14.121: Exam Notes

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1 Utility and Revealed Preferences

A utility function serves two purposes:

1. Positive – used to predict behavior (utility maximization)
2. Normative – used to determine what is 'good' (welfare analysis)

Definition (Choice Space)**.** *Let a choice space* $X \subset \mathbb{R}^N$ *be the set of states of the world (in an abstract sense).*

Definition (Utility Function)**.** *A utility function is a map* $u : X \to \mathbb{R}$.

Definition (Preference)**.** *A preference is a set* $P \subset X \times X$ *with* $x \succeq_p y$ *if* $(x, y) \in P$.

Definition. *The utlitity function* $u : X \to \mathbb{R}$ ***represents*** *a preference* P *if* $u(x) \geq u(y) \iff x \succeq_p y$.

Note: In general, representations are not unique. In fact, given any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ and some u representing P, $f \circ u$ also represents P.

Definition (Complete)**.** *A preference relation* $\succeq_p$ *is complete on* X *if* $\forall x, y \in X$ *either* $x \succeq_p y$ *or* $y \succeq_p x$.

Definition (Reflexive)**.** *A preference relation* $\succeq_p$ *is reflexive on* X *if* $\forall x \in X$, $x \succeq_p x$.

Definition (Transitive)**.** *A preference relation* $\succeq_p$ *is transitive on* X *if* $\forall x, y, z \in X$, $(x \succeq_p y) \wedge (y \succeq_p z) \implies x \succeq_p z$.

Proposition. *If* $\succeq_p$ *is represented by* u *on* X*, then* $\succeq_p$ *is complete, reflexive, and transitive.*

Proof. Prove each part separately:

1. Complete: $\forall x, y \in X, u(x) \geq u(y)$ or $u(y) \geq u(x)$ (property of R), $\implies$ $x \succeq_p y$ or $y \succeq_p x$.

2. Reflexive: $\forall x \in X, u(x) \geq u(x)$ (function well defined) $\implies x \succeq_p x$.

3. Transitive: $x \succeq_p y, y \succeq_p z \implies u(x) \geq u(y)$ and $u(y) \geq u(z)$ (because u is representative). Then by transitivity on $\mathbb{R}$ we have $u(x) \geq u(z)$, and again because u is representative, $x \succeq_p z$.

$\square$

Can preferences reasonably violate these properties? Consider each one individually:

- Intransitive preferences permit a cycle $x \to y \to z \to x$ such that an agent is being made better off with each exchange, so this would permit a small charge to be extracted at each exchange without ultimately changing the agent's choice, which is referred to as the "Dutch Book." It is thought that since we don't observe this in practice that such preferences must not exist, but it is possible that some other frictions exist.

- Anything without reflexivity makes not sense at all, so we don't consider it.

- Some preferences might not be complete, generally because of:

  1. moral issues – agents might refuse to establish a preference over certain alternatives for moral issues (e.g. killing one vs killing 100)
  2. bounded rationality – might not even be able to fully consider all possible states of the world, so can't have complete preferences

As an aside, consider a famous experiment that seems to isolate preferences which violate transitivity. Kahneman and Tversky (1984) do (basically) the following: Tell people they are going to MIT Coop to buy \$125 stereo and \$15 calculator. Then salesman either says that the calculator or stereo is on sale at the Harvard Coop for \$5 cheaper. Most people choose to travel to Harvard Coop to get \$5 if it is taken off calculator, but most don't if it is taken off stereo (they just buy the stuff at the MIT Coop). A third treatment has the salesman say that they are out of stock of both items, but if people go to the Harvard Coop (where the stuff is in stock), they can choose to take \$5 off of either item, and people are indifferent about which item to discount. Thus if we consider x to be getting \$5 off calculator by buying at Harvard Coop, y to be getting \$5 off stereo by buying at Harvard Coop, and z just buying both at MIT, the first two treatments give us that $x \succeq z$ and $z \succeq y$, but the third treatment gives us that $x\ y$, a contradiction (since transitivity) should have given us taht $x \succeq y$. The finiding is controversial, and it is often argued that people only make this mistake because they are uneducated about/unfamiliar with this situation. In fact, John List (Chicago) has a bunch of papers dealing with baseball card markets, and he finds that experience limits irrational/paradoxical behaviors. See also Gul and Pesendorfer, "The Case for Mindless Economics," which is a response to neuroeconomics.

Suppose we have the three properties we've been studying, then is the converse of the previous proposition true, i.e. can we always find a u that represents $\succeq_p$?

Proposition. *Let the choice space $X \subset \mathbb{R}^N$ be finite, P a preference on X with $\succeq_p$ complete, reflexive, and transitive on X, then there exists a utlity function u that represents P.*

Proof. Let $B(x) := \{z\ inX : x \succeq_p z\}$, with B meaning "below." Then we can define $u(x) = |B(x)|$, which is well defined since X is finite (note that the worst item $w \in X$ has $u(w) = |B(w)| = 1$ by reflexivity. Prove each direction of representation:

1. Show that $u(x) \geq u(y) \implies x \succeq_p y$. Note first that $u(x) \geq u(y) \implies |B(x)| \geq |B(y)|$. There are two possibilities:

   (a) $y \in B(x)$, meaning (by construction of B) that $x \succeq_p y$, or

   (b) $y \notin B(x) \implies |B(x)| = |B(x) \setminus \{y\}|$, and using reflexivity we have $y \in B(y) \implies |B(y)| - 1 = |B(y) \setminus \{y\}|$. Moreover, $|B(x)| \geq |B(y)| \implies |B(x) \setminus \{y\}| = |B(x)| > |B(y)| - 1 = |B(y) \setminus \{y\}|$. Hence there must be some $z \in X \setminus \{y\}$ such that $z \in B(x)$ but $z \notin B(y)$, i.e. $x \succeq_p z$ but $y \not\succeq_p z$. Using completeness of P, though, we therefore have $z \succeq_p y$, therefore by transitivity we have $x \succeq_p y$, meaning $y \in B(x)$ (by construction), a contradiction.

2. Show that $x \succeq_p y \implies u(x) \geq u(y)$. Suppose that in fact $x \succeq_p y$. Then $\forall z \in X, z \in B(y) \implies y \succeq_p z \implies x \succeq_p z$ (by transitivity) $\implies z \in B(x)$ (by construction), which in combination gives $z \in B(y) \implies z \in B(x)$, hence $B(y) \subseteq B(x)$ and therefore $|B(y)| \leq |B(x)|$, i.e. $u(y) \leq u(x)$.

$\square$

Definition (Monotone Preference Relation)**.** *A preference relation $\succeq$ on X is monotone if $\forall x, y \in X, y \geq x \implies y \succeq x$.*

Definition (Strictly Monotone Preference Relation)**.** *A preference relation $\succeq$ on X is strictly monotone if $\forall x, y \in X$ with $x \neq y$, $y \geq x \implies y \succ x$.*

Definition (Locally Non-Satiated Preference Relation)**.** *The preference relation $\succeq$ on X is locally non-satiated if $\forall x \in X$, and $\forall \delta > 0, \exists y \in X$ s.t. $||y - x|| < \delta$ and $y \succ x$.*

Definition (Convex Preference Relation)**.** *The preference relation $\succeq$ on X is convex if $\forall x, y \in X$, $(y \succeq x) \wedge (z \succeq x) \wedge (y \neq z) \implies \forall \alpha \in (0,1), \alpha y + (1-\alpha) z \succeq x$.*

Definition (Continuous Preference Relation)**.** *The preference relation $\succeq$ on X is continuous if it is preserved under limits: for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succeq y^n \forall n$, $x \succeq y$ where x, y are limit points of the respective sequences.*

Note: lexicographic preferences are not continuous. Consider the elements of the choice space with (ordered) components, $x = (x_1, x_2), y = (y_1, y_2)$. Then $\succeq$ is defined by $x \succeq y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. Then if we consider $x^n = (\frac{1}{n}, 0), y^n = (0, 1)$, we have $x^n \succeq y^n$ but $y(= (0, 1)) \succeq x(= (0, 0))$.

Theorem (Debreu's Theorem)**.** *Suppose the rational preference relation $\succeq$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succeq$.*

** TODO: do we need proof of Debreu?? **

Definition (Directly Revealed)**.** *At given price vector p, if consumption bundle x is chosen when y could have been chosen, we say x is directly revealed preferred to y, $xR^D y$.*

Definition (Indirectly Revealed)**.** *If a sequence of direct comparisons indicates that x is preferred to y, then x is revealed preferred to y. That is, if*

$$xR^D z_1, z_1 R^D z_2, \ldots, z_n R^D y$$

when we write xRy.

Definition (GARP)**.** *A set of consumption choices $\{(p^1, x^1), \ldots, (p^n, x^n)\}$ satisfies the general axiom of revealed preferences (GARP) if and only if*

$$x^i R x^j \implies \neg(x^j P^D x^i)$$

Proposition (Arfiat 1967)**.** *Let $\{(p^1, x^1), \ldots, (p^n, x^n)\}$ be a finite set of consumption choices. If a finite set of demand data violates GARP, then tehse data are inconsistent with choice according to locally nonsatiated, complete, and transitive preferences.*

Conversely, if a finite set of demand data satisfies GARP, then these data are consistent with choice according to strictly increasing, continuous, convex, complete, and transitive preferences.

Arfiat is huge because it gives succinct, testable conditions that a finite dataset must satisfy to be consistent with utility maximization.

2 Demand Theory

2.1 Consumer's Problem

World has L goods, with $X \in R^L_+$ the set of possible consumption vectors. The corresponding price vector $p \in R^L_{++}$. The budget set for a consumer with wealth w is

$$B_{p,w} = \{x \in X | p \cdot x \leq w\}$$

2.2 Utility Maximization Problem

Definition (UMP)**.** *The consumer's utility maximization problem (UMP) is*

$$\max_{x \in B_{p,w}} u(x) \quad s.t. \quad p \cdot x \leq w$$

Definition (Marshallian/Walrasian Demand)**.** *The Marshallian (Walrasian) Demand Correspondence $x(p,w) : X \times \mathbb{R} \to X$ is defined by*

$$x(p,w) = \{z \in B_{p,w} | u(z) = \max_{x \in B_{p,w}} u(x)\}$$

Proposition. *Suppose u is continuous and satisfies local nonsatiation, then:*

(a) The UMP problem has at least one solution

(b) $x(p,w)$ is homogenous of degree 0, i.e. $x(\alpha p, \alpha w) = x(p,w)$, $\forall \alpha > 0$

(c) $x(p,w)$ satisfies Walras' law, i.e. $p \cdot z = w$, $\forall z \in x(p,w)$

(d) If u is strictly quasi-concave, then $x(p,w)$ is a function, i.e. each $x(p,w)$ contains a single bundle

Definition (Elasticity of Demand)**.** *The elasticity of demand with income is*

$$\eta_i = \frac{\partial x_i}{\partial w} \cdot \frac{w}{x_i} = \frac{\partial \log x_i}{\partial \log w}$$

The elasticity of demand with price is

$$\epsilon_{ij} = \frac{\partial x_i}{\partial p_j} \cdot \frac{p_j}{x_i} = \frac{\partial \log x_i}{\partial \log p_j}$$

Proposition. *Let s_i be the budget share of good i, $s_i = \frac{p_i x_i(p,w)}{w}$, then*

(Engel aggregation) $\sum_{i=1}^n s_i \eta_i = 1$, *i.e. total expenditure must change by an amount equal to any wealth change*

(Cournot aggregation) $\sum_{i=1}^n s_i \epsilon_{ij} = -s_j$, *i.e. total expenditure cannot change in response to change in price*

Definition (Indirecty Utility)**.** *The indirect utility function* $v(p,w) : X \times R \to \mathbb{R}$ *is defined by*

$$v(p,w) = \max_{x \in B_{p,w}} u(x)$$

which in turn is equal to

$$v(p,w) = u(x(p,w))$$

Proposition. *Suppose* u *is continuous and satisfies local nonsatiation. Then the indirect utility function* $v(p,w)$ *is*

(a) Homogenous of degree 0*, i.e.* $v(\alpha p, \alpha w) = v(p,w)$

(b) Strictly increasing in w *and nonincreasing in* $p_i, \forall i$

(c) Quasi-convex, i.e. $\{(p,w)|v(p,w) \leq v)\}$ *is convex* $\forall v$

(d) Continuous in p *and* w

Proposition (Roy's Identity)**.** *Suppose that* u *is continuous utility function and preferences are locally nonsatiated and strictly convex. Suppose that the indirect utility function* $v(p,w)$ *is differentiable at* $(p^*, w^*) >> 0$*. Then* $\forall i = 1, \ldots, L$*,*

$$x_i(p^*, w^*) = -\frac{\frac{\partial v(p^*,w^*)}{\partial p_i}}{\frac{\partial v(p^*,w^*)}{\partial w}}$$

For examples, see 3.D.1 (p. 55 MWG).

2.3 Expenditure Minimization Problem

Definition (EMP)**.** *The consumer's expenditure minimization problem (EMP) is*

$$min_{x \in X} p \cdot x \quad s.t. \quad u(x) \geq u_0$$

Proposition (UMP-EMP Duality)**.** *Suppose* $u(x)$ *is a continuous utility function satisfying local nonsatiation and* $p >> 0$*, then*

(a) If x^* *is a solution to the UMP with wealth* w*, then* x^* *is a solution to the EMP with utility* $u(x^*)$

(b) if x^* *is a solution to the EMP with utility* u_0*, then* x^* *is a solution to the UMP with wealth* $p \cdot x^*$*.*

Definition (Expenditure Function)**.** *The expenditure function* $e : X \times \mathbb{R} \to \mathbb{R}$ *is defined by*

$$e(p,u) = \min_{x \in X : u(x) \geq u} p \cdot x$$

Definition (Hicksian Demand)**.** *The Hicksian demand correspodnence* $h : X \times \mathbb{R} \to X$ *is defined by*

$$h(p,u) = \{z \in X | p \cdot z = e(p,u) and u(z) \geq u\}$$

which yields the FOCs

$$p_i = \lambda \frac{\partial u(h(p,u))}{\partial x_i} \quad \textit{if} \quad h_i(p,u) > 0$$

and

$$p_i \geq \lambda \frac{\partial u(h(p,u))}{\partial x_i} \quad \textit{if} \quad h_i(p,u) = 0$$

Note that $e(p,u) = p \cdot z \quad \forall z \in h(p,u)$. The Hicksian demand is often called compensated demand since

$$h(p,u) = x(p, e(p,u))$$

that is, the Hicksian answers, "how would demand change if we changed prices and also gave wealth compensation so utility level is unchanged?"

