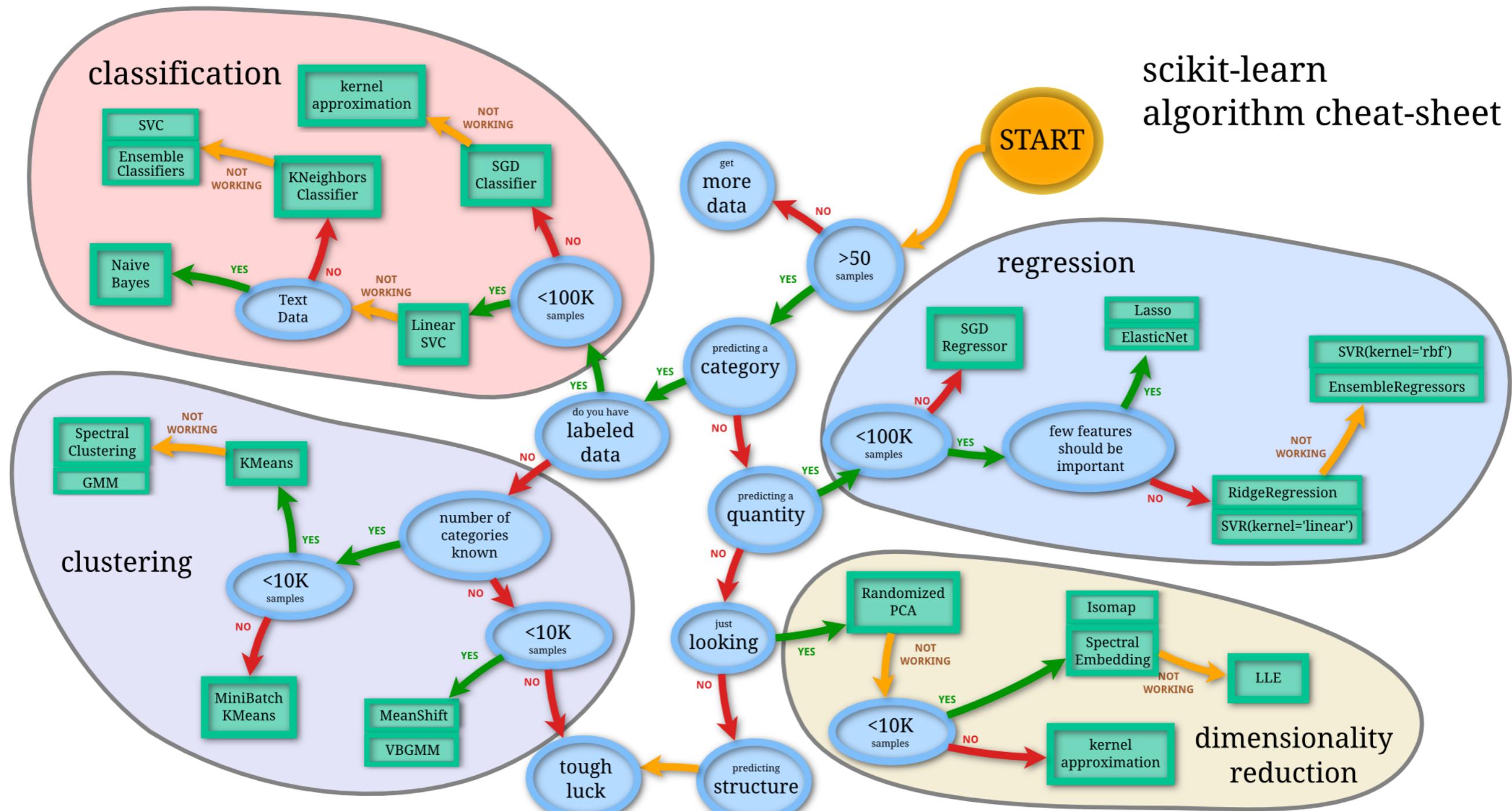




LINEAR ALGEBRA

Sourav Sen Gupta

scikit-learn algorithm cheat-sheet



PLAN OF ACTION

Part I — Geometry of Multivariate Data

1. Vector space modelling of multivariate data
2. Understanding Matrices as linear operators
3. Subspaces and Singular Value Decomposition

Part II — Applications in Multivariate Analysis

1. SVD and Principal Component Analysis
2. Eigenvalues and Eigenvectors in PageRank
3. Factor Analysis and SVD in Recommenders

MULTIVARIATE DATA

MULTIVARIATE DATA

p variables/features

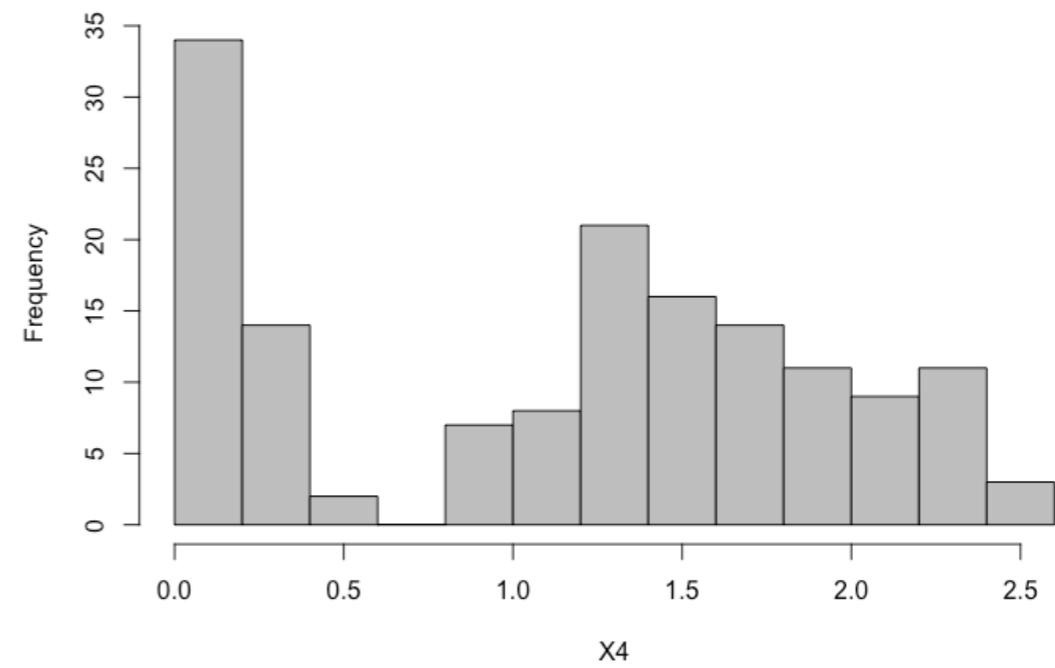
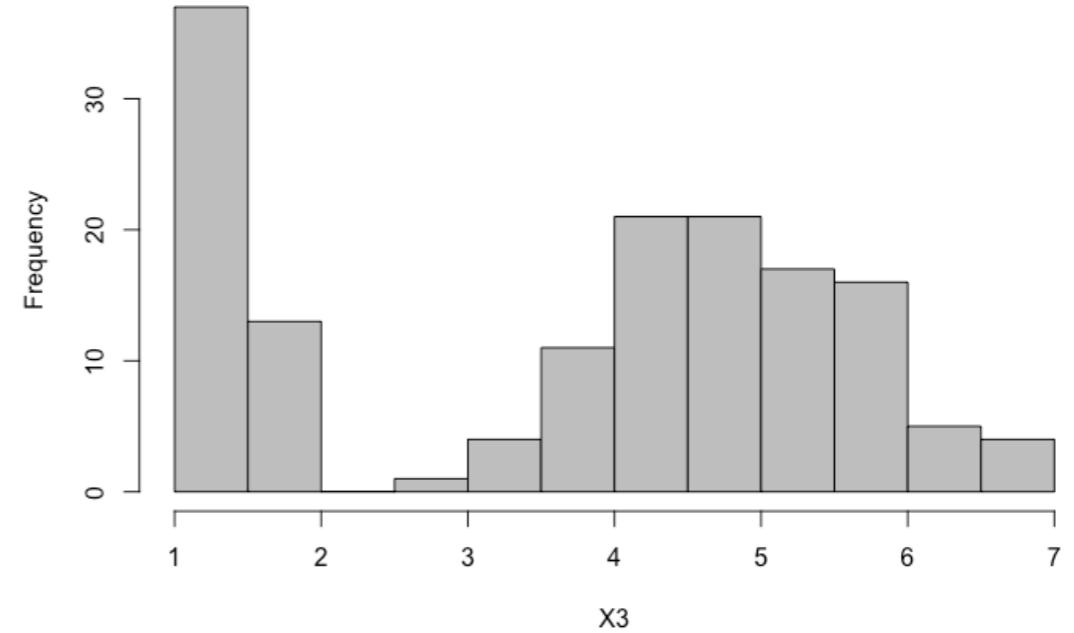
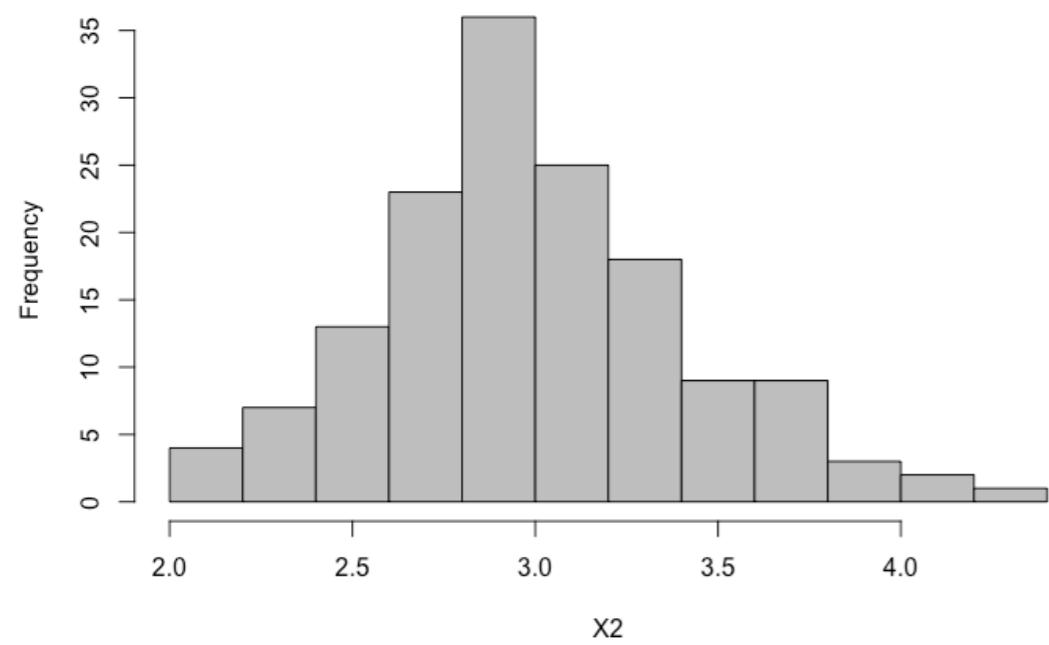
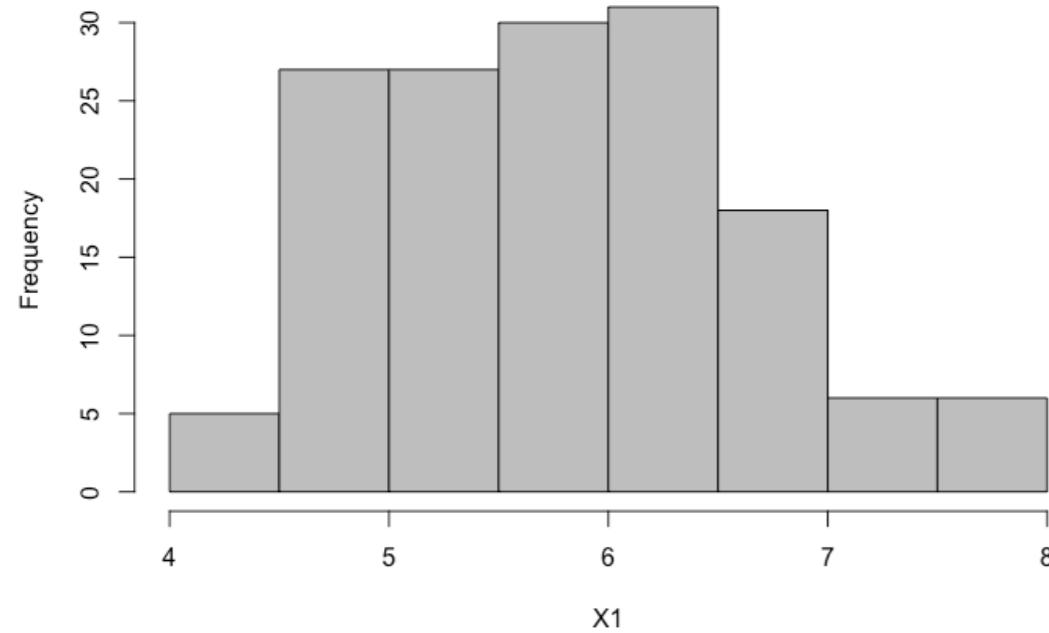
n samples/observations

