Applied Hodge Theory

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3. Summary

Topological & Geometric Methods in Data Analysis

- Differential Geometric methods: manifolds
  - data distribution: manifold learning/NDR, etc.
  - model space: information geometry (high-order efficiency for parametric statistics)
- Algebraic Geometric methods: polynomials/varieties
  - tensor (matrices etc.)
  - algebraic statistics
  - polynomial optimization (SOS)
- Algebraic Topological methods: complexes (graphs, etc.)
  - persistent homology (robust, slow)
  - Euler calculus (non-stable, fast)
  - Hodge theory (geometry$\leftrightarrow$topology via optimization/spectrum)

Helmholtz-Hodge Decomposition

Theorem (c.f. Marsden-Chorin 1992)

A vector field $\mathbf{w}$ *on a simply-connected* D *can be uniquely decomposed in the form*

$$\mathbf{w} = \mathbf{u} + \operatorname{grad} \phi$$

where $\mathbf{u}$ *has zero divergence and is parallel to* ∂D.

vector field = curl-free + div-free

Algebraic Elements of Hodge Decomposition

For inner product spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, consider

$$\mathcal{X} \xrightarrow{A} \mathcal{Y} \xrightarrow{B} \mathcal{Z}.$$

and $\Delta = AA^* + B^*B : \mathcal{Y} \to \mathcal{Y}$ where $(\cdot)^*$ is adjoint operator of $(\cdot)$. If

$$B \circ A = 0,$$

then $\ker(\Delta) = \ker(A) \cap \ker(B^*)$ and *orthogonal* decomposition

$$\mathcal{Y} = \mathrm{im}(A) + \ker(\Delta) + \mathrm{im}(B^*)$$

Note: $\ker(B)/\,\mathrm{im}(A) \simeq \ker(\Delta)$ is the (real) (co)-homology group ($\mathbb{R} \to$ rings; vector spaces$\to$module).

Classical Hodge Theory on Riemannian Manifolds

- de Rham complex: $d^2 = d_k \circ d_{k-1} = 0$

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0$$

  where M is a compact Riemannian manifold with k-differential forms $\Omega^k(M)$ and d is the exterior derivative operator whose adjoint, codifferential operator δ satisfies $\langle du, v\rangle = \langle u, \delta v\rangle$
- Laplacian $\Delta = d\delta + \delta d$ and Harmonic forms $\ker(\Delta)$
- Hodge decomposition (W.V.D. Hodge, 1903-1975)

$$\Omega^k(M) = \mathrm{im}(d_{k-1}) \oplus \ker(\Delta_k) \oplus \mathrm{im}(\delta_k)$$

  where $\ker(\Delta_k)$ is isomorphic to de Rham cohomology group $H^k(M) = \ker(d_k)/\,\mathrm{im}(d_{k-1})$.

Hodge Theory on Metric Spaces

- (Alexander-Spanier, Bartholdi-Schick-Smale-Smale, 2011) complex, $d^2 = 0$

$$0 \to L^2(X) \xrightarrow{d_0} L^2(X^2) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} L^2(X^n) \xrightarrow{d_n} \cdot$$

  - $L^2(X)$: square integral functions on metric space X
  - finite difference (Gilboa-Osher'08) $d : L^2(X^k) \to L^2(X^{k+1})$

$$(df)(x_0, \ldots, x_k) = \sum_{i=1}^{k} (-1)^i \prod_{j \neq i} \sqrt{K(x_i, x_j)} f(x_{-i})$$

  - adjoint operator $\delta : L^2(X^{k+1}) \to L^2(X^k)$

$$\delta g(x) = \sum_{i=0}^{k} (-1)^i \int_X \prod_{j=0}^{k-1} \sqrt{K(t, x_j)} g(x_0, \ldots, x_{i-1}, t, x_i, \ldots, x_{k-1}) dt$$

continued: Hodge Theory on Metric Spaces

(Bartholdi-Schick-Smale-Smale-Baker, 2011)

If X satisfies some regularity conditions, then Hodge decomposition holds

$$L^2(X^k) = \mathrm{im}(d_{k-1}) \oplus \ker(\Delta_k) \oplus \mathrm{im}(\delta_k)$$

In particular, if X is a compact Riemannian manifold with regularity conditions on convexity and curvature, there is a scale/kernel such that $\ker(\Delta_k)$ is isomorphic to the L^2-cohomology and de Rham cohomology.

