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HANDBOOK FOR

MATHEMATICAL OLYMPIADS

Dr. MRINAL NANDI

Ph.D. and M.Stat. from ISI

M.Sc. (Pure Math) and B.Ed. from C.U.

Assistant Professor, Statistics
West Bengal State University

Dr. SOUROV SEN GUPTA

Ph.D. (CS), M.Math. (Pure Math), B.E.

Lecturer, Cryptology and Security
Indian Statistical Institute

ACADEMIC PUBLISHERS
Kolkata 700073, www.academicpublishers.in

To our students, who taught us how to teach.

Preface

The Mathematical Olympiads are the toughest grounds for high-school students to exhibit their abilities in mathematics. These tests are not restricted to a specific grade; they allow students from class IX to XI to write the tests. In some cases, exceptionally strong class VIII students are also permitted to appear for the exams. Thus, the questions at the Olympiads are not bound to any set syllabus, and generally cover all aspects of mathematics that a bright high-school student should know. The material to be covered for these tests are often scattered over several classes, multiple books and some topics are not even taught in school as a part of the regular curriculum. This poses the hardest challenge for the students.

The Regional Mathematical Olympiad (RMO) is a state level examination, which acts as a stepping stone towards the national level Indian National Mathematical Olympiad (INMO), and subsequently to the International Mathematical Olympiad (IMO). In our experience of conducting numerous training camps for RMO and INMO over the last few years, we have come across many good books for IMO and a few good books targeted towards INMO. However, to the best our knowledge, there exists no textbook of mathematics that targets only the state-level RMO examinations. Quite often we have found the young students, especially from classes VIII to X, lost in the huge bulk of material required to prepare for INMO and IMO, resulting in intimidation, disinterest, and loss of focus. We attempt at consciously summarizing all relevant topics required to prepare for the RMO, and no more.

For the students interested in learning beyond the scope of the RMO examination, we provide motivating problems from past INMOs, but do not necessarily discuss all relevant materials to attempt those problems. One may consult the books suggested in the References section on the next page for further learning in this direction.

Any feedback, criticism and comments on this book is welcome! Please send us your valuable feedback at `mrinal.nandi1@gmail.com` and `sg.sourav@gmail.com`.

Kolkata, July 2014 — Mrinal Nandi and Sourav Sen Gupta

Acknowledgments

We extend our sincerest gratitude to Prof. Pradipta Bandyopadhyay, the Dean of Studies, Indian Statistical Institute, and Dr. Mridul Nandi, Applied Statistics Unit, ISI Kolkata, for kindly sharing interesting problems and instructional material from their RMO training repositories, later included in parts of this book.

The book would not have been possible without the encouragement and support from Prof. Bimal K. Roy, the Director, Indian Statistical Institute, and Prof. Subhamoy Maitra, Applied Statistics Unit, ISI Kolkata; we are grateful to them. We also convey our gratitude to the Joint Regional Coordinators of the Mathematical Olympiads from West Bengal – Prof. Rana Barua, Stat-Math Unit, ISI Kolkata, and Dr. Sumitra Purkayastha, Applied Statistics Unit, ISI Kolkata, for several invaluable discussions and consultations during the tenure of this project.

We thank our families and friends for their ardent support and inspiration, and for resting their unflinching faith in us at every step of our lives. We would especially like to convey our gratitude to Mr. Pradip K. Nandi, father of the first author, for a thorough proof-read of the manuscript, and for providing constructive feedback.

We are indebted to Mr. Bimal K. Dhar, Dr. Subhankar Dhar, Mr. Dipankar Dhar, and the team of Academic Publishers, without whose support this book would not have been a reality. Special thanks to our students Samantak, Saptarshi, Sagnik, Monosij, Dipranjan and Diptaran, who helped us in proof-reading the manuscript.

The experience of conducting numerous training camps for RMO form the backbone for this project. We would like to thank the organizers of many such training camps at Kolkata, Siliguri, Burdwan, Barasat, and Narendrapur for providing us with these opportunities. Last but not the least, we would like to thank all the students whom we have taught over years; and in turn, who have taught us how to teach.

References

The following books are highly recommended for students who are preparing for the Mathematical Olympiads, and in particular, for the RMO examination.

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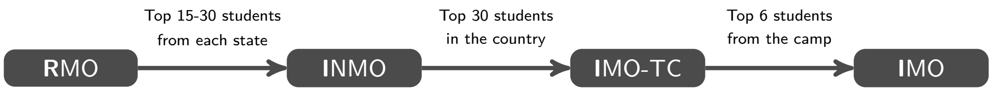
Contents

Before you begin . . .

The Mathematical Olympiads

International Mathematical Olympiad (IMO) is one of the toughest grounds for high-school students to exhibit their abilities in mathematics. Students of class IX to XI from all over the world, typically six representing each country, take part in this competition every year. To be a part of the six-member Indian team to take part in IMO, one has to qualify the Regional Mathematical Olympiad (RMO) of his/her respective state, and subsequently the Indian National Mathematical Olympiad (INMO) held uniformly across the country. Typically, 30 students are selected through INMO to take part in a month-long IMO Training Camp (IMO-TC), and finally the six-member team to represent India in the IMO is chosen from this camp.



Preparation for the Olympiads

The Olympiads are not restricted to a specific grade; they allow students from class IX to XI to write the tests. In some cases, exceptionally strong class VIII students are also permitted to appear for the exams. Thus, the questions at the Olympiads are not bound to any set syllabus, and generally cover all aspects of mathematics that a bright high-school student should know. The material to be covered for these tests are often scattered over several classes, multiple books and some topics are not even taught in school as a part of the regular curriculum. We attempt at summarizing the broad topics that are required for taking the RMO examination.

To the student readers

This is *not* a book that only presents and solves some Olympiad problems; this is meant to be a textbook for the Olympiads, where the *text* is of prime importance.

The art of solving problems and to build a solid foundation in basic mathematics should be the fundamental objective of any high-school student, and we want to promote this idea throughout the book. It is imperative that the students read the textual part of each chapter carefully, and closely study the problem solving methodology in the worked-out examples, before jumping into the exercise problems.

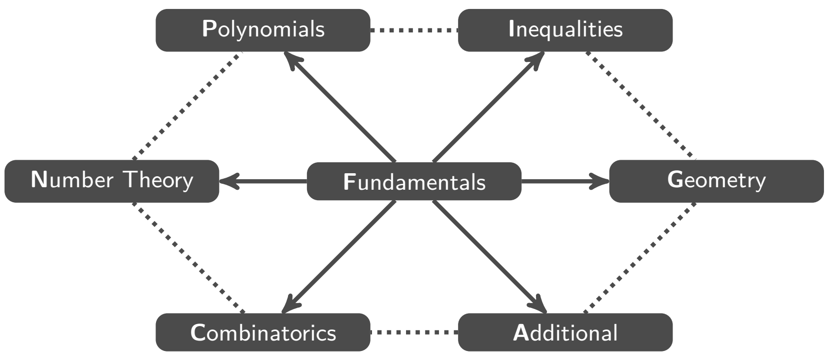
To start reading this book, students should have a basic understanding of school-level mathematics till class VIII, and it is expected that they would simultaneously cover the school-level mathematics syllabus of up to class X while reading this book. We shall not discuss in details any topic which is in the regular school-level syllabus, unless there is any ambiguity in different text books. Our objective is to prepare a high-school student (typically from class IX or X) for the Regional Mathematical Olympiad (RMO) by extending the limits of his/her mathematical knowledge beyond the regular school syllabus.

Organization of the book

The Mathematical Olympiads focus on six prime topics – Number Theory, Polynomials, Inequalities, Geometry, Combinatorics and a mixed bag of additional topics. In the RMO examination, the students are asked to solve around 6 to 7 problems as well, typically one from each topic.

The preparation for this six-part examination should also be in six phases; hence the name of the book, $hexad$ (set of six). In this competition, a student need not be the master of all six topics, but it is expected that he/she has mastered at least three of the six, and has a basic understanding of the other three. In classes IX to XI, it suffices to solve about 3 to 4 problems in order to qualify the RMO.

We have organized this book in six main chapters, one each for the topics mentioned above. In addition, we have included some fundamental notions in the first chapter to build the foundation. It is imperative that the student readers study the first chapter thoroughly before moving on to the main topics.



Chapter 1

Fundamentals

In this chapter we recall some fundamental results from (pre-calculus) School Mathematics and discuss some basic topics which will be used repeatedly in the subsequent chapters. The subsequent chapters may be read as independent pieces, but this first chapter forms the basis for all of them. It is imperative that the student readers study this chapter carefully before moving on to the rest of the book.

1.1. Mathematical Statement

Mathematics is a language. The numbers (e.g., $0, 1, 2, -3, \frac{1}{3}, 2.54347$, etc.), the variables denoted by English or Greek letters (e.g., $x, y, a, b, \alpha, \beta$, etc.), the symbols (e.g., $=, +, -, \leq, \neq$, etc.), and some logical connectors (e.g., if, then, but, therefore, etc.) behave as the words, while the sentences are equivalent to 'mathematical statements'. For example, '*If* $x = -2$*, then* $x^2 = 4$' is a mathematical statement consisting of some of the aforesaid words. Broadly speaking, a mathematical statement may either be a *definition*, or an *axiom*, or an *assumption*, or a *theorem*.

You cannot 'prove' a definition or an axiom to be true or false; it has to be accepted as it is. For example, 'x^2 *is defined as* x *multiplied by* x' is a definition, and one cannot prove this statement to be true or false.

An assumption is a statement that is taken for granted or assumed under certain circumstances; it does not make any sense to prove them true or false, but they may be supported or contradicted during a proof. For example, an assumption may look like '*Suppose that* $x = 5$'. This statement may later be supported or rejected in the course of a proof, but cannot be directly claimed to be true or false.

A theorem may be logically proved to be 'true' or 'false', based on the definitions and prior assumptions. For example, '*If* $x = -2$*, then* $x^2 = 4$' is a mathematical statement that can be proved true based on the definition 'x^2 *is defined as* x *multiplied by* x'.

Throughout this book, we shall encounter several mathematical statements, of all kinds. Remember that one should always accept the definitions, feel free to construct own assumptions during a proof, and try to prove the provable mathematical statements to be either true or false. Any provable mathematical statement needs to be either true or false; it may not be ambiguous or uncertain in nature.

It is important to remember . . .

- ◇ The statements $2 \leq 3$ and $3 \leq 3$ are both true! The notation '$\leq$' means 'less than or equal to', which is true by definition if the numbers are equal as well. Similar notion holds for '$\geq$' sign too.
- ◇ Division of any number by 0 is *not defined*. So it is completely meaningless to write terms like $\frac{1}{0}$ or $\frac{0}{0}$. Trying to find their values is of course absurd.

Within this book, we shall state several 'facts', which are mathematical statements proved to be true. However, it may not always be possible for us to write the proofs for these facts, and in such cases, one may find the proof in standard textbooks.

1.2. Numbers

The foundation of Mathematics is built upon the concept of numbers. By the word 'numbers' we usually mean Real Numbers ($\mathbb{R}$), infinite in number, and not so easy to understand. In this book we will mainly deal with Rational Numbers ($\mathbb{Q}$).

Rational numbers are infinite in number, and are also known as fractions. Examples of rational numbers are $\frac{1}{4}, -\frac{32}{5}, \frac{7}{1}$ etc. Rational numbers have decimal representation, either terminating (finite) or recurring (infinite but periodically repeating); for example, $\frac{1}{2} = 0.5$ is a terminating decimal while $\frac{1}{3} = 0.3333\cdots$ is a recurring decimal. Non-terminating as well as non-recurring decimal numbers are not rational.

Integers ($\mathbb{Z}$) form a special class of Rational numbers, quite commonly used in mathematics. The numbers $0, 1, -1, 2, -2, 3, -3 \ldots$ are integers. Zero (0) is a special integer. Some of the integers are positive (greater than 0) while some are negative (less that 0). Positive integers are also known as Natural numbers ($\mathbb{N}$).

The integers which are integral multiples of 2, i.e., integers representable in the form $2k$ for some integer k, are called even; the other integers are called odd. For example, $0, 2, -2, 4, -4, \ldots$ are even integers, while $1, -1, 3, -3, \ldots$ are odd. It is often helpful to represent the odd integers as $2k + 1$ for some integer k. Remember that the integer 0 is neither positive nor negative, but it is even by definition.

Within the scope of this book, we will not require any other form of numbers. If you are interested, refer to a standard number theory textbook to know more about the complete classification of numbers.

It is important to remember ...

- Square root of any positive number is *defined* to be positive. For example, $\sqrt{4} = 2$, and it is neither ± 2 nor -2.
- Similarly, $\sqrt{a^2} = |a|$ for any real number, and it is neither $\pm a$ nor a nor $-a$. By definition,

$$|a| = \begin{cases} a, & \text{if } a \text{ is positive or zero;} \\ -a, & \text{if } a \text{ is negative.} \end{cases}$$

Hence, $\sqrt{a^2}$ is always positive for any non zero real number a. For example, if $a = 2$, then $\sqrt{a^2} = \sqrt{2^2} = \sqrt{4} = 2 = a$. But if $a = -2$, then $\sqrt{a^2} = \sqrt{(-2)^2} = \sqrt{4} = 2 = -(-2) = -a$. Note that in both cases $\sqrt{a^2}$ is positive, irrespective of whether a is positive or not.

Some important notation

Definition 1.1 (Factorial notation)**.** Let n be a natural number (positive integer). Then $n!$ (factorial n) is defined as $1 \times 2 \times \cdots \times n$. In addition, $0!$ is defined as 1.

For example, $1! = 1$, $2! = 1 \times 2 = 2$, $3! = 1 \times 2 \times 3 = 6$, etc. Note that one may write $n! = n \times (n-1)!$. For example, $4! = 4 \times 3! = 4 \times 6 = 24$, $5! = 5 \times 4! = 5 \times 24 = 120$, etc.

Definition 1.2 (Summation notation)**.** If $a_1, a_2, \ldots, a_n$ are n real numbers then we denote the sum $a_1 + a_2 + \cdots + a_n$ by the summation notation $\sum_{i=1}^n a_i$. This is read as 'sum over a_i for i from 1 to n'.

For example, $\sum_{i=1}^n i = 1 + 2 + \cdots + n$, $\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2$, $\sum_{i=1}^n e^i = e + e^2 + \cdots + e^n$, etc. Note that the term a_i may be a constant as well. For example, $\sum_{i=1}^n 1 = 1 + 1 + \cdots + 1$ (for n times), and it evaluates as $\sum_{i=1}^n 1 = n$.

Definition 1.3 (Product notation)**.** If $a_1, a_2, \ldots, a_n$ are n real numbers then we denote the product $a_1 \times a_2 \times \cdots \times a_n$ by the product notation $\prod_{i=1}^n a_i$. This is read as 'product over a_i for i from 1 to n'.

For example, $\prod_{i=1}^n i = 1 \times 2 \times \cdots \times n = n!$, $\prod_{i=1}^n i^i = 1^1 \times 2^2 \times \cdots \times n^n$, and

$$\prod_{i=1}^n e^i = e \times e^2 \times \cdots \times e^n = e^{1+2+\cdots+n} = e^{\sum_{i=1}^n i}.$$

An example with constant a_i is $\prod_{i=1}^n \pi = \pi \times \pi \times \cdots \times \pi$ (for n times), where the product finally evaluates as $\prod_{i=1}^n \pi = \pi^n$.

Example 1.1. Let x be a non-zero real number such that $x^4 + \frac{1}{x^4}$ and $x^5 + \frac{1}{x^5}$ are both rational numbers. Prove that $x + \frac{1}{x}$ is a rational number.

Solution. For a natural number k let $T_k = x^k + \frac{1}{x^k}$. Note that $T_4T_2 = T_2 + T_6$ and $T_8T_2 = T_{10} + T_6$. Therefore $T_2(T_8 - T_4 + 1) = T_{10}$. Since $T_{2k} = T_k^2 + 2$ it follows that T_8, T_10 are rational numbers and hence T_2, T_6 are also rational numbers. Since $T_5T_1 = T_4 + T_6$ it follows that T_1 is a rational number. $\square$

Quick Exercise

1. Find the decimal representation of the following rational numbers: $\frac{1}{7}, \frac{3}{33}$.
2. Find all rational numbers which have a terminating decimal representation.
3. Simplify and evaluate: $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{5} i^2$.
4. Prove that $\sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.
5. Prove or disprove the following: $\sum_{i=1}^{n} a_i b_i = (\sum_{i=1}^{n} a_i) \cdot (\sum_{i=1}^{n} b_i)$.

1.3. Set Theory

Loosely speaking, a set is a 'well-defined' collection of 'well-defined' distinct objects.

Usually, sets are denoted by capital English letters, like A, B, C etc., but some standard sets are denoted by special scripts as well. For example, we denote the set of all natural numbers as $\mathbb{N}$, set of all integers as $\mathbb{Z}$, set of all positive integers as $\mathbb{Z}^+$, set of all negative integers as $\mathbb{Z}^-$, set of all real numbers as $\mathbb{R}$, and set of all rational numbers as $\mathbb{Q}$. If a is an element of a set A, then we write $a \in A$, read as 'a belongs to A'.

There are three methods to describe a set:

1. *Roster method:* We may denote a set as $\{1, 2, 3\}$ or {January, July, June}. The objects should be written inside curly brackets (braces), separated by comma, and no object should be repeated, i.e., $\{1, 2, 3, 1\}$ is not a set. Moreover, the ordering of objects in the set does not matter, i.e., the sets {January, July, June} and {January, June, July} are identical.
2. *Set builder method:* We may denote a set as $\{x$ is an integer such that $x > 0$ and $x < 4\}$. The roster form of this set is $\{1, 2, 3\}$. The term 'such that' is written as ':', and we may denote this set as $\{x \ : \ x \in \mathbb{Z}$ and $0 < x < 4\}$.
3. *Venn diagram:* We may also denote a set visually by the 'oval figures' or any bounded figures. We write the objects inside the oval. The shape may be circle, rectangle or any other form, and we can treat the sets visually.

Union and Intersection

The union of two sets A and B, denoted as $A \cup B$, is a set defined as

$$A \cup B = \{x \ : \ x \in A \text{ or } x \in B\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

The intersection of A and B, denoted as $A \cap B$, is a set defined as

$$A \cap B = \{x \ : \ x \in A \text{ and } x \in B\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 5\}$, then $A \cap B = \{2\}$.

Finite, Infinite and Empty sets

A set having finite number of elements is known as a finite set, a set having an infinite number of elements is known as an infinite set, and a set having no elements is known as an empty set or a null set or a void set (denoted by ϕ).

For example, $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{Q}$ and $\mathbb{R}$ are infinite sets, while the sets $\{1, 2, 3\}$ and $\{$January, June, July$\}$ are finite. The sets $\{x \ : \ x \in \mathbb{Z}$ and $x > 0$ and $x < -2\}$ and $\{x \ : \ x$ is an English month starting with K$\}$ are empty sets. Note that an empty set is a finite set by definition.

Cardinality of a finite set

The number of elements contained in a finite set is known as the cardinality of that set. The cardinality of a set A is denoted as $n(A)$ or $|A|$. For example, $n(\{1, 3, 5, 7\}) = 4$, $n(\{$January, June, July$\}) = 3$ and $n(\phi) = 0$.

Theorem 1.1. *For two finite sets A, B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.*

Proof. The term $n(A \cup B)$ counts all elements present in A or B, i.e., all elements present in A but not in B, all elements present in B but not in A, as well as all elements present in both A and B (that is, in $A \cap B$).

Now, number of elements present in A may be counted as a sum of the number of elements present in A but not in B and the number of elements present in both A and B. In other words, the number of elements present in A but not in B may be counted as $n(A) - n(A \cap B)$. Similarly, the number of elements present in B but not in A may be counted as $n(B) - n(A \cap B)$. Hence, $n(A \cup B)$ is given by

$$(n(A) - n(A \cap B)) + (n(B) - n(A \cap B)) + n(A \cap B) = n(A) + n(B) - n(A \cap B).$$

Note that any element that does not belong to A or B is not counted anywhere. $\square$

Subset and Superset

Set A is said to be a subset of set B if and only if every element of A is an element of B. If A is a subset of B then we say B is a superset of A. We write $A \subset B$ or $B \supset A$. For example, $\{1,2\} \subset \{1,2,3\}$, $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{R} \supset \mathbb{Q}$, etc. Note that, the empty set ϕ is a subset of any set by definition.

Equal sets and Proper subset

We say that two sets A and B are equal if and only if $A \subset B$ and $B \subset A$. In other words, every element belonging to A must belong to B, and every element that is in B must be in A as well. We write this as $A = B$. Set A is a proper subset of B if and only if $A \subset B$ and $A \neq B$.

Sequence

Sequence of numbers is a set of numbers with a well-defined ordering. Similar to sets, a sequence may be finite or infinite. For example $\{1, 2, \ldots, 100\}$ is a finite sequence of 100 numbers, whereas $\{1, 3, 5, 7, \ldots\}$ is an infinite sequence.

It is important to remember ...

- ◇ A set containing only one element is also known as a *singleton* set.
- ◇ The sets ϕ and $\{\phi\}$ are not the same. The first set ϕ is the empty set or the null set, containing no elements, while the second set $\{\phi\}$ is a set containing one element, ϕ, and thus it is a singleton set.
- ◇ Set disregards the ordering of its elements, while the exact ordering of the elements is crucial for a sequence.
- ◇ Two sets are equal if and only if all their elements are identical, but two sequences are equal if and only if their elements are identical as well as the ordering of the elements is the same. For example, $\{1, 2, \ldots, 99, 100\}$ and $\{100, 99, \ldots, 2, 1\}$ are equal as sets as their elements are identical, but they are not equal as sequences as the ordering is different.

Quick Exercise

1. Write down the roster form of the following sets:
   a) $\{x : x$ is a positive integer less than 5$\}$,
   b) $\{x : x$ is an English month starting with M$\}$,

c) $\{y : y \in \mathbb{N}$ and less than $4\}$.

2. Suppose that $A = \{1, 3, 5\}$, $B = \{2, 4, 5\}$ and $C = \{3, 6\}$ are three sets. Compute the sets $A \cap B$, $A \cap C$, $B \cap C$, $A \cup B$ and $A \cup C$.
3. Give some examples of finite and infinite sets.
4. Compute the sets $\mathbb{Z} \cap \mathbb{N}$, $\mathbb{Z} \cup \mathbb{N}$, $\mathbb{Z}^+ \cup \mathbb{Z}^-$ and $\mathbb{Z}^+ \cap \mathbb{Z}^-$. Are they finite, infinite or empty? Can you state a more general result?
5. Give some examples of subsets, supersets, equal sets and proper subsets.
6. Let $A = \{2, 3, 4, \{1, 2\}, \phi, \{3, 2\}\}$. State, with proper explanation, which of the following statements are correct:

   (a) $\phi \in A$ (b) $\phi \subset A$ (c) $\{2\} \subset A$ (d) $\{2\} \in A$

   (e) $2 \in A$ (f) $\{2, 3\} \subset A$ (g) $1 \in A$ (h) $n(A) = 4$
7. Let $A = \{1, 2, \{1, 2, 3\}\}$. State, with proper explanation, which of the following statements are correct:

   (a) $\phi \subset A$ (b) $\{1, 2\} \in A$ (c) $\{1, 2\} \subset A$

   (d) $\{1, 2, 3\} \subset A$ (e) $3 \in A$ (f) $1 \in A$
8. Draw the Venn diagrams for all the operations of sets – union, intersection, subsets, supersets. Find our what 'symmetric difference' means and draw Venn diagram for the operation.
9. There are 100 students in a class. In an examination, 50 of them failed in Mathematics, 45 failed in Physics, 40 failed in Statistics, and 32 failed in exactly two of these three subjects. Only one student passed in all the three subjects. Find the number of students who failed in all three subjects?
10. In a group of 120 persons, there are 80 Bengali and 40 Gujarati. 70 persons in the group are Muslims and the remaining are Hindus. Then the number of Bengali–Muslims in the group is

    (a) 30 or more (b) exactly 20

    (c) between 15 and 25 (d) between 20 and 25
11. Write down the first 5 terms of a sequence whose n-th term is $3n + 1$.
12. What is the n-th term of the sequence of odd integers $\{1, 3, 5, 7, \ldots\}$?
13. If the nth term of a sequence is a_n satisfying the relation $a_n = 2a_{n-1} + 1$, find a_n in terms of n.
14. Write down the first 10 terms of Fibonacci sequence where the nth term of the sequence satisfies the relation $a_n = a_{n-1} + a_{n-2}$ and $a_1 = a_2 = 1$.
15. If x, y, z, t satisfy the following equations:

    $$x+y+2z+3t = 0, x+2y+4z+5t = 10, 3x+2y+z+t = 10, 5x+4y+2z+t = -10$$

    Then find the value of $(x + t)(y + z)$.

1.4. Mathematical Induction

Mathematical induction is a powerful technique to prove statements for all elements of certain sequences. For example, this may be used to efficiently prove a statement true for all natural numbers, which may otherwise be quite hard to prove. There are three common forms of Induction that are used in practice.

Mathematical Induction: Suppose that there is a mathematical statement S_n for each natural number n. If we can show that
(i) S_1 is true; and
(ii) for any natural number k, if S_k is true then S_{k+1} is true;
then by mathematical induction, S_n is true for all natural numbers n.

General version of Mathematical Induction: Suppose that there is a mathematical statement S_n for each natural number n. If we can show that
(i) S_r is true for some natural number r, and
(ii) for any natural number $k \geq r$, if S_k is true then S_{k+1} is true;
then by mathematical induction, S_n is true for all natural numbers $n \geq r$.

Stronger version of Mathematical Induction: Suppose that there is a mathematical statement S_n for each natural number n. If we can show that
(i) S_1 is true; and
(ii) for any natural number k, if $S_1, S_2, \ldots, S_k$ is true then S_{k+1} is true;
then by mathematical induction, S_n is true for all natural numbers n.

Example 1.2. Prove that $2^n > n^2$ for all natural numbers $n > 4$.

Solution. Suppose that for any natural number $n > 4$, the n-th statement S_n is $2^n > n^2$. We shall use the general form of mathematical induction to prove the statement for all natural number $n > 4$. First, we prove that S_5 is true, i.e., $2^5 > 5^2$. By simplification, we see that this is evidently true, as $2^5 = 32 > 25 = 5^2$.

Next we assume that S_k, i.e., $2^k > k^2$, is true for some natural number $k > 4$, and try to prove that S_{k+1} is also true. For the natural number $(k + 1)$, the statement S_{k+1} is $2^{k+1} > (k + 1)^2$, and we have

$$\begin{aligned}
2^{k+1} &= 2 \times 2^k \\
&> 2 \times k^2 && \text{since } S_k \text{ is assumed to be true;} \\
&= k^2 + k^2 \\
&> k^2 + 3k && k > 4 \text{ and thus } k^2 > 4k > 3k; \\
&= k^2 + 2k + k \\
&> k^2 + 2k + 1 && \text{since } k > 4 > 1; \\
&= (k + 1)^2.
\end{aligned}$$

Hence S_{k+1} is true if S_k is true. By mathematical induction, the statement S_n, i.e., $2^n > n^2$, is true for all natural numbers $n > 4$. □

Quick Exercise

Prove the following using mathematical induction.

1. $n! > 2^n$ for all natural numbers $n \geq 4$.
2. $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ for all natural numbers n.
3. $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all natural numbers n.
4. $1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all natural numbers n.
5. 7 is a factor of $3^{2n+1} + 2^{n+2}$ for all natural numbers n.
6. $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1}n^2 = (-1)^{n-1}\left(\frac{n(n+1)}{2}\right)$ for all natural numbers n.
7. 3 is a factor of $2^{2n} - 1$ for all natural numbers n.
8. $(a + b)$ is a factor of $(a^n - b^n)$ for all even natural numbers n where $a, b \in \mathbb{N}$.
9. $(a + b)$ is a factor of $(a^n + b^n)$ for all odd natural numbers n where $a, b \in \mathbb{N}$.
10. $(a - b)$ is a factor of $(a^n - b^n)$ for all natural numbers n where $a, b \in \mathbb{N}$.
11. $a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d)$ for all natural numbers n where $a, d \in \mathbb{R}$.
12. $a + ar + ar^2 + \ldots + ar^n = a\left(\frac{r^{n+1}-1}{r-1}\right)$ for all natural numbers n where $a, r \in \mathbb{R}$ and $r \neq 1$.

1.5. Progression

Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers. If $a_{i+1} - a_i = d$ for all $i = 1, 2, \ldots, n-1$, where d is a real number which is independent of i, then the sequence of real numbers $a_1, a_2, \ldots, a_n$ is called Arithmetic Progression (or in short A.P.) and the real number d known as the common difference of the A.P. a_i is known as i-th term of the A.P.

For example, $1, 3, 5, 7, \ldots, 2n - 1$ is an A.P. with common difference 2 as $a_{i+1} - a_i = 2$ for all i, but $1, 2, 3, 5$ is not an A.P. as $5 - 3 \neq 3 - 2$.

Note that, $a_i = a_1 + (i - 1)d$ for all $i = 1, 2, \ldots, n - 1$.
The sum of the first k terms is $a_1 + a_2 + \cdots + a_k = \frac{k}{2}(2a_1 + (k - 1)d) = \frac{k}{2}(a_1 + a_k)$.

If $a_{i+1} = ra_i$ for all $i = 1, 2, \ldots, n - 1$, where r is a real number which is independent of i, then the sequence of real numbers $a_1, a_2, \ldots, a_n$ is called Geometric Progression (or in short G.P.) and the real number r known as the common ratio of the G.P. a_i is known as i-th term of the G.P.

For example, $1, 2, 4, 8$ is an G.P. with common ratio 2 but $1, 2, 4, 7, 14$ not a G.P. (why?)

Note that, $a_i = a_1r^{i-1}$ and the sum of the first k terms is $a_1 + a_2 + \cdots + a_k =$

$a_1\left(\frac{r^k-1}{r-1}\right)$, if $r \neq 1$. What happens if $r = 1$?

Example 1.3 (RMO 2014)**.** Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let
(i) $a_1^2+a_3^2+\cdots+a_{2n-1}^2 = x$, (ii) $a_2^2+a_4^2+\cdots+a_{2n}^2 = y$, and (iii) $a_n+a_{n+1} = z$. Express d in terms of x, y, z, n.

Solution. $y - x = (a_2^2 - a_1^2) + (a_4^2 - a_3^2) + \cdots + (a_{2n}^2 - a_{2n-1}^2)$
$= (a_2 - a_1)(a_2 + a_1) + (a_4 - a_3)(a_4 + a_3) + \cdots + (a_{2n} - a_{2n-1})(a_{2n} + a_{2n-1})$
$= (a_1 + a_2 + a_3 + \cdots + a_{2n})d = \frac{2n}{2}(a_1 + a_{2n})d = nd(2a_1 + (2n-1)d)$.
Also, $z = a_n + a_{n+1} = a_1 + (n-1)d + a_1 + nd = 2a_1 + (2n-1)d$.
It follows that $y - x = ndz$. Hence $d = (y-x)/nz$. □

Quick Exercise

1. Suppose five numbers are in arithmetic progression. If their sum is 40, find the middle term.

1.6. Pigeon Hole Principle

Pigeon Hole Principle (PHP), also known as Dirichlet's Box Principle, is an effective tool from combinatorics that is applicable towards a varied array of problems in quite unexpected ways. Although it is technically a part of combinatorics, the logic is quite straight-forward, and it is comprehensible without any background knowledge.

Theorem 1.2 (Pigeon Hole Principle)**.** *If more than n objects are distributed into n boxes, then at least one box must contain more than one object.*

Proof. If all the boxes contain at most one object and no more, then there are at most n balls in total, which is a contradiction. □

Theorem 1.3 (Generalized PHP)**.** *If there are k boxes containing n balls, with $n \geq kt+1$ for some natural number t, then at least one box contains at least $(t+1)$ many balls.*

Proof. If all the boxes contain at most t balls and no more, then there are at most kt balls in total. This means $n \leq kt$, which is a contradiction. □

Theorem 1.4 (Strong PHP)**.** *Suppose that $x_1, x_2, \ldots, x_k$ are k real numbers and m is the average of these k numbers, i.e., $m = \frac{1}{k}(x_1 + x_2 + \cdots + x_k)$. Then there exists at least one x_i $(1 \leq i \leq n)$ such that $x_i \leq m$, and there exists at least one x_j $(1 \leq j \leq n)$ such that $x_j \geq m$.*

Proof. The proof is similar to the proof for Generalized PHP. Try it yourself. □

Problems on Pigeon Hole Principle (PHP) can be classified into three types: problems based on simple reasoning, problems based on number theory, and problems based on geometry. Simple reasoning problem can be solved without using any deeper theory but other types require sound knowledge of number theory and geometry. In this chapter, we only discuss problems based on simple reasoning.

Example 1.4 (RMO 1990)**.** Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same color then at least two of them will always be of the same size (radius). Prove that there are at least 3 balls which lie in the same box, have the same color and have the same size (radius).

Solution. We shall use PHP repeatedly to prove the statement. Since $65 = 2\times32+1$, there exists one box that contains at least 33 balls. There are four different colors, since $33 = 4 \times 8 + 1$, there are 9 balls in the above box having the same color. It is given that if one takes any 5 balls of the same color at least two of them will always be of the same size; this implies that there are at most 4 different sizes. Let the number of different sizes be s, where $s \leq 4$. Now $9 = 2 \times 4 + 1 \geq 2s + 1$. Thus by PHP, there are at least 3 balls in the same box having the same color and the same size. □

Solution.(alternative) Let us solve the above problem in an alternative way. Let us define two balls to be of the same 'type' if they lie in the same box, have the same color and the same size (radius). Now there are 2 boxes, 4 different colors, and s different sizes, where $s \leq 4$ as per our discussion in the previous proof. Thus, there are $k = 2 \times 4 \times s$ many different 'types' of balls with $k = 8s \leq 8 \times 4 = 32$. Now $65 = 2 \times 32 + 1 \geq 2k + 1$. Thus by PHP, there are 3 balls of the same type, i.e., belonging to the same box, having the same color and the same size. □

Example 1.5. Suppose that the vertices of a regular polygon of 20 sides are coloured with three colours: red, blue and green, such that there are exactly three red vertices. Prove that there are three vertices A, B, C of the polygon having the same colour such that triangle ABC is isosceles.

Solution. Since there are exactly three vertices, among the remaining 17 vertices there are nine of them of the same colour, say blue. We can divide the vertices of the regular 20-gon into four disjoint sets such that each set consists of vertices that form a regular pentagon. Since there are nine blue points, at least one of these sets will have three blue points. Since any three points on a pentagon form an isosceles triangle, the statement follows. □

Example 1.6. Given any set of $mn + 1$ integers, there is a nondecreasing subsequence of size $m + 1$, or there is a non-increasing subsequence of size $n + 1$.

Solution. For each i, define a pair (l_i, k_i), where l_i is the maximum length of non-increasing, and k_i is the maximum length of non-decreasing subsequence till the i-th number, including this i-th number. Now show if $i \neq j$ then $(l_i, k_i) \neq (l_j, k_j)$.

Since there are $mn+1$ many distinct ordered pairs, one of the l_i's must be greater than or equal to $n+1$ or one of the k_j's must be greater than or equal to $m+1$. $\square$

Example 1.7. Let n be a positive integer and A be a subset of the set $\{1, 2, \cdots, 4n\}$ with $|A| = 3n+1$. Prove that A must contain three distinct elements a, b and c such that a divides b and b divides c.

Solution. Consider the set $\{2^i \times q : q \leq 4n$ and q is an odd natural number$\}$. Note that between $2n$ and $4n$ these sets are singletons. Remove these n many odds. So we have at least $2n+1$ with n choices of odd. $\square$

Example 1.8. Let A be a subset of the set $\{1, 2, \ldots, 200\}$ with $n(A) = 101$. Prove that A must contain at least two distinct elements a and b such that a divides b.

Solution. Each element of the set $\{1, 2, \ldots, 200\}$ may be represented as $2^i q$, where q is an odd natural number less than 200, and $i \geq 0$ is chosen appropriately. There are only 100 possible values of q, namely $\{1, 3, \ldots, 199\}$. Now the set A contains 101 elements from $\{1, 2, \ldots, 200\}$.

Thus by PHP, there exist at least two distinct elements of A that have the same q in their representation. These two elements may be represented as $2^a q$ and $2^b q$ for some $a, b \geq 0$, $a \neq b$, and $q \in \{1, 3, \ldots, 199\}$; hence the smaller of the two must divide the other.