	X1	X2	X3	X4
S1	5.1	3.5	1.4	0.2
S2	4.9	3	1.4	0.2
S3	4.7	3.2	1.3	0.2
S4	4.6	3.1	1.5	0.2
S5	5	3.6	1.4	0.2

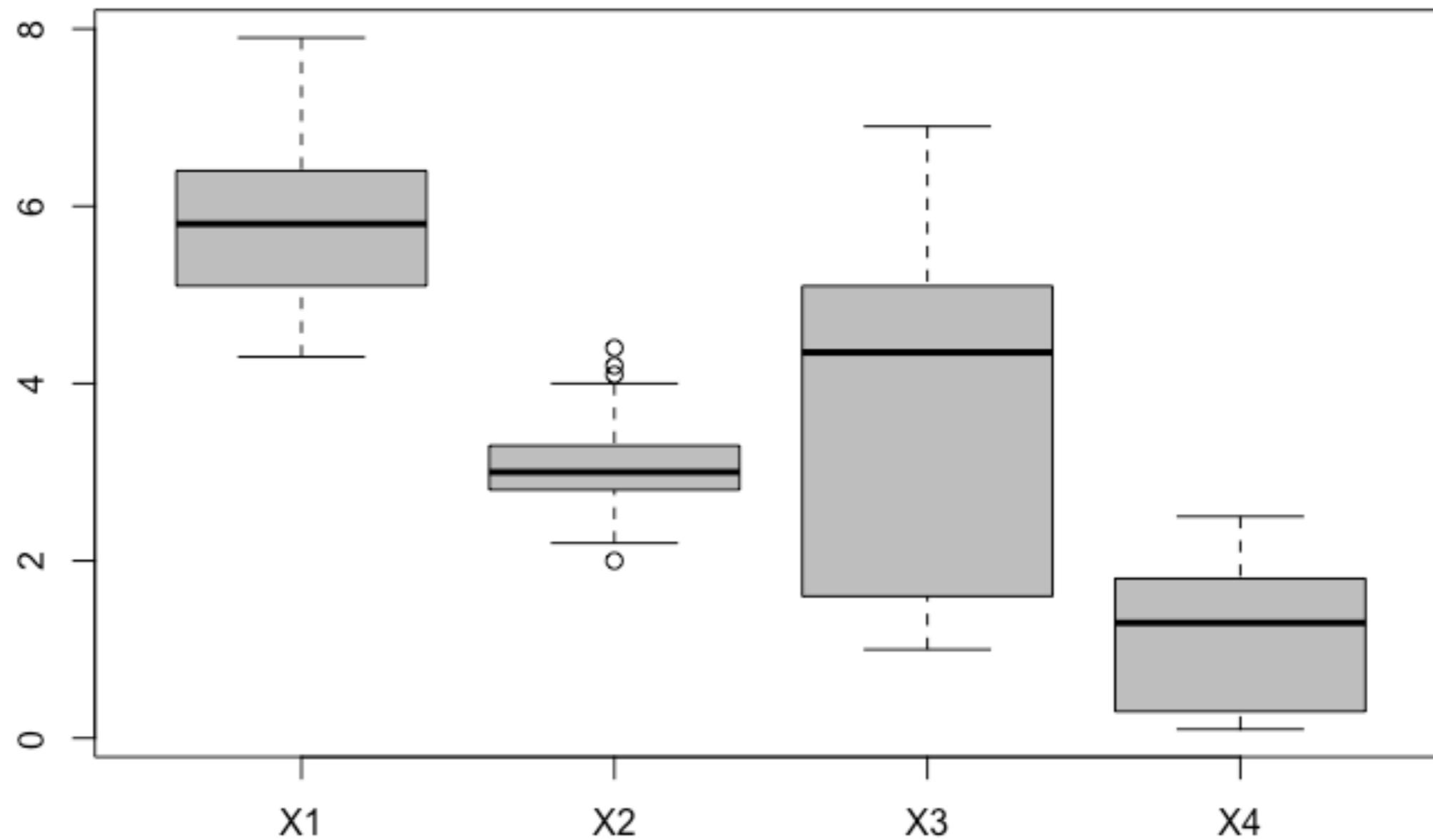
⋮ ⋮ ⋮ ⋮ ⋮

S146	6.7	3	5.2	2.3
S147	6.3	2.5	5	1.9
S148	6.5	3	5.2	2
S149	6.2	3.4	5.4	2.3
S150	5.9	3	5.1	1.8

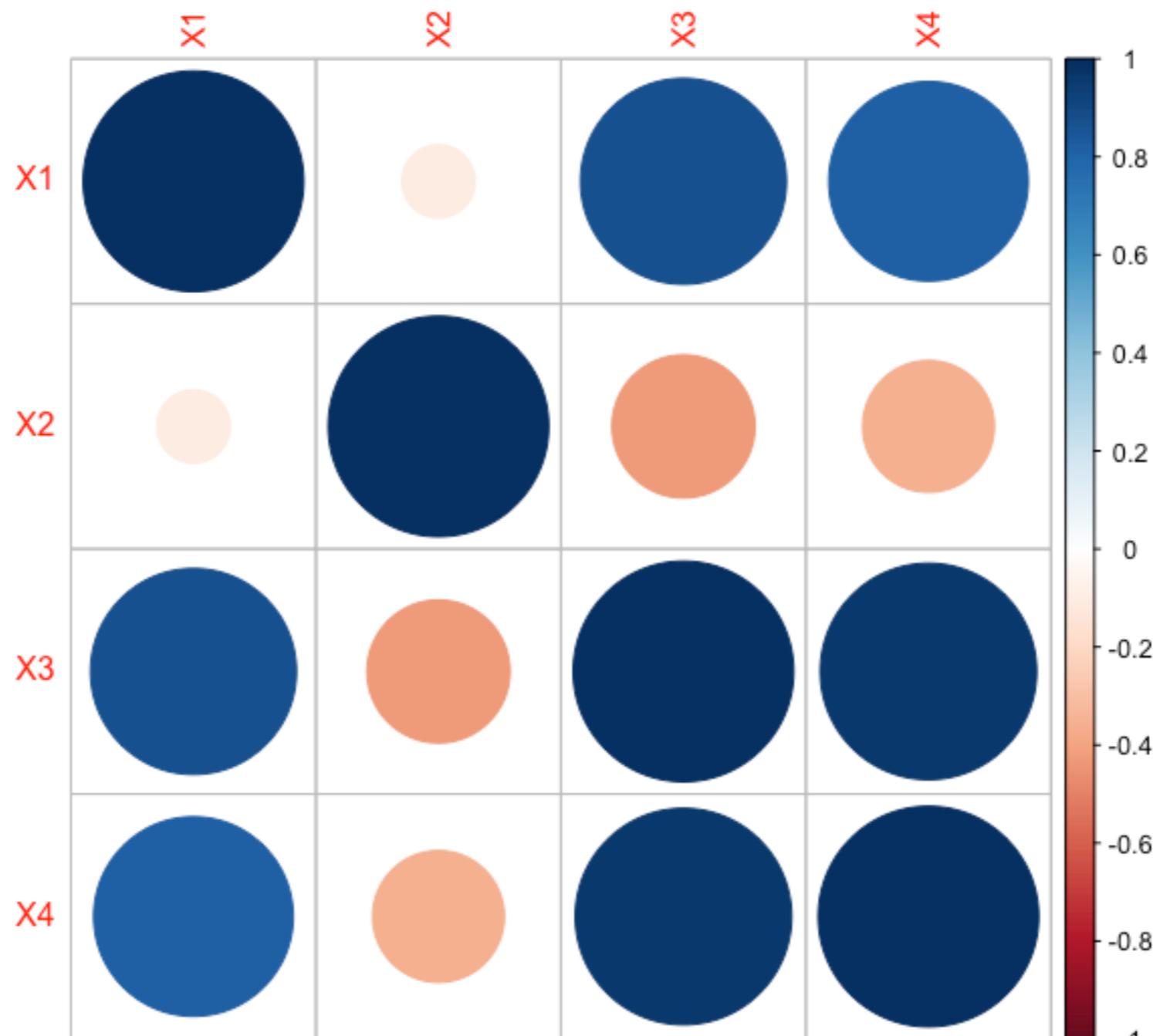
MULTIVARIATE DATA — HISTOGRAMS



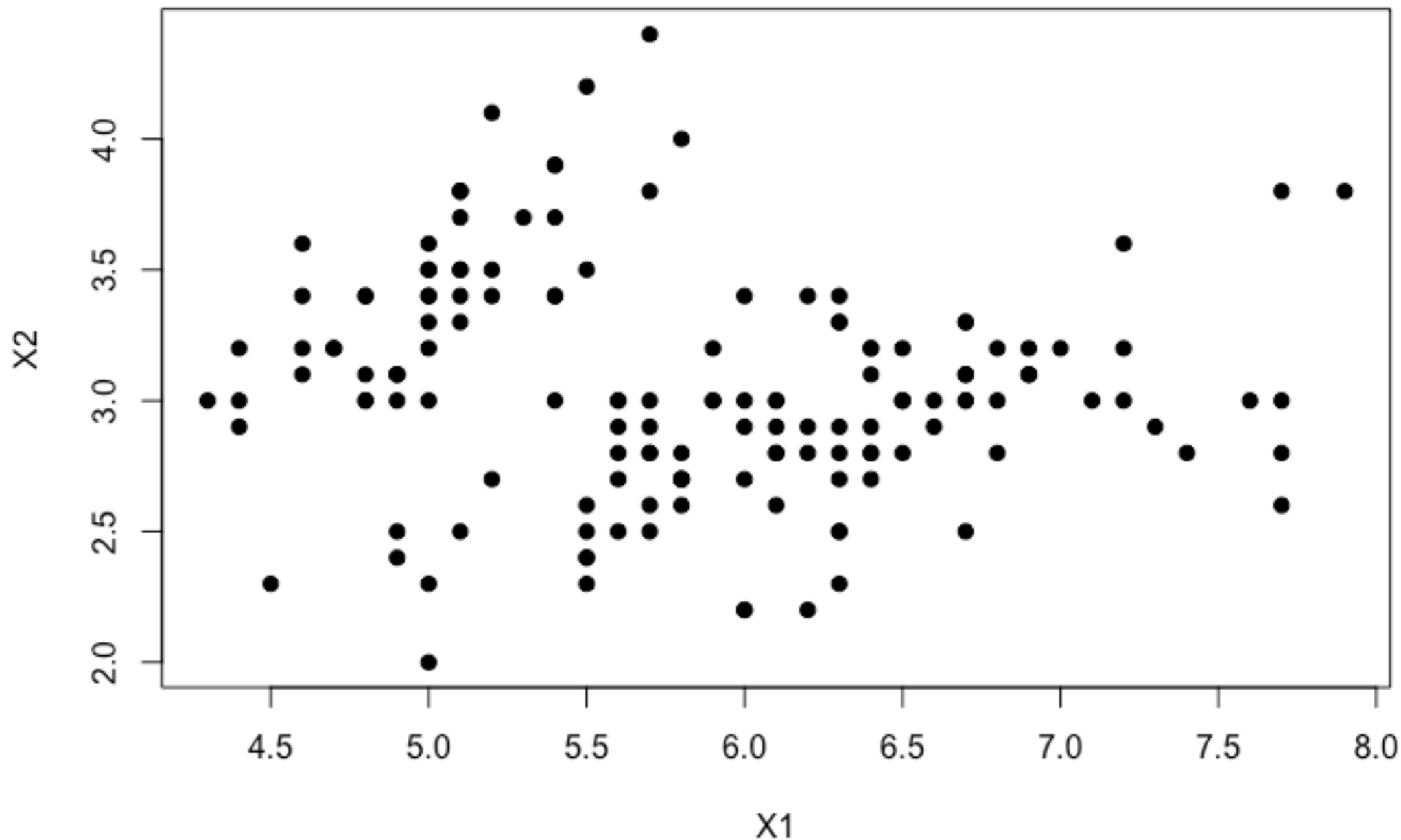
MULTIVARIATE DATA — BOXPLOTS



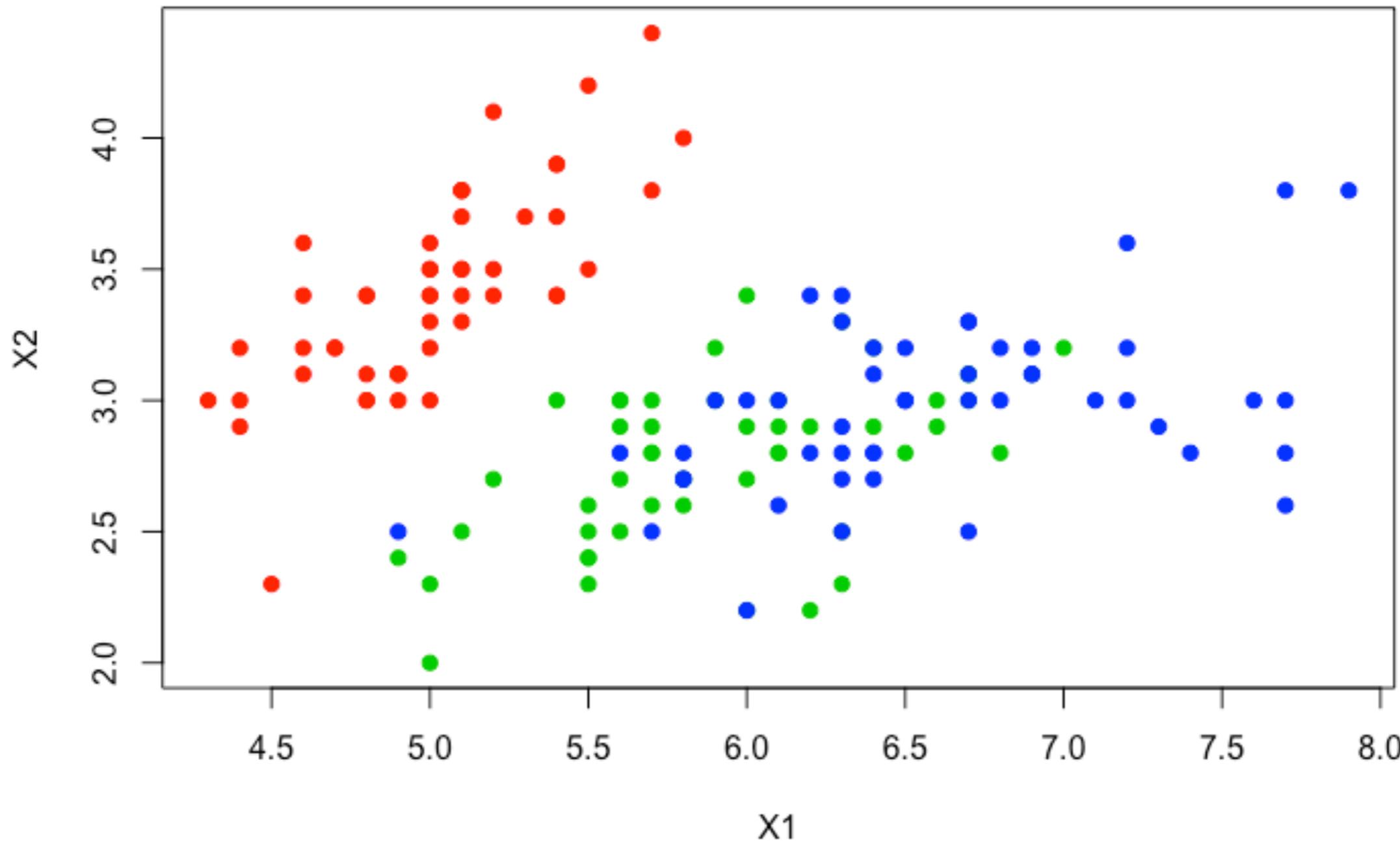
MULTIVARIATE DATA — CORR PLOT



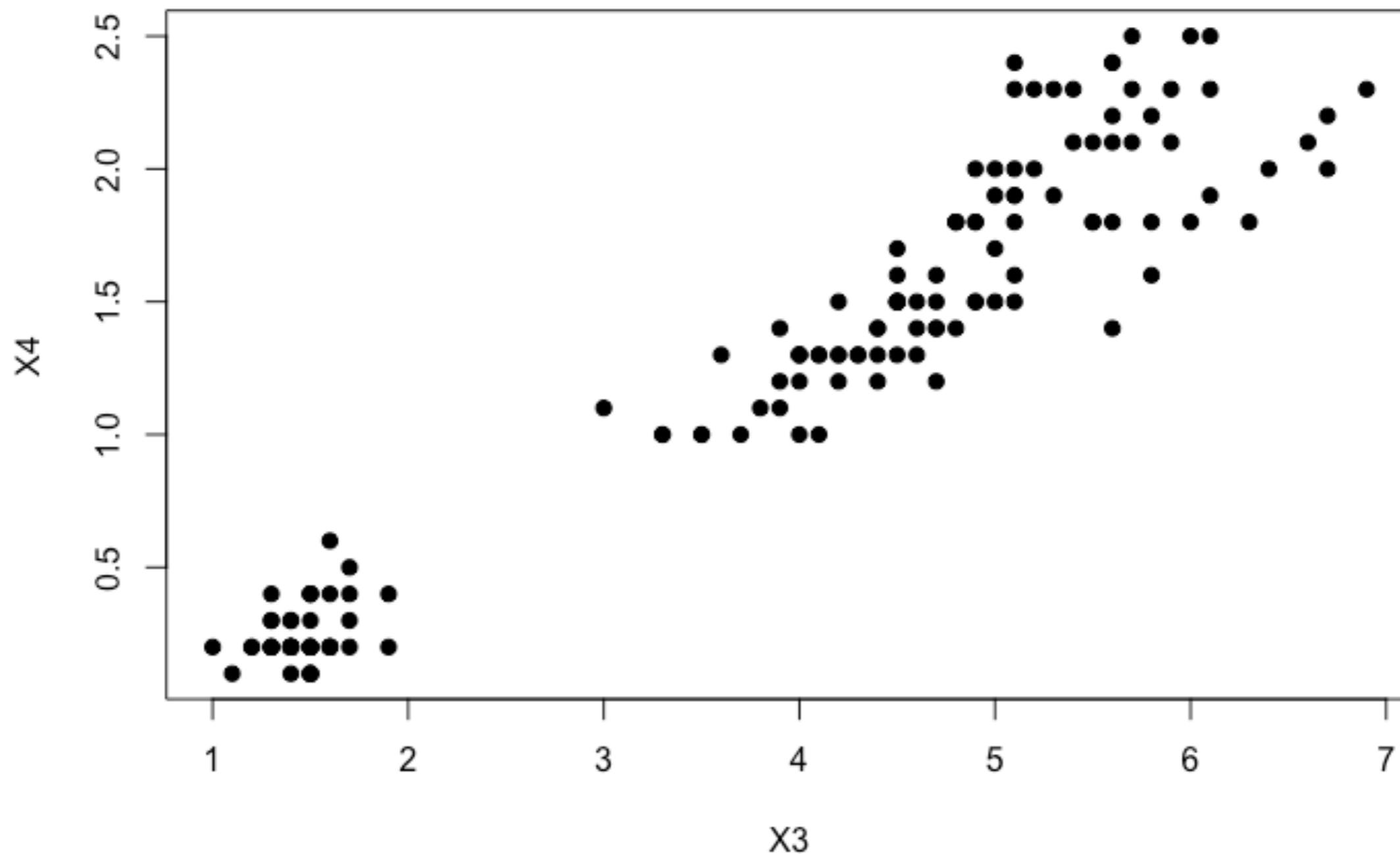
MULTIVARIATE DATA — SCATTERPLOT



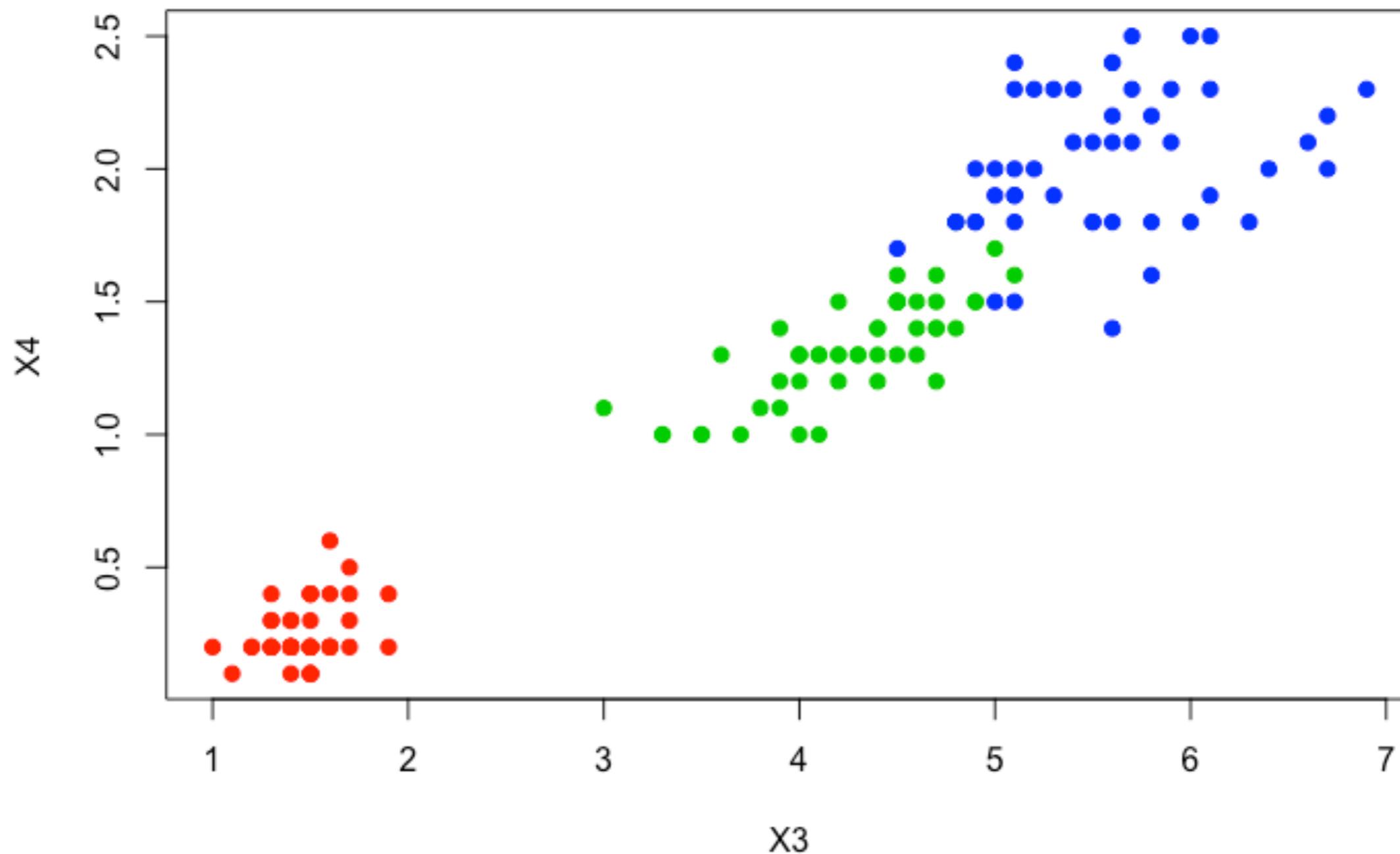
MULTIVARIATE DATA — SCATTERPLOT



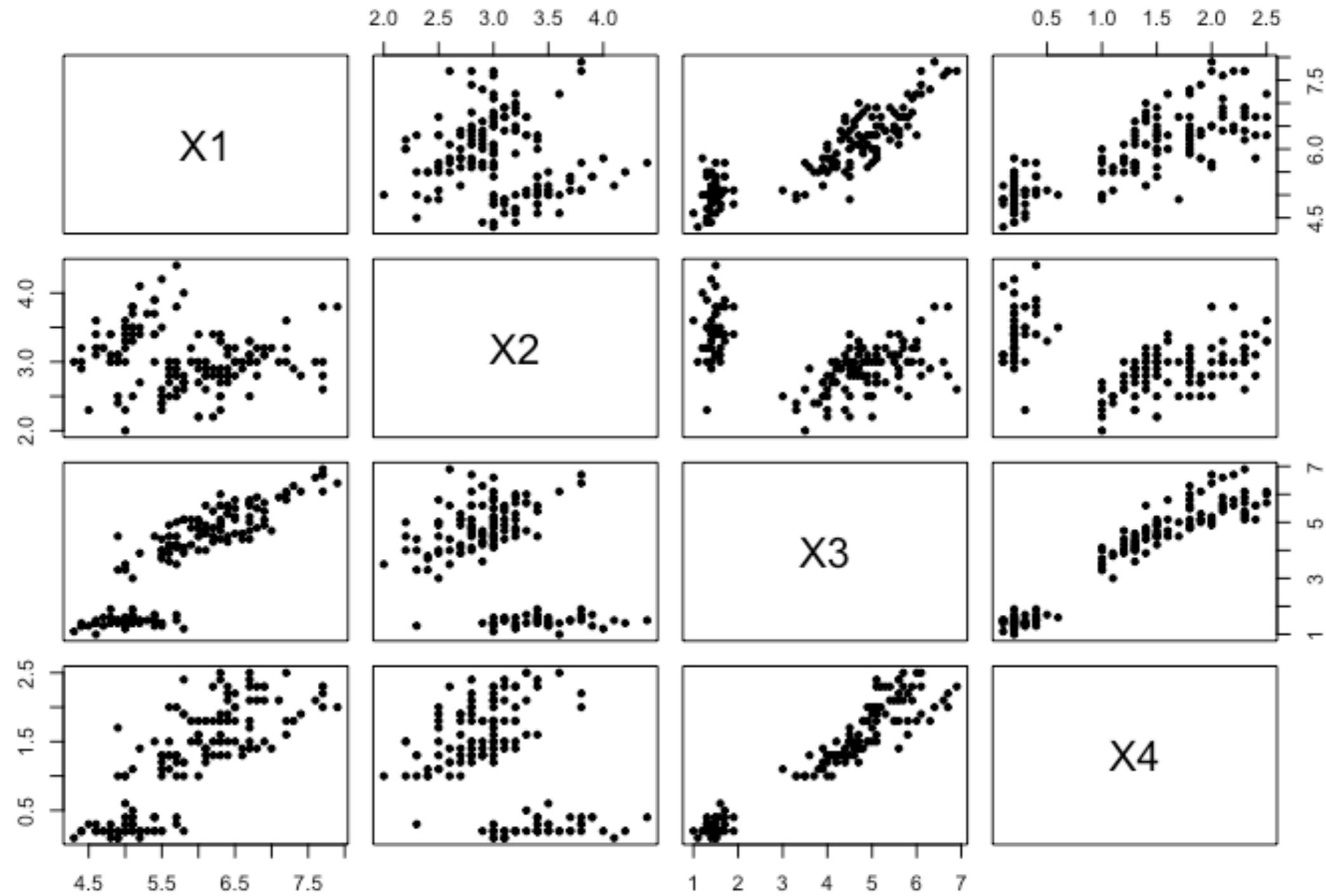
MULTIVARIATE DATA — SCATTERPLOT



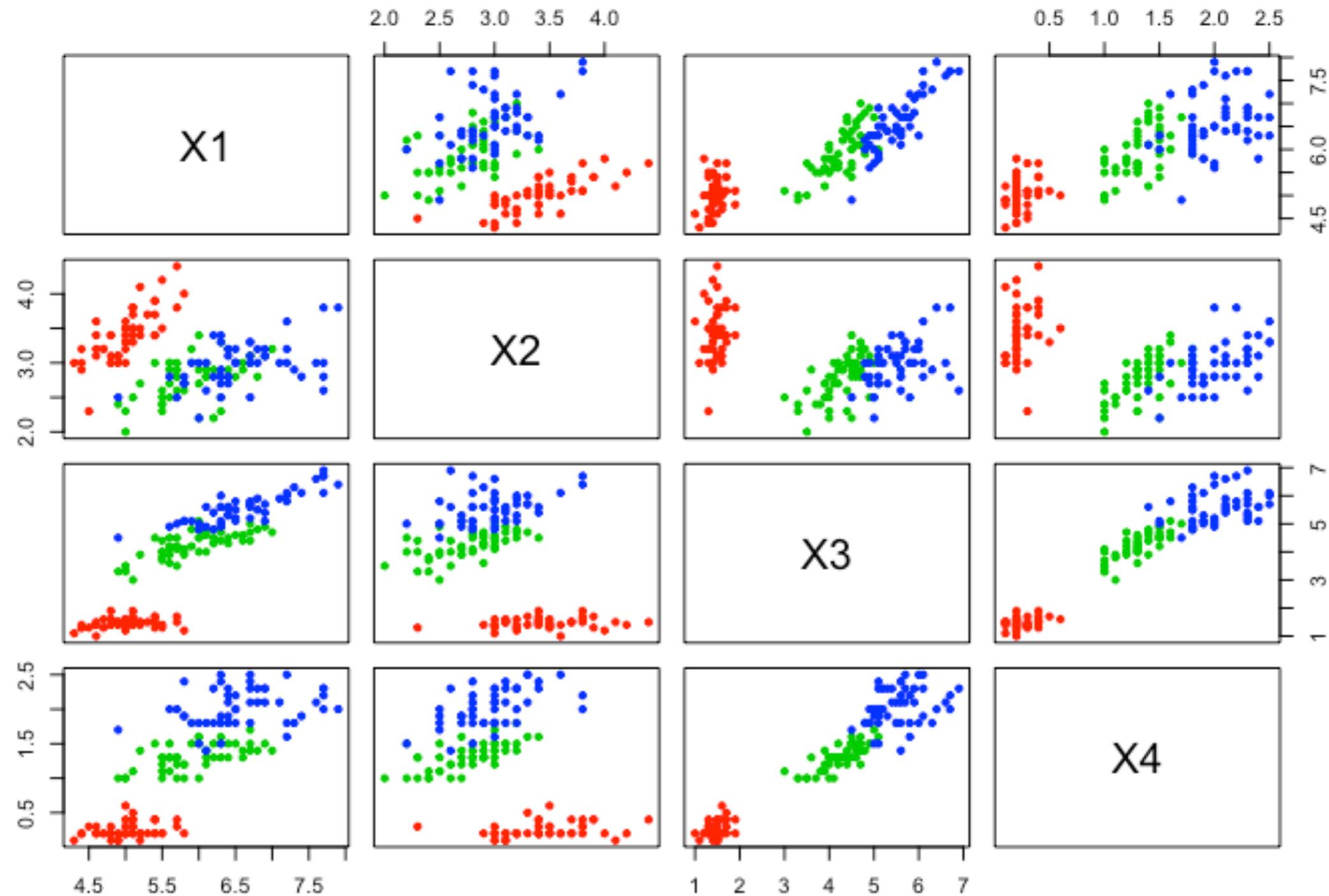
MULTIVARIATE DATA — SCATTERPLOT



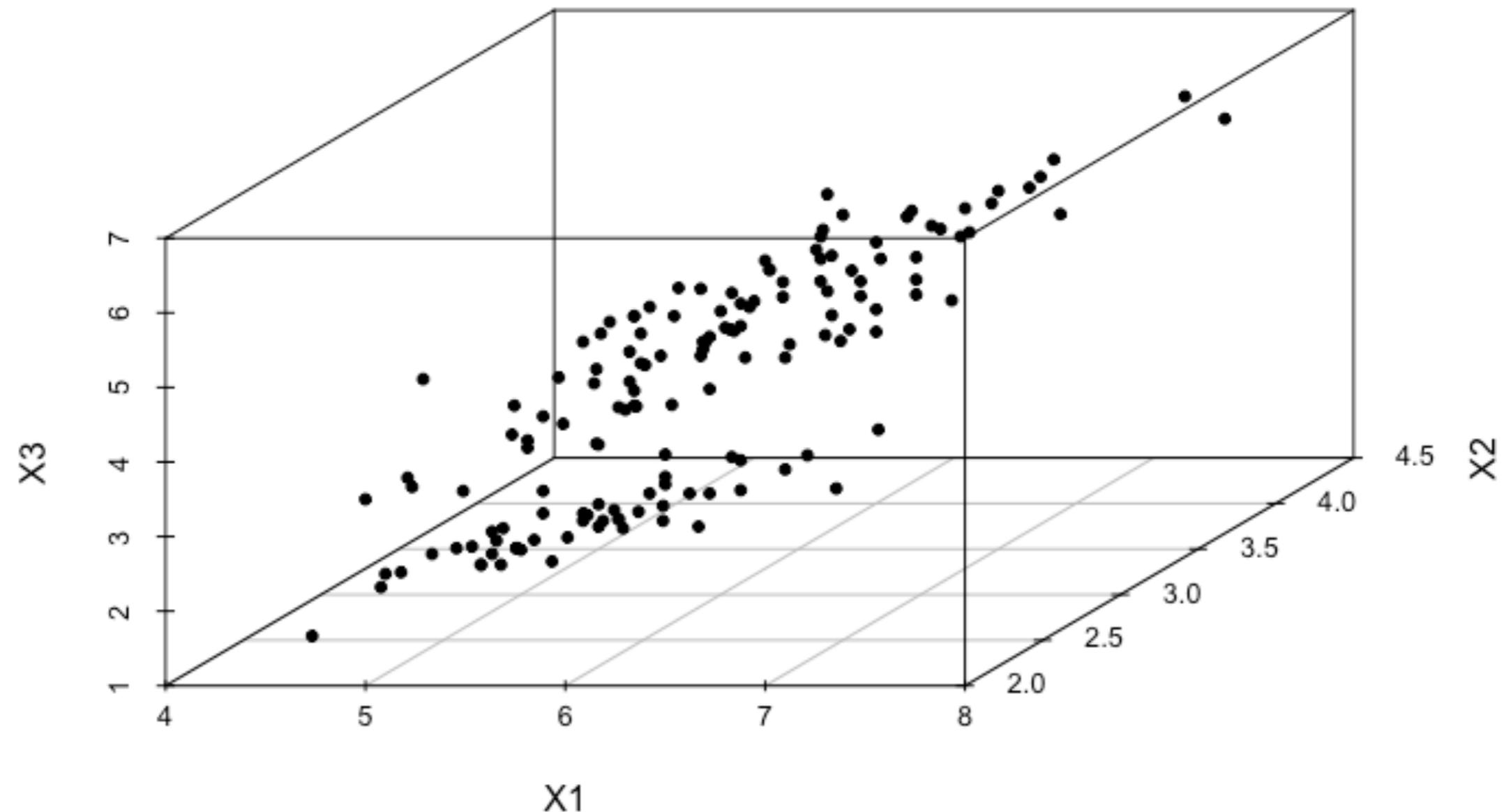