- $\Delta = d\delta + \delta d$
- for finite X, it essentially builds up a Čech complex for point cloud data at certain scale and applies combinatorial Hodge theory

Combinatorial Hodge Theory on Cell Complexes

- X is finite
- $\chi(X) \subseteq 2^X$ is a simplicial complex formed by X, such that $\tau \in \chi(X)$ and $\sigma \subseteq \tau$, then $\sigma \in \chi(X)$
- k-forms or cochains as alternating functions

$$\Omega^k(X) = \{u : \chi_{k+1}(X) \to \mathbb{R}, u_{i_{\sigma(0)},\ldots,i_{\sigma(k)}} = \text{sign}(\sigma) u_{i_0,\ldots,i_k}\}$$

  where $\sigma \in \mathfrak{S}_{k+1}$ is a permutation on $(0, \ldots, k)$.
- coboundary maps $d_k : \Omega^k(X) \to \Omega^{k+1}(X)$ are defined as the alternating difference operator

$$(d_k u)(i_0, \ldots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^{j+1} u(i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k+1})$$

Example: graph and clique complex

- $G = (X, E)$ is a undirected graph
- Clique complex $\chi_G \subseteq 2^X$ collects all complete subgraph of G
- k-forms or cochains $\Omega^k(\chi_G)$ as alternating functions:
  - 0-forms: $v : V \to \mathbb{R} \cong \mathbb{R}^n$
  - 1-forms as skew-symmetric functions: $w_{ij} = -w_{ji}$
  - 2-forms as triangular-curl:

  $z_{ijk} = z_{jki} = z_{kij} = -z_{jik} = -z_{ikj} = -z_{kji}$
- coboundary operators $d_k : \Omega^k(\chi_G) \to \Omega^k(\chi_G)$ as alternating difference operators:
  - $(d_0 v)(i, j) = v_j - v_i =: (\text{grad } v)(i, j)$
  - $(d_1 w)(i, j, k) = (\pm)(w_{ij} + w_{jk} + w_{ki}) =: (\text{curl } w)(i, j, k)$
- $d_1 \circ d_0 = \text{curl}(\text{grad } u) = 0$

continued: Combinatorial Hodge Theory on Cell Complexes

- So we have

$$0 \to \Omega^0(X) \xrightarrow{d_0} \Omega^1(X) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(X) \xrightarrow{d_n} \cdots$$

- $d_k \circ d_{k-1} = 0$
- combinatorial Laplacian $\Delta = d_{k-1} d_{k-1}^* + d_k^* d_k$
  - $k = 0$, $\Delta_0 = d_0^* d_0$ is the well-known graph Laplacian
  - $k = 1$, 1-Hodge Laplacian

$$\Delta_1 = \mathrm{curl} \circ \mathrm{curl}^* - \mathrm{div} \circ \mathrm{grad}$$

- Hodge decomposition holds for $\Omega^k(X)$
  - $\Omega^k(X) = \mathrm{im}(d_{k-1}) \oplus \ker(\Delta_k) \oplus \mathrm{im}(\delta_k)$
  - $\dim(\Delta_k) = \beta_k(\chi(X))$

Forgetful functors

Riemannian manifolds $\rightarrow$ Metric spaces $\rightarrow$ Cell complexes

- From differentiable to combinatorial structures, Hodge decomposition is functorial (invariant)
- Topological invariants (homology) are preserved in such coarse-grained functors
- Natural for data analysis, a connection between geometry and topology: harmonic basis
- More important than data itself, relations between data via functions, mappings, etc.