Proposition. *Suppose* u *is a continuous utility function satisfying local nonsatiation. Then the expenditure function* $e(p,u)$ *is*

(a) Homogenous of degree 1 *in* p*, i.e.* $e(\alpha p, u) = \alpha e(p,u)$

(b) strictly increasing in u *and nondecreasing in each* p_i

(c) concave in p

(d) continuous in p *and* u

Proposition. *Suppose* u *is a continuous utility function satisfying local nonsatiation. Then the Hicksian demand correspondence* $h(p,u)$ *is*

(a) Homogenous of degree 0 *in* p*, i.e.* $h(\alpha p, u) = h(p,u), \forall \alpha, p, u$

(b) no excess utility property: $u(x) = u \forall x \in h(p,u)$

(c) If u *is strictly quasi-concave, then* $h(p,u)$ *is a function*

Proposition (Compensated Law of Demand)**.** *Suppose* u *is a continuous, strictly quasiconcave utility function satisfying local nonsatiation, then* $\forall p', p''$,

$$(p'' - p') \cdot [h(p'', u_0) - h(p', u_0)] \leq 0$$

that is, the weighted average of price and demand movements (the dot product) is negative, i.e. demand goes down for products that have gone up in price.

Corollary. *If* p_i *increases and all otehr goods' prices are unchanged, then the Hicksian demand for good* i *weakly decreases.*

Proposition. *Suppose u is a continuous, strictly quasiconcave utility function satisfying local nonsatiation. Then for $i = 1, 2, \ldots, L$, we have*

$$h_i(p,u) = \frac{\partial}{\partial p_i} e(p,u)$$

As a corollary, if we differentiate once more we get cross partials, which must be equal, that is, provided h is continuously differentiable:

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

Definition (Slutsky Matrix)**.** *The Slutsky substitution matrix $S(p,w)$ is the $L \times L$ matrix with*

$$s_{ij} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j$$

which describes the change in the demand x_j resulting from changing p_j and giving the consumer extra wealth $x_j \cdot \partial p_j$ to make the old bundle affordable.

As a corollary, the Slutsky substitution matrix is symmetric and negative semi-definite. It is derived from $S(p,u) = D_p h(p,u) = D_p^2 e(p,u)$, which is where the symmetry comes from.

Proposition (Slutsky Equation)**.** *Suppose u is a continuous, locally nonsatiated, and strictly quasiconcave utility function, and let $w := e(p, u_0)$, then*

$$\frac{\partial h_i(p,u_0)}{\partial p_j} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w} x_j(p,w) = s_{ij}$$

which, after rearranging, gives

$$\frac{\partial x_i(p,w)}{\partial p_j} = \underbrace{\frac{\partial h_i(p,u_0)}{\partial p_j}}_{\textit{price effect}} - \underbrace{\frac{\partial x_i(p,w)}{\partial w} x_j(p,w)}_{\textit{income effect}}$$

2.4 EMP-UMP Duality

2.4.1 Duality

Proposition (UMP-EMP Duality)**.** *Suppose $u(x)$ is a continuous utility function satisfying local nonsatiation and $p >> 0$, then*

(a) If x^* is a solution to the UMP with wealth w, then x^* is a solution to the EMP with utility $u(x^*)$

(b) if x^* is a solution to the EMP with utility u_0, then x^* is a solution to the UMP with wealth $p \cdot x^*$.

2.4.2 Within UMP

- We get $v(p, w) = u(x(p, w))$
- We get $x(p, w)$ from $v(p, w)$ using Roy's Identity – as long as v is differentiable we get

$$x_i(p^*, w^*) = -\frac{\frac{\partial v(p^*, w^*)}{\partial p_i}}{\frac{\partial v(p^*, w^*)}{\partial w}}$$

2.4.3 From UMP to EMP

- We get $h(p, u) = x(p, e(p, u))$
- We can get derivative information about h from x through the Slutsky Equation:

$$\frac{\partial h_i(p, u_0)}{\partial p_j} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w) = s_{ij}$$

- ** TODO ** can go $v \to e$ but MWG wrong

2.4.4 Within EMP

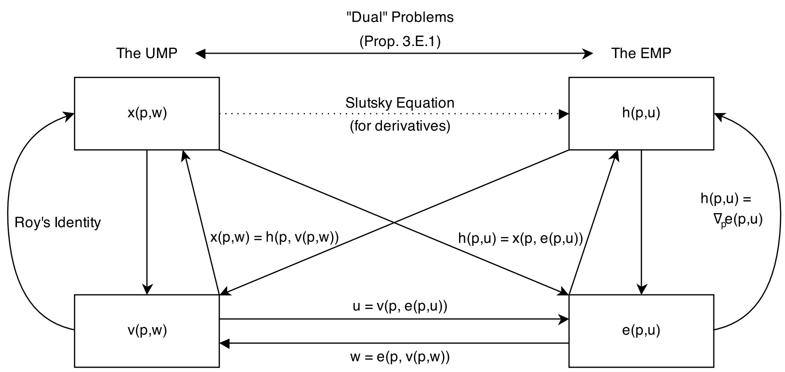
- We get $e(p, u) = p \cdot h(p, u)$
- We get

$$h_i(p, u) = \frac{\partial}{\partial p_i} e(p, u)$$

2.4.5 From EMP to UMP

- We get $x(p, w) = h(p, v(p, w))$
- ** TODO ** can go $e \to v$ but MWG wrong
- We can't go $h \to x$

2.4.6 Summary



"Dual" Problems
(Prop. 3.E.1)
The UMP
The EMP
x(p,w)
h(p,u)
Slutsky Equation
(for derivatives)
Roy's Identity
x(p,w) = h(p, v(p,w))
h(p,u) = x(p, e(p,u))
h(p,u) =
∇pe(p,u)
v(p,w)
e(p,u)
u = v(p, e(p,u))
w = e(p, v(p,w))

3 Applied Consumer Theory

3.1 Welfare Comparisons

Note: in what follows, the superscripts indicate time periods, while the subscripts denote individual goods. Hence p_y^x would be the price for good y in period x.

Definition (Equivalent Variation)**.** *Let $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$. Then the equivalent variation EV is*

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

(note that $e(p^0, u^0) = e(p^1, u^1) = w$).

This can be thought of as the change in the consumer's wealth that would be equivalent to the price change. That is, she would be indifferent between a price change from $p^0 \to p^1$ and a wealth change $EV(p^0, p^1, w)$, which of course can be negative.

Definition (Compensating Variation)**.** *Let $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$. Then the compensating variation CV is*

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$$

(note that $e(p^0, u^0) = e(p^1, u^1) = w$).

This can be thought of as the net revenue of the planner who must compensate the consumer for the price change after it occurs, to keep her utility exactly u^0 (again, this could be negative).

Theorem (Hicksian-Expenditure Theorem)**.** *Suppose only the price of good 1 changes, ceteris paribus, then*

$$EV(p^0, u^0, w) = -\int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^1) dp_1$$

and

$$CV(p^0, u^0, w) = -\int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^0) dp_1$$

where $\bar{p}_{-1}$ indicates the prices of all other goods, which are not changing.

To estimate the benefit of a new good we just take the above integrals out to ∞ (or some high price where demand would be zero).

The difficulties with the above is that they rely upon Hicksian demand, which we can't observe. So we have three tricks we can use to make welfare analysis tractable:

1. Assume no income effects in demand, i.e. $x(p, w) = x(p)$ (which may be true for small price changes). Then

$$EV = CV = -\int_{p_1^0}^{p_1^1} x_1(p_1, \bar{p}_{-1}) dp_1$$

2. Go from Marshallian to Hicksian demand nonparametrically (Slutsky lets us to Marshallian to derivatives of Hicksian, so if we solve ODE system we can recover Hicksian)
3. Convert from Marshallian to Hicksian by imposing tractable functional form

3.2 Price Indexes

Setup: At $t = 0$ we see price p^0 and consumption x^0. At $t = 1$ we see price p^1 and consumption x^1.

Definition (Laspeyres Price Index)**.** *The Laspeyres Price Index is*

$$LPI = \frac{p^1 \cdot x^0}{p^0 \cdot x^0}$$

Definition (Paasche Price Index)**.** *The Paasche Price Index is*

$$PPI = \frac{p^1 \cdot x^1}{p^0 \cdot x^1}$$

Proposition. *The LPI overstates inflation:*

$$\frac{p^1 \cdot x^0}{p^0 \cdot x^0} \geq \frac{e(p^1, u^0)}{e(p^0, u^0)}$$

The PPI understates inflation:

$$\frac{p^1 \cdot x^1}{p^0 \cdot x^1} \leq \frac{e(p^1, u^1)}{e(p^0, u^1)}$$

That is, the problem with both is that each has to fix consumption bundles, so neither allows substitution that makes consumers weakly better off in the EMP. Ideally we would just measure Hicksian expenditure changes, but of course we can never do so.

Definition (Fisher Ideal Index)**.** *The Fisher Ideal Index is given by*

$$FII = \sqrt{LPI \cdot PPI}$$

3.3 Choosing Functional Forms

To generate simpler functional forms, we often take one of two approaches:

1. Restrict preferences - impose that demand within group of goods is independent of prices outside the group. Thus the consumer is viewed as performing two step budgeting – first allocate group expenditures and then within-group expenditures.
2. Restrict price movements - assume prices within group move proportionately. This is related to the problem set question about composite goods.

3.4 Integrability

If we observe a set of demand functions, a natural question would be whether they are consistent with utility maximization. We already know that rationality implies

1. Marshallian demand x is homogenous of degree 0
2. Marshallian demand x satisfies Walras' law (i.e. $p \cdot x(p, w) = w$)
3. The Slutsky Matrix is symmetric and negative semidefinite

Proposition (Hurwicz-Uzawa)**.** *A set of continuously differentiable functions $x_i : X \times \mathbb{R} \to \mathbb{R}_+$ are the demand functions generated by some increasing, quasiconcave utility function u if the satisfy Walras' law and have a symmetric negative semidefinite Slutsky substitution matrix.*

Basically, this says that we need two very simple properties on x to ensure that we are playing by the "rules" of micro – i.e. that utility analysis is appropriate.

3.5 Demand Aggregation

Definition (Homothetic Preferences)**.** *Preferences are homothetic if $\forall x, y, \alpha > 0$,*

$$x \succeq y \implies \alpha x \succeq \alpha y$$

Proposition. *Homothetic preferences admit representation $u(x) = f(x)$ where $f(\alpha x) = \alpha f(x)$.*

Proposition. *If preferences are homothetic, then demand is homogenous of degree 1 in income*

$$x(p, \alpha w) = \alpha x(p, w)$$

In general, then, if we let $\tilde{x}(p) := x(p, 1)$, then we can write

$$x(p, w) = w\tilde{x}(p)$$

Moreover, the income elasticity of demand is 1, that is $\forall j$,

$$\frac{\partial \log(x_j)}{\partial \log(w)} = 1$$

Theorem (Chipman 1974)**.** *If individual preferences are homothetic but not necessarily identical, and incomes are proportional, then there exists a single preference ordering that generates aggregate demand.*

Proposition. *Aggregate demand is a function of aggregate wealth (i.e. typically aggregate demand is $X(p, w^1, \ldots, w^N)$, and we want to know when we can represent it as $X(p, \sum w^i)$) only if*

$$\frac{\partial x_i^k(p, w^k)}{\partial w^k} = \frac{\partial x_i^j(p, w^j)}{\partial w^j}$$

$\forall$ *goods* i, *and* $\forall$ *individuals* j, k. *The requirement is that wealth effects are the same across all individuals and across all wealth levels.*

In general, even if individuals have rational preferences that satisfy GARP, if we aggregate agents and look only at aggregate demand, often it will violate GARP. The easiest way to construct such an example is to establish a satiation point for one or more of the agents.

Moreover, we generally cannot treat problems involving the maximization over a set of agents' welfares instead as a problem involving the maximization of the sum of their welfares. The issue here is that the sum will depend upon the magnitudes of the individual utilities, which themselves have arbitrary magnitudes because they can be monotonically transformed and still represent preferences.

However, there is a very specifal case where we can do aggregation: Gorman form.

3.5.1 Gorman Polar Form

Proposition (Gorman Polar Form)**.** *Demand is a function of aggregate wealth if and only if preferences represented by indirect utility functions of the form*

$$v^k(p, w^k) = a^k(p) + w^k \cdot b(p)$$

where k *indexes the individual, hence* $b(p)$ *is uniform across consumers.*

In this case, aggregate demand is independent of the wealth distribution, and maximizes:

$$V(p, w) = \sum_k a^k(p) + w \cdot b(p)$$

where $w = \sum_k w^k$ so we don't care about distribution.

Now we can define a "representative consumer" who maximizes the utilitarian social welfare function:

$$W(u^1, \ldots, u^k) = \sum_k u^k$$

The intuition here is that there is some good with constant marginal utility that is the same for all consumers. So regardless of wealth each consumer will consume other goods until marginal utility drops below $b(p)$ and then will put the remainder of wealth into the given good. Then a representative consumer will consume each good until he consumes the sum of individual demands, then puts the rest in the constant marginal utility good.

And we can delve a bit more into the uniqueness proofs relating to Gorman Form:

Definition (Quasi-Homothetic Preferences)**.** *Quasi-homothetic preferences give demands as*

$$x^k(p, w^k) = \alpha^k(p) + w^k \beta(p)$$

where α^k is a person specific function but β is common/identical across people.

The aggregate demand for quasi-homothetic preferences itself quasi-homothetic, and is given by

$$X(p,w) = \alpha(p) + w \cdot \beta(p)$$

where $\alpha = \sum_k \alpha^k$ and $w = \sum_k w^k$.

