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Transition and saturation of traffic flow controlled by traffic lights

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Abstract

We study the traffic flow controlled by traffic lights on a single-lane roadway by using the optimal velocity model. The characteristic of traffic flow is clarified for the three different strategies of traffic light control: the simple synchronized, green wave, and random switching strategies. The current–density diagrams are calculated for the three different strategies. It is found that the saturation of current occurs at the critical density. The critical density of the dynamical transition depends on the cycle time of the traffic light and strategy. The value of the saturated current does not depend on the cycle time and the strategies. The density wave propagating backward appears when the current saturates. It is shown that the density wave is consistent with the spontaneous jam. A theoretical analysis is also presented for the dynamical transition.

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Keywords: Traffic flow; Dynamical transition; Density wave; Traffic light

1. Introduction

Recently, traffic flow has attracted considerable attention [1–5]. Traffic flow is a kind of many-body system of strongly interacting vehicles. Traffic jams are a typical signature of the complex behavior of traffic flow [6–20]. Traffic jams have been studied by several traffic models: car-following models, cellular automaton (CA) models, gas kinetic models, and hydrodynamic models [1–20]. Recent studies reveal physical

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phenomena such as the nonequilibrium phase transitions and the nonlinear waves [6–10]. It has been shown that the jamming transition is very similar to the conventional phase transitions and critical phenomena even if the traffic flow is a nonequilibrium system [1,11].

Mobility is nowadays one of the most significant ingredients of a modern society. The city traffic networks often exceed the capacity. In urban traffic, the flow is controlled by traffic lights. Recently, Brockfeld et al. have studied optimizing traffic lights in a CA model for city traffic [21]. They have shown that the flow throughout is improved by traffic light control strategies. They have also shown that the derivation of the optimal cycle times in the network can be reduced to a simpler problem of a single street with one traffic light for the synchronized traffic lights. However, one is often forced to question how the capacity is related to the jamming transition. Also, when the traffic lights operate as a bottleneck, how the saturation of traffic flow occurs? Is the saturation related to the bottleneck effect? Is the capacity of the traffic flow enhanced by the chosen traffic light control strategy?

In this paper, we study the traffic flow controlled by traffic lights on a single-lane roadway. We use the optimal velocity model. We clarify the characteristic of traffic flow controlled by traffic lights. We investigate the effect of different traffic control strategies on the traffic flow. We derive the flow-density diagram for three different strategies. We study the dynamical transition to the saturated flow. We show that the transition point depends on the strategies and the cycle time of the traffic lights. We show that the transition and saturation of traffic flow is connected closely to the occurrence of spontaneous jam. We present the theoretical analysis for the dynamical transition of traffic flow.

2. Model

We apply the optimal velocity model to the traffic flow on a single-lane roadway controlled by traffic lights. For later convenience, we summarize the optimal velocity model and its characteristic [1]. The optimal velocity model is described by the following equation of motion of car i:

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = a\left\{V(\Delta x_i) - \frac{\mathrm{d}x_i}{\mathrm{d}t}\right\} , \tag{1}$$

where $V(\Delta x_i)$ is the optimal velocity, $x_i(t)$ is the position of car i at time t, $\Delta x_i(t) \times (= x_{i+1}(t) - x_i(t))$ is the headway of car i at time t, and a is the sensitivity (the inverse of the delay time).

A driver adjusts the car velocity to approach the optimal velocity determined by the observed headway. The sensitivity a allows for the time lag $\tau = 1/a$ that it takes the car velocity to reach the optimal velocity when the traffic is varying. Generally, it is necessary that the optimal velocity function has the following properties: it is a monotonically increasing function and it has an upper bound (maximal velocity). The

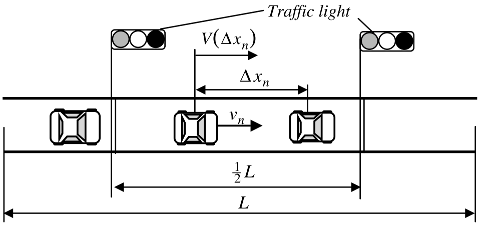

Fig. 1. Schematic illustration of the model. The two traffic lights are set on the single-lane roadway under the periodic boundary condition. The length of the roadway is L and the interval between the two traffic lights is $L/2$.

optimal velocity has been given by

$$V(\Delta x_i) = \frac{V_{\max}}{2}\left[\tanh(\Delta x_i - x_c) + \tanh(x_c)\right] , \tag{2}$$

where $V_{\max}$ is the maximal velocity and x_c is the safety distance.

If the average headway h satisfies the linear instability condition, the traffic jams appear. The transition point of the jamming is given by the following instability condition [1]:

$$a < 2V'(h) , \tag{3}$$

where $V'(h)$ is the derivative of the optimal velocity function. When the headway (density) is less (higher) than the critical value, the traffic flow becomes unstable and the jamming transition occurs. If the jamming transition occurs, the spontaneous jam appears and it propagates backward as the density wave. The kink solution of the density wave is given by

$$\Delta x_i(t) = x_c \pm \sqrt{\frac{5V'(x_c)(2V'(x_c)\tau - 1)}{|V'''(x_c)|}} \times \tanh\left[\sqrt{\frac{5(2V'(x_c)\tau - 1)}{2}}\left(\mathrm{i} - \frac{5V'(x_c)}{6}(2V'(x_c)\tau - 1)t\right)\right] . \tag{4}$$

The headways within and out of the jam are given as follows:

$$\Delta x_{jam} = x_c - \sqrt{\frac{5}{2}\left(\frac{a_c}{a} - 1\right)} \quad \text{and} \quad \Delta x_{free} = x_c + \sqrt{\frac{5}{2}\left(\frac{a_c}{a} - 1\right)} , \tag{5}$$

where $a_c = 1/\tau_c = V_{\max}$ [1].

