Dual problem of linear SVM

Primal problem (*)

$$\min_{w,b} \frac{1}{2}\|w\|^2 \quad \text{s.t.} \quad \forall i=1,\dots,N: \; c_i(w,b) \leq 0 \quad \text{where}$$

$$c_i(w,b) := 1 - y_i(\langle w, x_i\rangle + b)$$

Lagrangian

$$\mathcal{L}(\underbrace{w,b}_{\omega},\lambda) := \underbrace{\frac{1}{2}\|w\|^2}_{f \text{ } \underline{\text{convex}}} + \underbrace{\sum_{i=1}^{N} \lambda_i \left(1 - \underline{y_i}(\langle w, x_i\rangle + \underline{b})\right)}_{\underline{\text{affine}} \text{ in } (w,b)}$$

$\Rightarrow$ strong duality by Slater's condition

We can solve the dual problem

$$(*) \iff \max_{\lambda_i \geq 0} g(\lambda) \quad \text{where}$$

$$\underline{g(\lambda)} := \inf_{w,b} \mathcal{L}(w,b,\lambda)$$

$\Downarrow$

minimize $\mathcal{L}(w,b,\lambda)$ wrt w,b $\forall \lambda$

Find $w^*(\lambda), b^*(\lambda)$ satisfying

$$\frac{\partial \mathcal{L}}{\partial w_j}(w^*(\lambda), b^*(\lambda), \lambda) = 0 \quad \forall j \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial b}(w^*(\lambda), b^*(\lambda), \lambda) = 0 \qquad (2)$$

① $\frac{\partial \mathcal{L}}{\partial w_j} = w_j^* - \sum_{i=1}^{N} \lambda_i y_i (x_i)_j = 0 \Rightarrow w^*(\lambda) = \sum_{i=1}^{N} \lambda_i y_i x_i$

② $\frac{\partial \mathcal{L}}{\partial b} = -\sum_i \lambda_i y_i = 0 \Rightarrow \sum_{i=1}^{N} \lambda_i y_i = 0$

(both $\forall \lambda$)

$$\mathcal{L}(w^*(\lambda), b^*(\lambda), \lambda) = \frac{1}{2}\|w^*(\lambda)\|^2 + \sum_{i=1}^{N} \lambda_i \left(1 - y_i \left(\langle w^*(\lambda), x_i\rangle + b^*(\lambda)\right)\right)$$

$$\overset{①}{=} \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j \langle x_i, x_j\rangle + \sum_{i=1}^{N} \lambda_i$$

$$- \sum_{i,j} \lambda_i \lambda_j y_i y_j \langle x_j, x_i\rangle - \underbrace{\left(\sum_{i=1}^{N} \lambda_i y_i\right)}_{\overset{②}{=} 0} b^*(\lambda)$$

$$= \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j y_i y_j \langle x_i, x_j\rangle =: g(\lambda)$$

s.t. $\sum \lambda_i y_i = 0$