Theorem (Gorman 1961 Polar Form)**.** *The expenditure function is of the form*

$$e(p,u) = a(p) + u \cdot b(p)$$

if and only if Marshallian demand is quasi-homothetic of form

$$x(p,w) = \alpha(p) + w \cdot \beta(p)$$

where $a(p)$ and $b(p)$ are homogenous of degree 1 in price and

$$\beta(p) = \frac{1}{b(p)} \frac{\partial b(p)}{\partial p}$$

$$\alpha(p) = \frac{\partial a(p)}{\partial p} - \beta(p) a(p)$$

Given the expenditure function, we can derive the indirect utility function of the form

$$v(p,w) = \frac{w - a(p)}{b(p)}$$

4 Estimating Demand

Our goal is to estimate *indidivual* demand $x = f(p, w)$. Rationality imposes some restrictions (Slutsky symmetry, HOD 0, Walras' Law), but still goods space is very high dimensional. In general, we can think of goods in two ways:

(product space) consider specific types of products

(characteristic space) consider characteristics of goods to reduce dimensionality

4.1 Goods in Product Space

4.1.1 Product Grouping

We can divide goods into groups, and then assume the consumer employs multi-stage budgeting: first she allocates expenditures across groups, then she allocates expenditure within groups. If demand is weakly separable, i.e. $u(x) = U(u_1(g_1), \ldots u_n(g_n))$, then demand within a group is independent of prices outside the group (Deaton and Muelbauer (1980)).

We can also approach grouping goods as imposing price restrictions, as we did on the second problem set. In this case, we consider two groups of goods x and y with price vectors p and q, respectively. If we assume that prices q move in unison, i.e. $q = \alpha q_0$, then we can consider the group y as a *single composite good* $z = q_0 \cdot y$ and redefine the utility function to get

$$\tilde{u}(x, z) = \max_{y} u(x, y) \quad s.t. \quad q_0 \cdot y \leq z$$

The interpretation for the above is that z is the amount spent on group y, and for each level of expenditure z we maximize utility over within-group bundle y. Hence we have a two stage selection – first group-level expenditure then within group expenditure. In the problem set, we proved that under the assumptions this approach is dual to directly maximizing over x and y subject to the budget constraint, i.e. an allocation (x, y) is a solution to the compositive goods problem if and only if it is also a solution to the direct maximization problem

$$\max_{x,y} f(x, y) \quad s.t. \quad p \cdot x + q \cdot y \leq w$$

4.1.2 Restricting Preferences

We can also improve tractibility of demand estimation by imposing functional form assumptions on preferences. In particular, we consider the following:

Definition (Quasi-Homothetic Demand)**.** *Demand is quasi-homothetic if it takes the following form:*

$$x(p, w) = \alpha(p) + w \cdot \beta(p)$$

Proposition (Gorman 1961)**.** *Demand is quasi-homothetic if and only if the expenditure function takes the form:*

$$e(p, u) = a(p) + u \cdot b(p)$$

where $a(p)$ and $b(p)$ are homogenous of degree 1*.*

Note: Gorman form is particularly convenient because we have established an explicit relation between α, β and a, b (see section 3.5.1, hence once we estimate demand we are automatically handed the Hicksian without having to solve an ugly system of ODEs, thus allowing us to easily do welfare calculations.

A special case of quasi-homothetic preferences with gives explicit functional form is Stone-Geary Preferences:

Definition (Stone-Geary Preferences)**.** *Stone-Geary preferences are represented by a utility function of the form*

$$u(x_1, \ldots, x_L) = \prod_{i=1}^{L} (x_i - \gamma_1)^{\beta_i}$$

with

$$\sum_i \beta_i = 1$$

Note that Cobb-Douglas utility is a special case of the above with $\gamma_i = 0$.

Proposition. *Stone-Geary preferences generate Marshallian demand*

$$x(p, w) \ : \ p_i x_i = p_i \gamma_i + \beta_i (w - \sum_j p_j \gamma_j)$$

and Hicksian demand

$$h(p, u) \ : \ p_i h(p, u_0) = p_i \gamma_i + \beta_i \beta_0 \prod_j p_j^{\beta_j} u_0$$

*where β_0 is a function of the parameters. Note that the amount spent is **linear** in both price and wealth, hence this is also known as the linear demand system. The expenditure function, which can be used for welfare analysis, is*

$$e(p, u) = \sum_j p_j \gamma_j + \beta_0 \prod_j p_j^{\beta_j} u_0$$

4.1.3 Almost Ideal Demand System (AIDS)

Deaton and Muelbauer (1980) model expenditure functions as quasi-homothetic:

$$\log e(p, u) = a(p) + ub(p)$$

which implies shares of the form

$$s_i = \frac{p_i x_i}{w} = A_i(p) + B_i(p) \log w$$

Without getting into specifics, the AIDS model is good because it is very flexible: in general, the AIDS system is a **first-order approximation** to any demand system. Moreover, since it is a system of expenditure functions, it is easy to do welfare analysis, and is realtively easy to aggregate over consumers. The downside ("almost ideal") is that there are way too many parameters that show up once we write down Deaton and Muelbauer's proposed functional forms. The parameters γ_{ij} are cross price elasticities, thus if we can impose restrictions on these we can get a more tracible result.

4.2 Goods in Characteristic Space

Good are seen as bundles of characteristics and consumers derive utility from these characteristics and some idiosyncratic taste. That is, there are many products, with each product being a unique combination of different characteristics, and each sumer will choose the single option that maximizes utility. These models are often called discrete choice models (choosing either to buy or not to buy), or "random utility" models, because utility is described as a random variable reflecting unobserved taste differences. Luce first formalized this class of models with three axioms: (1) choice probability is strictly greater than zero for all choices, (2) IIA, (3) separability.

To gain tractibility, we consider **logit** models. The standard multinomial logit model assumes ϵ_{ij} is iid according to the Extreme Value Type I distribution, $F(\epsilon) = \exp\{-e^{-\epsilon}\}$. This is due to McFadden (1973) and allows us to use the Luce (logit) model with random utility, and thus we can estimate Luce values as a function of observed characteristics. If we suppose the direct utility individual i gets from product j is $U_{ij} = V_{ij} + \epsilon_{ij}$, and we let $d_{ij} = 1$ if the individual i chooses product j and 0 otherwise, we get the nice functional form

$$P(d_{ij} = 1) = \frac{\exp(V_{ij})}{\sum_k \exp(V_{ik})}$$

The issue is that logit relies very heavily on the Independence of Irrelevant Alternatives (IIA) property, i.e. that the odds of choosing j over k is independent of which other alternatives are available, i.e.

$$\frac{P(d_{ij} = 1)}{P(d_{ik} = 1)} = \frac{e^{V_{ij}}}{e^{V_{ik}}} \ \forall j, k$$

However, this is a very strong restriction, and is often violated. Suppose there are three options to get to school: (1) drive, (2) take the red bus, (3) take the blue bus. Assume further that the red/blue bus are otherwise identical (except for color) so they generate the same utility for me. Suppose then the choice probabilities are $(0.50, 0.25, 0.25)$ (I am indifferent between the busses, so I just split my $P(bus) = 0.5$ evenly among them). IIA would require

$$\left.\frac{P(car)}{P(redbus)}\right|_{3choices} \left.\frac{P(car)}{P(redbus)}\right|_{2choices} = \frac{2}{1}$$

thus when I remove the blue bus, it would require the choice probabilities be $(0.66, 0.33)$. But in reality because I didn't really care, we would intuitively expect them to be $(0.5, 0.5)$, i.e. now that the blue bus is gone I still want to ride the bus 50% of the time, but now I'll just ride the red bus whenever I decide to bus it (since I don't care about the color).

We can get away from IIA by trying different functional forms (GEV functional form), nested logits (partition goods into nests, then choices are MNL conditional on being in a nest, while choices across nests have logit form), or mixed logits (allow consumer preferences to deviate from broad population preferences – sometimes called random coefficients logit).

5 Monotone Methods

5.1 Motivation

We are now concerned with comparative statics, that is, given some maximization problem

$$\max_x f(x;\theta)$$

we want to know both how the maximum value (v) and the maximizing argument (x) depend upon θ. Assuming all derivatives exist and the solution is interior and unique (which we could get, e.g. if f is $\mathcal{C}^2$, and strictly convex), we can apply the implicit function theorem since the maximizer uniquely solves

$$f_x(x(\theta),\theta) = 0$$

and we can say locally that

$$\frac{\partial x(\theta)}{\partial \theta} = -\frac{f_{x\theta}(x(\theta),\theta)}{f_{xx}(x(\theta),\theta)}$$

but this breaks down if any of the above assumptions fails (e.g. differentiability, convexity), and we want our analysis to be robust to specifications that might give a different f. This introduces the idea of "Robust Comparative Statics" which seek to avoid calculus and find conditions which can be used even if functions are nondifferentiable, discontinuous, or even defined on discrete state spaces.

5.2 Preliminaries

Definition (Partial Order)**.** *A relation R that satisfies transitivity, reflexivity, and antisymmetry is called a partial order.*

Definition (Partially Ordered Set)**.** *A partially ordered set, or poset, is a partial order relation together with a set, (X, R). Note that a poset does not require completeness.*

Definition (Meet)**.** *The meet operator is defined as follows:*

$$x \wedge y = \max\{z | xRz, yRz\}$$

Definition (Join)**.** *The join operator is defined as follows:*

$$x \vee y = \min\{z | zRx, zRy\}$$

Definition (Lattice)**.** *A lattice is a poset that is closed under meet and join operations.*

Definition (Strong Set Order)**.** *The strong set order is given by the binary relation*

$$A \succeq_S B \iff a \vee b \in A, \quad a \wedge b \in B, \quad \forall a \in A, \forall b \in B$$

Definition (Increasing Differences)**.** *Let f be a function $f : X \times \Theta \to \mathbb{R}$ where both X and Θ are lattices. Consider $x^H, x^L \in X$, and $\theta^H, \theta^L \in \Theta$ with $x^H \geq_X x^L$ and $\theta^H \geq_\Theta \theta^L$. Then f has increasing differences if*

$$f(x^H, \theta^H) - f(x^L, \theta^H) \geq f(x^H, \theta^L) - f(x^L, \theta^L)$$

this is the same as saying that the difference $f(x^H, \theta) - f(x^L, \theta)$ is weakly increasing in θ.

Definition (Supermodularity)**.** *Let f be a function $f : X \to \mathbb{R}$ where X is a lattice. Then f is supermodular if*

$$f(x^H \vee x^L) + f(x^H \wedge x^L) \geq f(x^H) + f(x^L) \quad \forall x^H, x^L \in X$$

Definition (Submodularity)**.** *A function f is submodular if $-f$ is supermodular.*

Proposition. *A differentiable function f is supermodular if and only if*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \quad \forall i \neq j$$

Similarly, a differentiable function f is submodular if and only if

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0 \quad \forall i \neq j$$

Definition (Substitutes)**.** *Inputs x_i and x_j to an objective function f are substitutes at input vector x if f is submodular in these arguments when evaluated locally around x (i.e. all other inputs are fixed at their levels in x). In the special case where f is differentiable, this is captured by*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \leq 0$$

The intuition behind this definition of substitutes is that submodularity implies decreasing differences, so for a fixed level of output, I could slightly decrease one input, then increase the other input, and then the amount by which I would have to increase the first input in order to meet the fixed level of output would be lower than the amount by which I decreased it initially, hence I can get away with a lower absolute level of the first input by increasing the level of the second input, which is the very idea of substitutes.

** TODO see the proofs we did regarding supermodularity on the problem set! Those are especially useful here. **

5.3 Topkis

Theorem (Topkis Monotonicity Theorem)**.** *Consider an optimization problem of the form*

$$\max_x f(x, \theta)$$

and let $x^*(\theta)$ be the maximizer correspondence. If $f(x,\theta)$ has increasing differences, then $x^*(\theta)$ is ewakly increasing in θ.

Proposition. *Define*

$$x^{**}(\theta) = \sup\{\arg\max_x f(x,\theta) + g(x)\}$$

*Then $x^{**}(\theta)$ is weakly increasing in θ **for all** g if and only if f has increasing differences.*

Theorem (Topkis)**.** *Consider an optimization problem of the form*

$$\max_{x \in D} f(x,\theta)$$

where D is a lattice (with order $\succeq_X$) and does not depend on θ. If f exhibits increasing diferences in (x,θ) relative to $(\succeq_X, \succeq_\Theta)$ and is supermodular in x (relative to $\succeq_X$), then the optimal choice correspondence $x^*(\theta)$ will be increasing in the strong set order relative to $\succeq_X$ (where an increase in the parameter is relative to $\succeq_\Theta$).

Under the above conditions, we can conclude that $\forall \theta, \theta' \in \Theta$ with $\theta' \succeq_\Theta \theta$ and $\forall x \in x^*(\theta), x' \in x^*(\theta')$,

$$x \wedge x' \in x^*(\theta) \quad \textit{and} \quad x \vee x' \in x^*(\theta')$$

5.4 Milgrom-Shannon

Definition (Single Crossing)**.** *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is single crossing in (x,θ) if $\forall x' > x$ and $\theta' > \theta$,*

(i) $f(x',\theta) - f(x,\theta) \geq 0 \implies f(x',\theta') - f(x,\theta') \geq 0$

(ii) $f(x',\theta) - f(x,\theta) > 0 \implies f(x',\theta') - f(x,\theta') > 0$

We say the single crossing is strict if $f(x',\theta) - f(x,\theta) \geq 0 \implies f(x',\theta') - f(x,\theta') > 0$.

Note that unlike increasing differences, this condition is not symmetric in the variables (x,θ).

Proposition. *If f has increasing differences then it is also single crossing, but the converse is not true.*

Proposition (Milgrom-Shannon Monotonic Selection Theorem)**.** *Define*

$$x^*(\theta, \bar{X}) := \arg\max_{x \in X} f(x,\theta)$$

If f is single-crossing in (x,θ), then $x^*(\theta,\bar{X})$ is weakly increasing in θ $\forall \bar{X} \subset X$.

Moreover, if $x^*(\theta,\bar{X})$ is non-decreasing in θ for all finite sets $\bar{X} \subset \mathbb{R}$, then f is single-crossing in (x,θ).

The usefulness of the Monotonic Selection theorem is that we can make a statement about comparative statics that is robust to arbitrarily selecting/restricting the feasible set $\bar{X} \subset X$.

5.5 Long Run versis Short Run Response

5.5.1 Rough Idea

There are an interesting set of economic models where the long-run responses to price changes are larger than short run responses. The key idea is some sort of "positive feedback" argument.