We consider the traffic flow on the single-lane roadway with traffic lights. Fig. 1 shows the schematic illustration of the model. We set the two traffic lights on the single-lane roadway. The length of the roadway is L and the interval between the two traffic lights is $L/2$. We study the traffic flow under the periodic boundary condition. In synchronized strategy, the traffic lights are chosen to switch simultaneously after a fixed time period $T/2$. The traffic lights are assumed to flip periodically at regular time intervals $T/2$. Time T is called the cycle time. If the traffic light changes from green to red, the car closest to the traffic light changes its velocity according to Eq. (1) with changed optimal velocity. The optimal velocity changes from Eq. (2) to the following:

$$V(\Delta x_{closest}) = \frac{V_{\max}}{2}\left[\tanh(\Delta x_{closest} - x_c) + \tanh(x_c)\right] , \tag{6}$$

where $\Delta x_{closest}$ is the distance between the traffic light and the car closest to the traffic light. When the traffic light changes from red to green, the car closest to the traffic light changes its velocity according to Eq. (1) with changed optimal velocity. The optimal velocity changes from Eq. (6) to (2). Thus, the traffic flow controlled by the traffic lights can be simulated by the optimal velocity model with the extension described above.

In the simple synchronized strategy where all traffic lights change simultaneously from red (green) to green (red), the traffic flow does not depend on the number of traffic lights for the periodic boundary but is affected by the interval between the traffic lights. Therefore, the traffic problem reduces to the simple case of a single roadway with one traffic light in the synchronized strategy.

In the green wave strategy, the traffic light changes with a certain time delay T_{delay} between the traffic light phases of two successive intersections. The delay time T_{delay} is called the offset time. The change of traffic lights propagates backwards like a green wave. In such a case that the delay T_{delay} is given by $nT/2$ (n: positive integer), it is sufficient only to consider the traffic flow on the roadway with two traffic lights. The traffic flow with traffic lights more than two agrees with that with the two traffic lights because of the periodic boundary.

In the random switching strategy, the cycle time T changes randomly and the traffic lights change independently. In this case, the number of traffic lights may affect the traffic flow.

3. Simulation result

We now proceed to perform a computer simulation for the above model. We solve numerically Eq. (1) with optimal velocity functions (2) and (6) by using fourth-order Runge–Kutta method where the time interval is $\Delta t = 1/128$. Mainly, we study the traffic flow on the roadway with two traffic lights. Also, we investigate the effect of the number of traffic lights on the traffic flow. We take $L = 800$ and the interval 400 between two traffic lights.

We study the current–density diagram of traffic flow in the simple synchronized strategy. Fig. 2 shows the plot of the current against density for cycle times $T =$ 100, 200, 300, 400, and 500 and (a) maximal velocity $V_{\max} = 2.0$ and (b) $V_{\max} = 3.0$ where $a = 1.0$ and $x_c = 4.0$. The current (flow or flow rate) increases with increasing density. When the density is higher than the critical density, the current saturates at the constant value. When the density increases furthermore and is higher than the second

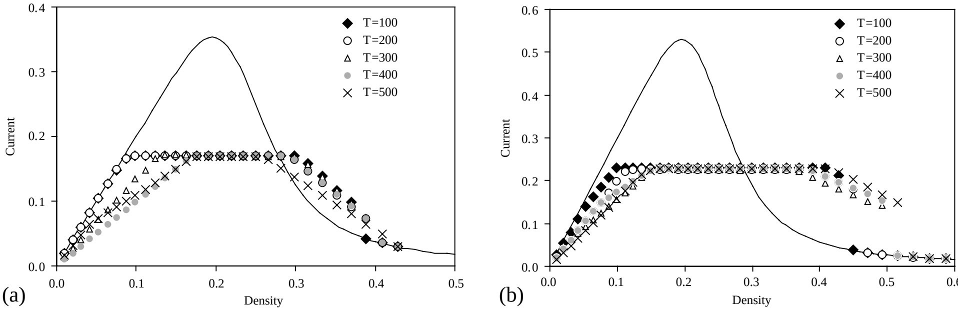

Fig. 2. Plot of the current against density in the synchronized strategy for cycle times $T = 100$, 200, 300, 400, and 500 and (a) maximal velocity $V_{\max} = 2.0$ and (b) $V_{\max} = 3.0$ where $a = 1.0$ and $x_c = 4.0$.

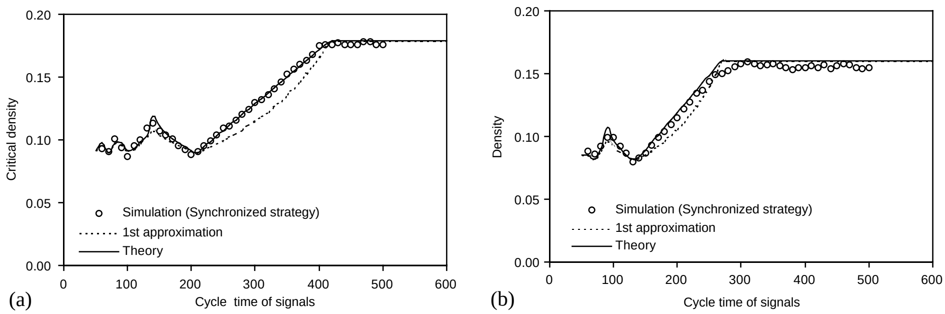

Fig. 3. Plot of the critical density against the cycle time in the synchronized strategy for (a) maximal velocity $V_{\max} = 2.0$ and (b) $V_{\max} = 3.0$. The circles indicate the simulation data. The dotted and solid lines indicate, respectively, the first approximation and theoretical result.

critical density, the current decreases with increasing density. The saturated current does not depend on the cycle time but the critical density changes with the cycle time.