Applications of Hodge Decomposition

- Boundary Value Problem (Schwarz, Chorin-Marsden'92)
- Computer vision
  - Optical flow decomposition and regularization (Yuan-Schnörr-Steidl'2008, etc.)
  - Retinex theory and shade-removal (Ma-Morel-Osher-Chien'2011)
  - Relative attributes (Fu-Xiang-Y. et al. 2014)
- Sensor Network coverage (Jadbabai et al.'10)
- Statistical Ranking or Preference Aggregation (Jiang-Lim-Y.-Ye'2011, etc.)
- Decomposition of Finite Games (Candogan-Menache-Ozdaglar-Parrilo'2011)

Optical Flow Decomposition and Regularization

- Rudin-Osher-Fatemi'1992: piecewise constant flows

$$\frac{1}{2}\|v-u\|_2^2 + TV(u), \quad u, v \in \mathbb{R}^2$$

$$TV(u) := \int \sqrt{(\text{grad } u_1)^2 + (\text{grad } u_2)^2},$$

- Yuan-Schnörr-Steidl'2007: piecewise harmonic flows

$$TV(u) \to R(u) = \int \sqrt{(\text{div } u)^2 + (\text{curl } u)^2}$$

Example: periodic motions are harmonic

Figure 2: Vector field comprising 7 components (left: vector plot; right: colour plot of vector directions): a smooth harmonic flow in the background, two medium-scale non-harmonic flows in the upper-left and lower-right part, respectively, and four small-scale constant flows.

Figure: Better motion separation with Hodge decomposition

Adelson's illusion in Computer Vision

Figure: Adelson's illusion: on the left the chess board is shadowed by a column such that the white square has the same illuminance intensity as the black square, proved by the right picture.

Retinex Theorey based on Approximation of Gradient Flows

- The edge information is a gradient field of intensity grad I
- Shade adds sparse noise $Y = \text{grad}\, I + E$
- Find sparse approximation of de-noised gradient field $\min_X \| \text{grad}\, X - T(Y) \|_1$

Figure: Ma-Morel-Osher-Chien 2011

Crowdsourcing QoE evaluation of Multimedia

Figure: (Xu-Huang-Y., et al. 11) Crowdsouring subjective Quality of Experience evaluation

Learning relative attributes: age

Figure: Age: a relative attribute estimated from paired comparisons (Fu-Y.-Xiang et al. 2013)

Collaborative Ideas Prioritization

Figure: Left: www.allourideas.org/wikipedia-banner-challenge, by Prof. Matt Salganik at Princeton; Right: www.crowdrank.net

Paired comparison data on graphs

Graph $G = (V, E)$

- V: alternatives to be ranked or rated
- $(i_\alpha, j_\alpha) \in E$ a pair of alternatives
- $y_{ij}^\alpha \in \mathbb{R}$ degree of preference by rater α
- $\omega_{ij}^\alpha \in \mathbb{R}_+$ confidence weight of rater α
- Examples: relative attributes, subjective QoE assessment, perception of illuminance intensity, sports, wine taste, etc.

Generalized Linear Models in Statistics: $l^2(E)$

- Majority voting (Condorcet'1785): inconsistency arises (Arrow's impossibility theorem 1950s)
- Statistical majority voting:
  - $\hat{Y}_{ij} = (\sum_\alpha \omega_{ij}^\alpha Y_{ij}^\alpha)/(\sum_\alpha \omega_{ij}^\alpha) = -\hat{Y}_{ji}$, $\omega_{ij} = \sum_\alpha \omega_{ij}^\alpha$
- $\hat{Y}$ from generalized linear models
  - *Uniform* model: $\hat{Y}_{ij} = 2\hat{\pi}_{ij} - 1$.
  - *Bradley-Terry* model: $\hat{Y}_{ij} = \log \frac{\hat{\pi}_{ij}}{1-\hat{\pi}_{ij}}$.
  - *Thurstone-Mosteller* model: $\hat{Y}_{ij} = \Phi^{-1}(\hat{\pi}_{ij})$.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-x/[2\sigma^2(1-\rho)]^{1/2}}^{\infty} e^{-\frac{1}{2}t^2} dt.$$

  - *Angular transform* model: $\hat{Y}_{ij} = \arcsin(2\hat{\pi}_{ij} - 1)$.
- Inner product induced on $\hat{Y} \in l^2_\omega(E)$, $\langle u, v \rangle_\omega = \sum u_{ij} v_{ij} \omega_{ij}$ where u, v skew-symmetric

Hodge Decomposition on Graphs [Jiang-Lim-Y.-Ye'11]