MULTIVARIATE DATA — SCATTERPLOTS



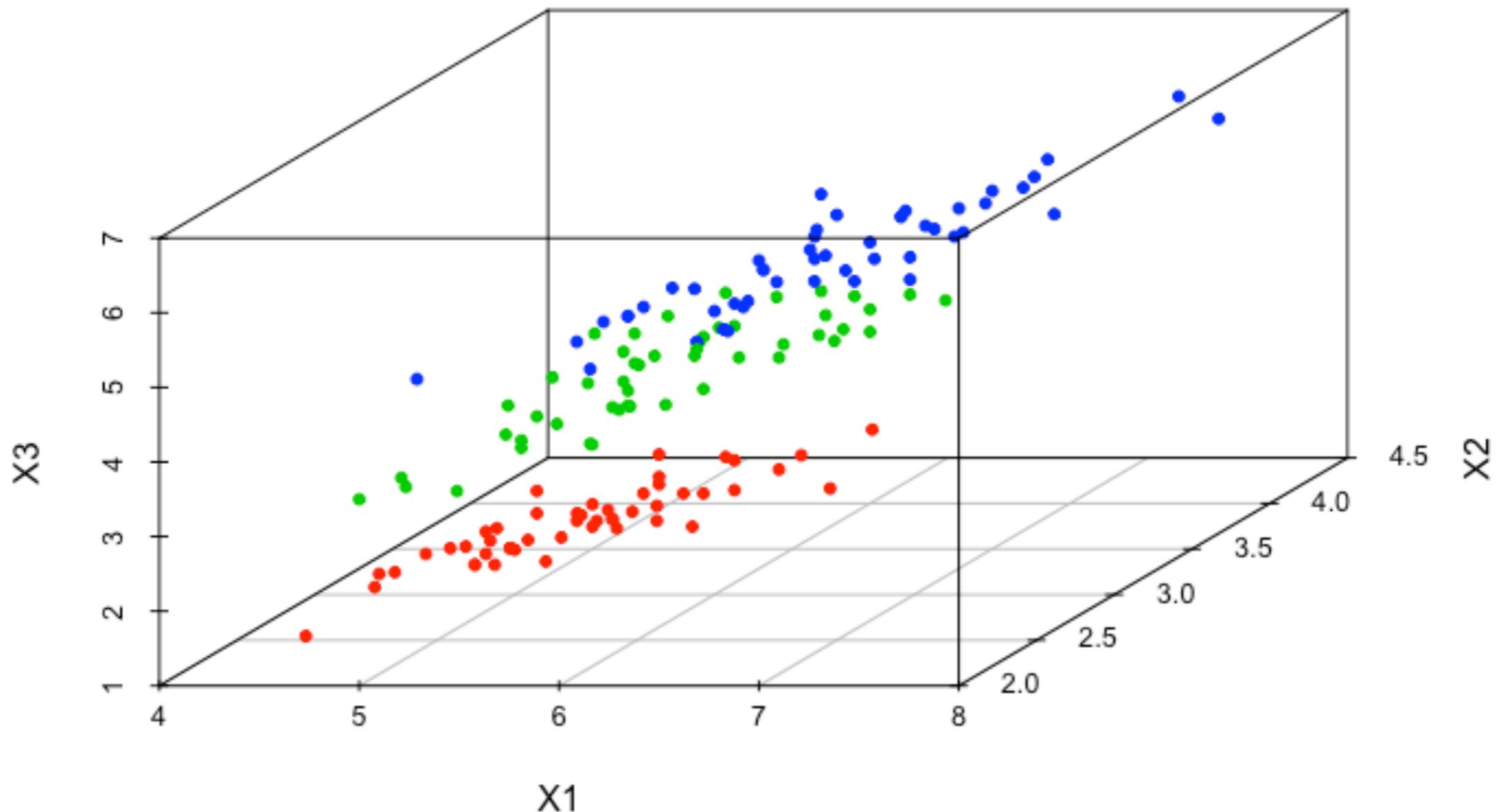
MULTIVARIATE DATA — SCATTERPLOTS



MULTIVARIATE DATA — 3D SCATTERPLOT



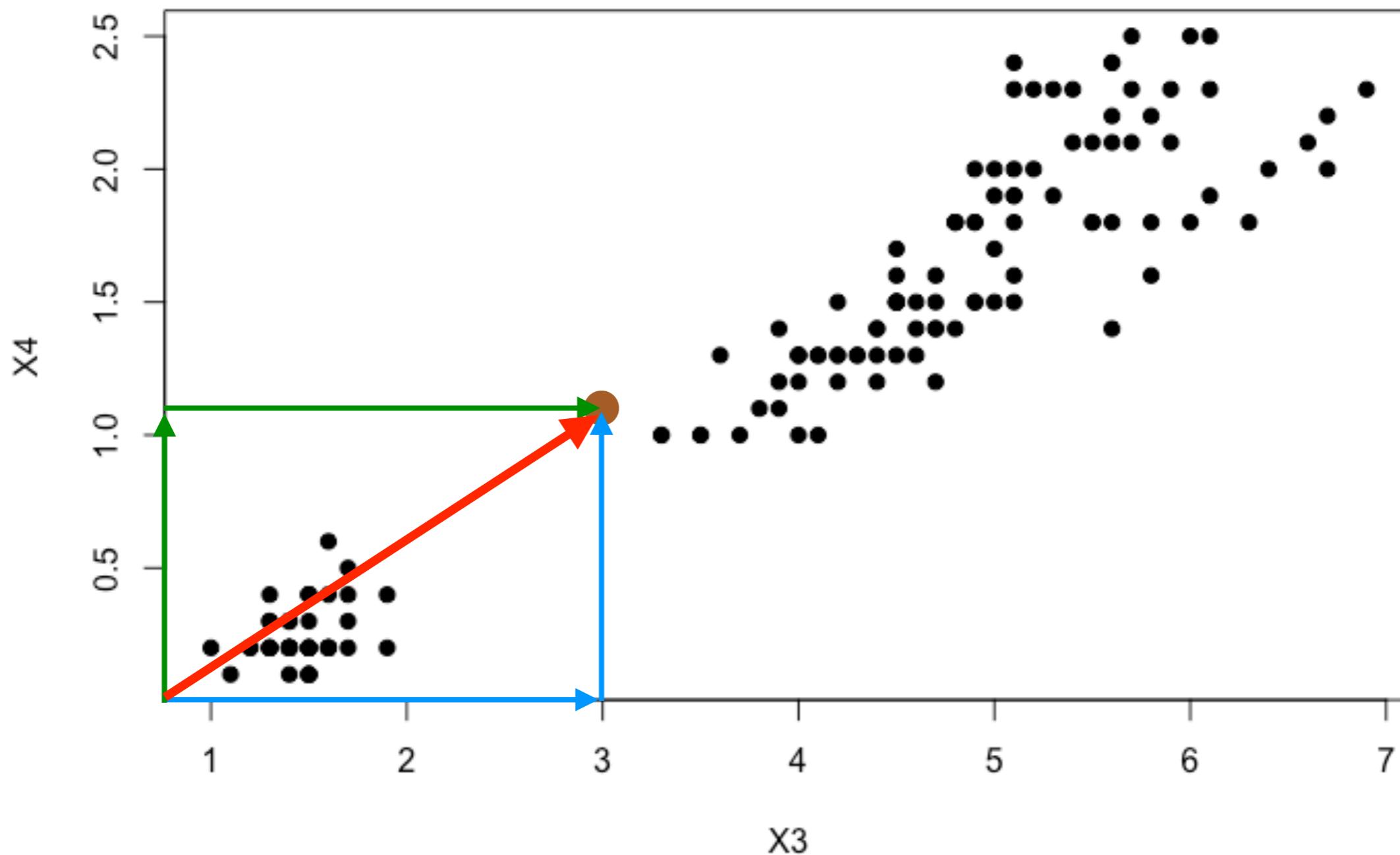
MULTIVARIATE DATA — 3D SCATTERPLOT



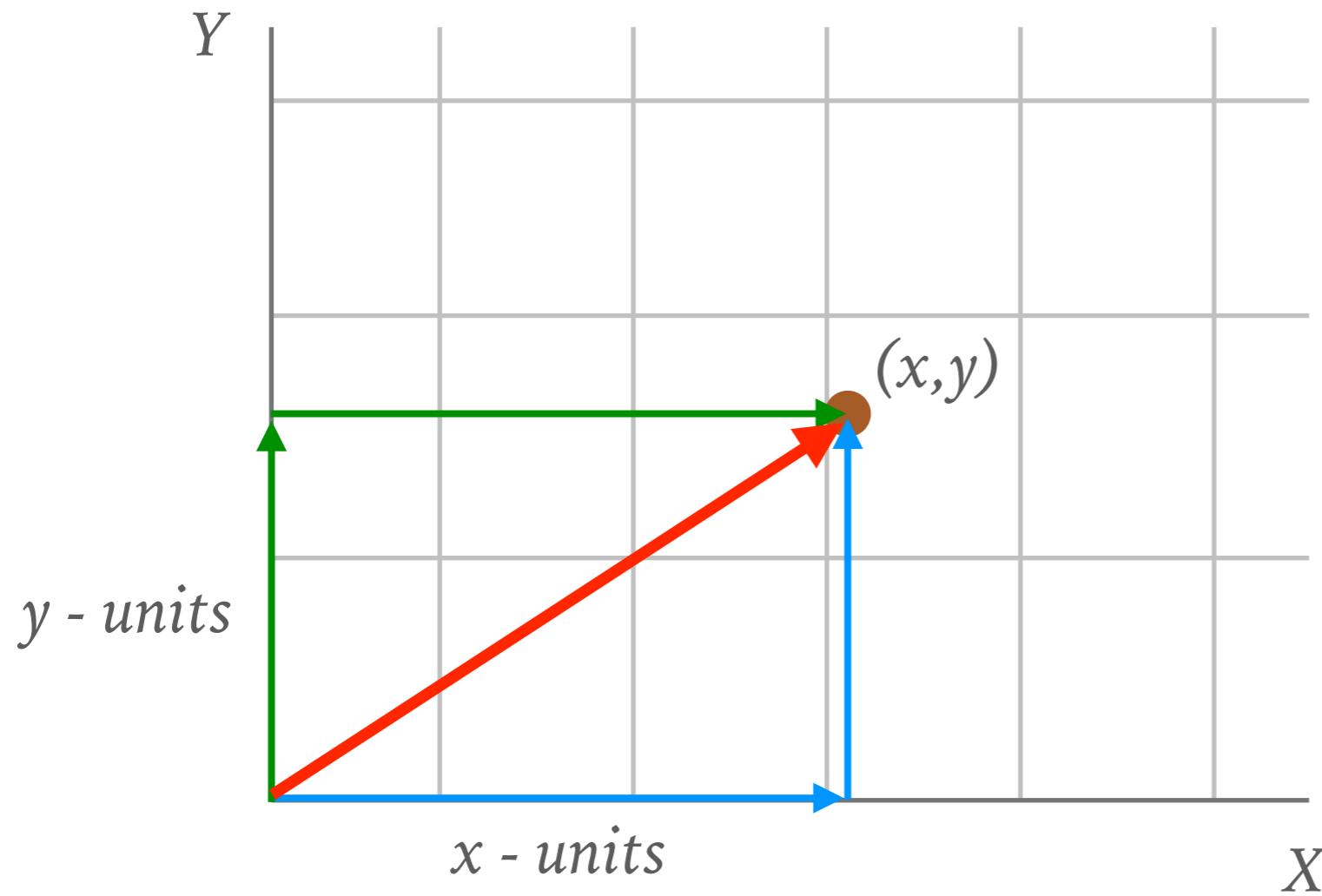


LET'S START WITH 2D

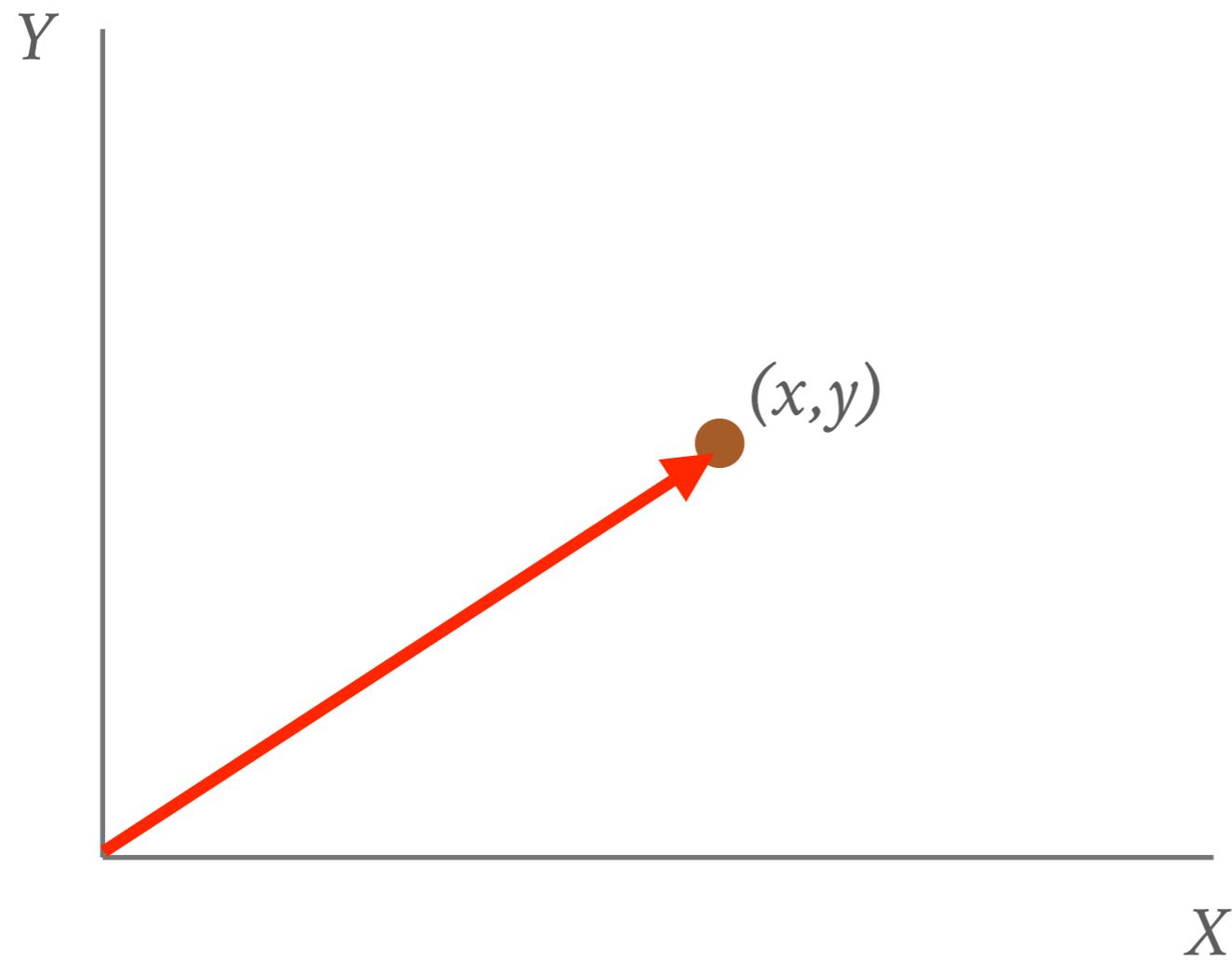
2D GEOMETRY



2D GEOMETRY



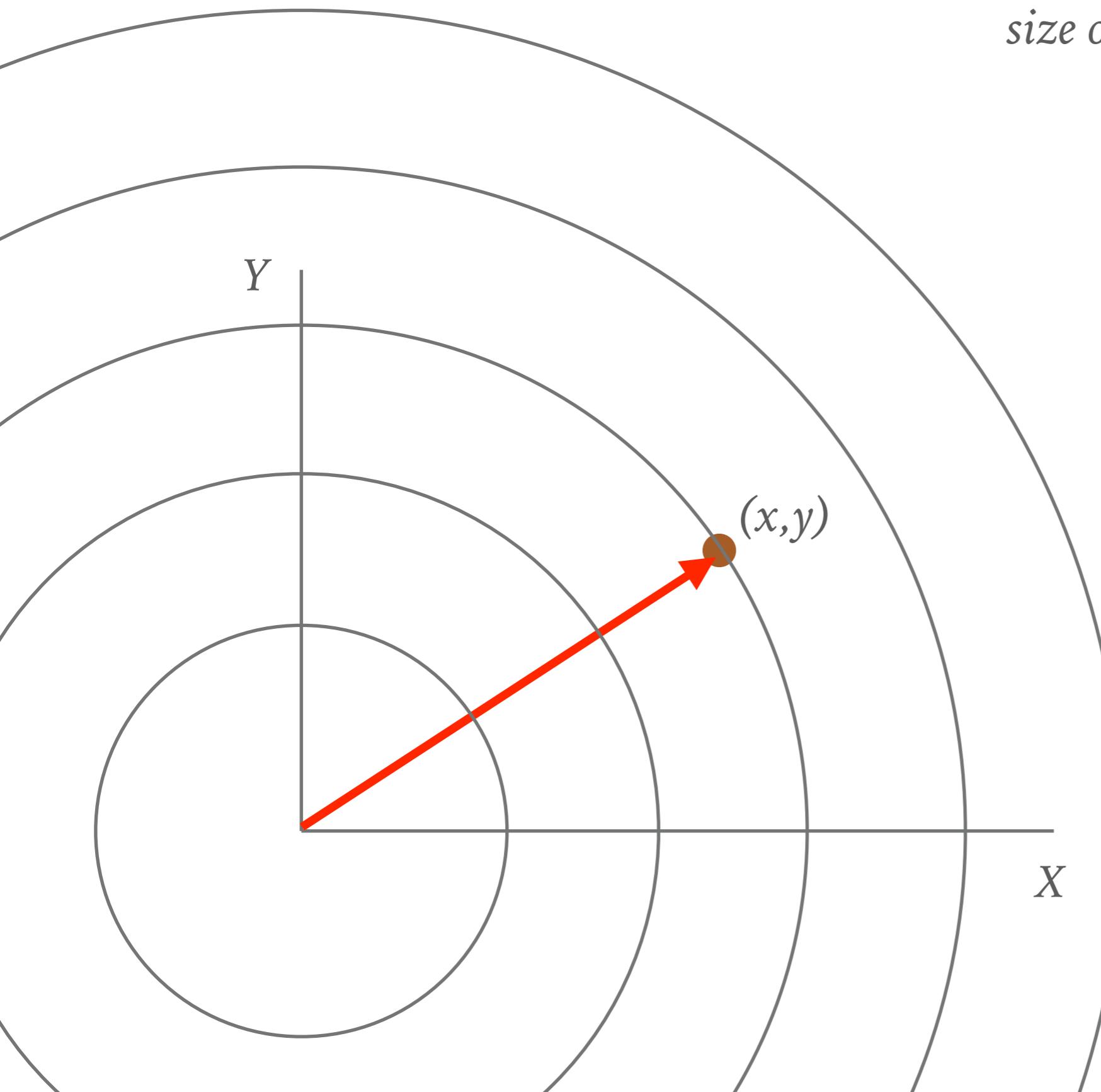
SIZE



SIZE

L2 Norm

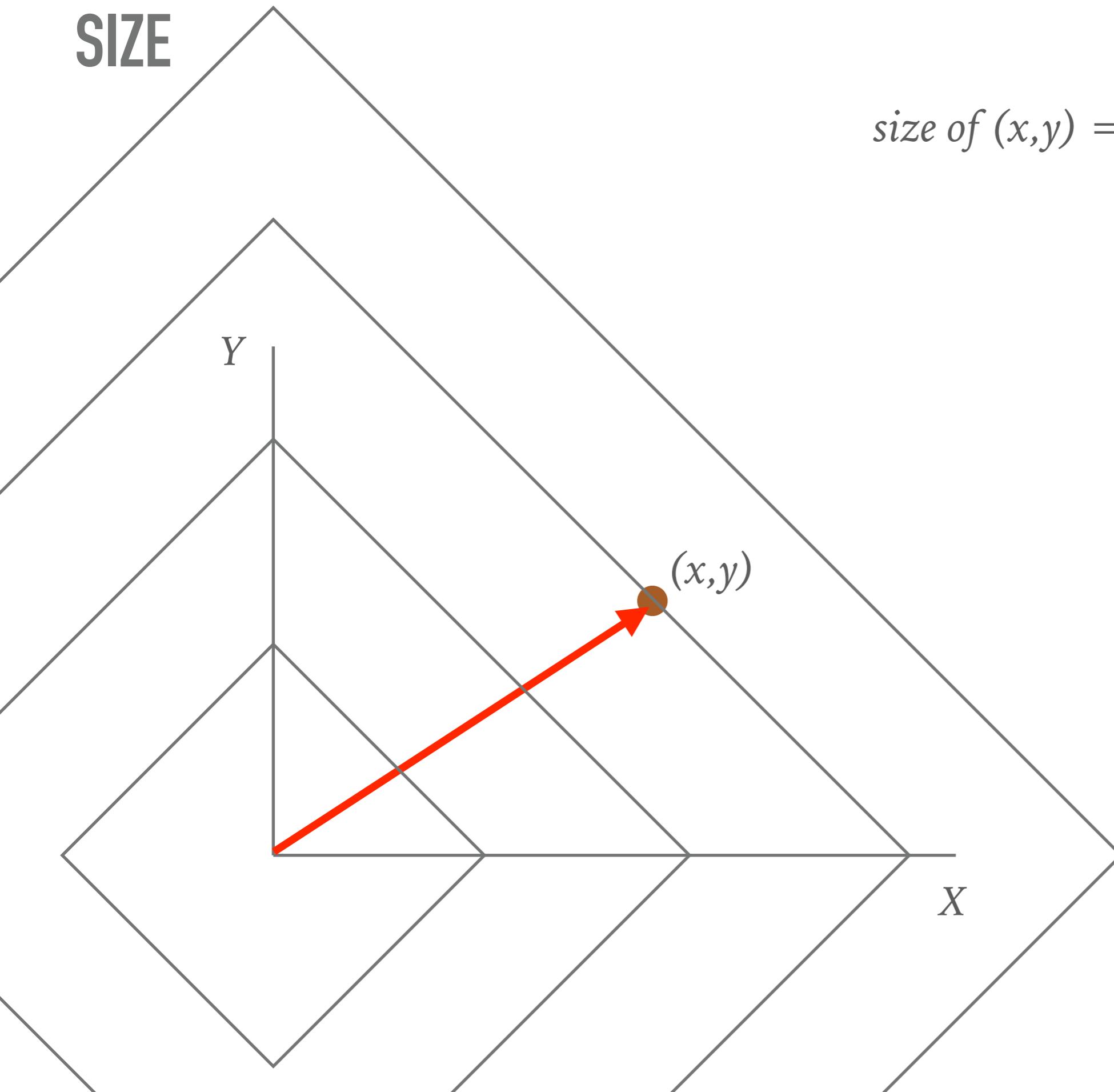
$$\text{size of } (x,y) = \sqrt{x^2 + y^2}$$



SIZE

L1 Norm

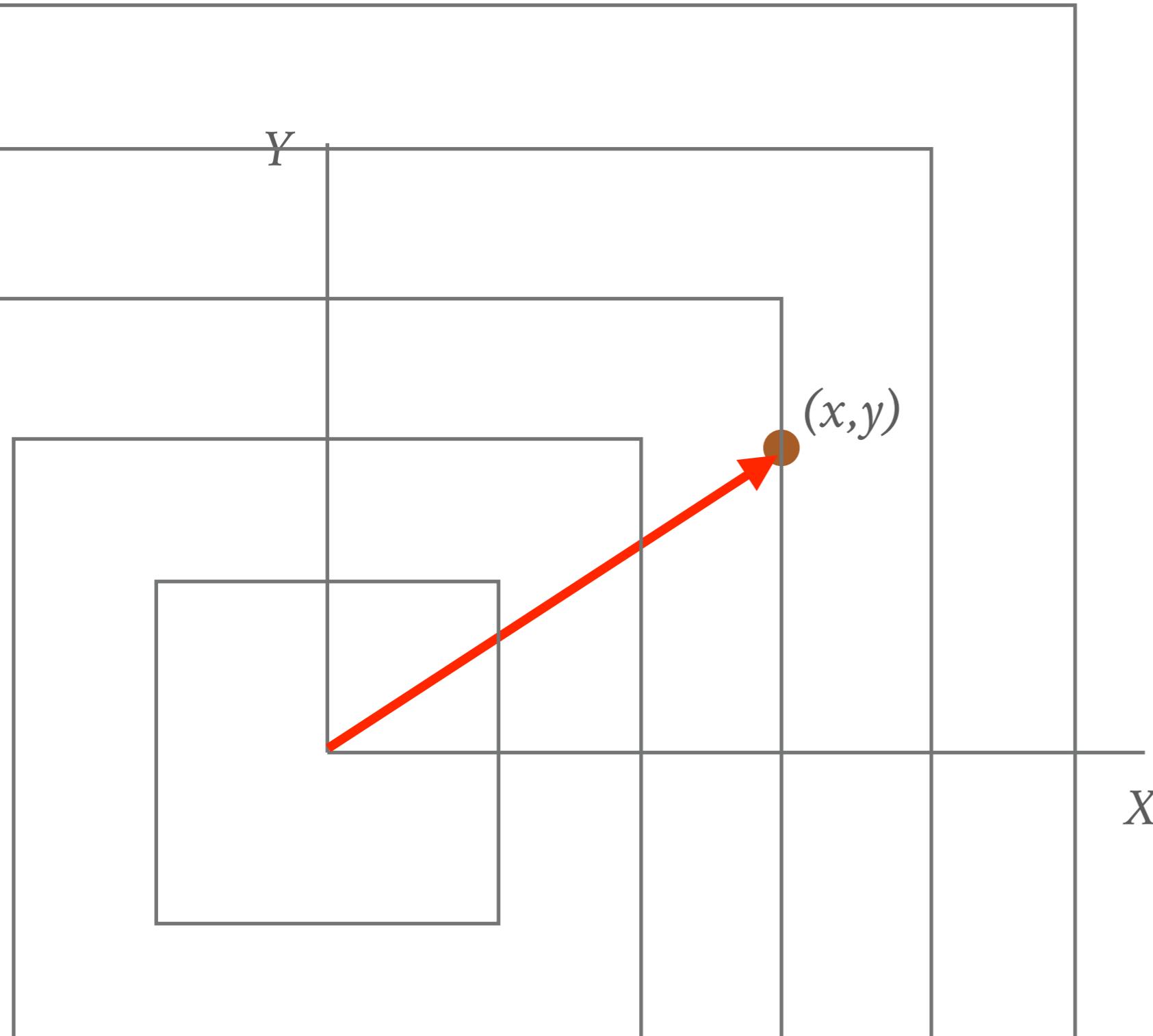
$$\text{size of } (x,y) = \text{abs}(x) + \text{abs}(y)$$



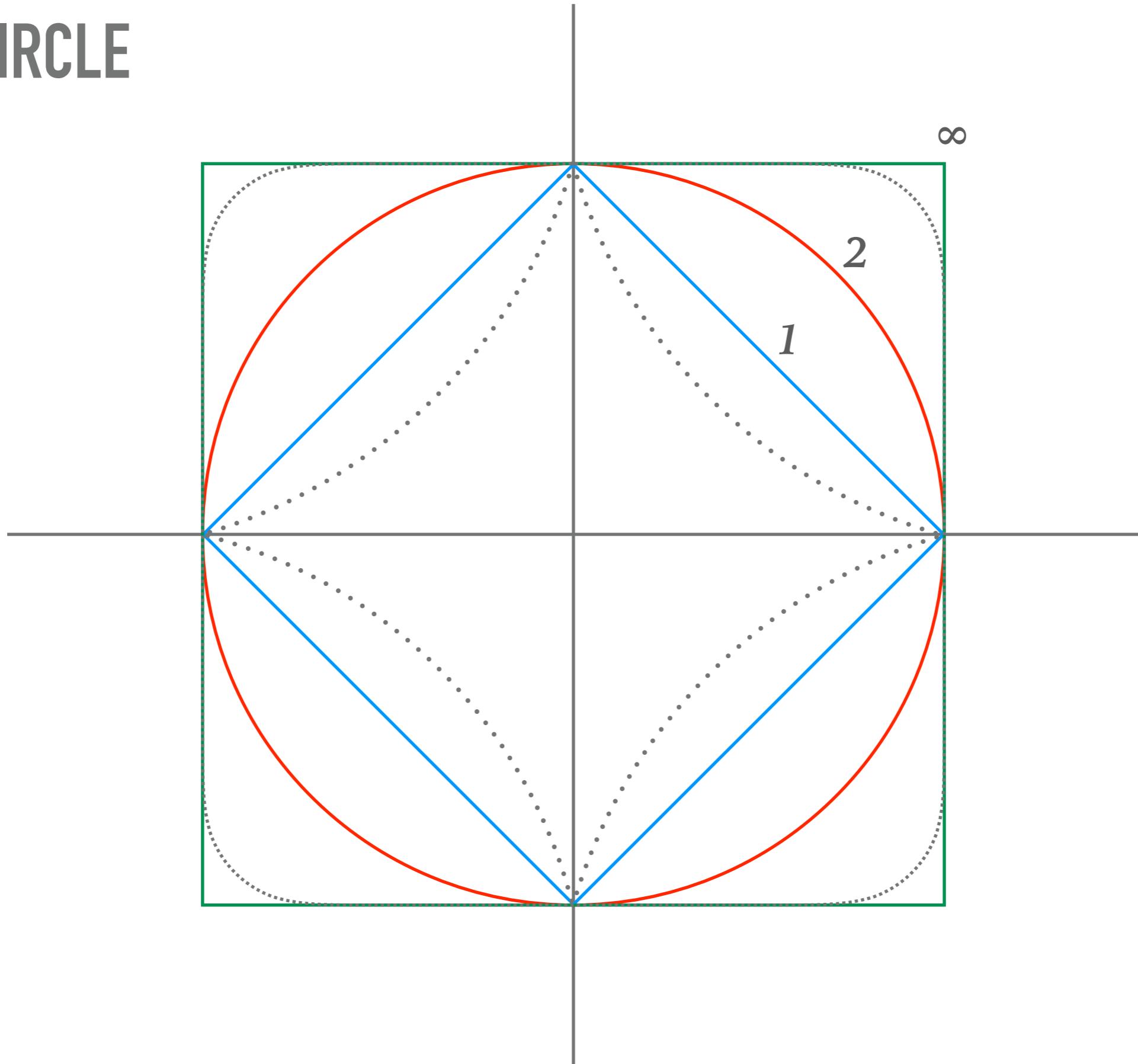
SIZE

L ∞ Norm

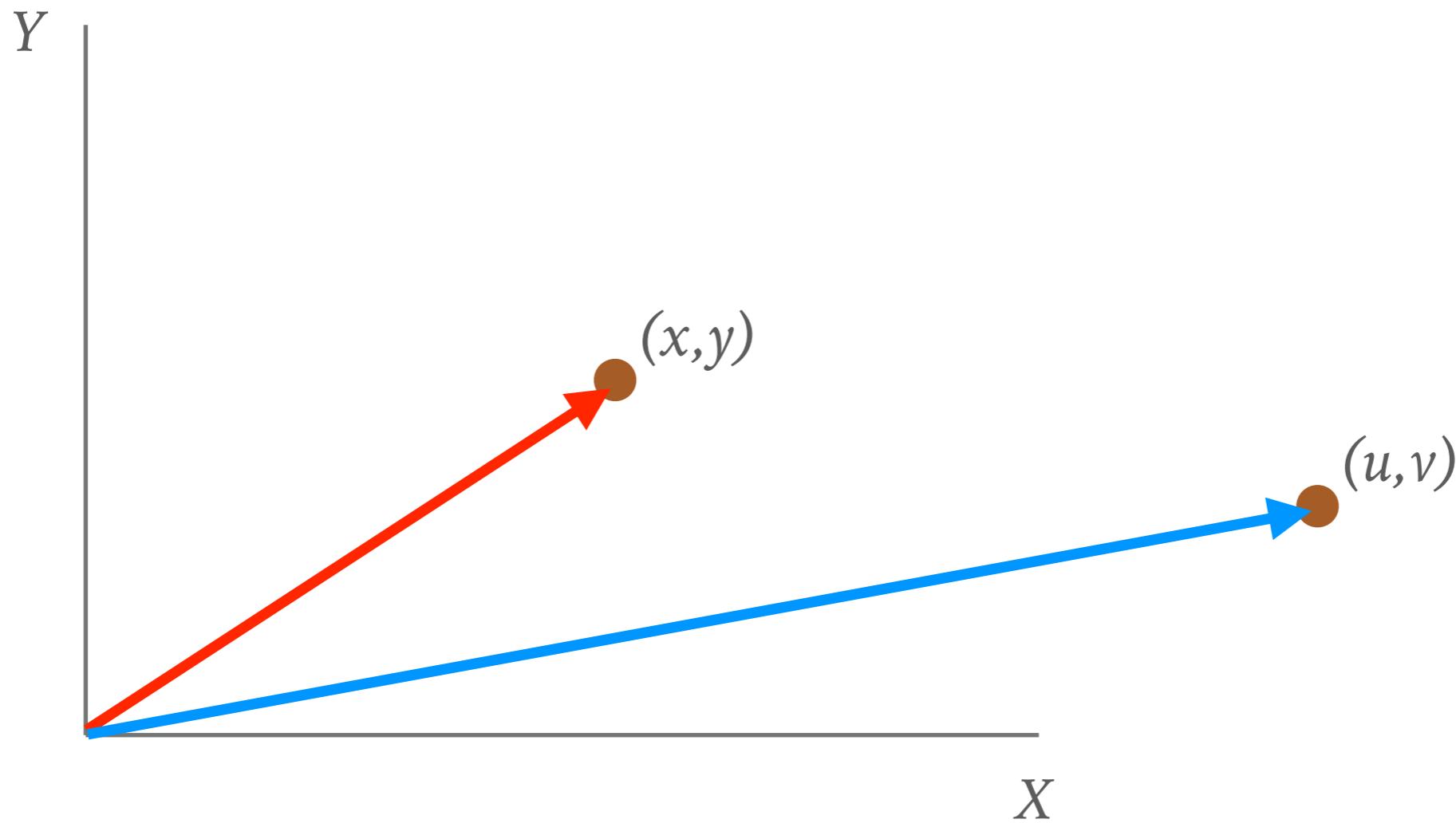
$$\text{size of } (x,y) = \max(\text{abs}(x), \text{abs}(y))$$