Fig. 3 shows the plot of the critical density against the cycle time in the synchronized strategy for (a) maximal velocity $V_{\max} = 2.0$ and (b) $V_{\max} = 3.0$. The circles indicate the simulation data. The dotted and solid lines indicate, respectively, the first approximation and theoretical result described in Section 4. The critical density changes periodically until $T = 200$ ($T = 130$) for $V_{\max} = 2.0$ ($V_{\max} = 3.0$). For $200 < T < 400$ ($130 < T < 270$) at $V_{\max} = 2.0$ ($V_{\max} = 3.0$), the critical density increases linearly with the cycle time. When the cycle time is higher than $T = 400$ ($T = 270$) for $V_{\max} = 2.0$ ($V_{\max} = 3.0$), the critical density becomes constant.

Fig. 4 shows the plot of the saturated current against the maximal velocity. The circles indicate the simulation result obtained in the synchronized strategy. The dotted line indicates half of the maximal current. The solid line indicates the theoretical result obtained in Section 4. The saturated current increases linearly with the maximal velocity until $V_{\max} = 2.0$. When the maximal velocity is higher than $V_{\max} = 2.0$, the saturated current deviates from the linear line.

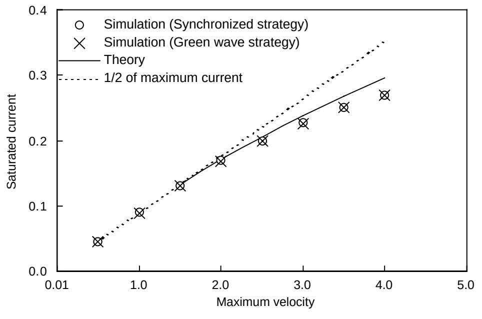

Fig. 4. Plot of the saturated current against the maximal velocity. The circles indicate the simulation result obtained in the synchronized strategy. The dotted line indicates the half of the maximal current. The solid line indicates the theoretical result.

We study the spatio-temporal pattern of the traffic flow induced by the traffic lights. Fig. 5 shows the time–space evolution of vehicles for cycle time $T = 100$ in the synchronized strategy where $V_{\max} = 2.0$, $x_c = 4.0$, and $a = 1$. The spatio-temporal pattern (a) is obtained for $\rho = 0.05$ and $0 \leqslant t \leqslant 1000$. At the early stage, vehicles stop instantly by the red traffic light and then restart. In time, all vehicles move during the green of traffic lights without stopping by the red. The spatio-temporal pattern (b) is obtained after a sufficiently large time. All vehicles move keeping a constant headway. Thus, for low density $\rho = 0.05$ and cycle time $T = 100$, the traffic lights do not affect the current. Therefore, the current agrees with that with no traffic lights until it saturates. The spatio-temporal pattern (c) is obtained for $\rho = 0.2$. The trajectories of vehicles are plotted every four vehicles. The vehicles stop at the red light. As soon as the traffic light changes to green, the vehicle closest to the traffic light restarts. The density wave propagates backwards from the traffic light. The current saturates at $\rho = 0.2$. When the current saturates, the density wave is always formed. The structure and propagating speed of the density wave agree with those of the traffic flow at a high density with no traffic lights. The headways within and out of the density wave are consistent with those of spontaneous jam in the optimal velocity model. The spatio-temporal pattern (d) is obtained for $\rho = 0.35$. The trajectories of vehicles are plotted every four vehicles. The traffic jam induced by one traffic light goes beyond the back traffic light. As a result, the current becomes less than the saturated current at high density $\rho = 0.35$.

Fig. 6 shows the time–space evolution of vehicles for cycle time $T = 400$ in the synchronized strategy where $V_{\max} = 2.0$, $x_c = 4.0$, and $a = 1$. The spatio-temporal pattern (a) is obtained for $\rho = 0.05$ and $0 \leqslant t \leqslant 1000$. At the early stage, vehicles stop instantly by the red of traffic light and then restart. In time, all vehicles move only during green traffic lights and stop always at the red. The spatio-temporal pattern (b) is obtained after a sufficiently large time. All vehicles stop at the red and go at the green. Thus, for low density $\rho = 0.05$ and cycle time $T = 400$, the traffic lights affect the current. The current is less than that with no traffic lights. The spatio-temporal pattern (c) is obtained for $\rho = 0.2$. The trajectories of vehicles are plotted every four vehicles. The vehicles stop at the red light. As soon as the traffic light changes to green, the vehicle