Paired comparison data $\hat{Y}_{ij} \in l^2_\omega(E)$ admits an orthogonal decomposition,

$$\hat{Y} = \hat{Y}^{(g)} + \hat{Y}^{(h)} + \hat{Y}^{(c)}, \tag{1}$$

where

$$\hat{Y}^{(g)}_{ij} = \hat{\beta}_i - \hat{\beta}_j, \quad \text{for some } \hat{\theta} \in \mathrm{R}^V, \tag{2}$$

$$\hat{Y}^{(h)}_{ij} + \hat{Y}^{(h)}_{jk} + \hat{Y}^{(h)}_{ki} = 0, \text{ for each } \{i,j,k\} \in T, \tag{3}$$

$$\sum_{j \sim i} \omega_{ij} \hat{Y}^{(h)}_{ij} = 0, \text{ for each } i \in V. \tag{4}$$

Global ranking and Local vs. Global Inconsistencies

$\hat{Y}^{(g)} = (\delta_0\beta)(i,j) := \hat{\beta}_i - \hat{\beta}_j$ where $\hat{\beta}$ solves

$$\min_{\beta\in\mathbb{R}^{|V|}} \sum_{\alpha,(i,j)\in E} \omega_{ij}^{\alpha}(\beta_i - \beta_j - Y_{ij}^{\alpha})^2 \Leftrightarrow \hat{\Delta}_0\hat{\beta} = \delta_0^T\hat{Y}$$

Residues $\hat{Y}^{(h)}$ and $\hat{Y}^{(c)}$ accounts for inconsistencies:

- $\hat{Y}^{(c)}$, the local inconsistency, triangular curls
  - $\hat{Y}_{ij}^{(c)} + \hat{Y}_{jk}^{(c)} + \hat{Y}_{ki}^{(c)} \neq 0$, $\{i,j,k\} \in T$
- $\hat{Y}^{(h)}$, the global inconsistency, harmonic ranking
  - harmonic ranking leads to circular coordinates (MVJ tutorial) on $V \Rightarrow$ fixed tournament issue

Topological Constraints

To get a faithful ranking, two topological conditions on the clique complex $\chi_G^2 = (V, E, T)$ are important:

- Connectivity: G is connected, then an unique global ranking is possible;
- Loop-free: harmonic ranking vanishes if χ_G^2 is loop-free, topology plays a role of obstruction of fixed-tournament
  - "Triangular arbitrage-free implies arbitrage-free"

Basic Problems in HodgeRank

- sampling method for crowdsourcing
  - passive, active, random graph theory, etc.
- reliability of data: inconsistency
  - outlier detection and robust ranking
- sequential or streaming data: online algorithms
  - persistent homology, online ranking

Random Graph Models for Crowdsourcing

- Recall that in crowdsourcing ranking on internet,
  - unspecified raters compare item pairs randomly
  - online, or sequentially sampling
- random graph models for experimental designs
  - P a distribution on random graphs, invariant under permutations (relabeling)
  - Generalized de Finetti's Theorem [Aldous 1983, Kallenberg 2005]: $P(i,j)$ (P ergodic) is an uniform mixture of

$$h(u,v) = h(v,u) : [0,1]^2 \to [0,1],$$

h unique up to sets of zero-measure
  - Erdös-Rényi: $P(i,j) = P(edge) = \int_0^1 \int_0^1 h(u,v)dudv =: p$
  - edge-independent process (Chung-Lu'06)

Phase Transitions of Large Random Graphs

For an Erdos-Renyi random graph $G(n, p)$ with n vertices and each edge independently emerging with probability $p(n)$,

- (Erdös-Rényi 1959) One phase-transition for β_0
  - $p << 1/n^{1+\epsilon}$ ($\forall \epsilon > 0$), almost always disconnected
  - $p >> log(n)/n$, almost always connected
- (Kahle 2009) Two phase-transitions for β_k ($k \geq 1$)
  - $p << n^{-1/k}$ or $p >> n^{-1/(k+1)}$, almost always β_k vanishes;
  - $n^{-1/k} << p << n^{-1/(k+1)}$, almost always β_k is nontrivial

For example: with $n = 16$, 75% distinct edges included in G, then χ_G with high probability is connected and loop-free. In general, $O(n \log(n))$ samples for connectivity and $O(n^{3/2})$ for loop-free.