UNIT CIRCLE



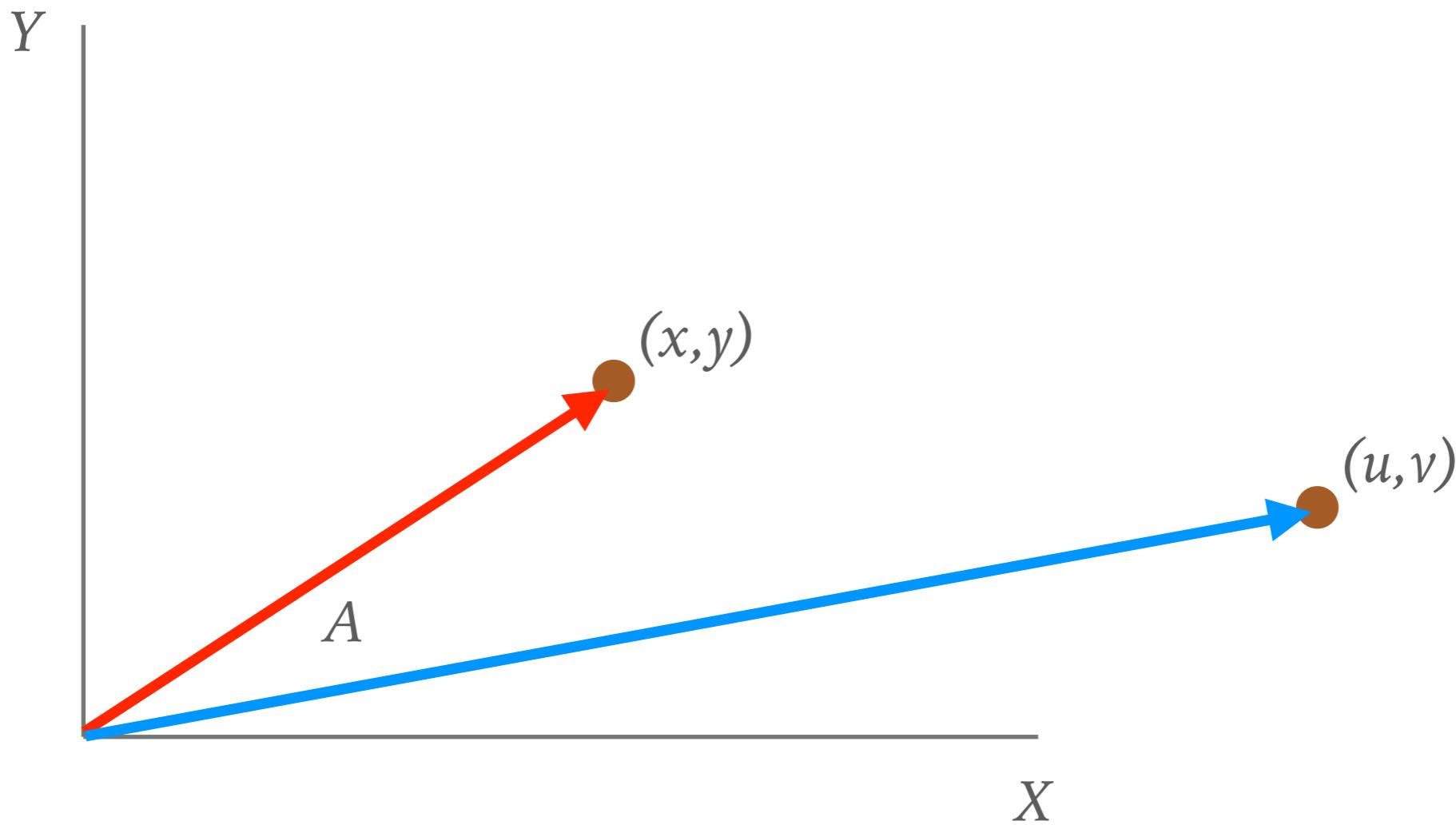
ANGLE



ANGLE

Dot Product

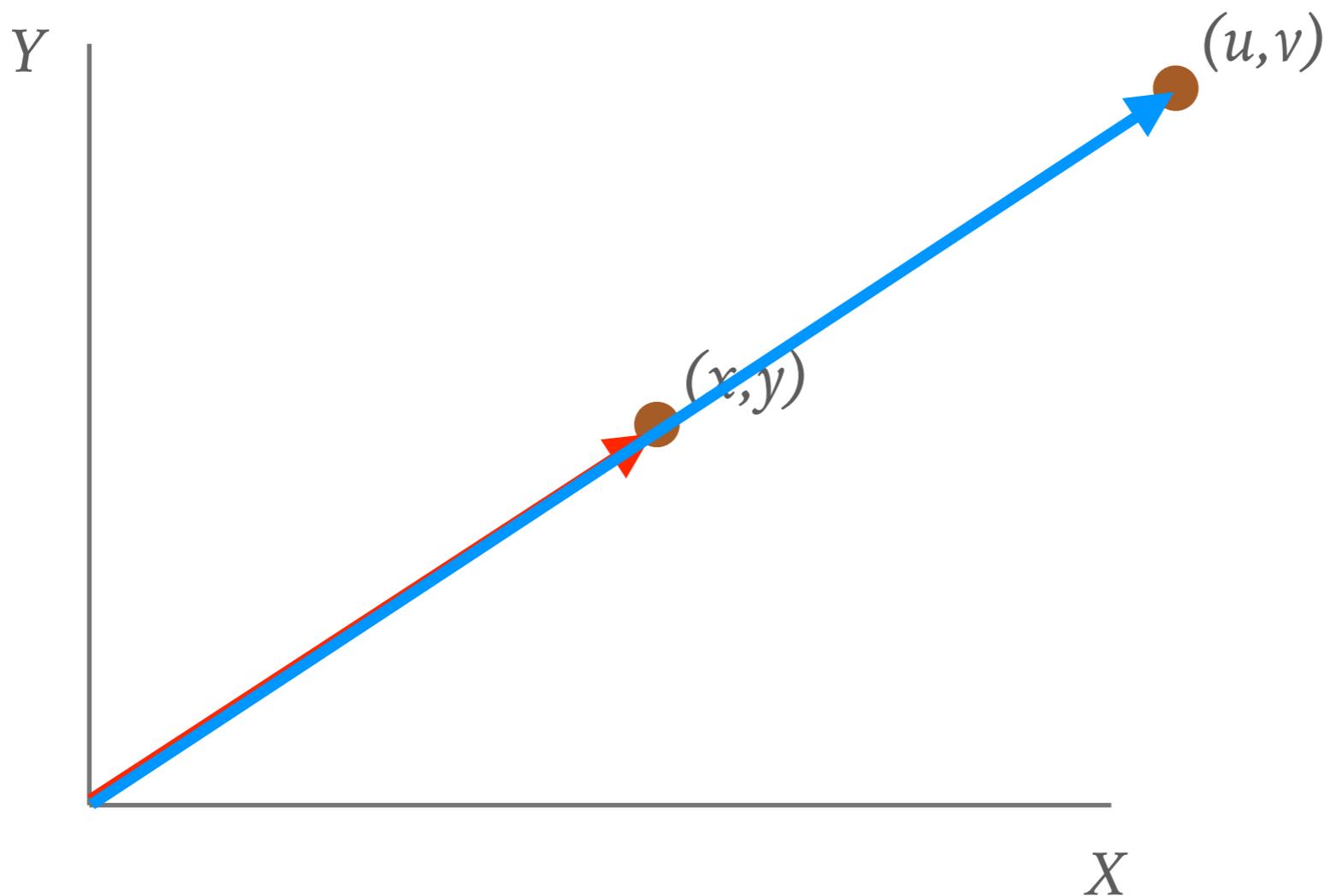
$$(x,y) \cdot (u,v) = |(x,y)| |(u,v)| \cos A$$



ANGLE

Dot Product

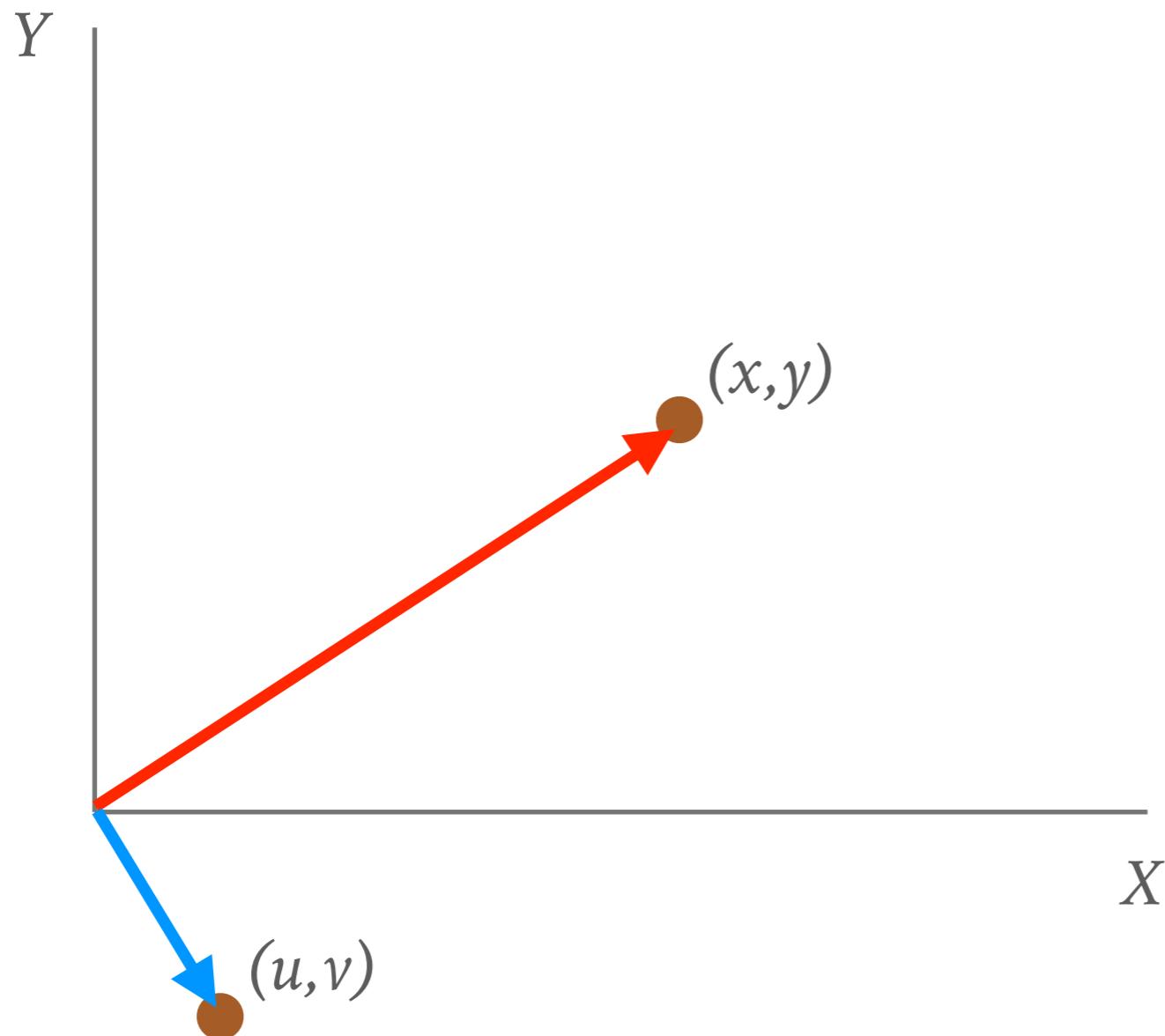
$$(x,y) \cdot (u,v) = |(x,y)| |(u,v)|$$



ANGLE

Dot Product

$$(x,y) \cdot (u,v) = 0$$



SIZE AND ANGLE CHEATSHEET

Notation for dot product : $v \cdot w$ or $v^T w = \sum_{i=1}^n v_i w_i$

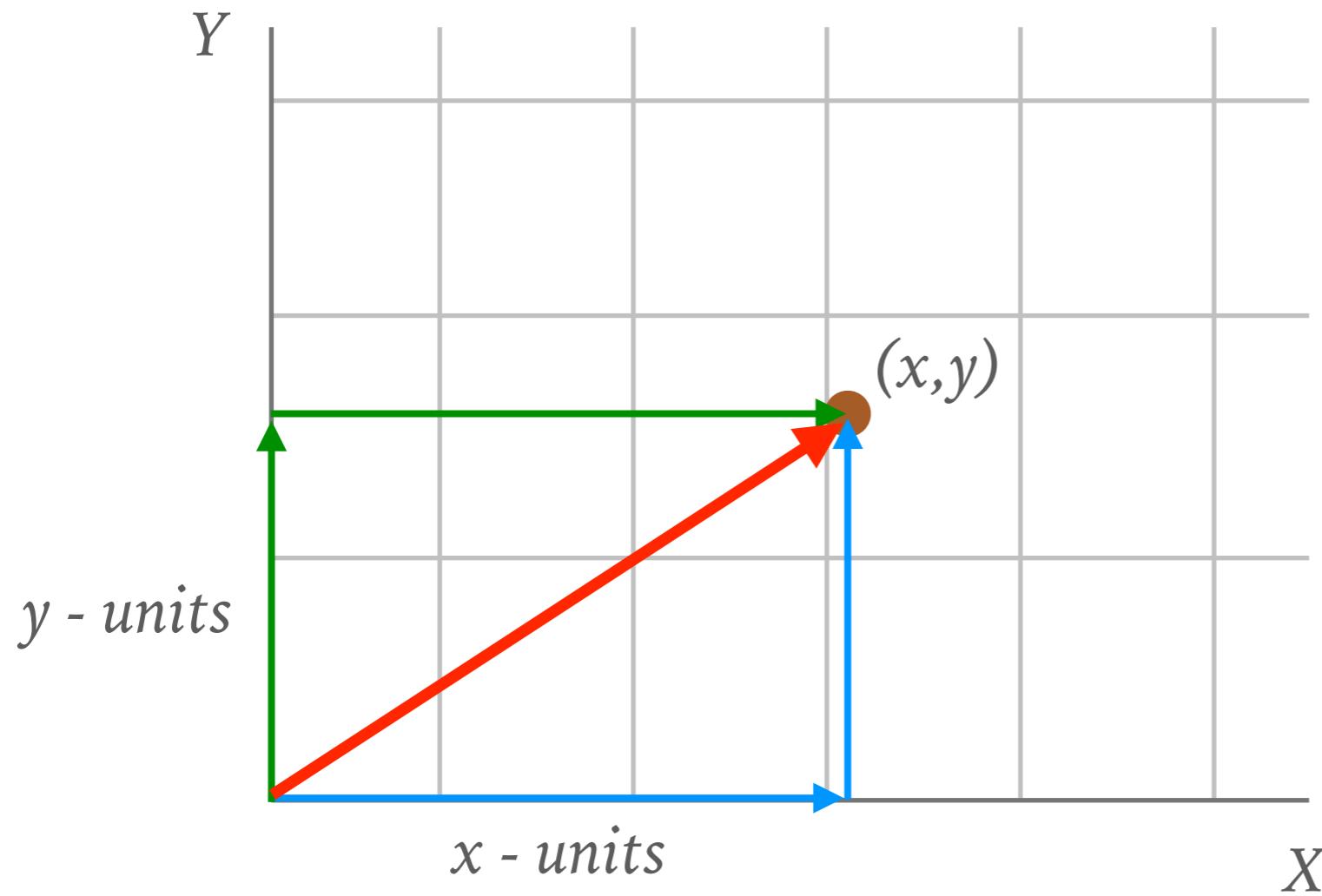
L2 norm of a vector : $v^T v = \sum_{i=1}^n v_i^2 = ||v||_2^2$

Distance between two vectors : $||v - w||_2 = \sqrt{\sum_{i=1}^n (v_i - w_i)^2}$

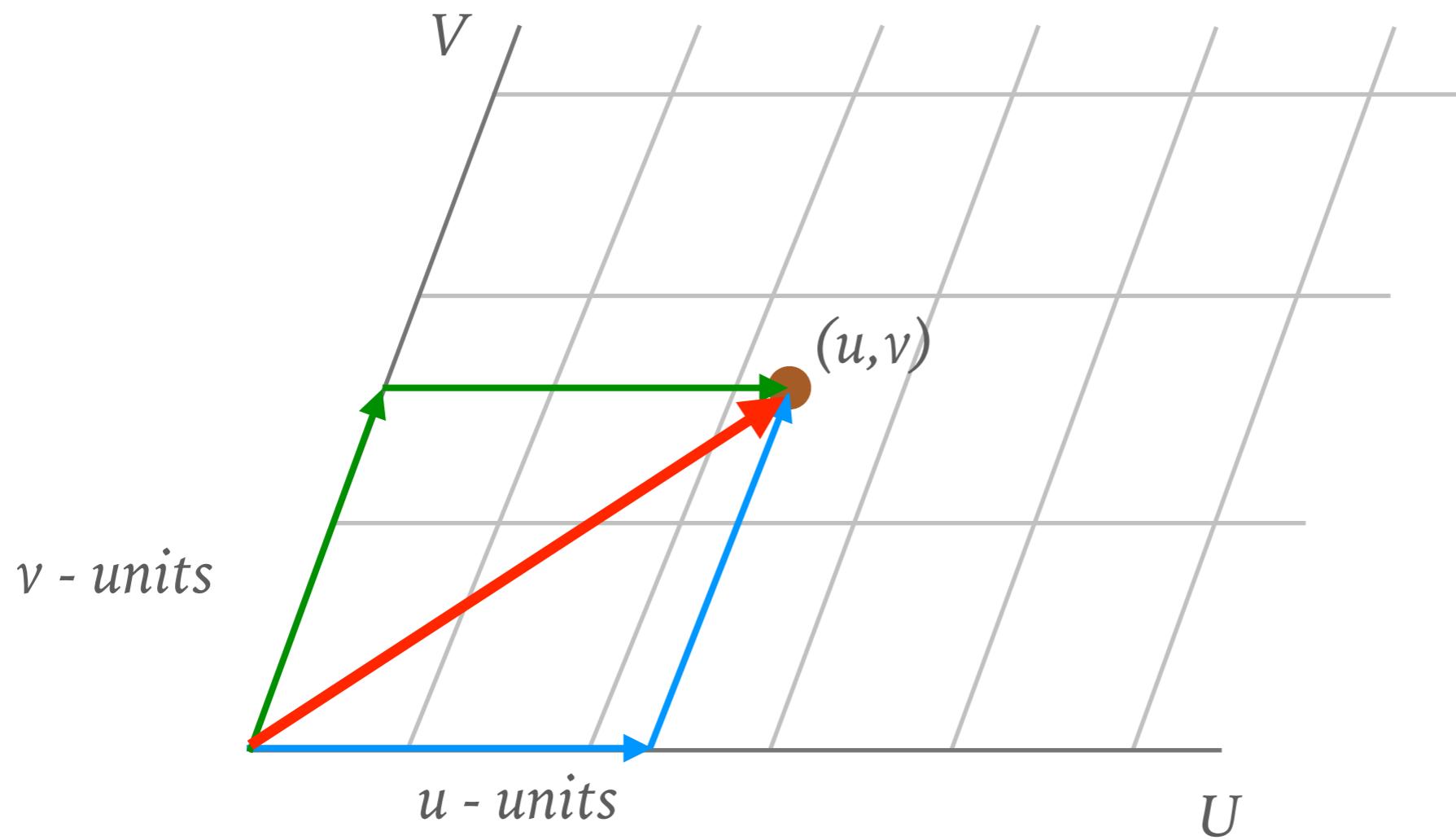
Do you see any connection with Statistics?