First, we start with some production function $f(k,l)$ where inputs are substitutes. We then assume taht in the short run k is fixed, so when wages increase labor falls in the short run (and so does output), but capital is fixed. In the long run, since the inputs are substitutes and before the wage change I was indifferent between an extra unit of k versus l, now with the wage change I prefer the additional unit of k so I will tend to increase k in the long run. But since they are substitutes, the long run increase in k lowers the marginal product of labor, so l further decreases in the long run. Hence the LR effect is larger than SR due to capital flexibility. Note that if k and l were complements the opposite effect would take place, but still LR effects are greater than in SR.

5.5.2 Formalization

Samuelson suggested that a firm would react more to input price changes in the long-run than in the short-run because it has more inputs it can adjust. We take the "short run" to be the period when k is effectively fixed, whereas the "long run" allows k to vary. We can be more precise about this relationship, though.

Proposition (Milgrom and Roberts)**.** *Suppose $X, Y = \mathbb{R}$ and define*

$$x(y,t) := \arg\max_{x \in X} F(x,y,t)$$

and

$$y(t) := \arg\max_{y \in Y} F(x(y,t),y,t)$$

Suppose further that $F : X \times Y \times \mathbb{R} \to \mathbb{R}$ is supermodular, that $t, t' \in \mathbb{R}$ with $t' \geq t$, and that the maximizes described below are unique for parameter values t and t'. Then

$$x(y(t'),t') \geq x(y(t),t') \geq x(y(t),t)$$

and

$$x(y(t'),t') \geq x(y(t'),t) \geq x(y(t),t)$$

The above may be useful to characterize long run response. For instance, if x and y happen to be complements, then the objective would be supermodular in x and y and the above would characterize growth paths.

Proposition (LeChatelier Principle)**.** *If production is $f(k,l)$ and wage $w > w_0$ increases, and if k and l are always complements or always substitutes, then*

$$l_{LR}(w) \leq l_{SR}(w_0, w) \leq l_{LR}(w_0)$$

Note that when we have always complements or always substitutes, we get supermodularity, and we can apply Milgrom-Roberts, which gives the result.

Note also that the above does not apply when we have sometimes complements and sometimes substitutes, which would likely be the case in many applications. For instance, perhaps capital is a complement to labor when labor is abundant but a substitute when labor is scarce.

6 Monopoly

The monopoly pricing problem is

$$\max_q \pi(q)$$

where $\pi(q) = q \cdot P(q) - c(q)$, and we assume it to be differentiable, concave, and having an interior solution. Thus the FOC is

$$q^m \cdot P'(q^m) + P(q^m) = c'(q^m)$$

Definition (Lerner Index)**.** *The Lerner Index is given by*

$$\frac{P(q^m) - c'(q^m)}{P(q^m)}$$

Proposition (Inverse Elasticity Rule)**.** *The percentave markup in the monopoly model is*

$$\frac{P(q^m) - c'(q^m)}{P(q^m)} = -\frac{1}{\epsilon}$$

where $\epsilon = \frac{dQ}{dP} \cdot \frac{P}{Q}$ *is the price elasticity of demand.*

6.1 Assumptions

Assume that

(A1) Indirect utility has the form

$$\begin{aligned} V_i(p, w) &= \max_{q : p \cdot q \leq w} u_i(q_1, \ldots, q_L) \\ &= \max_{q_1} f_i(q_1) + [w - p_1 q_1] \end{aligned}$$

which gives us that Marshallian demand has no income effect, so we can write $x(p)$ instead of $x(w, p)$ and we can make comparisons across consumers with different levels of wealth

(A2) $w_i = w_{0i} + s_i \pi(p)$ which means that firms are owned by consumers, where s_i is consumer i's share of the profits that the firm obtains

6.2 Monopolist Quantity

Proposition. *Given any depand curve* $P(q)$ *and cost function* $c(q, \theta)$*, define*

$$q^m = \sum \arg\max_q q \cdot p(q) - c(q, \theta)$$

If $c(q, \theta)$ *has increasing differences, then* q^m *is weakly increasing in* θ*.*

Proposition (Monopolistic Underprovision)**.** *Let*

$$q^{FB} = \sup \arg\max_q \int_0^q P(s)ds - c(q)$$

and

$$q^m = \sup \arg\max_q q \cdot P(q) - c(q)$$

If $P'(q) \leq 0$ then $q^m \leq q^{FB}$.

6.3 Monopolist Quality

Suppose a monopolist chooses quality s in addition to quantity, so now we can define

$$W(q,s) = \int_0^q P(x,s)dx - c(q,s)$$

and

$$\pi(q,s) = q \cdot P(q,s) - c(q,s)$$

Proposition. *Let*

$$s^{FB} = \sup \arg\max_s W(q,s)$$

and

$$s^m(q) = \sup \arg\max_s \pi(q,s)$$

If $P(q,s)$ has decreasing differences in q, then $s^{FB}(q) \geq s^m(q)$.

The intuition here is that the value of quality is less for the marginal consumer than for the average consumer. Since the monopolist only cares about the marginal consumer but the social planner cares about the average consumer, we get a distortion from optimum, known as the **Spence Distortion**. On the other hand, the IO literature notes that since the monopolist sells fewer units, the marginal monopolist consumer has a higher value than the social planner's marginal consumer. Thus in richer models where q can vary (in the above we've fixed q), this effect can offset and yield $s^m > s^{FB}$.

7 Externalities

7.1 Preliminaries

Consumers are assumed to have utility functions that take quasilinear form.

Definition (Quasilinear Utility). *A utility function is quasilinear with respect to a numeraire commodity if it takes the form*

$$u_i(x_i, h) = g_i(x_{-1i}, h) + x_{1i}$$

where x_1 is the numeraire and x_{-1} is all other goods. Thus the consumer gets linear utility in the numeraire and will always consumer other goods only until on the margin the numeraire is preferred, i.e the consumption decision for other goods $x_{-1i}(p, h)$ is constant as a function of wealth. We can then write the indirect utility function as

$$v_i(p, w, h) = g_i(x_{-1i}(p, h), h) + (w_i - p \cdot x_{-1i}(p, h))$$

there the final term is the amount of money left over for the numeraire. If we then let $\phi_i(p, h) := g_i(x_{-1i}(p, h), h) - p \cdot x_{-1i}(p, h)$ we get indirect utility of the form

$$v_i(p, w_i, h) = \phi_i(p, h) + w_i$$

which, as above, has no wealth effects.

We additionally assume that prices of traded goods are unaffected by any actions within our model, so we assume p is constant and suppress its notation. That is, we denote $\phi_i(p, h)$ as $\phi_i(h)$. We further assume that $\phi \in \mathcal{C}^2$ with $\phi_i''(\cdot) < 0$. In what follows, we study a two agent model, where the first consumer makes the choice about how much of the externality to consume.

The first consumer chooses h to solve

$$\max_{h \geq 0} \phi_1(h)$$

which gives, at the optimum,

$$\phi_1'(h^*) \leq 0 \quad \text{with equality if } h^* > 0$$

(because she cannot consume negative externality).

By contrast, any Pareto Optimal allocation h° would solve

$$\max_{h \geq 0} \phi_1(h) + \phi_2(h)$$

which gives, at the optimum

$$\phi_1'(h^\circ) \geq -\phi_2'(h^\circ) \quad \text{with equality if } h^\circ > 0$$

Note that unless $h^* = h^\circ = 0$, as long as $\phi_2(h) \neq 0$ the equilibrium level of h is not Pareto efficient.

7.2 Solutions to the Externality Problem

7.2.1 Quotas

Suppose that h generates a negative external effect, so that $h^\circ < h^*$. Then the government could impose that $h \leq h^\circ$ and indeed consumer one would set the level of the externality at h°.

7.2.2 Taxes

The government could either tax per unit of h or subsidize per unit reduction of h (on consumer 1). Suppose consumer 1 were forced to pay a tax t_h per unit of h. Then she maximizes

$$\max_{h \geq 0} \phi_1(h) - t_h \cdot h$$

thus if $t_h = -\phi_2'(h^\circ) > 0$, the competitive equilibrium will be $\phi_1'(h) = t_h = -\phi_2'(h)$, which is satisfied by h°. This type of tax is known as a Pigouvian corrective tax. In particular, the optimality-resotoring tax is set at the level of the marginal externality at the optimal solution.

We could also have used a subsidy above, which has consumer 1 maximize

$$\max_{h} \phi_1(h) + s_h(h^\circ - h)$$

which is effectively a lump sum transfer combined with a tax, and can be optimality restoring.

Note that in many cases the government lacks sufficient information about the costs and benefits of the externality, and also sometimes is unable to directly tax the externality and instead must apply quotas or taxes to total output. In these cases, we often cannot achieve an efficient outcome.

7.3 Coase

Coase viewed the externality problem as arising from a combination of

(i) lack of clearly specified property rights

(ii) transaction costs

Theorem (Coase)**.** *If property rights are clearly specified and there are no transaction costs, bargaining will lead to an efficient outcome no matter how the rights are allocated.*

Note that this does not say that distribution is unaffected, just that an efficient outcome is attained.

As an example, consider allocating to consumer 2 rights to externality free environment. She can make a take it or leave it offer of payment T (that consumer 1 would make to her) in order for 1 to have the rights to consume the externality at level h (but consumer 1 may reject the deal). Thus consumer 2 maximizes

$$\max_{h,T} \phi_2(h) + T \quad s.t. \quad \phi_1(h) - T \geq \phi_1(0)$$

but since the constraint is binding, we can solve for T and plug in to get the reframed problem for consumer 1:

$$\max_{h} \phi_2(h) + \phi_1(h) - \phi_1(0)$$

which yields the FOC that guarantees the socially optimal level.

If we switched the property rights, such that consumer 1 has the right to generate as much h as she wants. Still consumer 2 makes a take it or leave it offer to consumer 1 offering payment T in order to reduce the externality to h. Then consumer 1's problem is

$$\max_{h,T} \phi_2(h) - T \quad s.t. \quad \phi_1(h) + T \geq \phi_1(h^*)$$

again the constraint is binding so we can rewrite the above to

$$\max_{h} \phi_2(h) - (\phi_1(h^*) - \phi_1(h))$$

which again gives FOCs that generate the social optimum.

Note that in special case where two parties are firms, another solution to let the two firms merge. The resulting firm would fully internalize the externality when maximizing profits. Note also that unlike the tax and quota schemes, Coases approach only requires that the consumers know each others preferences, and not government.

7.4 Myerson Satterthwaite

Suppose h is now a discrete choice, $\{0, \bar{h}\}$ and θ, η random variables that influence each agent's utility, i.e. $\phi_1(h;\theta)$ and $\phi_2(h;\eta)$. Then we can define

$$b(\phi) = \phi_1(\bar{h};\theta) - \phi_1(0;\theta) > 0$$

and

$$s(\phi) = \phi_2(0;\eta) - \phi_2(\bar{h};\eta) > 0$$

thus when $b(\theta) > s(\eta)$ the Pareto optimal level is $\bar{h}$. Then if we let $G(b)$ and $F(s)$ be the CDFs induced by θ, η, and we give consumer 2 the right to an externality free environment, then she will always choose $h = 0$ which is inefficient if $b(\theta) > s(\eta)$.

Suppose now we allow consumer 1 to offer a transfer T for the ability to consume $\bar{h}$. And assume each consumer knows their own values but not the other person's. Under this transfer, we know consumer 2 will agree iff $T \geq s$, the probability of which is $F(T)$. Hence consumer 1 solves

$$\max_{T} \quad \underbrace{F(T)}_{P(2 \text{ accepts})} \; \underbrace{(b - T)}_{\text{payoff}}$$

and will make an offer T that is less than b (positive profits) but is greater than the minimum value of s (otherwise agent 2 never accepts). But suppose given the realization of η that

$$b(\theta) > s(\eta) > T$$

then in this case the efficient level is $\bar{h}$ since $b(\theta) > s(\eta)$, but the offer will be rejected. Hence there is a positive probability of inefficiency in this context even with transfers.

Proposition (Myerson-Sattherthwaite). *No bargaining procedure can lead to an efficient outcome for all values of b and s when they are private information and independently distributed.*

7.5 Missing Markets

Suppose we can construct a competitive market for the externality (which may not always be possible), with price per unit p_h. Then consumer 1 buys rights to h to solve

$$\max_{h_1} \phi_1(h_1) - p_h h_1$$

and consumer 2 sells rights to h to solve

$$\max_{h_2} \phi_2(h_2) + p_h h_2$$

which gives the two FOCS

$$\phi_1'(h_1) = p_h \quad \text{and} \quad \phi_2'(h_2) = -p_h$$

which clearly generateds the Pareto Optimal level of the externality in equilibrium (when we impose $h_1 = h_2 = h^{**}$.

7.6 Prices versus Quantities

We saw earlier that taxes and quotas generated efficient levels of the externality. Yet when we add uncertainty, the two are not equivalent any longer.

7.6.1 Setup

Suppose that firms generate externalities (and can be regulated) and consumers derive negative benefit from the externalities. Suppose there is uncertainty in the value of the externality, with the firm's profit given by $\pi(h, \theta)$ and the consumer's utility $\phi(h, \eta)$, where θ, η are random variables that are privately observed. However, the CDFs of θ, η are publicly known *ex ante*. We assume that $\pi(h, \theta)$ and $\phi(h, \eta)$ are strictly concave in $h, \forall \theta, \eta$.

If the government could obesrve realizations of θ and η and make its decision *ex post* then it would get the first best quota

$$h^{\circ}(\theta, \eta) = \arg\max_{h} \{\pi(h, \theta) + \phi(h, \eta)\}$$

or tax

$$t_h^\circ(\theta,\eta) = \frac{\partial \pi(h^\circ(\theta,\eta),\theta)}{\partial h} = \frac{\partial \phi(h^\circ(\theta,\eta),\theta)}{\partial h}$$

these are the first best solutions, and form the basis of comparison. Both will yield equilibrium externality levels of h°, given above.

However, the government is often unable to observe these values, or at least unable to craft policy in response to every realization. So now we assume the government must commit in advance to a tax or quota policy, knowing only the *ex ante* distributions of θ and η.

