Advanced Analytical Models: Over 800 Models and 300 Applications from the Basel II Accord to Wall Street and Beyond
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APPENDIX C

Understanding and Choosing the Right Probability Distributions

Plotting data is one method for selecting a probability distribution. The following steps provide another process for selecting probability distributions that best describe the uncertain variables in your spreadsheets.

To select the correct probability distribution, use the following steps:

1. Look at the variable in question. List everything you know about the conditions surrounding this variable. You might be able to gather valuable information about the uncertain variable from historical data. If historical data are not available, use your own judgment, based on experience, listing everything you know about the uncertain variable.
2. Review the descriptions of the probability distributions.
3. Select the distribution that characterizes this variable. A distribution characterizes a variable when the conditions of the distribution match those of the variable.

Alternatively, if you have historical, comparable, contemporaneous, or forecast data, you can use Risk Simulator's distributional fitting modules to find the best statistical fit for your existing data. This fitting process will apply some advanced statistical techniques to find the best distribution and its relevant parameters that describe the data.

PROBABILITY DENSITY FUNCTIONS, CUMULATIVE DISTRIBUTION FUNCTIONS, AND PROBABILITY MASS FUNCTIONS

In mathematics and Monte Carlo simulation, a probability density function (PDF) represents a *continuous* probability distribution in terms of integrals. If a probability distribution has a density of $f(x)$, then intuitively the infinitesimal interval of $[x, x + dx]$ has a probability of $f(x)\,dx$. The PDF therefore can be seen as a smoothed version of a probability histogram; that is, by providing an empirically large sample of a continuous random variable repeatedly, the histogram using very narrow ranges will resemble the random variable's PDF. The probability of the interval between $[a, b]$ is given by

$$\int_a^b f(x)dx$$

which means that the total integral of the function f must be 1.0. *It is a common mistake to think of f(a) as the probability of a.* This is incorrect. In fact, $f(a)$ can sometimes be larger than 1—consider a uniform distribution between 0.0 and 0.5. The random variable x within this distribution will have $f(x)$ greater than 1. The probability in reality is the function $f(x)dx$ discussed previously, where dx is an infinitesimal amount.

The cumulative distribution function (CDF) is denoted as $F(x) = P(X \leq x)$, indicating the probability of X taking on a less than or equal value to x. Every CDF is monotonically increasing, is continuous from the right, and at the limits, has the following properties:

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} F(x) = 1$$

Further, the CDF is related to the PDF by

$$F(b) - F(a) = P(a \leq X \leq b) = \int_a^b f(x)dx$$

where the PDF function f is the derivative of the CDF function F.

In probability theory, a probability mass function or PMF gives the probability that a *discrete* random variable is exactly equal to some value. The PMF differs from the PDF in that the values of the latter, defined only for continuous random variables, are not probabilities; rather, its integral over a set of possible values of the random variable is a probability. A random variable is discrete if its probability distribution is discrete and can be characterized by a PMF. Therefore, X is a discrete random variable if

$$\sum_u P(X = u) = 1$$

as u runs through all possible values of the random variable X.

DISCRETE DISTRIBUTIONS

Following is a detailed listing of the different types of probability distributions that can be used in Monte Carlo simulation. This listing is included in the appendix for the reader's reference.

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Bernoulli or Yes/No Distribution

The Bernoulli distribution is a discrete distribution with two outcomes (e.g., head or tails, success or failure, 0 or 1). The Bernoulli distribution is the binomial distribution with one trial and can be used to simulate Yes/No or Success/Failure conditions. This distribution is the fundamental building block of other more complex distributions. For instance:

- *Binomial distribution:* Bernoulli distribution with higher number of n total trials and computes the probability of x successes within this total number of trials.
- *Geometric distribution:* Bernoulli distribution with higher number of trials and computes the number of failures required before the first success occurs.
- *Negative binomial distribution:* Bernoulli distribution with higher number of trials and computes the number of failures before the xth success occurs.

The mathematical constructs for the Bernoulli distribution are as follows:

$$P(x) = \begin{cases} 1-p & \text{for } x = 0 \\ p & \text{for } x = 1 \end{cases}$$

or

$$P(x) = p^x(1-p)^{1-x}$$

$$Mean = p$$

$$Standard\ Deviation = \sqrt{p(1-p)}$$

$$Skewness = \frac{1-2p}{\sqrt{p(1-p)}}$$

$$Excess\ Kurtosis = \frac{6p^2-6p+1}{p(1-p)}$$

The probability of success (p) is the only distributional parameter. Also, it is important to note that there is only one trial in the Bernoulli distribution, and the resulting simulated value is either 0 or 1.

