08.12.2011
Sebastian Gottwald

Oscillatory integrals

Definition (amplitudes of order m). Let $m \geqslant 0$, then we call $a \in C^\infty(\mathbb{R}^n)$ an *amplitude of order* m, $a \in A^m$, if

$$\forall \alpha \in \mathbb{N}_0^n \ \ \exists C_\alpha : \ |\partial^\alpha a(x)| \leqslant C_\alpha (1+|x|^2)^{m/2} \quad \forall x \in \mathbb{R}^n$$

For fixed $m \geqslant 0$, we may define a family of norms $|||a|||_k$ on A^m by taking the maximum of the best constants C_α for $|\alpha| \leqslant k$, i.e. $|||a|||_k := \max_{|\alpha|\leqslant k} \|(1+|\cdot|^2)^{-m/2}\partial^\alpha a\|_\infty$.

Remark. Some simple properties are

(i) *If $k \leqslant m$, then $A^k \subset A^m$.*

(ii) *Derivatives and products of amplitudes are amplitudes (if $a \in A^m$ and $b \in A^k$, then $\partial^\alpha a \in A^m$, $ab \in A^{m+k}$).*

(iii) *The Schwartz space $\mathcal{S}$ is closed under multiplication by amplitudes.*

(Key) Lemma. *If q is a non-deg. real quadratic form, $\chi \in C_0^\infty(\mathbb{R}^n)$ s.th. $\chi \equiv 0$ on a neighbourhood of 0, $b \in C^\infty$ and $\mu \geqslant 1$, then $\forall N \in \mathbb{N}_0$ there is a constant C_N, which can be chosen independently of b and μ s.th.*

$$\left| \int e^{i\mu^2 q(y)} b(\mu y)\chi(y)\, dy \right| \leqslant C_N\, \mu^{-N} \sup_{\substack{y \in \operatorname{supp}\chi \\ |\alpha| \leqslant N}} |\partial^\alpha b(\mu y)|$$

(We can find a PDO of first order $\mathcal{L}$ with smooth coefficients on $\operatorname{supp}\chi$, s.th. $\mathcal{L}q \equiv 1$. Therefore the integral on the left-hand side remains unchanged, if we apply the operator $(i\mu^2)^{-N}\mathcal{L}^N$ on the exponential. The estimate then follows after performing N integrations by parts and using the product rule.)

Auxiliary lemma 1. *Let $\Omega \subset \mathbb{R}^k$ be open with $0 \in \overline{\Omega}$ and let $(\gamma_j)_{j\in\mathbb{N}_0}$ be a seq. of functions on Ω s.th. $\exists\, g \in C(\mathbb{R}^k)$ and $C \geqslant 0$ with $|\gamma_j(\eta)-\gamma_{j-1}(\eta)| \leqslant C g(\eta) 2^{-j}$ for all $\eta \in \Omega$ and $j \in \mathbb{N}$. If we set $\gamma(\eta) := \lim_{j\to\infty}\gamma_j(\eta)$ $\forall \eta \in \Omega$, then*

$$|\gamma(\eta) - \gamma_j(\eta)| \leqslant C\, g(\eta)\, 2^{-j} \qquad \forall \eta \in \Omega, j \in \mathbb{N}_0$$

and therefore it holds:

(i) $|\gamma(\eta)| \leqslant C\, g(\eta) + |\gamma_0(\eta)|$

(ii) $|\gamma(\eta) - \gamma_j(\eta)| \xrightarrow{j\to 0} 0$ *unif. in η on each compact set $\subset \Omega$.*

(The statements (i)-(iii) are direct consequences of the estimate on $|\gamma(\eta) - \gamma_j(\eta)|$, which follows from the assumption on γ_j after using a telescoping sum and the geometric series formula.)

