Pólya Urn Models

Stefan Grosskinsky
Warwick Mathematics Institute

September 21, 2019

These notes and supplementary material are available on
https://github.com/sgrosskinsky/urns

Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$

Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$

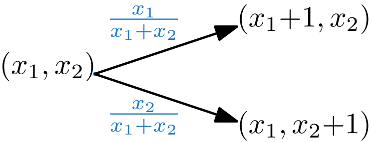
$$\begin{tikzcd}
 & (x_1+1, x_2) \\
(x_1, x_2) \arrow[ur, "\frac{x_1}{x_1+x_2}"] \arrow[dr, "\frac{x_2}{x_1+x_2}"'] & \\
 & (x_1, x_2+1)
\end{tikzcd}$$

$= \mathbb{P}\big[\underline{X}(n+1) = (x_1, x_2+1) \big| \underline{X}(n) = \underline{x}\big]$

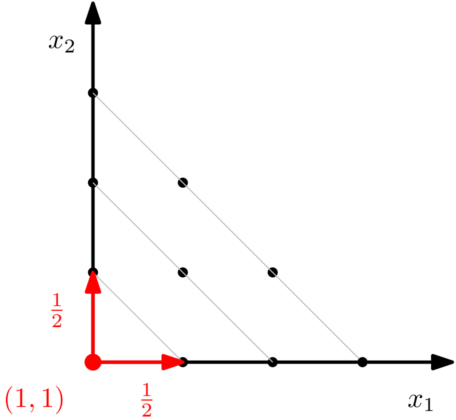
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$



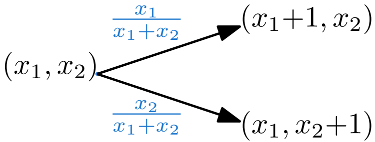
$= \mathbb{P}\big[\underline{X}(n+1) = (x_1, x_2+1) \big| \underline{X}(n) = \underline{x}\big]$



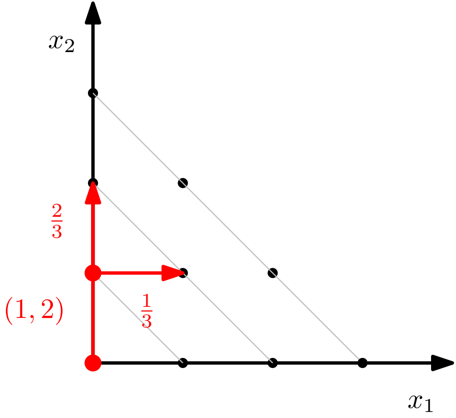
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \ldots\big)$ with $\underline{X}(0) = (1, 1)$



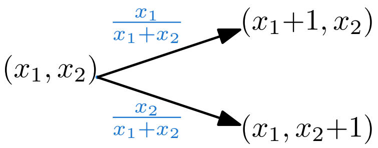
$= \mathbb{P}\big[\underline{X}(n+1) = (x_1, x_2+1) \big| \underline{X}(n) = \underline{x}\big]$



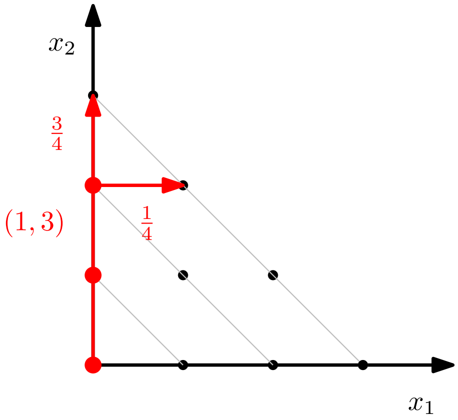
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$



$= \mathbb{P}\big[\underline{X}(n+1) = (x_1, x_2+1) \big| \underline{X}(n) = \underline{x}\big]$



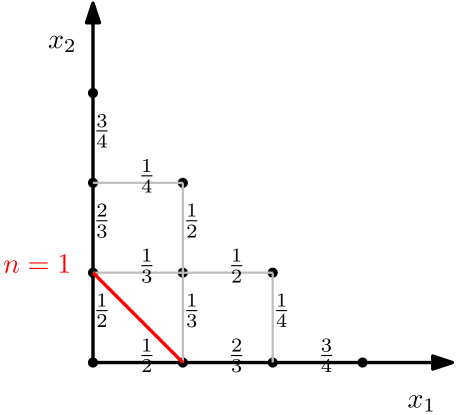
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$

$$\begin{tikzcd}[column sep=large]
 & (x_1{+}1, x_2) \\
(x_1, x_2) \arrow[ur, "\frac{x_1}{x_1+x_2}"] \arrow[dr, "\frac{x_2}{x_1+x_2}"'] & \\
 & (x_1, x_2{+}1)
\end{tikzcd}$$