Input requirements:

Probability of success > 0 and < 1 (that is, $0.0001 \le p \le 0.9999$)

Binomial Distribution

The binomial distribution describes the number of times a particular event occurs in a fixed number of trials, such as the number of heads in 10 flips of a coin or the number of defective items out of 50 items chosen.

The three conditions underlying the binomial distribution are:

1. For each trial, only two outcomes are possible that are mutually exclusive.
2. The trials are independent—what happens in the first trial does not affect the next trial.

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3. The probability of an event occurring remains the same from trial to trial.

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)} \text{ for } n > 0;\ x = 0, 1, 2, \ldots n; \text{ and } 0 < p < 1$$

$$Mean = np$$

$$Standard\ Deviation = \sqrt{np(1-p)}$$

$$Skewness = \frac{1-2p}{\sqrt{np(1-p)}}$$

$$Excess\ Kurtosis = \frac{6p^2 - 6p + 1}{np(1-p)}$$

The probability of success (p) and the integer number of total trials (n) are the distributional parameters. The number of successful trials is denoted x. It is important to note that probability of success (p) of 0 or 1 are trivial conditions and do not require any simulations, and, hence, are not allowed in the software.

Input requirements:

Probability of success > 0 and < 1 (that is, $0.0001 \leq p \leq 0.9999$).

Number of trials ≥ 1 or positive integers and $\leq 1{,}000$ (for larger trials, use the normal distribution with the relevant computed binomial mean and standard deviation as the normal distribution's parameters).

Discrete Uniform

The discrete uniform distribution is also known as the *equally likely outcomes* distribution, where the distribution has a set of N elements, and each element has the same probability. This distribution is related to the uniform distribution, but its elements are discrete and not continuous.

The mathematical constructs for the discrete uniform distribution are as follows:

$$P(x) = \frac{1}{N} \text{ ranked value}$$

$$Mean = \frac{N+1}{2} \text{ ranked value}$$

$$Standard\ Deviation = \sqrt{\frac{(N-1)(N+1)}{12}} \text{ ranked value}$$

$$Skewness = 0 \text{ (that is, the distribution is perfectly symmetrical)}$$

$$Excess\ Kurtosis = \frac{-6(N^2+1)}{5(N-1)(N+1)} \text{ ranked value}$$

Input requirements:

Minimum < Maximum and both must be integers (negative integers and zero are allowed)

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Geometric Distribution

The geometric distribution describes the number of trials until the first successful occurrence, such as the number of times you need to spin a roulette wheel before you win.

The three conditions underlying the geometric distribution are:

1. The number of trials is not fixed.
2. The trials continue until the first success.
3. The probability of success is the same from trial to trial.

The mathematical constructs for the geometric distribution are as follows:

$$P(x) = p(1-p)^{x-1} \text{ for } 0 < p < 1 \text{ and } x = 1, 2, \ldots, n$$

$$Mean = \frac{1}{p} - 1$$

$$Standard\ Deviation = \sqrt{\frac{1-p}{p^2}}$$

$$Skewness = \frac{2-p}{\sqrt{1-p}}$$

$$Excess\ Kurtosis = \frac{p^2 - 6p + 6}{1-p}$$

The probability of success (p) is the only distributional parameter. The number of successful trials simulated is denoted x, which can only take on positive integers.

Input requirements:

Probability of success > 0 and < 1 (that is, $0.0001 \leq p \leq 0.9999$). It is important to note that probability of success (p) of 0 or 1 are trivial conditions and do not require any simulations, and, hence, are not allowed in the software.

Hypergeometric Distribution

The hypergeometric distribution is similar to the binomial distribution in that both describe the number of times a particular event occurs in a fixed number of trials. The difference is that binomial distribution trials are independent, whereas hypergeometric distribution trials change the probability for each subsequent trial and are called *trials without replacement*. For example, suppose a box of manufactured parts is known to contain some defective parts. You choose a part from the box, find it is defective, and remove the part from the box. If you choose another part from the box, the probability that it is defective is somewhat lower than for the first part because you have removed a defective part. If you had replaced the defective part, the probabilities would have remained the same, and the process would have satisfied the conditions for a binomial distribution.

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The three conditions underlying the hypergeometric distribution are:

1. The total number of items or elements (the population size) is a fixed number, a finite population. The population size must be less than or equal to 1,750.
2. The sample size (the number of trials) represents a portion of the population.
3. The known initial probability of success in the population changes after each trial.

