 ; Springer 1986

[B-G] P.Bérard-S.Gallot , Inégalités isopérimétriques pour l'équation de la chaleur et application a l'estimation de quelques invariants , Séminaire Goulaouic-Meyer-Schwartz 1983—1984

[B-G-M] M. Berger- P.- Gauduchon - E. Mazet , Le spectre d'une variété riemannienne, Lect. Notes in Math. 194, (1971)

[BU1] P. Buser, On Cheeger Inequality $\lambda_1 \geq h^2/4$, Proc. of Symp. in Pure Math. 36 p. 29-77 (1980)

[BU2] P. Buser, A note on the isoperimetric constant, Ann. Scient. Ec. Norm. Sup. Série4, t.15, (1982) P.213-230

[C-G1] J.Cheeger, J.- Gromov, M., Collapsing Riemanniann manifolds while keeping their curvature bounded I, J. diff. Geom. 23 (1986) 309-346

[C-G2] J. Cheeger, M. Gromov : Collapsing Riemanniann manifold while keeping their curvature bounded II, Preprint

[CR] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian, problems in Analysis, A symposium in honour of Bochner, Princeton University Press, N.J. p. 195-199 (1970)

[CS] G.Courtois, Thesis , Université de Grenoble , 1987

[GA1] S. Gallot : Inégalités isopérimétriques courbure de Ricci et invariants géométriques I, C.A. Acad. Sci. Paris 296, série I, 1983, p. 333-336

[GA2] S. Gallot : Inégalités isopérimétriques, courbure de Ricci et invariants géométriques II, C.R. Acad. Sci. Paris 296, série I, 1983, p. 365-368

[GA3] S. Gallot , A Sobolev inequality and some geometric applications, Spectra of Riemannian manifolds (actes du colloque de Kyoto 1981), Kayai Publ. 1983, p. 45-55

[GH] P. Ghanaat , Almost Lie groups of type $\mathbb{R}^n$, to appear at Journal für die reine und angewandte Mathematik

[G-M] S.Gallot.- D.Meyer, Opérateur de courbure et Laplacien des formes différentielles d'une variété riemannienne, J. Math., pures et appl. 54 (1975) 259-284

[GR1] M. Gromov, Hyperbolic manifolds according to Thurston and Jorgensen, Séminaire Bourbaki n° 546 (1979-80)

[GR2] M.Gromov, Volume and bounded cohomology , I.H.E.S. 1982 , P.5-100

[GR3] M. Gromov , Structures métriques pour les variétés riemanniennes, Réd. par J. Lafontaine et P. Pansu, Paris 1981

[K] A. Katsuda , Gromov's convergence theorem and its application, Nagoya, Math. J. 100 (1985) p.11-48

[Li] P.Li , On the Sobolev constant and the p-spectrum of a compact Riemannian manifold . Ann. scient .Ec.Norm.Sup . Série 4 , t.13 , (1980) 451-469

[MI] J.Milnor , On the 3-dimensional Brieskorn manifolds M(p,q,r) . In knots, groups and 3-manifolds, Ann. Math., Studies, Princeton University Press 84, (1975)

[PA] P. Pansu , Dégénérescence des variétés riemanniennes, d'après J. Cheeger et M. Gromov, Astérisque 121 (1985)

[PE] S. Peters , Convergence of Riemanniann manifolds compositio, Math. 62 (1987) p. 1-16

[SN] R. Schoen, A lower bound for the first eigenvalue of a negatively curved manifold, J. Differential Geometry 17 (1982) P.233-238

[ST] P.Scott , The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983) 401-487

[T] N. Teleman , The index of signature operators on Lipschitz manifolds, Pub. Math. IHES 58 (1983) p: 39-78

[TH] W. Thurston, The geometry and topology of 3-manifolds, Lecture notes from Princeton University, (1977-78)

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