ANOTHER LOOK AT 2D

2D GEOMETRY



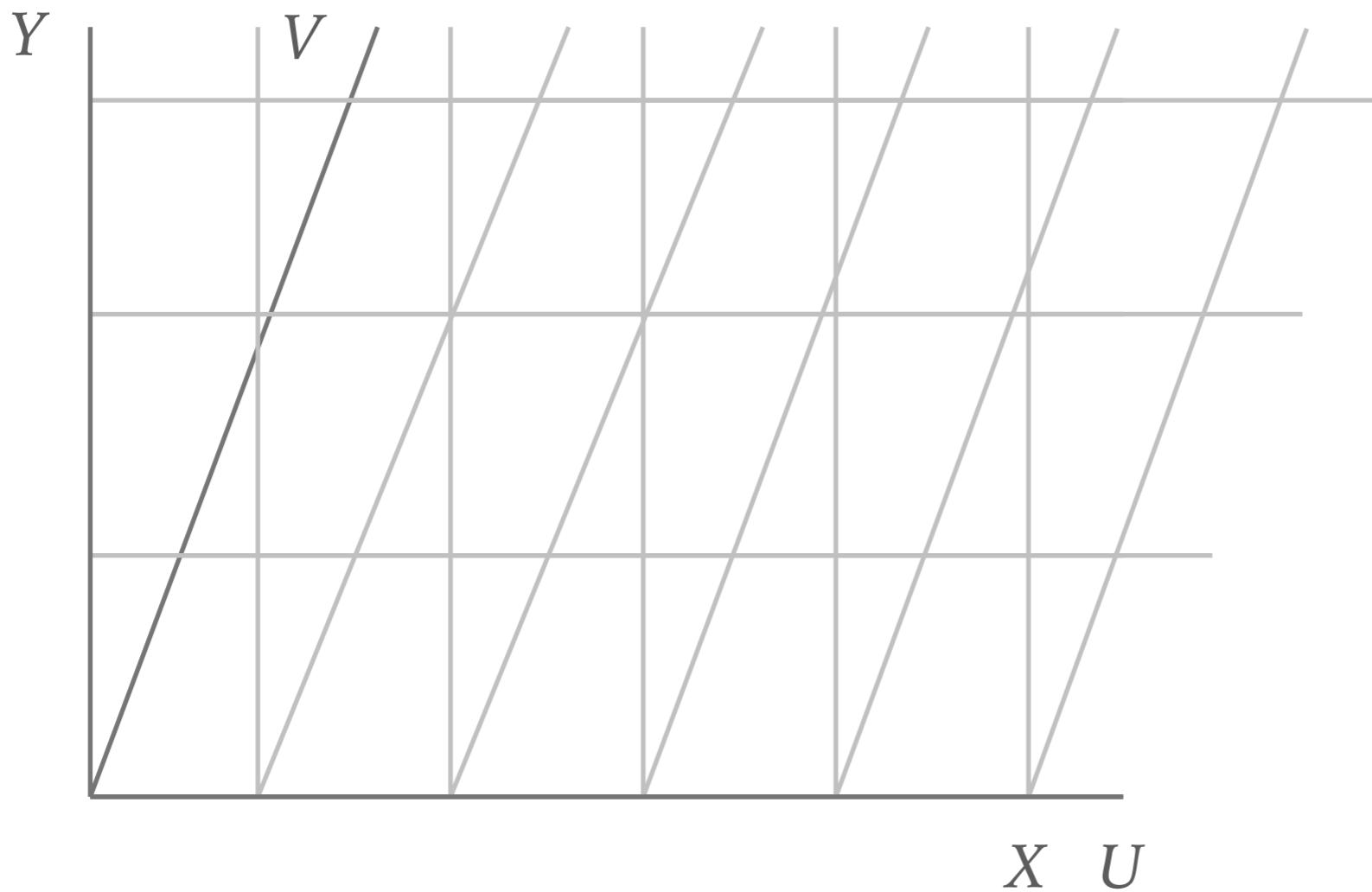
2D GEOMETRY



2D GEOMETRY

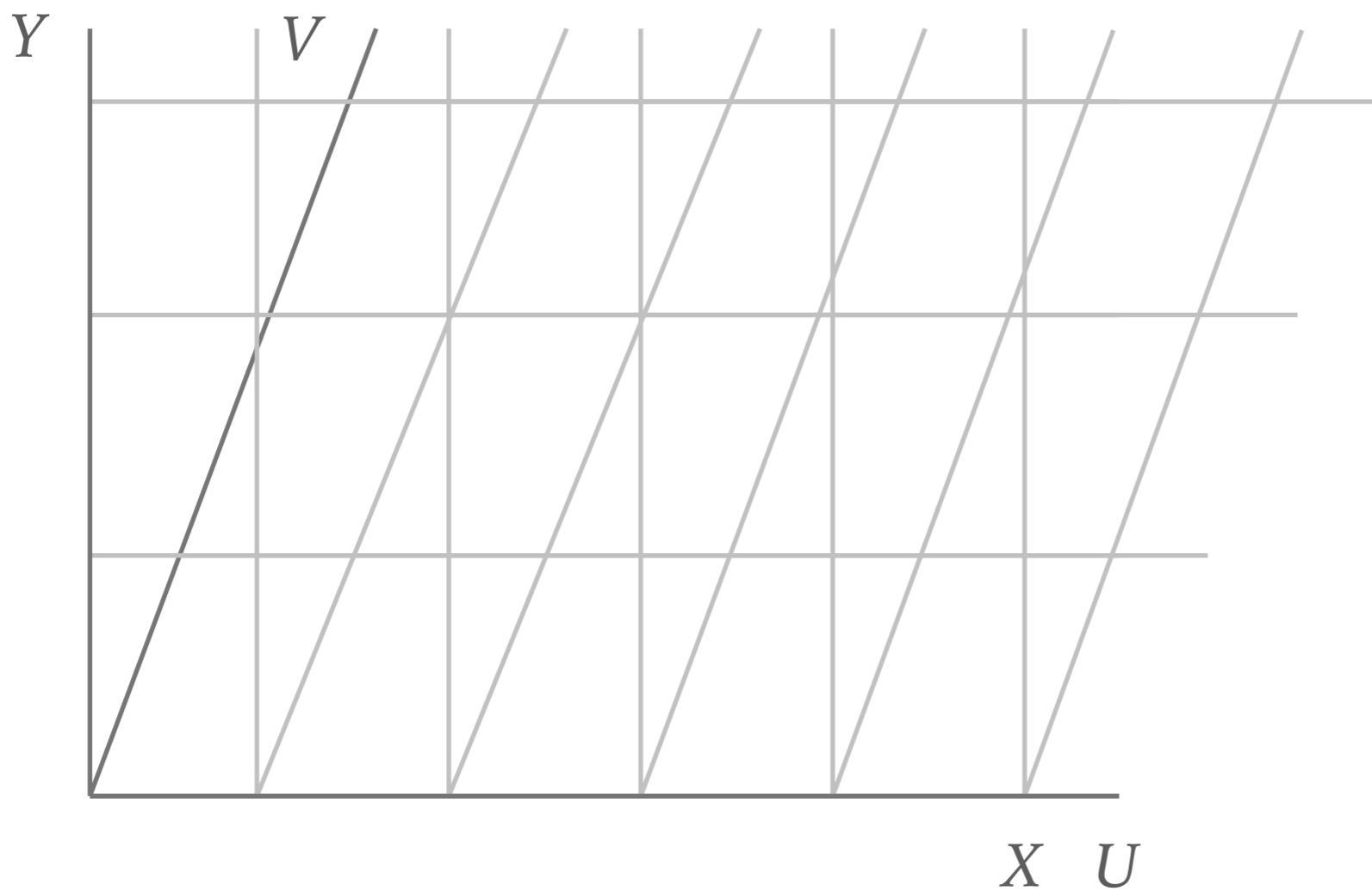
$$U = aX + bY$$

$$V = cX + dY$$



2D GEOMETRY

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

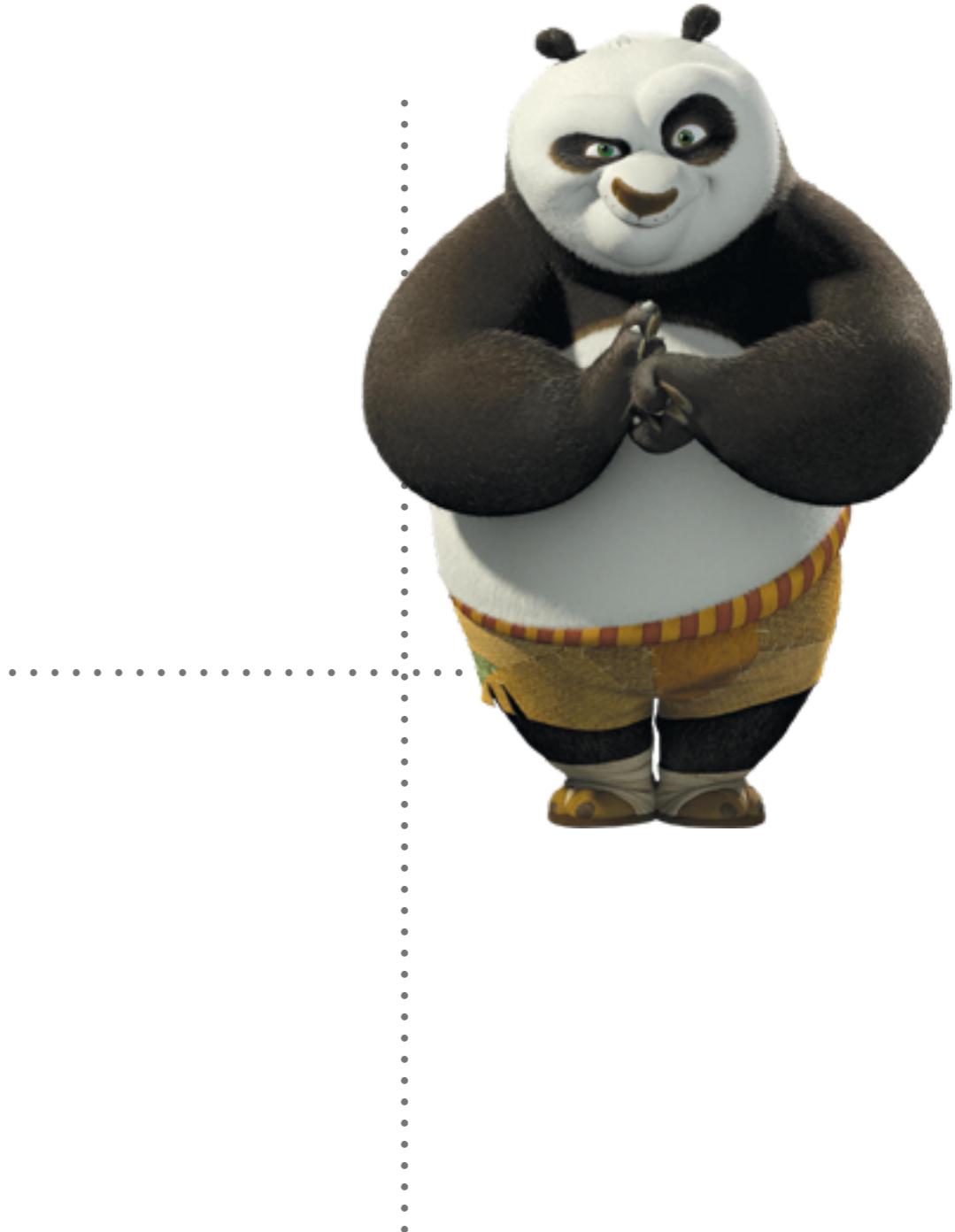


TRANSFORMATIONS

TRANSLATION



$$\begin{bmatrix} p \\ q \end{bmatrix}$$



ROTATION



$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$



SCALING



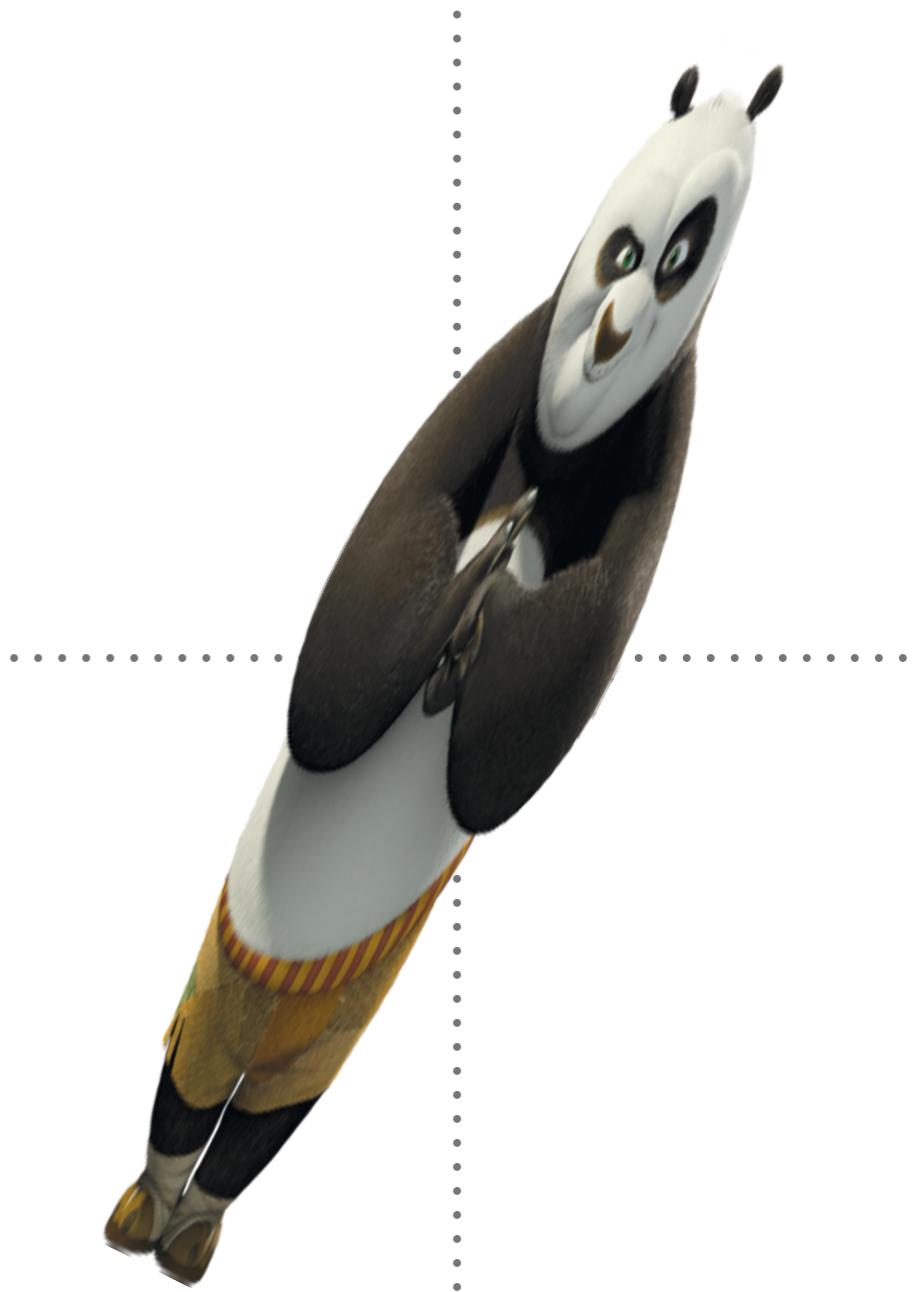
$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$



GENERIC

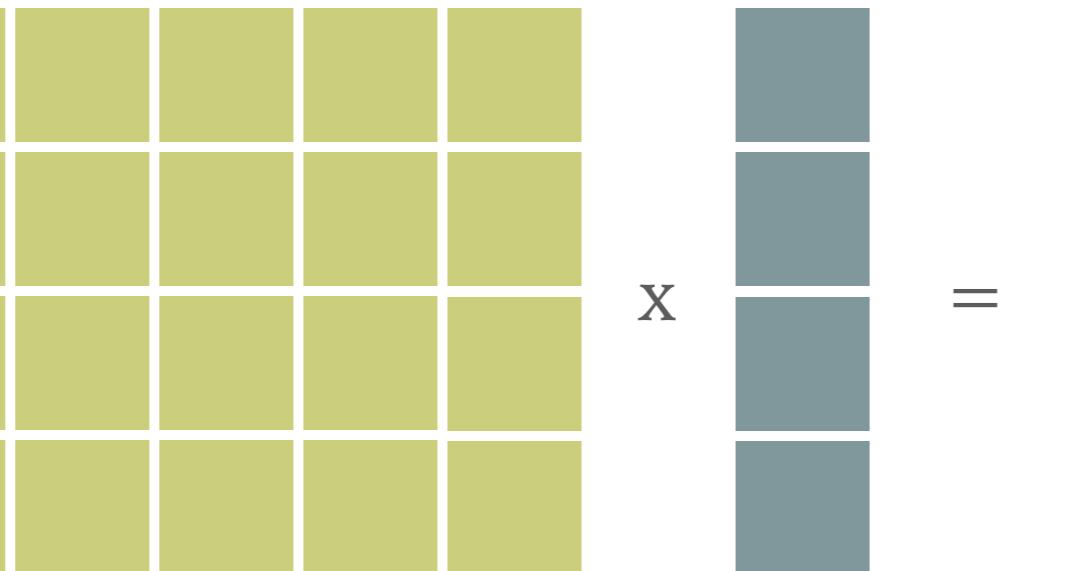


$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

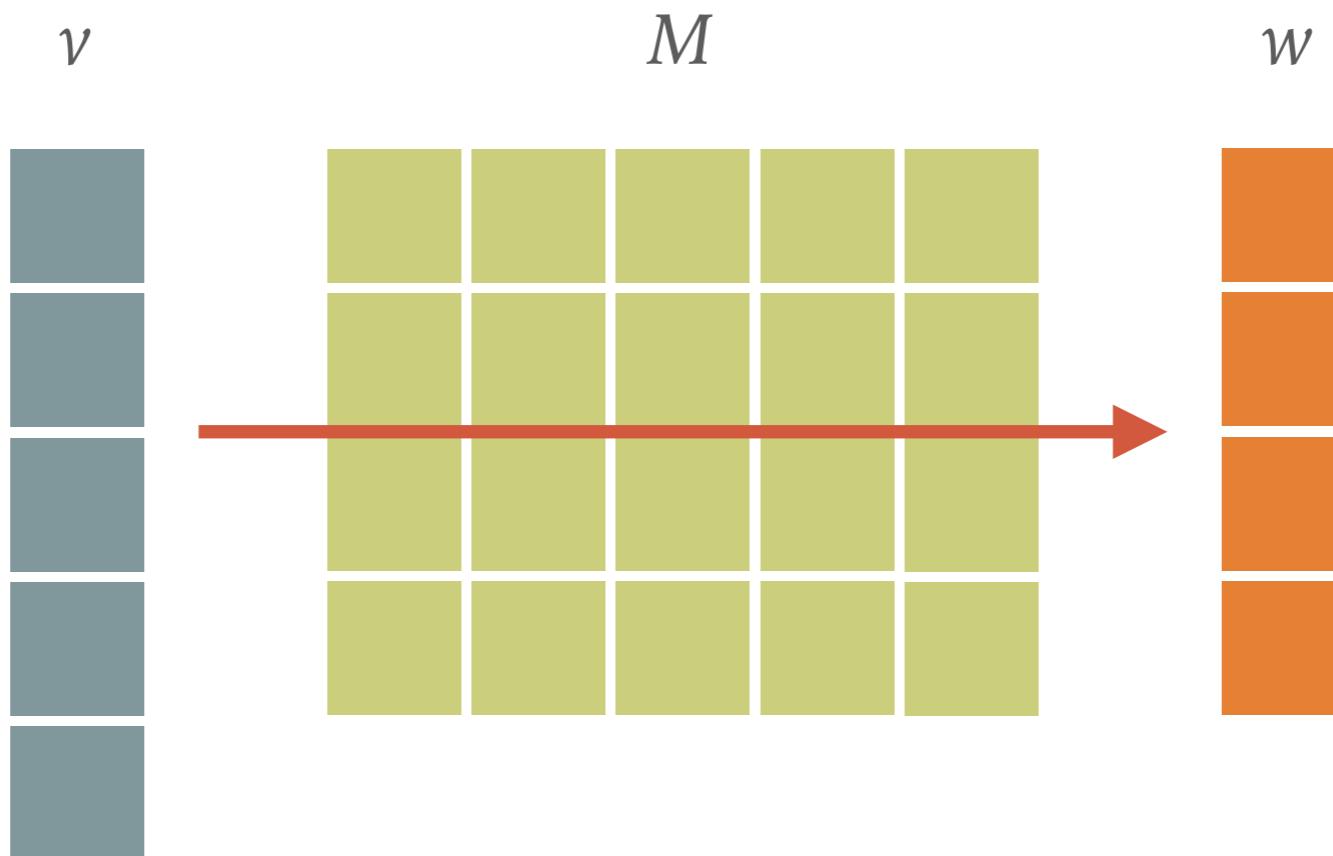


ACTION OF A MATRIX

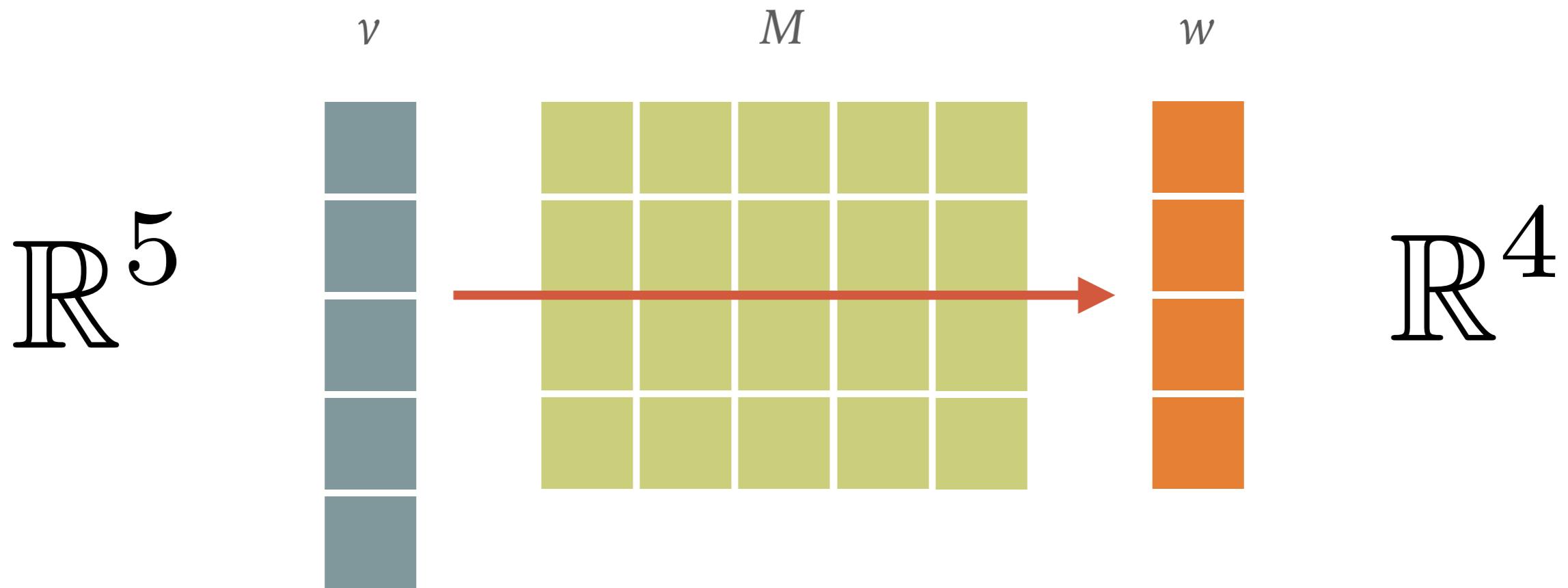
MATRIX-VECTOR MULTIPLICATION

$$M \nu = w$$


LINEAR MAP



LINEAR MAP



$$\mathbb{R}^n \xrightarrow{M_{m \times n}} \mathbb{R}^m$$

Domain of the function? — Where does the input originate from?

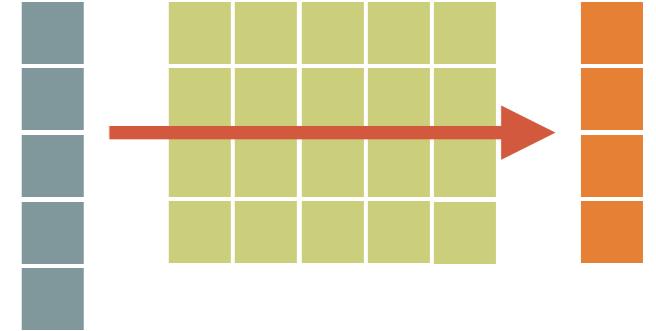
Codomain of the function? — What are possible values of output?

Range of the function? — What are the actual values of output?

Kernel of the function? — Which inputs map to zero in the range?

SUBSPACES

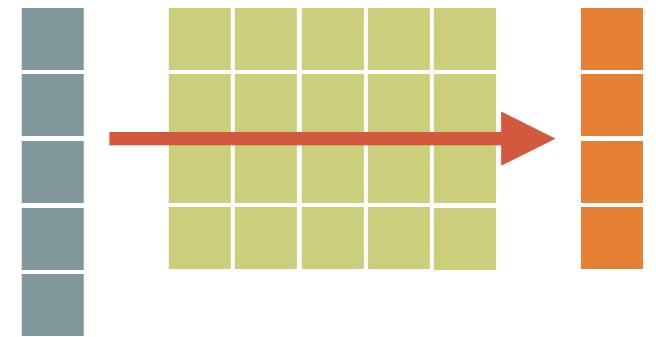
RANGE



Must be the linear combination of Columns

$$\begin{array}{c} \text{blue square} \\ \text{green square} \\ \text{green square} \end{array} + \begin{array}{c} \text{blue square} \\ \text{green square} \\ \text{green square} \end{array} + \begin{array}{c} \text{blue square} \\ \text{green square} \\ \text{green square} \end{array} + \begin{array}{c} \text{blue square} \\ \text{green square} \\ \text{green square} \end{array} + \begin{array}{c} \text{blue square} \\ \text{green square} \\ \text{green square} \end{array} = \begin{array}{c} \text{orange square} \\ \text{orange square} \\ \text{orange square} \end{array}$$

KERNEL



Must be the orthogonal to all Rows

$$\begin{matrix} \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} \\ \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} \\ \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} \\ \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} & \text{yellow} \end{matrix} \cdot \begin{matrix} \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} \\ \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} \end{matrix} = \begin{matrix} \text{orange} \\ \text{orange} \\ \text{orange} \end{matrix}$$

A diagram illustrating matrix multiplication. On the left, a 4x5 matrix of yellow squares is multiplied by a 4x5 matrix of blue squares. Between the two matrices are four black dots, indicating they are multiplied together. To the right of the multiplication is an equals sign followed by a vertical vector of three orange squares.

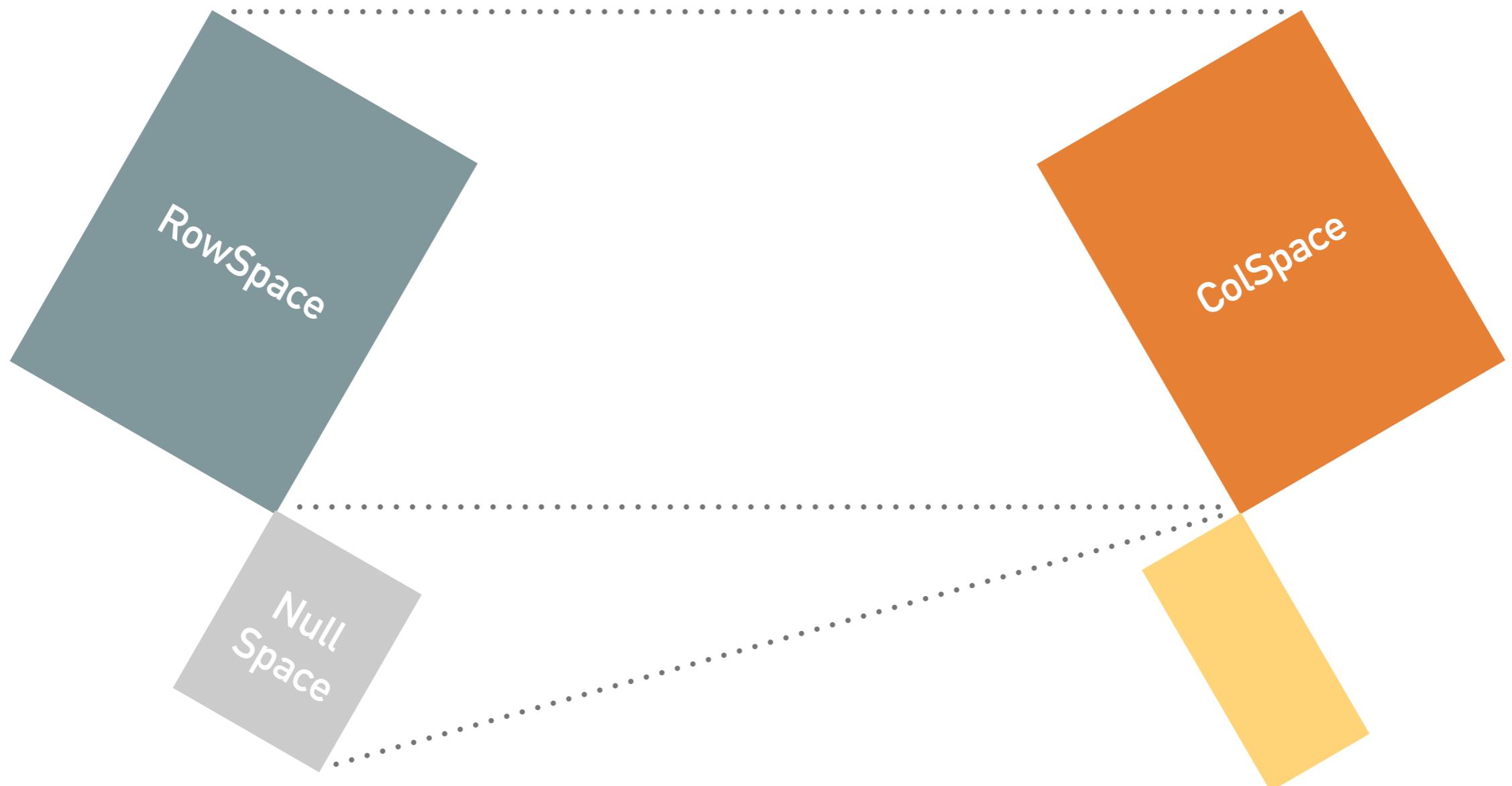
$$\mathbb{R}^n \xrightarrow{M_{m \times n}} \mathbb{R}^m$$

Column Space of the Matrix = Linear span of all Columns = Range

Row Space of the Matrix = Linear span of Rows = Non-zero output

Null Space of the Matrix = Map to zero = Orthogonal to Row Space

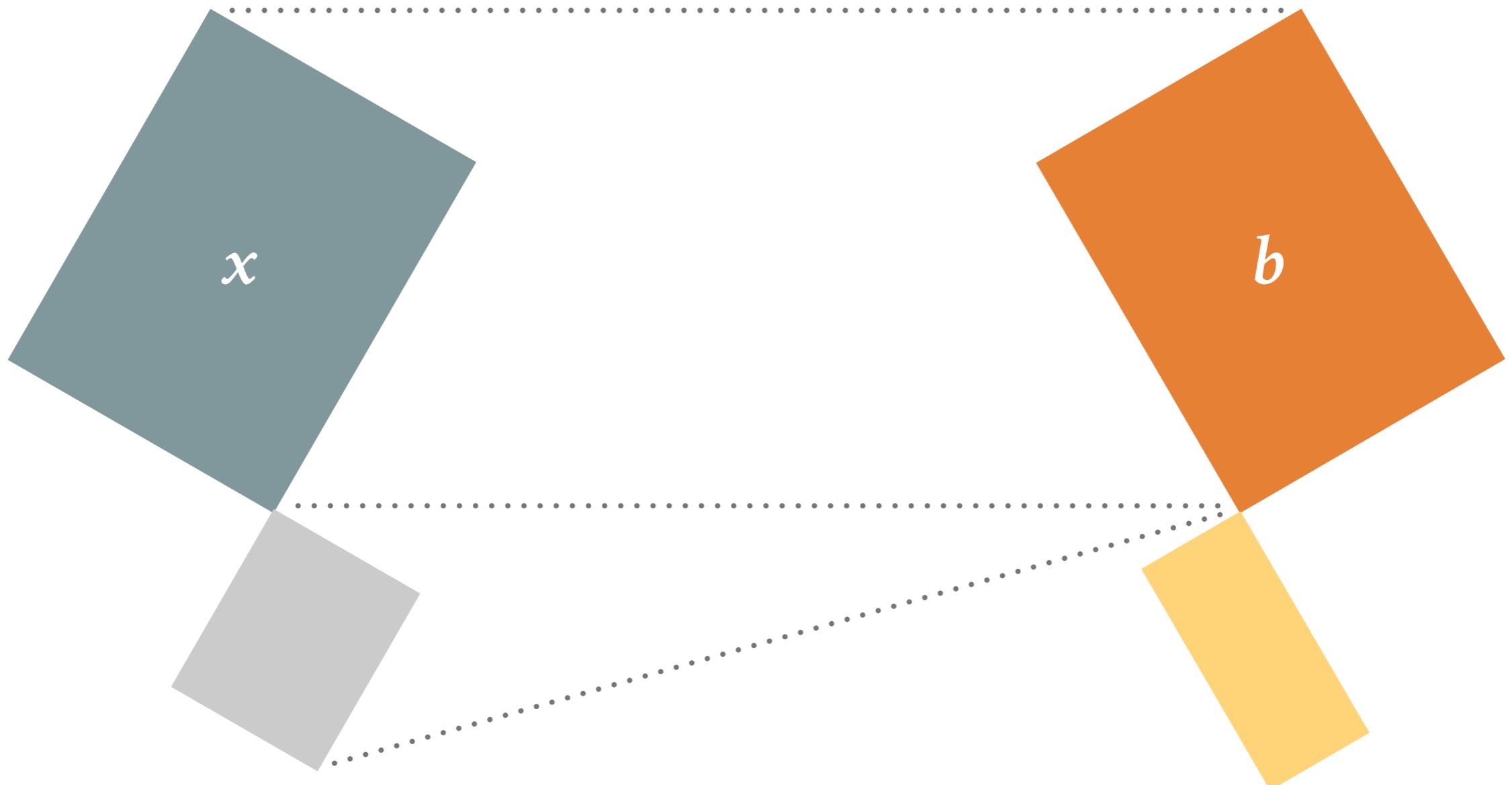
$$\mathbb{R}^n \xrightarrow{M_{m \times n}} \mathbb{R}^m$$