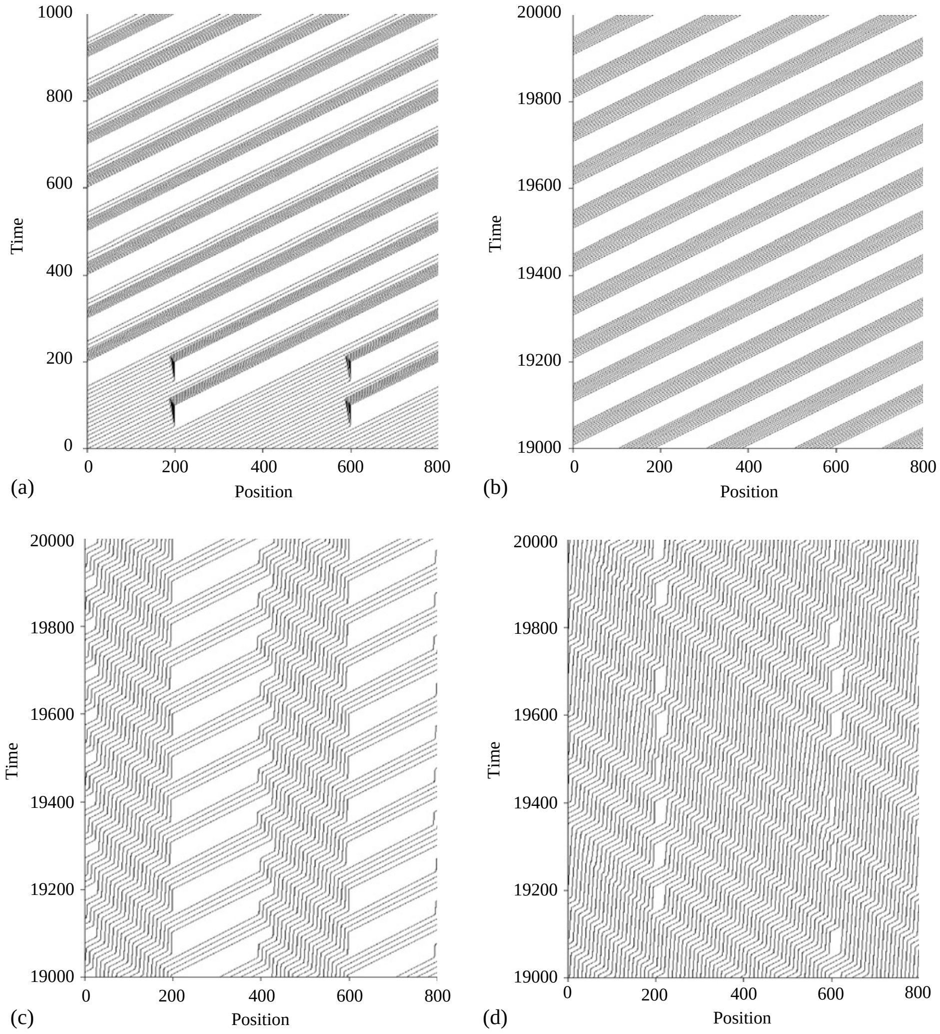

Fig. 5. Time–space evolution of vehicles for cycle time $T = 100$ in the synchronized strategy where $V_{\max} = 2.0$, $x_c = 4.0$, and $a = 1$. (a) The spatio-temporal pattern is obtained at $\rho = 0.05$ and $0 \leqslant t \leqslant 1000$. (b) The spatio-temporal pattern is obtained after a sufficiently large time at low density $\rho = 0.05$. (c) The spatio-temporal pattern is obtained for $\rho = 0.2$. (d) The spatio-temporal pattern is obtained for $\rho = 0.35$.

closest to the traffic light restarts. The density wave propagates backward from the traffic light. The current saturates at $\rho = 0.2$. When the current saturates, the density wave is always formed. The structure and propagating speed of the density wave agree with those of the traffic flow at a high density with no traffic lights. The headways within and out of the density wave are consistent with those of spontaneous jam in the optimal velocity model. The spatio-temporal pattern (d) is obtained for $\rho = 0.35$.

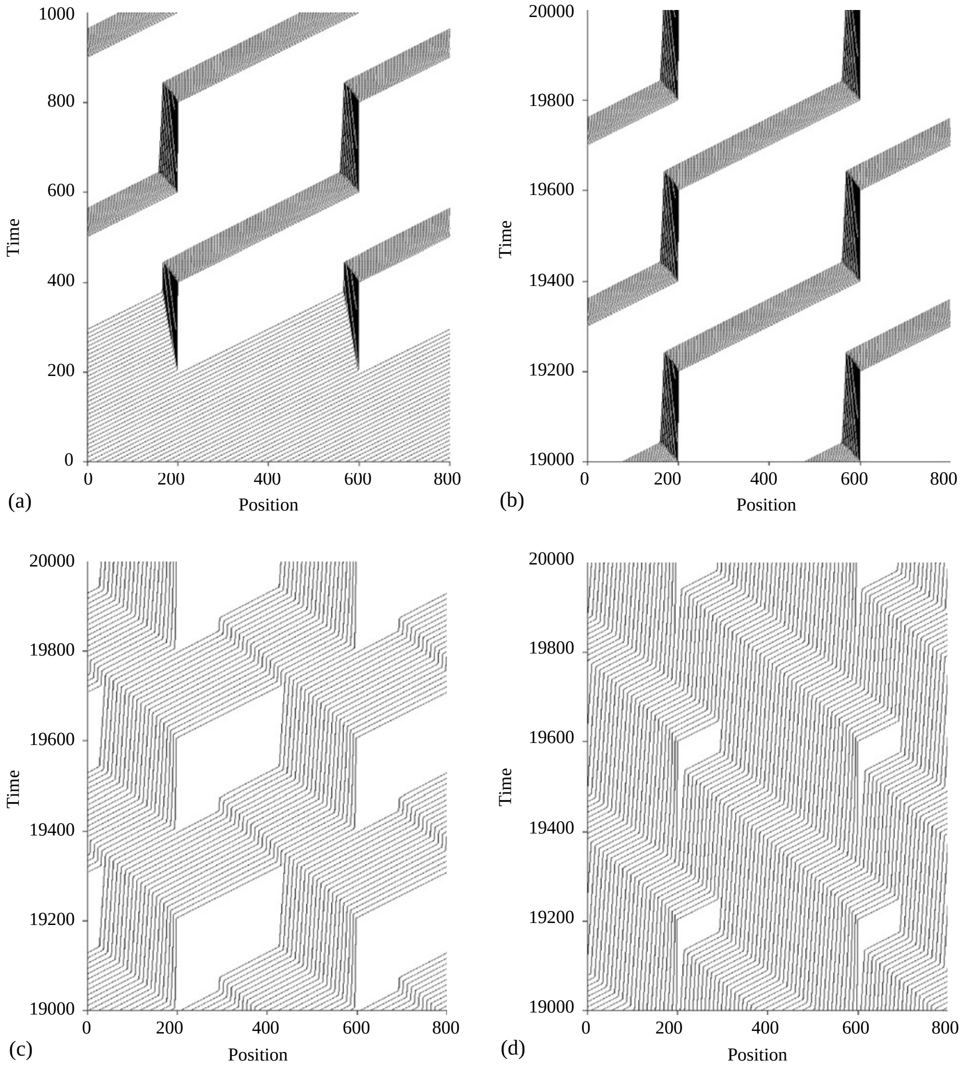

Fig. 6. Time–space evolution of vehicles for cycle time $T = 400$ in the synchronized strategy where $V_{\max} = 2.0$, $x_c = 4.0$, and $a = 1$. (a) The spatio-temporal pattern is obtained at $\rho = 0.05$ and $0 \leqslant t \leqslant 1000$. (b) The spatio-temporal pattern is obtained after a sufficiently large time at low density $\rho = 0.05$. (c) The spatio-temporal pattern is obtained for $\rho = 0.2$. (d) The spatio-temporal pattern is obtained for $\rho = 0.35$.