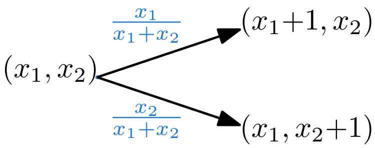
$= \mathbb{P}\big[\underline{X}(n{+}1) = (x_1, x_2{+}1) \big| \underline{X}(n) = \underline{x}\big]$



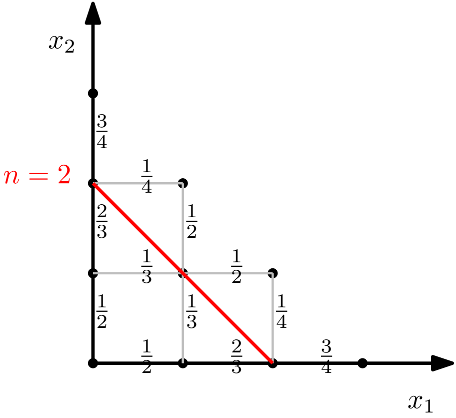
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$



$= \mathbb{P}\big[\underline{X}(n+1) = (x_1, x_2+1) \big| \underline{X}(n) = \underline{x}\big]$



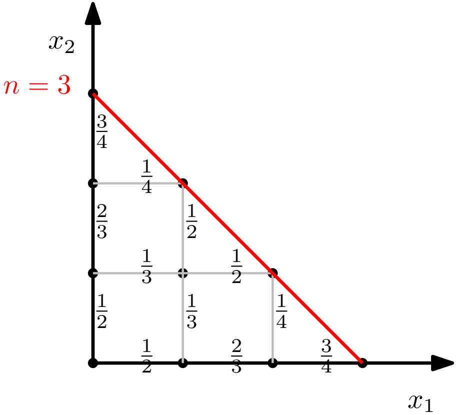
Pólya urn model

State space $\mathbb{N} \times \mathbb{N}$ **Configurations** $\underline{x} = (x_1, x_2) \in \mathbb{N}^2$

Random process $\big(\underline{X}(n) : n = 0, 1, \dots\big)$ with $\underline{X}(0) = (1, 1)$

$$\begin{tikzcd}[column sep=huge] & (x_1{+}1, x_2) \\ (x_1, x_2) \arrow[ur, "\frac{x_1}{x_1+x_2}"] \arrow[dr, "\frac{x_2}{x_1+x_2}"'] & \\ & (x_1, x_2{+}1) \end{tikzcd}$$

$= \mathbb{P}\big[\underline{X}(n{+}1) = (x_1, x_2{+}1) \big| \underline{X}(n) = \underline{x}\big]$



Pólya urn model

Distribution at time n

For all $n \geq 0$ we have $\mathbb{P}\big[\underline{X}(n) = \underline{x}\big] = \dfrac{1}{n+1}\,\delta_{|\underline{x}|,n+2}$.

Induction. $\mathbb{P}\big[\underline{X}(n+1) = (k, n+3-k)\big] = \dfrac{1}{n+1}\left[\dfrac{k-1}{n+2} + \dfrac{n+2-k}{n+2}\right] = \dfrac{1}{n+2}$

Pólya urn model

Distribution at time n

For all $n \geq 0$ we have $\mathbb{P}\big[\underline{X}(n) = \underline{x}\big] = \dfrac{1}{n+1}\,\delta_{|\underline{x}|,n+2}$.

Induction. $\mathbb{P}\big[\underline{X}(n+1) = (k, n+3-k)\big] = \dfrac{1}{n+1}\left[\dfrac{k-1}{n+2} + \dfrac{n+2-k}{n+2}\right] = \dfrac{1}{n+2}$

Fraction of type i: $\xi_i(n) = \dfrac{X_i(n)}{|\underline{X}(n)|} = \dfrac{X_i(n)}{n+2}$

Pólya urn model

Distribution at time n

For all $n \geq 0$ we have $\mathbb{P}\big[\underline{X}(n) = \underline{x}\big] = \dfrac{1}{n+1}\,\delta_{|\underline{x}|,n+2}$.