Can you prove the same if $|A| = 68$. $\square$

Example 1.9. Prove that among any 101 rectangles whose lengths and breadths are all positive integers less than or equal to 100, there exist three rectangles A, B, C such that A fits within B and B fits within C.

Solution. Let l be that length and b be the breadth of a rectangle. Consider 50 categories of rectangles, with the difference between the length and the breadth as follows: $l - b = 2k$ or $2k+1$, where $0 \leq k \leq 49$.

Note that if two rectangles with dimensions (l_1, b_1) and (l_2, b_2) belong to the same category, then without loss of generality, $l_1 - b_1 = l_2 - b_2$, i.e., $l_1 - l_2 = b_1 - b_2$, or $l_1 - b_1 = l_2 - b_2 + 1$, i.e., $l_1 - l_2 = b_1 - b_2 + 1$. If $l_1 = l_2$, then we must have $b_1 = b_2$ or $b_1 = b_2 - 1$, which implies that rectangle (l_1, b_1) fits within rectangle (l_2, b_2). If $l_1 > l_2$, then we have $b_1 > b_2$ or $b_1 > b_2 - 1$ (i.e., $b_1 \geq b_2$), which implies that rectangle (l_2, b_2) fits within rectangle (l_1, b_1). Thus, if two rectangles belong to the same category, then one must fit within the other.

Now we have $101 = 2 \times 50 + 1$ rectangles in total, while there are only 50 categories. Thus by PHP, there exist at least one category with three rectangles in it; these rectangles satisfy the condition of the problem. $\square$

Quick Exercise

1. Prove the first two versions of PHP from Strong PHP.

2. Some 46 squares are randomly chosen from a 9×9 chess-board and are colored red. Show that there exists a 2×2 block of 4 squares of which at least three are colored red. INMO 2006
3. There is a $2n \times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeros. Show that it is possible to remove all the zeros by deleting some n rows and some n columns.

   [Note: A $m \times n$ array is a rectangular arrangement of mn numbers in which there are m horizontal rows and n vertical columns.] INMO 2006
4. There are 10 objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the 10 objects can be divided into two groups that balance each other when placed on the two pans of a balance. INMO 1991

   Hint: Order $1 = a_1 = \cdots = a_k < a_{k+1} \leq \cdots \leq a_{10}$. Prove that there is a subset with sum 10.
5. Given any set of 12 integers, show that there are two whose difference is divisible by 11.
6. In any group of five people, show that there are two who have the same number of friends in the group.
7. Five lattice points are chosen in the plane. Show that one can always choose two of these such that the line segment joining them passes through a lattice point (a lattice point has both integer coordinates).
8. The digits $1, 2, \ldots, 9$ are divided into three groups. Prove that the product of the numbers in at least one of the groups must exceed 71.
9. a) Show that there is a natural number whose decimal representation consists entirely of the digit 1 that is divisible by 1987.

   b) Show that there is a number of the form $2^n - 1$ that is divisible by 1987.
10. Show that a real number is rational if and only if its decimal representation is a terminating or recurring decimal.
11. In an island, there are 6 cities. Every two of them are connected either by bus or by train, but not by both. Show that there are three cities which are mutually connected by the same mode of transport.
12. 100 balls distributed among 14 students such that each student gets at least one ball. Show that there are at least two students who have received same number of balls.

1.7. Function

Consider two sets A and B. A function, denoted by f, is an input–output machine, which takes an element of A as an input and gives a unique element of B as an output. The set A is known as the *domain* and the set B is known as the *co-domain* of the function f. The input is known as the *pre-image* and the output is known as the *image* under f.

For example, consider the domain and the co-domain of a function f to be $\mathbb{R}$, and let pre-image x give image x^2 for all $x \in \mathbb{R}$. We write this as $f(x) = x^2$. In general, if the function is f, with domain A and co-domain B, and if the pre-image is x, then we write the the image as $f(x)$ and the function as '$f : A \to B$'.

Every element of the domain should have a *unique* image in the co-domain; multiple outputs of a single input is not permitted for a function. However, the converse may not be true, i.e., all elements of co-domain may not have a pre-image in the domain of the function. In the above example, $f(x) = x^2$ for $x \in \mathbb{R}$, every real number has an image, but the negative real numbers have no pre-image.

Onto and One-One functions

If every element of the co-domain has at least one pre-image (in the domain), then the function is called an 'onto' function or a *surjection*. If every element of the co-domain has at most one pre-image (in the domain), then the function is called a 'one-one' function or an *injection*.

A function which is both onto (surjection) and one-one (injection) is known as a 'bijective' function, often simply referred to as a *bijection*. A bijection $f : A \to B$, where $A = B = \{1, 2, \ldots, n\}$, is known as permutation on the set $\{1, 2, \ldots, n\}$.

Examples of the different classes of functions are as follows.

- ◇ $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x$, is a bijection, i.e., both one-one and onto.
- ◇ $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 4x^2 - 5$, is neither one-one nor onto.
- ◇ $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3 - x$, is onto but not one-one.
- ◇ $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2^x$, is one-one but not onto.

Theorem 1.5. *Let the function $f : A \to B$ be one to one, where A and B are finite sets. Then $n(A) \le n(B)$.*

Proof. Prove the statement using PHP. □

Theorem 1.6. *Let $f : A \to B$ be a function, where A and B are finite sets. If any two of the following statements hold, then the third also holds: (a) f is onto; (b) f is one-one; (c) $n(A) = n(B)$.*

Proof. Try it yourself, using the idea from the previous proof. □

Quick Exercise

1. Justify the examples provided above for each class of functions, and give more examples for each type of functions.
2. What happens to the statement of Theorem 1.5 if f is an onto function, or if f is a bijection?

1.8. Solved Examples

Example 1.10 (RMO 1990)**.** A census-man on duty visited a house in which the lady inmates declined to reveal their individual ages, but said, "we do not mind giving you the sum of the ages of any two ladies you may choose". Thereupon the census-man said, "In that case please give me the sum of the ages of every possible pair of you". They gave the sums as follows: 30, 33, 41, 58, 66, 69. The census-man took these figures and happily went away. How did he calculate the individual ages of the ladies from these figures? Ages are in positive integers.

Solution. Clearly there are four women in the house. Let $x_1 \leq x_2 \leq x_3 \leq x_4$ be the ages of the women in ascending order. Then $x_1+x_2 = 30, x_1+x_3 = 33, x_2+x_4 = 66$ and $x_3 + x_4 = 69$. Therefore $x_1 + x_2 + x_3 + x_4 = 99$. Now $x_1 + x_4$ may be 41 or 58. But if $x_1 + x_4 = 41$ then, $3x_1 + x_2 + x_3 + x_4 = 104$. So $2x_1 = 5$, since x_1 is a positive integer, this is not possible, hence, $x_1 + x_4 = 58$. From these relations one can find the ages. □

Exercise: Observe that we cannot find the values of x_1, x_2, x_3 and x_4 from only the first four equations. Do you know why?

Solution.(alternative) Let x_1, x_2, x_3 and x_4 be the ages of the women. Then $x_1 + x_2 + x_3 + x_4 = 99$. Let $x_1 + x_2 = 30$, so $x_3 + x_4 = 69$. Now one of x_3, x_4 is odd and other is even. Without loss of generality let x_3 be even. Now if both x_1, x_2 are even, then x_4 is the only odd. Therefore, $(x_1 + x_4) + (x_2 + x_4) + (x_3 + x_4) = 33 + 41 + 69 = 143$ implies $x_4 = 22$ a contradiction. Hence both x_1, x_2 are odd, therefore, $(x_1+x_3)+(x_2+x_3)+(x_3+x_4) = 33+41+69 = 143$ implies $x_3 = 22$. □

Exercise: In this problem, the ages were all positive integers. What happens if the unknowns x_1, x_2, x_3, x_4 are not positive integers?

Example 1.11 (RMO 1992)**.** Solve the following system in terms of parameter A.

$$\begin{aligned}
(x+y)(x+y+z) &= 18 \\
(y+z)(x+y+z) &= 30 \\
(z+x)(x+y+z) &= 2A
\end{aligned}$$

Solution. Adding three equations we get, $(x + y + z)^2 = 24 + A$. If $A < -24$ there is no solution of the system. Let $A \geq -24$ then $x + y + z = \sqrt{24 + A}$ or

$x+y+z=-\sqrt{24+A}$. If $x+y+z=\sqrt{24+A}$ from the first equation we get $x+y=\frac{18}{\sqrt{24+A}}$ and hence $z=\sqrt{24+A}-\frac{18}{\sqrt{24+A}}=\frac{6+A}{\sqrt{24+A}}$.

If $x+y+z=-\sqrt{24+A}$ from the first equation we get $x+y=-\frac{18}{\sqrt{24+A}}$ and hence $z=-\sqrt{24+A}+\frac{18}{\sqrt{24+A}}=-\frac{6+A}{\sqrt{24+A}}$. Similarly we can find x and y.

□

Example 1.12 (INMO 1993). Let f be a bijective (1-1 and onto) function from $A=$ $\{1,2,3\ldots,n\}$ to itself. Show that there is positive number $M\geq 1$ such that

$$f^M(i)=f(i), \text{ for each } i \text{ in } A.$$

f^M denotes the composite function $f\circ f\circ\cdots\circ f$ (composition for M times).

Solution. There exists M_i such that $f^{M_i}(i)=f(i)$. Now $M=\operatorname{lcm}\{M_i\}$. □

1.9. Exercise Problems

1. (RMO 1994). A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf. **Note:** This problem is wrong! Why?
2. (RMO 1991). The 64 squares of an 8×8 chess-board are filled with positive integers in such a way that each integer is the average of the integers on the neighboring squares. (Two squares are neighbors if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbors depending on its position). Show that all the 64 integer entries are in fact equal.
3. (RMO 1993). In a group of ten persons, each person is asked to write the sum of the ages of all the other 9 persons. If all the ten sums form the 9-element set {82, 83, 84, 85, 87, 89, 90, 91, 92} find the individual ages of the persons (assuming them to be whole numbers of years).
4. (RMO 1993). I have 6 friends and during a vacation I met them during several dinners. I found that I dined with all the 6 exactly on 1 day; with every 5 of them on 2 days; with every 4 of them on 3 days; with every 3 of them on 4 days; with every 2 of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?
5. (RMO 2006). Let X be the set of all positive integers greater than or equal to 8 and let $f:X\to X$ be a function such that $f(x+y)=f(xy)$ for all $x\geq 4$, $y\geq 4$. If $f(8)=9$, determine $f(9)$.

6. (RMO 2001). Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

   Suppose each row consists of the n numbers $1, 2, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ are $1, 2, \ldots, n$ in some order. What happens if n is even.

7. (RMO 2001). Prove that the product of the first 1000 positive even differs from the product of the first 1000 odd by a multiple of 2001.

1.10. Miscellaneous Puzzles and Problems

Puzzles and brain-teasers are probably the most effective avenue to build a logical sense of word problems and an analytical approach towards solving them. We have compiled a set of puzzles comprising of common word problems from arithmetic as well as brain-teasers famous in the community for decades.

Before diving into the more serious part of this book, the student readers are encouraged to attempt the following puzzles to sharpen their problem-solving skills.

1. The egg basket problem(Due to Brahmagupta, 628),
   Suppose that a basket has n eggs in it. If the eggs are taken from the basket 2,3,4,5 and 6 at a time, there remain 1,2,3,4 and 5 eggs in the basket, respectively. If the eggs are removed from the basket 7 at a time, no eggs remain in the basket. What is the smallest value of n such that the above could occur?
2. A contractor constructed a brick wall. He employed one bricklayer who could build the wall in ten hours, and another who could build the wall in nine hours. Working together, the bricklayers laid 10 fewer bricks per hour than when they had worked independently. It took them exactly 5 hours to build the wall. How many bricks are in the wall?
3. Two ferry-boats started at the same instant from opposite sides of a river, traveling across the water at right angles to the shore. They first met 720 yards from the nearest shore and, on reaching the opposite side, spent 10 minutes in the slip before starting back. They met again 400 yards from the nearest shore. What is the width of the river?
4. A little boy liked to play the 'up and down' in an escalator of a department store. When he walked up the escalator, he counted 10 steps and the trip took 20 seconds. When he ran down, he counted 50 steps and the trip took 30 seconds. How many steps of the escalator were visible at one time?

5. Jack tore out several consecutive pages from a book. The number on the first page he tore out was 183 and the number on the last page has the same digits in a different order. Find the range of the page numbers torn.

6. There are 10 cigarette packets, each containing 10 cigarettes. Among these 10 packets, 9 packets contain cigarettes of weight 10 gm each. Remaining one contains cigarettes of weight 9 gm each. All the cigarettes of a particular packet are identical, but cigarettes of different packets looks different. Identify the odd packet by a single weighing using a spring balance. Can you identify the odd ones if there are 11 packets each containing 10 cigarettes?

   If all the packets contains 1000 cigarettes and any number of packet may be counter fit the how one find the counter fit packets using a single weighing.

7. A new tyre of a two-wheeler can go 1000 km. If you start with two new tyres and another new tyre as spare, find the maximum distance you can travel before you need to buy new tyres.

8. In a particular month of some year, there are three Mondays which have even dates. On which day of the week does the 15th of that month fall?

9. There are three boxes; the first box contains 10 red balls, the second box contains 10 blue balls, and the third box contains 5 red and 5 blue balls. The boxes are wrongly labeled as red, blue and red-blue. You can draw one ball from a box without looking. You have to tell the color of the balls in each box; how would you do this?

10. There are two ropes. Each takes one hour to burn, but not at a uniform rate. How do you measure 45 minutes using these two ropes? First try 30 minutes with one rope!

11. An island is inhabited by two tribes. Members of Tribe I always tell the truth. Members of Tribe II always tell lies. Now a tourist visited the island and came across three persons, A, B and C walking together. He asked A – "To which tribe do you belong?"; A replied, but the tourist didn't hear it properly. So he asked B and C – "What was A's answer?". "He said he belongs to Tribe I" – said B; "No, he said he belongs to Tribe II" – replied C. Given these replies, can you say to which tribe B and C belong?

12. In a college, there are 100 professors. They specialize in one of the three subjects – Mathematics, Statistics and Economics. And some of them always tell the truth while the others always tell lies! An interviewer asked all of them whether they specialized in Mathematics; 40 of them said 'Yes'. Another interviewer asked all of them whether they specialized in Statistics; again, 40 of them said 'Yes'. A third interviewer asked all of them whether they specialized in Economics; once again, 40 of them said 'Yes'. Can you say how many of them are liars?

13. There are two sand-clocks, one can measure 4 minutes and the other can measure 7 minutes. How do you measure 9 minutes using these clocks? Can you say which times that can be measured using these clocks?

14. There are 25 racehorses. In the race-course, there are only five lanes (i.e., at most 5 horses can compete together). Find the least number of races needed to find the first, second and third among these 25 horses. You may assume that the speed of a horse does not vary over different races and that there is no tie in a race.
15. There are nine coins; eight are identical in all respect, and the remaining one is slightly lighter than the others, but identical in any other respect. Identify the coin by two weighings using a simple balance (no weights will be provided with the balance).
16. There are twelve coins; eleven are identical in all respect, remaining one is slightly heavier or lighter than the others, but identical in any other respect. Identify the coin (you also have to say whether it is heavier or lighter than the others) by three weighings using a simple balance (no weights will be provided with the balance).
17. Divide a 40 kg stone into four parts in such a way that one can measure every unit of weight from 1 kg to 40 kg using these four parts and a simple balance.
18. Four persons try to cross an old bridge at night; they can individually cross the bridge in 1, 2, 5 and 10 minutes respectively. They have only one torch with them, and that torch is necessary to cross the bridge. Only two persons can cross the bridge at a time. How can they cross the bridge in total 17 minutes?
19. The strength of a mug (made of glass) is defined as follows – there is a hundred-stored building, and the mug is said to possess strength l $(1 \leq l < 100)$ units if it does not break when it is dropped from the l-th floor, but it breaks when it is dropped from the $(l+r)$-th floor for $r \geq 0, l+r \leq 100$. The strength of a mug is known to be always less than 100. If you are given only one mug, you can determine its strength by dropping it successively from 1st, 2nd, 3rd floors until it breaks. Thus, if the strength of the mug is l, then the number of times you need to perform this dropping experiment is $l+1$. Note that we may use the same mug for many experiments until it breaks. Now consider that instead of one, you are given two mugs of the same strength. Plan a scheme to determine the strength of these mugs with minimum number of experiments, and report the exact number of experiments you need to perform according to your scheme.
20. There are 10 pairs of black gloves and 15 pairs of white gloves scattered piece-wise in a box. You have to draw a complete pair of gloves without looking inside the box. What is the minimum number of gloves that you have to draw to confirm a complete pair?
21. Five cards are taken from a pack of cards. The assistant of a magician looks at these five cards, closes (face-down) one card on a table and opens (face-up) other four cards on the table in a specific order. The magician does not know the closed card, but just by looking at the four open cards, the magician tells exactly which card is closed. How is it possible?

22. Two friends want to divide a cake into two equal parts, one for each. Plan a scheme that both of them will be completely satisfied with their shares. What will be the scheme if there are n friends sharing the cake?

23. There are n petrol pumps on a circular road; the length of the road is l km. If the total amount of petrol in n petrol pumps is l liters and 1 liter petrol is needed for a car to go 1 km, prove that there is a petrol pump from where the car can start with an empty petrol tank and after traveling l km can reach the same petrol pump.

24. Consider the following game with n persons in a team; the rule of the game is as follows. All the persons in the team stand in a queue, and a cap is placed on the head of each person. Nobody is allowed to see these caps while they are placed. After placing these caps the persons are allowed to see the caps only in front of him/her. That is, the person in the front (first person in the queue) cannot see any cap. The second person can only see one cap, that of the first person. The third person can see two caps in front of him/her. In general, the i-th person can see $i - 1$ caps in front of him/her. The caps are of two colors, white or black (which is known to the n persons before the game).

    After the setup is complete, the referee asks the first person: "What is the color of your cap, white or black?". Everybody in the queue can hear the answer of the first person, and the referee will not say whether the answer is right or wrong. The referee asks the same question to the second person, and again everyone hears the answer and as previous referee will not say whether the answer is right or wrong. This continues with the referee asking the same question to every person in the queue. The score of the team is the number of correct answers to the question. If the team knows the rules of the game, what strategy should they follow to score maximum points? What is the maximum possible point to be scored in this game?

    What will be the strategy if there are caps of k different colors?

25. In a village there are $2n$ many persons ($n \in \mathbb{N}$). Among these $2n$ villagers, n are always truthful and the other n are always liars; all the villagers know this fact. An outsider meets one of these villagers at a cross road, one of which goes to the station and other goes to the bus stop. The outsider want to know which road goes to the station. The outsider also knows the whole scenario. He/she can ask only one question to the villager and the villager will answer only in 'yes' or 'no'. Which question will be asked by the outsider?

26. There are 3 monks and 3 devils, trying to cross a river. There is only one boat in the river and only two of them can board the boat at a time. All the monks can drive the boat, while only one devil can drive it. While crossing the river, the number of devils cannot be greater than that of the monks on either bank of the river at any point of time; otherwise the monks lesser in number will be killed by the devils. When the boat reaches any bank, the boarder(s) of the boat are considered to be members of that bank. Devise a strategy so that all of them can safely cross the river. (Hint: first try with the case that all the

devil can drive the boat.)

27. There are 8 persons trying to cross a river – father, mother, two daughters, two sons, a police and a criminal. There is only one boat in the river and only two of them can board the boat at a time. Father, mother and police can drive the boat. The rule is that at any point of time, no daughter can stay at either banks of the river or in the boat with the father unless the mother is also there, and no son can stay at either banks of the river or in the boat with the mother unless the father was also there. In addition, the criminal can not stay with anybody unless the police was also there; however the criminal can stay alone at either bank of the river. Devise a strategy so that they all can safely cross the river.
28. Form a 3×3 'magic square' with the digits 1,2,3,4,5,6,7,8 and 9; that is, place the digits in the boxes of 3×3 table so that the sums of the numbers along the three rows, three columns and the two diagonals are all equal.
29. Three friends are riding in a train; the train passes through a tunnel for several minutes and they are plunged into darkness. When they came out from the tunnel, each of them notices that the faces of the other two are black with the soot that flew in through the open window. They start laughing at each other, but all of a sudden, the most intelligent friend realizes that his face must be soiled too. How does he arrive at this conclusion? Assume that all the friends are intelligent, and all of them know this fact.
30. Can a knight start at square $A1$ (bottom left square) of a chess board, and go to square $H8$ (top right square), visiting each of the remaining 62 squares exactly once on the way?
31. A closed path (loop) is made up of 11 line segments. Each point where two or more line segments join is called a vertex. Can you draw another line, not containing any vertex of the path, such that it intersects each of the 11 segments?
32. The product of 22 integers is equal to 1. Show that their sum cannot be zero.
33. Can you form a 4×4 'magic square' out of the first 16 prime numbers? What about a 2×2 magic square with the first 4 prime numbers? What about an $n \times n$ magic square with the first n^2 prime numbers ($n \in \mathbb{N}$)?
34. What is the maximum number of kings one can place on a chess board such that no king can attack another one? What about knight, rook, bishop and pawn? What about placing queens?
35. Do you know how many years are there in 2^{64} seconds? Can you guess what will be the weight of 2^{64} many rice seeds if a single rice seed weighs 1 milligram? Can you imagine what is the total value of 2^{30} Indian paisa.
36. How can you measure 14 liters of milk using two mugs of capacity 10 liters and 6 liters respectively? Can you measure 11 liters of milk using only these two mugs?

37. 25 prisoners will be kept in different rooms so that they can not communicate after the first day. On each day a randomly selected prisoner let enter into a room where there are two switches initially they are set on. They can be called for more than one times in any order. They are allowed to flip exactly one switch. If a prisoner can claim correctly that all 25 prisoners have been called to this room then all will be set free. What is the strategy they will fix on the first day so that they will be set free eventually.

38. List all the primes from 1 to 200.

39. Can you say the birth day of your friends knowing the date and year only?

40. In a monastery, there are n many monks. Only at night they meet in a circular dining table so each can see the face of the others but they do not talk or they do not communicate in any manner. They can not see their own face also. One day their master came and tell them that from next day the face of some of them will be black and whenever a monk realizes that his face is black he drop to come at dining room. How and when a monk realizes?

41. There are seven flat stone consecutively in a row on a pond. On consecutive three stone there are three frog of type A and on consecutive three stone there are three frog of type B. The middle one is empty and only one frog allow to sit on one stone. Type A can move only from left to right and type B can move only from right to left. Each frog can move to the next stone or to the next to next stone if they are empty. For example, if the second stone from left is empty then the frog on the first stone from left can move to the second stone. Again if the third stone is empty and there is a frog on the second stone then the frog on the first stone can move to the third stone. No two frog can move at a time, i.e., in each step only one move is allowed. Problem is how the two types interchanged their position.

Hint and solution of Miscellaneous Puzzles and Problems:

1. 119 5. 183 to 318 6. Take one from first packet, two from second, three from third and so on and the weight; if there are 11 packet do not take any cigarette from the last one. Take 2^{i-1} cigarettes from the i-th packet 7. 1500 km.

Chapter 2

Number Theory

2.1. Introduction

In these chapter we shall discuss on the different properties of integers. In our school book we learned some results about positive integers or natural numbers. We shall recapitulate all the results and discuss more results on integers.

Lemma 2.1 (Well Ordering Principle)**.** *Any non-empty subset S of natural numbers has a smallest element, i.e., there is a natural number $a \in S$ such that $a \leq s$ for any $s \in S$.*

2.2. Divisibility

Definition 2.1 (Divisibility)**.** An integer b is said to be divisible by a non-zero integer a if there exists an integer x such that $b = ax$. In such a case, we write $a|b$, and say that a divides b, or a is a divisor of b, or b is a multiple of a. If b is not divisible by a, we write $a \nmid b$.

For example, $3|6$, $(-3)|3$, $(-2)|2$ and $(-4)|12$, but $3 \nmid 5$.

Quick Exercise

Suppose that a, b, c, m, x, y are integers. Prove that

1. If $a|b$ then $a|bc$ for any integer c.
2. If $a|b$ and $b|c$ then $a|c$.
3. If $a|b$ and $a|c$ then $a|(bx + cy)$ for any integers x, y.
4. If $a|b$ and $b \neq 0$ then $|a| \leq |b|$.

5. If $a|b$ and $b|a$ then $a = b$ or $a = -b$.
6. If $m \neq 0$ then $a|b$ if and only if $ma|mb$.

Theorem 2.1 (Division Algorithm). *Given integers a and b with $a \neq 0$, there exist unique integers q and r such that $b = qa + r$, with $0 \leq r < |a|$. If a does not divide b then $0 < r < |a|$.*

Theorem 2.2 (Divisibility test for $7, 11, 13$). *If $x = a_n a_{n-1} \dots a_1 a_0 = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10a_1 + a_0$ is an $(n+1)$-digit number, where $a_0, a_1, \dots, a_{n-1}, a_n$ are the digits, then x is divisible by $7, 11, 13$ if and only if $(a_0 a_1 a_2 - a_3 a_4 a_5 + a_6 a_7 a_8 - \cdots)$ is divisible by $7, 11, 13$, respectively. Moreover, x is divisible by 11 if and only if $(a_0 - a_1 + a_2 - a_3 + \cdots)$ is divisible by 11.*

Exercise: Find out the rules for divisibility by $2, 3, 4, 5, 6, 8, 9, 10, 12$.

Theorem 2.3 (Finding the last digit of a^n). *The last digit of a^n is the same as the last digit of b^m, where b is the last digit of a, and $m = 4$ if $4|n$, or otherwise m is the remainder when n is divided by 4.*

Definition 2.2 (Greatest Common Divisor). Integer d is called a common divisor of a and b if $d|a$ and $d|b$. If a, b are integers, not both zero, then a positive integer g is said to be the 'greatest common divisor' or GCD of a and b if and only if

- ◇ g is a common divisor of a and b, and
- ◇ if d is a common divisor of a and b then $d|g$.

The greatest common divisor of a and b is denoted by $\gcd(a, b)$ or simply by (a, b). It can be proved that $\gcd(a, b)$ is unique for two fixed integers a, b.

Theorem 2.4 (Bezout's Theorem). *If a, b are two integers, not both zero, then $\gcd(a, b)$ exists, and there exist integers x_0, y_0 such that $\gcd(a, b) = ax_0 + by_0$.*

The integers x_0, y_0 are not unique and can be found by the *Euclidean algorithm*. Note that the converse of the Bezout's theorem is not true. One may illustrate this using a counterexample $2 \times 3 + 4 \times 1 = 10 \neq \gcd(2, 4)$.

Definition 2.3 (Co-prime Integers). Two integers a, b, not both zero, are called 'relatively prime' or 'co-prime' if $\gcd(a, b) = 1$.

Theorem 2.5. *If a, b are two integers, not both zero, then $\gcd(a, b) = 1$ if and only if there exist integers x_0, y_0 such that $ax_0 + by_0 = 1$.*

Theorem 2.6. *If $\gcd(a, b) = d$, then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.*

Theorem 2.7 (Euclid's Lemma). *For integers a, b, n, if $n|ab$ and $\gcd(a, n) = 1$, then $n|b$.*

2.3. Prime Numbers

Definition 2.4 (Prime Number)**.** An integer $p > 1$ is called a prime number if it has no divisor d such that $1 < d < p$.

If a natural number greater than 1 is not a prime, then it is called a *composite number*. For example, 2 is prime number but 4 is composite.

Theorem 2.8. *An integer p is prime number if and only if $p|ab$ implies $p|a$ or $p|b$.*

Corollary 2.1. *If p is a prime and $p|a_1a_2\cdots a_n$, then p divides at least one factor a_i of the product.*

Corollary 2.2. *For integers a, b, m, if $a|m$, $b|m$ and $\gcd(a,b) = 1$, then $ab|m$.*

Theorem 2.9 (Fundamental Theorem of Arithmetic)**.** *Every positive integer $n > 1$ can be expressed as product of primes in a unique way except for the ordering of the prime factors.*

Corollary 2.3. *Every integer $n > 1$ can be uniquely written as $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, where p_i's are primes with $p_1 < p_2 < \cdots < p_r$ and a_i's are positive integers.*

In the above representation of positive integers, the a_i's are known as the *exponents* of primes p_i's in n, and this representation is known as the *canonical representation* of the positive integer.

Theorem 2.10. *Suppose that two positive integers a and b have the following canonical representations: $a = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ and $b = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$. Then*

$$\gcd(a,b) = {p_1}^{\min\{a_1,b_1\}} \cdot {p_2}^{\min\{a_2,b_2\}} \cdots {p_r}^{\min\{a_r,b_r\}}$$
$$\operatorname{lcm}(a,b) = {p_1}^{\max\{a_1,b_1\}} \cdot {p_2}^{\max\{a_2,b_2\}} \cdots {p_r}^{\max\{a_r,b_r\}}$$

Quick Exercise

1. Prove all theorems and corollaries stated in this section.
2. Prove that the product of any consecutive n positive integers is divisible by $n!$.
3. Show that the square of any integer is of the form $4k$ or $8k+1$, where k is a positive integer.
4. Given a positive integer n, prove that there exist n consecutive composite integers.
5. Show that $\gcd(am, bm) = m\gcd(a,b)$ for any positive integer m.
6. If $d|a$, $d|b$ and $d > 0$, then show that $\gcd(\frac{a}{d}, \frac{b}{d}) = \frac{\gcd(a,b)}{d}$.
7. Find all possible four digit numbers satisfying all of the following properties – (a) it is a perfect square, (b) its first two digits are identical, and (c) its last two digits are identical.

8. Find the number of rational numbers, m/n for integers $m, n \neq 0$, such that: (i) $0 < m/n < 1$, (ii) $\gcd(m, n) = 1$, and (iii) $mn = 25!$.
9. Call a positive integer n 'good', if there are n integers, not necessarily distinct, such that their sum and product are both equal to n. For example, 8 is 'good', since 8 is equal to the product as well as the sum of $\{4, 2, 1, 1, 1, 1, -1, -1\}$. Show that integers of the form $4k+1$ for $k \geq 0$, and $4l$ for $l \geq 2$ are 'good'.
10. Find all positive integer triples (a, b, c) such that $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3$.
11. Prove that the in-radius of an integer-sided right-angled triangle is an integer.
12. If $23! = 2585201 \star \star 38884976640000$, find the two missing digits.

2.4. Congruence and Modular Operations

Definition 2.5. Let m be a non-zero integer. The integers a and b are said to be congruent modulo m if and only if $m|(a-b)$, and we write $a \equiv b \pmod{m}$.

For example, $19 \equiv 1 \pmod{9}$, $2 \equiv -5 \pmod{7}$, $8 \equiv 8 \pmod{8}$, $3 \not\equiv 5 \pmod{3}$, $4 \not\equiv 9 \pmod{3}$, etc. We shall confine our attention to a positive modulus.

Lemma 2.2. *For integers a, b, c, d, x, y and positive integers m, k,*

1. $a \equiv b \pmod{m}$, $b \equiv a \pmod{m}$, *and* $a - b \equiv 0 \pmod{m}$ *are equivalent.*
2. *If* $a \equiv b \pmod{m}$ *and* $b \equiv c \pmod{m}$, *then* $a \equiv c \pmod{m}$.
3. *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then* $ax + cy \equiv bx + dy \pmod{m}$.
4. *If* $a \equiv b \pmod{m}$ *and* $c \equiv d \pmod{m}$, *then* $ac \equiv bd \pmod{m}$. *In particular, if* $a \equiv b \pmod{m}$ *and k is a positive integer, then* $a^k \equiv b^k \pmod{m}$.
5. *If* $a \equiv b \pmod{m}$ *and* $d|m$, *then* $a \equiv b \pmod{d}$.
6. *For integer polynomial* $f(x)$, *if* $a \equiv b \pmod{m}$, *then* $f(a) \equiv f(b) \pmod{m}$.
7. *If* $ax \equiv ay \pmod{m}$ *and* $\gcd(a, m) = 1$, *then* $x \equiv y \pmod{m}$.
8. $ax \equiv ay \pmod{m}$ *if and only if* $x \equiv y \pmod{\frac{m}{\gcd(a,m)}}$.
9. $x \equiv y \pmod{m_i}$ *for* $i = 1, \ldots, r$ *if and only if* $x \equiv y \pmod{\operatorname{lcm}(m_1, \ldots, m_r)}$, *where* $m_1, m_2, \ldots, m_r$ *are integers.*
10. $x \equiv y \pmod{a}$, $x \equiv y \pmod{b}$ *and* $\gcd(a, b) = 1$ *then* $x \equiv y \pmod{ab}$.
11. *If* $a \equiv b \pmod{m}$ *then* $\gcd(a, m) = \gcd(b, m)$.

Example 2.1. In any set of 181 square integers, prove that one can always find a subset of 19 numbers such that the sum of these numbers is divisible by 19.

Solution. First prove that there are 10 residue classes for square integers modulo 19. Then use this fact along with the Pigeon Hole Principle to obtain the solution. □

Definition 2.6 (Euler's ϕ function)**.** For any positive integer n, the integer $\phi(n)$ is defined as the number of positive integers less than or equal to n and relatively prime to n.

For example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(6) = 2$, $\phi(8) = 4$ etc.

Example 2.2. For a prime p, $\phi(p) = p - 1$ and for two distinct primes p, q, $\phi(pq) = (p-1)(q-1)$.

Solution. For a prime p, all the natural numbers $1, 2, \ldots, p-1$ are coprime to p. Prove the second part. □

Theorem 2.11 (Formula for $\phi(n)$)**.** *If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where p_i's are distinct primes and a_i's are positive integers, then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$.*

Theorem 2.12 (Fermat's little theorem)**.** $a^p \equiv a \pmod{p}$ *for any prime p and any integer a. Equivalently, $a^{p-1} \equiv 1 \pmod{p}$ for prime p and integer a with $p \nmid a$.*
Alternative form of Fermat's little theorem:
$p|(a^p - a)$ for any prime p and any positive integer a. Equivalently, $p|(a^{p-1} - 1)$ for prime p and integer a with $p \nmid a$.

Exercise. Prove that the two statements for Fermat's little theorem are equivalent.

Theorem 2.13 (Euler's theorem)**.** *If a, m are any two integers with $\gcd(a, m) = 1$, then $a^{\phi(m)} \equiv 1 \pmod{m}$.*

Theorem 2.14 (Wilson's theorem)**.** *If p is a prime, then $(p-1)! \equiv (-1) \pmod{p}$.*

Definition 2.7 (Greatest Integer Function)**.** For a real number x, the notation $[x]$ denotes the greatest integer less than or equal to x.

For example, $[1.1] = 1$, $[2.99999] = 2$, $[5.00001] = 5$, $[-5.8] = -6$, $[2] = 2$, $[-3] = -3$, $[\pi] = 3$, $[-\pi] = -4$.

Lemma 2.3 (De Polignac's Formula)**.** *If p is a prime and e is the largest exponent such that p^e divides $n!$ (i.e., $p^e|n!$ and $p^{e+1} \nmid n!$), then $e = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]$.*

Note that $\left[\frac{n}{p^i}\right] \neq 0$ only for finitely many exponents i. In other words, for any integer n, there exists a positive integer k such that $\left[\frac{n}{p^i}\right] = 0$ for all $i \geq k$.

Quick Exercise

1. For any real number x, prove that $x - 1 < [x] \leq x < [x] + 1$ and $0 \leq x - [x] < 1$.
2. For any real number x, if m is an integer, prove that $[x + m] = [x] + m$.

3. For any real number x, if m is a positive integer, prove that $[[x]/m] = [x/m]$.
4. Prove that $-[-x]$ is the least integer greater than or equal to x.
5. For any real number x, prove that $[x + 0.5]$ is the nearest integer to x, and if two integers are equally near to x, then $[x + 0.5]$ is the larger of the two.

2.5. Arithmetic Functions

Definition 2.8 (Functions $d(n)$ and $\sigma(n)$)**.** The number of positive divisors of n is denoted by $d(n)$, and the sum of all the positive divisors of n is denoted by $\sigma(n)$.

For example, $d(1) = 1$, $\sigma(n) = 1$, $d(2) = 2$, $\sigma(2) = 3$, $d(4) = 3$, $\sigma(4) = 7$.