The trajectories of vehicles are plotted every four vehicles. The traffic jam induced by one traffic light goes beyond the back traffic light. As the result, the current becomes less than the saturated current at high density $\rho = 0.35$.

We study the traffic flow controlled by the green wave strategy. We restrict ourselves to the case of delay time $T_{delay} = T/2$ between two successive traffic lights. Fig. 7 shows

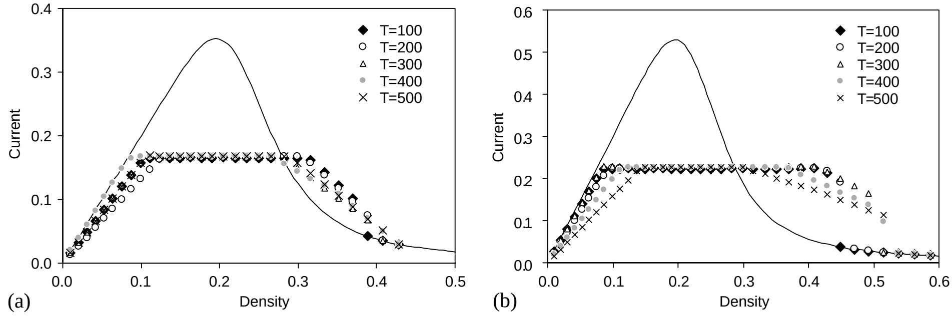

Fig. 7. Current–density diagrams in the green wave strategy for (a) $V_{\max}=2.0$ and (b) $V_{\max}=3.0$ at various cycle times $T = 100,\ 200,\ 300,\ 400$, and 500 where $x_c = 4$ and $a = 1$.

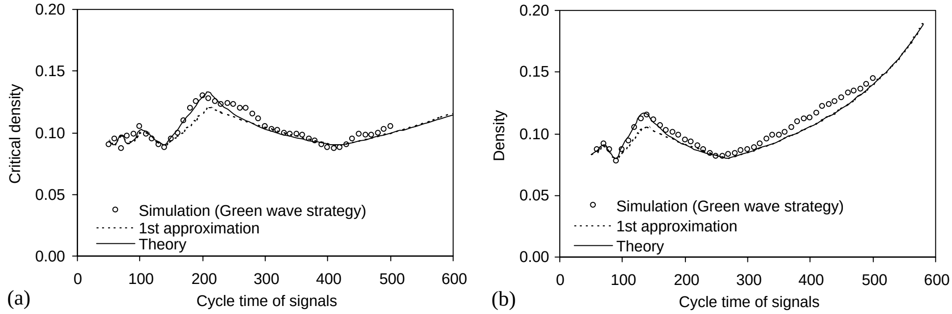

Fig. 8. Plot of the critical density against the cycle time in the green wave strategy for (a) maximal velocity $V_{\max}=2.0$ and (b) $V_{\max}=3.0$. The circles indicate the simulation data. The dotted and solid lines indicate, respectively, the first approximation and theoretical result.

the current–density diagram for (a) $V_{\max}=2.0$ and (b) $V_{\max}=3.0$ at various cycle times $T=100,\ 200,\ 300,\ 400$, and 500 where $x_c=4$ and $a=1$. The current increases linearly with the density. The proportional coefficient depends highly on the cycle time. When the density is higher than the critical density, the current saturates to the constant value. Thus, the dynamical transition to the saturation occurs. The transition is similar to that of synchronized strategy. The saturated current is consistent with that of synchronized strategy. The saturated current does not depend on the cycle time but the critical density changes with cycle time. If the density is higher than the second critical density, the current decreases with increasing density. The solid line indicates the theoretical curve for the case that no jams occur.

Fig. 8 shows the plots of the first critical density against the cycle time for (a) $V_{\max}=2.0$ and (b) $V_{\max}=3.0$. The circles indicate the simulation data. The dotted and solid curves represent, respectively, the first approximation and theoretical results obtained in Section 4. The critical density exhibits the complex dependency on the cycle time. Fig. 8 is compared with Fig. 3 in the synchronized strategy. The critical

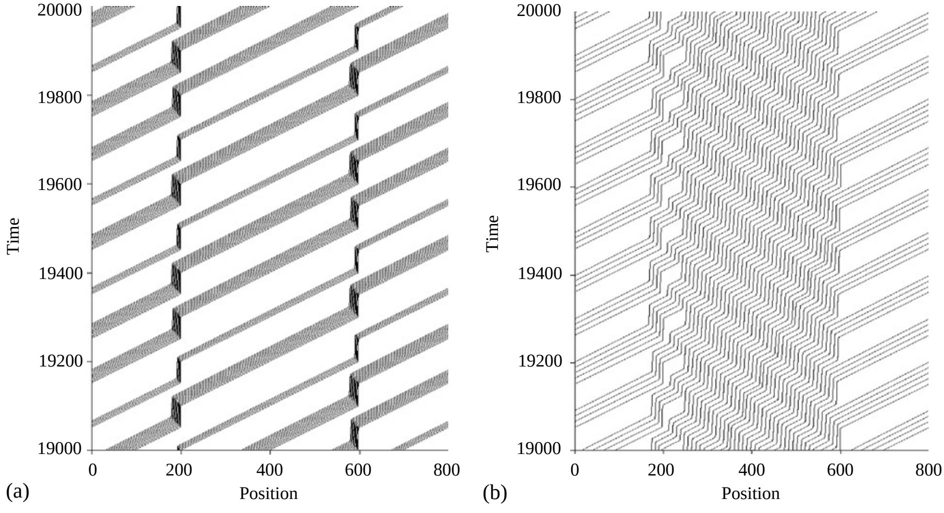

Fig. 9. Space–time evolution of vehicles for cycle time $T = 100$ at (a) $\rho = 0.05$ and (b) $\rho = 0.2$ in the green wave strategy. The space–time patterns (a) and (b) correspond, respectively, to the traffic states before and after the first transition.

density in Fig. 8(a) has the peak at $T = 200$, then decreases with increasing cycle time, and increases again when the cycle time is higher than $T = 400$.