LINEAR EQUATIONS

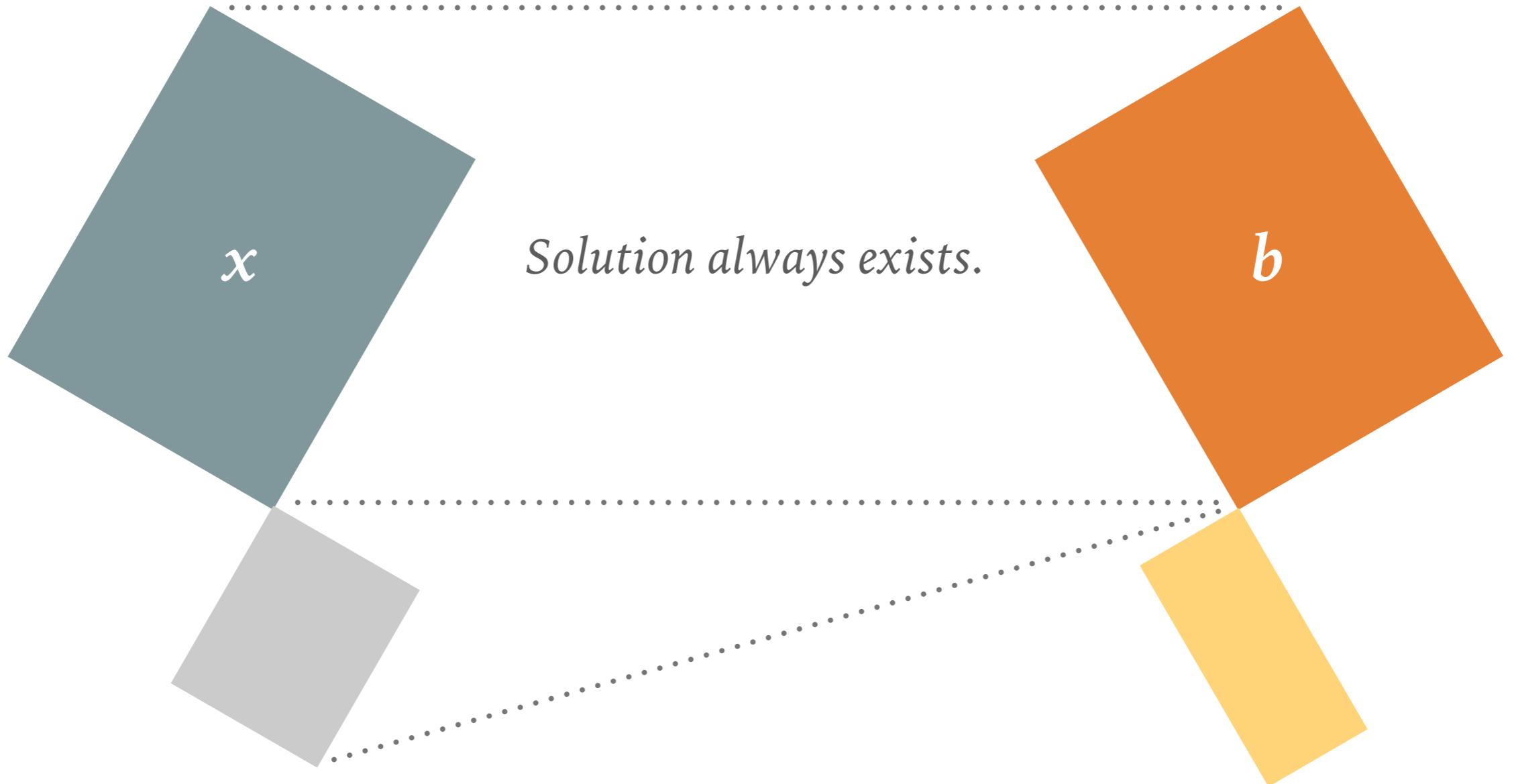
SOLVABILITY OF LINEAR EQUATION

$$Mx = b$$



SOLVABILITY OF LINEAR EQUATION

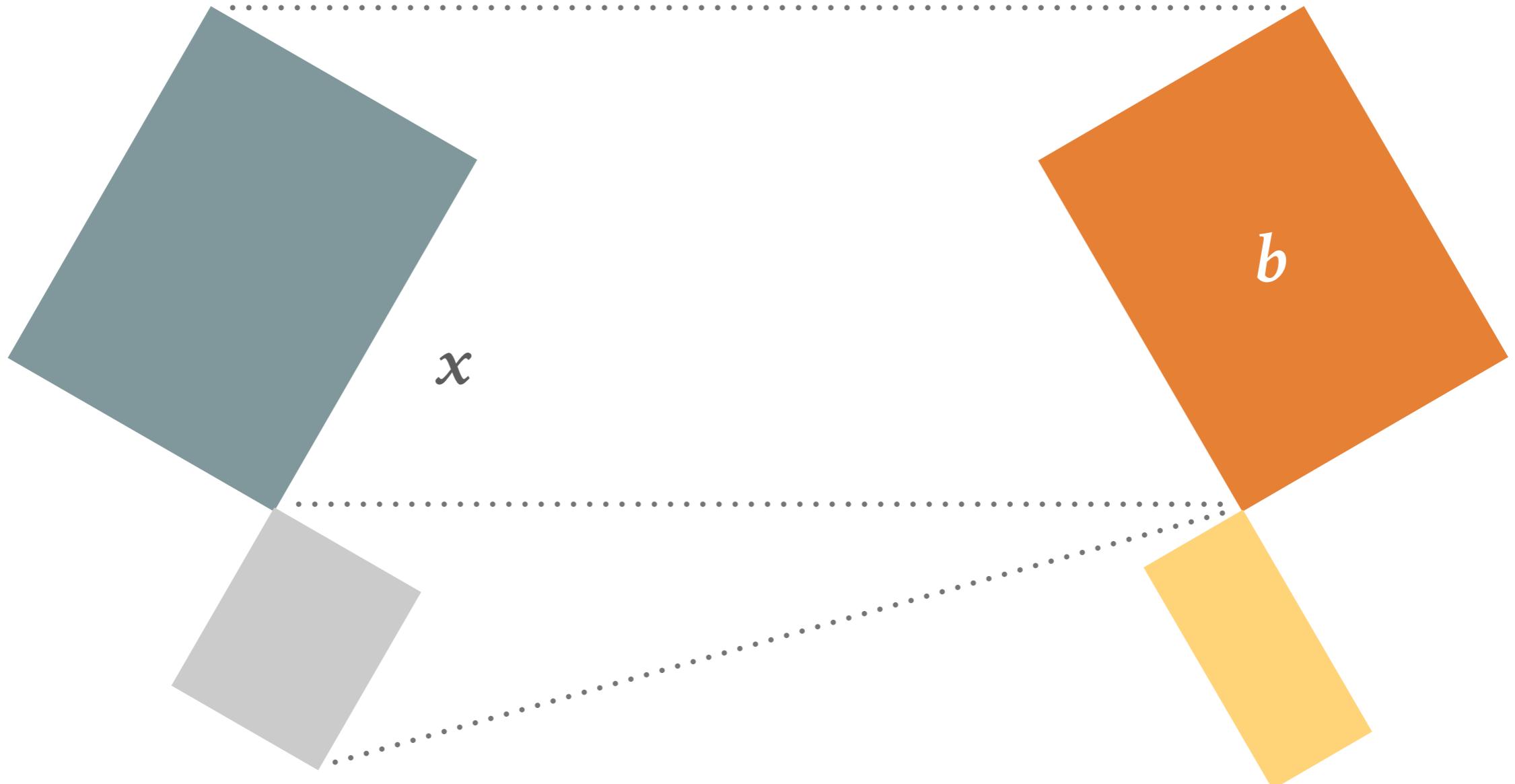
$$Mx = b$$



Solution is unique as $Mx = b$ and $My = b$ implies $(x-y) = 0$.

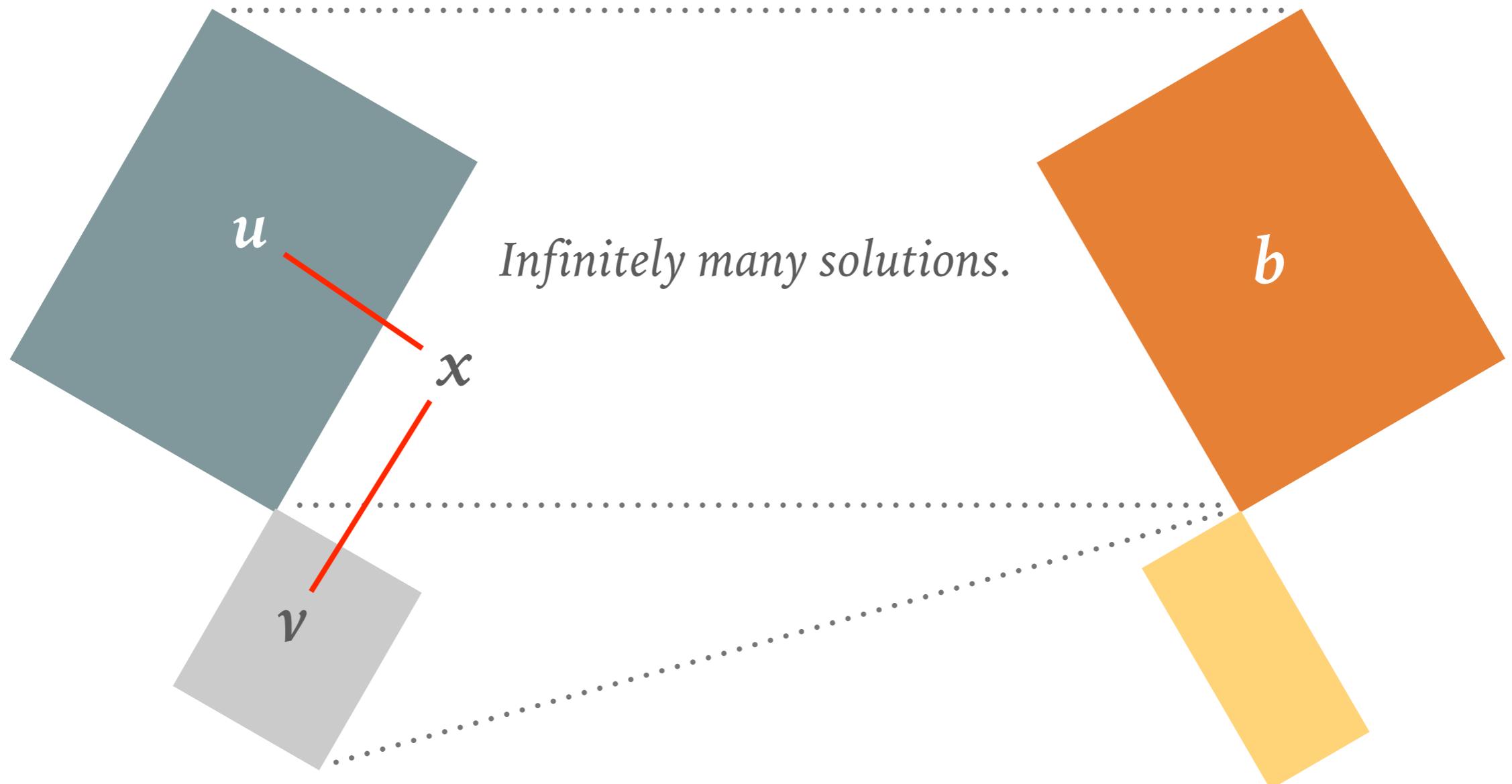
SOLVABILITY OF LINEAR EQUATION

$$Mx = b$$



SOLVABILITY OF LINEAR EQUATION

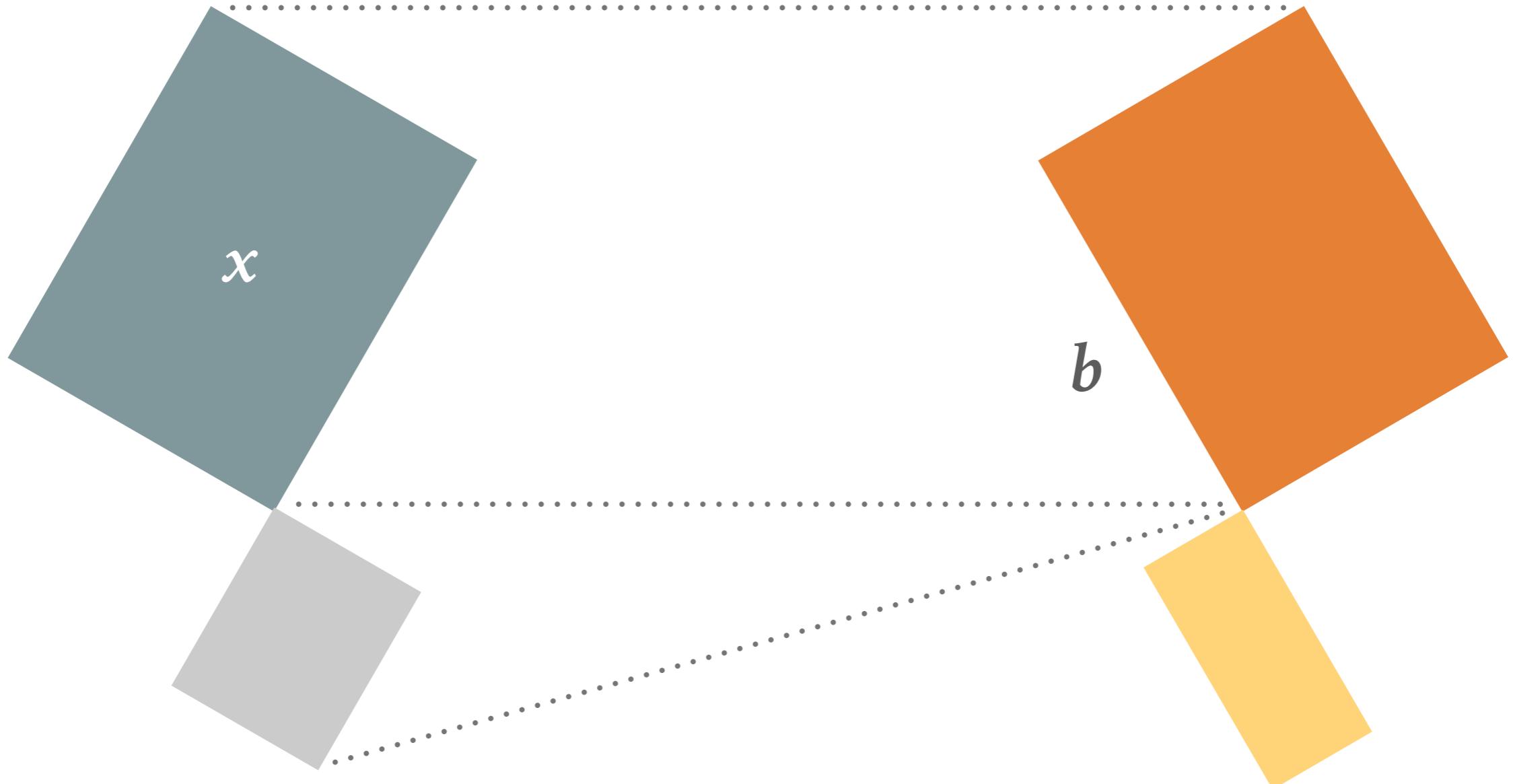
$$Mx = b$$



If $Mu = b$, then $M(u+v) = b$, for any v from Null Space.

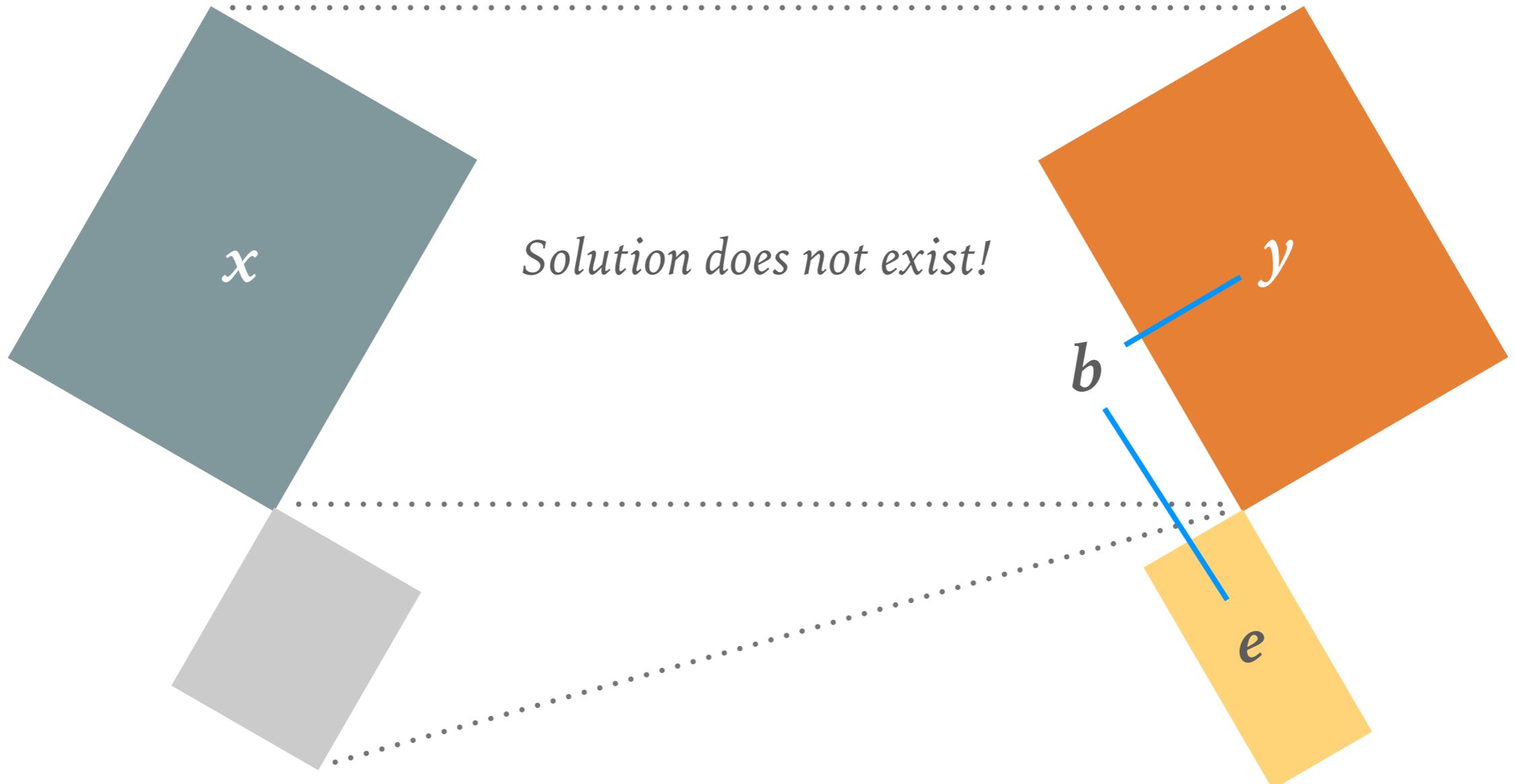
SOLVABILITY OF LINEAR EQUATION

$$Mx = b$$



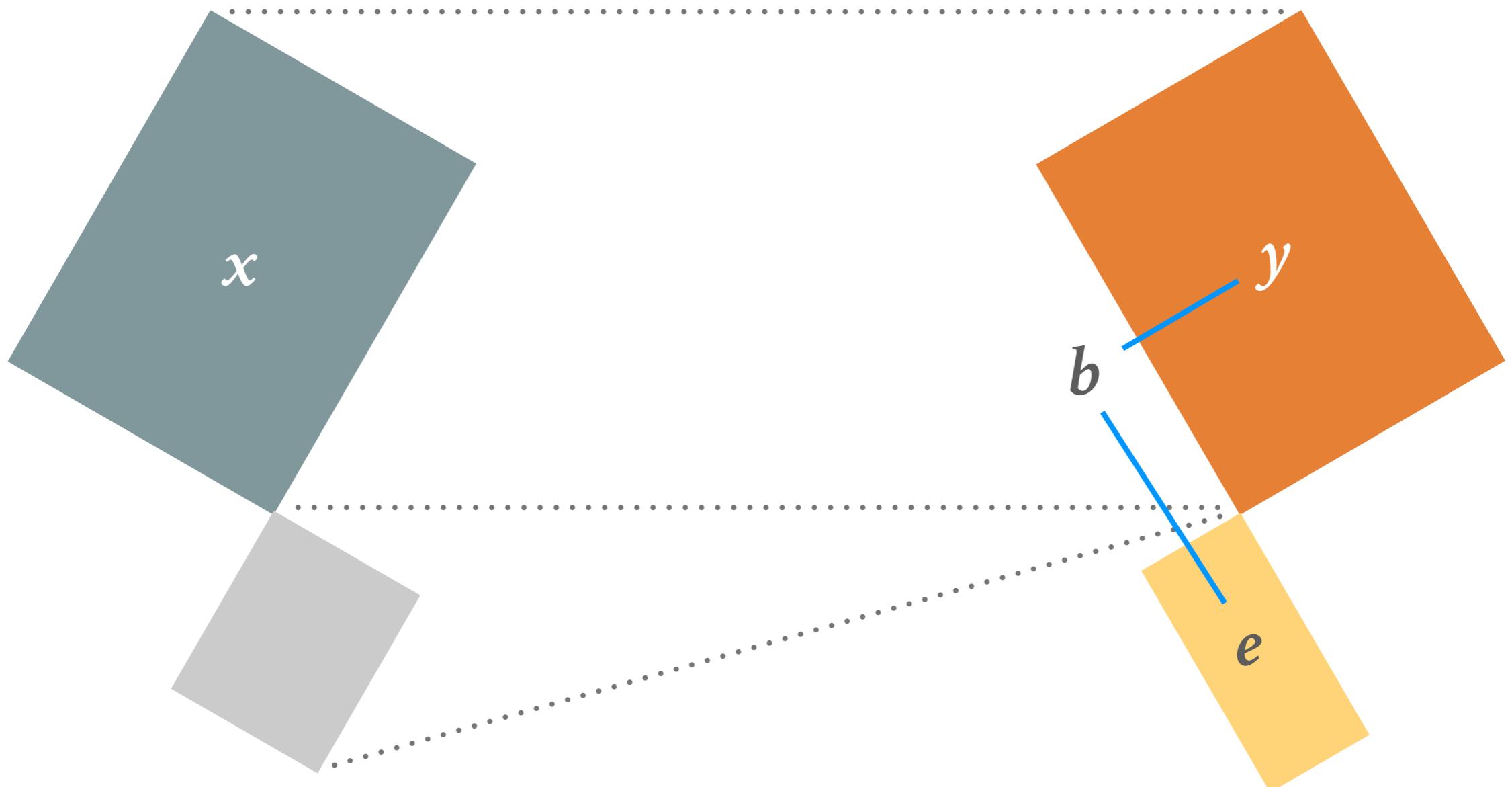
SOLVABILITY OF LINEAR EQUATION

$$Mx = b$$

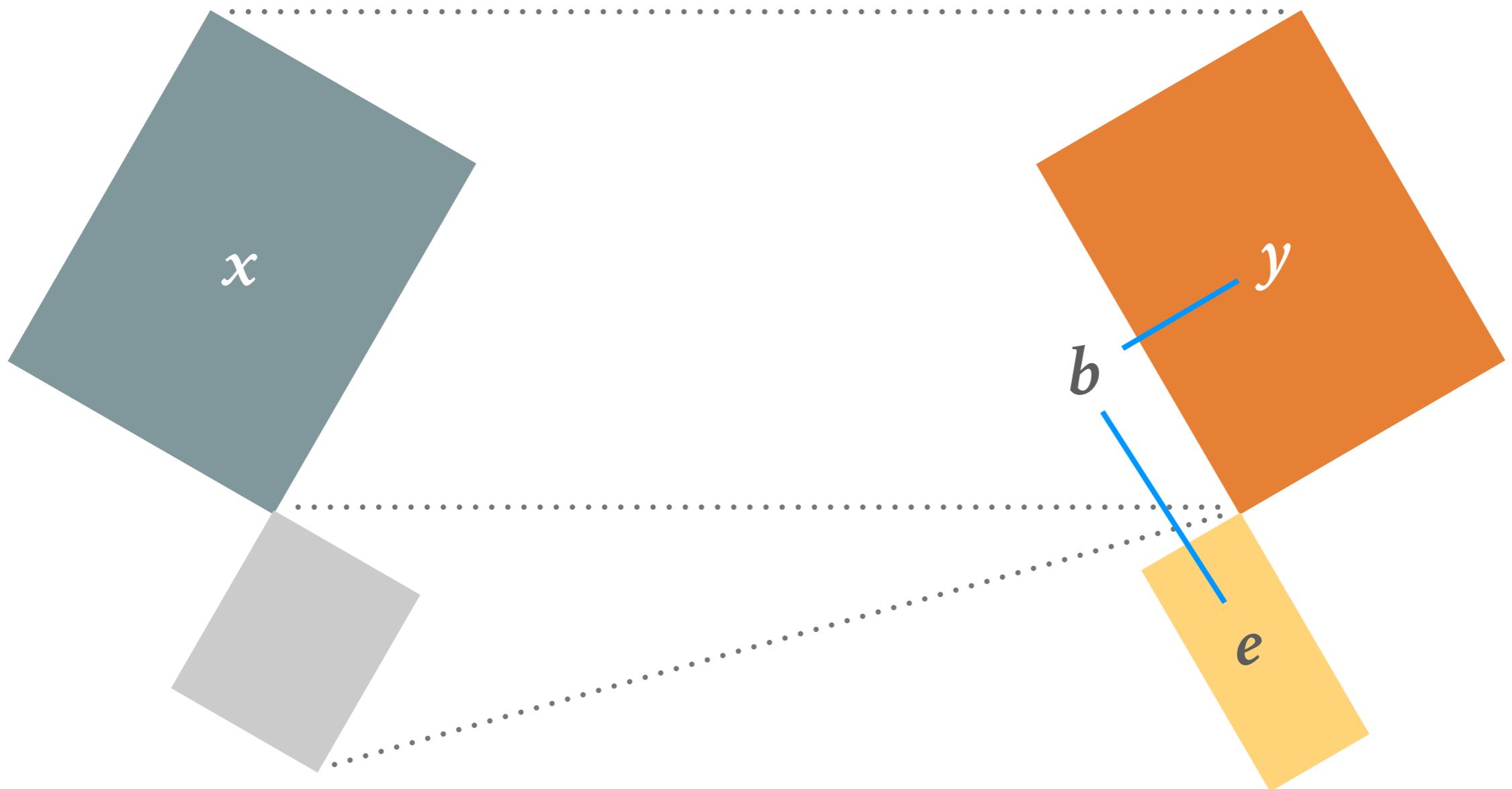


If $Mx = y$, then $Mx \sim b + e$, with irreducible error e .

$$\mathbb{R}^n \xleftarrow{\hspace{1cm}} M_{m \times n}^T \xrightarrow{\hspace{1cm}} \mathbb{R}^m$$



$$\mathbb{R}^n \xleftarrow{M_{m \times n}^T} \mathbb{R}^m$$



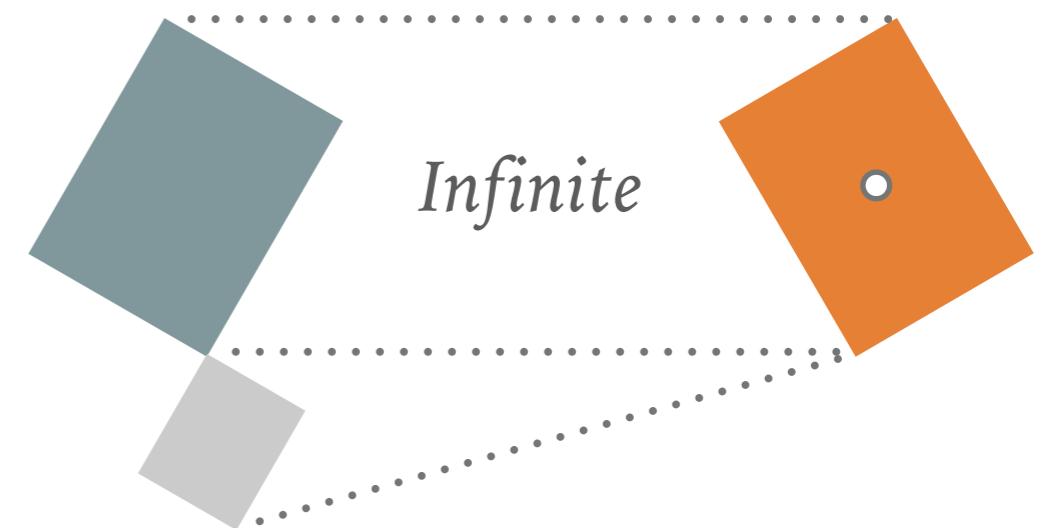
$M^T M x = M^T b = M^T y$ provides solution $x = (M^T M)^{-1} M^T y$