Theorem 2.15 (Formula for $d(n)$ and $\sigma(n)$)**.** *If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where p_i's are distinct primes and a_i's are positive integers, then $d(n) = (a_1+1)(a_2+1)\cdots(a_r+1)$ and $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1}-1}{p_i-1}$.*

Definition 2.9 (Pythagorean Triples)**.** If $x^2 + y^2 = z^2$ for positive integers x, y, z, then the triple (x, y, z) is called a Pythagorean triple. If $\gcd(x, y, z) = 1$, then such a triple is called a primitive Pythagorean triple.

For example, $(3, 4, 5)$, $(5, 12, 13)$, $(6, 8, 10)$, $(15, 36, 39)$ are Pythagorean triples, out of which $(3, 4, 5)$, $(5, 12, 13)$ are primitive Pythagorean triples.

Exercise. If (x, y, z) is a Pythagorean triple, then prove that at least one of x, y is even. If (x, y, z) is a primitive Pythagorean triple, then prove that at least one of x, y is even and the other one is odd.

Theorem 2.16. *A primitive Pythagorean triple (x, y, z) with even y can be represented as $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$, where r, s are arbitrary positive integers of opposite parity (one even and other odd), with $r > s$ and $\gcd(r, s) = 1$.*

2.6. Representation of Positive Integers

Any positive integer n can be uniquely represented as $n = n_s b^s + n_{s-1} b^{s-1} + \cdots + n_1 b + n_0$, where $b > 1$ is an integer, $0 \le n_i < b$ for $i = 0, 1, \ldots, s$, and $n_s \neq 0$.

We often denote this representation by $n = (n_s, n_{s-1}, \ldots, n_1, n_0)_b$, known as the digital representation of n in base b, where the coefficients n_i are called digits. For any $k > s$, we may write $n = (n_k, n_{k-1}, \ldots, n_1, n_0)_b$, where $n_k = \ldots = n_{s+1} = 0$.

2.7. Solved Examples

Example 2.3 (RMO 1990)**.** Find the remainder when 2^{1990} is divided by 1990.

Solution. Since 199 is prime, $199|2^{199} - 2$, by Fermat's little theorem. Therefore, $199|2^{1990} - 2^{10}$ (why?). Also $10|2^{1990} - 2^{10}$ (see Theorem 2.2). Since 199 and 10 are prime to each other $1990|2^{1990} - 2^{10}$. Hence the remainder is $2^{10} = 1024$. □

Example 2.4 (RMO 1990)**.** N is a 50 digit decimal number. All digits except the 26th digit from the left are 1. If N is divisible by 13, find the 26th digit from left.

Solution. Observe that 111111 is divisible by 13. Let the 26th digit (from the left) is d. Put $a = 111...111(24 \text{ times}) = 111111 \times (10^{18} + 10^{12} + 10^{6} + 1)$, therefore $N = a \times 10^{26} + (10 + d) \times 10^{24} + a$, hence, $(10 + d) \times 10^{24} = N - a \times (10^{26} + 1)$ is divisible by 13. Now 13 is a prime and 10^{24} is not divisible by 13, so $13|(10 + d)$, which shows that $d = 3$. □

Solution.(alternative) Using rule for divisibility by 13 (see Theorem 2.1). □

Example 2.5 (INMO 1990)**.** Determine the number of three subsets of the set $\{1, 2, \ldots, 300\}$ for which the sum of the elements is multiple of 3.

Solution. When the sum is multiple of 3. All are same modulo 3 or all have different modulo 3. There are three different residue classes, each having 100 many integers. We can choose 3 integers with same modulo in $3\binom{100}{3}$ ways and 3 integers with different modulo in $\left(\binom{100}{1}\right)^3$ ways.

□

Example 2.6 (RMO 1991)**.** Prove that $n^4 + 4^n$ is not a prime number for any integer $n > 1$.

Solution. If n is even, then $n^4 + 4^n$ is divisible by 4 for $n > 1$.
If n is odd, let $n = 2k + 1$ for some positive integer k, then we have,
$n^4 + 4^n = n^4 + 4^{2k+1} = n^4 + 4 \cdot 4^{2k} = n^4 + 4 \cdot (2^k)^4$. Since, it is of the form $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$,
we have the factorization: $n^4 + 4^n = (n^2 + 2n(2^k) + 2(2^k)^2)(n^2 - 2n(2^k) + 2(2^k)^2)$.
Note that $n^2 - 2n(2^k) + 2(2^k)^2 > n^2 - 2n(2^k) + 2(2^k)^2 = (n - 2^k)^2 + (2^k)^2 > 1$ for $n > 1$ equivalently, $k > 0$. □

Example 2.7 (RMO 2004)**.** Let $\langle p_1, p_2, p_3, \ldots, p_n, \ldots \rangle$ be a sequence of primes defined by $p_1 = 2$ and p_{n+1} being the largest prime factor of $(p_1 p_2 \ldots p_n + 1)$ for $n > 1$. Prove that $p_n \neq 5$ for any n.

Solution. Note that, $p_1 = 2, p_2 = 3, p_3 = 7$. Clearly, $2 \nmid p_n$ for $n > 1$.
Now, if possible, assume that $p_n = 5$ for some $n > 3$. Then, $p_1 p_2 \cdots p_{n-1} + 1$ does not have any prime divisor > 5. Also $p_1 p_2 \cdots p_{n-1} + 1$ does not have prime divisors 2 and 3. So, $p_1 p_2 \cdots p_{n-1} + 1 = 5^k$ for some positive integer k.
Thus, we have $2p_2 p_3 \ldots p_{n-1} = 5^k - 1$, which is a contradiction since the R.H.S. is a multiple of 4, while the L.H.S. (although even) is not. □

Example 2.8. If both n and $\sqrt{n^2 + 204n}$ are positive integers, find the maximum value of n.

Solution. For some positive integer m, we have $n^2 + 204n = m^2$. Considering this a quadratic in n, note that the discriminant $204^2 + 4m^2$ must be an even perfect square (why?), say $4a^2$. So we have $a^2 - m^2 = 102^2 = 2^2 \cdot 3^2 \cdot 17^2$.

Since $(a+m)$ and $(a-m)$ are of same parity, so both are even, and that leads to: $\frac{a-m}{2} \cdot \frac{a+m}{2} = 3^2 \cdot 17^2$. We can see only four solutions for $(\frac{a-m}{2}, \frac{a+m}{2})$ are $(1, 3^2 17^2), (3, 3 \cdot 17^2), (17, 3^2 17)$ and $(3^2, 17^2)$. To obtain the maximum n, we require that m is also the maximum, so $m = 3^2 17^2 - 1 = 2600 \implies n = 2500$. □

Example 2.9 (RMO 2013)**.** Find all 4-tuples (a, b, c, d) of natural numbers with $a \le b \le c$ and $a! + b! + c! = 3^d$.

Solution. Note that if $a > 1$ then the left-hand side is even, and therefore $a = 1$. If $b > 2$ then 3 divides $b! + c!$ and hence 3 does not divide the left-hand side (as $a = 1$). Therefore $b = 1$ or $b = 2$.

If $b = 1$ then $c! + 2 = 3^d$, so $c < 2$ and hence $d = 1$. If $b = 2$ then $c! = 3^d - 3$. Note that $d = 1$ does not give any solution. If $d > 1$ then $9 \nmid c! = 3(3^{d-1} - 1)$, so $c < 6$. By checking the values for $c = 2, 3, 4, 5$ we see that $c = 3$ and $c = 4$ are the only two solutions. Thus $(a, b, c, d) = (1, 1, 1, 1), (1, 2, 3, 2)$ or $(1, 2, 4, 3)$. □

Example 2.10 (RMO 1992 and extension)**.** Consider three positive integers a, b, c with no common factor other than 1 (i.e., there is no prime p such that $p \mid a$, $p \mid b$ and $p \mid c$) that satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. For example, note that $(n+1, n(n+1), n)$; $n \in \mathbb{N}$ fit the conditions. Prove that:

1. $a + b$ is a square and $a + b - 2c$ is a sum of two squares.
2. $\gcd(a, b) > 1$.
3. If $\gcd(a, c) = 1, \gcd(b, c) = 1$, then show that $a = 2, b = 2, c = 1$.

Solution.

1. $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} \implies \frac{a+b}{ab} = \frac{1}{c} \implies (a+b)c = ab \implies (a-c)(b-c) = c^2$. Let p be a prime factor of c. Then p divides $a - c$ or $b - c$, but not both (as a, b, c do not have any common divisor). Assume that, $p \mid (a - c)$. Now, since p^2 is a factor of c^2, p^2 divides $a - c$ and $p \nmid (b - c)$. Thus, $a - c$ and $b - c$ are both squares (otherwise, as their product is a square, there will be common prime factor of them, which is a contradiction). Now assume that $a - c = k^2$, $b - c = l^2$, with the condition that $\gcd(k, l) = 1$. Therefore, $c^2 = k^2 l^2 \implies c = kl$. Now, it is easy to see that $a + b - 2c = k^2 + l^2$ and $a + b = (k + l)^2$.
2. Assume, if possible, that $\gcd(a, b) = 1$. Now, $c(a + b) = ab \implies a \mid c(a + b) \implies a \mid cb \implies a \mid c$. But $\frac{1}{a} < \frac{1}{c} \implies a > c$, a contradiction.
3. The previous part shows that, $a \mid cb$. If $\gcd(a, c) = 1$, then $a \mid b$. A similar argument leads to: $b \mid a$. So, $a = b$. Now, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} \implies a = 2c$. Now, $\gcd(a, c) = \gcd(2c, c) = c \implies c = 1$.

□

Example 2.11 (RMO 2014). Suppose for some positive integers r and s, the number 2^r is obtained by permuting the digits of the number 2^s in decimal expansion. prove that $r = s$.

Solution. Suppose $s \leq r$. If $s < r$ then $2^s < 2^r$. Since the number of digits in 2^s and 2^r are the same, we have $2^r < 10 \times 2^s < 2^{s+4}$. Thus we have $2^s < 2^r < 2^{s+4}$ which gives $r = s+1$ or $r = s+2$ or $r = s+3$. Let a be the sum of the digits of 2^s, then a be the sum of the digits of 2^r also. We know that the difference between an positive integer and sum of it's digits is divisible by 9, hence, $2^r - a$ and $2^s - a$ is divisible by 9. Therefore $2^r - 2^s$ is divisible by 9. Since $r = s+1$ or $r = s+2$ or $r = s+3$ we have $2^r - 2^s = 2^s$ or 3×2^s or 7×2^s, but none of these three numbers is divisible by 9. We conclude that $s < r$ is not possible, hence, $r = s$. □

Example 2.12 (RMO 2014). Is it possible to write the numbers $17, 18, \ldots, 32$ in a 4×4 grid of unit squares, with one number in each square, such that the product of the numbers in each 2×2 sub-grids $AMRG, GRND, MBHR$ and $RHCN$ is divisible by 16?

Solution. It is not possible. The product of all the numbers in all the four sub-squares is divisible by 2^{16} but not divisible by 2^{17} as $17 \times 18 \times 19 \times \cdots \times 32 = 2^{16}k$, for some odd integer k. Observe that there is $32 = 2^5$ and it must appear in some sub-square. Hence there will be 2^{11} available for the product of the remaining three sub-squares. So other three sub-squares can not be divisible by 16 each as then product of all the 12 numbers of these sub-squares must be divisible by 2^{12}. □

Example 2.13 (INMO 1995). Show that the number of 3-element subset $\{a, b, c\}$ of $\{1, \cdots, 63\}$ with $a+b+c < 95$ is less the number of those with $a+b+c > 95$.

Solution. Consider $c' = 64 - c$, $b' = 64 - b$, and $a' = 64 - a$. If $a+b+c < 95$ then $a' + b' + c' > 95$. □

Example 2.14 (INMO 1997). Find the number of 4×4 arrays whose entries are from the set $\{0, 1, 2, 3\}$ and which are such that the sum of numbers in each of the four rows and each of the four columns is divisible by 4. (An $m \times n$ array is an arrangement of mn numbers in m rows and n columns.)

Solution. Prove that the fourth row and the fourth column is uniquely determined by the other 3×3 elements. □

Example 2.15 (INMO 2001). Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a+b-c-d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

Solution. If $a - c$ and $b - d$ are divisible by 20, we are done. If not, there must exist at most three integers with same remainder modulo 20, and others with all different remainders. Look at all possible sums $a+b$ in this case, and apply PHP over the residue classes modulo 20.

In the second case, use the same approach with at most three integers with same remainders modulo 20, and the other five with all different remainders. Then use PHP, as before. $\square$

Example 2.16. Let a, b, c be positive integers such that$a$ divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a+b+c)^7$.

Solution. If a prime p divides a, then $p^2|b^2$ and hence $p|b$. This implies that $p|c^2$ and hence $p|c$. Thus every prime dividing a also divides b and c. By symmetry, this is true for b and c as well. We conclude that a, b, c have same set of prime divisors. We write $p^x||a$ to mean $p^x|a$ and $p^{x+1} \not| a$. Let $p^x||a, p^y||b$ and $p^z||c$. Without loss of generality we may assume $\min\{x, y, z\} = x$. Now $b|c^2$ implies that $y \leq 2z; c|a^2$ implies that $z \leq 2x$. We obtain $y \leq 2z \leq 4x$.
Thus $x+y+z$âĽď$x+2x+4x = 7x$. Hence the maximum power of p that divides abc is $x+y+z \leq 7x$. Since x is the minimum among x, y, z, p^x divides a, b, c. Hence p^x divides $a+b+c$. This implies that p^{7x} divides $(a+b+c)^7$. Since $x+y+z \leq 7x$, it follows that p^{x+y+z} divides $(a+b+c)^7$. This is true of any prime p dividing a, b, c. Hence abc divides $(a+b+c)^7$.

$\square$

Example 2.17. Find all triples (p, q, r) of primes such that $pq = r+1$ and $2(p^2 + q^2) = r^2 + 1$.

Solution. If p and q are both odd, then $r = pq - 1$ is even so $r = 2$. But in this case $pq \geq 3 \times 3 = 9$ and hence there are no solutions. This proves that either $p = 2$ or $q = 2$. If $p = 2$ then we have $2q = r+1$ and $8 + 2q^2 = r^2 + 1$. Multiplying the second equation by 2 we get $2r^2 + 2 = 16 + (2q)^2 = 16 + (r+1)^2$. Rearranging the terms, we have $r^2 - 2r - 15 = 0$, or equivalently $(r+3)(r-5) = 0$. This proves that $r = 5$ and hence $q = 3$. Similarly, if $q = 2$ then $r = 5$ and $p = 3$. Thus the only two solutions are $(p, q, r) = (2, 3, 5)$ and $(p, q, r) = (3, 2, 5)$. $\square$

Example 2.18. Let a_1, b_1, c_1 be natural numbers. We define
$a_2 = \gcd(b_1, c_1), b_2 = \gcd(c_1, a_1), c_2 = \gcd(a_1, b_1)$, and $a_3 = \operatorname{lcm}(b_2, c_2), b_3 = \operatorname{lcm}(c_2, a_2), c_3 = \operatorname{lcm}(a_2, b_2)$. Show that $\gcd(b_3, c_3) = a_2$.

Solution. For a prime p and a natural number n we shall denote by $v_p(n)$ the power of p dividing n. Then it is enough to show that $v_p(a_2) = v_p(\gcd(b_3, c_3))$ for all primes p. Let p be a prime and let $\alpha = v_p(a_1), \beta = v_p(b_1)$ and $\gamma = v_p(c_1)$. Because of symmetry, we may assume that $\alpha \leq \beta \leq \gamma$. Therefore, $v_p(a_2) = \min\{\beta, \gamma\} = \beta$ and similarly $v_p(b_2) = v_p(c_2) = \alpha$. Therefore $v_p(b_3) = \max\{\alpha, \beta\} = \beta$ and similarly $v_p(c_3) = \max\{\alpha, \beta\} = \beta$. Therefore $v_p(\gcd(b_3, c_3)) = v_p(a_2) = \beta$. This completes the solution. $\square$

2.8. Exercise Problems

1. The 64 squares of an 8×8 chess board are filled with positive integers in such a way that each integer is the average of the integers on the neighboring square. (Two squares are neighbors if they share a common edge or vertex. Thus a square can have $8, 5$ or 3 neighbors depending on its position). Show that all the 64 integers entries are in fact equal.
2. What is the digit in the unit position of the integer $1! + 2! + 3! + \cdots + 99!$.
3. Prove that if $p \geq 7$ is a prime, then $p^4 \equiv 1 \pmod{240}$.
4. Let $d_1, d_2, \ldots, d_k$ be all the factors of a positive integer n including 1 and n. If $d_1 + d_2 + \cdots + d_k = 72$, then find the value of $1/d_1 + 1/d_2 + \cdots + 1/d_k$ in terms of n.
5. If $\gcd(a, b) = 1$, then prove that $\gcd(a^2, b^2) = 1$.
6. Find all positive integers x, y such that $7^x - 3^y = 4$ [IMOTC 1995].
7. Let k be any given integer. Prove that there are infinitely many triples a, b, c of positive integers such that $bc - k, ca - k, ab - k$ are all squares. [IMOTC 1995]
8. Four person A, B, C and D think of one natural number. A says it consists of two digits. B says it is a divisor of 150. C says it is not 150. D says it is divisible by 25. One of them is not telling the truth and others tell true. Which one of them is not telling the truth? Give reason.
9. For a natural number n, let $a_n = n^2 + 20$, If $d_n = \gcd(a_n, a_{n+1})$, Then show that, d_n divides 81.
10. A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?
11. Prove that there exists infinitely many prime numbers.
12. Prove that there exists infinitely many prime numbers of the form $4n - 1$.
13. Prove that there exists infinitely many prime numbers of the form $4n + 1$.
14. Prove that any prime p divides $\binom{p}{r}$.
15. Show that there exists a positive integer n such that $n!$ ends exactly in 1993 zeros.
16. Prove that the product of any k consecutive positive integers is divisible by $k!$.
17. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.
18. Find all positive integers n, not a perfect square, such that $[\sqrt{n}]^3$ divides n^2.
19. Prove that if a positive integer is constructed using only the two digits 0 and 6, then it can not be a perfect square. What if it is constructed only using 0 and 8?
20. Prove that N^5 and N have the same decimal digit in its units place.

21. Prove that for any prime $p > 5$, $p^4 - 1$ divisible by 240.
22. For two distinct primes p, q, prove that pq divides $p^{q-1} + q^{p-1} - 1$.
23. Prove that if $(n-1)! + 1$ is divisible by n, then n is a prime number.
24. Find all primes p for which the quotient $(2^{p-1} - 1)/p$ is a perfect square.
25. Let $a_1, a_2, \ldots, a_n$ be n integers, not necessarily distinct. Then we can find some among them whose sum is divisible by n.
26. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots, 541, 551\}$ having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements. [INMO-1989]
27. There are 10 objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the 10 objects can be divided into two groups that balance each other when placed on the two pans of a balance. [INMO-1991]
28. Let $a_1, a_2, \ldots, a_n$ be n real numbers. Prove that there is a real number k such that $a_1 + k, a_2 + k, \ldots, a_n + k$ are all irrational numbers.
29. Given any set of 12 integers, show that there are two whose difference is divisible by 11.
30. Given 8 natural numbers, none greater than 15, show that at least three pairs of them have the same difference.
31. Five lattice points are chosen in the plane. Show that one can always choose two of these such that the line joining them passes through a lattice point.
32. The digits $1, 2, \ldots, 9$ are divided into three groups. Prove that the product of the numbers in at least one of the groups must exceed 71.
33. Show that there is a natural number whose decimal representation consists entirely of the digit 1 that is divisible by 1987.
34. Show that there is a power of three whose decimal representation ends in 001.
35. Let $a_1, a_2, \ldots a_n$ be n integers, not necessarily distinct. Then we can find some among them whose sum is divisible by n.
36. Given a set of $n+1$ positive integer, none greater than $2n$, show that at least one member of the set divides another member of the set.
37. Suppose six points are placed arbitrarily in a 3×4 rectangle. Then we can find two among them such that the distance between them is not more than $\sqrt{5}$ units.
38. Let N be a 16-digit natural number. Show that one can find some consecutive digits of N such that the product of these digits is a square.
39. Let there be a set of five distinct points with integer co-ordinates in the $X-Y$ plane. Show that the mid-point of the line joining at least one pair of these points has integer co-ordinates.

40. Two disks, one smaller than the other, are each divided into 200 congruence sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is either painted red or blue with no condition on the number of red or blue sectors. The smaller disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of the sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least 100 (Principle of counting in two different ways).
41. Let twenty one distinct points be arbitrarily chosen on a circle. Prove that there exist at least one hundred arcs that subtend at most degrees at the center of the circle. [IMOTC 1995]
42. Let $(a_1, \cdots, a_n)$ be a permutation of $(1, 2, \cdots, n)$ with n odd. Then $(a_1 - 1) \cdots (a_n - n)$ is even.
43. Let a, b, c be positive integers such that$a$ divides b^4, b divides c^4 and c divides a^4. Prove that abc divides $(a + b + c)^2 1$.

2.9. RMO Problems

1991 A four-digit number has the following properties – it is a perfect square, its first two digits are equal to each other, its last two digits are equal to each other. Find all such four-digit numbers.

1991 There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

- ◇ remove an equal number of balls simultaneously from the urns, and
- ◇ double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty.

1991 Find all integer values of a such that the quadratic expression

$$(x + a)(x + 1991) + 1$$

can be factored as a product $(x + b)(x + c)$ where b and c are integers.

1992 Determine the set of integers n for which $n^2 + 19n + 92$ is a square of an integer.

1992 Determine the largest 3-digit prime factor of the integer $\binom{2000}{1000}$.

1993 Prove that the ten's digit of any power of 3 is even. [e.g. the ten's digit of $3^6 = 729$ is 2].

1993 Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162.

1994 Find all 6-digit natural numbers $a_1a_2a_3a_4a_5a_6$ formed by using the digits 1, 2, 3, 4, 5, 6, once each such that the number $a_1a_2 \ldots a_k$ is divisible by k, for

$1 \le k \le 6$.

1994 Let A be a set of 16 positive integers with the property that the product of any two distinct numbers of A will not exceed 1994. Show that there are two numbers a and b in A which are not relatively prime.

1994 Find the number of all rational numbers $\frac{m}{n}$ such that

- ◇ $0 < \frac{m}{n} < 1$
- ◇ m and n are relatively prime
- ◇ $mn = 25!$

1995 Call a positive integer n **good** if there are n integers, positive or negative, and not necessarily distinct, such that their sum and product are both equal to n (e.g. 8 is **good** since

$$8 = 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1(-1)(-1) = 4 + 2 + 1 + 1 + 1 + 1 + (-1) + (-1)).$$

Show that integers of the form $4k + 1$ $(k \ge 0)$ and $4l$ $(l \ge 2)$ are **good**.

1995 Prove that among any 18 consecutive 3-digit numbers there is at least one number which is divisible by the sum of its digits.

1996 Find all triples (a, b, c) of positive integers such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 3.$$

1996 Suppose N is an n-digit positive integer such that

- ◇ all the n-digits are distinct; and
- ◇ the sum of any three consecutive digits is divisible by 5.

Prove that n is at most 6. Further, show that starting with any digit one can find a six-digit number with these properties.

1996 Given any positive integer n show that there are two positive rational numbers a and b, $a \ne b$, which are not integers and which are such that $a - b$, $a^2 - b^2$, $a^3 - b^3, \ldots, a^n - b^n$ are all integers.

1996 If A is a fifty-element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100 show that A contains a square.

1997 For each positive integer n, define $a_n = 20 + n^2$, and $d_n = \gcd(a_n, a_{n+1})$. Find the set of all values that are taken by d_n and show by examples that each of these values are attained.

1997 Solve for real x:

$$\frac{1}{[x]} + \frac{1}{[2x]} = 9(x) + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x and $(x) = x - [x]$, [e.g. [3.4] = 3 and (3.4) = 0.4].

1998 Let n be a positive integer and $p_1, p_2, \ldots, p_n$ be n prime numbers all larger than 5 such that 6 divides $p_1^2 + p_2^2 + \cdots + p_n^2$. Prove that 6 divides n.

1998 Find the minimum possible least common multiple (lcm) of twenty (not necessarily distinct) natural numbers whose sum is 801.

1999 Prove that the inradius of a right-angled triangle with integer sides is an integer.

1999 Find the number of positive integers which divide 10^{999} but not 10^{998}.

1999 Find all solutions in integers m, n of the equation

$$(m-n)^2 = \frac{4mn}{m+n-1.}$$

2000 All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000th number in this list.

2000 Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab? If $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab?

2001 Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.

2001 Find the number of positive integers x which satisfy the condition $[x/99] = [x/101]$.

2002 Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$ and $b_1 > b_2 > b_3 > b_4 > b_5$.

- ◇ Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$, is at least 6.
- ◇ Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

2002 Find all integers a, b, c, d satisfying the following relations:

- ◇ $1 \le a \le b \le c \le d$;
- ◇ $ab + cd = a + b + c + d + 3$.

2002 Let a, b, c be positive integers such that a divides b^2, b divides c^2, and c divides a^2. Prove that abc divides $(a+b+c)^7$.

2003 If n is an integer greater than 7, prove that $\binom{n}{7} - [n/7]$ is divisible by 7.

2003 Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint non-empty subsets A and B of X such that: (i) $A \cup B = X$, (ii) prod(A) is divisible by prod(B), and (iii) the quotient prod(A)/prod(B) is as small as possible.

2003 Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

- ◇ $x \le y \le z$;
- ◇ $x + y + z \le 100$.

2004 Positive integers are written on all faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the three faces that meet at that corner is written. The sum of all these eight numbers at the corners is 2004. Find the sum of all six numbers written on the faces of the cube.

2005 If x, y are integers, and 17 divides both the expressions $x^2-2xy+y^2-5x+7y$ and $x^2-3xy+2y^2+x-y$, then prove that 17 divides $xy-12x+15y$.

2005 Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)

2005 Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a+b+c+ab+bc+ca=abc+1.$$

2006 A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always **two** congruent rectangles.

2006 Find the least possible value of $(a+b)$, where a, b are positive integers, such that 11 divides $a+13b$, and 13 divides $a+11b$.

2006 Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in at least two different ways.

2007 Let a, b, c be three natural numbers such that $a<b<c$ and $\gcd(c-a, c-b)=1$. Suppose there exists an integer d such that $(a+d, b+d, c+d)$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c+d=l^2+m^2$.

2008 Three nonzero real numbers a, b, c are said to be in harmonic progression if $1/a+1/c=2/b$. Find all three-term harmonic progressions a, b, c strictly increasing positive integers in which $a=20$ and b divides c.

2008 Prove that there exist two infinite sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of positive integers such that the following conditions holds simultaneously:

$$(i)\ 1<a_1<a_2<a_3<\cdots; \qquad (ii)\ a_n<b_n<a_n^2, \text{ for all } n\geq 1;$$

$$(iii)\ a_n-1 \text{ divides } b_n-1, \text{ for all } n\geq 1;$$

$$(iv)\ a_n^2-1 \text{ divides } b_n^2-1, \text{ for all } n\geq 1.$$

2009 Show that there is no integer a such that $a^2-3a-19$ is divisible by 289.

2010 For each integer $n\geq 1$, define $a_n=[n/[\sqrt{n}]]$. Find the number of all such n in the set $\{1,2,3,\ldots,2010\}$ for which $a_n>a_{n+1}$.

2011 A natural number n is chosen strictly between two consecutive perfect squares. The smaller of the two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n. Prove that $n-kl$ is a perfect square.

2012 Let a, b, c positive integers such that a divides b^5, b divides c^5 and c divides a^5. Prove that abc divides $(a+b+c)^{31}$.

2012 Find all positive integers n such that $3^{2n} + 3n^2 + 7$ is a perfect square.

Chapter 3

Inequalities

3.1. Introduction

There exists a natural ordering amongst the real numbers. Given any two real numbers, one can determine whether they are equal, and in case they are not, one may also ascertain which one is the bigger of the two. Formally, we may state:

> For any two real numbers a and b, either of the following relations is true: $a > b$ or $a = b$ or $a < b$. This is known as the law of trichotomy.

For real numbers a, b, c and d,

- ◇ If $a < b$ and $b < c$, then $a < c$.
- ◇ If $a < b$, then $a + c < b + c$.
- ◇ If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c < 0$, then $ac > bc$.
- ◇ If $a < b$ and $c < d$, then $a + c < b + d$.
- ◇ If $a < b$, $c < d$ and $a, b, c, d > 0$, then $ac < bd$.
- ◇ For a positive integer n, if $a, b > 0$, then $a > b$ implies $a^n > b^n$.
- ◇ For a negative integer n, if $a, b > 0$, then $a > b$ implies $a^n < b^n$.
- ◇ For positive integers $n > m$, if $1 > a > 0$, then $a^n < a^m$.
- ◇ For positive integers $n > m$, if $a > 1$, then $a^n > a^m$.

Quick Exercise

1. Prove all the above results, and provide examples for each. What will happen in each case if the sign '$>$' is replaced by '$\geq$' in the above statements?
2. For real numbers a, b, prove that $a^2 + b^2 \geq (a + b)^2/2$.
3. For two distinct real numbers a, b, determine if $3ab^2 > a^3 + 2b^3$ or otherwise.

3.2. Important Inequalities

Theorem 3.1 (AM-GM-HM inequality)**.** *For positive real numbers $a_1, a_2, \ldots, a_n$,*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \;\geq\; (a_1 a_2 \cdots a_n)^{1/n} \;\geq\; \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

The following result related to the AM-HM inequality is also quite useful:

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \;\geq\; n^2.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.
That means the following four statements are equivalent:

$$(1)\ AM = GM \quad (2)\ AM = HM \quad (3)\ GM = HM \quad (4)\ a_1 = a_2 = \cdots = a_n.$$

Theorem 3.2 (weighted AM-GM-HM inequality)**.** *For positive real numbers $a_1, a_2, \ldots, a_n$, and $w_1, w_2, \ldots, w_n$ (known as weights);*

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n} \;\geq\; (a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \cdots + w_n}} \;\geq\; \frac{w_1 + w_2 + \cdots + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + \cdots + \frac{w_n}{a_n}}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 3.3. *(RMS-AM inequality) For real numbers $x_1, x_2, \ldots, x_n$,*

$$\frac{x_1^2 + x_2^2 + \cdots x_n^2}{n} \;\geq\; \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2.$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.4. *(Bernoulli's inequality) For any real number $x > -1$,*

$$(1+x)^a \;\geq\; 1 + ax \quad \text{for } a \geq 1 \quad \text{and} \quad (1+x)^a \;\leq\; 1 + ax \quad \text{for } 0 < a < 1.$$

Theorem 3.5. *(Rearrangement inequality) Suppose that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are two sequences sorted in the same direction. Then for any rearrangement or permutation (c_n) of the numbers (b_n),*

$$\sum a_i b_i \;\geq\; \sum a_i c_i \;\geq\; \sum a_i b_{n-i+1}.$$

Theorem 3.6. *For $a > 0$, we have $f(x) = ax^2 + bx + c \geq 0$ for all real numbers x if and only if $b^2 - 4ac \leq 0$.*

Proof. The proof follows from the observation

$$f(x) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right)$$

□

Exercise: Prove the RMS-AM inequality and the Bernoulli's inequality.

3.3. Solved Examples

Example 3.1. If a, b, c are positive reals, then prove that $6abc \le bc(b+c) + ca(c+a) + ab(a+b)$.

Solution. $b^2c + a^2c \ge 2\sqrt{b^2ca^2c} = 2abc$. Similarly, $bc^2 + a^2b \ge 2abc$ and $ab^2 + ac^2 = 2abc$. ☐

Example 3.2. Prove that, (a) $2^{48} < 3^{32}$, (b) $2^{50} < 3^{32}$ (c) $2^{49} < 3^{31}$.

Solution. (a) $\dfrac{3^{32}}{2^{48}} = \dfrac{(3^2)^{16}}{(2^3)^{16}} = \left(\frac{9}{8}\right)^{16} > 1.$

(b) $\dfrac{3^{32}}{2^{48}} = \dfrac{(3^2)^{16}}{(2^3)^{16}} = \left(\frac{9}{8}\right)^{16} = \left(1 + \frac{1}{8}\right)^{16} > 1 + \frac{16}{8} + \frac{120}{64} > 4$ (using binomial theorem).

(c) Try to show $\dfrac{3^{32}}{2^{48}} > 6.$ ☐

Example 3.3. If $x, y, z > 0$ and $x + y + z = 1$ then show that $2 < (1+x)(1+y)(1+z) < \dfrac{64}{27}$.

Solution. Using A.M.-G.M. inequality on three positive numbers $(1+x), (1+y)$ and $(1+z)$ we get, $\dfrac{(1+x)+(1+y)+(1+z)}{3} > \sqrt[3]{(1+x)(1+y)(1+z)}$

$\implies (1+x)(1+y)(1+z) < \left(\dfrac{3+(x+y+z)}{3}\right)^3.$

To prove the other inequality, note that, $(1+x)(1+y)(1+z) = 1+x+y+z+xy+xz+yx+xyz > 1+x+y+z = 2.$ ☐

Example 3.4. Given $a, b, c > 0$ and $(a+b)(b+c)(c+a) = 8$, show that

$$\frac{a+b+c}{3} \ge \sqrt[27]{\frac{a^3+b^3+c^3}{3}}.$$

Solution. Since $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(a+c) = a^3+b^3+c^3+24$, and it follows that, $(a+b+c)^3 = (a^3+b^3+c^3)+(3)+(3)+(3)+(3)+(3)+(3)+(3)+(3)$ $\ge 9\sqrt[9]{(a^3+b^3+c^3)\times 3^8}$ (after applying A.M-G.M. inequality).
Therefore, $(a+b+c)^3 \ge 27\sqrt[9]{(a^3+b^3+c^3)/3}$, i.e., $\left(\frac{a+b+c}{3}\right)^3 \ge \sqrt[9]{\frac{a^3+b^3+c^3}{3}}$. ☐

Example 3.5. If $x, y, z > 0$ and $x+y+z = a$, then show that: $8xyz \le (a-x)(a-y)(a-z) \le \dfrac{8}{27}a^3$.

Solution. Apply A.M. $\ge$ G.M. on $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ separately and multiply to yield: $\dfrac{(x+y)(y+z)(z+x)}{8} \ge \sqrt{x^2y^2z^2} \implies (a-x)(a-y)(a-z) \ge 8xyz.$

On the other hand, applying A.M. $\geq$ G.M. on $(x+y),(y+z),(z+x)$ yields:

$$\frac{(x+y)+(y+z)+(z+x)}{3} \geq \sqrt[3]{(x+y)(y+z)(z+x)}$$
$$\implies \frac{2a}{3} \geq \sqrt[3]{(a-x)(a-y)(a-z)}.$$

□

Example 3.6. (partly INMO 1990)

If a, b, c are sides of a triangle, prove that $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}<2.$

Solution.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{2a}{(b+c)+(b+c)}+\frac{2b}{(c+a)+(c+a)}+\frac{2c}{(a+b)+(a+b)}$$
$$<\frac{2a}{(b+c)+(a)}+\frac{2b}{(c+a)+(b)}+\frac{2c}{(a+b)+(c)}=2 \text{ [since } a<b+c \text{ and so on].}$$

□

Example 3.7. Let $a, b, c>0$, prove that $\dfrac{a^2+b^2+c^2}{ab+bc+ca}+\dfrac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$

Solution. Without loss of generality, assume that $a \geq b \geq c$.

Now, we have $2b(a+c)^2-(a+b)(b+c)(a+c)=(a+c)(a-b)(b-c) \geq 0$ $\implies \dfrac{8abc}{(a+b)(b+c)(a+c)} \geq \dfrac{4ac}{(a+c)^2}.$

So it is sufficient to prove, $\dfrac{a^2+b^2+c^2}{ab+bc+ac}+\dfrac{4ac}{(a+c)^2} \geq 2.$

Since, $\dfrac{a^2+b^2+c^2}{ab+bc+ac}=\dfrac{(a+b+c)^2}{ab+bc+ac}-2$, it is equivalent to show;

$$\frac{(a+b+c)^2}{ab+bc+ac} \geq 4-\frac{4ac}{(a+c)^2}=\frac{4(a^2+ac+c^2)}{(a+c)^2}$$

i.e., to show, $4(a^2+ac+c^2)(ab+bc+ac) \leq (a+c)^2(a+b+c)^2$

Now, using A.M.-G.M. inequality on (a^2+ac+c^2) and $(ab+bc+ac)$, we have, $4(a^2+ac+c^2)(ab+bc+ac) \leq (a^2+ac+c^2+ab+bc+ac)^2=[(a+c)^2+b(a+c)]^2=(a+c)^2(a+b+c)^2$, as required. □

Example 3.8. Given $a, b, c>0$ such that $a^2+b^2+c^2=\frac{5}{3}$. Prove that, $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}<\frac{5}{6abc}$.