We must first consider the firm's response to taxes or quantity regulation. In the case of quantity regulation with quata $\hat{h}$, the firm solves

$$\max_{h\geq 0} \pi(h,\theta) \quad s.t. \quad h \leq \hat{h}$$

and we denote the optimal choice $h^q(\hat{h},\theta)$. In the case of a tax, the firm solves

$$\max_{h\geq 0} \phi(h,\theta) - th$$

and we denote the optimal choice $h^t(t,\theta)$.

We now consider the optimal tax or quota from the government's perspective. Under quota regulation the planner solves

$$\hat{h}^* = \arg\max_{\hat{h}} E\left[\pi(h^q(\hat{h},\theta),\theta) + \phi(h^q(\hat{h},\theta),\eta)\right]$$

and in the case of taxation the planner solves

$$t^* = \arg\max_{t} E\left[\pi(h^t(t,\theta),\theta) + \phi(h^t(t,\theta),\eta)\right]$$

Note that these optimizations are performed in expectation, so in general it will not be the case that for any given realization they are first best. Thus we will want to study the loss in aggregate surplus under given realizations of (θ,η). In the case of a quota, the loss is

$$(\pi(h^q(\hat{h},\theta),\theta) + \phi(h^q(\hat{h},\theta),\eta)) - (\pi(h^\circ(\theta,\eta),\theta) + \phi(h^\circ(\theta,\eta),\eta))$$

which is equivalent to

$$\int_{h^\circ(\theta,\eta)}^{h^q(\hat{h},\theta)} \left(\frac{\partial \pi(h,\theta)}{\partial h} + \frac{\partial \phi(h,\eta)}{\partial h}\right) dh$$

and in the case of a tax the loss is

$$(\pi(h^t(t,\theta),\theta) + \phi(h^t(t,\theta),\eta)) - (\pi(h^\circ(\theta,\eta),\theta) + \phi(h^\circ(\theta,\eta),\eta))$$

which is equivalent to

$$\int_{h^\circ(\theta,\eta)}^{h^t(t,\theta)} \left(\frac{\partial \pi(h,\theta)}{\partial h} + \frac{\partial \phi(h,\eta)}{\partial h}\right) dh$$

Weitzman makes functional form simplifications which are similar to second order Taylor approximations around the point $\hat{h}^*$ to give

$$\pi(h, \theta) = b(\theta) + (\pi' + \beta(\theta))(h - \hat{h}^*) + \frac{\pi''}{2}(h - \hat{h}^*)^2$$

and

$$\phi(h, \theta) = a(\eta) + (\phi' + \alpha(\eta))(h - \hat{h}^*) + \frac{\phi''}{2}(h - \hat{h}^*)^2$$

with

$$E[\beta(\theta)] = E[\alpha(\eta)] = 0$$

and he defines the advantage of a tax over quota regulation as

$$\Delta = E[\{\pi(h^t(t, \theta), \theta) + \phi(h^t(t, \theta), \eta)\} - \{\pi(\hat{h}, \theta) + \phi(\hat{h}, \eta)\}]$$

to give the result

Proposition (Weitzman)**.** *The benefit of price regulation over quantity regulation, as given above, is*

$$\Delta = \frac{\sigma^2}{2(\phi'')^2}(\pi'' - \phi'') = (+)[(-) + (+)]$$

where σ^2 is the variance of ϕ at $\hat{h}$. This means that price regulation is better than quantity regulation if and only if $\phi'' > |\pi''|$. The intuition is that prices are better when the firm faces more uncertainty, since they allow variability here, while quantities are better when the consumer faces more uncertainty, since it imposes a hard cap on the externality.

7.7 Equilibrium Number of Boats

This is an example that is addressed both in lecture and in the problem set, so see either for full exposition and solution. Basically the planner cares about average cost of sending out additional boats while the individual fisherman only care about whether they can make positive profit on the margin (they don't internalize the externality that they impose on the other fishermen by reducing the number of fish per capita or something). Thus the planner's number of boats is always smaller than if we let the fishermen decide on their own.

** TODO: do we need to know Lindahl equilibrium? I think it was on a problem set, and is addressed briefly on MWG pp. 363–364 **

** TODO: write up definitions of depletable and non-depletable externalities MWG section 11.D **

8 General Equilibrium: Main Issues

8.1 Setup

We study here an exchange economy with no production. There are a finite number of agents i and commodities l. A consumption bundle is a bundle of commodities $x \in \mathbb{R}^L_+$. Each agent has an endowment $\omega^i \in \mathbb{R}^L_+$ and a utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$. We denote an economy as a set of endowments and utlity functions

$$\mathcal{E} = \{(u^i, \omega^i)\}$$

thus at prices p the agent has a budget set $B^i(p) = \{x : p \cdot x \leq p \cdot \omega^i\}$ hence each agent solves

$$\max_{x \in B^i(p)} u^i(x)$$

Definition (Walrasian Equilibrium)**.** *A Walrasian equilibrium for economy $\mathcal{E}$ is a pair $(p, \{x^i\}_{i \in I})$ such that*

(i) Agents maximize utility under prevailing prices:

$$\forall i \in I, x^i \in \arg \max_{x \in B^i(p)} u^i(x)$$

(ii) Markets clear for each good:

$$\forall l \in \mathcal{L}, \sum_{i \in I} x^i_l = \sum_{i \in I} \omega^i_l$$

Definition. *An allocation $\{x^i\}_{i \in I}$ is feasible if*

$$\forall l \in \mathcal{L}, \sum_{i \in I} x^i_l \leq \sum_{i \in I} \omega^i_l$$

Note that the equilibrium only cares about relative, not absolute prices.

Proposition. *If the market for one clears, then the other markets clear as well.*

Definition (Pareto Optimal)**.** *Given an economy $\mathcal{E}$, a feasible allocation x is Pareto optimal/efficient if there is no other feasible allocation $\hat{x}$ such that*

(i) $\forall i \in I, u^i(\hat{x}^i) \geq u^i(x^i)$

(ii) $\exists i \in I : u^i(\hat{x}^i) > u^i(x^i)$

8.2 The Four Assumptions

We make the following assumptions on preferences on each agent i, termed ***the four assumptions***:

1. u^i is continuous

2. u^i is increasing in x
3. u^i is concave
4. $\omega^i >> 0$

The first three are similar to what we've always used. The last is important, new, and strong. It can be used to eliminate odd corner solutions.

8.3 A 2×2 Exchange Economy

For simplicity we now study the pure exchange economy described above in the limited case where there are two consumers and two goods. This lends itself to easy study by use of an Edgeworth box. We use the following definitions in the context of this exchange economy

Definition (Allocation). *An allocation x assigns a nonnegative consumption vector to each consumer*

$$x = (x^1, x^2) = ((x_1^1, x_2^1), (x_1^2, x_2^2))$$

Definition (Feasible). *An allocation is feasible if $\forall l = 1, 2$*

$$x_l^1 + x_l^2 \leq \omega_l^1 + \omega_l^2$$

Definition (Non-Wasteful). *An allocation is nonwasteful if*

$$x^1 + x^2 = \omega$$

Definition (Budget Set). *The budget set for consumer i is given by*

$$B^i(p) = \{x \in \mathbb{R}_+^2 : p \cdot x \leq p \cdot \omega^i\}$$

where, in general, the budget set may extend outside of the Edgeworth box.

Definition (Offer Curve). *Given a fixed exchange economy $\mathcal{E}$ (fixed endowments and utility functions), the offer curve for consumer i is generated by adjusting p and for each p tracing out the feasible allocation that maximizes consuemr i's utility, thus generating a curve.*

Note that the consumer's endowment will always be affordable, so the offer curve must lie above the indifference curve passing through the endowment.

Definition (Excess Demand). *Total agent demand for a commodity is greater than what is available in the economy (corner solution).*

Definition (Pareto Set). *Allocations where indifference curves of the two consumers are tangent. They are characterized by*

$$\frac{\frac{\partial u^1}{\partial x^1}}{\frac{\partial u^1}{\partial x^2}}(x^1) = \frac{\frac{\partial u^2}{\partial x^1}}{\frac{\partial u^2}{\partial x^2}}(\omega - x^1)$$

Definition (Contract Curve). *The section of the Pareto set where each consumer does at least as well as her initial endowment.*

8.4 Welfare Theorems

Theorem (First Welfare Theorem)**.** *Let $(p, \{x^i\}_{i\in I})$ be a Walrasian equilibrium for economy $\mathcal{E}$. If utility is increasing, then allocation $\{x^i\}_{i\in I}$ is Pareto optimal.*

This says that even without explicit coordination, decentralized markets where agents simply maximize their utilities given prices are efficient. However, it should be emphasized that the model has a number of heroic assumptions; e.g., agents face the same prices, agents are price takers, markets exist for all goods, and we havent said where prices come from.

Theorem (Second Welfare Theorem)**.** *Let $\mathcal{E}$ satisfy our four assumptions. If $\{\omega^i\}_{i\in I}$ is Pareto optimal, then $\exists p \in \mathbb{R}^L_+$ such that $(p, \{\omega^i\}_{i\in I}$ is a Walrasian equilibrium for $\mathcal{E}$.*

Note that the SWT does not say that starting from a given endowment, every Pareto optimal allocation is a Walrasian equilibrium; rather, it says if we start from a given endowment, then for any Pareto optimal allocation, there is a way to redistribute resources and prices that makes the allocation a Walrasian equilibrium outcome. That is, once we find an efficient allocation, we could support it with a price equilibrium. But in that case, why not simply implement it directly without prices?

We now proceed with a sketch of the proofs for the Welfare Theorems. We use the four assumptions given above, and additionally make the simplifying assumption that $\nabla u^i(x^i) > 0$ and that $u^i(0) = 0$.

Definition (Utility Possibility Set)**.** *The utility possibility set (UPS) is given by*

$$\mathcal{U} = \{u^1(x^1), \ldots, u^l(x^l) : \sum x^i \leq \sum \omega^i\}$$

Note that Pareto optimal allocations are those on the "northeast" frontier of the UPS. Now consider the following program

$$\begin{aligned} \max_{x^1} u^1(x^1) \quad s.t. \quad & u^i(x^i) \geq \bar{u}^i & \qquad i = 2, \ldots, I \\ & \sum x^i_l \leq \sum \omega^i_l & \qquad l = 1, \ldots, L \end{aligned}$$

then Pareto optimal allocations solve this program for different values of $(\bar{u}^2, \ldots, \bar{u}^l) \geq 0$. That is, we maximize the utility of our first consumer subject to other consumers getting at least some prespecificed level of utility each. Since we can vary the required levels $\bar{u}$ we can recover the full set of Pareto optimal outcomes. If we restrict our attention to $\bar{u} >> 0$, then under our first three assumptions the utility constraints are binding; if they weren't, we could reduce allocation to the slack constraint and give it to consumer 1 whose utility is increasing in x^1. Then we can apply KT to get the Lagrangian

$$\sum_{i\in I} \lambda^i u^i(x^i) + \sum_{l\in L} \mu_l \left(\sum_{i\in I} \omega^i_l - \sum_{i\in I} x^i_l \right)$$

where by convention $\lambda^1 = 1$. And since all constraints bind, $\lambda^i > 0$ and $\mu_l > 0$. So our FOCs are

1. $\lambda^i \frac{\partial u^i(x^i)}{\partial x_l^i} - \mu_l \leq 0$
2. $\left(\lambda^i \frac{\partial u^i(x^i)}{\partial x_l^i} - \mu_l\right) x_l^i \leq 0$
3. $x_l^i \geq 0$
4. $\sum_i x^i = \sum_i \omega^i$

(note that we could also interpret this Lagrangian as arising from maximizing a social welfare function subject to resource constraints).

Now we turn our attention not to Pareto optimal outcomes, but competitive outcomes under Walrasian equilibria. Each consumer solves the UMP under prices p:

$$\max_{x^i} u^i(x^i) \quad s.t. \quad p \cdot x^i \leq p \cdot \omega^i$$

which is a constrained optimization problem, so we use KT and let γ^i be the multiplier on the budget constraint. Then under the FOCs are

1. $\frac{\partial u^i(x^i)}{\partial x_l^i} - \gamma^i p_l \leq 0$
2. $\left(\frac{\partial u^i(x^i)}{\partial x_l^i} - \gamma^i p_l\right) x_l^l = 0$
3. $x_l^i \geq 0$

then x can be supported as a price equilibrium if and only if we can find vectors p and γ such that the above KT conditions hold. But these KT conditions are exactly the same as on the Pareto optimal outcome! This observation implies the two welfare theorems.

Proof of FWT. If ω and p are given and each agent solves UMP, then at consumption bundles $\{x^i\}$ the UMP KT conditions hold. Then if we let $\bar{u}^i = u^i(x^i), i = 2, \ldots, I$, and we define $\mu_l = p_l$ and $\lambda^i = (\gamma^i)^{-1}$ then the KT conditions for the Pareto problem are also met. Hence any Walrasian equilibrium is Pareto optimal.

Proof of SWT. If KT condtions for the Pareto program are satisfied then we can define prices $p_l = \mu_l$ and multipliers $\gamma^i = (\lambda^i)^{-1}$ and we therefore meet the KT conditions for the UMP, hence (p, x) Walrasian equilibrium with endowment $\omega = x$.

Note that the above proof of the SWT gives the interpretation that the supporting price vector p is the vector of shadow prices μ on the aggregate endowment of each good.

8.5 Robinson-Crusoe: One-Producer, One-Consumer Economy

Setup:

- two goods, leisure (1) and consumption good (2)
- wage for labor w and price of consumption good p taken as exogenous
- single consumer with endowment $\bar{L}$ of leisure and 0 of the consumption good
- single firm with strictly concave technology $f(z)$
- consumer owns the firm and receives all profits

The firm therefore solves

$$\max_{z} pf(z) - wz$$

and we denote the optimal labor demand $z(p, w)$, the optimal output level $q(p, w)$, and the optimal profits $\pi(p, w)$.