Fig. 9 shows the space–time evolution of vehicles for cycle time $T = 100$ at (a) $\rho = 0.05$ and (b) $\rho = 0.2$ in the green wave strategy. The space–time patterns (a) and (b) correspond, respectively, to the traffic states before and after the first transition. The trajectory of each vehicle is indicated by the solid line in Fig. 9(a). In Fig. 9(b), the trajectories are indicated every four vehicles. At (a) density $\rho = 0.05$, all vehicles stop instantly at each traffic light and restart when the traffic lights change to green. At (b) density $\rho = 0.2$, the density wave appears at the traffic lights and it propagates backward. Fig. 10 shows the space–time evolution of vehicles for cycle time $T = 400$ at (a) $\rho = 0.05$ and (b) $\rho = 0.2$ in the green wave strategy. At (a) low density $\rho = 0.05$, all vehicles pass successfully over the traffic lights without stopping. At (b) $\rho = 0.2$ where the current saturates, the density wave appears at the traffic lights and it propagates backward.

We study the traffic flow controlled by the random switching strategy. We change the cycle time randomly. The cycle time is generated by the use of the uniform random variable from 0 to $2T$. Fig. 11 shows the current–density diagram for various values of mean cycle time $T = 100$, 200, 300, 400, and 500 where $V_{max} = 2.0$, $x_c = 4.0$, and $a = 1$. For low densities, the current increases as the density increases. When the density is higher than the critical density, the current saturates at a constant value. The saturated current does not depend on the mean cycle time but the increased rate of current against density depends on the mean cycle time. Fig. 12 shows the space–time evolution of vehicles for mean cycle time $T = 100$ at (a) $\rho = 0.05$ and (b) $\rho = 0.2$ in the random switching strategy. The space–time patterns (a) and (b) correspond, respectively, to

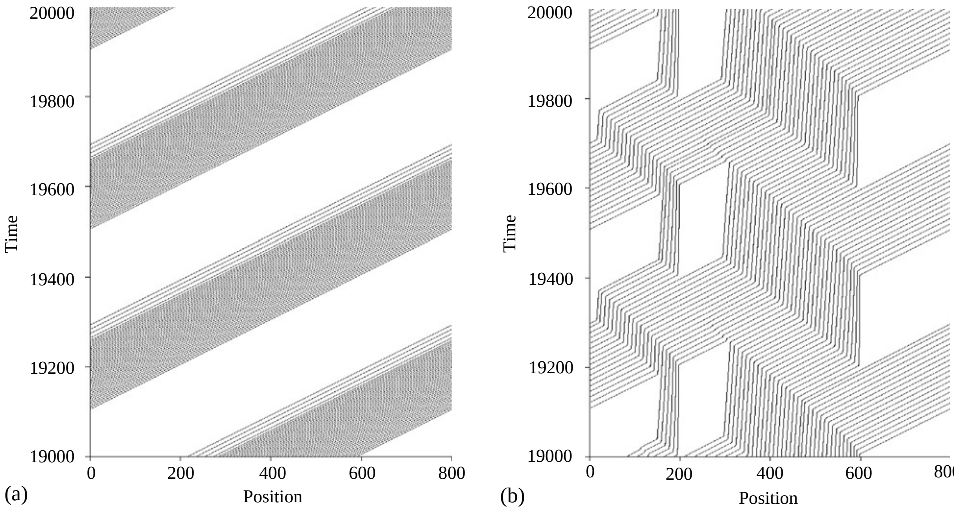

Fig. 10. Space–time evolution of vehicles for cycle time $T=400$ at (a) $\rho=0.05$ and (b) $\rho=0.2$ in the green wave strategy. The space–time patterns (a) and (b) correspond, respectively, to the traffic states before and after the first transition.

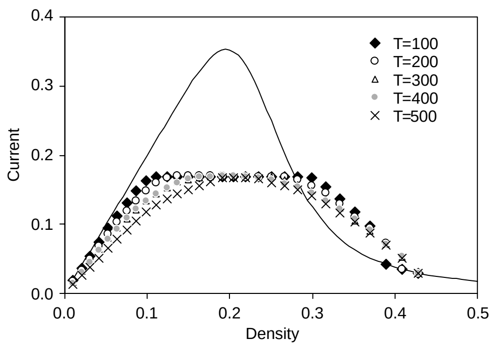

Fig. 11. Current–density diagram in the random switching strategy for various values of mean cycle time $T=100$, 200, 300, 400, and 500 where $V_{\max}=2.0$, $x_c=4.0$, and $a=1$.

the traffic states before and after the first transition. The trajectory of each vehicle is indicated by the solid line in Fig. 12(a). In Fig. 12(b), the trajectories are indicated every four vehicles. At (a) density $\rho=0.05$, vehicles stop sometimes, instantly at each traffic light and restart when the traffic lights change to green. However, vehicles do not stop always at the traffic lights but pass sometime over the green traffic lights. At (b) density $\rho=0.2$, the density wave appears at the traffic lights and it propagates backwards.