Solution. $c(a+b) \leq \frac{c^2+(a+b)^2}{2} = \frac{a^2+b^2+c^2}{2} + ab = \frac{5}{6} + ab \implies \frac{1}{a} + \frac{1}{b} \leq \frac{5}{6abc} + \frac{1}{c}$ □

Example 3.9. Consider three positive reals x, y, z such that $xyz = 1$. Prove that: $\dfrac{x^4y^4}{x^2+y^2} + \dfrac{y^4z^4}{y^2+z^2} + \dfrac{x^4z^4}{x^2+z^2} \geq \dfrac{3}{2}$

Solution. Dividing by $x^2y^2z^2 (= 1)$ reduces the inequality to: $\dfrac{x^2y^2}{x^2z^2+y^2z^2} + \dfrac{y^2z^2}{x^2y^2+x^2z^2} + \dfrac{x^2z^2}{x^2y^2+y^2z^2} \geq \dfrac{3}{2}$. Now apply *Nesbitt's inequality*. □

Example 3.10. Prove the following inequality: $\dfrac{a^3}{a+b} + \dfrac{b^3}{b+c} + \dfrac{c^3}{c+a} \geq \dfrac{ab+bc+ca}{2}$, for a, b, c real positive numbers.

Solution.

$$\frac{a^3}{a+b} + \frac{b^3}{b+c} + \frac{c^3}{c+a} = \frac{a^4}{a^2+ab} + \frac{b^4}{b^2+bc} + \frac{c^4}{c^2+ac}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+bc+ca+ab} \text{ (by Engel's form of Cauchy-Schwarz inequality)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)}$$

$$\Big[\text{since, } 2(a^2+b^2+c^2) - 2(ab+bc+ca) = (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$

$$\implies ab+bc+ca \leq a^2+b^2+c^2\Big]$$

Therefore, $\dfrac{a^3}{a+b} + \dfrac{b^3}{b+c} + \dfrac{c^3}{c+a} \geq \dfrac{a^2+b^2+c^2}{2} \geq \dfrac{ab+bc+ca}{2}$.

□

Example 3.11. Let x, y be positive reals with $xy \geq 1$. Show that,

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} \geq \frac{2}{1+xy}.$$

Solution. The given inequality is equivalent to

$$\frac{2+x^2+y^2}{(1+x^2)(1+y^2)} \geq \frac{2}{1+xy},$$

which is equivalent to, $(2+x^2+y^2)(1+xy) \geq 2(1+x^2+y^2+x^2y^2)$, which is equivalent to, $2xy + x^3y + xy^3 \geq x^2 + y^2 + 2x^2y^2$ which is equivalent to, $(x-y)^2(xy-1) \geq 0$, which is true. □

Example 3.12. Find the maximum value of $24x - 8 - 9x^2$.

Solution. $24x - 8 - 9x^2 = 8 - (3x-4)^2 \leq 8$. Maximum attained for $x = 4/3$. □

Example 3.13. Show that, if $a, b, c > 0$, then $\frac{a^3+b^3}{2} \geq \left(\frac{a+b}{2}\right)^3$ and $\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2$.

Solution.
$$\frac{a^3+b^3}{2} \geq \left(\frac{a+b}{2}\right)^3$$
is equivalent to $4(a^3+b^3) \geq (a+b)^3$, which is equivalent to $a^3+b^3-a^2b-ab^2 \geq 0$. Now $a^3+b^3-a^2b-ab^2 = (a^2-b^2)(a-b) = (a+b)(a-b)^2 \geq 0$.

We have, $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$, therefore $a^2+b^2+c^2 \geq ab+bc+ca$ and hence $3(a^2+b^2+c^2) \geq (a+b+c)^2$. Hence the result. □

Example 3.14. If a, b, c are sides of a triangle such that $a \neq b \neq c$, then show that
$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} \geq \frac{9}{a+b+c}.$$

Solution. Using A.M.-H.M. inequality on three positive numbers $b+c-a, c+a-b, a+b-c$ we get the result. □

Example 3.15. If a, b, c are sides of a triangle, prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

Solution. By symmetry we may assume that $a \leq b \leq c$. In that case
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{a}{a+c} + \frac{c}{c+a} + \frac{c}{a+b} = 1 + \frac{c}{a+b} < 2.$$
□

Example 3.16 (INMO 1991)**.** If x, y, z are positive proper fractions satisfying $x+y+z=2$, prove that
$$\frac{xyz}{(1-x)(1-y)(1-z)} \geq 8.$$

Solution. We have $\frac{x}{2} = \frac{(1-y)+(1-z)}{2} \geq \sqrt{(1-y)(1-z)}$ and so on. Multiplying all three inequalities, we get $xyz/8 \geq (1-x)(1-y)(1-z)$. □

Example 3.17. If a, b, c are all positive, prove $a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b) \geq 0$.

Solution. We shall prove this inequality by case study.

- ◇ Case-I: When $a=b=c$ then equality holds.
- ◇ Case-II: When two of them are equal. Since the expression is symmetric with respect to a, b, c, it is enough to prove the inequality for $a=b$. When $a=b$ then the given inequality reduce to $c(c-a)^2 \geq 0$, which is trivially true.

◇ Case-III: When all the three numbers are distinct. Without loss of generality, we may assume that $a > b > c$. Then $a-b, b-c, a-c > 0$ and therefore $a(a-b)(a-c) > b(a-b)(b-c)$ which is equivalent to $a(a-b)(a-c)+b(b-a)(b-c) > 0$. Also $c(c-a)(c-b) \geq 0$.

□

Example 3.18. If a, b, c, d are positive numbers prove that $\sqrt{(a+c)(b+d)} \geq \sqrt{ab}+\sqrt{cd}$.

Solution. The given inequality is equivalent to $(a+c)(b+d) \geq ab+2\sqrt{abcd}+cd$, as the both sides are positive, which is equivalent to $(\sqrt{ab}-\sqrt{cd})^2 \geq 0$. □

Example 3.19. If $a, b > 0$ and $a+b=1$, show that $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}$.

Solution. Equivalent to show $(a^2+b^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq \frac{17}{2}$. Now $a^2+b^2 \geq (a+b)^2/2 = 1/2$. Hence enough to show $\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq 8$. Since $\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = \frac{a^2+b^2}{a^2b^2}$, it is enough to show $\frac{1}{a^2b^2} \geq 16$, i.e., enough to show $ab \leq 1/4$. We have $4ab = (a+b)^2-(a-b)^2 = 1-(a-b)^2 \leq 1$, hence the result.
Alternative proof: $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{1}{2}\left(a+\frac{1}{a}+b+\frac{1}{b}\right)^2 = \frac{1}{2}\left(1+\frac{1}{a}+\frac{1}{b}\right)^2$
Now $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$, hence, $\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.
Therefore, $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{1}{2}(1+4)^2$. □

Example 3.20. Show for every integer $n \geq 2$, $n! < \left(\frac{n+1}{2}\right)^n$ and $1 \cdot 3 \cdot 5 \cdots (2n-1) < n^n$.

Solution. Applying A.M.-G.M. inequality on n distinct positive numbers $1, 2, \ldots, n$, we get $(n!)^{1/n} < (1+2+\cdots+n)/n$, i.e., $(n!)^{1/n} < (n+1)/2$.

Again applying A.M.-G.M. inequality on n distinct positive numbers $1, 3, \ldots, (2n-1)$, we get

$$(1 \cdot 3 \cdot 5 \cdots (2n-1))^{\frac{1}{n}} < \frac{1+3+\cdots+(2n-1)}{n} = n.$$

□

Example 3.21. Let $a_1, a_2, \ldots, a_n$ are real numbers. If $a_1+a_2+\cdots+a_n \leq 1/2$, each $a_i \geq 0$, prove that $(1-a_1)(1-a_2)\cdots(1-a_n) \geq 1/2$.

Solution. We have $(1-a_1)(1-a_2) = 1-(a_1+a_2)+a_1a_2 \geq 1-(a_1+a_2)$.
By induction we can show that $(1-a_1)(1-a_2)\cdots(1-a_n) \geq 1-(a_1+a_2+\cdots+a_n)$.
Since $a_1+a_2+\cdots+a_n \leq 1/2$ we have the result. □

Example 3.22 (RMO 1990)**.** For all positive real numbers a, b, c prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

This inequality is known as Nesbitt's inequality.

Solution. Using A.M.-H.M. inequality on three positive real numbers $b+c, c+a$ and $a+b$ we get

$$(b+c+c+a+a+b)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 9$$

therefore,

$$\left(\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\right) \geq \frac{9}{2}$$

□

Example 3.23 (RMO 1991)**.** If a, b, c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4.$$

Solution. Using A.M.-G.M. inequality on four positive real numbers $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}$, we get the result. □

Example 3.24 (RMO 1992)**.** Prove that

$$1 < \frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001} < \frac{4}{3}.$$

Solution. Using A.M.-G.M. inequality on 2001 positive numbers $1001, 1002, \ldots, 3001$, we get

$$(1001+1002+\cdots+3001)\left(\frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001}\right) > 2001^2$$

Therefore,

$$\frac{2001(1001+3001)}{2}\left(\frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001}\right) > 2001^2$$

Hence,

$$\frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001} > 1.$$

Now,

$$\frac{1}{1000}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{1499} < \frac{1}{1000}+\frac{1}{1000}+\frac{1}{1000}+\cdots+\frac{1}{1000}$$

Implies,

$$\frac{1}{1000}+\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{1499} < 1/2.$$

Similarly,

$$\frac{1}{1500}+\frac{1}{1501}+\frac{1}{1502}+\cdots+\frac{1}{1999} < 1/3$$

and

$$\frac{1}{2000}+\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2999} < 1/2.$$

Hence,

$$\frac{1}{1000}+\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{2999}<\frac{4}{3}.$$

Since

$$\frac{1}{3000}+\frac{1}{3001}<\frac{1}{1000},$$

We have

$$\frac{1}{1001}+\frac{1}{1002}+\frac{1}{1003}+\cdots+\frac{1}{3001}<\frac{4}{3}.$$

□

Example 3.25 (RMO 1993)**.** If a, b, c, d are four positive real numbers such that $abcd=1$, prove that

$$(1+a)(1+b)(1+c)(1+d)\geq 16.$$

Solution. $(1+a)(1+b)=1+(a+b)+ab\geq 1+2\sqrt{ab}+ab=(1+\sqrt{ab})^2$. Similarly, $(1+c)(1+d)=(1+\sqrt{cd})^2$. Hence,

$$(1+a)(1+b)(1+c)(1+d)\geq(1+\sqrt{ab})^2(1+\sqrt{cd})^2\geq(1+\sqrt{abcd})^4=16.$$

□

Example 3.26 (RMO 1994)**.** If a, b and c are positive real numbers such that $a+b+c=1$, prove that

$$(1+a)(1+b)(1+c)\geq 8(1-a)(1-b)(1-c).$$

Solution. $1+a=(1-b)+(1-c)\geq 2\sqrt{(1-b)(1-c)}, 1+b=(1-a)+(1-c)\geq 2\sqrt{(1-a)(1-c)}$ and $1+c=(1-b)+(1-a)\geq 2\sqrt{(1-b)(1-a)}$. □

Example 3.27 (RMO 1995)**.** Show that for any real number x,

$$x^2\sin x+x\cos x+x^2+\frac{1}{2}>0.$$

Solution. We have, $x^2\sin x+x\cos x+x^2+\frac{1}{2}$

$$=x^2\sin x+x\cos x+\frac{x^2}{2}+\frac{x^2\sin^2 x}{2}+\frac{x^2\cos^2 x}{2}+\frac{1}{2}$$

$$=\frac{x^2}{2}(\sin x+1)^2+\frac{1}{2}(x\cos x+1)^2>0$$

. □

Example 3.28 (RMO 1998)**.** Prove the following inequality for every natural number n:

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\frac{1}{2n}\right).$$

Solution. First we shall show that for $r < n$

$$\frac{1}{n+1}\left(\frac{1}{r}+\frac{1}{2n-r}\right) > \frac{1}{n}\left(\frac{1}{r+1}+\frac{1}{2n-r+1}\right).$$

Which is equivalent to $n^2(r+1)(2n-r+1) > r(n+1)^2(2n-r)$.
Which is equivalent to $2n^3+n^2 > 4n^2r+2nr-2nr^2-r^2$.
Which is equivalent to $(n-r)^2(2n+1) > 0$. Which is true.

Hence, $\frac{1}{n+1}(\frac{1}{1}+\frac{1}{2n-1}) > \frac{1}{n}(\frac{1}{2}+\frac{1}{2n})$ and so on.

If n is even summing all these inequalities for $r = 1, 3, 5, \ldots, n-1$, we get the desired result. If n is odd summing all these inequalities for $r = 1, 3, 5, \ldots, n-2$, we get the desired result. Note that in that case for $r = n$ terms in both side are equal.

Observe that $\frac{1}{n+1}(\frac{1}{r}) >$ or $=$ or $< \frac{1}{n}(\frac{1}{r+1})$ according as $r <$ or $=$ or $> n$. So for adjustment we take two terms at a time. □

Example 3.29 (RMO 2002)**.** For any natural number $n > 1$, prove the following:

$$\frac{1}{2} \;<\; \frac{1}{n^2+1}+\frac{2}{n^2+2}+\frac{3}{n^2+3}+\cdots+\frac{n}{n^2+n} \;<\; \frac{1}{2}+\frac{1}{2n}.$$

Solution.

$$\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} \;>\frac{1}{n^2+n}+\frac{2}{n^2+n}\cdots+\frac{n}{n^2+n}=\frac{1}{2}.$$

Again,

$$\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} \;<\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}=\frac{1}{2}+\frac{1}{2n}.$$

□

Solution.(alternative solution for first part) Applying weighted A.M.-H.M. inequality on n distinct positive numbers $n^2+1, n^2+2, \ldots, n^2+n$ with weights $1, 2, \ldots, n$ respectively, we get

$$\frac{(n^2+1)+2(n^2+2)+\cdots+n(n^2+n)}{1+2+\cdots+n} > \frac{1+2+\cdots+n}{\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}},$$

i.e.,

$$\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} > \frac{(1+2+\cdots+n)^2}{(n^2+1)+2(n^2+2)+\cdots+n(n^2+n)},$$

$$\text{i,e., } \frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} > \frac{n^2(n+1)^2}{2n^3(n+1)+2n(n+1)(2n+1)/3},$$

$$\text{i,e., } \frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} > \frac{1}{2}\times\frac{3n(n+1)}{3n^2+2n+1} > 1/2.$$

□

Example 3.30 (RMO 2006)**.** If a, b, c are three positive real numbers, then prove that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \;\geq\; 3.$$

Solution. As $x^2 + 1 \geq 2x$ for all real x we have

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \;\geq \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b}.$$

Now, by A.M.-H.M. inequality $\left(\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}\right) \times$

$$\left(\frac{b+c}{a+b+c} + \frac{c+a}{a+b+c} + \frac{a+b}{a+b+c}\right) \geq 9.$$

Hence, $\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} \geq \frac{9}{2}.$

Therefore, $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$

□

Example 3.31 (RMO 2007)**.** Prove that

1. $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;
2. $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;
3. $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solution. We prove each item separately.

1. Applying weighted A.M.-G.M. inequality on three distinct positive numbers $\sqrt{5}, \sqrt[3]{5}, \sqrt[4]{5}$, we get

   $$\sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} > 3\left(\sqrt{5} \times \sqrt[3]{5} \times \sqrt[4]{5}\right)^{(1/3)} = 3(5^{13/36}) > 3(\sqrt[3]{5})\ .$$

   Now $3^3 > 5^2$ implies, $5 > (5/3)^3$ therefore, $3(\sqrt[3]{5}) > 5$.
2. $4 > \sqrt{8}, 2 = \sqrt[3]{8}$ and $2 > \sqrt[4]{8}$.
3. $n/3 \geq \sqrt{n} > \sqrt[3]{n} > \sqrt[4]{n}$ for all integers $n \geq 9$.

□

Example 3.32 (RMO 2011)**.** Find all real pairs (x, y) such that $16^{x^2+y} + 16^{x+y^2} = 1$.

Solution. Applying weighted A.M.-G.M. inequality on two positive real numbers $16^{x^2+y}, 16^{x+y^2}$, we get

$$16^{x^2+y} + 16^{x+y^2} \geq 2\sqrt{16^{x^2+y} \times 16^{x+y^2}} = 2 \times 4^{x^2+y^2+x+y}.$$

Therefore, $4^{x^2+y^2+x+y} \leq 4^{-\frac{1}{2}}$, hence $x^2 + y^2 + x + y \leq -\frac{1}{2}$, which implies $(x + 1/2)^2 + (y + 1/2)^2 \leq 0$. Which gives $x = y = -1/2$, clearly these values of x, y satisfy the given equation. Hence the solution is $x = y = -1/2$. □

Example 3.33 (RMO 2012)**.** Let a and b be positive real numbers such that $a+b = 1$. Prove that $a^a b^b + a^b b^a \leq 1$.

Solution. Applying weighted A.M.-G.M. inequality on two positive real numbers a and b with weights a and b respectively, we get $a^a b^b \leq a^2+b^2$ and applying weighted A.M.-G.M. inequality on two positive real numbers a and b with weights b and a respectively, we get $a^b b^a \leq 2ab$, hence the result. $\square$

Example 3.34 (RMO 2012)**.** Let $x_1, x_2, \ldots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that $K \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq 1$.

Solution.

$$\sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} = \sum_{j=1}^{2014} \frac{x_j^2 - 1}{1-x_j} + \sum_{j=1}^{2014} \frac{1}{1-x_j} = \sum_{j=1}^{2014}(-1-x_j) + \sum_{j=1}^{2014} \frac{1}{1-x_j}.$$

Using AM-HM inequality we have,

$$\sum_{j=1}^{2014} \frac{1}{1-x_j} \geq \frac{2014^2}{\sum_{j=1}^{2014}(1-x_j)} = \frac{2014^2}{2013}.$$

Thus we obtain

$$\sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq \sum_{j=1}^{2014}(-1-x_j) + \frac{2014^2}{2013} = -2014 - 1 + \frac{2014^2}{2013} = \frac{1}{2013}.$$

Hence, $\sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq \frac{1}{2013}$ and equality holds if and only if all x_j's are equal and equal to $\frac{1}{2014}$. Thus we get smallest K such that $K \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} \geq 1$ to be 2013. Otherwise if we take, $K < 2013$ then for all x_j's are equal to $\frac{1}{2014}$, we have $K \sum_{j=1}^{2014} \frac{x_j^2}{1-x_j} < 1$. $\square$

Example 3.35. If u, v, x, y, z are real numbers such that $u+v+x+y+z = 8$ and $u^2+v^2+x^2+y^2+z^2 = 16$, determine the maximum and minimum values of z.

Solution. $u+v+x+y = 8-z$ and $u^2+v^2+x^2+y^2 = 16-z^2$. Using R.M.S.-A.M. inequality on u, v, x, y, we get

$$\frac{u^2+v^2+x^2+y^2}{4} \geq \left(\frac{u+v+x+y}{4}\right)^2,$$

equality holds if and only if $u = v = x = y$.

Therefore we have $\frac{16-z^2}{4} \geq \left(\frac{8-z}{4}\right)^2$, which is equivalent to $0 \leq z \leq 16/5$. Hence maximum value of z is $16/5$ and in that case $u = v = x = y = 6/5$. Minimum value of z is 0 and in that case $u = v = x = y = 2$. $\square$

Example 3.36. Show that if u, v, x, y are positive then

$$\frac{uv}{u+v} + \frac{xy}{x+y} \leq \frac{(u+x)(v+y)}{u+v+x+y}.$$

Solution. The inequality is equivalent to

$$(uv(x+y) + (u+v)xy)(u+v+x+y) \leq (u+v)(x+y)(u+x)(v+y),$$

which is equivalent to $2uvxy \leq u^2y^2 + v^2x^2$, which is trivially true. $\square$

Example 3.37. Find the maximum value of $4x + 2y$ if $4x^2 + 2y^2 = 1$.

Solution. Put $x = \frac{1}{2}\sin\theta$ then $y = \frac{1}{\sqrt{2}}\cos\theta$. Now $4x + 2y = 2\sin\theta + \sqrt{2}\cos\theta$ and the maximum value of $2\sin\theta + \sqrt{2}\cos\theta$ is $\sqrt{2^2 + \sqrt{2}^2} = \sqrt{6}$. $\square$

Example 3.38. Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1, \frac{1}{yz} = \frac{y}{x} + 1, \frac{1}{zx} = \frac{z}{y} + 1.$$

Solution. The given system reduces to $z = x^2y + xyz, x = y^2z + xyz, y = z^2x + xyz$. Hence $z - x^2y = x - y^2z = y - z^2x$. If $x = y$, then $y^2z = z^2x$ and hence $x^2z = z^2x$. This implies that $z = x = y$. Similarly, $x = z$ implies that $x = z = y$.
Hence if any two of x, y, z are equal, then all are equal. Suppose no two of x, y, z are equal. We may take x is the largest among x, y, z so that $x > y$ and $x > z$. Here we have two possibilities: $y > z$ and $z > y$. Suppose $x > y > z$.
Now $z - x^2y = x - y^2z = y - z^2x$ shows that $y^2z > z^2x > x^2y$.But $y^2z > z^2x$ and $z^2x > x^2y$ give $y^2 > zx$ and $z^2 > xy$. Hence $y^2z^2 > (zx)(xy)$.
This gives $yz > x^2$. Thus $x^3 < xyz = (xz)y < (y^2)y = y^3$. This forces $x < y$ contradicting $x > y$. Similarly, we arrive at a contradiction if $x > z > y$.
The only possibility is $x = y = z$. For $x = y = z$, we get only one equation $x^2 = 1/2$. Since $x > 0, x = 1/\sqrt{2} = y = z$. $\square$

Example 3.39. Given real numbers $a, b, c, d, e > 1$ prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20$$

Solution. Note that $(a-2)^2 \geq 0$ and hence $a^2 \geq 4(a-1)$. Since $a > 1$ we have $\frac{a^2}{a-1} \geq 4$. By applying AM-GM inequality we get

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 5\sqrt[5]{\frac{a^2b^2c^2d^2e^2}{(a-1)(b-1)(c-1)(d-1)(e-1)}} \geq 20$$

$\square$

3.4. Exercise Problems

1. If $x_1, x_2, \ldots, x_n$ are positive and if $x_1 x_2 \cdots x_n = 1$, then show that
$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq 2^n.$$
2. If $a, b, c > 0$ and $a+b+c = x$, then show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{x}$.
3. If $a, b, c > 0$, then show that $(a+b)(b+c)(c+a) \geq 8abc$.
4. Find the greatest value of $a^2 b^3$ where a, b are positive real numbers satisfying $a+b = 10$. Determine the values of a, b for which the greatest value is attained.
5. If $a, b, c > 0$ and $a+b+c = 1$, then show that $ab(a+b)^2 + bc(b+c)^2 + ca(c+a)^2 \geq 4abc$.
6. If a, b, c are sides of a triangle, then show that
$$\frac{1}{2} < \frac{ab+bc+ca}{a^2+b^2+c^2} \leq 1.$$
7. If a, b, c are positive numbers show that
$$\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \geq a+b+c.$$
8. If a, b, c are positive numbers satisfying $4ab + 6bc + 8ca = 9$, find the maximum value of abc, also find a, b, c for which abc is maximum.

   $a = 3/4, b = 1, c = 1/2$.
9. Find the maximum possible product of an arbitrary number of positive integers whose sum is a fixed positive integer n.
10. If a, b, c are positive numbers, prove that $(a+b+c)(bc+ca+ab) \geq 9abc$.

    Hint: Apply A.M.-G.M. inequality separately on $\{a, b, c\}$ and $\{ab, bc, ca\}$, then multiply.
11. If $a, b, c > 0$ and $a+b+c = 1$, then show that $8abc \leq (1-a)(1-b)(1-c) \leq 8/27$.
12. If $a, b, c, d > 1$ prove that $(a+1)(b+1)(c+1)(d+1) < 8(abcd+1)$.
13. If x, y, z are unequal positive numbers, prove that $(1+x^3)(1+y^3)(1+z^3) > (1+xyz)^3$.
14. For any real numbers a, b, c show that
$$-(a^2+b^2+c^2)/2 \leq ab+bc+ca \leq a^2+b^2+c^2.$$
15. For any positive integer p, prove that
$$2(\sqrt{p+1} - \sqrt{p}) < 1/\sqrt{p} < 2(\sqrt{p} - \sqrt{p-1}).$$

    Hence or otherwise compute the integral part of $1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{1000}$.

16. If $a, b, c \geq 0$ and $a + b + c = 1$, then show that
$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{9}{10}.$$

17. If $a, b, c, d > 0$ then show that
$$1/a^3 + 1/b^3 + 1/c^3 + 1/d^3 \geq 1/abc + 1/bcd + 1/cda + 1/dab.$$

18. If a, b, c denotes three sides of a triangle, prove that
$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 4abc > a^3 + b^3 + c^3.$$

19. If a, b, c are positive numbers, prove that
$$a(1+b) + b(1+c) + c(1+a) \geq 6\sqrt{abc}.$$

20. If a, b, c be positive real numbers, and n be a positive rational number, prove that
$$a^n(a-b)(a-c) + b^n(b-a)(b-c) + c^n(c-a)(c-b) \geq 0.$$

21. If a, b, c, x, y, z be reals and $a^2 + b^2 + c^2 = x^2 + y^2 + z^2 = 1$ prove that $-1 \leq ax + by + cz \leq 1$.

22. If a, b, c be positive real numbers, not all equal, prove that
$$2(a^3 + b^3 + c^3) > a^2(b+c) + b^2(c+a) + c^2(a+b) > 6abc.$$

23. If $a, b > 0$ and $a + b = 4$, show that $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}$.

24. Find the minimum value of $3x+2y$ when x, y are positive real numbers satisfying the condition $x^2y^3 = 48$.

25. If a, b, c be positive real numbers, prove that
$$a^4 + b^4 + c^4 \geq abc(a+b+c).$$

26. If a, b, c be positive real numbers, prove that
$$\left(\frac{a+b+c}{3}\right)^3 \geq a\left(\frac{b+c}{2}\right)^2.$$

27. If the perimeter of a triangle remains constant, prove that the area of the triangle is greatest when the triangle is equilateral.

28. What is the maximum possible product of arbitrary number of positive integers, whose sum is 100. What happen if the sum is a fixed positive integer, say n.

Hint and answer

3.5. RMO Problems

1995 Show that for any triangle ABC, the following inequality is true :

$$a^2 + b^2 + c^2 > \sqrt{3}\max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\},$$

where a, b, c are, as usual, the sides of the triangle.

1997 Let x, y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left|\frac{x}{y} - \frac{y}{x}\right|, \quad \left|\frac{y}{z} - \frac{z}{y}\right|, \quad \left|\frac{z}{x} - \frac{x}{z}\right|$$

can be the lengths of the sides of a triangle.

1999 If a, b, c are the sides of a triangle prove the following inequality:

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3.$$

2000 Suppose that $\langle x_1, x_2, \ldots, x_n, \ldots\rangle$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$, and for all n,

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \ \leq\ 1.$$

Show that for all k, the following is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \ \leq\ 3.$$

2001 If x, y, z are sides of a triangle, then prove that

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| \ <\ xyz.$$

2003 Let a, b, c be three positive real numbers such that $a+b+c = 1$. Prove that among the three numbers $a - ab$, $b - bc$, $c - ca$, there is one which is at most $1/4$ and there is one which is at least $2/9$.

2004 Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Then prove that the following inequalities are true: (a) $y < x < 1$, and (b) $x^2 + y^2 < 1$.

2005 If a, b, c are three real numbers such that $|a-b| \geq c$, $|b-c| \geq a$, $|c-a| \geq b$, then prove that one of a, b, c is the sum of the other two.

2008 Prove that for every positive integer n and a non-negative real number a,

$$n(n+1)a + 2n \ \geq\ 4\sqrt{a}\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right).$$

2011 Find the largest real constant α such that

$$\frac{\alpha abc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2,$$

for all positive real numbers a, b, c.

Chapter 4

Polynomials

4.1. Introduction

For an integer $n \geq 0$, the expression $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where a_i's are real numbers, is called a polynomial in x with real coefficients, or simply a polynomial in x over $\mathbb{R}$. If the coefficients are rational numbers, integers or natural numbers, then we say $p(x)$ is a polynomial in x over $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{N}$, respectively.

If $a_n \neq 0$, then $p(x)$ is called polynomial of degree n and a_n is known as its leading coefficient. If all the coefficients are zero, then $p(x)$ is called the zero polynomial and is denoted by $0(x)$ or simply 0. If all the coefficients other than a_0 are zero, then $p(x)$ is called the constant polynomial and is denoted by a_0. The degree of the non-zero constant polynomial is *defined* to be 0. A polynomial is called monic if $a_n = 1$. Two polynomials are called identical or equal if the degrees are same and the coefficients of each respective power of x in them are equal.

For example

- ◇ $2x^3 + 6x - 7$ is a degree 3 polynomial in x over $\mathbb{N}$.
- ◇ $\frac{2}{3}x^4 - x^2 + (6.7)x$ is a degree 4 polynomial in x over $\mathbb{Q}$.
- ◇ $x^3 + \sqrt{2}$ is a degree 3 monic polynomial in x over $\mathbb{R}$.
- ◇ $x^2 + \frac{1}{x}$ is not a polynomial in x.

4.2. Division Algorithm

Let $p(x)$ and $f(x)$ be polynomials over $\mathbb{R}$ and let $f(x)$ be non-zero. Then there exist unique polynomials $q(x)$ and $r(x)$ over $\mathbb{R}$ such that $p(x) = f(x)q(x) + r(x)$, where $r(x)$ is either the zero polynomial or a non-zero polynomial of degree less than the degree of $f(x)$. Here $f(x), p(x), q(x)$ and $r(x)$ are known as divisor, dividend, quotient and remainder respectively. If $r(x)$ is the zero polynomial, we say that $p(x)$

is divisible by $f(x)$ over $\mathbb{R}$ or that $f(x)$ is a factor of $p(x)$ over $\mathbb{R}$.

Theorem 4.1 (Remainder theorem)**.** *For any polynomial $p(x)$, the remainder on dividing $p(x)$ by $(x-a)$ is $p(a)$.*

Theorem 4.2 (Factor Theorem)**.** *For any polynomial $p(x)$, if $p(a)=0$ then $(x-a)$ is a factor of $p(x)$ and vice-versa.*

In case $p(a)=0$, the real number a is called a root of the equation $p(x)=0$.

Theorem 4.3 (Fundamental Theorem of Algebra)**.** *If $p(x)$ is a polynomial of degree $n\geq 0$, with complex coefficients, then it has exactly n complex roots.*

Note that in the fundamental theorem of algebra, any real number is considered as a complex number, and the roots are not necessarily assumed to be distinct.

Theorem 4.4. *If the roots of the polynomial equation $p(x)=0$ are $\alpha_1,\alpha_2,\ldots,\alpha_n$, then $p(x)$ has the factorization $p(x)=a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, where a_n is the leading coefficient of $p(x)$.*

Theorem 4.5. *If a is a real number, then*

- ⋄ *$(x-a)$ is factor of x^n-a^n for all natural numbers n.*
- ⋄ *$(x+a)$ is factor of x^n-a^n for all even numbers n.*
- ⋄ *$(x+a)$ is factor of x^n+a^n for all odd numbers n.*

If a,b are integers, then $(a-b)|(a^n-b^n)$ for all natural numbers n, $(a+b)|(a^n-b^n)$ for all even natural numbers n, and $(a+b)|(a^n+b^n)$ for all odd natural numbers n.

Theorem 4.6 (Sophie Germain's identity)**.**

$$a^4+4b^4=(a^2+2ab+2b^2)(a^2-2ab+2b^2)$$

Example 4.1. Prove that x^5+x^4+1 is a composite number for $x>1$.

Solution. Note that x^5+x^4+1 can be written as

$$\begin{aligned}&(x^5+x^4+x^3+x^2+x+1)-(x^3+x^2+x)=\frac{x^6-1}{x-1}-x\cdot\frac{x^3-1}{x-1}\\&=\frac{(x^3-1)(x^3+1-x)}{x-1}=(x^2+x+1)(x^3-x+1)\end{aligned}$$

Since both the factors are greater than 1, we have the result. □

Example 4.2. Prove that x^4+4^x is composite for any natural number $x>1$.

Solution. Note that x^4+4^x is even (hence composite as $x^4+4^x>2$) if x is even. If x is odd, write $x=2k+1$, where k is an natural number. Then $x^4+4^x=(2k+1)^4+4(2^k)^4$. Now use Sophie Germain's identity to factorize and show smaller factor is greater than 1. □

Quick Exercise

1. Find quotient and remainder when x^5+4x^3-6x+1 is divided by x^2-7x+1.
2. Show that $x-1$ is a factor of x^4+4x-5 over $\mathbb{Z}$.
3. Show that $3^{12}+2^6$ is a composite number.
4. Show that $3^{2008}+4^{2009}$ can be written as product of two positive integers, each of which is larger than 2009^{182}.

4.3. Roots and Coefficients

Suppose that $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial of degree n and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $p(x) = 0$. Then the elementary symmetric polynomials in the roots are given by

- ◇ $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$
- ◇ $\alpha_1\alpha_2\cdots\alpha_n = (-1)^n\frac{a_0}{a_n}$.
- ◇ $\sum_{1\le i<j\le n}\alpha_i\alpha_j = \frac{a_{n-2}}{a_n}$, etc.

Theorem 4.7. *If the real quadratic surd $\alpha+\sqrt{\beta}$ is a root of the polynomial equation $p(x) = 0$, with $p(x)$ over $\mathbb{Q}$, then $\alpha-\sqrt{\beta}$ is also a root of the equation.*

Theorem 4.8. *If the non-real complex number $\alpha + i\beta$ is a root of the polynomial equation $p(x) = 0$, with $p(x)$ over $\mathbb{R}$, then $\alpha - i\beta$ is also a root of the equation.*

For example, if a quadratic equation $ax^2+bx+c = 0$, with $a \neq 0$, has roots α, β, then $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. If a cubic equation $ax^3 + bx^2 + cx + d = 0$, with $a \neq 0$, has roots α, β, γ, then $\alpha+\beta+\gamma = -\frac{b}{a}$, $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$, and $\alpha\beta\gamma = -\frac{d}{a}$.

Theorem 4.9 (Descartes' rule of signs)**.** *The number of positive roots of the polynomial equation $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $p(x)$. If the number of positive roots is less than the number of variations of sign, it is less by an even number.*

This says that if v be the number of variations of signs in the sequence of the coefficients of $p(x)$, and r be the number of positive roots of the polynomial equation $p(x) = 0$, then $v = r + 2h$, where h is a non-negative integer.

Theorem 4.10. *The number of positive roots of the polynomial equation $p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n = 0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $p(-x) = a_0 + (-a_1)x + a_2x^2 + \cdots + (-1)^na_nx^n$. If the number of negative roots is less than the number of variations, it is less by an even number.*

Note that one should write the polynomial in a descending order or an ascending order of the power of x before counting the variations in sign. If the equation is $p(x) = x^3 + x - 2x^2 - 2 = 0$, then the number of variations is not 1, it is 3, since we have to write $p(x)$ as $x^3 - 2x^2 + x - 1$ and then count the variations in sign.

Theorem 4.11. *Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ be a polynomial equation with integer coefficients. If $\frac{p}{q}$ is a rational root of this equation, with $\gcd(p, q) = 1$, then $p|a_0$ and $q|a_n$. If $a_n = 1$ or -1, then all the rational roots of this polynomial equation must be integers, and each such root must divide a_0.*

Theorem 4.12. *For polynomial $f(x)$, if $f(a)$ and $f(b)$ have opposite signs for $a < b$, then there exists a number c such that $a < c < b$ and $f(c) = 0$.*

Roots of a Quadratic Equation

If $ax^2 + bx + c = 0$ is a quadratic equation with real coefficients $a \neq 0$ and b, c, then it has two roots $\frac{1}{2a}(-b + \sqrt{b^2 - 4ac})$ and $\frac{1}{2a}(-b - \sqrt{b^2 - 4ac})$.