The mathematical constructs for the hypergeometric distribution are as follows:

$$P(x) = \frac{\dfrac{(N_x)!}{x!(N_x - x)!}\dfrac{(N - N_x)!}{(n - x)!(N - N_x - n + x)!}}{\dfrac{N!}{n!(N - n)!}}$$

for $x = Max(n - (N - N_x), 0), \ldots, Min(n, N_x)$

$$Mean = \frac{N_x n}{N}$$

$$Standard\ Deviation = \sqrt{\frac{(N - N_x)N_x n(N - n)}{N^2(N - 1)}}$$

$$Skewness = \frac{(N - 2N_x)(N - 2n)}{N - 2}\sqrt{\frac{N - 1}{(N - N_x)N_x n(N - n)}}$$

$$Excess\ Kurtosis = \frac{V(N, N_x, n)}{(N - N_x)N_x n(-3 + N)(-2 + N)(-N + n)} \text{ where}$$

$$\begin{aligned} V(N, Nx, n) &= (N - N_x)^3 - (N - N_x)^5 + 3(N - N_x)^2 N_x - 6(N - N_x)^3 N_x \\ &\quad + (N - N_x)^4 Nx + 3(N - N_x)N_x^2 - 12(N - N_x)^2 N_x^2 + 8(N - N_x)^3 N_x^2 + N_x^3 \\ &\quad - 6(N - N_x)N_x^3 + 8(N - N_x)^2 N_x^3 + (N - N_x)N_x^4 - N_x^5 - 6(N - N_x)^3 N_x \\ &\quad + 6(N - N_x)^4 N_x + 18(N - N_x)^2 N_x n - 6(N - N_x)^3 N_x n + 18(N - N_x)N_x^2 n \\ &\quad - 24(N - N_x)^2 N_x^2 n - 6(N - N_x)^3 n - 6(N - N_x)N_x^3 n + 6N_x^4 n + 6(N - N_x)^2 n^2 \\ &\quad - 6(N - N_x)^3 n^2 - 24(N - N_x)N_x n^2 + 12(N - N_x)^2 N_x n^2 + 6N_x^2 n^2 \\ &\quad + 12(N - N_x)N_x^2 n^2 - 6N_x^3 n^2 \end{aligned}$$

The number of items in the population (N), trials sampled (n), and number of items in the population that have the successful trait (N_x) are the distributional parameters. The number of successful trials is denoted x.

Input requirements:

Population ≥ 2 and integer

Trials > 0 and integer

Successes > 0 and integer

Population $>$ Successes

Trials $<$ Population

Population $< 1,750$

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Negative Binomial Distribution

The negative binomial distribution is useful for modeling the distribution of the number of trials until the rth successful occurrence, such as the number of sales calls you need to make to close a total of 10 orders. It is essentially a *superdistribution* of the geometric distribution. This distribution shows the probabilities of each number of trials in excess of r to produce the required success r.

The three conditions underlying the negative binomial distribution are:

1. The number of trials is not fixed.
2. The trials continue until the rth success.
3. The probability of success is the same from trial to trial.

The mathematical constructs for the negative binomial distribution are as follows:

$$P(x) = \frac{(x + r - 1)!}{(r - 1)!x!} p^r (1 - p)^x \text{ for } x = r, r + 1, \ldots; \text{ and } 0 < p < 1$$

$$Mean = \frac{r(1 - p)}{p}$$

$$Standard\ Deviation = \sqrt{\frac{r(1 - p)}{p^2}}$$

$$Skewness = \frac{2 - p}{\sqrt{r(1 - p)}}$$

$$Excess\ Kurtosis = \frac{p^2 - 6p + 6}{r(1 - p)}$$

The probability of success (p) and required successes (r) are the distributional parameters.

Input requirements:

Successes required must be positive integers > 0 and < 8,000.

Probability of success > 0 and < 1 (that is, $0.0001 \leq p \leq 0.9999$). It is important to note that probability of success (p) of 0 or 1 are trivial conditions and do not require any simulations, and, hence, are not allowed in the software.

Poisson Distribution

The Poisson distribution describes the number of times an event occurs in a given interval, such as the number of telephone calls per minute or the number of errors per page in a document.

The three conditions underlying the Poisson distribution are:

1. The number of possible occurrences in any interval is unlimited.
2. The occurrences are independent. The number of occurrences in one interval does not affect the number of occurrences in other intervals.
3. The average number of occurrences must remain the same from interval to interval.