We study the effect of the number of traffic lights on the traffic flow. The interval between the successive traffic lights is kept constant. Fig. 13 shows the current–density

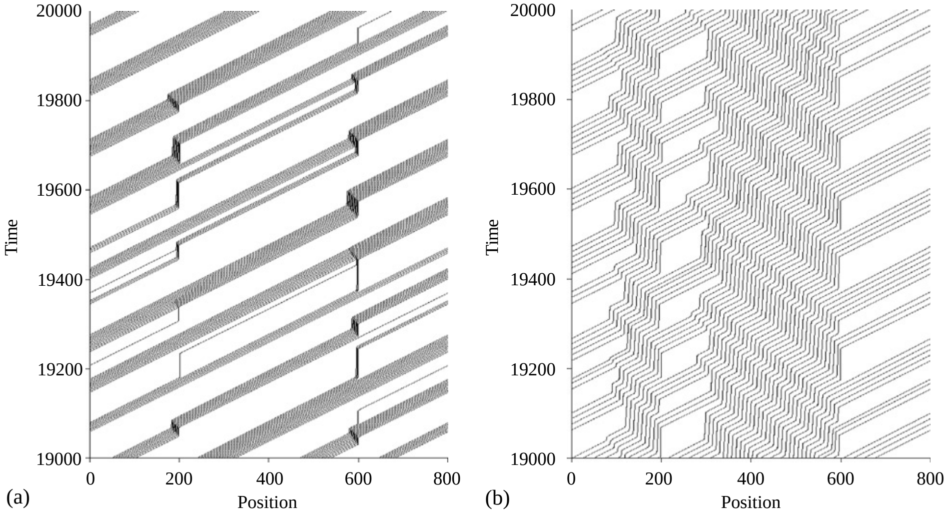

Fig. 12. Space–time evolution of vehicles in the random switching strategy for mean cycle time $T = 100$ at (a) $\rho = 0.05$ and (b) $\rho = 0.2$.

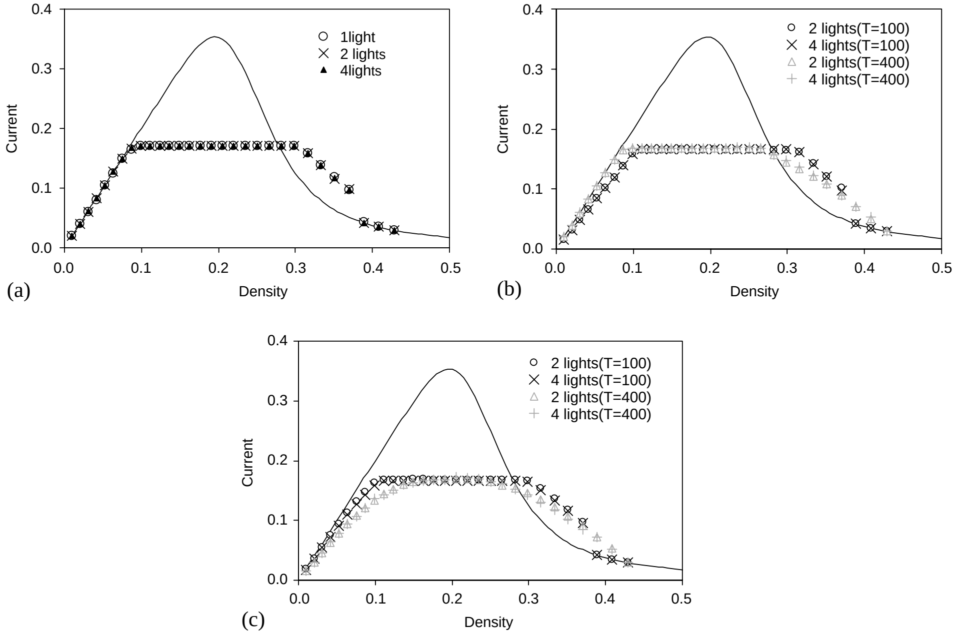

Fig. 13. Current–density diagrams for the traffic flow with different number of traffic lights in the three different strategies. (a) The synchronized strategy. (b) The green wave strategy. (c) The random switching strategy.

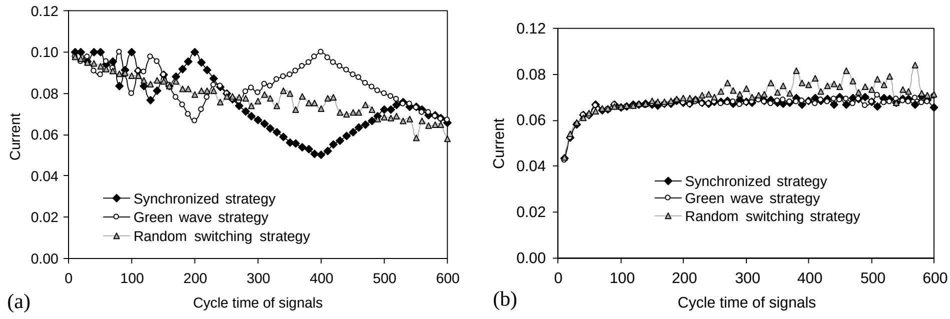

Fig. 14. (a) Plots of the current against the cycle time at $\rho = 0.05$ before the current saturates. Full diamonds, open circles, and full triangles indicate, respectively, the currents obtained for the synchronized, green wave, and random switching strategies. (b) Plots of the current against the cycle time at $\rho = 0.2$ after the current saturates.