SOLVABILITY OF LINEAR EQUATION

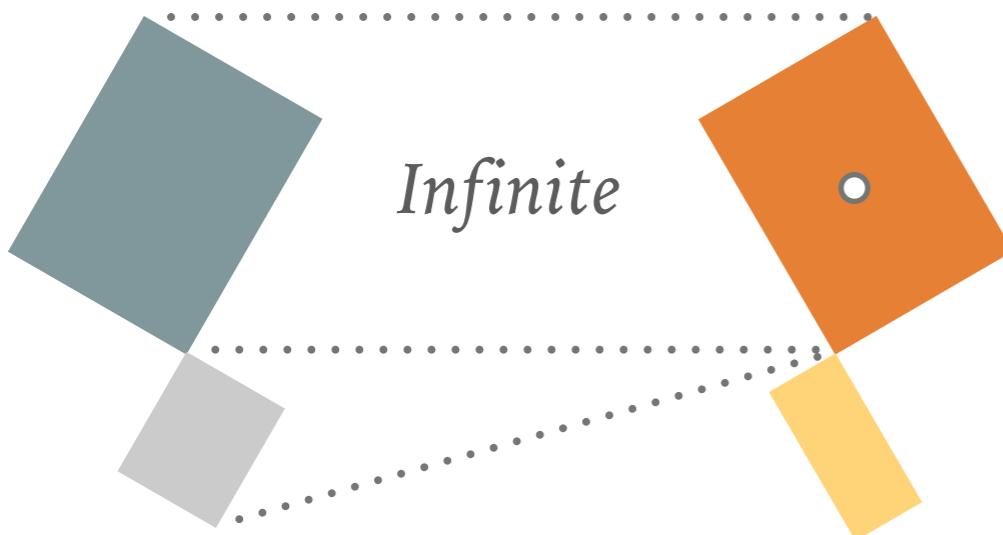
$$Mx = b$$



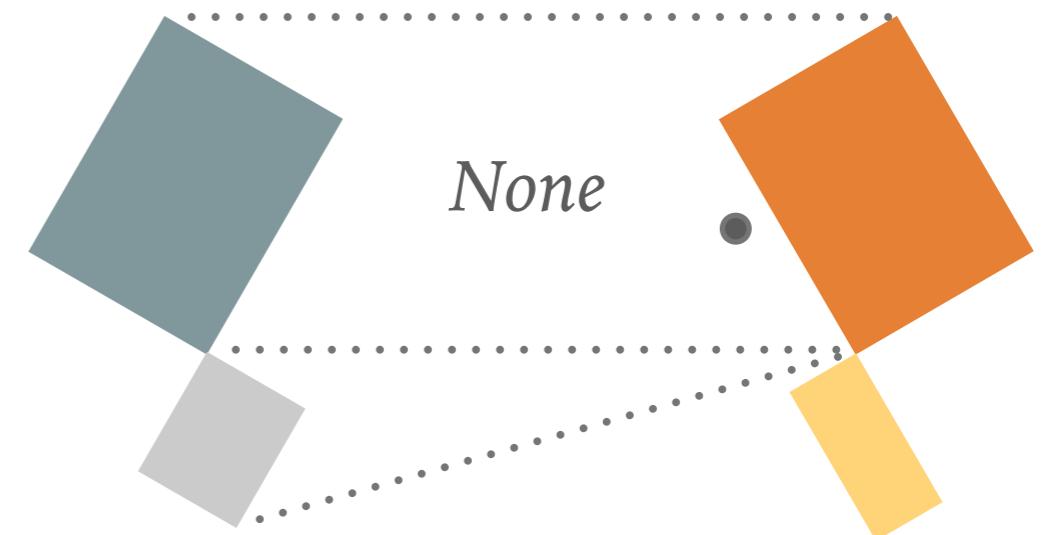
Unique



Infinite



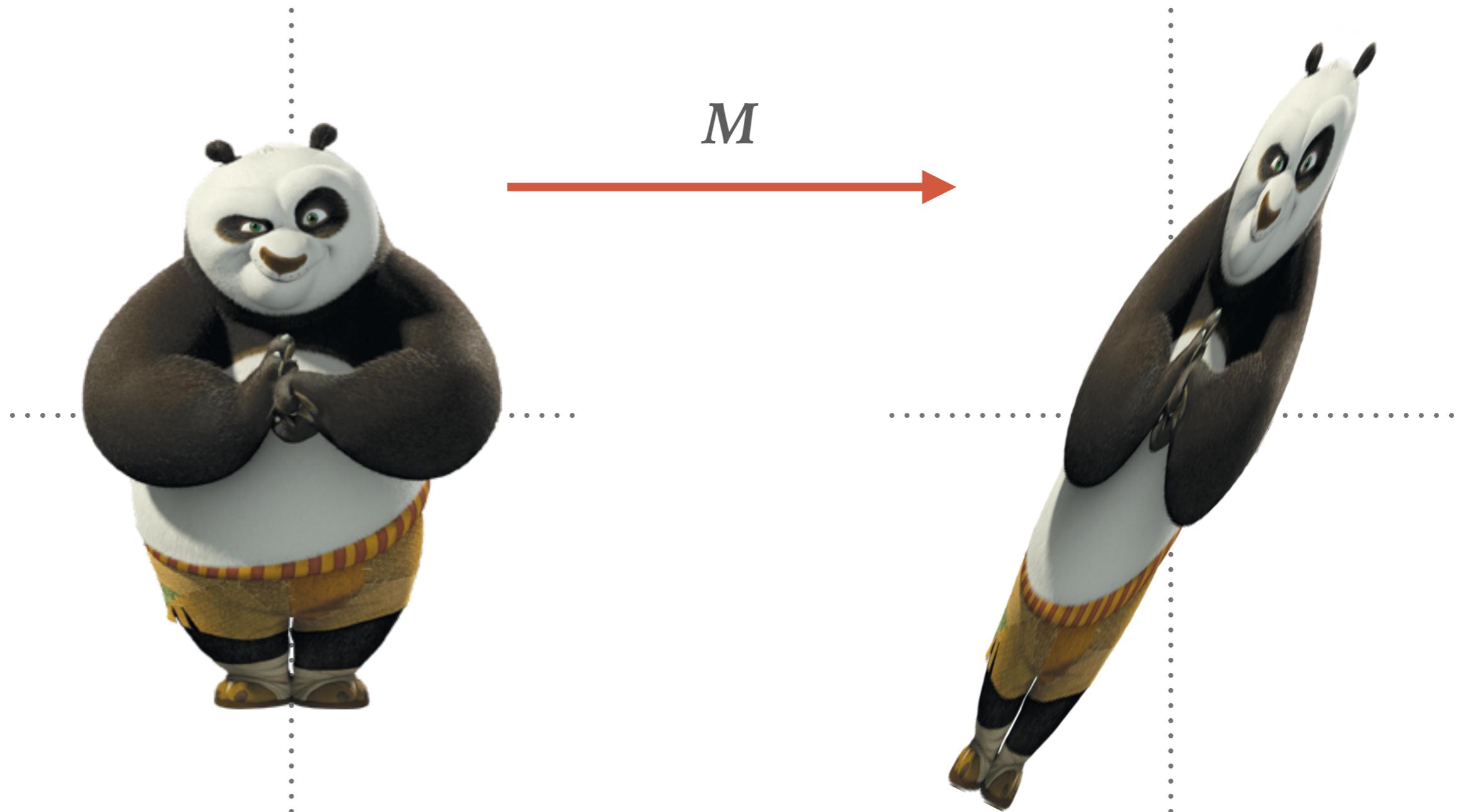
Infinite



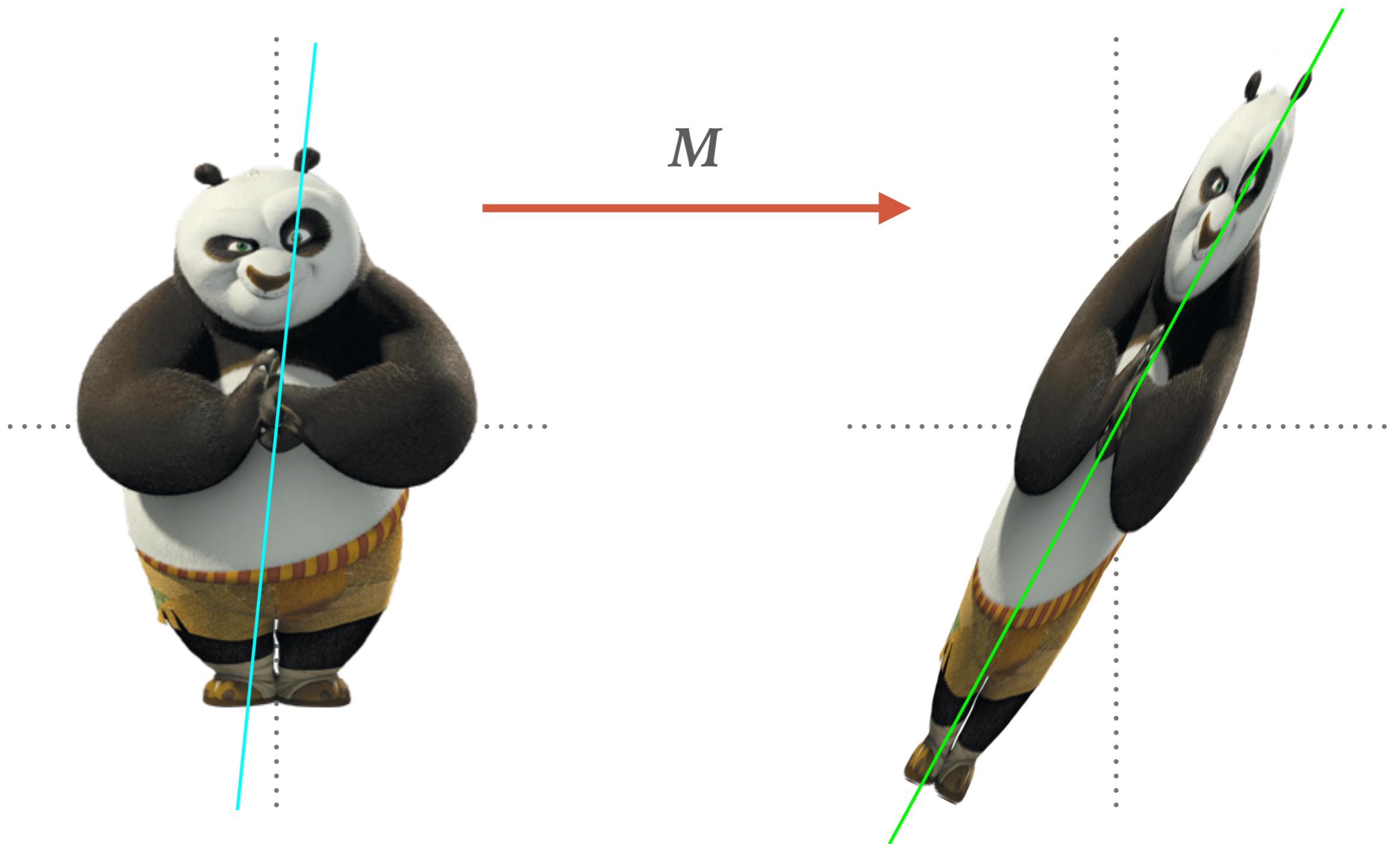
None

SINGULAR VALUES

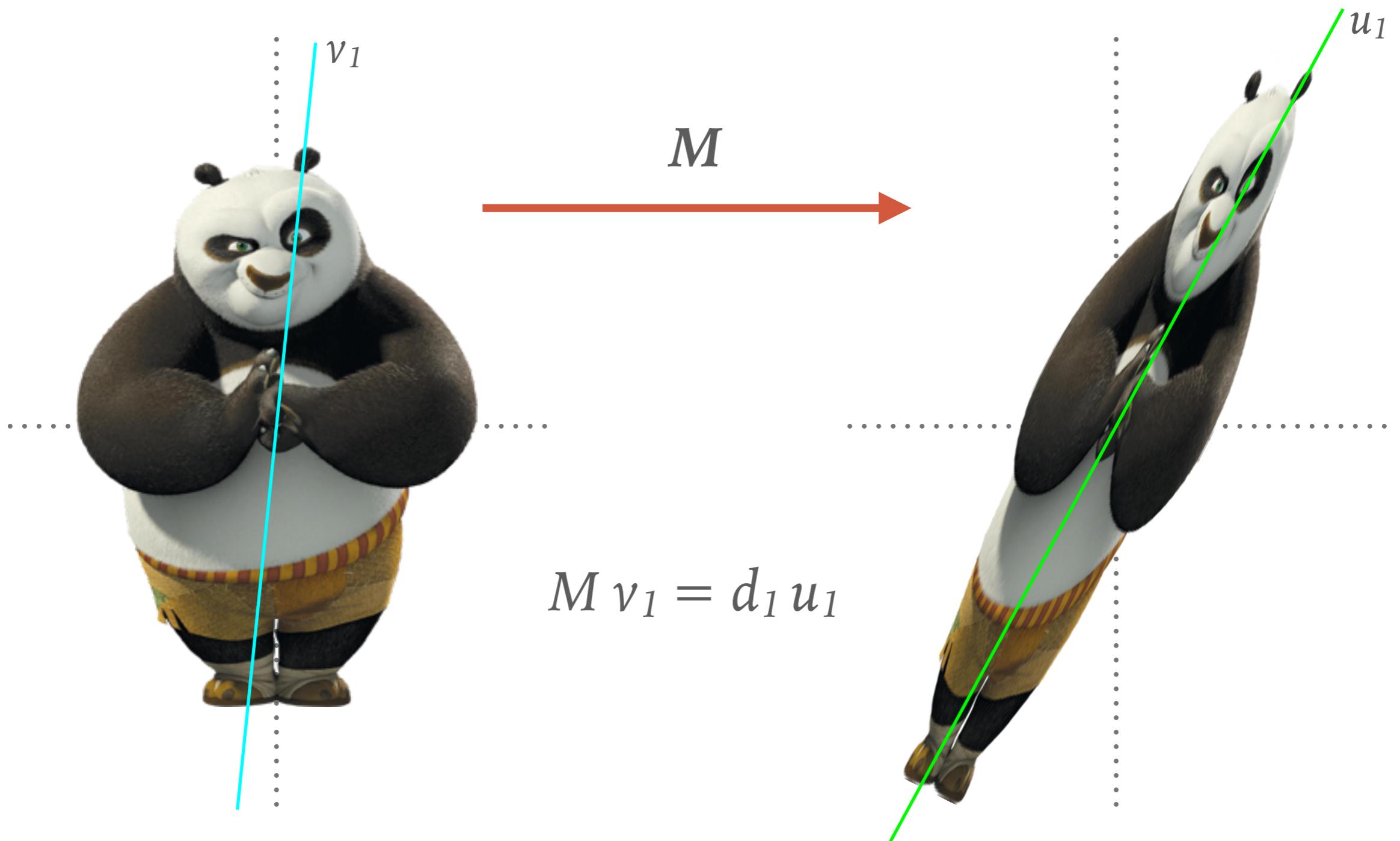
ACTION OF A MATRIX



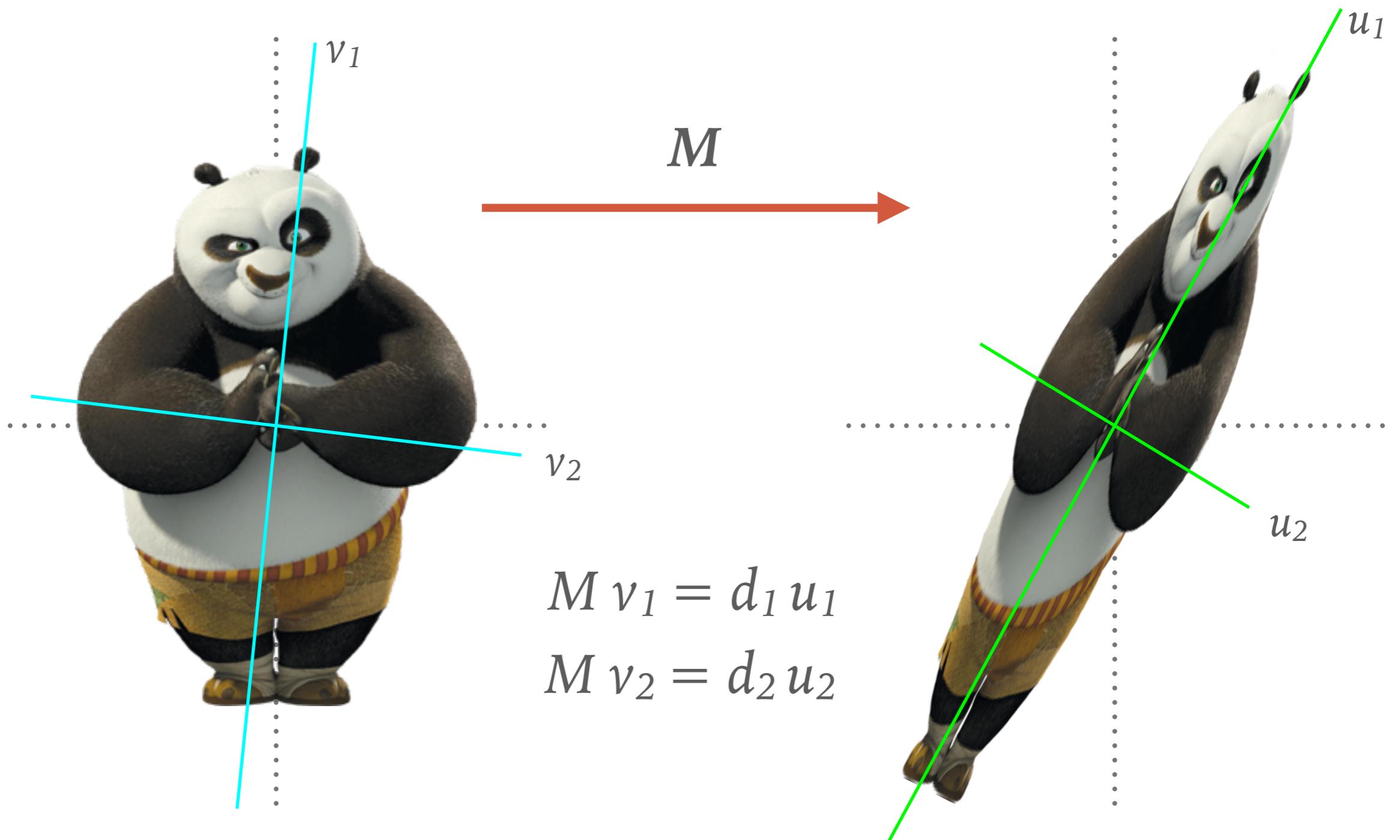
ACTION OF A MATRIX



ACTION OF A MATRIX



ACTION OF A MATRIX



SINGULAR VECTORS

$$M = v_1 \ v_2 \ v_3 \ v_4 \ v_5 \times u_1 \ u_2 \ u_3 \ u_4 \ u_5 = d_1 \ d_2 \ d_3 \ d_4 \ d_5$$

The diagram illustrates the Singular Value Decomposition (SVD) of a matrix M . The matrix M is shown as a 5x5 grid of light green squares. It is multiplied by the transpose of the matrix V , which is a 5x5 grid of dark grey squares. This product is then multiplied by the transpose of the matrix U , which is a 5x5 grid of orange squares. Finally, the result is multiplied by the diagonal matrix D , which has blue entries d_1, d_2, d_3, d_4, d_5 on its diagonal and white entries elsewhere.

SINGULAR VECTORS

$$M = u_1 \ u_2 \ u_3 \ u_4 \ u_5 \times d_1 \ d_2 \ d_3 \ d_4 \ d_5 \times v_1 \ v_2 \ v_3 \ v_4 \ v_5$$

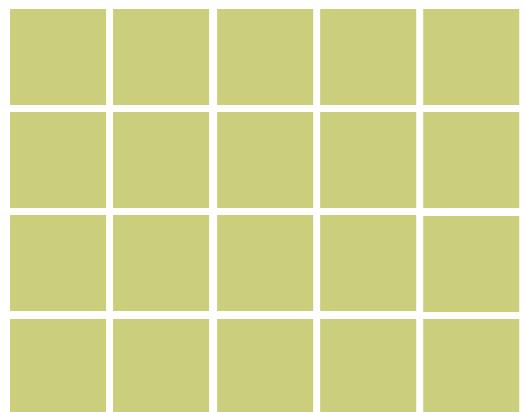
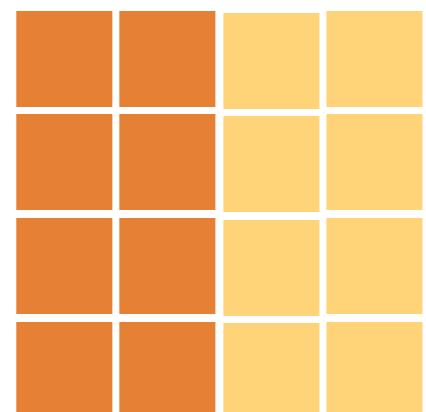
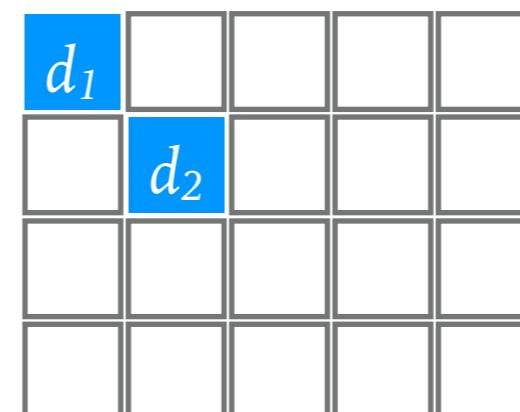
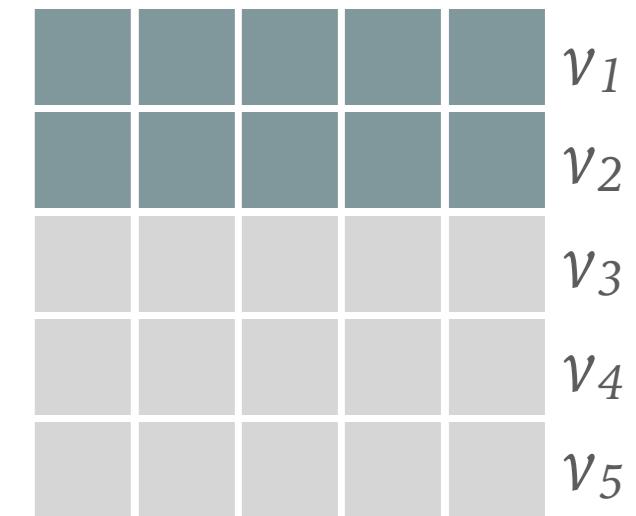
The diagram illustrates the Singular Value Decomposition (SVD) of a matrix M . The matrix M is shown as a 5x5 grid of light green squares. It is equal to the product of three matrices: U , D , and V . Matrix U is represented by a 5x5 grid of orange squares. Matrix D is a 5x5 diagonal matrix where the main diagonal elements are labeled d_1, d_2, d_3, d_4, d_5 in blue. Matrix V is represented by a 5x5 grid of dark grey squares. To the right of the matrices, the singular values $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are listed vertically, corresponding to the diagonal elements of D .

SINGULAR VALUE DECOMPOSITION

$$M = u_1 \ u_2 \ u_3 \ u_4 \times d_1 \ d_2 \times v_1 \ v_2 \ v_3 \ v_4 \ v_5$$

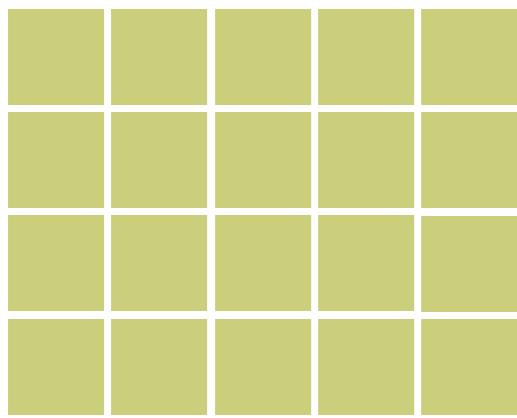
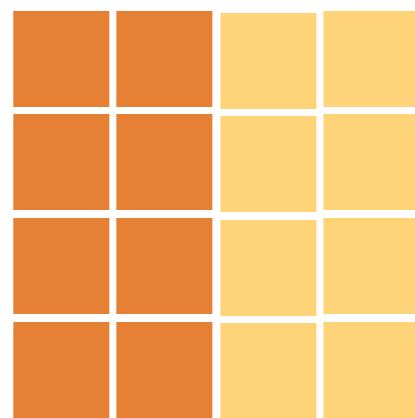
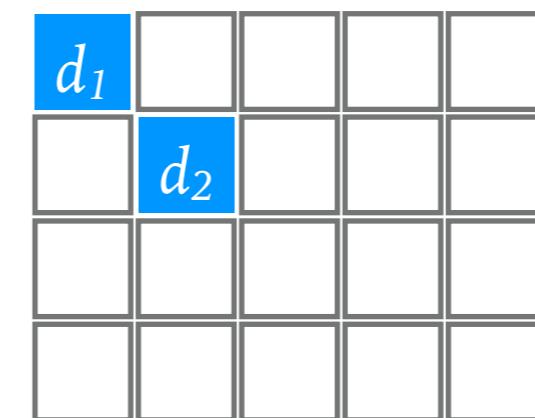
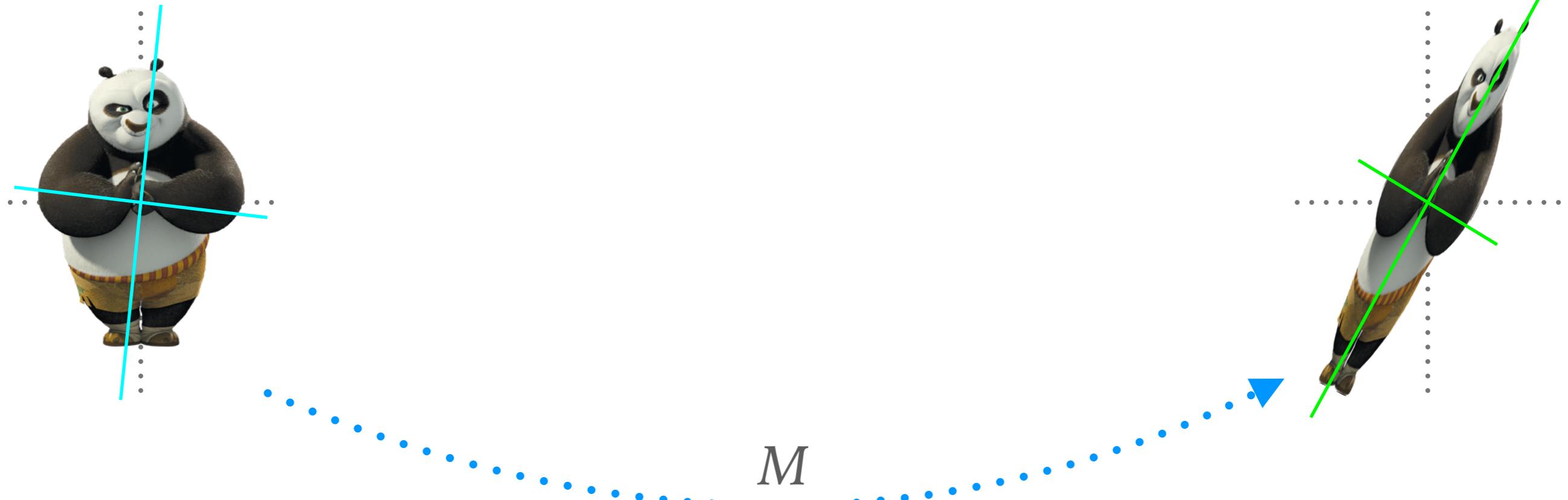
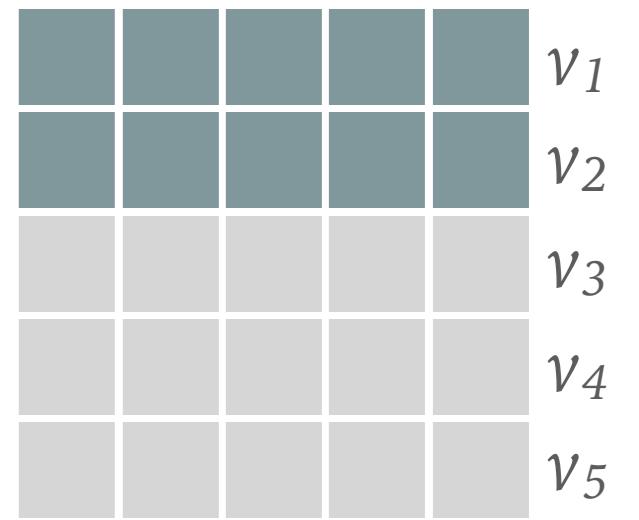
The diagram illustrates the Singular Value Decomposition (SVD) of a matrix M . The matrix M is shown as a 4x5 grid of light green squares. It is decomposed into three components: U , Σ , and V^T . The matrix U is represented by a 4x4 grid of orange squares, labeled $u_1 \ u_2 \ u_3 \ u_4$ above it. The matrix Σ is represented by a 4x5 grid where the first two columns are blue (labeled d_1 and d_2) and the remaining three columns are white. The matrix V^T is represented by a 5x5 grid of grey squares, labeled $v_1 \ v_2 \ v_3 \ v_4 \ v_5$ to its right.