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The mathematical constructs for the Poisson are as follows:

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \text{ for } x \text{ and } \lambda > 0$$
$$Mean = \lambda$$
$$Standard\ Deviation = \sqrt{\lambda}$$
$$Skewness = \frac{1}{\sqrt{\lambda}}$$
$$Excess\ Kurtosis = \frac{1}{\lambda}$$

Rate (λ) is the only distributional parameter.
Input requirements:

Rate > 0 and $\leq 1{,}000$ (that is, $0.0001 \leq$ rate $\leq 1{,}000$)

CONTINUOUS DISTRIBUTIONS

Beta Distribution

The beta distribution is very flexible and is commonly used to represent variability over a fixed range. One of the more important applications of the beta distribution is its use as a conjugate distribution for the parameter of a Bernoulli distribution. In this application, the beta distribution is used to represent the uncertainty in the probability of occurrence of an event. It is also used to describe empirical data and predict the random behavior of percentages and fractions, as the range of outcomes is typically between 0 and 1.

The value of the beta distribution lies in the wide variety of shapes it can assume when you vary the two parameters, alpha and beta. If the parameters are equal, the distribution is symmetrical. If either parameter is 1 and the other parameter is greater than 1, the distribution is J-shaped. If alpha is less than beta, the distribution is said to be positively skewed (most of the values are near the minimum value). If alpha is greater than beta, the distribution is negatively skewed (most of the values are near the maximum value).

The mathematical constructs for the beta distribution are as follows:

$$f(x) = \frac{(x)^{(\alpha-1)}(1-x)^{(\beta-1)}}{\left[\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right]} \text{ for } \alpha > 0;\ \beta > 0;\ x > 0$$
$$Mean = \frac{\alpha}{\alpha+\beta}$$
$$Standard\ Deviation = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}}$$

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$$Skewness = \frac{2(\beta - \alpha)\sqrt{1 + \alpha + \beta}}{(2 + \alpha + \beta)\sqrt{\alpha\beta}}$$

$$Excess\ Kurtosis = \frac{3(\alpha + \beta + 1)[\alpha\beta(\alpha + \beta - 6) + 2(\alpha + \beta)^2]}{\alpha\beta(\alpha + \beta + 2)(\alpha + \beta + 3)} - 3$$

Alpha (α) and beta (β) are the two distributional shape parameters, and Γ is the gamma function.

The two conditions underlying the beta distribution are:

1. The uncertain variable is a random value between 0 and a positive value.
2. The shape of the distribution can be specified using two positive values.

Input requirements:

Alpha and beta > 0 and can be any positive value

Cauchy Distribution or Lorentzian Distribution or Breit–Wigner Distribution

The Cauchy distribution, also called the Lorentzian distribution or Breit–Wigner distribution, is a continuous distribution describing resonance behavior. It also describes the distribution of horizontal distances at which a line segment tilted at a random angle cuts the x-axis.

The mathematical constructs for the Cauchy or Lorentzian distribution are as follows:

$$f(x) = \frac{1}{\pi} \frac{\gamma/2}{(x - m)^2 + \gamma^2/4}$$

The Cauchy distribution is a special case where it does not have any theoretical moments (mean, standard deviation, skewness, and kurtosis) as they are all undefined.

Mode location (m) and scale (γ) are the only two parameters in this distribution. The location parameter specifies the peak or mode of the distribution, while the scale parameter specifies the half-width at half-maximum of the distribution. In addition, the mean and variance of a Cauchy or Lorentzian distribution are undefined.

In addition, the Cauchy distribution is the Student's t distribution with only 1 degree of freedom. This distribution is also constructed by taking the ratio of two standard normal distributions (normal distributions with a mean of zero and a variance of one) that are independent of one another.

Input requirements:

Location can be any value

Scale > 0 and can be any positive value

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Chi-Square Distribution

The chi-square distribution is a probability distribution used predominantly in hypothesis testing, and is related to the gamma distribution and the standard normal distribution. For instance, the sums of independent normal distributions are distributed as a chi-square (χ^2) with k degrees of freedom:

$$Z_1^2 + Z_2^2 + \ldots + Z_k^2 \overset{d}{\sim} \chi_k^2$$

The mathematical constructs for the chi-square distribution are as follows:

$$f(x) = \frac{2^{(-k/2)}}{\Gamma(k/2)} x^{k/2-1} e^{-x/2} \text{ for all } x > 0$$

$$Mean = k$$

$$Standard\ Deviation = \sqrt{2k}$$

$$Skewness = 2\sqrt{\frac{2}{k}}$$

$$Excess\ Kurtosis = \frac{12}{k}$$

The gamma function is written as Γ. Degrees of freedom k is the only distributional parameter.