The equation has two distinct real roots if $b^2 > 4ac$, it has equal real roots if $b^2 = 4ac$, and it has two complex roots if $b^2 < 4ac$. The term $(b^2 - 4ac)$ is known as the discriminant of this quadratic equation. For example

- $x^2 + 4x - 1 = 0$ has two distinct real roots as its discriminant is 20.
- $x^2 + 2x + 1 = 0$ has two equal real roots as its discriminant is 0.
- $x^2 + x + 1 = 0$ has no real roots as its discriminant is -3.

4.4. Symmetric Polynomial

A polynomial in the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ is said to be symmetric if it remains unchanged under any permutation of the roots. That is, the polynomial f is symmetric if $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = f(\sigma(\alpha_1, \alpha_2, \ldots, \alpha_n))$, where σ is any permutation.

For an example, the polynomials $\alpha^2 + \beta^2 + \gamma^2$ and $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$ are symmetric in α, β, γ, but the polynomials $\alpha + \beta + \gamma^2$ and $\alpha + \beta - \gamma$ are not.

Definition 4.1 (Elementary symmetric polynomial)**.** The elementary symmetric polynomials in n variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ are defined as

- $e_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{1 \leq j \leq n} \alpha_i$
- $e_2(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j$

and so on, ending with $e_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = \prod_{1 \leq i \leq n} \alpha_i$.

For example, the elementary symmetric polynomials in α, β, γ are $e_1(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$, $e_2(\alpha, \beta, \gamma) = \alpha\beta + \beta\gamma + \gamma\alpha$ and $e_3(\alpha, \beta, \gamma) = \alpha\beta\gamma$.

Theorem 4.13 (Newton)**.** *Every symmetric polynomial in the roots can be expressed as a polynomial in terms of the elementary symmetric polynomials in the roots.*

For example, $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = e_1^2 - 2e_2$.

4.5. Transformation of Polynomials

Let the roots of the polynomial equation $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$, and let a bijective transformation $y = g(x)$ be applied on this polynomial. Then the polynomial $p(g^{-1}(y)) = 0$ has roots $g(\alpha_1), g(\alpha_2), \ldots, g(\alpha_n)$.

For example, if $p(x) = ax^3+bx^2+cx+d = 0$ has roots α, β, γ, then the equation $p(1/y) = a(1/y)^3+b(1/y)^2+c(1/y)+d = 0$ or equivalently, $dy^3+cy^2+by+a = 0$ has roots $1/\alpha, 1/\beta, 1/\gamma$. To find the equation whose roots are $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}, \frac{\gamma}{1+\gamma}$, we apply the transformation $y = \frac{x}{1+x}$ or $x = \frac{y}{1-y}$ and obtain the required equation as

$$a\left(\frac{y}{1-y}\right)^3 + b\left(\frac{y}{1-y}\right)^2 + c\left(\frac{y}{1-y}\right) + d = 0.$$

4.6. Solved Examples

Example 4.3. Find all integers x satisfying $3x^6 + 5x^3 - 9x + 1 = 0$.

Solution. If p is a integral solution then $p|1$. Hence either $p = 1$ or $p = -1$. But $x = -1$ does not satisfy the equation. Since $x = 1$ satisfies the equation, 1 is the only integral solution. □

Example 4.4. Find c, d such that the roots of $x^2 + cx + d = 0$ are c, d.

Solution. We have $c+d = -c$ and $cd = d$. If $d \neq 0$ then $c = 1$, and hence $d = -2$. If $d = 0$ then $c = -c$, hence $c = 0$. Therefore, $c = d = 0$ and $c = 1, d = -2$ are the solutions for c, d. □

Example 4.5. Suppose α, a, b are integers and $b \neq -1$. Show that if α satisfies the equation $x^2 + ax + b + 1 = 0$, then $a^2 + b^2$ is composite.

Solution. Let β be the other root of the equation $x^2 + ax + b + 1 = 0$, therefore, $\alpha+\beta = -a$ and $\alpha\beta = b+1$. Now $a^2+b^2 = (\alpha+\beta)^2+(\alpha\beta-1)^2 = (\alpha^2+1)(\beta^2+1)$. Since $b \neq -1$, both the roots α, β are non zero and since α, a are integers and $\alpha+\beta = -a$, therefore β is an integer. Hence both the factor (α^2+1) and (β^2+1) of a^2+b^2 are positive integers greater than 1. □

Example 4.6. Find all rational roots of $2x^3 - 5x^2 + 5x - 3 = 0$.

Solution. Let $\frac{p}{q}$ be a rational root of $2x^3 - 5x^2 + 5x - 3 = 0$. Then $p|3$ and $q|2$. Hence the possible rational roots are $\frac{3}{2}, -\frac{3}{2}, 3, -3, \frac{1}{2}, -\frac{1}{2}, 1, -1$. Also there will be no negative root. Since only $\frac{3}{2}$ satisfies the equation, the rational root is $\frac{3}{2}$. □

Example 4.7. If a, b, c are odd integers, prove that the roots of the quadratic equation $ax^2 + bx + c = 0$ cannot be rational numbers.

Solution. The discriminant $b^2 - 4ac$ is odd, and the roots are rational if and only if it is a square. Let us assume that the roots are rational, that is, $b^2 - 4ac = k^2$, where k is an odd integer. Then, $ac = \frac{b+k}{2} \cdot \frac{b-k}{2}$, which implies that both $\frac{b+k}{2}$ and $\frac{b-k}{2}$ are odd integers, as a, c are odd. However, $\frac{b+k}{2} = \frac{b-k}{2} + k$, and as k is odd, $\frac{b+k}{2}$ and $\frac{b-k}{2}$ must be of opposite parity. Thus, by contradiction, we prove the result. □

Example 4.8. If a, b, c, d are any four real numbers, not all equal to zero, prove that the roots of the equation $x^6 + ax^3 + bx^2 + d = 0$ cannot all be real.

Solution. Let $f(x) = x^6 + ax^3 + bx^2 + d$ and p, n be the variations of sign in the sequence of coefficients of $f(x)$ and $f(-x)$ respectively. Then whatever may be the sign of a, b, c may be $p + n \leq 4$. Hence all the roots cannot be real. □

Example 4.9. $x^3 + ax + 1 = 0$ has a real root α, prove that $a^2 \geq 4\alpha$.

Solution. Since $x^3 + ax + 1 = 0$ has a real root α, we have $\alpha^3 + a\alpha + 1 = 0$. Now consider the quadratic equation $\alpha x^2 + ax + 1 = 0$. α is a real root of this quadratic equation, therefore, discriminant of this quadratic equation is nonnegative. Hence the result. □

Example 4.10. Discuss the nature of the roots of the equation $x + x^4 - x^3 - 2x^2 = 0$

Solution. The given equation is $p(x) = x^4 - x^3 - 2x^2 + x = 0$. Here the variation of signs in $p(x)$ is 2. Hence number of positive root is 2 or 0. Now $p(-x) = x^4 + x^3 - 2x^2 - x$, hence the variation of signs in $p(-x)$ is 1. Therefore, number of negative root is 1 and there is a zero root. □

Example 4.11. Let a, b, c, x, y, z be positive real numbers such that $a + b + c = x + y + z$ and $abc = xyz$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.

Solution. Consider the polynomial $f(t) = (t-x)(t-y)(t-z) - (t-a)(t-b)(t-c)$. Then $f(t) = kt$ for some constant k. Note that $ka = f(a) = (a-x)(a-y)(a-z) \leq 0$, and hence $k \leq 0$.
Similarly, $kc = f(c) = (c-x)(c-y)(c-z) \leq 0$, and hence $k \geq 0$.
This follows that $k = 0$ and hence $f(a) = f(c) = 0$.
Therefore $(a-x)(a-y)(a-z) = 0 = (c-x)(c-y)(c-z)$.
Now $a < y < z$ and $x < y < c$, which follows $a = x, c = z$ and hence $b = y$.

□

Example 4.12. Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

Solution. For $i \geq 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increase or decrease the constant term). Since $f(2) = 2020$ so for some i, we have $0 \leq f_i(2) \leq 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

□

Example 4.13. Let $p(x)$ and $q(x)$ be two monic polynomials with complex coefficients such that $p(p(x)) = q(q(x))$ for all complex x. Prove that $p(x) = q(x)$ for all x.

Solution. Let the degree of p and q be n and m respectively.
Then degree of $p(p(x))$ and $q(q(x))$ is n^2 and m^2 respectively, hence $n = m$.
Let $p(x) = x^n + r(x)$ and $q(x) = x^n + s(x)$, where r and s have degree less than or equal to $n - 1$.
Note that $p(p(x)) = (p(x))^n + r(p(x))$ and $q(q(x)) = (q(x))^n + s(q(x))$.
So $(p(x))^n - (q(x))^n + r(p(x)) - s(q(x)) = 0$ for all x.
Since the degree of $r(p(x))$ and $s(q(x))$ is less than or equal to $n(n-1) = n^2 - n$, degree of $(p(x))^n - (q(x))^n$ is also $n^2 - n$. Now
$(p(x))^n - (q(x))^n = (p(x) - q(x))\left((p(x))^{n-1} + (p(x))^{n-2}q(x) + \cdots + (q(x))^{n-1}\right)$ and degree of $\left((p(x))^{n-1} + (p(x))^{n-2}q(x) + \cdots + (q(x))^{n-1}\right)$ is $n^2 - n$ (with leading coefficient n), hence $p(x) - q(x)$ is a constant polynomial.
Let $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ and $q(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + b_n$.
So $p(p(x)) - q(q(x)) = (p(x))^n - (q(x))^n + a_1(p(x))^{n-1} + a_2(p(x))^{n-2} + \cdots + a_{n-1}p(x) + a_n - a_1(q(x))^{n-1} + a_2(q(x))^{n-2} + \cdots + a_{n-1}q(x) + b_n = (a_n - b_n)\left((p(x))^{n-1} + (p(x))^{n-2}q(x) + \cdots + (q(x))^{n-1}\right) + a_1(p(x))^{n-1} - a_1(q(x))^{n-1} + t(x)$, where degree of $t(x)$ is less than or equal to $n(n-2)$.
Now the coefficient of x^{n^2-n} in $p(p(x)) - q(q(x))$ is $n(a_n - b_n)$, but $p(p(x)) - q(q(x))$ is an zero polynomial and hence $a_n = b_n$, therefore $p(x) = q(x)$. □

Example 4.14. Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution. Consider the equation whose roots are $x, y, z:$
$(t - x)(t - y)(t - z) = 0$. This gives $t^3 - 3t - \lambda = 0$, where $\lambda = xyz$. Since x, y, z are roots of this equation, we have
$x^3 - 3x - \lambda = 0, y^3 - 3y - \lambda = 0, z^3 - 3z - \lambda = 0.$
Multiplying the first by y, the second by z and the third by x, we obtain

$$x^3y - 3xy - \lambda y = 0, y^3z - 3yz - \lambda z = 0, z^3x - 3zx - \lambda x = 0.$$

Adding we obtain $x^3y + y^3z + z^3x - 3(xy + yz + zx) - \lambda(x + y + z) = 0$.
This simplifies to $x^3y + y^3z + z^3x = -9$.

Exercise: Also solve for y and z in terms of x and substitute these values in $x^3y + y^3z + z^3x$ to get -9. □

Example 4.15. Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Solution. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$. Observe none of these is zero since their product is $1/a$.
Then the roots of $x^4+x^3+x^2+bx+a = 0$ are $\beta_1 = \frac{1}{\alpha_1}, \beta_2 = \frac{1}{\alpha_2}, \beta_3 = \frac{1}{\alpha_3}, \beta_4 = \frac{1}{\alpha_4}$.

$$\text{We have } \beta_1 + \beta_2 + \beta_3 + \beta_4 = -1 \quad \text{and} \quad \sum_{1\le i<j\le 4} \beta_i\beta_j = 1.$$

$$\text{Hence } \sum_{i=1}^{4} \beta_i^2 = \left(\sum_{i=1}^{4} \beta_i\right)^2 - 2\left(\sum_{1\le i<j\le 4} \beta_i\beta_j\right) = 1 - 2 = -1.$$

This shows that not all β_j can be real. Hence not all α_j's can be real. □

4.7. Exercise Problems

1. Find the cubic polynomial in x which vanishes when $x = 2$ and $x = -1$ and has values 3 and 4 when $x = 0$ and $x = 1$ respectively.
2. Find all polynomials $f(x)$ and $g(x)$ such that $f(g(x)) = x$ for all real x.
3. Find all positive integer a, b such that each of the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ has distinct positive integer roots.
4. If $a \neq b, c \neq 0$ and if the equations $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a common root, then show that their other roots satisfy the equation $x^2 + cx + ab = 0$.
5. Find all integer roots of $x^5 - 29x^4 - 31x^3 + 31x^2 - 32x + 60 = 0$.
6. Find all rational roots of $x^4 + 4x^3 - 7x^2 - 22x + 24 = 0$.
7. Solve the given system of equations: $x + y + z = 1, x^2 + y^2 + z^2 = 29, xyz = -24$.
8. Let a, b, c be real numbers. Consider the equation

$$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0.$$

   Prove that the roots of this equation are always real. Further, show that the roots are equal if and only if $a = b = c$.
9. Find the remainder when $x^6 + 5x$ is divided by $x^3 + x - 1$.
10. Let $f(x)$ be a polynomial having integer coefficients and let $f(0) = 1989$ and $f(1) = 9891$. Prove that $f(x)$ has no integer roots.

11. Find the roots of $4x^3 - 12x^2 - x + 3 = 0$, given that one root is the negative of another.
12. Find the all integers k such that the equation $x^3 - 3x + k = 0$, has three integer roots.
13. Express the following symmetric polynomial in α, β, γ as a polynomial in the elementary symmetric polynomial in α, β and γ:
    a) $\alpha^3 + \beta^3 + \gamma^3$
    b) $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$
14. Find all rational roots of $2x^3 - 5x^2 + 4x - 1 = 0$.
15. Discuss the nature of roots of the equation $x^5 - x^2 + x^3 - x = 0$.
16. If α, β, γ are the roots of $x^3 - 2x - 1 = 0$, compute the value of $\frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta} + \frac{1-\gamma}{1+\gamma}$.
17. Given that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has four real positive roots, prove that: (i) $pr - 16s \geq 0$, and (ii) $q^2 - 36s \geq 0$.
18. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a, b are integers. Given any integer n, show that there is an integer M such that $p(n)p(n+1) = p(M)$.
19. If $p(x)$ is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have $p(a) = b$, $p(b) = c$ and $p(c) = a$.
20. Find all cubic polynomials $p(x)$ such that $(x-1)^2$ is a factor of $p(x) + 2$, and $(x+1)^2$ is a factor of $p(x) - 2$.
21. Find all positive integers n for which the quadratic equation
$$a_{n+1}x^2 - 2x\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + (a_1 + a_2 + \cdots + a_n) = 0$$
has real roots for all real $a_1, a_2, \ldots, a_{n+1}$.
22. Prove that if a polynomial $p(x)$ with integer coefficients takes on the value 7 for four integer values of x, then it cannot have the value 14 for any integer value of x.
23. Suppose $p(x), q(x)$ are polynomials which satisfy the identity $p(q(x)) = q(p(x))$ for all $x \in \mathbb{R}$. If the equation $p(x) = q(x)$ has no real solution, show that the equation $p(p(x)) = q(q(x))$ has no real solution.
24. Prove that if the integers $a_1, a_2, \ldots, a_n$ are all distinct then the polynomial $(x-a_1)^2(x-a_2)^2 \ldots (x-a_n)^2 + 1$ cannot be written as the product of two other polynomials with integer coefficients.
25. Prove that the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients has odd values for $x = 0$ and $x = 1$, then the equation $p(x) = 0$ cannot have integer roots.
26. Prove that if $p(t)$ is a polynomial with integer coefficients and there exists

positive integer k such that none of the integers $p(1), p(2), \ldots, p(k)$ is divisible by k, then $p(t)$ has no integer zeros.

27. Determine all values of the parameters a and b for which the polynomial $x^4 + (2a+1)x^3 + (a-1)^2x^2 + bx + 4$ can be factored into a product of two monic polynomials $p(x)$ and $q(x)$ such that the equation $q(x) = 0$ has two different roots r and s with $p(r) = s$ and $p(s) = r$.

28. Let $a_1, a_2, \ldots, a_n$ be nonzero real with $a_1 < a_2 < \ldots < a_n$. Show that the following equation hold for n real values of x:
$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = n$$
if all the a_i have the same sign.

29. Let $u, v, w,$ be distinct constants. Solve
$$\frac{x}{a+u} + \frac{y}{b+u} + \frac{z}{c+u} = 1, \quad \frac{x}{a+v} + \frac{y}{b+v} + \frac{z}{c+v} = 1$$
$$\text{and} \quad \frac{x}{a+w} + \frac{y}{b+w} + \frac{z}{c+w} = 1.$$

30. Is there a set of real numbers u, v, w, x, y, z satisfying $u^2 + v^2 + w^2 + 3(x^2 + y^2 + z^2) = 6$ and $ux + vy + wz = 2$.

31. Find all ordered triplets (x, y, z) of real numbers such that
$$5\left(x + \frac{1}{x}\right) = 12\left(y + \frac{1}{y}\right) = 13\left(z + \frac{1}{z}\right) \quad \text{and} \quad xy + yz + zx = 1.$$

32. If $f(x)$ be a polynomial in x and a, b are unequal, show that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is
$$\frac{(x-b)f(a) - (x-a)f(b)}{a-b}.$$

33. Show that
$$1 - \frac{x}{1!} + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \cdots + (-1)^n \frac{x(x-1)\ldots(x-n+1)}{n!}$$
$$= \frac{(-1)^n}{n!}(x-1)(x-2)\ldots(x-n).$$

34. If the polynomial $x^n - qx^{n-m} + r$ has a factor of the form $(x-a)^2$ show that
$$\left(\frac{q}{n}(n-m)\right)^n = \left(\frac{r}{m}(n-m)\right)^m.$$

35. If $x^2 + px + 1$ be a factor of $ax^3 + bx + c$ prove that $a^2 - c^2 = ab$. Show that in this case $x^2 + px + 1$ is also a factor of $cx^3 + bx^2 + a$.

36. If ax^4+bx^3+c has a factor of the form x^2+kx+1 prove that $(a+c)(a-c)^2 = b^2c$.

37. Prove that x^2+px+p^2 is a factor of $(x+p)^n - x^n - p^n$, if n be odd and not divisible by 3.

38. Prove that $x^2+y^2+z^2-xy-yz-zx$ is a factor of $(x-y)^n+(y-z)^n+(z-x)^n$, if n is not divisible by 3.

39. Solve $x^3+6x-3x-18=0$, given that sum of two of the roots is zero.

40. Solve $x^4+2x^3+5x^2+4x+3=0$, given that the product of two of the roots is 1.

41. If $a+b+c=1$, $a^2+b^2+c^2=3$ and $a^3+b^3+c^3=7$, prove that $a^4+b^4+c^4=11$.

42. If α,β,γ are the roots of $x^3-x-1=0$, compute the value of
$$\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}.$$

43. For positive integer n, suppose that α_n, β_n are the roots of $x^2+(2n+1)x+n^2=0$. Then, find the value of
$$\frac{1}{(\alpha_3+1)(\beta_3+1)}+\frac{1}{(\alpha_4+1)(\beta_4+1)}+\cdots+\frac{1}{(\alpha_{20}+1)(\beta_{20}+1)}.$$

44. Suppose a,b are positive real numbers such that the roots of the equation $x^3-ax+b=0$ are real. If α is a root with minimal absolute value, then prove that $b/a \le \alpha \le 3b/2a$.

45. If a,b,c are the roots of $x^3-x^2+4x+7=0$, find a cubic equation whose roots are $a+b, b+c, c+a$.

46. Let a,b,c be real numbers such that $a+b+c, ab+bc+ca, abc$ are all positive. Prove that a,b,c are positive.

47. Prove that the equation $x^5-2x^2+7=0$ has at least two complex roots.

48. For positive real numbers q,r,s, prove that $f(x)=x^4+qx^2+rx-s=0$ has exactly one positive real, one negative real and two complex roots.

49. For real numbers a,b,c, suppose that $29c+10b+2a=0$. Then prove that the equation $f(x)=ax^3+bx+c=0$ has at least one real root between 0 and 1.

50. If $\alpha^3+a\alpha+b=0$ for some real α, prove that $a^2 \ge 4b\alpha$.

4.8. RMO Problems

1994 Solve the system of equations for real x and y :
$$5x\left(1+\frac{1}{x^2+y^2}\right)=12 \text{ and } 5y\left(1-\frac{1}{x^2+y^2}\right)=4.$$

1995 Show that the quadratic expression

$$x^2 + 7x - 14(q^2 + 1) = 0,$$

where q is an integer, has no integer root.

1996 Solve for real number x and y:

$$xy^2 = 15x^2 + 17xy + 15y^2 \text{ and } x^2y = 20x^2 + 3y^2.$$

1999 If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2x - r^3 = 0$, show that $p = q = r$.

2000 Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y.

2000 Find all real values of a for which $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

2002 Solve the following equation for real x: $(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3$.

2003 Find all real numbers a for which the equation: $x^2 + (a-2)x + 1 = 3|x|$ has exactly three distinct real solutions

2004 Let α and β be the roots of $x^2 + mx - 1 = 0$, where m is an odd integer, and $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for all $n \geq 0$:
(i) λ_n is an integer, and
(ii) $\gcd(\lambda_n, \lambda_{n+1}) = 1$.

2005 Let a, b, c be three positive real numbers such that $a + b + c = 1$, and $\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}$. Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

2007 Find all pairs of real numbers (a, b) such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

2008 Suppose a, b are real numbers such that the roots of the equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that:
(i) $0 < 3ab \leq 1$, and
(ii) $b \geq \sqrt{3}$.

2010 Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

2013 A polynomial is called a Fermat polynomial if it can be written as the sum of the squares of two polynomials with integer coefficients. Suppose that $f(x)$ is a Fermat polynomial such that $f(0) = 1000$. Prove that $f(x) + 2x$ is not a Fermat polynomial.

Chapter 5

Geometry

5.1. Introduction

It is expected that a student preparing for the Olympiads will be familiar with all theorems in geometry that appear in generic textbooks up to class ten. There are many theorems in school geometry, which we do not repeat in this chapter. We only state the theorems which are not so likely to appear in school level textbooks, but the proofs of which are quite elementary. Note that we do not discuss the proofs of the theorems, and leave those as exercises for the students.

Definition 5.1 (Directed Line Segment)**.** When we associate a direction with a line segment, we call it a directed line segment.

The line segment from A to B, denoted by AB, and the line segment from B to A, denoted by BA, are not the same. The relation between AB and BA can be represented as $AB + BA = 0$, or $AB = -BA$.

When a point P on line AB lies between A and B, we say that P divides AB internally, and the ratio $AP : PB$ is defined to be positive. If a point P on line AB lies outside the segment AB, then it is said to divide AB externally, and the ratio $AP : PB$ is defined to be negative.

5.2. Interesting Results

Theorem 5.1 (Apollonius Theorem)**.** *Let ABC be a triangle and D be the mid point of BC. Then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.*

Theorem 5.2 (Angle Bisector Theorem)**.** *In a triangle ABC, the internal bisector of $\angle BAC$ divides the opposite side internally in the ratio $AB : AC$, and the external bisector of $\angle BAC$ divides the opposite side externally in the ratio $-AB : AC$.*

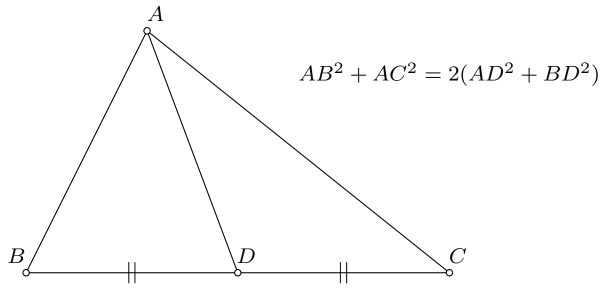


Figure 5.1.: Apollonius Theorem

If $AB = AC$ then external bisector of $\angle BAC$ is parallel to BC and if $AB \neq AC$ then external bisector of $\angle BAC$ is not parallel to BC.

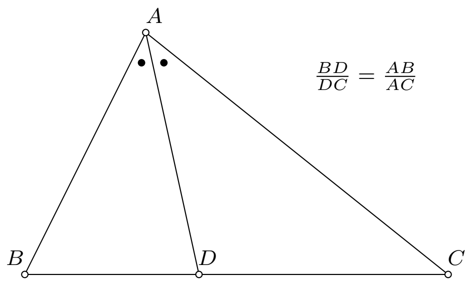


Figure 5.2.: Angle Bisector Theorem

Theorem 5.3 (Excenter and Excircle)**.** *Two external angle bisectors and one internal angle bisector of any triangle are concurrent. The point of concurrency is known as an excenter of the triangle. The circle which touch all the sides (one internally and two externally) and whose center is an excenter, known as excircle. There are three excenters and excircles of any triangle.*

Theorem 5.4 (Euler Line)**.** *Circumcenter O, centroid G and orthocenter H of any triangle are collinear, and G divides OH internally with a ratio $GH : OG = 2 : 1$. The line OH is called the Euler line of the triangle.*

Theorem 5.5 (Simson Line)**.** *The feet R, Q, P of the perpendiculars drawn from any point X on the circumcircle of a triangle ABC, upon the sides BC, CA, AB, are collinear. The line RQP is called the Simson line or the pedal line of the point X with respect to the triangle ABC.*

Theorem 5.6. *If a point P is inside, outside, or on a circle, and any two lines through P cut the circle at points A, A' and B, B', then $PA \cdot PA' = PB \cdot PB'$. If P is outside the circle, if any line through P cuts the circle at points A, A', and if PT is a tangent from P to the circle touching it at T, then $PA \cdot PA' = PT^2$.*

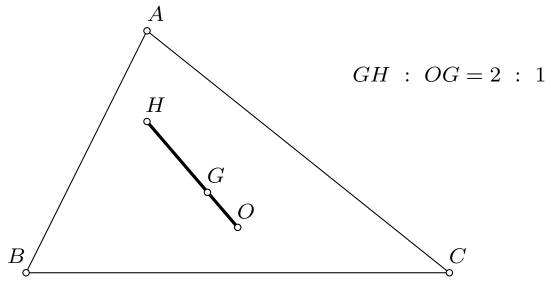


Figure 5.3.: Euler Line

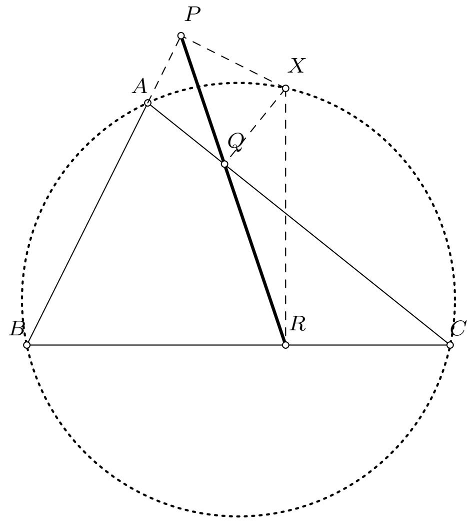


Figure 5.4.: Simson Line

Definition 5.2 (Power of a Point)**.** If a point P is inside, outside, or on a circle, and if any line through P cuts the circle at points A, A', then $PA \cdot PA'$ is known as the power of the point P with respect to the circle.

Note that the power of a point as per the above definition is positive, negative or zero according to whether the point lies outside, inside, or on the circle.

Theorem 5.7 (Apollonius Circle)**.** *If there are two fixed points A, B, and P is a moving point such that the ratio $PA : PB$ is constant and not equal to 1, then the locus of P is a circle. This circle is called the circle of Apollonius.*

Theorem 5.8 (Ptolemy's Theorem)**.** *In a cyclic quadrilateral, the product of its diagonals is equal to the sum of the products of its two pairs of opposite sides.*

Theorem 5.9 (Extended Ptolemy's Theorem)**.** *Let $ABCD$ be a quadrilateral which is not cyclic. Then $AC \cdot BD < AB \cdot CD + BC \cdot AD$.*

Theorem 5.10 (Brahmagupta's Theorem)**.** *In triangle ABC, if AD is the altitude and AE the diameter of the circumcircle through A, then $AB \cdot AC = AD \cdot AE$.*

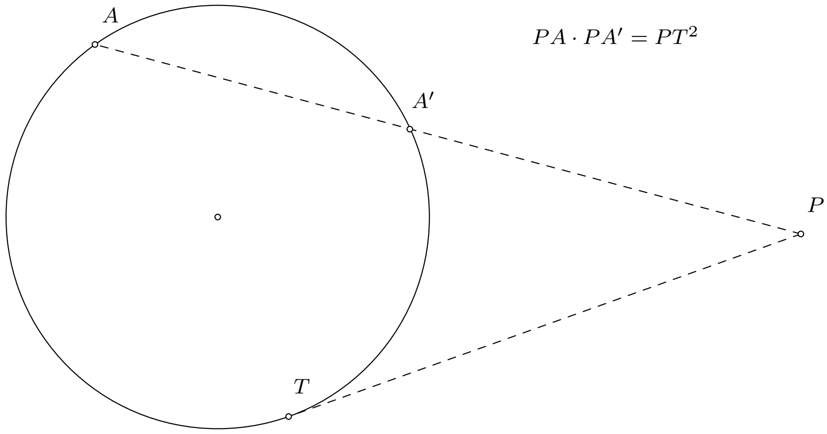


Figure 5.5.: Power of a Point

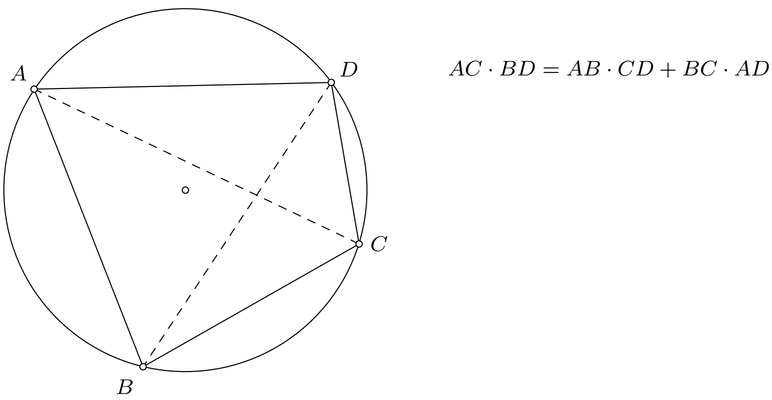


Figure 5.6.: Ptolemy's Theorem

Theorem 5.11 (Heron's Formula)**.** *In triangle ABC, let a, b, c denote the lengths of the sides BC, CA, AB, respectively. Then the area of the triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.*

Theorem 5.12 (Pappus' Area Theorem)**.** *Let ABC be any triangle, and let two parallelograms $BADE, BCFG$ be placed on the sides BA, BC of the triangle. The lines DE, FG, parallel to the sides AB, BC of the triangle, may be extended to meet at point H. Then the sum of the areas of parallelograms $BADE$ and $BCFG$ is equal to the area of the parallelogram $ACIJ$, drawn on the third side AC of the triangle, and having sides CI, AJ equal to and parallel to the line segment BH.*

Theorem 5.13 (Ceva's Theorem)**.** *In triangle ABC, let the points D, E, F be on sides BC, CA, AB, respectively. Then AD, BE, CF are concurrent if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{AE} = 1,$$

equivalently,
$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} = 1.$$

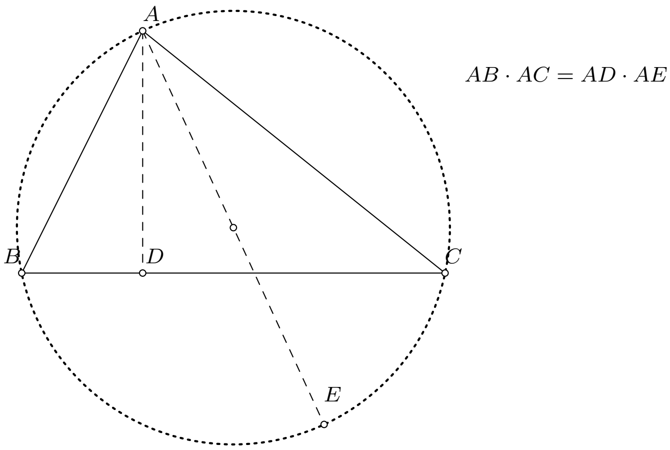


Figure 5.7.: Brahmagupta's Theorem

Theorem 5.14. *In a quadrilateral two opposite sides are equal and two opposite angles are equal prove that the quadrilateral is a parallelogram.*

What is wrong in the following proof:
Consider the quadrilateral $ABCD$ with $AB = CD$ and $\angle ABC = \angle CDA$. Draw the diagonal AC. Now between two triangles ABC and CDA; $AB = CD$, AC common and $\angle ABC = \angle CDA$, hence $\Delta ABC \equiv \Delta CDA$. Therefore, $BC = DA$, hence $ABCD$ is a parallelogram.
Exercise Give a correct proof.

5.3. Solved Examples

Example 5.1. Let ABC be a triangle. Let BE and CF be internal angle bisectors of $\angle B$ and $\angle C$ respectively with E on AC and F on AB. Suppose X is a point on the segment CF such that AX is perpendicular to CF ; and Y is a point on the segment BE such that AY is perpendicular to BE. Prove that $XY = (b+c-a)/2$ where $BC = a, CA = b$ and $AB = c$.

Solution. Produce AX and AY to meet BC is X' and Y' respectively. Since BY bisects $\angle ABY'$ and BY is perpendicular to AY it follows that $BA = BY'$ and $AY = YY'$. Similarly, $CA = CX'$ and $AX = XX'$. Thus X and Y are mid-points of AX and AY respectively. By mid-point theorem $XY = X'Y'/2$. But $X'Y' = X'C + Y'B - BC = AC + AB - BC = b + c - a$. Hence $XY = (b+c-a)/2$. □

Example 5.2. Let ABC be a triangle and D be a point on the segment BC such that $DC = 2BD$. Let E be the mid-point of AC. Let AD and BE intersect in P. Determine the ratios BP/PE and AP/PD.

Solution. Let F be the midpoint of DC, so that D, F are points of trisection of BC. Now in triangle CAD, F is the mid-point of CD and E is that of CA. Hence

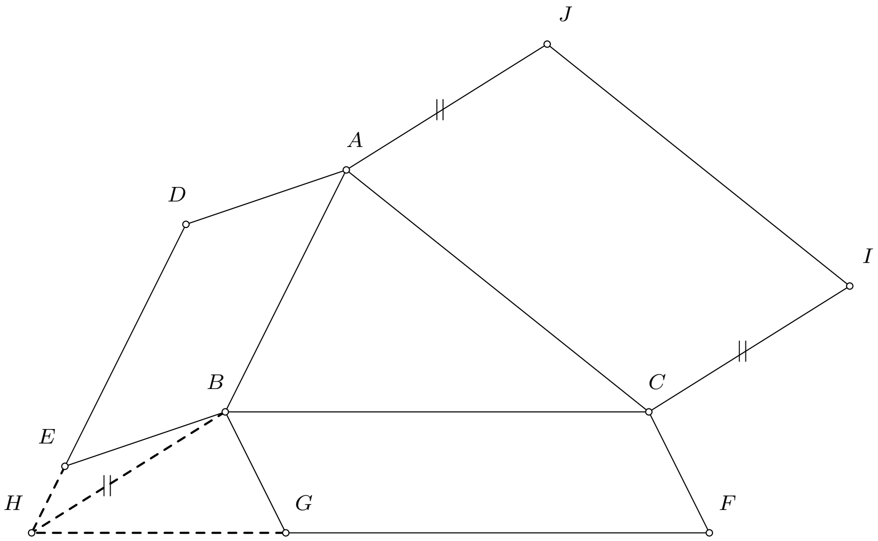


Figure 5.8.: Pappus' Area Theorem

$CF/FD = 1 = CE/EA$. Thus $EF||AD$. Hence we find that $EF||PD$. Hence $BP/PE = BD/DF$. But $BD = DF$. We obtain $BP/PE = 1$. In triangle ACD, since $EF||AD$ we get $EF/AD = CF/CD = 1/2$. Thus $AD = 2EF$. But $PD/EF = BD/BF = 1/2$. Hence $EF = 2PD$. This gives $AP = AD - PD = 3PD$. We obtain $AP/PD = 3$. □

Example 5.3. Let ABCD be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle S touching the arc AC internally, the arc BD internally and also touching the side AB. Find the radius of the circle S.