Other sampling models

- Random k-regular graphs
  - Kim-Vu sandwich theorem/conjecture: coupling with Erdös-Rényi if edges are dense enough
- Preferential-attachment random graphs
  - online but dependent (active) sampling
  - coupling with edge-independent process (Chung-Lu'06)
- Geometric random graphs
  - ranking items from Euclidean feature space
- Active sampling?
  - Osting, Brune, and Osher, *ICML* 2013
  - Osting, Xiong, Xu, and Y., 2014

Robust Ranking with Sparse Outliers

For each $(i, j) \in E$,

$$y_{\alpha ij} = \beta_0 + \beta_i - \beta_j + z_{\alpha ij} \tag{5}$$

where

- β_V: global ranking score on V
- β_0: head-advantage (home- in NBA, white- in chess)
- z_{ij} error

$$z_{\alpha ij} = \gamma_{\alpha ij} + \varepsilon_{\alpha ij}$$

  - [A0a] $\gamma_{\alpha ij}$ symmetric p-sparse (zero w.p. p and median 0)
  - [A0b] $\varepsilon_{\alpha ij} \sim \mathcal{N}(0, \sigma^2/w_{ij})$

Huber's LASSO [Xiong-Cheng-Y.'13, Xu-Xiong-Huang-Y.'13]

- Robust ranking can be formulated as a Huber's LASSO problem (Gannaz'07, She-Owen'09, Fan-Tang-Shi'12)
- Sparse outliers are sparse approximation of cyclic rankings (curl+harmonic)
- Exact recovery is possible without Gaussian noise
- Outlier detection is possible against Gaussian noise, provided
  - Irrepresentable condition (e.g. random graph)
  - Outliers have large enough magnitudes

Exact Recover against pure Sparse Outliers

Theorem (Xiong-Cheng-Y.'2013)

Let $G(n,q)$ be an Erdös-Rényi Random Graph with n nodes and each edge drawn independently with probability $q \in (0,1]$.
(A) Suppose that paired comparison data y is collected on $G(n,q)$ subject to the linear model with symmetric p-sparse outliers ($p \in [0,1]$). Then with probability tending to one the L1 solution exactly recovers the global ranking β^* if

$$p \gg O\left(\sqrt{\frac{\log n}{nq}}\right).$$

Note: no method can recover if

$$p \ll O\left(\frac{1}{\sqrt{nq}}\right).$$

Persistent Homology: online algorithm for topology tracking (e.g Edelsbrunner-Harer'08)

Figure: Persistent Homology Barcodes

- vertice, edges, and triangles etc. sequentially added
- online update of homology
- $O(m)$ for surface embeddable complex; and $O(m^{2.xx})$ in general (m number of simplex)

Online HodgeRank as Stochastic Approximations

Robbins-Monro (1951) algorithm for $\bar{A}x = \bar{b}$

$$x_{t+1} = x_t - \gamma_t(A_t x_t - b_t), \quad \mathbb{E}(A_t) = \bar{A},\ \mathbb{E}(b_t) = b$$

Now consider $\Delta_0 s = \delta_0^* \hat{Y}$, with new rating $Y_t(i_{t+1}, j_{t+1})$

$$\begin{aligned} s_{t+1}(i_{t+1}) &= s_t(i_{t+1}) - \gamma_t[s_t(i_{t+1}) - s_t(j_{t+1}) - Y_t(i_{t+1}, j_{t+1})] \\ s_{t+1}(j_{t+1}) &= s_t(j_{t+1}) + \gamma_t[s_t(i_{t+1}) - s_t(j_{t+1}) - Y_t(i_{t+1}, j_{t+1})] \end{aligned}$$

Note:

- updates only occur locally on edge $\{i_{t+1}, j_{t+1}\}$
- initial choice: $s_0 = 0$ or any vector $\sum_i s_0(i) = 0$
- step size
  - $\gamma_t = a(t+b)^{-\theta}$ $(\theta \in (0, 1])$
  - $\gamma_t = const(T)$, .e.g. $1/T$ where T is total sample size