The chi-square distribution can also be modeled using a gamma distribution by setting the shape parameter as $k/2$ and scale as $2S^2$ where S is the scale.

Input requirements:

Degrees of freedom > 1 and must be an integer < 1,000

Exponential Distribution

The exponential distribution is widely used to describe events recurring at random points in time, such as the time between failures of electronic equipment or the time between arrivals at a service booth. It is related to the Poisson distribution, which describes the number of occurrences of an event in a given interval of time. An important characteristic of the exponential distribution is the "memoryless" property, which means that the future lifetime of a given object has the same distribution, regardless of the time it existed. In other words, time has no effect on future outcomes.

The mathematical constructs for the exponential distribution are as follows:

$$f(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0; \lambda > 0$$

$$Mean = \frac{1}{\lambda}$$

$$Standard\ Deviation = \frac{1}{\lambda}$$

$Skewness = 2$ (this value applies to all success rate λ inputs)

$Excess\ Kurtosis = 6$ (this value applies to all success rate λ inputs)

Success rate (λ) is the only distributional parameter. The number of successful trials is denoted x.

The condition underlying the exponential distribution is:

1. The exponential distribution describes the amount of time between occurrences.

Input requirements:

Rate > 0 and ≤ 300

Extreme Value Distribution or Gumbel Distribution

The extreme value distribution (Type 1) is commonly used to describe the largest value of a response over a period of time, for example, in flood flows, rainfall, and earthquakes. Other applications include the breaking strengths of materials, construction design, and aircraft loads and tolerances. The extreme value distribution is also known as the Gumbel distribution.

The mathematical constructs for the extreme value distribution are as follows:

$$f(x) = \frac{1}{\beta} z e^{-z} \text{ where } z = e^{\frac{x-m}{\beta}} \text{ for } \beta > 0; \text{ and any value of } x \text{ and } m$$

$$Mean = m + 0.577215\beta$$

$$Standard\ Deviation = \sqrt{\frac{1}{6}\pi^2\beta^2}$$

$$Skewness = \frac{12\sqrt{6}(1.2020569)}{\pi^3}$$

$$= 1.13955 \text{ (this applies for all values of mode and scale)}$$

$$Excess\ Kurtosis = 5.4 \text{ (this applies for all values of mode and scale)}$$

Mode (m) and scale (β) are the distributional parameters.

There are two standard parameters for the extreme value distribution: mode and scale. The mode parameter is the most likely value for the variable (the highest point on the probability distribution). The scale parameter is a number greater than 0. The larger the scale parameter, the greater the variance.

Input requirements:

Mode can be any value

Scale > 0

F Distribution or Fisher–Snedecor Distribution

The F distribution, also known as the Fisher–Snedecor distribution, is another continuous distribution used most frequently for hypothesis testing. Specifically, it is used to test the statistical difference between two variances in analysis of variance tests and likelihood ratio tests. The F distribution with the numerator degree of

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freedom n and denominator degree of freedom m is related to the chi-square distribution in that:

$$\frac{\chi_n^2/n}{\chi_m^2/m} \overset{d}{\sim} F_{n,m} \text{ or } f(x) = \frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{m}\right)^{n/2} x^{n/2-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left[x\left(\frac{n}{m}\right)+1\right]^{(n+m)/2}}$$

$$Mean = \frac{m}{m-2}$$

$$Standard\ Deviation = \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)} \text{ for all } m > 4$$

$$Skewness = \frac{2(m+2n-2)}{m-6}\sqrt{\frac{2(m-4)}{n(m+n-2)}}$$

$$Excess\ Kurtosis = \frac{12(-16+20m-8m^2+m^3+44n-32mn+5m^2n-22n^2+5mn^2)}{n(m-6)(m-8)(n+m-2)}$$

The numerator degree of freedom n and denominator degree of freedom m are the only distributional parameters.

Input requirements:

Degrees of freedom numerator and degrees of freedom denominator both > 0 integers

Gamma Distribution (Erlang Distribution)

The gamma distribution applies to a wide range of physical quantities and is related to other distributions: lognormal, exponential, Pascal, Erlang, Poisson, and chi-square. It is used in meteorological processes to represent pollutant concentrations and precipitation quantities. The gamma distribution is also used to measure the time between the occurrence of events when the event process is not completely random. Other applications of the gamma distribution include inventory control, economic theory, and insurance risk theory.