Solution. Let O be the centre of S. By symmetry O is on the perpendicular bisector of AB. Draw OE perpendicular to AB. Then $BE = AB/2 = 1/2$. If r is the radius of S, we see that $OB = 1 - r$, and $OE = r$. Using Pythagoras' theorem $(1 - r)^2 = r^2 + (1/2)^2$. Simplification gives $r = 3/8$. □

Example 5.4. Let ABC be an isosceles triangle with $AB = AC$ and let Γ denote its circumcircle. A point D is on the arc AB of Γ not containing C and a point E is on the arc AC of Γ not containing B such that $AD = CE$. Prove that BE is parallel to AD.

Solution. We note that triangle AEC and triangle BDA are congruent. Therefore $AE = BD$ and hence $\angle ABE = \angle DAB$. This proves that AD is parallel to BE. □

Example 5.5. In a triangle ABC, points D and E are on segments BC and AC such that $BD = 3DC$ and $AE = 4EC$. Point P is on line ED such that D is the midpoint of segment EP. Lines AP and BC intersect at point S. Find the ratio BS/SD.

Solution. Let F denote the midpoint of the segment AE. Then it follows that DF is parallel to AP. Therefore, in triangle ASC we have $CD/SD = CF/FA = 3/2$. But $DC = BD/3 = (BS + SD)/3$. Therefore $BS/SD = 7/2$. $\square$

Example 5.6. Let ABC be an acute angled triangle. The circle Γ with BC as diameter intersects AB and AC again at P and Q, respectively. Determine $\angle BAC$ given that the orthocenter of triangle APQ lies on Γ.

Solution. Let K denote the orthocenter of triangle APQ. Since triangles ABC and AQP are similar it follows that K lies in the interior of triangle APQ. Note that $\angle KPA = \angle KQA = 90^o - \angle A$. Since $BPKQ$ is a cyclic quadrilateral it follows that $\angle BQK = 180^o - \angle BPK = 90^o - \angle A$, while on the other hand $\angle BQK = \angle BQA - \angle KQA = \angle A$ since BQ is perpendicular to AC. This shows that $90^o - \angle A = \angle A$, so $\angle A = 45^o$. $\square$

Example 5.7. Let ABC be a triangle with $\angle A = 90^o$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 3 : 5 : 4$. Prove that $\angle DAE = 45^o$.

Solution. Rotating the configuraiton about A by 90^o, the point B goes to the point C. Let P denote the image of the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45^o$, so ECP is a right-angled triangle with $CE : CP = 4 : 3$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45^o$. $\square$

Example 5.8. Let ABC be a triangle with $\angle A = 90^o$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 1 : 2 : \sqrt{3}$. Prove that $\angle DAE = 45^o$.

Solution. Rotating the configuraiton about A by 90^o, the point B goes to the point C. Let P denote the image of the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45^o$, so ECP is a right-angled triangle with $CE : CP = \sqrt{3} : 1$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45^0$. $\square$

Example 5.9. In a triangle ABC, AD is the altitude from A, and H is the orthocentre. Let K be the centre of the circle passing through D and tangent to BH at H. Prove that the line DK bisects AC.

Solution. Note that $\angle KHB = 90^o$. Therefore $\angle KDA = \angle KHD = 90^o - \angle BHD = \angle HBD = \angle HAC$. On the other hand, if M is the midpoint of AC

then it is the circumcenter of triangle ADC and therefore $\angle MDA = \angle MAD$. This proves that D, K, M are collinear and hence DK bisects AC.

$\square$

Example 5.10. Let Γ be a circle with centre O. Let Λ be another circle passing through O and intersecting Γ at points A and B. A diameter CD of Γ intersects Λ at a point P different from O. Prove that $\angle APC = \angle BPD$.

Solution. Suppose that A' is a point on Λ such that $\angle APC = \angle BPD$. Then the segments OA' and OB subtends same angle in the respective minor arcs, so $OA' = OB$. This shows that A lies on Γ and hence $A' = A$. This proves that $\angle APC = \angle BPD$.

$\square$

Example 5.11. In a triangle ABC, let H denote its orthocentre. Let P be the reflection of A with respect to BC. The circumcircle of triangle ABP intersects the line BH again at Q, and the circumcircle of triangle ACP intersects the line CH again at R. Prove that H is the incentre of triangle PQR.

Solution. Since $RACP$ is a cyclic quadrilateral it follows that $\angle RPA = \angle RCA = 90^o - \angle A$. Similarly, from cyclic quadrilateral $BAQP$ we get $\angle QPA = 90^o - \angle A$. This shows that PH is the angular bisector of $\angle RPQ$.
We next show that R, A, Q are collinear. For this, note that $\angle BPC = \angle A$. Since $\angle BHC = 180^o - \angle A$ it follows that $BHCP$ is a cyclic quadrilateral. Therefore $\angle RAP + \angle QAP = \angle RCP + \angle QBP = 180^o$. This proves that R, A, Q are collinear. Now $\angle QRC = \angle ARC = \angle APC = \angle PAC = \angle PRC$. This proves that RC is the angular bisector of $\angle PRQ$ and hence H is the incenter of triangle PQR. $\square$

5.4. Exercise Problems

1. Let ABC be a triangle. Let D, E be a points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC. Let BF intersect AD in P and AE in Q respectively. Determine BP/PQ.
2. Let ABC be a triangle. Let E be a point on the segment BC such that $BE = 2EC$. Let F be the mid-point of AC. Let BF intersect AE in Q. Determine BQ/QF.
3. Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as end points of the arc. Inscribe a circle Γ touching the arc AC externally, the arc BD internally and also touching the side AD. Find the radius of the circle Γ.
4. From Heron's formula show that among all the triangles having the same perimeter, equilateral triangle has the maximum area.

5. The straight line that passes through the point of intersection of the diagonals of a trapezium and through the point of intersection of its non-parallel sides, bisects each of the parallel sides of the trapezium.

6. Let a, b and c be the sides of a right angled triangle. Let θ be the smallest angle of this triangle. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are also the sides of a right angled triangle then show that, $\sin\theta = \frac{\sqrt{5}-1}{2}$.

7. Let M be a point in the triangle ABC such that, $\triangle ABM = 2.\triangle ACM$. Show that the locus of all such points is a straight line.

8. Let ABC and DEF be two triangles such that $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle BAC = \angle EDF$. Show that, the two triangles are similar.

9. Let $ABCD$ be a quadrilateral such that the sum of a pair of opposite sides equal to the sum of the other pair of opposite sides i.e. $AB+CD = AD+BC$. Prove that the circles inscribed in the triangles ABC and ACD touch each other.

10. Sides AB, BC and CA of are cut by a transversal at points Q, R and S respectively. The circumcircles of $\triangle ABC$ and $\triangle SCR$ intersect at P. Prove that $APSQ$ is a cyclic quadrilateral.

11. Let in $\triangle ABC, AB = AC$. D is the midpoint of BC, E is the foot of the perpendicular from D to AB. F is the midpoint of DE. Prove that, AF perpendicular to CE.

12. Let G be the centroid of the triangle ABC. AG is produced to P so that $AG = GP$. Parallels through P to CA, AB and BC meet BC, CA and AB at L, M and N respectively. Prove that L, M and N are collinear.

13. In triangle ABC, let I denotes the point of concurrence of the lines that join A, B, C to the points of contact of respective ex-circles with the sides BC, CA, AB. Suppose that I lies on the in-circle of triangle ABC. Prove that the sum of two of the sides of the triangle is equal to three times the third side.

14. An angle $\angle ABC$ was drawn on a sheet of paper. Suppose due to a rat, the vertex portion of the angle is found missing in the diagram. How can you draw the angle bisector of this rat- eaten angle?

5.5. RMO Problems

1990 A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. Prove that the crease will divide BC in the ratio $5 : 3$.

1990 P is any point inside a triangle ABC. The perimeter of the triangle $AB + BC + CA = 2s$. Prove that

$$s < AP + BP + CP < 2s.$$

1990 If the circumcenter and centroid of a triangle coincide, prove that the triangle must be equilateral.

1991 Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively. Show that

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

1991 Take any point P_1 on the side BC of a triangle ABC and draw the following chain of lines : P_1P_2 parallel to AC (P_2 on AB), P_2P_3 parallel to BC, P_3P_4 parallel to AB, P_4P_5 parallel to CA, and P_5P_6 parallel to BC. Here P_2, P_5 lie on AB; P_3, P_6 lie on CA; and P_4 on BC. Show that P_6P_1 is parallel to AB.

1992 $ABCD$ is a cyclic quadrilateral with $AC \perp BD$; AC meets BD at E. Prove that

$$EA^2 + EB^2 + EC^2 + ED^2 = 4R^2,$$

where R is the radius of the circumscribing circle.

1992 $ABCD$ is a cyclic quadrilateral; x, y, z are the distances of A from the lines BD, BC, CD respectively. Prove that

$$\frac{BD}{x} = \frac{BC}{y} + \frac{CD}{z}.$$

1992 $ABCD$ is a quadrilateral and P, Q are mid-points of CD, AB respectively. Let AP, DQ meet at X, and BP, CQ meet at Y. Prove that

$$\text{area of } ADX + \text{area of } BCY = \text{area of quadrilateral } PXQY.$$

1992 The cyclic octagon $ABCDEFGH$ has sides a, a, a, a, b, b, b, b respectively. Find the radius of the circle that circumscribes $ABCDEFGH$ in terms of a and b.

1993 Let ABC be an acute-angled triangle and CD be the altitude through C. If $AB = 8$ and $CD = 6$, find the distance between the mid-points of AD and BC.

1993 Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 is the radius of the circle passing through B and C and touching AD. Show that

$$r_1 + r_2 \geq \frac{5}{8}(a + b).$$

1994 In the triangle ABC, the incircle touches the sides BC, CA and AB respectively at D, E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers, find the sides of the triangle ABC.

1994 Let AC and BD be two chords of a circle with center O such that they intersect at right angles inside the circle at the point M. Suppose K and L are the mid-points of the chord AB and CD respectively. Prove that $OKML$ is a parallelogram.

1995 In triangle ABC, K and L are points on the side BC (K being closer to B than L) such that $BC \cdot KL = BK \cdot CL$ and AL bisects $\angle KAC$. Show that AL is perpendicular to AB.

1995 Let $A_1A_2A_3 \dots A_{21}$ be a 21-sided regular polygon inscribed in a circle with center O. How many triangles $A_iA_jA_k$, $1 \leq i < j < k \leq 21$, contain the point O in their interior.

1996 The sides of a triangle are three consecutive integers and its inradius is four units. Determine the circumradius.

1996 Let ABC be a triangle and h_a the altitude through A. Prove that

$$(b+c)^2 \geq a^2 + 4h_a^2.$$

(As usual a, b, c denote the sides BC, CA, AB respectively.)

1997 Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 13$, find $[AFPE]$. (Here $[\cdot]$ denotes the area of a triangle or a quadrilateral, as the case may be.)

1997 In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that

- ⋄ $AD \cdot BC \geq AB \cdot CD$;
- ⋄ $AD + BC \geq AB + CD$.

1998 Let $ABCD$ be a convex quadrilateral in which $\angle BAC = 50^\circ$, $\angle CAD = 60^\circ$, $\angle CBD = 30^\circ$, and $\angle BDC = 25^\circ$. If E is the point of intersection of AC and BD, find $\angle AEB$.

1998 Let ABC be a triangle with $AB = BC$ and $\angle BAC = 30^\circ$. Let A' be the reflection of A in the line BC; B' be the reflection of B in the line CA; C' be the reflection of C in the line AB. Show that A', B', C' form the vertices of an equilateral triangle.

1999 Let $ABCD$ be a square and M, N points on sides AB, BC, respectably, such that $\angle MDN = 45^\circ$. If R is the midpoint of MN show that $RP = RQ$ where P, Q are the points of intersection of AC with the lines MD, ND.

2000 Let AC be a line segment in the plane and B a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC such that $\angle APB = \angle BQC = 120^\circ$ and an isosceles triangle RAC on the other side of AC such that $\angle ARC = 120^\circ$. Show that PQR is an equilateral triangle.

2000 The internal bisector of angle A in a triangle ABC with $AC > AB$, meets the circumcircle Γ of the triangle in D. Join D to the center O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD.

2001 Let BE and CF be the altitudes of an acute triangle ABC, with E on AC

and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

2001 In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

2002 In an acute triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \qquad \frac{AE}{AF} = \frac{AB}{AC}, \qquad \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ABC.

2002 The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

2003 Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90°$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45°$.

2003 Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios, where $d(X, YZ)$ denotes the perpendicular distance from a point X to the line YZ.

2004 Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.

2004 Let $ABCD$ be a quadrilateral, X, Y be the midpoints of AC, BD, respectively, and the lines through X and Y, parallel to BD, AC, respectively, meet at O. Let P, Q, R, S be the midpoints of AB, BC, CD, DA, respectively. Prove that

- ◇ the quadrilaterals $APOS$ and $APXS$ have the same area, and
- ◇ the areas of the quadrilaterals $APOS, BQOP, CROQ, DSOR$ are equal.

2005 Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

2005 In triangle ABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

2006 Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are *collinear*.

2006 Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD; $AB = 3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

2007 Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$.

2007 Suppose that a trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with center O. Suppose that the diagonals AC, BD of the trapezium intersect at M, and that $OM = 2$.

a) If $\angle AMB$ is $60°$, find and prove the value of the $|AB - CD|$.

b) If $\angle AMD$ is $60°$, find and prove the value of the $|AB - CD|$.

2008 Let ABC be an acute angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular from B to BC meet in N. Prove that ND is equal to the circumradius of ABC.

2011 Let ABC be an acute scalene triangle with circumcenter O and orthocenter H. If M is the mid-point of BC, prove that AO and HM meet on the circumcircle.

2011 Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be the mid-points of AB, BC, CD, DA respectively. Suppose AC, BD, EG and FH concur at a point O. Prove that $ABCD$ is a parallelogram.

2012 Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as center and B, D as end points of the arc. Similarly, draw a quadrant of a circle with B as center and A, C as end points of the arc. Inscribe a circle Γ touching the arcs AC and BD both externally and also touching the side CD. Find the radius of the circle Γ.

2012 Let ABC be a triangle. Let D, E be two points on the segment BC such that $BD = DE = EC$. Let F be the mid-point of AC. Let BF intersect AD in P and AE in Q respectively. Determine the ratio of the area of the triangle APQ to that of the quadrilateral $PDEQ$.

2013 In an acute-angled triangle ABC with $AB < AC$, the circle Γ touches AB at B and passes through C intersecting AC again at D. Prove that the orthocenter of the triangle ABD lies on Γ if and only if it lies on the perpendicular bisector of BC.

2013 Let ABC be a triangle which is not right-angled. Define a sequence of triangles $A_iB_iC_i$ with $i \geq 0$, as follows: $A_0B_0C_0$ is the triangle ABC; and, for $i \geq 0$, $A_{i+1}, B_{i+1}, C_{i+1}$ are the reflections of the orthocenter of the triangle $A_iB_iC_i$ in the sides B_iC_i, C_iA_i, A_iB_i, respectively. Assume that $\angle A_m = \angle A_n$ for some distinct natural numbers m, n. Prove that $\angle A = 60^o$.

2013 Let $n \geq 4$ be a natural number. Let $A_1A_2 \cdots A_n$ be a regular polygon and $X = \{1, 2, \ldots, n\}$. A subset $\{i_1, i_2, \ldots, i_k\}$ of X, with $k \geq 3$ and $i_1 < i_2 < \cdots < i_k$, is called a good subset if the angles of the polygon $A_{i_1}A_{i_2} \cdots A_{i_k}$, when arranged in the increasing order, are in an arithmetic progression. If n is a prime, show that a proper good subset of X contains exactly four elements.

2014 Let ABC be an acute-angled triangle and let H be its ortho-center. For any point P on the circumcircle of triangle ABC, let Q be the point of intersection of the line BH with the line AP. Show that there is a unique point X on the circumcircle of ABC such that for every point $P \neq A, B$, the circumcircle of HPQ pass through X.

2014 Let ABC be a triangle and let AD be the perpendicular from A on to BC. Let K, L, M be points on AD such that $AK = KL = LM = MD$. Also $PK||QL||RM||BD$. If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that $BD : DC = 2 : 1$.

Chapter 6

Combinatorics

6.1. Introduction

Combinatorics at the high-school level mainly deals with the number of arrangements of a set of objects in a set of boxes. The objects and the boxes may be distinct, or identical, or partially identical. We generally have to count the possible arrangements with specific restrictions. There are three basic counting principles, as follows.

Addition principle

If a finite set A of objects is divided into two disjoint subsets A_1, A_2, then the number of objects in A can be determined by finding the number of objects in A_1 and A_2, as follows, $|A| = |A_1| + |A_2|$.

Exercise: Can you extend this result for n disjoint subsets of A.

Multiplication principle

If an action A has k different outcomes and a second action B has l different outcomes for each possible outcome of the first action A, then on performing the actions A and B together, in that order, we get kl composite outcomes, that is, kl pairs of outcomes with the first outcome for the action A and the second outcome for the action B.

Exercise: Can you extend this result for more than two actions.

Example 6.1. How many numbers can be formed from some or all of the digits $1, 2, 3, 4$ if no number is to have repeated digits?

Solution. The number can be of $1, 2, 3$ or 4 digits. A number of 1 digit can not be same as a number of 2 digits or a number of 3 digits or a number of 4 digits. Similar things happen with the number of other digits. Let A_i be the set of numbers of i digits, $i = 1, 2, 3, 4$. Then A_i's are pairwise disjoint for $i = 1, 2, 3, 4$. Here we use addition principle of counting. So we have to find the number of elements in sets A_i for $i = 1, 2, 3, 4$.

There are 4 one-digit numbers. Hence number of elements in A_1 is 4. To find number of elements in A_2 we have to use multiplication principle. Here the first action is choosing of the first digit and second action is choosing of the second digit. The first action has 4 outcomes and for each outcomes there are 3 outcome for the second action since there are no repetitions of digits. For an example, if the outcome of the first action is 1 then for second action there are three outcomes $2, 3, 4$. Hence by multiplication principle there are $4 \times 3 = 12$ two digits numbers, i.e., $|A_2| = 12$. Similarly, $|A_3| = 4 \times 3 \times 2 = 24$ and $|A_4| = 4 \times 3 \times 2 \times 1 = 24$. So, by addition principle, there are $|A_1| + |A_2| + |A_3| + |A_4| = 64$ many numbers which can be formed from some or all of the digits $1, 2, 3, 4$ and no number is to have repeated digits. $\square$

Exercise: Find the number of 3-digit numbers that are odd and have different digits.

Bijection principle

If two sets A and B have a one-to-one correspondence with each other, i.e., if there exists a bijection between them, then they contain the same number of elements, that is, $|A| = |B|$.

By this principle, the number of elements in a given set A can be computed by finding a set B which is in one-to-one correspondence with A, such that counting the number of elements in B is easy.

Example 6.2. Consider a set $S = \{a_1, a_2, \ldots, a_n\}$ that has n elements. Let A be the set of all subsets of S, i.e., $A = \{T : T \subseteq S\}$. Find $|A|$.

Solution. Let B be the set of all binary sequences of length n, that is, $B = \{(x_1, x_2, \ldots, x_n) \ : \ x_i = 0 \text{ or } 1\}$. By multiplication principle $|B| = 2^n$. Now we define a one-to-one correspondence between A and B as follows: a subset $T \subseteq S$ corresponds to the binary sequence $t = (x_1, x_2, \ldots, x_n)$ where $x_i = 1$ if $a_i \in T$ and $x_i = 0$ if $a_i \notin T$. Hence $|A| = |B| = 2^n$. $\square$

Exercise: List all subsets of the set $S = \{1, 2, 3, 4, 5\}$.

6.2. Permutation and Combination

Permutation

By an r-permutation of a set $A = \{1, 2, \ldots, n\}$, we mean an ordered arrangement of r elements from the n elements of A, that is, $(x_1, x_2, \ldots, x_r)$, where $x_i \in A$ for all $i = 1, 2, \ldots, r$.

The number of all possible r-permutations of a set A containing n different objects is denoted by $P(n,r)$ or $(n)_r$ or ${}^nP_r$, and is computed as

$$(n)_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!},$$

where n and r are positive integers with $r \le n$. By definition, $(n)_0 = (0)_0 = 1$. In case $r = n$, we simply call it a permutation, instead of n-permutation. Number of possible permutations of n different elements is $n!$.

For example, all 3-permutations on the set $S = \{1,2,3,4\}$ are

$$\begin{aligned}&(1,2,3),(1,2,4),(1,3,2),(1,3,4),(1,4,2),(1,4,3),(2,1,3),(2,1,4),\\&(2,3,1),(2,3,4),(2,4,1),(2,4,3),(3,1,2),(3,1,4),(3,2,1),(3,2,4),\\&(3,4,1),(3,4,2),(4,1,2),(4,1,3),(4,2,1),(4,2,3),(4,3,1),(4,3,2),\end{aligned}$$

and the number of 3-permutations of S is $(4)_3 = 4 \times 3 \times 2 = 24$.

Quick Exercise

1. List all 3-permutations from the set $S = \{1,2,3,4,5\}$.
2. How many 4-permutations are there for a 6-element set?

Combination

By an r-combination of a set $A = \{1,2,\ldots,n\}$, we mean an unordered arrangement of r elements from the elements of A.

The number of r-combinations from a set A containing n different objects is denoted by $C(n,r)$ or $\binom{n}{r}$ or ${}^nC_r$, and is computed as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

where n and r are positive integers with $r \le n$. By definition, $\binom{n}{0} = \binom{0}{0} = 1$. In case $r = n$, we simply call it a combination, instead of n-combination. Number of possible combinations of n elements is $\binom{n}{n} = 1$. For example, all 3-combinations on the set $S = \{1,2,3,4\}$ are all subsets of S with 3 elements.

Quick Exercise

1. List all the 3-combinations from the set $S = \{1,2,3,4,5\}$.
2. For any two non-negative integers n, r with $r \le n$, prove that
   - ◇ ${}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n$
   - ◇ ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$

◇ $r.^nC_r = n.^{n-1}C_{r-1}$

◇ $\sum_{r=0}^{k} {^nC_r} {^mC_{k-r}} = {^{m+n}C_k}$.

Some important results

1. Number of all subsets of a set containing m elements is 2^m.
2. The number of ways in which m distinguishable balls can be put in n distinguishable boxes is n^m.
3. The number of ways in which m balls can be put in m boxes, such that no cell remains empty, is $m!$.
4. The number of m-tuples with distinct elements from an m-set is $m!$.
5. $(m)_r = m(m-1)\ldots(m-r+1)$ denotes
   ◇ The number of r-tuples with distinct elements from an m-set.
   ◇ The number of injective functions mapping from an r-set to m-set.
   ◇ The number of ways in which r balls can be put in m boxes, such that no cell gets more than one ball.
6. $\binom{m}{r} = \frac{(m)_r}{r!}$, also known as the binomial coefficient, denotes
   ◇ The number of r-subsets of an m-set.
   ◇ The number of m-ordered tuples consisting of r identical objects of one type and $(m-r)$ identical objects of another type.

Permutation with repetition

The number of r-permutations, with repetition, of a set having n different objects is n^r. Here repetition means that an object from the set may be chosen any number of times, including zero or none. For example, all possible 2-permutations, with repetition, of the 3-set $\{a, b, c\}$ are $\{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$. The number of such 2-permutations is $3^2 = 9$.

Permutations of identical objects

Suppose that there are n objects, of which n_i are identical of the i-th kind, such that $n = \sum n_i$. Then the number of permutations of these n objects is $\frac{n!}{\prod n_i!}$. For example, number of permutations of 3 black, 4 red and 2 blue balls is $\frac{9!}{3!\times 4!\times 2!}$.

Circular permutation

By a circular permutation of a set $A = \{1, 2, \ldots, n\}$, we mean an ordered arrangement of the elements of A around a circle. The number of distinct circular permutations of n different objects is $(n-1)!$.

Note that a necklace or garland is circular, but it has an additional level of symmetry, which is reflective. Thus, the number of different circular necklaces that can be made from n beads $(n > 2)$ is $\frac{(n-1)!}{2}$.

Combination with repetition

The number of r-combinations, with repetition, of a set having n different objects is $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$. Here repetition means that an object from the set may be chosen any number of times. This is related to the well known ball and bar problem:

> In how many ways can r identical balls be distributed among n children (or in n distinct boxes), such that some box(es) may be left empty?

To solve the problem, construct a bijection between the solution set of this problem and the set of all permutations of r identical balls and $n-1$ identical bars.

Example 6.3. The number of non-negative solutions of $(x_1, \ldots, x_n)$ for equation $x_1 + \cdots + x_n = r$ is $A_{r,n} = \binom{n+r-1}{n-1}$. The number of positive solutions is $\binom{r-1}{n-1}$.

Solution. Consider the r on the right hand side of the equation as a set of r identical 1's. Then the non-negative case is a bijective transformation of the problem

> In how many ways r indistinguishable 1's (balls) can be placed in n different variables x_i (boxes), such that some box(es) may remain empty?

From the ball and bar problem, this produces $A_{r,n} = \binom{n+r-1}{n-1}$.

For the positive solutions, we transform the equation by $x_i = y_i + 1$, such that it becomes $y_1 + \cdots + y_n = r - n$. Now the problem is identical to counting non-negative solutions of $(y_1, \ldots, y_n)$, and thus the number of solutions is $A_{r-n,r} = \binom{r-1}{n-1}$. □

Quick Exercise

1. Find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 9$ with the conditions $x_1 > 0$, $x_2 > 0$, $x_3 \geq 0$ and $x_4 > 1$.
2. Given a regular 8×8 chess board, determine – (i) how many squares of all sizes are there on it, and (ii) how many rectangles of all sizes are there on it.
3. In how many ways can 12 identical white and 12 identical black pawns be placed on the black squares of a regular 8×8 chess board?
4. How many necklaces can be made using 7 beads, of which 5 are identically red, and 2 are identically blue?

6.3. Binomial Theorem

For n any position integer, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$.

Note that to evaluate a product of the form $(a+b)(c+d)(e+f)\ldots$, we take one term from each factor and multiply. Thus we have two choices of elements from each factor (e.g., a or b), and we finally add all possible combinations of the terms. So for n factors, there are 2^n many product terms added in the expression.

6.4. Counting in Two Different Ways

Counting in two different ways is a very useful tool for counting problems in high-school combinatorics. We present a few examples to illustrate the idea.

Example 6.4 (De Polignac's Formula)**.** If p is a prime and e is the largest exponent such that p^e divides $n!$ (i.e., $p^e|n!$ and $p^{e+1}\nmid n!$), then

$$e=\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]=\sum_{i=1}^{\infty}e_i,$$

where $e_1=\left[\frac{n}{p}\right]$ and $e_n=\left[\frac{e_{n-1}}{p}\right]$ for $n>1$.

Solution. Count the number of factors p in $n!$ in two different ways:

- ◇ multiples of p less than or equal to n, plus multiples of p^2 less than or equal to n, plus multiples of p^3 less than or equal to n, etc.
- ◇ multiples of p less than or equal to n, multiples of p^2 within the multiples of p less than or equal to n, multiples of p^3 within the multiples of p^2 less than or equal to n, etc.

The first way produces $\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$, and the second way produces $\sum_{i=1}^{\infty}e_i$. □

Example 6.5. Let $p_k(n)$ be the number of permutations on $\{1,\ldots,n\}$ with exactly k many fixed points. Prove that $\sum_{k=1}^{n}k\cdot p_k(n)=n!$.

Solution. Count the total number of fixed points in two different ways:

- ◇ for every $k=1,\ldots,n$, there are $p_k(n)$ permutations, each with k fixed points
- ◇ for every element $i=1,\ldots,n$, there are $(n-1)!$ permutations that fix i

The first way produces $\sum_{k=1}^{n}k\cdot p_k(n)$, and the second way gives $n\cdot(n-1)!=n!$. □

Example 6.6. A 10×10 square grid is tiled with unit squares, and each vertex is colored either blue or yellow. There are 80 blue points, two of them at the corners, and 25 of them on the sides of the big square. The sides of the unit squares are colored blue if both the end vertices are blue, colored yellow if both the end vertices are yellow, and colored green if the end vertices are of different colors. Given that there are 140 green sides, find out how many blue sides are there.

Solution. For the 10×10 grid, the total number of unit sides is $10\times11\times2=220$. Given 140 green sides, there are 80 sides that are either blue or yellow. Let there be b blue sides, and $(80-b)$ yellow sides. Now, we will count the total number of blue vertices, appearing at the end of the unit sides, in two different ways.

◇ Each blue side has two blue vertices, and each green side has one. Thus the total number of appearances of blue vertices at the end of unit sides is $2b+140$.

◇ 2 blue vertices at the corners of the big square are end-points of 2 unit sides each, 25 blue vertices at the sides of the big square are end-points of 3 unit sides each, and the remaining $(80-2-25)=53$ blue vertices within the big square are end-points of 4 unit sides each. Thus the total number of appearances of blue vertices at the end of unit sides is $2\times 2+25\times 3+53\times 4=256$.

Thus we obtain $2b+140=256$, that is, $b=58$. □

Quick Exercise

For non-negative integers m,n,k, prove that

$$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}=\binom{n+m}{k}, \qquad \sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}, \qquad \sum_{i=0}^{n}\binom{n}{i}=2^n.$$

6.5. Recurrence Relation

A recurrence relation for the sequence $x_1, x_2, \ldots$ is an equation that provides a relationship between x_n and its predecessors $x_1, x_2, \ldots, x_{n-1}$. That is, a recurrence relation can be used to find the value of x_n from the values of $x_1, x_2, \ldots, x_{n-1}$.

Rabbit Problem and Fibonacci Sequence

Suppose that we are given a pair of newborn rabbits on January first, and assume that they reach maturity at two months of age. Suppose further that they give birth to another pair of rabbit on the March first and continually produce a new pair on the first of the every succeeding month. Each newborn pair takes two months to mature and produces a new pair on the first day of the third month of their life, and on the first day of every succeeding month. If no rabbits ever die, how many pairs of rabbits are there on the first day of the nth month?

If F_n be the number of pairs of rabbits are there on the first day of the nth month then the recurrence relation is given by $F_1=1$, $F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for $n>2$. The sequence is $1,1,2,3,5,8,13,21,\ldots$. This sequence is known as Fibonacci Sequence.

Note that, $\frac{F_n}{F_{n-1}}=1+\frac{F_{n-1}}{F_{n-2}}$. If $\frac{F_n}{F_{n-1}}\to\phi$ as $n\to\infty$, then $\phi=1+\frac{1}{\phi}$ or $\phi=\frac{1+\sqrt{5}}{2}=\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$. This number is known as golden ratio.

Matching Problem and Derangement Sequence

Let there be n letters and n envelopes. Each letter has a correct envelope. Problem is to find the number of ways one can put all the letters in envelopes that no one goes to the correct one.

Let D_n be the required number. We consider two disjoint cases. Case-1, when the first letter goes to the second envelope and second letter goes to the first one, then number of ways to put all the letters in wrong envelope is D_{n-2}. Case-2, when the first letter goes to the second envelope but second letter do not go to the first one, then number of ways to put all the letters in wrong envelope is D_{n-1}. So if the first letter goes to the second envelope, then number of ways to put all the letters in wrong envelope is $D_{n-1} + D_{n-2}$. Hence the recurrence relation is given by $D_n = (n-1)(D_{n-1} + D_{n-2})$ for $n > 2$ and $D_1 = 0$, $D_2 = 1$.

Exercise: Find few terms of this sequence.

Tower of Hanoi

There are three vertical posts and n different-sized disks with holes in the center. In the initial setup, the disks are placed concentrically on one of the posts, in order of size, such that no disk is placed above a smaller one. The objective of the game is to move all the disks from the given post (initial setup) to another one, subject to the two following rules at every move:

◇ only one disk may be moved during a specific move, and

◇ a disk can never be placed over a smaller one during any move.

What is the minimum number of moves needed to complete this task?

Solution. Observe that the solution set of this problem follows a recurrence relation: $C_1 = 1$ and $C_n = 2C_{n-1} + 1$ for $n > 1$, where C_n is the minimum number of moves needed to move n disks from the given post to another. Therefore, the solution is computed as $C_n + 1 = 2(C_{n-1} + 1)$, that is, $C_n + 1 = 2^n$, or $C_n = 2^n - 1$. □

Quick Exercise

1. Consider n lines in the plane, such that no two of them are parallel, and no three of them are concurrent. Find the number of regions in which these n lines divide the plane.
2. Find the number of binary sequences of length eight having no '01' in them.
3. Determine the number of regions into which the plane is divided by r circles, where each pair of circles intersect in exactly two points and no three circles meet in a single point.
4. There are m necklaces, such that the first necklace contains 3 beads, the second necklace contains 7 beads, and in general the i-th necklace contains $2i$ beads

more than the number of beads in the $(i-1)$-th necklace. Find the total numbers of beads in all the necklaces.

6.6. Inclusion and Exclusion

Let $A_1, A_2, \ldots, A_k$ be subsets of a finite set A. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = n_1 - n_2 + n_3 - n_4 + \cdots + (-1)^{n-1} n_k,$$

where $n_r = \sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}|$, and the sum is taken over all integers $i_1, i_2, \ldots, i_r$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq k$.

Quick Exercise

1. By using principle of inclusion-exclusion, prove that $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$.
   (Hint: Let $A_1 = \cdots = A_n = \{x\}$ and then apply inclusion-exclusion.)
2. Prove that $|\bigcap_{i=1}^{m} A_i| \geq \sum_{i=1}^{m} |A_i| \; - \; (m-1)|\bigcup_{i=1}^{m} A_i|$.
   (Hint: Prove it by induction on m, and apply inclusion-exclusion for $m = 2$.)
3. Find the number of natural numbers between 1 and 2014, which are not divisible by any of 2,3 and 5.

Derangement Problem

In combinatorics, the derangement is defined as a permutation on a set of elements such that no element appears at its original position.

Derangement is also called matching problem. Different equivalent versions of the derangement problems can be found in literature. For example,

- ◇ In how many ways m letters can be placed in m envelopes so that no letter goes to the correct destination?
- ◇ In how many ways m passengers can sit in an airplane so that no passenger sits on his/her own seat?
- ◇ In a party, every person leaves his/her hat at the corner. In how many ways can the hats be given back so that nobody receives his/her own hat?

A permutation $\pi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ fixes x if $\pi(x) = x$, and x is called a fixed point of π. In terms of permutations, a derangement is a special kind of permutation that has no fixed points in the set of elements it is defined on.

Theorem 6.1. *The number of derangements on* $\{1, 2, \ldots, m\}$ *is*

$$m! \cdot \sum_{i=2}^{m} \frac{(-1)^i}{i!} = m! \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^m}{m!} \right).$$

Proof. Define A_i as the set of permutations that fix the element $i \in \{1, 2, \ldots, m\}$. Then the set of all permutations which has at least one fixed point is $A = \cup_{i=1}^{m} A_i$. In this case, $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}$ signify the set of all permutations that fix the r-element subset $I_r = \{i_1, i_2, \ldots, i_r\}$, and there are $(m-r)!$ such permutations. So, in the principle of inclusion-exclusion, we have $n_r = \frac{m}{r}(m-r)!$, and thus

$$|A| = \binom{m}{1}(m-1)! - \binom{m}{2}(m-2)! + \cdots = m! \cdot \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots\right).$$

Therefore, the number of derangement, or the number of permutations that have no fixed points is $m! - |A| = m! \cdot \sum_{i=2}^{m} \frac{(-1)^i}{i!} = m! \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^m}{m!}\right)$. □

Example 6.7. How many n-length sequences of $\{H, T\}$ contain at least one run of k number of consecutive H's?