The consumer solves

$$\max_{x_1, x_2} u(x_1, x_2) \quad s.t. \quad px_2 \leq w(\bar{L} - x_1) + \pi(p, w)$$

A Walrasian equilibrium in this economy is a price vector (p^*, w^*) at which both markets clear, i.e.

$$x_2(p^8, w^*) = q(p^*, w^*) \qquad \text{consumption market clears}$$
$$z(p^*, w^*) = \bar{L} - x_1(p^*, w^*) \text{labor market clears}$$

8.6 A 2×2 Production Economy

Setup:

- 2 goods and 2 factors
  - each firm produces a single good, i.e. two firms
  - factor 1 is labor, factor 2 capital, with input prices w and r, respectively
  - total endowment is L and K, respectively
  - goods prices p_1, p_2
- each firm has constant returns to scale (CRS) technology, i.e. $f(\lambda k, \lambda l) = \lambda f(k, l)$ and $c(w, r, q) = qc(w, r, 1)$

Each firm j then solves

$$\max_{k_j,l_j} p_j f_j(k_j,l_j) - c_j(w,r,q_j)$$

and under CRS the FOC is

$$p = \frac{\partial c(w,r,q)}{\partial q} = \frac{\partial[qc(w,r,1)]}{\partial q} = c(w,r,1)$$

however, this does not gives us a level of output (due to CRS). So we move to per-unit analysis, and we get total cost for firm j to produce q^j units is

$$C^j(w,r,q^j) = \underbrace{\min\{wl + rk \mid f(k,l) \geq 1\}}_{c^j(w,r)=\text{"unit cost"}} \cdot q^j$$

then we can denote $a_l^j(w,r)$ and $a_k^j(w,r)$ to be the per-unit factor demands at given factor prices that solve the above cost minimization problem.

Lemma (Shepard's Lemma)**.** *Under the above,*

$$\frac{\partial c^j(w,r)}{\partial w} = a_l^j(w,r)$$

and

$$\frac{\partial c^j(w,r)}{\partial r} = a_k^j(w,r)$$

For a given quantity q^j, firm j therefore uses $a_l^j q^j$ units of labor. "Full employment conditions" mandate that all available labor and capital is employed, that is, with two firms:

$$a_l^1 q^1 + a_l^2 q^2 = L$$

and

$$a_k^1 q^1 + a_k^2 q^2 = K$$

Definition (Specialization)**.** *If only one good is produced is equilibrium, we say the economy is specialized.*

If we assume that there is no specialization and both goods are produced, we have an interior solution and equilibrium factor prices are determined by the FOCs

$$p^1 = c^1(c^*,r^*) \quad p^2 = c^2(c^*,r^*)$$

Definition (Intensive Production)**.** *We say that production of good* 1 *is relative more intensive in labor than production of good* 2 *if*

$$\frac{a_l^1(w,r)}{a_l^2(w,r)} > \frac{a_k^1(w,r)}{a_k^2(w,r)} \quad \forall(w,r)$$

Theorem (Samuelson 1949). *Two countries with identical technologies facing the same product prices will have the same factor prices even if factor endowment differs (version of "factor price equalization (FPE)" theory by Samuelson).*

Theorem (Stolper-Samuelson Theorem). *Under the above economy, if p_j increases, the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases (assuming interior equilibria both before and after the price change).*

Theorem (Rybcszynski Theorem). *If the endowment of a factor increases, then the production of the good that uses this factor relatively more intensively increases, while the production of the other good decreases (assuming interior equilibria both before and after the price change).*

8.7 Public Goods

We now return to Chapter 11 in MWG for discussion of public goods.

Definition (Public Good). *A public good is a good for which use of a unit by one consumer does not preclude its use by others.*

Definition (Non-depletable/non-rivalrous/public). *A good is non-depletable if one agent's consumption of the good does not subtract from any other agent's consumption of said good.*

Definition (Excludability). *A public good is non-excludable if, once it has been provided to one consumer, it is impossible (or prohibitively costly) to prevent others from consuming it (e.g. a lighthouse).*

Definition (Impure Public Good). *An impure public good, or public good with congestion, is a good that can be shared, but eventually additional consumers impose negative externalities on others (e.g. a swimming pool).*

Public goods are interesting to study because common sense suggests that they will be under-provided by the usual market mechanism.

8.7.1 Neo-Classical Theory of Public Goods

Setup:

- n consumers indexed by $i = 1, \ldots, n$
- two goods – a numeraire private good x and public good G
- consumer i has utility $u^i(x^i, G)$ which is differentiable, increasing in both arguments, quasi-concave, and $u^i(x^i, 0) > 0$ (which guarantees an interior solution)
- consumer i has endowment of numeraire w^i, and we denote $W = \sum_i w^i$
- public good endowment is zero

- $G = f(z)$, where z is total units of private goods used as inputs

Definition (Allocation). *An allocation for this community consists of*

(i) *A level public good* G

(ii) *An allocation of private goods* $x = (x^1, \ldots, x^n)$

Definition (Feasible). *An allocation for this community is feasible if* $\exists z \geq 0$ *s.t.*

(i) $z + \sum_i x^i \leq W$

(ii) $G \leq f(z)$

Definition (Pareto Optimality/Lindahl-Samuelson Condition). *An allocation* (x, G) *is Pareto optimal if and only if*

$$\sum_{i=1}^{n} \frac{\frac{\partial u^i(x^i,g)}{\partial G}}{\frac{\partial u^i(x^i,G)}{\partial x^i}} = \frac{1}{f'(z)}$$

The above is know as the *Lindahl-Samuelson Condition.* It says that the sum of the marginal rates of substitutions between the agents is equal to the amount of private good required to produce an additional unit of the public good.

8.7.2 Market Provision of Public Goods

Setup:

- public good has price p
- agents choose how much to buy, g^i
- single price-taking, profit-maximizing firm that produces public good

The issue is that we have a *strategic problem*, since the amount demanded by each agent depends on what she expects other agents to demand (note that all agents uniformly enjoy total G that the firm produces).

Definition (Competitive Equilibrium). *A competitive equilibrium in this contetx consists of* p^* *and* $G^* = (g^{1*}, \ldots, g^{n*})$ *s.t.*

(i) *Each agent's choice* g^i *maximizes utility given* p^* *and* g^*_{-i}

(ii) *The firm optimizes*

$$\max_{z \geq 0} p^* f(z) - z$$

which is equivalent to

$$p^* = \frac{1}{f'(f^{-1}(\sum g^{i*})}$$

Proposition. *In the above competitive equilibrium, there is under-provision of the public good relative to the level prescribed by the Lindahl-Samuelson condition.*

The intuition is that we have a *free-rider* problem, i.e. each agent free rides on others' provisions, and does not consider the benefit to other agents of the output that she purchased.

8.7.3 Lindahl Equilibrium and Personalized Prices

In this model, we assume that somehow the public good is excludable and can be offered to each agent at a different "personalized price." We also assume that agents own the firm that produces the public good, with agent i's share in the firm's profits given by $s^i \in [0,1]$ with $\sum_i s^i = 1$.

Definition (Personalized Price)**.** *An agent's personalized price is a price p^i that the agent faces to purchase one unit of the public good, and can differ across agents.*

We can select personalized prices such that agents all agree on the level of the public good G. This fictitious economy has $n+1$ goods, the private good (which is the same for everybody) plus the n personalized public goods.

Definition (Lindahl Equilibrium)**.** *A Lindahl equilibrium is a vector $p^* = (p^{1*}, \ldots, p^{n*})$ and allocation $x^{1*}, \ldots, x^{n*}, G^*$ s.t.*

(i) The firm maximizes total profits:

$$G^* = \arg\max_{G \geq 0} \left(\sum_i p^{i*} \right) G - f^{-1}(G)$$

(ii) Consumers maximize utility subject to their budget constaints:

$$(x^{i*}, G^*) = \arg\max_{x^i, G} u^i(x^i, G) \quad s.t. \quad x^i + p^{i*} G^* \leq w^i + s^i \pi(G^*)$$

(iii) Market clears:

$$\sum_i x^{i*} + f^{-1}(G^*) \leq \sum_i w^i$$

where $f^{-1}(G)$ is the cost in inputs to produce G.

Proposition. *Any Lindahl equilibrium is Pareto optimal.*

Note: by the definition of personalized price in the Lindahl equilibrium, an agent will not behave competitively. He will have an incentive to misreport his desire for the public good. Moreover, this equilibrium requires perfect excludability and also perfect ability to price discriminate on the public good. Thus this should be thought of as a normative benchmark, and the conclusion should be that market mechanisms are unlikely to provide public goods efficiently; this provides a rationale for government involvement.

9 GE: Existence and Other Properties

9.1 Existence of Walrasian Equilibrium

We begin with a few preliminaries

Theorem (Brouwer Fixed Point)**.** *Suppoes $A \subset \mathbb{R}^L$ is nonempty, convex, compact, and $f : A \to A$ is continuous. Then f has a fixed point:*

$$\exists x^* \in A \ : \ f(x^*) = x^*$$

Theorem (Kakutani Fixed Point)**.** *Suppose $A \subset \mathbb{R}^L$ is nonempty, compact, convex, and $f : A \to A$ is a non-empty, convex-valued, upper hemi-continuous correspondence. Then f has a fixed point:*

$$\exists x^* \in A \ : \ x^* \in f(x^*)$$

Proof. What we really want to do is to find a continuous function g that falls inside of f and then apply Brouwer, but unfortunately we can't do this in general. Instead, for each $n \in \mathbb{N}$, find a continuous function g_n whose graph is everywhere within $\frac{1}{n}$ of f.

For each such function g_n, we can, by Brouwer, find fixed point (a_n^*, a_n^*) in the graph of g_n. Thus $\exists (x_n, y_n)$ in the graph of f such that

$$|a_n^* - x_n| < \frac{1}{n} \quad \text{and} \quad |a_n^* - y_n| < \frac{1}{n}$$

Since A is compact, $\exists$ convergent subsequence $a_{n_k}^* \to a^*$ for some $a^* \in A$, hence both x_{n_k} and y_{n_k} also converge to a^*. And since the sequence (x_{n_k}, y_{n_k}) is in the graph of f and f is UHC it must be the case that the limit point (a^*, a^*) is also in the graph of f, hence $a^* \in f(a^*)$. $\square$

Definition (Excess Demand)**.** *The excess demand of consumer i at prices p is*

$$z^I(p) = x^i(p, p \cdot \omega^i) - \omega^i$$

Definition (Aggregate Excess Demand)**.** *The aggregate excess demand at prices p is*

$$z(p) = \sum_i z^i(p)$$

Proposition. *A price vector p satisfies $z(p) = 0$ if and only if $(p, \{x^i\}_{i \in I})$ is a Walrasian equilibrium.*

Thus our goal to establish existsnce will be to show that a solution to $z(p) = 0$ exists.

In order to simplify our study, we study prices on the simplex (since only relative prices matter), i.e. prices sum to 1. We denote these prices on the simplex as Δ (when prices are nonnegative) or Δ^0 (when prices are strictly positive).

We continue to use the four assumptions from section 8.2, but strengthen the monotonicity condition. For simplicity, they are repeated below (with strong monotonicity added). For each agent i,

1. u^i is continuous
2. u^i is strongly monotone in x (i.e. incrementing *any* element of x increases u)
3. u^i is concave
4. $\omega^i >> 0$

The reason we add strong monotonicity is so consumers strictly value increases in any single commodity, thus market clearing will be exact. If we didn't add this, we could have all consumers value only a single commodity which would permit many equilibria (because nobody cares about the other commodities).

Proposition. *In an exchange economy, if preferences satisfy our four assumptions and are strongly monotone, then $z(p)$ defined for $p \in \Delta^0$ satisfies:*

(i) z is a continuous function (we use this to establish that it must cross 0 in the later proof of existence)

(ii) homogeneity of degree 0 (i.e. we can arbitrarily normalize prices)

(iii) Walras' law: $p \cdot z(p) = 0 \; \forall p$

(iv) bounded below: $z(p) \geq -\bar{\omega}$

(v) boundary condition: a sequence p^n of prices that converges to a limit p with some nonzero entries and at least one zero entry will have

$$\lim_{n \to \infty} \max\{z_1(p^n), \ldots, z_L(p^n)\} = \infty$$

that is, under the above prices, at least one consumer will have strictly positive wealth, and given the strongly monotone preferences, this consumer will consume unboundedly higher levels of at least one of the goods whose price is going to zero (if more than one are going to zero, the demands for all need not go to $+\infty$!)

We use the proposition in the following way to prove existence (what follows is just a rough sketch, though). For simplicity, we consider a 2 commodity market.

1. Homogeneity of degree 0 allows us to normalize $p_2 = 1$ and consider only $p = p_1$.
2. Walras' Law allows us to check market clearning in $L-1$ markets (since the final market must then clear). Hence in our case we look only at the parket for good 1 and consider solutions to the single equation $z_1(p, 1) = 0$.
3. We then try to establish two different prices p such that $z_1(p, 1) > 0$ and $z_1(p, 1) < 0$:

(a) First we let p be very small. Walras Law gives that $p \cdot z_1(p,1) + 1 \cdot z_2(p,1) = 0$ and the boundary condition gives that $\max\{z_1(p,1), z_2(p,1)\} \to \infty$. Since z_1 is bounded away from $-\infty$ by the boundary condition, it cannot be the case that $z_2 \to \infty$, hence we must have $z_1 \to \infty$, i.e. we've estalished that at this tiny p, $z_1(p,1) > 0$

(b) Next we let p be huge. We use h.o.d. zero to rewrite $z_1(p,1) = z_1(1,1/p)$ so effectively we're letting p_2 become tiny. By the same argument as above (Walras) since p_2 is effectively tiny, then z_2 must be huge and the product is strictly postive, hence $p \cdot z_1$ must be strictly negative. But p i s strictly positive, hence we know $z_1(p,1) < 0$ for this very large p.

4. Finally, since we found prices that make $z_1 < 0$ and $z_1 > 0$, since z_1 is by the proposition continuous, it must cross zero, i.e. $\exists p \,:\, z(p) = 0$.

Proposition. *If $z : \Delta \to \mathbb{R}^L$ is a continuous function that satisfies Walras' law, then $\exists p^* \in \Delta \,:\, z(p^*) \leq 0$.*

We now turn our attention to proving the existence of a Walrasian equilibrium.

Proof. The following is just a rough outline of how we would complete the proof.