Induction. $\mathbb{P}\big[\underline{X}(n{+}1) = (k, n{+}3{-}k)\big] = \dfrac{1}{n{+}1}\left[\dfrac{k{-}1}{n{+}2} + \dfrac{n{+}2{-}k}{n+2}\right] = \dfrac{1}{n{+}2}$

Fraction of type i: $\xi_i(n) = \dfrac{X_i(n)}{|\underline{X}(n)|} = \dfrac{X_i(n)}{n+2}$

Scaling limit

$$(\xi_1, \xi_2)(n) \to (U, 1-U) \quad \text{in distribution as } n \to \infty\ ,$$

where U is a uniform random variable on $[0, 1]$.

Proof. $\mathbb{P}\big[\xi_1(n) \leq u\big] = \mathbb{P}\big[X_1(n) \leq u(n{+}2)\big] = \frac{1}{n+1}\lfloor u(n{+}2)\rfloor \to u\ , \quad u \in (0, 1).$

Extensions/Applications

[H.M. Mahmoud, Pólya urn models. CRC Press. 2008]

- $k > 2$ **types** of balls $\rightarrow$ $k + n$ balls at time $n \geq 0$
- different **initial conditions** $\underline{a} \in \mathbb{N}^k$ $\rightarrow$ Dirichlet multinomial

$$\mathbb{P}\left[\underline{X}(n) - \underline{a} = \underline{x}\right] \propto \prod_{i=1}^{k} \frac{\Gamma(x_i + a_i)}{x_i! \Gamma(a_i)} \delta_{|\underline{x}|,n}$$

- generalized **schemes** for adding balls
- a simple model for growth processes with **linear reinforcement** can describe biological growth, evolution of market shares, …

Extensions/Applications

[H.M. Mahmoud, Pólya urn models. CRC Press. 2008]

- $k > 2$ **types** of balls $\rightarrow$ $k + n$ balls at time $n \geq 0$
- different **initial conditions** $\underline{a} \in \mathbb{N}^k \quad \rightarrow \quad$ Dirichlet multinomial

$$\mathbb{P}\big[\underline{X}(n) - \underline{a} = \underline{x}\big] \propto \prod_{i=1}^{k} \frac{\Gamma(x_i + a_i)}{x_i!\Gamma(a_i)} \delta_{|\underline{x}|,n}$$

- generalized **schemes** for adding balls
- a simple model for growth processes with **linear reinforcement** can describe biological growth, evolution of market shares, ...

Generalized Pólya Urn/Balls in boxes $\big(\underline{X}(n) : n \geq 0\big)$ on $\mathbb{N}^k$

$$\mathbb{P}\big[\underline{X}(n+1) = \underline{x} + \underline{e}_i \big| \underline{X}(n) = \underline{x}\big] = \frac{f_i x_i^\gamma}{\sum_{j=1}^{k} f_j x_j^\gamma}$$

with **fitness** $f_i > 0$ and **reinforcement parameter** $\gamma \geq 0$

[Oliviera, RS&A **34**(4): 454 (2009); Jiang et al., Proc. 2016 ACM SIGMETRICS, arxiv:1604.02097]

Generalized Pólya Urn Models

- **No reinforcement** $\gamma = 0, p_i = f_i / \sum_j f_j \quad \rightarrow \quad$ Multinomial

$$\mathbb{P}\left[\underline{X}(n) - \underline{X}(0) = \underline{x}\right] = \frac{n!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n} \, \delta_{|\underline{x}|,n}$$

Generalized Pólya Urn Models

- **No reinforcement** $\gamma = 0, p_i = f_i / \sum_j f_j \quad \rightarrow \quad$ Multinomial

$$\mathbb{P}\left[\underline{X}(n) - \underline{X}(0) = \underline{x}\right] = \frac{n!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n} \, \delta_{|\underline{x}|,n}$$

weak LLN and CLT

For all $i = 1, \ldots, k$, as $n \to \infty$ in distribution

$$\xi_i(n) = \frac{X_i(n)}{|\underline{X}(n)|} \to p_i \quad \text{and} \quad \frac{X_i(n) - p_i n}{\sqrt{n p_i (1 - p_i)}} \to \mathcal{N}(0, 1) \, .$$

Proof. $X_i(n) = X_i(0) + \sum_{t=1}^n Y_t$ with $Y_1, Y_2, \ldots$ i.i.d. $\mathrm{Be}(p_i)$