The gamma distribution is most often used as the distribution of the amount of time until the rth occurrence of an event in a Poisson process. When used in this fashion, the three conditions underlying the gamma distribution are:

1. The number of possible occurrences in any unit of measurement is not limited to a fixed number.
2. The occurrences are independent. The number of occurrences in one unit of measurement does not affect the number of occurrences in other units.
3. The average number of occurrences must remain the same from unit to unit.

The mathematical constructs for the gamma distribution are as follows:

$$f(x) = \frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta} \text{ with any value of } \alpha > 0 \text{ and } \beta > 0$$

$$Mean = \alpha\beta$$

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$$Standard\ Deviation = \sqrt{\alpha\beta^2}$$

$$Skewness = \frac{2}{\sqrt{\alpha}}$$

$$Excess\ Kurtosis = \frac{6}{\alpha}$$

Shape parameter alpha (α) and scale parameter beta (β) are the distributional parameters, and Γ is the gamma function.

When the alpha parameter is a positive integer, the gamma distribution is called the Erlang distribution, used to predict waiting times in queuing systems, where the Erlang distribution is the sum of independent and identically distributed random variables each having a memoryless exponential distribution. Setting n as the number of these random variables, the mathematical construct of the Erlang distribution is:

$$f(x) = \frac{x^{n-1}e^{-x}}{(n-1)!} \text{ for all } x > 0 \text{ and all positive integers of } n$$

Input requirements:

Scale beta > 0 and can be any positive value

Shape alpha ≥ 0.05 and any positive value

Location can be any value

Logistic Distribution

The logistic distribution is commonly used to describe growth, that is, the size of a population expressed as a function of a time variable. It also can be used to describe chemical reactions and the course of growth for a population or individual.

The mathematical constructs for the logistic distribution are as follows:

$$f(x) = \frac{e^{\frac{\mu-x}{\alpha}}}{\alpha\left[1+e^{\frac{\mu-x}{\alpha}}\right]^2} \text{ for any value of } \alpha \text{ and } \mu$$

$$Mean = \mu$$

$$Standard\ Deviation = \sqrt{\frac{1}{3}\pi^2\alpha^2}$$

$$Skewness = 0 \text{ (this applies to all mean and scale inputs)}$$

$$Excess\ Kurtosis = 1.2 \text{ (this applies to all mean and scale inputs)}$$

Mean (μ) and scale (α) are the distributional parameters.

There are two standard parameters for the logistic distribution: mean and scale. The mean parameter is the average value, which for this distribution is the same as the mode, because this distribution is symmetrical. The scale parameter is a number greater than 0. The larger the scale parameter, the greater the variance.

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Input requirements:

Scale > 0 and can be any positive value

Mean can be any value

Lognormal Distribution

The lognormal distribution is widely used in situations where values are positively skewed, for example, in financial analysis for security valuation or in real estate for property valuation, and where values cannot fall below zero.

Stock prices are usually positively skewed rather than normally (symmetrically) distributed. Stock prices exhibit this trend because they cannot fall below the lower limit of zero but might increase to any price without limit. Similarly, real estate prices illustrate positive skewness and are lognormally distributed as property values cannot become negative.

The three conditions underlying the lognormal distribution are:

1. The uncertain variable can increase without limits but cannot fall below zero.
2. The uncertain variable is positively skewed, with most of the values near the lower limit.
3. The natural logarithm of the uncertain variable yields a normal distribution.

Generally, if the coefficient of variability is greater than 30 percent, use a lognormal distribution. Otherwise, use the normal distribution.

The mathematical constructs for the lognormal distribution are as follows:

$$f(x) = \frac{1}{x\sqrt{2\pi}\ln(\sigma)} e^{-\frac{[\ln(x)-\ln(\mu)]^2}{2[\ln(\sigma)]^2}} \text{ for } x > 0; \mu > 0 \text{ and } \sigma > 0$$

$$\mathit{Mean} = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

$$\mathit{Standard\ Deviation} = \sqrt{\exp(\sigma^2 + 2\mu)[\exp(\sigma^2) - 1]}$$

$$\mathit{Skewness} = \left[\sqrt{\exp(\sigma^2) - 1}\right](2 + \exp(\sigma^2))$$

$$\mathit{Excess\ Kurtosis} = \exp(4\sigma^2) + 2\exp(3\sigma^2) + 3\exp(2\sigma^2) - 6$$

Mean (μ) and standard deviation (σ) are the distributional parameters.