Minimax Optimal Convergence Rates (Lim-Y.'13, Xu-Xiong-Huang-Y.'13)

- Choose $\gamma_t \sim t^{-1/2}$ (e.g. a=$1/\lambda_1(\Delta_0)$ and b large enough)
- In this case, s_t converges to s^* (population solution), with probability $1-\delta$, in the (optimal) rate of t

$$\|s_t - s^*\| \leq O\left(t^{-1/2} \cdot \kappa^{3/2}(\Delta_0) \cdot \log^{1/2}\frac{1}{\delta}\right)$$

- Dependence on $\kappa^{3/2}$ can be improved to κ by Ji Liu (U Wisc-Madison) (optimal order of κ?)

Some reference

- Random graph sampling models: Erdös-Rényi and beyond
  - Xu, Jiang, Yao, Huang, Yan, and Lin, *ACM Multimedia*, 2011, IEEE Trans Multimedia, 2012
- Online algorithms
  - Xu, Huang, and Yao, *ACM Multimedia* 2012
- l_1-norm ranking
  - Osting, Darbon, and Osher, 2012
- Robust ranking: Huber's Lasso
  - Xiong, Cheng, and Yao, 2013
  - Xu, Xiong, Huang, and Yao, *ACM Multimedia* 2013
- Active sampling
  - Osting, Brune, and Osher, *ICML* 2013
  - Osting, Xiong, Xu, and Yao, 2014

Strategic Simplicial Complex for Flow Games

	O	F
O	3, 2	0, 0
F	0, 0	2, 3

(a) Battle of the sexes

$$\begin{tikzcd}
(O,O) & (O,F) \arrow[l, "2"'] \arrow[d, "2"] \\
(F,O) \arrow[u, "3"] \arrow[r, "3"] & (F,F)
\end{tikzcd}$$

- n person game, each one has utility with strategy profile $u_k(s_1, s_2, \ldots, s_n)$;
- Every strategy vector $(s_1, s_2, \ldots, s_n)$ is a node
- Comparable strategies connected by edge: $(s_{-k}, s_k) = (s_1, \ldots, s_k, \ldots, s_n)$ and $(s_{-k}, s'_k) = (s_1, \ldots, s'_k, \ldots, s_n)$
- Edge flow: $u_k(s_{-k}, s_k) - u_k(s_{-k}, s'_k)$

Hodge Decomposition of Finite Games

- Every game can be mapped to a flow preserving its Nash equilibrium
- Game flow = potential + harmonic

STRATEGIES	B Cooperate	B Defect
A Cooperate	(3,3)	(0,5)
A Defect	(5,0)	(1,1)

$$\begin{tikzcd} (c,c) \arrow[r, "2"] \arrow[d, "2"'] & (c,d) \arrow[d, "1"] \\ (d,c) \arrow[r, "1"] & (d,d) \end{tikzcd}$$

Note: Prisoner's dilemma is a potential game to its Nash equilibrium, not efficient! So we want new way for flow construction...

Candogan-Menache-Ozdaglar-Parrilo, 2010, Flows and Decompositions of Games: Harmonic and Potential Games, arXiv: 1004.2405v1, May 13, 2010.

Note: Shapley-Monderer Condition $\equiv$ Harmonic-free $\equiv$ quadrangular-curl free

Bimatrix Games

For bi-matrix game (A, B),

- potential game is decided by $((A + A')/2, (B + B')/2)$
- harmonic game is zero-sum $((A - A')/2, (B - B')/2)$
- Computation of Nash Equilibrium:
  - each of them is tractable;
  - however direct sum is NP-hard;
  - approximate potential game leads to approximate Nash-equilibrium;
- a special case of Leontief Equilibrium for Exchange Market

Summary

Hodge Decomposition Theorem finds new applications in the following fields

- Statistical ranking: where every paired comparison data is decomposed into
  - gradient flow (global ranking)
  - harmonic flow (global inconsistency)
  - curl flow (local inconsistency)
- Game Theory: every finite game can be decomposed into
  - potential game
  - harmonic game
- Computer Vision: shade-removal, optical flow decomposition etc.
- more are coming ...

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