Generalized Pólya Urn Models

- **No reinforcement** $\gamma = 0, p_i = f_i / \sum_j f_j \quad \rightarrow \quad$ Multinomial

$$\mathbb{P}\big[\underline{X}(n) - \underline{X}(0) = \underline{x}\big] = \frac{n!}{x_1! \cdots x_n!} p_1^{x_1} \cdots p_n^{x_n} \, \delta_{|\underline{x}|,n}$$

weak LLN and CLT

For all $i = 1, \ldots, k$, as $n \to \infty$ in distribution

$$\xi_i(n) = \frac{X_i(n)}{|\underline{X}(n)|} \to p_i \quad \text{and} \quad \frac{X_i(n) - p_i n}{\sqrt{n p_i (1 - p_i)}} \to \mathcal{N}(0, 1)\ .$$

Proof. $X_i(n) = X_i(0) + \sum_{t=1}^n Y_t$ with $Y_1, Y_2, \ldots$ i.i.d. $\mathrm{Be}(p_i)$

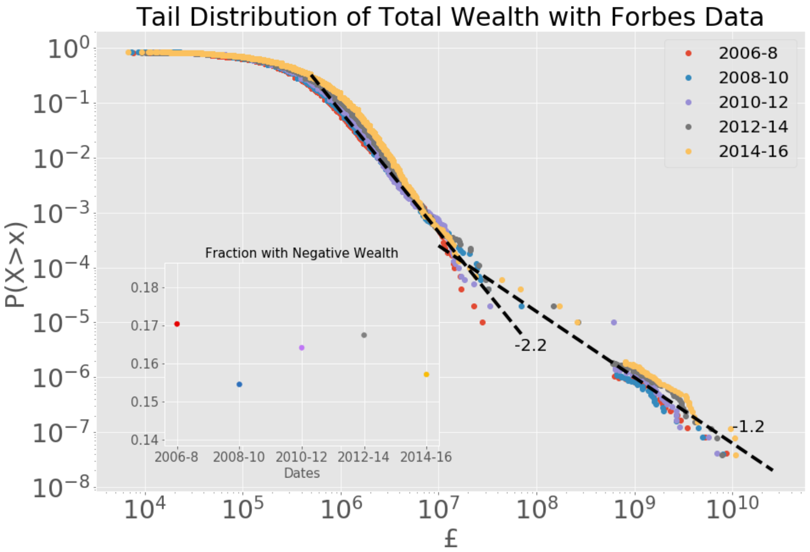
- **Superlinear reinforcement** $\gamma > 1, f_i = 1$

limiting behaviour

$$m(n) := \max_i X_i(n) \to \infty \quad \text{and} \quad n - m(n) \to Y < \infty$$

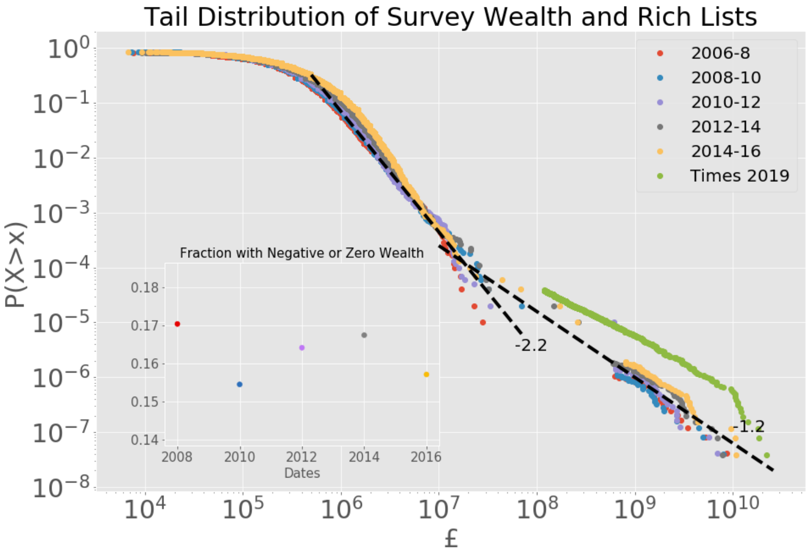
in distribution as $n \to \infty$.

Wealth distribution in UK



PhD project with Sam Forbes, Warwick

Wealth distribution in UK



PhD project with Sam Forbes, Warwick