Input requirements:

Mean and standard deviation both > 0 and can be any positive value

Lognormal Parameter Sets By default, the lognormal distribution uses the arithmetic mean and standard deviation. For applications for which historical data are available, it is more appropriate to use either the logarithmic mean and standard deviation, or the geometric mean and standard deviation.

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Normal Distribution

The normal distribution is the most important distribution in probability theory because it describes many natural phenomena, such as people's IQs or heights. Decision makers can use the normal distribution to describe uncertain variables such as the inflation rate or the future price of gasoline.

The three conditions underlying the normal distribution are:

1. Some value of the uncertain variable is the most likely (the mean of the distribution).
2. The uncertain variable could as likely be above the mean as it could be below the mean (symmetrical about the mean).
3. The uncertain variable is more likely to be in the vicinity of the mean than further away.

The mathematical constructs for the normal distribution are as follows:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for all values of } x \text{ and } \mu; \text{ while } \sigma > 0$$

$$Mean = \mu$$

$$Standard\ Deviation = \sigma$$

$$Skewness = 0 \text{ (this applies to all inputs of mean and standard deviation)}$$

$$Excess\ Kurtosis = 0 \text{ (this applies to all inputs of mean and standard deviation)}$$

Mean (μ) and standard deviation (σ) are the distributional parameters.
Input requirements:

Standard deviation > 0 and can be any positive value

Mean can be any value

Pareto Distribution

The Pareto distribution is widely used for the investigation of distributions associated with such empirical phenomena as city population sizes, the occurrence of natural resources, the size of companies, personal incomes, stock price fluctuations, and error clustering in communication circuits.

The mathematical constructs for the Pareto are as follows:

$$f(x) = \frac{\beta L^{\beta}}{x^{(1+\beta)}} \text{ for } x > L$$

$$Mean = \frac{\beta L}{\beta - 1}$$

$$Standard\ Deviation = \sqrt{\frac{\beta L^2}{(\beta-1)^2(\beta-2)}}$$

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$$Skewness = \sqrt{\frac{\beta - 2}{\beta}} \left[\frac{2(\beta + 1)}{\beta - 3} \right]$$

$$Excess\ Kurtosis = \frac{6(\beta^3 + \beta^2 - 6\beta - 2)}{\beta(\beta - 3)(\beta - 4)}$$

Location (L) and shape (β) are the distributional parameters.

There are two standard parameters for the Pareto distribution: location and shape. The location parameter is the lower bound for the variable. After you select the location parameter, you can estimate the shape parameter. The shape parameter is a number greater than 0, usually greater than 1. The larger the shape parameter, the smaller the variance and the thicker the right tail of the distribution.

Input requirements:

Location > 0 and can be any positive value

Shape ≥ 0.05

Student's t Distribution

The Student's t distribution is the most widely used distribution in hypothesis testing. This distribution is used to estimate the mean of a normally distributed population when the sample size is small, and is used to test the statistical significance of the difference between two sample means or confidence intervals for small sample sizes.

The mathematical constructs for the t distribution are as follows:

$$f(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{r\pi}\Gamma[r/2]}(1 + t^2/r)^{-(r+1)/2}$$

where $t = \dfrac{x - \bar{x}}{s}$ and Γ is the gamma function

$Mean = 0$ (this applies to all degrees of freedom r except if the distribution is shifted to another nonzero central location)

$$Standard\ Deviation = \sqrt{\frac{r}{r-2}}$$

$Skewness = 0$ (this applies to all degrees of freedom r)

$$Excess\ Kurtosis = \frac{6}{r-4} \text{ for all } r > 4$$

Degree of freedom r is the only distributional parameter.

The t distribution is related to the F-distribution as follows: The square of a value of t with r degrees of freedom is distributed as F with 1 and r degrees of freedom. The overall shape of the probability density function of the t distribution also resembles the bell shape of a normally distributed variable with mean 0 and variance 1, except that it is a bit lower and wider or is leptokurtic (fat tails at the ends and peaked center). As the number of degrees of freedom grows (say,

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above 30), the t distribution approaches the normal distribution with mean 0 and variance 1.

Input requirements:

Degrees of freedom ≥ 1 and must be an integer

Triangular Distribution

The triangular distribution describes a situation where you know the minimum, maximum, and most likely values to occur. For example, you could describe the number of cars sold per week when past sales show the minimum, maximum, and usual number of cars sold.

The three conditions underlying the triangular distribution are:

1. The minimum number of items is fixed.
2. The maximum number of items is fixed.
3. The most likely number of items falls between the minimum and maximum values, forming a triangular-shaped distribution, which shows that values near the minimum and maximum are less likely to occur than those near the most likely value.