1. Define a correspondence $f : \Delta^0 \to \Delta$ by

$$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \ \forall q' \in \Delta\}$$

this correspondence identifies the goods with highest excess demand

2. Extend domain of f to Δ in order to make it UHC

3. Verify that $p^* \in f(p^*) \implies p^* \in \Delta^0,\ z(p^*) = 0$, that is, any fixed point of the correspondence f should have strictly positive prices and zero excess demand

4. Check that f satisfies the hypotheses of Kakutani and apply it to conclude that $\exists p^* \in \Delta \,:\, p^* \in f(p^*)$, hence by the above $p^* \in \Delta^0$ and $z(p^*) = 0$

□

9.2 Large Economies

Convexity of preferences has been central throughout our treatment of GE. However, we might be able to get away with lack of convexity with large numbers of agents. Suppose we have finitely many types of agents, say I, and we have r agents of each type. We can approximate convexity with a fixed I by blowing up r:

Theorem (Shapley-Folkman-Starr)**.** *(Very roughly): the sum of a large number of arbitrary sets in a finite dimensional vector space will be close to convex.*

9.3 General Facts about GE

Because z is homogenous of degree 0, we only care about relative prices, hence we only need to check market clearing in $L-1$ markets. That is, the final market is fully determined by the other $L-1$.

** TODO: what is the "implication of homogeneity" doing on p. 14 of lecture 9? not clear where he gets the "when $\alpha = 1$" thing from **

Definition (Regular Prices)**.** *An equilibrium price vector p^* is regular if the Jacobian $Dz(p^*)$ has rank $L-1$. Whenever such an equilibrium is regular, we call the economy regular as well.*

If we have regularity, then we can trim off the final row and column to get an $(L-1)\times(L-1)$ matrix that is full rank. We term this $D\hat{z}(p)$. Moreover, if we fix $p_L^* = 1$ and let q give the *relative* prices, and if we (for convenience) denote $\Omega = \{\omega^i\}$, then we can define

$$\hat{z}(q_1, \ldots, q_{L-1}, \underbrace{1}_{p_L^*}; \Omega)$$

which we can totally differentiate to get (in equilibrium, of course)

$$D_q\hat{z}(p^*) \cdot dq + D_\Omega \hat{z}(p^*) \cdot d\Omega = 0$$

which we can solve to get the comparative static result

$$\frac{\partial q}{\partial \Omega} = -[D_q\hat{z}(p)]^{-1}[D_\Omega \hat{z}(p)]$$

Hence regularity is the key condtion for comparative statics. It allows us to invert the matrix and solve directly for comparative statics results.

Definition (Locally Unique)**.** *An equilibrium price vector $p \in \Delta^0$ is locally unique if $\exists \epsilon > 0$ such that $\forall p' \in B_\epsilon(p), z(p') \neq 0$.*

Theorem. *Any regular equilibrium is locally unique.*

Proof. Apply the inverse function theorem, and if p is not locally unique then p is not regular (i.e. we can't invert the matrix, because if we could we would get a linear system around p, making it unique locally). □

Theorem. *A regular economy has a finite number of equilibria.*

Proof. By contradition: suppose not, then there is an infinite sequence of equilibrium price vectors in the simplex. Since the simplex is compact, we can find converging subsequence, and since z is continuous, the limit point of the sequence will not be locally unique (similar to the standard proof of Bolzano-Weierstrass theorem, where we make sequential cuts in the compact set; if there are infinitely many limit points, no matter how many cuts we make we have infinitely many points inside at least one tiny little area, so we don't have local uniqueness). Thus, p is not regular. □

Theorem (Debreu 1970). *For almost every vector of initial endowments, the exchange economy is regular.*

Theorem (Sonnenschein-Mantel-Debreu). *Suppose z is continuous, homogenous of degree 0, and satisfies Walras' Law. Then for any $\epsilon > 0$, $\exists k$ consumers with continuous, strictly convex, and non-decreasing preferences such that z is the aggregate excess demand for those k consumers, for all p such that $\frac{p_i}{||p||} > \epsilon \; \forall i$.*

The above is the "Anything Goes" result – any potential excess demand function can be in fact a demand function. In Arrow's words (1991), "in the aggregate, the hypothesis of rational behavior has in general no implications." The issue is that aggregation dissipates the restrictions that rationality imposes (in the individual case, these were very strong properties on the Slutsky matrix).

The theorem says: give me an excess demand function $z(p)$ that satsifies continuity, homogeneity of degree 0 and Walras law. Then I can construct an economy (i.e. consumer preferences and endowments) that generates this excess demand function. One interpretation is that there are no testable implications of rationality (outside continuity, homogeneity of degree 0, and Walras law) when looking at the agreggate economy. Why is this? *The punchline is that this is all due to income effects. We do have a lot of testable implications for individual demand.*

However, if we impose additional functional form restrictions, we can get something more tractible.

10 GE: Microfoundations

Recall that SWT requires such great information to construct supporting prices, that we question why given so much information why we wouldn't just control the economy directy (why do we need prices?). Market mechanism works by using prices to give people concise sufficient statistics allowing them to make coordinated choices and arrive at a socially optimal allocation.

The general approach regarding prices is to formalize the statement that it is impossible to have a mechanism verifying a Pareto optimal allocation without revealing a supporting Walrasian equilibrium (or common marginal rates of substitution, which gives supporting prices) This provides a formalization of the necessity of price revelation for informational reasons.

10.1 Microfoundations and the Core

Definition (Core). *The core is the set of all allocations such that no coalition (set of agents) can improve on or block the allocation (make all of its members better off) by seceding from the economy and only trading among its members.*

Note that the core is an institution free concept, and does not require prices.

Our central result is that for economies with a large number of agents, core allocations are "approximately Walrasian." This gives a positive foundation for price taking. Core convergence and nonconvergence allows us to identify situations in which price-taking is more or less reasonable.

Definition (Coalition). *In an exchange economy, a coalition is a set $S \subset I$ of agents.*

Definition (Blocking Coalition). *A coalition S blocks/improves on an allocation x by $\hat{x}$ if*

$$\forall i \in S,\ \hat{x}^i \succeq_i x^i$$

with at least one preference strict, subject to the constraint

$$\sum_{i \in S} \hat{x}^i = \sum_{i \in S} \omega^i$$

Definition (Core). *The core (by weak domination) is the set of all allocations which cannot be blocked/improved on by any coalition.*

Proposition. *In an exchange economy, every core allocation is Pareto optimal.*

Proposition. *In an exchange economy, every Walrasian Equilibrium lies in the core.*

Note that this adds strength to the FWT, since it says that no group (not just individual) can be made better off!

Recall that in the two consumer Edgeworth box, every allocation in the contract curve is in the core, but only one is a Walrasian allocation. However, as we increase the size of the economy, the non-Walrasian allocations gradually drop from the core until in the limit only the Walrasian allocations are left.

10.2 Core Convergence

Definition (Type). *The set of types of consumers is* $\mathcal{I} = \{1, \ldots, I\}$.

Definition (R-Replica Economy). *The* R*-replica economy* $\mathcal{E}_R$ *is one with* R *consumers ofeach type, hence the tptal number of consumers is* $R \cdot I$.

Definition (Allocation). *An allocation in the* R*-replica economy is a set* $(x^{i,r}) \in R_+^{LRI}$*, where* $x^{i,r}$ *is the bundle of the* r^{th} *consumer of type* i*. The budget constraint in this case is*

$$\sum_{i \in \mathcal{I}} \sum_{r=1}^{R} x^{i,r} = R \sum_{i \in \mathcal{I}} \omega^i$$

Proposition (Equal Treatment). *Suppose consumers have strictly convex preferences. If* x *is a core allocation in* $\mathcal{E}_R$*, then* $x^{i,r} = x^{i,r'}$ $\forall r, r'$ *(i.e. consumers of the same type consume the same bundle).*

Hence we can regard core allocations as vectors of fixed size LI, irrespective of the replica we are concerned with.

Definition (Type Allocation). *The type allocation for any replica* R *is given by*

$$(x_1, \ldots, x_I) \in \mathbb{R}_+^{LI}$$

where x_i *is the equal treatment allocation for each consumer of type* i.

10.2.1 Constructing a Walrasian Equilibrium from the Core

Theorem (Debreu-Scarf 1963). *Let* $C_R \subset \mathbb{R}_+^{LI}$ *denote the set of type allocations corresponding to the equal treatment core allocations in the* R*-replica economy. Then*

$$\cap_R C_R = W$$

i.e. if $x^* \in C_R$ $\forall R$*, then* x^* *is Walrasian.*

Proof. Let C_R be as above, and let $W_R \subset \mathbb{R}_+^{LI}$ denote the type allocations corresponding to the equal treatment Walrasian allocations of the R-replica economy.

Observation 1. Suppose coalition S has an objection against a type allocation x in the R-replica. Then the same is true in the $R+1$ replica, which implies

$$C_{R+1} \subseteq C_R \; \forall R$$

Observation 2. If a type coalition x is Walrasian in the R-replica, it is also $Walrasian$ in the $R+1$ replica and vice versa:

$$W_R = W_{R+1} = W \; \forall R$$

Observation 3. By our earlier proposition

$$W \subset C_R \; \forall R$$

In combination, these observations generate

$$W \subseteq \cdots \subseteq C_{R+1} \subseteq C_R \subseteq \cdots \subseteq C_1$$

$\square$

A Mathematical Definitions

A.1 Utility Functional Forms

Definition (Quasi-Concave)**.** *A function $f : X \to \mathbb{R}$ is (strictly) quasiconcave if every upper level set of f is convex, i.e. $\forall a \in \mathbb{R}$,*

$$P_a := \{x \in S \mid f(x) \geq a\}$$

is (strictly) convex.

Proposition. *The function f is quasiconcave if and only if $\forall x \in S$, $\forall x' \in S$, $\forall \lambda \in [0, 1]$,*

$$f(x) \geq f(x') \implies f((1-\lambda)x + \lambda x') \geq f(x')$$

that is, the line segment connecting any two level curves lies nowhere below the level curve corresponding to the lower value of the objective. The function is strictly quasiconcave if the inequality is strict.

Proposition. *A concave function is quasiconcave. A convex function is quasiconvex. (The converse is not necessarily true).*

Proposition. *If f is quasiconcave and $g : \mathbb{R} \to \mathbb{R}$ is increasing, then $g \circ f$ is quasiconcave.*

Why do we care about quasiconcavity? Ah, good question! The reason is that we want every upper level set of the utility function (i.e. the set bounded below by an *indifference curve*) to be convex. If we did not have this condition, then in general we would not get interior solutions. Consider quasiconvex indifference curves (i.e. level curves of the utility function). Then we would *always* get a corner solution for optimal utility under a linear budget constraint.

For completeness, we include definitions also of quasiconvexity, though we tend not to use these at all.

Definition (Quasi-Convex)**.** *A function $f : X \to \mathbb{R}$ is quasiconvex if every lower level set of f is convex, i.e. $\forall a \in \mathbb{R}$,*

$$P_a := \{x \in S \mid f(x) \leq a\}$$

is convex.

Proposition. *If f is quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ is decreasing, then $g \circ f$ is quasiconvex.*

Definition (Homogenous Function)**.** *A function $f : V \to W$ between vector spaces over a field F if homogenous of degree k ($k \in \mathbb{Z}_+$) if*

$$f(\alpha v) = \alpha^k f(v)$$

$\forall \alpha \in F, \alpha \neq 0, v \in V$.

Definition (Homothetic Preferences)**.** *A set of preferences is homothetic if for any bundles* (x, y) *and* (x', y')*, and any* $\alpha > 0$*,*

$$(x, y) \sim (x', y') \implies (\alpha x, \alpha y) \sim (\alpha x', \alpha y')$$

Proposition (Homothetic Utility)**.** *Preferences are homothetic if and only if they admit representation by a utility function with the following property:*

$$u(x, y) = u(x', y') \implies u(\alpha x, \alpha y) = u(\alpha x', \alpha y')$$

$\forall \alpha > 0$*.*

Proposition. *Homothetic preferences admit representation by a utility function with the following property:*

$$u(\alpha x, \alpha y) = \alpha u(x, y)$$

$\forall \alpha > 0$*. Note that unlike the previous result, this is not bidirectional; many utility functions without the above property represent homothetic preferences.*

Proof. We proceed by construction. Construct u such that $u(z, z) = z$. Now consider any point (x, y) and find the (z, z) such that $(x, y) \sim (z, z)$ and note then that $u(x, y) = z$. Then it must be the case that $(\alpha x, \alpha y) \sim (\alpha z, \alpha z)$. Hence $u(\alpha x, \alpha y) = \alpha z = \alpha u(z, z) = \alpha u(x, y)$, competing the proof. $\square$

Note: Cobb-Douglas and Leonteif utility functions are homothetic.

A.2 Walrasian Equilibrium Existence Preliminaries

Note: Parag put these in lecture 8 but never used them to prove existence.

Theorem (Separating Hyperplane Theorem)**.** *If* $B, C \subseteq \mathbb{R}^L$ *are convex and nonempty, and* $B \cap C = \emptyset$*, then* $\exists p \in \mathbb{R}^L, p \neq 0$ *such that*

$$\sup_{b \in B} p \cdot b \leq \inf_{c \in C} p \cdot c$$

Theorem (Supporting Hyperplane Theorem)**.** *Suppose that* $B \subseteq \mathbb{R}^L$ *is conved and that* $x \notin \mathring{B}$*. Then* $\exists p \in \mathbb{R}^L, p \neq 0$ *such that*

$$p \cdot y \geq p \cdot y \quad \forall y \in B$$

B Calculus

Definition (Total Derivative). *Consider the function $f(x(t), t)$, where one (or more) of the arguments varies in another argument. The total derivative of the function is given by*

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x(t)}\frac{dx(t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x(t)}\frac{dx(t)}{dt}$$

Definition (Chain Rule). *In order to differentiate a composite function $f(g(x))$, the chain rule gives*

$$[Df \circ g(x)] = [Df(g(x))][Dg(x)]$$

where the first term is the derivative of f evaluated at $g(x)$ and the second term is the derivative of g evaluated at x.