Solution. Let A_i denote the event where a run of at least k number of consecutive H's start at position $i+1$ in the sequence. So, our goal is to compute $|\cup_{i=0}^{n-k} A_i|$. In this case, determine the significance of $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}$, and use the principle of inclusion-exclusion to count the number of sequences. □

Quick Exercise

1. Prove that the sum $\binom{m}{1} - \binom{m}{2} + \ldots + (-1)^{r-1}\binom{m}{r}$ is non-negative (or non-positive) if r is odd (or even).
2. Let $0 \le a_0 < a_1 < a_2 < \cdots < a_k \le a_{k+1} > a_{k+2} > \cdots > a_m$ be a set of numbers. If we define $N_i = a_0 - a_1 + \cdots + (-1)^i a_i$ for $0 \le i \le r$, then prove that $0 \le N_0 < N_2 < \cdots$ and $N_1 < N_3 < \cdots \le 0$.
3. Find out the number of ways in which n passengers sit in n seats, such that
   - ◇ the first person sits in any random seat, and
   - ◇ the i-th person sits in the i-th seat if it is empty, otherwise, sits in a random seat.
4. Count the number of integers less than 10^n, which contains odd number of 1's.
5. Let $n > 2$ be a positive integer, and $1 < a_1 < a_2 < \cdots < a_k \le n$ be integers such that no a_i divides the product of the remaining a_j's. Show that k cannot exceed the number of primes less than or equal to n.
6. Prove that for any natural number n, there is a Fibonacci number which has n many 0's at end.

6.7. Solved Examples

Example 6.8. Let $X = \{1, 2, 3, \ldots, 11\}$. Find the the number of pairs $\{A, B\}$ such that $A \subset X, B \subset X, A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

Solution. Let $A \cup B = Y, B \setminus A = M, A \setminus B = N$ and $X \setminus Y = L$. Then X is the disjoint union of M, N, L and $A \cap B$. Now $A \cap B = \{4, 5, 7, 8, 9, 10\}$ is fixed. The remaining 5 elements $1, 2, 3, 6, 11$ can be distributed in any of the remaining sets M, N, L. This can be done in 3^5 ways. Of these if all the elements are in the set L, then $A = B = \{4, 5, 7, 8, 9, 10\}$ and this case has to be omitted. Hence the total number of pairs $\{A, B\}$ such that $A \subset X, B \subset X, A \neq B$ is $3^5 - 1$. □

Example 6.9. A finite non-empty set S of integers is called $3 - good$ if the the sum of the elements of S is divisble by 3. Find the number of $3 - good$ non-empty subsets of $\{0, 1, 2, \ldots, 9\}$.

Solution. Let A be a $3 - good$ subset of $\{0, 1, \ldots, 9\}$. Let $A_1 = A \cap \{0, 3, 6, 9\}, A_2 = A \cap \{1, 4, 7\}$ and $A_3 = A \cap \{2, 5, 8\}$. Then there are three possibilities:

- ◇ $|A_2| = 3, |A_3| = 0;$
- ◇ $|A_2| = 0, |A_3| = 3;$
- ◇ $|A_2| = |A_3|.$

Note that there are 16 possibilities for A_1. Therefore the first two cases correspond to a total of 32 subsets that are $3 - good$. The number of subsets in the last case is $16(1^2 + 3^2 + 3^2 + 1^2) = 320$. Note that this also includes the empty set. Therefore there are a total of 351 non-empty $3 - good$ subsets of $\{0, 1, 2, \ldots, 9\}$. □

Example 6.10. Prove that $\frac{(nk)!}{(n!)^k}$ is an integer for any pair of natural numbers n, k.

Solution. Note that the number of arrangements (permutations) of nk elements, divided into k groups of n identical elements each, is given by the formula

$$\frac{(nk)!}{(n!)(n!) \cdots k \text{ times} \cdots (n!)} = \frac{(nk)!}{(n!)^k}.$$

As this number of permutations denote an integer, this is an integer. □

Example 6.11. Let $r, n > 0$ be positive integers, and let $a_1, a_2, \ldots, a_r$ be given integers. Then the number of integer solutions of the equation $x_1 + x_2 + \cdots + x_r = n$, such that $x_i > a_i$ for $1 \leq i \leq r$, is

$$\binom{n - a_1 - a_2 - \cdots - a_n - 1}{r - 1}.$$

Solution. Consider the set of transformed variables $y_i = x_i - a_i - 1$ for $1 \leq i \leq r$. With the set of these new variables, the equation may be rewritten as

$$y_1 + y_2 + \cdots + y_r = n - r - a_1 - a_2 - \cdots - a_r.$$

Now the question transforms to 'finding non-negative integer solutions for y_i in the above equation', and we know from the Balls-and-Bars problem that the number of all such solutions is

$$\binom{n - r - a_1 - a_2 - \cdots - a_r + r - 1}{r - 1}$$

Hence the result. □

Example 6.12. Find the number of integer solutions of $x_1 + x_2 + x_3 = 20$ subject to the conditions $1 \le x_1 \le 5$, $10 \le x_2 < 16$, and $-3 < x_3 < 9$.

Solution. Set $y_1 = x_1 - 1$, $y_2 = x_2 - 10$, $y_3 = x_3 + 2$, so that the original equation in (x_1, x_2, x_3) reduces to $y_1 + y_2 + y_3 = 11$, where y_1, y_2, y_3 are nonnegative integers with respective bounds $y_1 \le 4$, $y_2 \le 5$, $y_3 \le 10$.

Let A, B, C be respectively the sets of all solutions of (y_1, y_2, y_3) such that $y_1 > 4$, $y_2 > 5$, $y_3 > 10$. Then $A \cap B$ is the set of all solutions with $y_1 > 4$, $y_2 > 5$, and $A \cap C$, $B \cap C$ and $A \cap B \cap C$ carry similar meanings.

Using the Balls-and-Bars problem, we know that the number of nonnegative integer solutions of $y_1 + y_2 + y_3 = 11$ is $\binom{13}{2} = 78$. Moreover, $|A| = \binom{8}{2} = 28$, $|B| = \binom{7}{2} = 21$, $|C| = \binom{2}{2} = 1$, $|A \cap B| = \binom{2}{2} = 1$, and $|A \cap C| = |B \cap C| = |A \cap B \cap C| = 0$. By the principle of inclusion and exclusion, we also know that $|(A \cup B \cup C)| = 28 + 21 + 1 - 1 - 0 - 0 + 0 = 49$. So, the required number of solutions is $|A^c \cap B^c \cap C^c| = |(A \cup B \cup C)^c| = 78 - |(A \cup B \cup C)| = 29$. □

Example 6.13 (Multinomial Theorem)**.** Let n, k be positive integers. Then the coefficient of $x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$ in the expression $(x_1 + x_2 + \cdots + x_k)^n$ is

$$\frac{n!}{r_1! r_2! \cdots r_k!},$$

where $r_1 + r_2 + \cdots + r_k = n$ and $r_1, r_2, \ldots, r_k$ are nonnegative integers.

Solution. The expression is the product of n factors, each equal to $(x_1 + x_2 + \cdots + x_k)$, and every term in the expression is formed by taking one item from $\{x_1, x_2, \ldots, x_k\}$ out of each of these n factors.

Therefore the number of ways in which any term $x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$ will appear in the final expression is equal to the number of ways of arranging n items from $\{x_1, x_2, \ldots, x_k\}$, where r_1 of them are x_1, r_2 of them are x_2, and so on. This mode of counting results in the desired coefficient $\frac{n!}{r_1! r_2! \cdots r_k!}$. □

6.8. Exercise Problems

1. Let $X = \{1, 2, 3, \ldots, 12\}$. Find the the number of pairs $\{A, B\}$ such that $A \subset X, B \subset X, A \ne B$ and $A \cap B = \{2, 3, 4, 5, 7, 8\}$.
2. Find the coefficient of $a^2 b^3 c^4 d$ in the expression $(a + b + c + d)^{10}$.
3. Find the number of functions from a set containing n elements to a set containing m elements.
4. Find the number of injective functions from a set containing n elements to a set containing m elements.

5. Find the number of onto functions from a set containing n elements to a set containing m elements.
6. Find the sum of all numbers greater than 10000 formed by using the digits $1, 2, 5, 7, 9$, no digit being repeated more than once.
7. Find the number of positive integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ subject to the conditions $x_1 < x_2 < x_3 < x_4 < x_5$.
8. The number of ways of choosing 2 boys and 2 girls in a class is 1620. Find the number of ways of choosing 3 students from the class.
9. Find the sum of all 3-digit numbers whose digits are (i) all even, or (ii) all odd.
10. How many diagonals are there of a n-sided polygon?
11. All the 6-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6$ exactly once, and not divisible by 5, are arranged in increasing order. Find the 290-th number.

6.9. RMO Problems

1993 Suppose $A_1A_2\ldots A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?

1997 Find the number of unordered pairs $\{A, B\}$ (i.e., the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions: $A \neq B$ and $A \cup B = X$. For example, if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}$, $\{\{a\}, \{b, c, d\}\}$, $\{\emptyset, \{a, b, c, d\}\}$ are some admissible pairs.

1998 Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

1999 Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

- ◇ a, b, c are distinct;
- ◇ $a, b, c \in \{1, 2, 3, \ldots, 1999\}$ and
- ◇ $x + 1$ divides $ax^2 + bx + c$.

2004 Prove that the number of triples (A, B, C), where $A, B, C \subseteq \{1, 2, \cdots, n\}$, such that $A \cap B \cap C = \emptyset$, $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, is $7^n - 2 \cdot 6^n + 5^n$.

2007 How many 6-digit numbers are there such that:

a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$;

b) any digit that appears in the number appears at least twice?

2008 Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits $\{0, 1, 2, 3\}$ occurs at least once in them.

2008 Find the number of all integer sided isosceles obtuse angled triangles with perimeter 2008.

2011 Find the number of 4-digit numbers with distinct digits chosen from the set $\{0, 1, 2, 3, 4, 5\}$ in which no two adjacent numbers are even.

2012 Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{5, 7, 8\}$.

2013 Find the number of eight-digit numbers the sum of whose digits is 4.

Chapter 7

Additional Topics

7.1. Pythagorean Triples

Definition 7.1 (Pythagorean Triples)**.** If $x^2 + y^2 = z^2$ for positive integers x, y, z, then the triple (x, y, z) is called a Pythagorean triple. If $\gcd(x, y, z) = 1$, then such a triple is called a primitive Pythagorean triple.

Quick Exercise

Let (x, y, z) be a primitive Pythagorean triple, then

1. Show that x, y cannot be of same parity, i.e., both even or both odd.
2. If y is even, then show that $\frac{z+x}{2}, \frac{z-x}{2}$ are both perfect squares.
3. If $\frac{z+x}{2} = r^2$ and $\frac{z-x}{2} = s^2$, then show that r and s are arbitrary positive integers of opposite parity, with $r > s$ and $\gcd(r, s) = 1$.
4. If y is even, then show that $x = r^2 - s^2$, $y = 2rs$ and $z = r^2 + s^2$.
5. Show that exactly one of x, y, z is divisible by 3.
6. show that xyz divisible by 60.
7. Show that $xy(x^2 - y^2)$ divisible by 84.

7.2. Mobius function

The Mobius function $\mu(n)$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } a^2 | n \text{ for some } a > 1, \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ for distinct primes } p_i. \end{cases}$$

Note that if $p_1, \dots p_r$ are distinct primes then $\mu(p_1p_2\cdots p_r) = \prod_{i=1}^{r}\mu(p_i)$. Since $(1+\mu(p_1))\cdot(1+\mu(p_2))\cdots(1+\mu(p_r)) = 0$, and $\mu(d) = 0$ whenever d has a square factor, we obtain $1 + \sum_{d|n, d\neq 1}\mu(d) = 0$. Hence, $\sum_{d|n}\mu(d) = 0$.

Mobius Inversion Formula

If $F(n) = \sum_{d|n} f(d)$ for every positive integer n, then $f(n) = \sum_{d|n}\mu(d)F(n/d)$.

7.3. Functional Equation

An equation in which the unknowns are functions, is called a functional equation. In such cases, we have to find all functions satisfying some given condition(s).

For example, if we try to find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(-x) = f(x)$, then the solutions are known as even functions. A functional equation possesses large number of solutions. So quite often there are more conditions, like $f(x+y) = f(x) + f(y)$ or $f(x+y) = f(x)f(y)$ or $f(xy) = f(x)f(y)$ for all real x, y.

Example 7.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

- ◇ $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, and
- ◇ $f(1/x) = f(x)/x^2$ for all $x \neq 0$.

Solution. For $x = y = 0$, we get $f(0) = 0$. We also obtain $f(2) = 2 \cdot f(1)$ from the first condition. For $y = -x$, we get $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Hence $f(x-y) = f(x) - f(y)$ for all $x, y \in \mathbb{R}$. Therefore, for all $x \neq 0, 1$,

$$\begin{aligned}
& f\left(\frac{1}{x(x-1)}\right) = f\left(\frac{1}{x-1} - \frac{1}{x}\right) = f\left(\frac{1}{x-1}\right) - f\left(\frac{1}{x}\right) \\
\implies & \frac{f(x(x-1))}{x^2(x-1)^2} = \frac{f(x-1)}{(x-1)^2} - \frac{f(x)}{x^2} \\
\implies & f(x^2 - x) = x^2 f(x-1) - (x-1)^2 f(x) \\
\implies & f(x^2) + x^2 f(1) = 2x f(x).
\end{aligned}$$

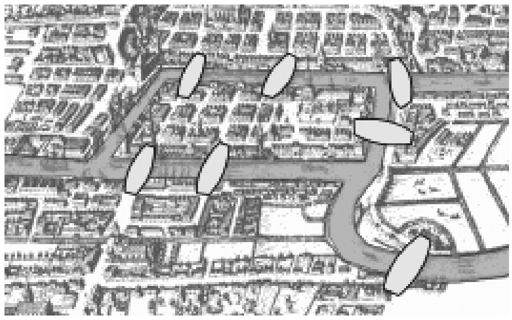
Replacing x by $x + \frac{1}{x}$, we get

$$\begin{aligned}
& f\left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x^2 + 2 + \frac{1}{x^2}\right) f(1) = 2\left(x + \frac{1}{x}\right) f\left(x + \frac{1}{x}\right) \\
& \implies f(x^2) + f(2) + f\left(\frac{1}{x^2}\right) + x^2 f(1) + 2f(1) + \frac{1}{x^2} f(1) \\
& = 2\left(x + \frac{1}{x}\right)\left(f(x) + f\left(\frac{1}{x}\right)\right) \\
& \implies f(2) + 2f(1) = 2xf\left(\frac{1}{x}\right) + \frac{2}{x} f(x) \\
& \implies 4f(1) = \frac{4}{x} f(x).
\end{aligned}$$

Hecne, $f(x) = xf(1)$ for all $x \neq 0, 1$. But this relation remain true for $x = 0, 1$ as well. Hence, $f(x) = cx$ for some real number c. □

7.4. Graph Theory

Till 1736, there was a famous unsolved problem known as the Koenigsberg Bridge problem. It is fascinating to note that Euler invented a new area of mathematics known as Graph Theory to solve this problem. The original problem, however, is quite naive in its statement, pictorially depicted as follows.



There are two islands in the Pregel River, linked to each other by a bridge. One of the two islands is linked to the two banks of the Pregel River by four bridges, two on each bank, and the other island is linked to the two banks of the Pregel River by two bridges, one on each bank. This creates four land areas, two islands and two banks, as shown in the picture. The problem is to begin at any one of the four land areas, walk across each of the seven bridges exactly once, and return to the starting point.

Euler proved that the answer to the above problem is negative. One may replace each land area by a point (vertex) and each bridge by a line (edge) joining the corresponding points, thus producing a mathematical object called graph.

Now the Koenigsberg Bridge problem is equivalent to the problem of drawing the corresponding graph with a pen on a paper, without lifting the pen from the paper and without overwriting any line.

Exercise: Try to draw this graph and justify why it is not possible.

Example 7.2 (Gomory's Theorem)**.** Regardless of the positions, if one white and one black square are cut off from a regular 8×8 chess-board, then the reduced board can always be covered exactly with 31 dominoes of dimensions 2×1.

Solution. The 8×8 chess-board can be thought of as a 64×1 circular arc, and the removal of any two squares of different colors will make this either into a single row of 62×1 (if the two squares were side-by-side on the arc), or into two even length rows. Moreover, the dominoes in the chess-board clearly represent dominoes in the circular arc, and hence completely cover even length rows. □

7.5. Inequalities

Cauchy-Schwarz. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of n real numbers,

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \ \ge \ \left(\sum a_i b_i\right)^2$$

Engel. The Engel's form related to Cauchy-Schwarz inequality is as follows

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \ \ge \ \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

Cirtoaje. The Cirtoaje's form related to Cauchy-Schwarz inequality is as follows

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \ \le \ \frac{3}{\sqrt{2}}$$

Tchebycheff. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of n real numbers, such that $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \le \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n}$$

Equality holds if and only if either all a_i's are equal or all b_i's are equal.

7.6. Geometry

Theorem 7.1 (Ceva's Theorem)**.** *If points D, E, F are taken on sides BC, CA, AB of triangle ABC, respectively, then AD, BE, CF are concurrent if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Theorem 7.2 (Menelaus' Theorem)**.** *If points D, E, F are taken on (suitably extended) sides BC, CA, AB of triangle ABC, respectively, then D, E, F are collinear if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

Theorem 7.3 (Viviani's Theorem)**.** *For any point P inside an equilateral triangle ABC, the sum of the length of perpendiculars from P to the sides of ABC is equal to the altitude of ABC.*

Example 7.3 (Fermat's problem to Torricelli)**.** Let P be a point inside the triangle ABC. The minimum possible value of $PA + PB + PC$ is attained at a point P such that $\angle APC = \angle CPB = \angle BPA = 120^{\circ}$.

Solution. Construct the Fermat's point P casting 120^o to each side. Then draw the equilateral triangle XYZ circumscribing ABC, for which PA, PB and PC act as perpendiculars on the sides. Prove that the sum is minimum in this case. □

Solution.(alternative). Rotate APB about the vertex B outwardly by 60^o, such that $A \to C'$ and $P \to P'$. Prove that the sum $PA + PB + PC$ is equal to the path $CP + PP' + P'C'$ during rotation, and it is minimum when it is straight line. □

Corollary 7.1. *If ABC', BCA' and ACB' are equilateral triangles drawn outwardly, then AA', BB' and CC' are concurrent at the Fermat's point P.*

Theorem 7.4 (Napoleon's Theorem)**.** *If ABC', BCA' and ACB' are equilateral triangles drawn outwardly to the triangle ABC, then their centers X, Y, Z form an equilateral triangle.*

Proof. Instead of drawing triangles, draw the circles. Any triangle passing through A, B, C must be an equilateral triangle and the biggest one is double of XYZ. □

Example 7.4 (Covering problem)**.** Every plane set of diameter 1 can be completely covered with an equilateral triangle of sides $\sqrt{3}$. A weaker statement is that every plane set of diameter 1 can be completely covered with a circle of radius $\frac{\sqrt{3}}{2}$.

Solution. (Logical reasoning) Draw two circles C_1 and C_2 with centers O_1 and O_2 from the set with radius one. Note that the set should be contained in $C_1 \cap C_2$. The circle with center at the mid point of C_1C_2 and with radius $\frac{\sqrt{3}}{2}$ covers $C_1 \cap C_2$.

Alternatively, given set can be fit inside parallel lines with gap 1. Choose three such parallel line pairs, making 60^o angles with each other. The set is now inside a hexagon. Next, pick a point P and draw perpendiculars. □

Example 7.5. ABD, BCE and CAF are outwardly drawn triangles such that $\angle D + \angle E + \angle F = 180^o$. Prove that the circumcircles of the triangles are concurrent, and that the triangle formed by the circumcenters is similar to DEF.

Solution. Observe that the line between the centers of two circles is always perpendicular to their common chord. The proof should follow. □

7.7. Trigonometry

Trigonometric ratios

For any two angles A and B,

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$

- ⋄ $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- ⋄ $\sin(\frac{\pi}{2} + A) = \cos A$ and $\cos(\frac{\pi}{2} + A) = -\sin A$.

Properties of triangles

For any triangle ABC with the sides $AB = c, BC = a, CA = b$, internal angles $\angle ABC = B, \angle BCA = C, \angle CAB = A$, circum-radius R, in-radius r, area Δ, and semi-perimeter $s = (a + b + c)/2$,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = 2R$$
$$a = b\cos C + c\cos B,\ b = c\cos A + a\cos C,\ c = a\cos B + b\cos A$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - c^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\Delta^2 = s(s-a)(s-b)(s-c), \qquad \Delta = rs, \qquad R = \frac{abc}{4\Delta}$$

7.8. Solved Examples

Example 7.6. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all the five numbers is positive.If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed : the numbers x, y, z are replaced by $x + y, -y, y + z$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine if the procedure comes to an end after finite number of steps.

Solution. Define function $(x-z)^2 + (y-w)^2 + (z-t)^2 + (w-x)^2 + (t-y)^2$ where $x, y, \ldots, t$ are in order. Note that it is positive and decreasing. □

Example 7.7 (Sylvester's Theorem)**.** Suppose n points on a plane satisfying the following properties : Given any straight line joining through two points among n points contains another point. Prove that all the n points lie on a straight line. Note that this is not true for infinitely many points.

Solution. (Fermat's descent) Draw all lines and all perpendiculars from points not in the line (assume that they are not in a straight line). So there are finitely many (also positive) perpendiculars. Choose the minimum perpendicular length and prove that it is not possible as there are always a perpendicular less than minimum. □

Example 7.8. Consider the set $S = \{(i, j) : 1 \leq i, j \leq 13\}$ of 169 points. For any subset T of size 53, some 4 points of T will be the vertices of a rectangle. Given a square grid of 7×7 S, a subset T of k points is selected. The problem is to find the maximum value of k such that no 4 points of T determine a rectangle R having sides parallel to the sides of S.

Solution. (Logical reasoning) Let a_i be the number of points in ith row. Total number of pairs (i,j) is $\binom{13}{2}$ where as the total number of pairs from 53 points are $\sum_{i=1}^{13}\binom{a_i}{2}$ where $\sum_{i=1}^{13} = 53$. Now an easy calculation shows that $\sum_{i=1}^{13}\binom{a_i}{2} > \binom{13}{2}$ (by Cauchy-Schwartz's inequality). Make similar analysis for the second part. □

Example 7.9. If a cell is removed from a 8×8 chess-board, prove that the remainder can be tiled with L-trominoes. In general prove it for $2^n \times 2^n$ chess board.

Solution. (Logical reasoning) Divide the chess-board into four parts. Draw a L trominoes at the center so that it uses a square from three remaining parts. Now use induction. □

Example 7.10. Let any $2n$ points be chosen in the plane so that no three are collinear, and let any n of them be colored red and the other n blue. Prove that it is always possible to pair up the n red points and with the n blue ones in on-to-one fashion (R, B) so that no two of the n segments RB, which connect the members of a matched pair, intersect.

Solution. (Fermat's descent) Look for joining where the total sum of the length segments is minimum. This has the required property. □

Example 7.11. There are three group of matches each containing 20 matches. Now we perform the following operations as long as all groups are nonempty: We add one match in a group and remove one match each from the remaining two groups. Is it possible to end up with three groups with number of matches $0, 0, 1$.

Solution. (Logical reasoning). The differences between the number of matches in groups always even and hence it is not possible. □

Example 7.12. There are n chairs arranged around a table, numbered $1, 2, ..., n$. Seated on these chairs are n persons, each bearing a distinct numbers from the set $\{1, 2, \cdots, n\}$.

1. Prove that if n is even, there exists an integer $k \geq 0$ such that if each person shifts k chairs to the right no person's number matches with the number of chair upon which he is seated.
2. If n is odd find an initial arrangement for which the above statement is false for every integer k.

Solution. $\sigma(i)$: the person number seating on chair i. For each k, $\sigma(i) + k = \alpha(i)$ for some permutation α. Sum over all we arrive a contradiction for even n. For odd n, $\sigma(i) = i(n+1)/2$. □

Example 7.13. In a party with 2007 persons, among any group of four, there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else.

Solution. Knowing a asymmetric relation : No one. Put them in a circle and i knows everybody except $(i-1)$. Knowing a symmetric relation: Suppose A and B do not know each other. For any other C and D, they should known each other. If C does not know any one of A and B then for any D, he should know everybody. □

Example 7.14. From a committee of n members n distinct subcommittee are formed Show that there is a member of the committee such that if he resigns from all the subcommittees on which he is serving, the resulting subcommittees are distinct (one of them possibly empty).

Solution. If not we can form a chain. Edge is in between two committee such that removing a person from one committee we obtain the other one. □

7.9. Exercise Problems

1. Angles ABC and CBD are each 60^{o}. From the point P anywhere in the triangle ABC, perpendiculars PX, PY, PZ are drawn to AB, BC and BD respectively. Prove that $PZ = PX + PY$.
2. $ABCD$ is a parallelogram with no angle 60^o. ADE and DCF are equilateral drawn outwardly. Then BEF is equilateral.
3. Let ΔABC be a triangle such that the length of the angle bisectors of $\angle A$ and $\angle B$ are equal, then ΔABC is an isosceles triangle.
4. Find out the flaw in the proof of "For every triangle ΔABC, $AB = CA$".

   Proof. Let the perpendicular bisector of BC and the angle bisector of $\angle A$ meet at P. Let Y and X be the foot of the perpendiculars drawn from P to AC and AB respectively. Let D be the mid point of BC. By SAS rule $\Delta DPB \equiv \Delta DPC$. Similarly by ASA rule $\Delta AXP \equiv \Delta AYP$. Thus, $AX = AY$. Now by RHS rule $\Delta YPC \equiv \Delta XBP$ and hence $YC = XB$. So, $AB = AC$. □

5. A set S of $2n+3$ points is given in the plane, no 3 on a line and no 4 on a circle. Prove that it is always possible to find a circle C which goes through exactly three points of S and splits the other $2n$ in half, i.e., n on the inside and n on the outside.
6. Around equilateral triangle ABC circumscribe a rectangle in any direction you like (say $PBQR$). In general, each side of ABC cuts off a right triangle from the rectangle. Prove that the areas of the two smaller triangles always add up to the the area of the largest one.
7. P is a variable point on the arc of a circle cut by the chord AB, Prove that the sum of the chords $AP + PB$ is maximum when P is the midpoint of arc AB.
8. We say a quadrilateral $ABCD$ is equilic if $AD = BC$ and $\angle A + \angle B = 120^{\text{o}}$.

a) Prove that the midpoints P, Q, R of the diagonals and the side CD always determines an equilateral triangle.

b) If equilateral triangle PCD is drawn outwardly on CD, then prove that the triangle PAB is also equilateral.

9. Given a point P, construct an equilateral triangle ABC such that P occurs inside it, and is x units from A, y units from B and z units from C.

10. Outwardly on the sides of triangle ABC, three triangles ABD, BCE, CAF are drawn such that $\angle D + \angle E + \angle F = 180^{\text{o}}$. Prove that the circumcircles of the latter three triangles are concurrent.

11. (Nine-point circle theorem). Feet of the perpendiculars, midpoints of the sides and the midpoints of line segments between vertices and centroid of any triangle lie on a circle. The center of the nine point circle N, orthocenter H and the centroid G lie on a straight line with $HN : NG = 3 : 1$.

12. Among all polygons with a fixed number of sides n which circumscribe a given circle K, the regular n-gon has the least area.

13. Suppose three equal circles C_1, C_2, C_3 of same radius r pass through a point O and have second point of intersection P_1, P_2, P_3. Then the circumcircles of $\Delta P_1P_2P_3$ is a fourth circle of the same radius r.

14. (Hungarian MO 1914) Let A and B be points on a circle k. Suppose that an arc k' of another circle ℓ cuts at A and B and divides the area of k into equal parts. Prove that $k' > \text{diam}(k)$.

15. (Hungarian MO 1914)
The circle k intersects the ΔABC at $A_1, A_2, B_1, B_2, C_1, C_2$. The perpendiculars to the sides through A_1, B_1, C_1 meet at a point M. Prove that the perpendiculars through other points are also concurrent (to N, say).

16. (Ramsey's Theorem). Given an infinite complete graph with each edge colored red or blue, prove that there exists an infinite subgraph with all edges either blue or red.

17. Two persons are playing a game, where each player can successively subtract a factor $a < n$ of the remaining integer n. The person who reaches 1, wins the game. If the game starts from $n = 100$, determine which player wins.

18. Given a circular board with a circular hole, two players are successively drawing either a circle or a line segment on the board so that it does not intersect the previously drawn figures. Determine a strategy for the second player to win.

19. Let A, B be disjoint finite sets of integers with the property that if $x \in A \cup B$, then either $x + 1 \in A$ or $x - 2 \in B$. Prove that $|A| = 2|B|$.

20. Find all positive integers n for which every integer whose decimal representation contains $(n-1)$'s 1 and 1 zero is prime.

21. Call an ordered pair (S, T) of subsets $\{1, \cdots, n\}$ of S admissible, if $s > |T|$ for all $s \in S$ and $t > |S|$ for all $t \in T$. How many admissible ordered pairs of

admissible of $\{1, \cdots, n\}$.

22. Suppose $A = \{a_1, \ldots, a_{101}\}$ is a set of 101 real numbers such that for each $1 \leq i \leq 101$, it is possible to divide the set $A \backslash \{a_i\}$ into two equal groups (each of size 50) such that the sum are equal. Then prove that $a_1 = \cdots = a_{101}$.
23. Assume that the points of the plane is colored red or blue. Prove that the one of the colors contains pair of the points at every mutual distances.
24. Let A be a set of 16 positive integers with the property that the product of any two distinct numbers from A does not exceed 1994. Show that there are numbers a and b such that $gcd(a, b) > 1$.
25. Given a set of n positive integers $\{a_1, \cdots a_n\}$ so that it is always possible to choose a subsets of these integers such that the sum of its elements is divisible by n.
26. Let $0 < a_1 < \cdots < a_n$, $\epsilon_i = \pm 1$. Prove that $\sum_{i=1}^{n} \epsilon_i a_i$ assumes at least $\binom{n+1}{2}$ distinct values as ϵ_i ranges 2^n combinations of signs.
27. Suppose $-1 \leq u \leq 1$. Prove that each root of the equation $x^{n+1} - ux^n + ux - 1$ has modulus 1.
28. Given a finite collection of closed squares of total area 3 squares unit. Prove that they can be arranged to cover an unit square.
29. Given a finite collection of closed squares of total area $1/2$ square unit. Prove that they can be arranged in a unit square with no overlap.
30. Find the number of coloring n^2 points of a $n \times n$ grid of points if each of the point colored red or blue and if each of the $(n-1)^2$ squares has some two vertices colored red and other two colored blue.
31. Let m and n be positive integers. If $x_1, \cdots, x_m$ are positive integers whose average is less than $n+1$. If $y_1, \cdots, y_n$ are positive integers whose average is less than $m+1$. Prove that some sum of one or more x's equal some of sum of one or more y's.
32. If the number of elements of a set X is $n > 1$, find the number of all functions $f : X \to X$ such that $f(f(x)) = a$ for every $x \in X$, where a is a fixed element chosen from set X.
33. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f\left(f(m) + f(n)\right) = m + n,$$

for every pairs of natural numbers (positive ingeters) m and n.

Appendix A

INMO Problems

Indian National Mathematical Olympiad (INMO) is open only for those students who are selected through the corresponding RMO examination, and those who have received an INMO certificate of merit in the previous year. The INMO examination is a four hour written test, on the basis of which, top 30-35 students in merit from all over the country are chosen as INMO awardees. In addition to INMO awardees, the next 45-50 students who are in class X or lower and have done well in INMO, but have not qualified as INMO awardees, are awarded the INMO certificate of merit. These students are eligible to appear for the next year's INMO test directly without qualifying through the corresponding RMO.

This book is primarily targeted towards the RMO examination. However, for the interested readers, we provide a compilation of previous INMO problems, sorted according to the topics discussed in this book.

Number Theory

1989 Determine with proof, all the positive integers n for which:

- n is not the square of any integer; and
- $[\sqrt{n}]$ divides n^2.

1989 For positive integers n, define $A(n)$ to be $\frac{(2n)!}{(n!)^2}$. Determine the sets of positive integers n for which

- $A(n)$ is an even number,
- $A(n)$ is a multiple of 4.

1990 Determine all non-negative integral pairs (x, y) for which

$$(xy - 7)^2 = x^2 + y^2.$$

1990 Let f be a function defined on the set of non-negative integers and taking values in the same set. Given that

- $\diamond$ $x - f(x) = 19[x/19] - 90[f(x)/90]$ for all non-negative integers x;
- $\diamond$ $1900 < f(1990) < 2000$,

find the possible values that $f(1990)$ can take.

1991
- $\diamond$ Determine the set of all positive integers n for which

$$3^{n+1} \text{ divides } 2^{3^n} + 1$$

- $\diamond$ Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

1991 For any positive integer n, let $S(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

(for instance, $S(2) = 3$). Determine the set of positive integers n for which $S(n) = 5$.

1992 Determine all pairs (m, n) of positive integers for which

$$2^m + 3^n$$

is a perfect square.

1992 Find the remainder when 19^{92} is divided by 92.

1992 Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for 5 distinct integers a_1, a_2, a_3, a_4 and a_5 one has

$$f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2.$$

Show that there does not exist an integer b such that $f(b) = 9$.

1993 Let $P(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that

$$P(n) \cdot P(n+1) = P(M).$$

1993 Show that there is a natural number n such that $n!$ when written in decimal notation (that is, in base 10) ends exactly in 1993 zeros.

1993 Let $A = \{1, 2, 3, \ldots, 100\}$ and B be a subset of A having 53 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

1994 In any set of 181 square integers, prove that one can always find a subset of 19 numbers, sum of whose elements is divisible by 19.

1995 Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic functions $x^2 + ax + b = 0$ and $x^2 + 2ax + b = 0$ have integer roots.

1995 Find all primes p for which the quotient

$$\frac{(2^{p-1}-1)}{p}$$

is a square.

1996
- ◇ Given any positive integer n, show that there exist distinct positive integers x and y such that $x+j$ divides $y+j$ for $j = 1, 2, 3, \cdots, n$.
- ◇ If for some positive integers x and y, $x+j$ divides $y+j$ for all positive integers j, prove that $x = y$.

1996 Define a sequence $(a_n)_{n\geq 1}$ by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = 2a_{n+1} - a_n + 2$ for $n \geq 1$. Prove that for any m, $a_m a_{m+1}$ is also a term in the sequence.

1997 Show that there do not exist positive integers m and n such that

$$\frac{m}{n} + \frac{n+1}{m} = 4.$$

1998 Let a and b be two positive rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is also a rational number. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

1998 Let p, q, r, s be four integers such that s is not divisible by 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is divisible by 5, prove that there is an integer b such that $sb^3 + rb^2 + qb + p$ is also divisible by 5.

1998 It is desired to choose n integers from the collection of $2n$ integers, namely, $0, 0, 1, 1, 2, 2, \ldots, n-1, n-1$ such that the average (that is, the arithmetic mean) of these n chosen integers is itself an integer and as minimum as possible. Show that this can be done for each positive integer n and find this minimum average for each n.

1999 In a village 1998 persons volunteered to clean up, for a fair, a rectangular field with integer sides and perimeter equal to 3996 feet. For this purpose, the field was divided into 1998 equal parts. If each part had an integer area (measured in sq.ft.), find the length and breadth of the field.

1999 For which positive integer values of n can the set $\{1, 2, 3, 4, \ldots, 4n\}$ be split into n disjoint 4-element subsets $\{a, b, c, d\}$ such that in each of these sets $a = \dfrac{b+c+d}{3}$?

2000 Solve for integers x, y, z:

$$x + y = 1 - z, \qquad x^3 + y^3 = 1 - z^2.$$

2000 For any natural numbers n, $(n \geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e. g., $f(3) = 1$, $f(4) = 0$, $f(7) = 2$). Show that $f(1999) > f(1996)$ and $f(2000) = f(1997)$.

2001 Show that the equation

$$x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$$

has infinitely many solutions in integers x, y, z.

2002 Determine the least positive value taken by the expression $a^3 + b^3 + c^3 - 3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

2002 Suppose the n^2 numbers $1, 2, 3, \cdots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order. Denote by a_{jk} the number in the jth row and kth column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \leq j \leq n$. Prove that

$$b_1 + b_2 + b_3 + \cdots + b_n \leq \frac{n}{3}(n^2 - 3n + 5).$$

(Example : In the case $n = 3$, the only numbers which can occur as a_{22} are $4, 5$ or 6 so that $b_2 = 3$.)

2003 Find all primes p and q, and even numbers $n > 2$, satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

2003 Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.

2003 In a lottery tickets are given nine-digit numbers using only the digits $1, 2, 3$. They are also colored red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colors. Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the color of the ticket bearing the number 123123123.

2004 Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

2004 Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of T.