The mathematical constructs for the triangular distribution are as follows:

$$f(x) = \begin{cases} \dfrac{2(x - Min)}{(Max - Min)(Likely - Min)} & \text{for } Min < x < Likely \\ \dfrac{2(Max - x)}{(Max - Min)(Max - Likely)} & \text{for } Likely < x < Max \end{cases}$$

$$Mean = \frac{1}{3}(Min + Likely + Max)$$

$$Standard\ Deviation = \sqrt{\frac{1}{18}(Min^2 + Likely^2 + Max^2 - MinMax - MinLikely - MaxLikely)}$$

$$Skewness = \frac{\sqrt{2}(Min + Max - 2Likely)(2Min - Max - Likely)(Min - 2Max + Likely)}{5(Min^2 + Max^2 + Likely^2 - MinMax - MinLikely - MaxLikely)^{3/2}}$$

$Excess\ Kurtosis = -0.6$ (this applies to all inputs of Min, Max, and $Likely$)

Minimum value (Min), most likely value ($Likely$), and maximum value (Max) are the distributional parameters.

Input requirements:

$Min \leq Likely \leq Max$ and can also take any value

However, $Min < Max$ and can also take any value

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Uniform Distribution

With the uniform distribution, all values fall between the minimum and maximum and occur with equal likelihood.

The three conditions underlying the uniform distribution are:

1. The minimum value is fixed.
2. The maximum value is fixed.
3. All values between the minimum and maximum occur with equal likelihood.

The mathematical constructs for the uniform distribution are as follows:

$$f(x) = \frac{1}{Max - Min} \text{ for all values such that } Min < Max$$

$$Mean = \frac{Min + Max}{2}$$

$$Standard\ Deviation = \sqrt{\frac{(Max - Min)^2}{12}}$$

$$Skewness = 0 \text{ (this applies to all inputs of } Min \text{ and } Max\text{)}$$

$$Excess\ Kurtosis = -1.2 \text{ (this applies to all inputs of } Min \text{ and } Max\text{)}$$

Maximum value (*Max*) and minimum value (*Min*) are the distributional parameters.

Input requirements:

$Min < Max$ and can take any value

Weibull Distribution (Rayleigh Distribution)

The Weibull distribution describes data resulting from life and fatigue tests. It is commonly used to describe failure time in reliability studies as well as the breaking strengths of materials in reliability and quality control tests. Weibull distributions are also used to represent various physical quantities, such as wind speed.

The Weibull distribution is a family of distributions that can assume the properties of several other distributions. For example, depending on the shape parameter you define, the Weibull distribution can be used to model the exponential and Rayleigh distributions, among others. The Weibull distribution is very flexible. When the Weibull shape parameter is equal to 1.0, the Weibull distribution is identical to the exponential distribution. The Weibull location parameter lets you set up an exponential distribution to start at a location other than 0.0. When the shape parameter is less than 1.0, the Weibull distribution becomes a steeply declining curve. A manufacturer might find this effect useful in describing part failures during a burn-in period .

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The mathematical constructs for the Weibull distribution are as follows:

$$f(x) = \frac{\alpha}{\beta}\left[\frac{x}{\beta}\right]^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

$$Mean = \beta\Gamma(1+\alpha^{-1})$$

$$Standard\ Deviation = \beta^2\left[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})\right]$$

$$Skewness = \frac{2\Gamma^3(1+\beta^{-1}) - 3\Gamma(1+\beta^{-1})\Gamma(1+2\beta^{-1}) + \Gamma(1+3\beta^{-1})}{\left[\Gamma(1+2\beta^{-1}) - \Gamma^2(1+\beta^{-1})\right]^{3/2}}$$

$$Excess\ Kurtosis =$$
$$\frac{-6\Gamma^4(1+\beta^{-1}) + 12\Gamma^2(1+\beta^{-1})\Gamma(1+2\beta^{-1}) - 3\Gamma^2(1+\beta^{-1}) - 4\Gamma(1+\beta^{-1})\Gamma(1+3\beta^{-1}) + \Gamma(1+4\beta^{-1})}{\left[\Gamma(1+2\beta^{-1}) - \Gamma^2(1+\beta^{-1})\right]^2}$$

Location (L), shape (α), and scale (β) are the distributional parameters, and Γ is the gamma function.

Input requirements:

Scale $>$ 0 and can be any positive value

Shape $\geq$ 0.05

Location can take on any value

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