Proposition (Total Derivative of Parametrized Function). *Consider the function parametrized by α, $f(x; \alpha)$, and let*

$$\phi(\alpha) = \max_x f(x; \alpha)$$

$$x(\alpha) = \arg\max_x f(x; \alpha)$$

then the total derivative of ϕ with respect to each component of α is given by

$$\frac{d\phi(\alpha)}{d\alpha_i} = \frac{\partial f(x(\alpha); \alpha)}{\partial a_i} + \sum_j \frac{\partial f(x(\alpha); \alpha)}{\partial x_j}\frac{\partial x_j(\alpha)}{\partial \alpha_i}$$

Proof. This follows directly from applying the complete derivative to ϕ and then the chain rule when differentiating x. □

C Optimization

Theorem (Kuhn-Tucker Necessary Conditions)**.** *Suppose $\bar{x} \in \mathbb{R}^N$ is a local maximizer of the following that meets the given constraints:*

$$\max_{x \in \mathbb{R}^N} f(x) \quad \textit{s.t.} \quad g_1(x) = b_1, \ldots, g_m(x) = b_m \qquad \textit{(equality constraints)}$$

$$h_1(x) = c_1, \ldots, h_k(x) = c_k \qquad \textit{(inequality constraints)}$$

then $\exists$ multipliers $\lambda_m \in \mathbb{R}$, $\lambda_k \in \mathbb{R}_+$ such that

(1) $\forall n \in 1, \ldots, N$,

$$\frac{\partial f(\bar{x})}{\partial x_n} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(\bar{x})}{\partial x_n} + \sum_{k=1}^{K} \lambda_k \frac{\partial h_k(\bar{x})}{\partial x_n}$$

(2) $\forall k \in 1, \ldots, K$,

$$\lambda_k (h_k(\bar{x}) - c_k) = 0$$

(i.e. $\lambda_k = 0$ for any $h_k(\bar{x}) < c_k$).

Proposition (Bang for the Buck)**.** *For a given utility function $u(x)$, prices that generate an interior solution x^* must satisfy*

$$\frac{\frac{\partial u(x^*)}{\partial x_j}}{\frac{\partial u(x^*)}{\partial x_i}} = \frac{p_j}{p_i}$$

Proof. Suppose x^* is an interior solution, and consider a feasible perturbation x' where the agent spends δ less on j and δ more on i. Then locally

$$u(x') \approx u(x^*) + \frac{\partial u(x^*)}{\partial x_j} \left(-\frac{\delta}{p_j} \right) + \frac{\partial u(x^*)}{\partial x_i} \left(\frac{\delta}{p_i} \right)$$

and since x^* is the interior maximimum, moving to x' cannot increase utility, hence (for small δ)

$$u(x') \leq u(x^*) \implies \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j} \geq \frac{\frac{\partial u(x^*)}{\partial x_i}}{p_i}$$

and vice versa if we spend δ mroe in j and δ less on i. Thus we have two inequalitys which in combination give the desired result:

$$\frac{\frac{\partial u(x^*)}{\partial x_j}}{\frac{\partial u(x^*)}{\partial x_i}} = \frac{p_j}{p_i}$$

□

Proposition (Envelope Theorem without Constraints)**.** *Let $\phi(\alpha) = \max_x f(x;\alpha)$ and let $x(\alpha)$ be the maximizing argument. Suppoes that α is differentiable at $\bar{\alpha}$ and that $\phi(\alpha)$ is uniquely maximized at $\bar{\alpha}$, then*

$$\frac{d\phi}{d\alpha_i} = \frac{\partial f(x(\bar{\alpha});\bar{\alpha})}{\partial \alpha_i}$$

Proof. We get from the total derivative of a parametrized function (section B) that

$$\frac{d\phi(\alpha)}{d\alpha_i} = \frac{\partial f(x(\alpha);\alpha)}{\partial a_i} + \sum_j \frac{\partial f(x(\alpha);\alpha)}{\partial x_j}\frac{\partial x_j(\alpha)}{\partial \alpha_i}$$

but since $x(\bar{\alpha})$ uniquely maximizes f,

$$[Df(x(\bar{\alpha});\bar{\alpha})] = [0]$$

so the first term in the sum is always zero, giving our desired result. □

Theorem (Envelope Theorem with Constraints)**.** *Suppose we want to keep track of some subset of parameters, $q \in \mathbb{R}^S$. The we can define*

$$v(q) := \max_{x\in\mathbb{R}^N} f(x;q) \quad s.t. \quad g_1(x) = b_1, \ldots, g_m(x) = b_m \quad \textit{(equality constraints)}$$

with $x(q)$ the arg max *of the above. And let $\lambda_1, \ldots, \lambda_m$ be the lagrange multipliers associated with the solution $v(q)$, then $\forall s = 1, \ldots, S$,*

$$\frac{\partial v(q)}{\partial q_s} = \frac{\partial f(x(q);q)}{\partial q_s} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(q);q)}{\partial q_s}$$

Proof. Again we get from the total derivative of a parametrized function (section B) that

$$\frac{d\phi(\alpha)}{d\alpha_i} = \frac{\partial f(x(\alpha);\alpha)}{\partial a_i} + \sum_j \frac{\partial f(x(\alpha);\alpha)}{\partial x_j}\frac{\partial x_j(\alpha)}{\partial \alpha_i}$$

but now because we have constraints, we don't necessarily have that the derivative of f evaluated at $x(\bar{\alpha})$ is zero. Rather, we have our KT multipliers:

$$\frac{\partial f(x(\bar{\alpha});\bar{\alpha})}{\partial x_j} = \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{\alpha});\alpha)}{\partial x_j}$$

which we can substitute into the above and switch the order of the sums to get

$$\frac{d\phi(\alpha)}{d\alpha_i} = \frac{\partial f(x(\alpha);\alpha)}{\partial a_i} + \sum_{m=1}^{M} \lambda_m \left(\sum_j \frac{\partial g_m(x(\bar{\alpha});\alpha)}{\partial x_j}\frac{\partial x_j(\alpha)}{\partial \alpha_i} \right)$$

because our constraints hold with equality, we have $g_m(x(\bar{\alpha});\alpha) = b_m$, hence we can totally differentiate to get

$$0 = \frac{\partial g_m(x(\bar{\alpha});\alpha)}{\partial \alpha_i} + \sum_j \frac{\partial g_m(x(\bar{\alpha});\alpha)}{\partial x_j}\frac{dx_j}{d\alpha_i}$$

which we can then substitute in to get

$$\frac{\partial \phi(\bar{\alpha})}{\partial \alpha_i} = \frac{\partial f(x(\bar{\alpha});\alpha)}{\partial \alpha_i} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x(\bar{\alpha});\alpha)}{\partial \alpha_i}$$

$\square$

Definition (LHC)**.** *A correspondence* $\Gamma : X \to Y$ *is lower hemicontinuous (LHC) if* $\forall x$ *and* $\forall y \in \Gamma(x)$*, for any sequence* $x_n \to x$ *there exists a corresponding sequence* $y_n \in \Gamma(x_n)$ *with* $y_n \to y$*. (Any point* (x, y) *in correspondence has sequence* (x_n, y_n) *the converges to* (x, y)*).*

Definition (UHC)**.** *A correspondence* $\Gamma : X \to Y$ *is upper hemicontinuous (UHC) if for every sequence* (x_n, y_n) *in the correspondence, with* $y_n \in \Gamma(x_n)$ *such that* $x_n \to x$*, then* $y_n \to y \in \Gamma(x)$*. (Any sequence in the correspondence converges to a limit point in the correspondence).*

Theorem (Berge/Theorem of the Maximum)**.** *Let* $X \subseteq \mathbb{R}^L$*,* $Y \subseteq \mathbb{R}^m$*,* $f : X \times Y \to \mathbb{R}$ *continuous,* $\Gamma : X \to Y$ *compact-valued and continuous correspondence. Then:*

(1) $h(x) := \max_{y \in \Gamma(x)} f(x, y)$ *is continuous*

(2) $G(x) := \{y \in \Gamma(x) : \ f(x, y) = h(x)\}$ *is nonempty, compact-valued, and UHC*

D Common Utility Functions

D.1 Cobb-Douglas

Definition (Cobb-Douglas)**.** *The n-good Cobb-Douglas utility function is*

$$u(x) = A \prod_{i=1}^{N} x_i^{\alpha_i}$$

where $A > 0$ and $\sum_i \alpha_i = 1$.

D.1.1 Deriving Marhsallian Demand

Note first that log is monotone, so maximizing $\log u$ is equivalent to maximizing u. Hence our UMP is

$$\max_{\{x_i\}} \sum \alpha_i \log x_i \quad s.t. \quad \sum_i p_i x_i = w$$

Note: the budget constraint holds with strict equality since u is increasing. Moreover, since $\log 0 = -\infty$, we will have an interior solution – nonzero consumption of x_i $\forall i$.

Our Lagrangian for the constrainted maximization is

$$\mathcal{L} = \sum \alpha_i \log x_i - \lambda(\sum_i p_i x_i - w)$$

which gives the FOCs

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\alpha_i}{x_i} - \lambda p_i = 0$$

Thus we have, $\forall i, k$

$$p_i x_i = \frac{\alpha_i}{\alpha_k} p_k x_k$$

which we can substitute into our budget constraint

$$\begin{aligned}
\sum_i p_i x_i &= w \\
\sum_i \frac{\alpha_i}{\alpha_k} p_k x_k &= w && \text{substituting from above} \\
\frac{p_k x_k}{\alpha_k} \sum \alpha_i &= w \\
\frac{p_k x_k}{\alpha_k} &= w && \sum_i \alpha_i = 1 \text{ by assumption} \\
x_k(p, w) &= \frac{\alpha_k w}{p_k}
\end{aligned}$$

which is true for each of our goods. It is immediately clear that this is homogenous of degree 1 in both prices and income. Additionally, we can quickly verify that the income elasticity of demand is 1:

$$\frac{\partial x_j(p,w)}{\partial w}\frac{w}{x_k(p,w)} = \frac{\alpha_k}{p_k}\frac{w}{\frac{\alpha_k w}{p_k}} = 1$$

D.1.2 Deriving Indirect Utility

We obtain the indirect utility function by simple substituting the Marshallian demand into the direct utility function, i.e.

$$\begin{aligned}
v(x(p,w)) &= \prod_i x_i^{\alpha_i} \\
&= \prod_i \left(\frac{\alpha_i w}{p_i}\right)^{w} \\
&= \prod_i w^{\alpha_i}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \\
&= w^{\sum_i \alpha_i}\prod_i \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \\
&= w\prod_i \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} && \sum_i \alpha_i = 1
\end{aligned}$$

D.1.3 Computing Expenditure Function

Consider a fixed price vector p. And let

$$C := \prod_i \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$$

which is a constant at prevailing prices. Then the utility I get at each level of expenditure/wealth is

$$v(w) = Cw$$

hence the minimum expenditure w to get u utility is $\frac{u}{C}$. Hence

$$\begin{aligned}
e(p,u) &= \frac{u}{C} \\
&= u\prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}
\end{aligned}$$

D.1.4 Computing Hicksian Demand

We know that $h(p,u) = x(p, e(p,u))$. Hence

$$\begin{aligned} h_i(p,u) &= x_i(p, e(p,u)) \\ &= \frac{\alpha_i e(p,u)}{p_i} \\ &= \frac{\alpha_i}{p_i} \cdot u \prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i} \end{aligned}$$

D.2 Leontief

Definition (Leontief Preferences)**.** *Leontief preferences are given by*

$$x'' \succeq x' \iff \min\{x_i''\} \geq \min\{x_i'\}$$

Note: Leontief preferences are a case of preferences that are representable by a continuous utility function, but no differentiable utility function can represent them. The issue is at the kink where $x_i = x_j$ for all components. In the case of two components, the indifference curves are L shaped. Note that Leontief preferences *are* quasiconcave, but are not strictly quasiconcave.

D.3 CES

Definition (CES Utility)**.** *The CES (Constant Elasticity of Substitution) utility function is given by*

$$u(x) = \sum_{i=1}^{n} \alpha_i x_i^\gamma$$

with $\gamma \in [0,1]$ *and* $\alpha_i > 0$.

D.3.1 Deriving the Marshallian Demand

Now the consumer's problem is

$$\max_{\{x_i\}} \sum \alpha_i x_i^\gamma \quad s.t. \quad \sum_i p_i x_i = w$$

which gives the Lagrangian

$$\mathcal{L} = \sum \alpha_i x_i^\gamma - \lambda \left[\sum_i p_i x_i - w\right]$$

(since utility is increasing in x we have strict equality on the constraint).

We can differentiate to get the FOC on each x_i

$$\gamma \alpha_i x_i^{\gamma-1} - \lambda p_i = 0 \implies \lambda = \frac{\gamma \alpha_i}{p_i} x_i^{\gamma-1}$$

hence

$$\begin{aligned}
\lambda &= \lambda \\
\frac{\gamma\alpha_i}{p_i}x_i^{\gamma-1} &= \frac{\gamma\alpha_k}{p_k}x_k^{\gamma-1} \\
x_i^{\gamma-1} &= \left(\frac{\alpha_i}{p_i}\right)^{-1}\left(\frac{\alpha_k}{p_k}\right)x_k^{\gamma-1} \\
x_i &= \left(\frac{\alpha_i}{p_i}\right)^{\frac{-1}{\gamma-1}}\left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{\gamma-1}}x_k
\end{aligned}$$

which we can substitute into our budget constraint

$$\begin{aligned}
\sum_i p_i x_i &= w \\
\sum_i p_i \left(\left(\frac{\alpha_i}{p_i}\right)^{\frac{-1}{\gamma-1}}\left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{\gamma-1}}x_k\right) &= w && \text{substituting from above} \\
x_k\left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{\gamma-1}}\sum_i p_i\left(\frac{\alpha_i}{p_i}\right)^{\frac{-1}{\gamma-1}} &= w \\
x_k(p,w) &= \frac{\left(\frac{p_k}{\alpha_k}\right)^{\frac{1}{\gamma-1}}}{\sum_i p_i\left(\frac{p_i}{\alpha_i}\right)^{\frac{1}{\gamma-1}}}
\end{aligned}$$

which gives our Marshallian demand.

Note that the general trick here is that we set the Lagrange multipliers equal to themselves (we only have a single constraint), then then solve for a general x_i in terms of some fixed x_k. We then substitute this into our budget constraint to get x_k only as a function of prices, wealth, and the parameters of the utility function.

For more information, or for further functional forms analysis, see the file "Consumer_Theory_Functional_Forms.pdf" in the parent directory of these notes. These are not, however, required for the exam.