2004 Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

is an odd integer.

2005 Let x_1 be a given positive integer. A sequence $(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

- ◇ the sequence has an even number;
- ◇ the sequence has infinitely many even numbers.

2006 Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a.$$

2006
- ◇ Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < (1 + \frac{1}{2005})n$.
- ◇ Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that $n < l^2 < (1+\frac{1}{2005})n$.

2007 Let n be a natural number such that $n = a^2+b^2+c^2$, for some natural numbers a, b, c. Prove that

$$9n = (p_1a + q_1b + r_1c)^2 + (p_2a + q_2b + r_2c)^2 + (p_3a + q_3b + r_3c)^2,$$

where p_j's, q_j's, r_j's are all **nonzero** integers. Further, of 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

2007 Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha]+[m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

2008 Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

2008 Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

2013 Find all positive integers m, n and primes $p \geq 5$ such that $m(4m^2+m+12) = 3(p^n - 1)$.

2013 Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.

2013 Let n be a positive integer. Call a nonempty subset S of $\{1, 2, 3, \ldots, n\}$ good if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, 3, \ldots, n\}$. Prove that t_n and n are both odd or both even.

2014 Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots + \left[\frac{n}{n}\right] + [\sqrt{n}]$$

is even.

2014 Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their g.c.d. and l.c.m. is divisible by $a + b$. Prove that the quotient is at most $(a + b)/4$. When is this quotient exactly equal to $(a + b)/4$?

2015 From a set of 11 square integers, show that one can choose 6 numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2$âĽą$d^2 + e^2 + f^2 (\mod 12)$.

2015 For any natural number $n > 1$, write the infinite decimal expansion of $1/n$ (for example, we write $1/2 = 0.499...$ as its infinite decimal expansion, not 0.5). Determine the length of the non-periodic part of the (infinite) decimal expansion of $1/n$.

Inequalities

1990 Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{(b+c)} + \frac{b}{(c+a)} + \frac{c}{(a+b)}$$

must lie between the limits $3/2$ and 2. Can equality hold at either limits?

1991 Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ and $a + b + c = 2$. Prove that :

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

1992 If x, y and z are three real numbers such that $x+y+z = 4$ and $x^2+y^2+z^2 = 6$, then show that each of x, y and z lies in the closed interval $[2/3, 2]$, that is $2/3 \leq x \leq 2$, $2/3 \leq y \leq 2$ and $2/3 \leq z \leq 2$. Can x attain the extreme value $2/3$ or 2 ?

1993 If a, b, c, d are 4 non-negative real numbers and $a + b + c + d = 1$, show that $ab + bc + cd \leq 1/4$.

1994 If $x^5 - x^3 + x = a$, prove that $x^6 \geq 2a - 1$.

1995 Let $a_1, a_2, a_3, \ldots, a_n$ be n real numbers all greater than 1 and such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n - 1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

1996 Solve the following system of equations for real numbers a, b, c, d, e.

$$3a = (b + c + d)^3, 3b = (c + d + e)^3, 3c = (d + e + a)^3, 3d = (e + a + b)^3,$$

$$\text{and } 3e = (a + b + c)^3.$$

2001 If a, b, c are positive real numbers such that $abc = 1$, prove that $a^{b+c}b^{c+a}c^{a+b} \leq 1$.

2005 Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

2007 If x, y, z are positive real numbers, prove that

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

Polynomials

1989 Prove that the polynomial

$$f(x) = x^4 + 26x^3 + 56x^2 + 78x + 1989$$

cannot be expressed as a product of two polynomials with integral coefficients and with degree less than 4.

1989 Let a, b, c and d be any four real numbers, not all equal to zero. Prove that the roots of the polynomial

$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot be real.

1990 Given the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

has four real, positive roots, prove that $pr - 16s \geq 0$ and $q^2 - 36s \geq 0$ with equality in each case holding if and only if the four roots are equal.

1991 Solve the following system of equations for real x, y, z:

$$\begin{aligned} x+y-z &= 4 \\ x^2-y^2+z^2 &= 4 \\ xyz &= 6 \end{aligned}$$

1997 If a, b, c are three distinct real numbers and

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a} = t$$

for some real number t, prove that $abc + t = 0$.

1997 Suppose a and b are two positive real numbers such that the roots of the cubic equation

$$x^3 - ax + b = 0$$

are all real. If α is a root of this cubic with minimum absolute value, prove that

$$\frac{b}{a} < \alpha \leq \frac{3b}{2a}.$$

1998 Suppose a, b, c are three real numbers such that the quadratic equation

$$x^2 - (a+b+c)x + (ab+bc+ca) = 0$$

has roots of the form $\alpha \pm i\beta$ where $\alpha > 0$ and $\beta \neq 0$ are real numbers [here $i = \sqrt{-1}$]. Show that

- ◇ the numbers a, b, c are all positive;
- ◇ the numbers $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form the sides of a triangle.

1999 Show that there do not exist polynomials $p(x)$ and $q(x)$ each having integer coefficients and of degree greater than or equal to 1 such that

$$p(x)q(x) = x^5 + 2x + 1.$$

1999 Given any four distinct real numbers, show that one can choose three numbers, say, A, B, C from among them such that all the three quadratic equations

$$Bx^2 + x + C = 0, \quad Cx^2 + x + A = 0, \quad Ax^2 + x + B = 0$$

have only real roots or all the three equations have only imaginary roots.

2000 Solve for integers x, y, z: $x + y = 1 - z$, and $x^3 + y^3 = 1 - z^2$.

2000 Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a (real or complex) root of the cubic equation $x^3 + ax^2 + bx + c = 0$, then $|\lambda| \leq 1$.

2001 Show that the equation $x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$ has infinitely many integer solutions x, y, z.

2003 Show that for every real number a, the equation $8x^4 - 16x^3 + 16^2 - 8x + a = 0$ has at least one non-real root and find the sum of all the non-real roots of the equation.

2004 If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$.

2005 Let p, q, r be positive real numbers, not all equal, such that some two of the equations $px^2 + 2qx + r = 0$, $qx^2 + 2rx + p = 0$, $rx^2 + 2px + q = 0$, have a common root, say α. Prove that: (i) α is real and negative, and (ii) the third equation has non-real roots.

2007 Let m, n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α, β are integers if and only if $[m\alpha] + [m\beta]$ is a perfect square.

2008 Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that: (i) $P(x)Q(x)$ is a polynomial in x^2, and (ii) $P(x)R(x)$ is a polynomial in x^3.

2009 Find all real numbers x such that: $[x^2 + 2x] = [x]^2 + 2[x]$.

2010 Find all non-zero real numbers x, y, z which satisfy the system of equations:
$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = xyz$
$(x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) = x^3y^3z^3$

2011 Consider two polynomials $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$ with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_nb_0 - a_0b_n \neq 0$. Suppose there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that r is an integer.

Geometry

1990 Triangle ABC is scalene with angle A having a measure greater than 90^o. Determine the set of points D that lie on the extended line BC, for which

$$|AD| = \sqrt{|BD||CD|}$$

where $|BD|$ refers to the (positive) distance between B and D.

1990 Let ABC be an arbitrary acute angled triangle. For any point P lying within the triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all possible positions of the point P for which the triangle DEF is isosceles. For which position of P will the triangle DEF become equilateral?

1991 Given any acute-angled triangle ABC, let points A', B', C' be located as follows : A' is the point where altitude from A on BC meets the outwards facing semi-circle drawn on BC as diameter. Points B', C' are located similarly. Prove that

$$[BCA']^2 + [CAB']^2 + [ABC']^2 = [ABC]^2,$$

where $[ABC]$ denotes the area of triangle ABC, etc.

1991 Given a triangle ABC, define the quantities x, y, z as follows:

$$\begin{aligned} x &= \tan((B-C)/2)\tan(A/2) \\ y &= \tan((C-A)/2)\tan(B/2) \\ z &= \tan((A-B)/2)\tan(C/2). \end{aligned}$$

Prove that : $x + y + z + xyz = 0$.

1991 Triangle ABC has incenter I. Let points X, Y be located on the line segment AB, AC respectively so that :

$$BX \cdot AB = IB^2 \text{ and } CY \cdot AC = IC^2$$

Given that the points X, I, Y lie on a straight line, find the possible values of the measure of angle A.

1991 Triangle ABC has incenter I, its incircle touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets the sides AB, AC at points C', B' respectively. Prove that the triangle $AB'C'$ is similar to triangle ABC.

1992 In a triangle ABC, $\angle A$ is twice $\angle B$. Show that

$$a^2 = b \cdot (b + c).$$

1992 Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the mid-point of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the mid-point of XZ.

1992 Let $A_1 A_2 A_3 \ldots A_n$ be an n-sided regular polygon such that

$$\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}.$$

Determine n, the number of sides of the polynomial.

1993 The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H, P, O are collinear.

1993 Let ABC be a triangle in a plane Σ. Find the set of all points P (distinct from A, B, C) in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP have the same radii.

1993 Let ABC be triangle right-angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB and AC and the circle S internally. Further let S_2 be the circle touching the lines AB and AC, and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2 respectively, show that

$$r_1 \cdot r_2 = 4(\text{area } \Delta ABC).$$

1993 Show that there exists a convex hexagon in the plane such that

- ◇ all its interior angles are equal,
- ◇ all its sides are 1, 2, 3, 4, 5, 6 in some order.

1994 Let G be the centroid of a triangle ABC in which the angle C is obtuse and AD and CF be the medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that

$$\frac{AC}{BC} > \sqrt{2}.$$

If further P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangle ABC and GAP are similar.

1994 A circle passes through a vertex C of a triangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units, find the area of the rectangle $ABCD$.

1995 In an acute-angled triangle ABC, $\angle A = 30^o$, H is the orthocenter and M is the midpoint of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.

1995 Let ABC be triangle and a circle Γ' be drawn inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show the ratio of the radii of the circles Γ' and Γ is equal to $\tan^2\left(\frac{\pi-A}{4}\right)$.

1996 Let C_1 and C_2 be two concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocenter of any triangle inscribed in circle C_1 lies in the *interior* of circle C_2. Conversely, show that also every point in the interior of C_2 is the orthocenter of some triangle inscribed in C_1.

1997 Let $ABCD$ be a parallelogram. Suppose a line passing through C and lying outside the parallelogram meets AB and AD produced at E and F respectively. Show that

$$AC^2 + CE \cdot CF = AB \cdot AE + AD \cdot AF.$$

1997 In a unit square one hundred segments are drawn from the center to the sides dividing the square into one hundred parts (triangles and possibly quadrilaterals). If all the parts have equal perimeter p show that $1.4 < p < 1.5$.

1998 In a circle C_1 with center O, let AB be a chord that is not a diameter. Let M be the midpoint of AB. Take a point T on the circle C_2 with OM as diameter. Let the tangent to C_2 at T meet C_1 in P. Show that

$$PA^2 + PB^2 = 4PT^2.$$

1998 Suppose $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius one unit. If

$$AB \cdot BC \cdot CD \cdot DA \geq 4,$$

prove that $ABCD$ is a square.

1999 Let ABC be an acute angled triangle in which D, E, F are points on BC, CA, AB respectively such that AD is perpendicular to BC; $AE = EC$; and CF bisects $\angle C$ internally. Suppose CF meets AD and DE in M and N respectively. If $FM = 2$, $MN = 1$, $NC = 3$, find the perimeter of the triangle ABC.

1999 Let Γ and Γ' be two concentric circles. Let ABC and $A'B'C'$ be any two equilateral triangles inscribed in Γ and Γ' respectively. If P and P' are any two points on Γ and Γ' respectively, show that

$$P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2.$$

2000 The incircle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of the triangle ABC.

2000 In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SPQ = 30^\circ$. Prove that

$$\angle PQR - \angle QRS = 90^\circ.$$

2001 Let ABC be a triangle in which *no* angle is 90°. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements :

- $\diamond$ If P is the incenter or an excenter of ABC, then P is the circumcenter of $A_1B_1C_1$;
- $\diamond$ If P is the circumcenter of ABC, then P is the orthocenter of $A_1B_1C_1$;
- $\diamond$ If P is the orthocenter of ABC, then P is either the incenter or an excenter of $A_1B_1C_1$.

2001 Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15^\circ$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcenter of ADC, prove that triangle AOD is equilateral.

2002 For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements :

$$\begin{array}{llll} (a_1) & AB \text{ is parallel to } DE; & (a_2) & AE = BD; \\ (b_1) & BC \text{ is parallel to } EF; & (b_2) & BF = CE; \\ (c_1) & CD \text{ is parallel to } FA; & (c_2) & CA = DF. \end{array}$$

- $\diamond$ Show that if all the six statements are true, then the hexagon is cyclic (*i.e.*, it can be inscribed in a circle).
- $\diamond$ Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

2003 Consider an acute triangle ABC and let P be an interior point of ABC. Suppose the lines BP and CP, when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC. Show that DK bisects $\angle EKF$.

2003 Let ABC be a triangle with sides a, b, c. Consider a triangle $A_1B_1C_1$ with sides equal to $a + \frac{b}{2}$, $b + \frac{c}{2}$, $c + \frac{a}{2}$. Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where $[XYZ]$ denotes the area of the triangle XYZ.

2004 Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose

- $\diamond$ BD bisects KM at Q;
- $\diamond$ $QA = QB = QC = QD$; and
- $\diamond$ $LK/LM = CD/CB$.

Prove that $ABCD$ is a square.

2004 Let R denote the circumradius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

◇ $a > b$ and $a > c$;

◇ $2R > r_b$ and $2R > r_c$.

2006 In a non equilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incenter and circumcenter of the triangle respectively.

◇ Prove that IO is perpendicular to BI.

◇ Suppose BI extended meets AC in K, and D, E are the midpoints of BC, BA respectively. Prove that I is the circumcenter of triangle DKE.

2006 In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120°$, and $\angle ABD = 30°$. Prove that

◇ $c \geq a + b$;

◇ $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

2007 Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of the perpendicular from P on AC. If $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \lambda$, $\dfrac{BD}{AD} = m$ and $z = m^2(1+\lambda)$, prove that

$$z^2 - (\lambda^3 - \lambda^2 - 2)z + 1 = 0.$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

2008 All the points with integer coordinates in the xy-plane are colored using three colors, red, blue and green, each color being used at least once. It is known that the point $(0, 0)$ is colored red and the point $(0, 1)$ is colored blue. Prove that there exist three points with integer coordinates of distinct colors which form the vertices of a right-angled triangle.

2008 Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB and BC; and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circumcenter O and the in-centre I of triangle ABC passes through the centre of Γ.

2008 Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circumcircle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

2013 Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to Γ_2 at P and passing through the centre O_1 of Γ_1. Similarly, let l_2 be a line which is tangent to Γ_1 at Q and passing through the centre O_2 of Γ_2. Suppose l_1 and l_2 are not parallel and intersect at K. If $KP = KQ$, prove that the triangle PQR is equilateral.

2013 In an acute angled triangle ABC, O is the circumcenter, H the orthocenter and G the centroid. Let OD be the perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the mid-point of AB. Suppose the areas of the triangle ODC, HEA and GFB are equal. Find all the possible values of $\angle C$.

2014 In a triangle ABC, let D be a point on the segment BC such that $AB+BD = AC+CD$. Suppose that the points B, C and the centroid of the triangles ABD and ACD lies on a circle. Prove that $AB = AC$.

2014 In an acute angled triangleABC, a point D lies on the segmentBC. Let O_1, O_2 denote the circumcenters of the triangles ABD and ACD, respectively. Prove that the line joining the circumcenter of triangle ABC and the orthocenter of the triangle O_1O_2D is parallel to BC.

2015 Let ABC be a right-angled triangle with $\angle B = 90^o$. Let BD be the altitude from B on to AC. Let P, Q and I be the incentres of triangles ABD, CBD and ABC respectively. Show that the circumcentre of the triangle PIQ lies on the hypotenuse AC.

2015 Let $ABCD$ be a convex quadrilateral. Let the diagonals AC and BD intersect in P. Let PE, PF, PG and PH be the altitudes from P onto the sides AB, BC, CD and DA respectively. Show that $ABCD$ has an incircle if and only if $\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}$.

Combinatorics

1991 Find the number of positive integers n such that $n \leq 1991$ and 6 is a factor of $n^2 + 3n + 2$.

1992 Find the number of permutations $(P_1, P_2, P_3, P_4, P_5, P_6)$ of $(1, 2, 3, 4, 5, 6)$ such that for any k, $1 \leq k \leq 5$. $(P_1, P_2, \ldots P_k)$ does not form a permutation of $\{1, 2, \ldots k\}$. That is $P_1 \neq 1$; (P_1, P_2) is not permutation of $\{1, 2\}$; (P_1, P_2, P_3) is not a permutation of $\{1, 2, 3\}$, etc.

1992 Find the number of ways in which one can place the numbers $1, 2, 3, \ldots n^2$ on the n^2 squares of $n \times n$ chess-board, one on each, such that the numbers in each row and each column are in arithmetic progression. (Assume $n \geq 3$).

1994 Find the number of non degenerate triangles whose vertices lie in the set of points (s, t) in the plane such that $0 \leq s \leq 4$, $0 \leq t \leq 4$, with s and t integers.

1996 Let X be a set containing n elements. Find the number of all ordered triples (A, B, C) of subsets of X such that A is a subset of B and B is a *proper* subset of C.

2005 All possible 6-digit numbers, in each of which the digits occur in non-increasing order (from left to right, e.g., 877550) are written as a sequence in increasing order. Find the 2005th number in this sequence.

2007 Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of $(1, 2, 3, \ldots, n)$. A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (Example:

In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, $(3, 1)$.) How many permutations of $(1, 2, 3, \ldots, n)$, $(n \geq 3)$, have exactly two inversions.

2014 Let n be a natural number and $X = \{1, 2, 3, \ldots, n\}$. For subsets A and B of X we define $A\Delta B$ to be the set of all those elements of X which belong to exactly one of A and B. Let $\mathcal{F}$ be a collection of subsets of X such that for any two distinct elements A and B in $\mathcal{F}$ the set $A\Delta B$ has at least two elements. Show that $\mathcal{F}$ has at most 2^{n-1} elements. Find all such collections $\mathcal{F}$ with 2^{n-1} elements.

2015 There are four basket-ball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after seven passes? (For example $A \to C \to B \to D \to A \to B \to C \to A$ and $A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to A after seven passes.)

Functional Equations

1992 Determine all functions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ satisfying the functional relation

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)},$$

where x is a real number different from 0 and 1. (Here $\mathbb{R}$ denotes the set of all real numbers.)

1994 If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the properties

- ⋄ $f(-x) = -f(x)$,
- ⋄ $f(x + 1) = f(x) + 1$,
- ⋄ $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$, for $x \neq 0$,

prove that $f(x) = x$ for all real values of x. Here $\mathbb{R}$ denotes the set of all real numbers.

2001 Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all x, y in $\mathbb{R}$.

2005 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

for all x, y, z in $\mathbb{R}$. (Here $\mathbb{R}$ denotes the set of all real numbers.)

2006 Let X denote the set of all triples (a, b, c) of integers. Define a function $f : X \to X$ by

$$f(a, b, c) = (a + b + c, ab + bc + ca, abc).$$

Find all triples (a, b, c) in X such that $f(f(a, b, c)) = (a, b, c)$.

2015 Find all real functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the relation $f(x^2 + yf(x)) = xf(x + y)$.

Appendix B

Practice Tests

In this chapter, we prescribe a few practice tests to acquaint the readers with the level of RMO. Starting 2013, the RMO in West Bengal has changed in format[1]. It now consists of two tests – the first of duration 2 hours and the second of 3 hours.

- ◇ The first test is based on multiple-choice questions (MCQ), and may be thought of as a qualifying stage. All answer-scripts of the MCQ-based test are graded.
- ◇ The second test is analogous to a generic RMO paper, based on 6 to 7 subjective problems. This test is graded only for those who qualify in the MCQ paper.

However, the students appearing for the RMO are required to take both tests, as they are held on the same day. Thus it is imperative that the students prepare not only for the MCQ-based paper, but also for the advanced subjective paper.

Multiple Choice Questions

Time: 2 hours **Full Marks: 100**

1. Which one is true,
   a) $625^{0.25} > 7 > 216^{1/3} > 5^{1.5}$
   b) $625^{0.25} > 216^{1/3} > 7 > 5^{1.5}$
   c) $625^{0.25} > 5^{1.5} > 7 > 216^{1/3}$ d) none of these.
2. What is the average of first 100 natural numbers?
   a) 50 b) 51 c) 49.5 d) 50.5
3. July 3, 1977, was a Sunday. Then July 4, 1970 was a
   a) Wednesday b) Saturday c) Thursday d) Monday
4. The product of the first 40 positive integers ends with
   a) 8 zeros b) 9 zeros c) 10 zeros d) 11 zeros.

[1]The reader may refer to `www.isical.ac.in/~rmo` for further details.

5. The number of perfect cubes among the first 3999 positive integers is
   a) 16 b) 15 c) 14 d) 13
6. The number of multiples of 4 among all 10 digit numbers is
   a) 25×10^8 b) 25×10^7 c) 225×10^7 d) 25×10^7
7. For triangle ABC and PQR, it is given that $AB = PQ, BC = QR$, and the $\angle ACB = \angle QRP$. Then, the triangles ABC and PQR are
   a) not congruent b) congruent
   c) need not be congruent but must be similar
   d) need not be similar but, if they are, then must be congruent.
8. The average of scores of 11 students is 75. The highest score is 80. Then, the minimum possible lowest score must be
   a) 25 b) 20 c) 30 d) None of these
9. The unit place of the integer $1! + 2! + 3! + \cdots + 15!$ is
   a) 5 b) 0 c) 3 d) 7
10. Let $x_1, x_2, \ldots, x_{15}$ be positive integers such that $x_i + x_{i+1} = 10$ for all i. If $x_9 = 1$, then the value of x_1 is
    a) 10 b) 1 c) 9 d) None of these
11. How many ways are there to arrange the letters in the word GARDEN with the vowels in alphabetical order?
    a) 360 b) 240 c) 120 d) 480
12. The number of ways of distributing 8 identical balls in 3 different boxes so that none of the boxes is empty is
    a) 38 b) 21 c) 5 d) None of these
13. In a particular month of some year, there are three Sundays which have even dates. On which day of the week does the 16th day of the month fall?
    a) Sunday b) Monday c) Saturday d) none of the above.
14. The number of diagonals of a convex decagon is
    a) 20 b) 30 c) 35 d) 40
15. Let a, b, c be three distinct real numbers, and let

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

    Then $f(1) + f(0)$ is
    a) 2 b) 3 c) 5 d) 0
16. Which of the following numbers is divisible by 5?
    a) 2^{100} b) $2^{100} + 1$ c) $2^{100} - 1$ d) $2^{100} + 2$
17. The number of positive integers which divide 10^{14} but not 15^{14} equals
    a) 205 b) 210 c) 212 d) 216
18. Let L_1 and L_2 be two intersecting straight lines such that the angle between them is 60°. The locus of moving point P such that the sum of the distances

between P and L_1 and that between P and L_2 is a constant is
a) a circle   b) an ellipse   c) a hyperbola   d) a parallelogram.

19. Let the roots of the equation $x^4+2x^2+4=0$ be $\alpha,\beta,\gamma,\delta$. An equation whose roots are $\alpha^2,\beta^2,\gamma^2,\delta^2$ is
a) $x^2+2x+4=0$ b) $x^4+2x^2+4=0$
c) $x^4+4x^3+12x^2+16x+16=0$ d) none of the above

20. Let $f:\mathbb{R}\setminus\{5\}\to\mathbb{R}$ be defined by $f(x)=\frac{(x-1)(x-3)}{(x-5)}$. Then f is
a) neither 1-1 nor onto   b) onto but not 1-1
c) 1-1 but not onto   d) both 1-1 and onto.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. For non-cyclic quadrilateral $ABCD$, prove that $BC\cdot AD+AB\cdot CD > AC\cdot BD$.
2. Show that there do not exist positive integers m and n such that $\frac{m}{n}+\frac{n+1}{m}=4$.
3. Show that if $3n+1$ and $4n+1$ are both perfect squares then $56|n$.
4. Prove that no set of 7 integers below 25 can have sums of all subsets different.
5. For all positive numbers $a,\ b,\ c$, prove that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$.
6. Find the number of isosceles triangles with integer sides not exceeding 2012.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. Let a, b, c be positive real numbers such that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP, and $a^2+b^2, b^2+c^2, c^2+a^2$ are in GP. Prove that $a=b=c$.
2. Let A be a subset of $\{1,11,21,31,\ldots 541,551\}$, such that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.
3. Triangle ABC has incenter I, and the incircle touches BC, CA at points D, E respectively. Let BI meet DE at G. Show that AG is perpendicular to BC.
4. Let A be one of the two points of intersection of two circles with centers X, Y respectively. The tangents at A to the two circles meet the circles again at B, C respectively. Let a point P be located so that $PXAY$ is a parallelogram. Show that P is the circumcenter of triangle ABC.
5. Let x, y be positive reals such that $x+y=2$. Prove that $x^3y^3(x^3+y^3)\leq 2$.
6. Let n be a positive integer such that $2n+1$ and $3n+1$ are perfect squares. Prove that $5n+3$ is a composite number.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. There are n necklaces, such that the first necklace contains 4 beads, the second necklace contains 7 beads, and in general the i-th necklace contains $i+1$ beads more than the number of beads in the $(i-1)$-th necklace. Find the total numbers of beads in all the necklaces.
2. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H, P, O are collinear.
3. Find the number of positive integers n for which $n < 2015$ and 6 is a factor of $n^2 + 3n + 2$.
4. In a triangle ABC, angle A is twice of angle B, show that $a^2 = b(b+c)$.
5. Let $< a_1, a_2, \ldots, a_n, \ldots >$ be a sequence of positive integers such that $a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. If $a_7 = 120$, find a_8.
6. If a, b, c are three distinct real numbers such that $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$, then show that $\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2} = 0$.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?
2. If a, b, c, x are real numbers such that $abc \neq 0$ and
$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c}$$
then prove that $a = b = c$.
3. Do there exist three distinct positive real numbers a, b, c such that the numbers $\{a, b, c, b+c-a, c+a-b, a+b-c, a+b+c\}$ form an AP in some order?
4. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$.
5. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, where K is nearer to A than L. If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $(5 : 10 : 13)$ in some order.
6. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that $\frac{5}{2} < \frac{AB}{BC} < 3$.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. Prove that in any group of five people, there are two who must have the same number of friends in the group.
2. Let $ABCD$ be a quadrilateral where $AB = CD$ and $\angle BAD = \angle BCD$. Prove that $ABCD$ is a parallelogram.
3. If n is odd, then prove that 24 is a factor of $n(n^2 - 1)$.
4. Find integers a, b such that $\operatorname{lcm}(a, b) = 168$ and $a + b = 52$.
5. If a, b, c, d satisfy $a+7b+2c+6d = 0$, $8a+4b+3c+d = 0$, $a+3b+4c+8d = 16$, and $6a + 2b + 7c + d = 16$, then find the value of $(a + d)(b + c)$.
6. Prove that the medians of a triangle divide it into six parts of equal area.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. Determine the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.
2. Let a, b be real numbers and, let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$.
3. Prove that there do not exist natural numbers x and y, with $x > 1$, such that $\frac{x^7-1}{x-1} = y^5 + 1$.
4. Consider the expression $2013^2+2014^2+2015^2+\cdots+n^2$. Prove that there exists a natural number $n > 2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.
5. For a natural number n, let $T(n)$ denote the number of ways we can place n objects of weights $1, 2, \ldots, n$ on a balance such that the sum of the weights in each pan is the same. Prove that $T(100) > T(99)$.
6. Let $n \geq 3$ be a natural number and let P be a polygon with n sides. Let $a_1, a_2, \ldots, a_n$ be the lengths of the sides of P and let p be its perimeter. Prove that
$$\frac{a_1}{p - a_1} + \frac{a_2}{p - a_2} + \cdots + \frac{a_n}{p - a_n} < 2$$
.

RMO Practice Test

Time: 3 hours **Full Marks: 100**

1. Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.
2. Find the number of 10-tuples $(a_1, a_2, \dots, a_{10})$ of integers such that $|a_1| \leq 1$ and $a_1^2 + a_2^2 + a_3^2 + \cdots + a_{10}^2 - a_1a_2 - a_2a_3 - a_3a_4 - \cdots - a_9a_{10} - a_{10}a_1 = 2$.
3. Find all primes p and q such that p divides $q^2 - 4$ and q divides $p^2 - 1$.
4. Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:
   $(a) f(1) = 0;$
   (b) the roots of $g(x)$ are squares of the roots of $f(x)$.
   Find the value of $a^{2013} + b^{2013} + c^{2013}$.
5.
6.

Pre-RMO Test (2014)

The test was based on MCQ's. However, the questions are NOT given in MCQ form here. Time: 2 hours.

1. Suppose five numbers are in arithmetic progression. If their sum is 30, find the middle term.
2. Suppose $r_1 > r_2 > r_3 \geq 0$ and $q_1 > 0, q_2 > 0$ are integers satisfying $r_0 = q_1r_1 + r_2$ and $r_1 = q_2r_2 + r_3$. Decide if $\gcd(r_0, r_1) = \gcd(r_1, r_2)$. If not, find which one is larger.
3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = (x^2 - x + 4)/x$. Is f 1 to 1? Is it onto?
4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and $a_1 \neq 0$. Suppose $f(x) = 0$ has exactly one real root and that root belongs to $(0, 1)$. Decide if irrespective of what f is, the quantity $f(0) \times f(2014)$ is always zero or positive or negative.
5. Let x be real number satisfying $|x^2 - 2x + 5| = 20$. Find the set of possible value(s) of $x^3 + 2x + 1$.
6. Let $A = \{x \in \mathbb{R} : ||x - 5/2| - 3/2| = |x^2 - 5x + 4|\}$. Find A.
7. For $a > 0$, assuming all the logarithms to be well-defined, we define $S(a) = \{x \in \mathbb{R} : \log_a(x^2 - x - 2) > log_a(3 + 2x - x^2)\}$. Suppose $a + 1 \in S(a)$. Find $S(a)$.

8. The number 23104x791 is divisible by 63. Find the set of possible value(s) of the missing digit x.
9. Suppose $\angle AOB = 75^o, \angle AOC = 30^o$. Let P move along OC and F_A, F_B be the feet of the perpendiculars PF_A, PF_B from P to OA and OB, respectively. Let Q be the mid-point of F_AF_B. Find the set of all such Q's.
10. Suppose $\angle AOB = 90^o$. Let C_1 and C_2 be two moving circles of equal radii such that they touch each other, C_1 touch OA and C_2 touch OB. Denote by P the point where C_1 and C_2 touch each other. Find the set of all such P's.
11. Suppose A and B are two distinct fixed points and P is a moving point such that 2length$(PA) >$ length(PB). Find the set of all such P's.
12. Find the maximum value of $4x - 3y - 2z$ subject to $2x^2 + 3y^2 + 4z^2 = 1$.
13. Find the maximum value of $\sum_{i=1}^{20} i^2(21-i)a_i$ subject to $a_1, \ldots, a_{20} \geq 0$ and $a_1 + \cdots + a_{20} = 1$.
14. If the cubic equation $x^3 + ax + b = 0$ has two identical roots, show that $4a^3 + 27b^2 = 0$.
15. Let A be the set of all real numbers a such that $x^2 + (2a-1)x - a - 3 = 0$ has two distinct real roots x_1, x_2 and $x_1^3 + x_2^3 = -72$. Find A.
16. Let $P(x)$ be a polynomial of degree 5. Suppose $P(k) = k/(k+1)$ for $k = 0, \ldots, 5$. Find $P(8)$.
17. Consider all the 5-digit numbers containing each of the digits 1, 2, 3, 4, 5 exactly once, and not divisible by 6. Find the sum of all these numbers.
18. Find the sum of reciprocals of all positive divisors of 360.
19. Ten identical balls are put into four boxes. Find the number of ways this can be done so that no box is empty.
20. In how many ways can you distribute 100 identical chocolates among 10 children so that the number of chocolates everyone gets is a multiple of 3, allowing some chocolates to be undistributed? [In solving this problem, you should take 0 to be a multiple of 3.]

B.1. Numbers

There are several types of numbers such as Integers, Natural numbers, Odd numbers, Even numbers, Primes, Composites. We discussed them in the Number Theory chapter. Here we introduce some other types of positive integers.

Pseudoprime

Fermat's theorem stated that if n is prime then $n|2^n - 2$. But the converse is not true as, $341|2^{341} - 2$, but $341 = 11.31$ is not prime.
A composites integer n is called pseudoprime if $n|2^n - 2$. There are infinitely many pseudoprimes, the smallest four are 341,561,645 and 1105. The smallest even pseudoprime is 161038. There are 245 pseudoprimes smaller than one million, where as there are 78492 primes.

Absolute pseudoprime or Carmichael number

A composite number n is called Carmichael number if for all integers a, $n|a^n - a$, i.e., $a^n \equiv a(\mod n)$.
A Carmichael number is a pseudoprime, but converse is not true, as 561 is not a Carmichael number. The smallest Carmichael number is 561. There are infinitely many Carmichael numbers. There are just 43 absolute pseudoprimes smaller than one million.

Perfect number

A positive integer n is said to be perfect if n is equal to the sum of all it's positive divisors, excluding n itself, i.e., $\sigma(n) = 2n$. The smallest four are 6,28,496,8128. It is not known that whether there are infinitely many perfect numbers. An even perfect number end with 6 or 28. So far no odd perfect number has been produced.

Deficient number and abundant number

A positive integer n is said to be a deficient number if $\sigma(n) < 2n$ and an abundant number if $\sigma(n) > 2n$. Prove that there are infinitely many abundant numbers as well as deficient numbers.

Super perfect number

A positive integer n is super perfect if $\sigma(\sigma(n)) = 2n$. Check that 16 and 64 are super perfect numbers.

k-perfect number or multiply perfect

A positive integer n is k-perfect for $k \geq 3$ if $\sigma(n) = kn$. 523,776 is a 3-perfect (prove!).

multiplicatively perfect

A positive integer n is multiplicative perfect if $\Pi_{d|n} d = n^2$. Find all multiplicative perfect numbers.

Mersenne number and Mersenne prime

Any integer of the form $2^n - 1$ is called Mersenne number. A Mersenne number which is also prime is called Mersenne prime. Clearly if n is composite then $2^n - 1$ is composite (prove!). $2^n - 1$ is a Mersenne prime if $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127$ and is composite for all other prime $n < 257$.

Fermat number and Fermat prime

A number $F_n = 2^{2^n} + 1$, where n is a non-negative integer, is known as Fermat number. $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$ are all primes but F_5 is not a prime, which is divisible by 641. Two Fermat numbers are coprime (prove!).

If a Fermat number is prime then that number is called Fermat prime. F_7, F_8 are composite. The smallest prime divisor of F_8 is 1,238,926,361,552,897, the other factor of F_8 is 62 digit long and it is prime!! F_{14} is composite but no prime factor is known up to 1993.

Repunit

It is an integer written in decimal notation as a string of 1's. For example, 1,11,111,1111, etc. The n-th repunit is $R_n = 111 \ldots 111(n$- many 1's). $R_2, R_{19}, R_{19}, R_{23}, R_{317}, R_{1031}$ are only detected primes for $n \leq 10000$. Clearly R_n is composite if n is composite. Converse is not true as $R_5 = 41.271$.

Triangular number

Any number of the form $1+2+3+\cdots+n$ is called triangular number.

Palindrome

A palindrome is a number that reads the same backward as forward, for example, 373 and 521125.

Lucas number

Consider the recursion relation $a_n = a_{n-1} + a_{n-2}$. If $a_1 = a_2 = 1$ then a_n is Fibonacci number and if $a_1 = 1, a_2 = 3$ then a_n is Lucas number. The first few Lucas numbers are 1,3,4,7,11,18.

Amicable numbers

Amicable pair m and n is such that each number is equal to the sum of all the positive divisors of the other, not counting the number itself, i.e., $\sigma(m) = \sigma(n) = m + n$. For example, 220 and 284 are amicable pair.

Euclidean number

For a prime p, define $p^{\#}$ to be the product of all primes that are less than or equal to p. Numbers of the form $p^{\#}+1$ is known as Euclidean number. $2^{\#}+1, 3^{\#}+1, 5^{\#}+1, 7^{\#}+1, 11^{\#}+1$ are all primes, but, $13^{\#}+1, 17^{\#}+1, 19^{\#}+1$ are composite. It is not known whether there are infinitely many Euclidean numbers.

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