diagrams for the traffic flow with different number of traffic lights in the three different strategies. Fig. 13(a) shows the current–density diagram for cycle time $T = 100$ in the synchronized strategy. The circles, crosses, and triangles indicate, respectively, the currents obtained from one traffic light, two traffic lights, and four traffic lights. All data collapse on a single curve. The traffic flow in the synchronized strategy does not depend on the number of traffic lights. The traffic flow problem can be reduced to the simple one with one traffic light in the synchronized strategy. Fig. 13(b) shows the current–density diagram for cycle times $T = 100$ and 400 in the green wave strategy. The circles and crosses indicate, respectively, the currents obtained from two and four traffic lights for $T = 100$. Data collapse on a single curve. The triangles and Greek crosses indicate, respectively, the currents obtained from two and four traffic lights for $T = 400$. Data collapse on a single curve. The traffic flow in the green wave strategy does not change for $2, 4, 6, \ldots$ traffic lights. The traffic flow problem can be reduced to the simple one with two traffic lights in the green wave strategy. Fig. 13(c) shows the current–density diagram for mean cycle times $T = 100$ and 400 in the random switching strategy. The circles and crosses indicate, respectively, the currents obtained from two and four traffic lights for $T = 100$. Data collapse almost on a single curve. The triangles and Greek crosses indicate, respectively, the currents obtained from two and four traffic lights for $T = 400$. Data collapse on a single curve. The traffic flow in the random switching strategy changes little for $2, 4, 6, \ldots$ traffic lights. The traffic flow problem can be reduced to the simple one with two traffic lights in the random switching strategy.

We study the dependency of the current on the cycle time in three different strategies. Fig. 14(a) shows the plots of the current against the cycle time at $\rho = 0.05$ before the current saturates. Full diamonds, open circles and full triangles indicate, respectively, the currents obtained for the synchronized, green wave, and random switching strategies. The currents in the synchronized and green wave strategies change periodically and alternately with increasing cycle time. At $T = 200$, the current in the synchronized strategy is higher than that in the green wave strategy. However, at $T = 400$, the current in the green wave strategy becomes higher than that in the synchronized strategy. The current in the random switching strategy agrees nearly with the mean value of the currents in the synchronized and green wave strategies. Thus, one should select the different strategy of traffic lights by changing the cycle time.

Fig. 14(b) shows the plots of the current against the cycle time at $\rho = 0.2$ after the current saturates. Full diamonds, open circles, and full triangles indicate, respectively, the currents obtained in the synchronized, green wave, and random switching strategies. Until $T = 200$, the currents agree each other. For $T > 200$, the current in the random switching strategy is a little bit higher than those in the synchronized and green wave strategies. Thus, the current depends little on the strategy at high density after the current saturates.

4. Theoretical analysis

We present the theoretical analysis of the dynamical transition. We compare the theoretical result with the simulation result. If the current saturates to the constant value, the density wave appears and propagates backward. The density wave has the same structure as the spontaneous jam [1]. The density and velocity out of the spontaneous jam are given by Eq. (5) and Eq. (2) at $\Delta x = \Delta x_{free}$. The vehicles flow when the traffic light is green. Therefore, one obtains the saturated current

$$q_{sat} = \frac{V(\Delta x_{free})}{\Delta x_{free}} \frac{G}{T} , \tag{7}$$

where G is the time period during the green and $\Delta x_{free} = x_c + \sqrt{\frac{5}{2}((a_c/a) - 1)}$.

The saturated current (7) is indicated by the solid line in Fig. 4. The simulation result agrees with the theoretical result (7).

At a low density, the dynamical transition occurs when the current equals Eq. (7). The current depends highly on the cycle of the traffic lights at the low density. Before the current saturates, the traffic current is given by

$$q = \rho V_{\max} \frac{G}{T} A . \tag{8}$$

Here, A represents the fraction of vehicles passing over the traffic lights without stopping. The vehicles move with maximal velocity except for stopping at the red. Also, the optimal velocity at $\Delta x = \Delta x_{free}$ equals nearly the maximal velocity. Therefore, the critical density is obtained from Eqs. (7) and (8):

$$\rho_c = \frac{1}{A \Delta x_{free}} . \tag{9}$$

We derive the fraction A. When the first vehicle passing over the green light is blocked by the red of the next traffic light, the stopping time of the vehicle is given by

$$R_{stop} = \left| \frac{L_{sig}}{V_{\max}} - \frac{T}{2} N \right| , \tag{10}$$

where L_{sig} is the interval between the traffic lights, $N = S_{int}$ for an even number of S_{int}, $N = S_{int} + 1$ for an odd number of $S_{int}, S_{int} = int[S]$, and $S = (L_{sig}/V_{\max})/(T/2)$ in the synchronized strategy. In the green wave strategy, $N = S_{int} + 1$ for an even number of S_{int}, $N = S_{int}$ for an odd number of S_{int}. S_{int} represents the changed number of the traffic light during the moving time of the vehicle between the traffic light interval L_{sig}.

When $S < 1$, the fraction A is given by

$$A = 1 - \frac{R_{stop}}{ST} . \tag{11}$$

When $S \geqslant 1$, the fraction A is given by

$$A = 1 - \frac{R_{stop}}{S_{int}T} \quad \text{in the synchronized strategy} . \tag{12}$$

In the green wave strategy, when $S \geqslant 1$, the fraction A is given by

$$A = 1 - \frac{R_{stop}}{(S_{int} - 0.5)T} \quad \text{for even } S_{int} , \tag{13}$$

$$A = 1 - \frac{R_{stop}}{(S_{int} + 0.5)T} \quad \text{for odd } S_{int} . \tag{14}$$

Eq. (11) is considered as a first approximation. The first approximation (11) is indicated by the dotted line in Fig. 3 for the synchronized strategy and in Fig. 8 for the green wave strategy. The theoretical result (11)–(14) is indicated by the solid line in Fig. 3 for the synchronized strategy and in Fig. 8 for the green wave strategy. The theoretical result is consistent with the simulation result.

5. Summary

We have studied the traffic flow controlled by the traffic lights on a single-lane roadway by using the optimal velocity model. We have derived the current–density diagrams in the three different strategies of traffic light control. We have classified the characteristic of the traffic flow in the three different strategies. We have shown that the dynamical transition to the current saturation occurs when the density is higher than the critical density. We have found that the value of the saturated current does not depend on the cycle time and the strategies of traffic light control. We have presented the theoretical analysis to the dynamical transition. We have derived the theoretical values of the saturated current and of the critical point. We have shown that the theoretical result agrees with the simulation result.

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