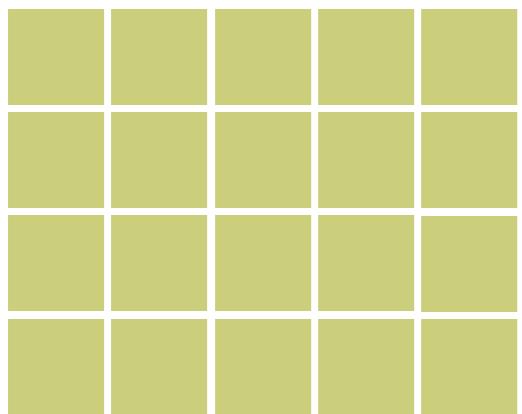
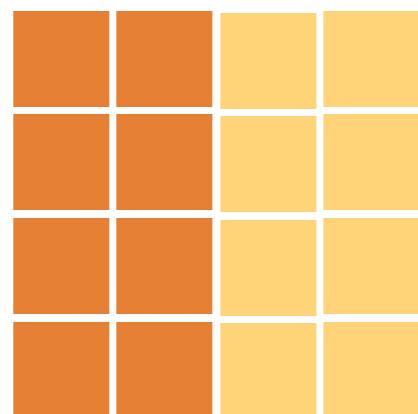
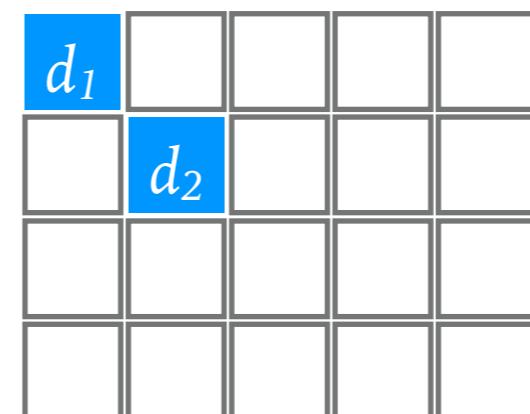
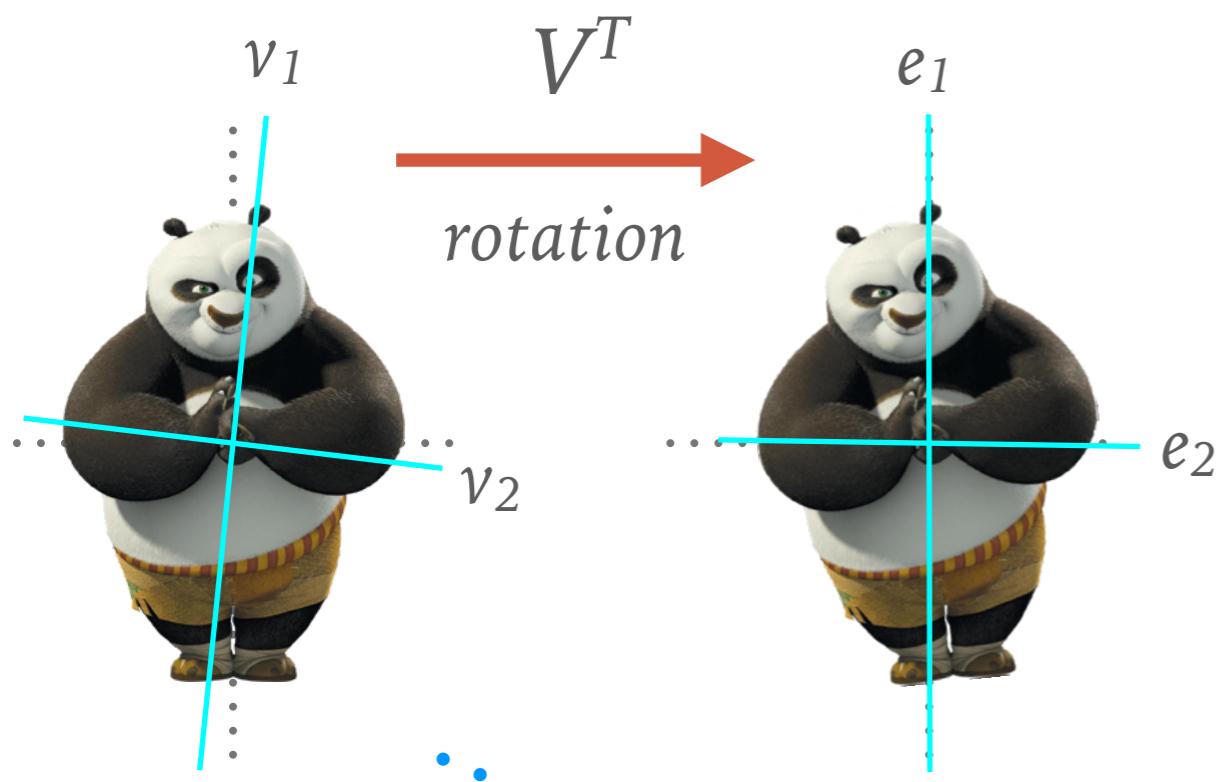
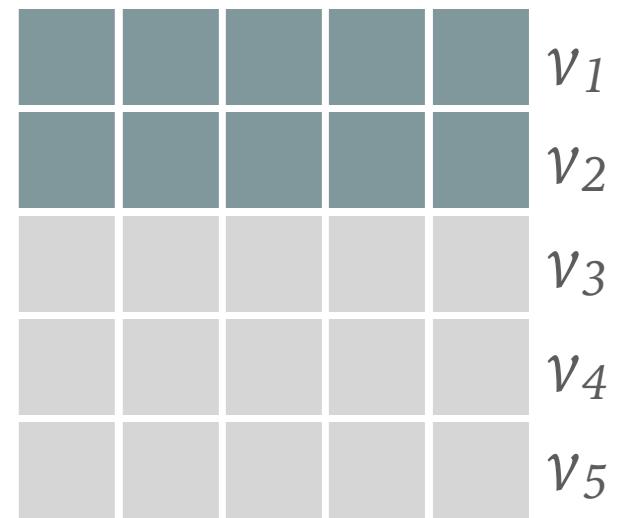
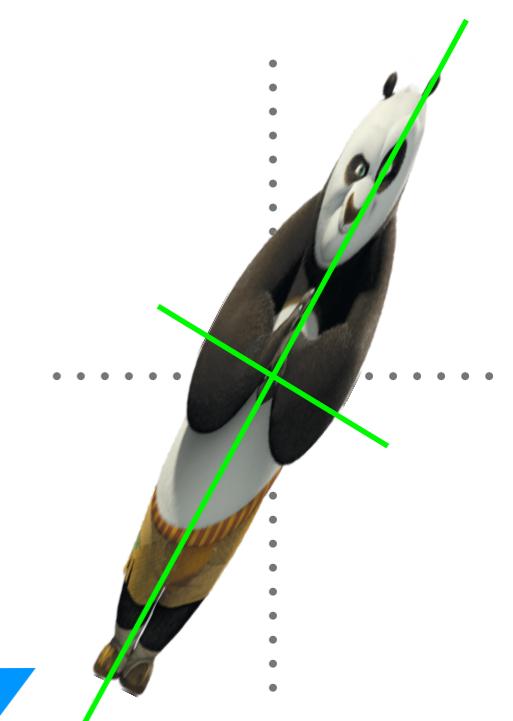
$$M = U\Sigma V^T$$

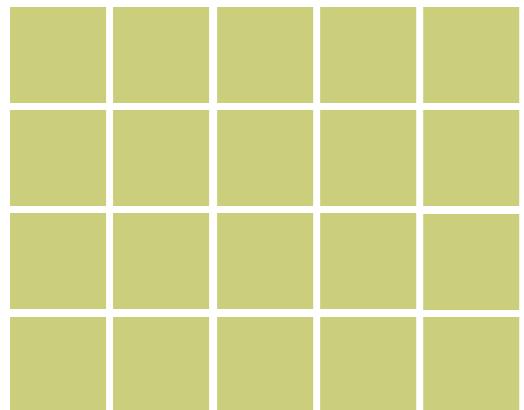
M  $u_1 \ u_2 \ u_3 \ u_4$ $=$  X  X  v_1
 v_2 *RowSpace* v_3
 v_4
 v_5

$$M = U\Sigma V^T$$

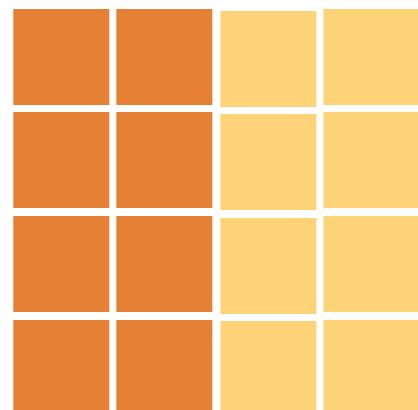
 u_1
 u_2 *ColSpace* u_3
 u_4

M  $u_1 \ u_2 \ u_3 \ u_4$ $=$  \times  \times 

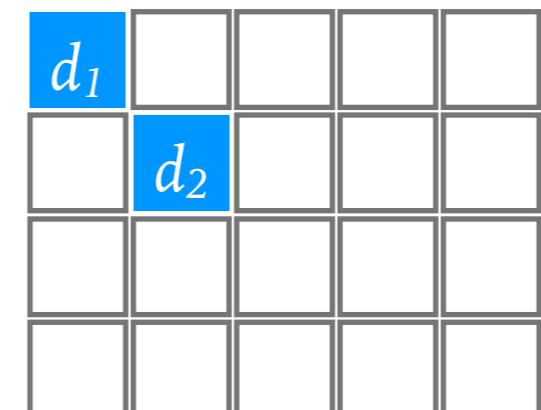
M  $u_1 \ u_2 \ u_3 \ u_4$ $=$  \times  \times  M 

M $u_1 \ u_2 \ u_3 \ u_4$ 

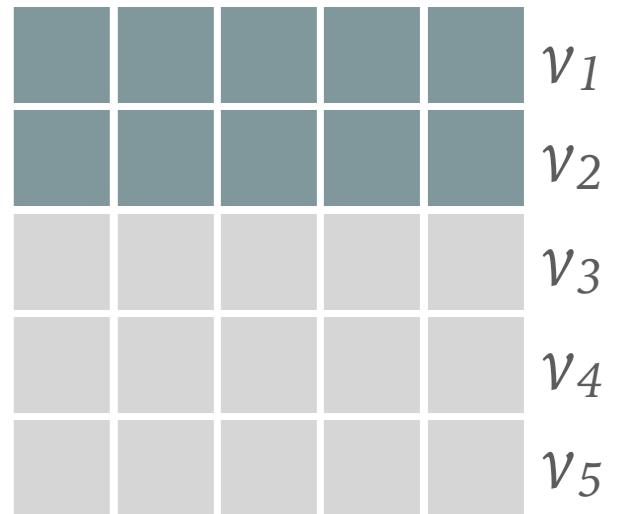
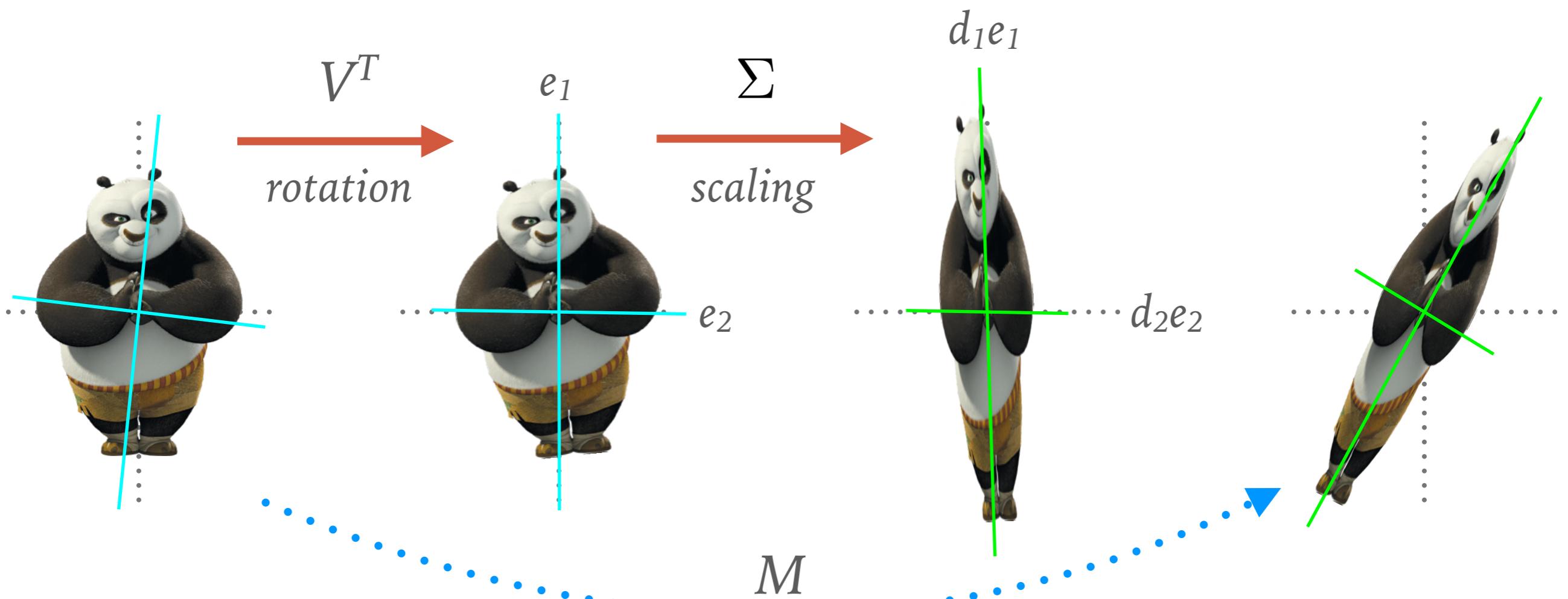
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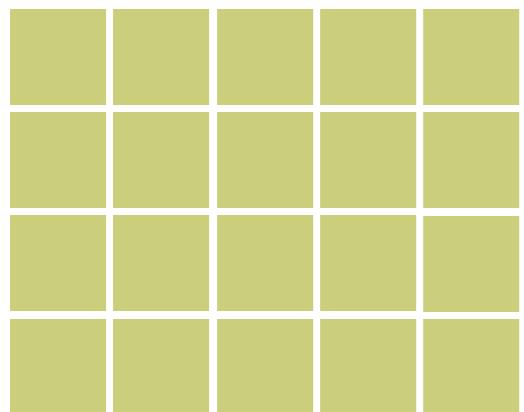


X

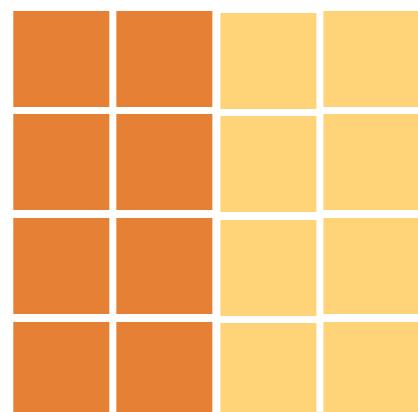


X

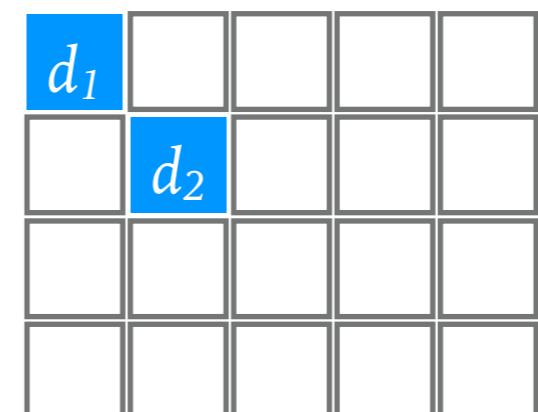
 v_1
 v_2
 v_3
 v_4
 v_5 

M $u_1 \ u_2 \ u_3 \ u_4$ 

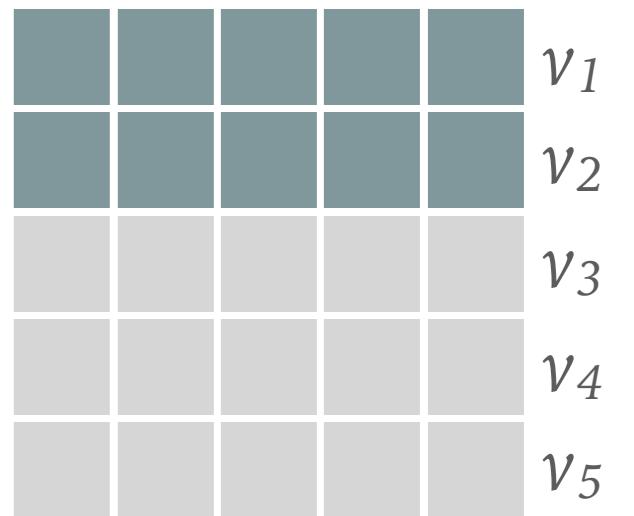
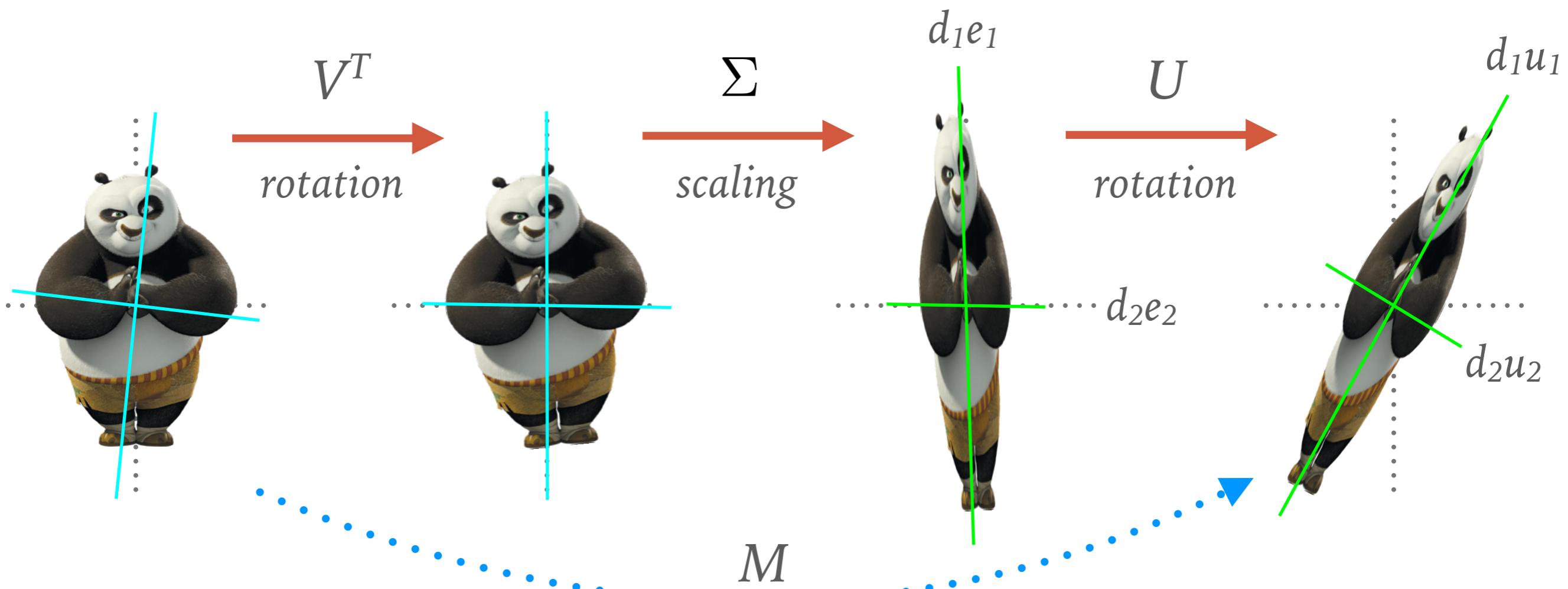
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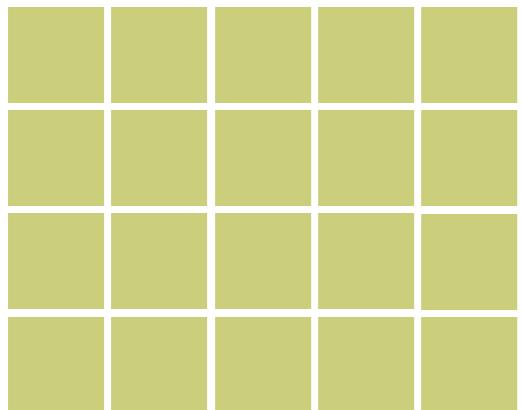


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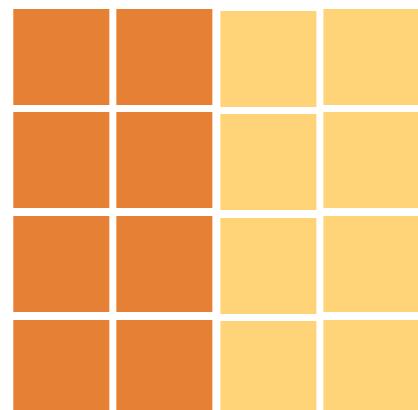


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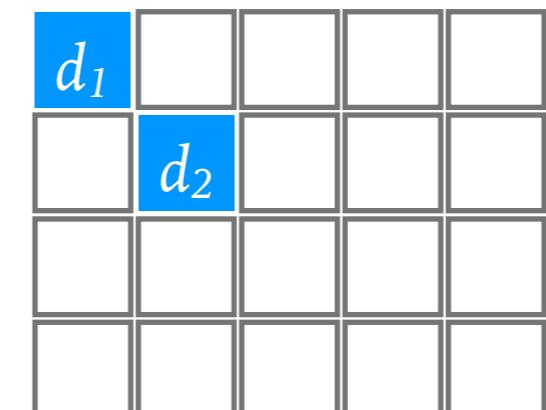
 v_1
 v_2
 v_3
 v_4
 v_5 

M $u_1 \ u_2 \ u_3 \ u_4$ 

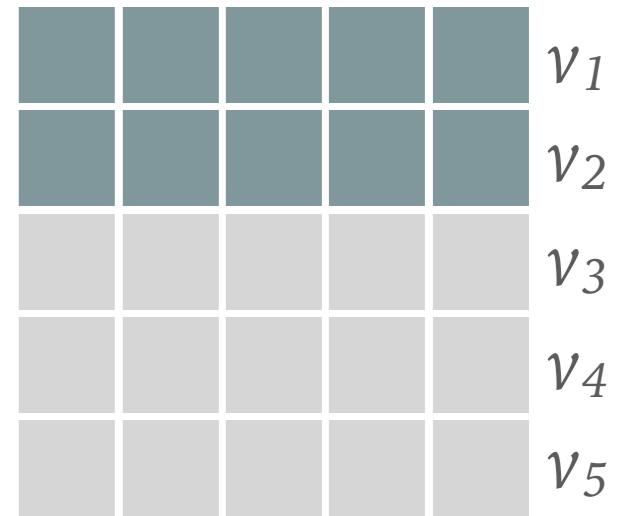
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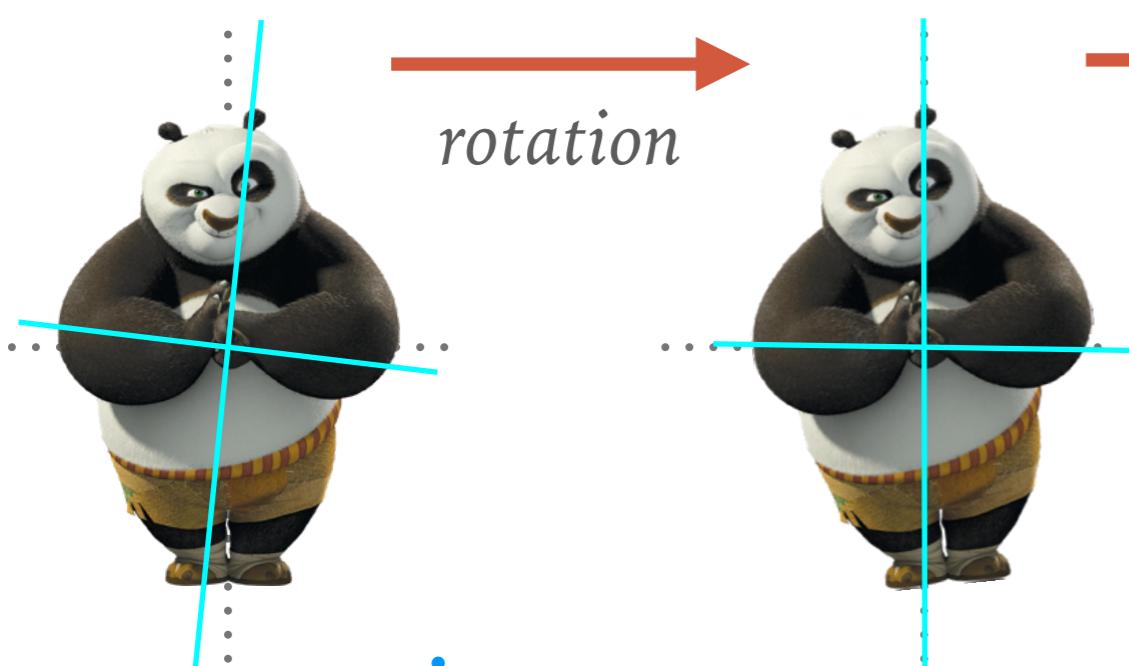
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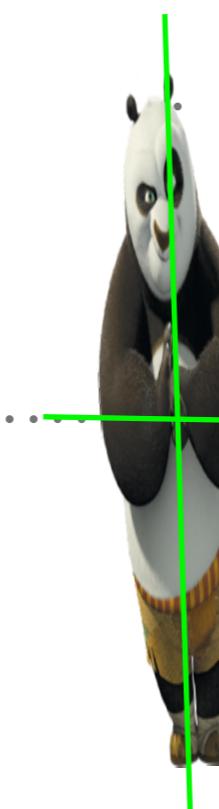
X

 v_1
 v_2
 v_3
 v_4
 v_5 V^T 

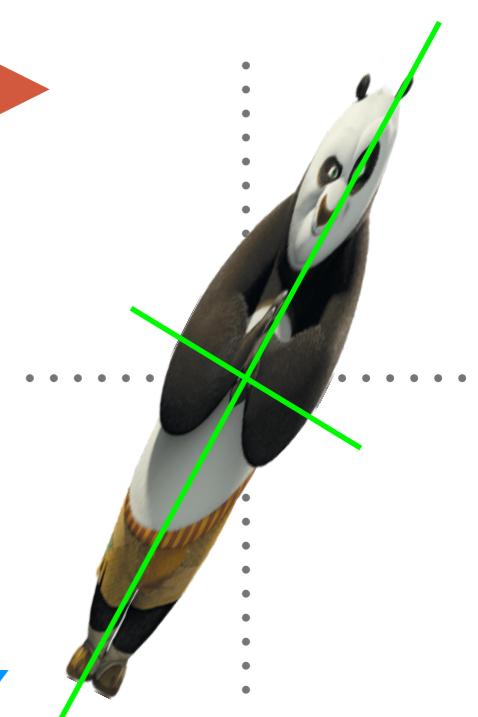
rotation

 Σ 

scaling

 U 

rotation

 M

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