Auxiliary lemma 2. *Let $(f_j)_{j\in\mathbb{N}}$ be a seq. in $C^\infty(\mathbb{R}^n)$ that converges pointwise to some f and assume that each $x \in \mathbb{R}^n$ has a neighbourhood U, s.th. $\partial^\alpha f_j \to h$ unif. on U as $j \to \infty$ (for some $\alpha \in \mathbb{N}_0^n$ and $h : \mathbb{R}^n \to \mathbb{C}$). Then it follows $\partial^\alpha f = h$, i.e. for all $x \in \mathbb{R}^n$ it holds*

$$\partial^\alpha \lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} \partial^\alpha f_j(x)$$

(The proof is elementary and similar versions can be found in many textbooks on analysis. A proof using the fund. theorem of calculus and Lebesgue dominated convergence can be found in *Giga M., Giga Y., Saal. Nonlinear partial differential equations, 2010.*)

Theorem (def. of oscillatory integrals). *If q is a non-degenerate real quadratic form on $\mathbb{R}^n$, a is an amlitude of any order m and $\varphi \in \mathcal{S}_0 := \{\varphi \in \mathcal{S} | \varphi(0) = 1\}$, then:*

(i) *The following limit exists and is indep. of $\varphi \in \mathcal{S}_0$*

$$\lim_{\varepsilon\to 0} \int e^{iq(x)} a(x)\varphi(\varepsilon x)\, dx \ \ =: I[a]$$

(ii) $\exists\ C_{q,m} \geqslant 0$ *s.th.* $|I[a]| \leqslant C_{q,m} |||a|||_{m+n+1}$.

(iii) *For $a \in \mathcal{L}^1$, $I[a] = \int e^{iq(x)} a(x) dx$.*

(Using the notation $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ and $f_j(x) := f(2^{-j}x)$, where $f \in C_0^\infty$ is fixed s.th. $f|_{B_1} \equiv 1$ and $f|_{\mathbb{R}^n\setminus B_2} \equiv 0$, the claim (i) follows by showing first that the sequence $(I[af_j])_{j\in\mathbb{N}_0}$ is a Cauchy seq. by applying the key lemma to $I[af_j] - I[af_{j-1}]$ and then, that $I[a\varphi_\varepsilon]$ converges to the same limit as $\varepsilon \to 0$ for any $\varphi \in \mathcal{S}_0$. This can be done by applying aux. lemma 1 to $\gamma_j(\varepsilon) := I[a(1-\varphi_\varepsilon)f_j]$. (ii) then just follows from aux. lemma 1.i and (iii) is an easy application of dominated convergence.)

Remark. (i) and (iii) in the previous theorem imply, that we may continue to denote $I[a]$ by $\int e^{iq(x)}a(x)dx$ for any $a \in A^m$ and the following proposition and theorem show that this *oscillatory integral* shares some useful properties with the Lebesgue integral.

Corollary. *If $a, b \in A^m$ and $A \in GL(n, \mathbb{R})$ then*

(i) *Linear change of variables:*

$$\int e^{iq(x)} a(x) dx = \int e^{iq(Ay)} a(Ay) |\det A| dy$$

(ii) *Integration by parts: For any $\alpha \in \mathbb{N}_0^n$*

$$\int e^{iq}\, a\, \partial^\alpha b = (-1)^{|\alpha|} \int \partial^\alpha (e^{iq} a)\ b$$

(It is straightforward to check that all integrals involved are oscillatory. (i) then follows from the usual transformation law and (ii) makes use of integration by parts in the def. of the right-hand side and the Leibniz product rule on $b\varphi_\varepsilon$. After taking the limit $\varepsilon \to 0$, only the term of order ε^0 contributes in the sum.)

Theorem. *Let $a \in A^m(\mathbb{R}^k \times \mathbb{R}^l)$.*

(i) *$\forall \alpha \in \mathbb{N}_0^l$ and $y \in \mathbb{R}^l$, we have $\partial_y^\alpha a(\cdot, y) \in A^m(\mathbb{R}^k)$ and moreover*

$$\partial_y^\alpha I[a(\cdot,y)] = I[\partial_y^\alpha a(\cdot,y)]$$

(ii) *The function $y \mapsto I[a(\cdot,y)]$ is in $A^m(\mathbb{R}^l)$ and for q, r being non-deg. real quadratic forms, it holds*

$$\int e^{ir(y)} I[a(\cdot,y)] dy = \int e^{i(q(x)+r(y))} a(x,y)\, dxdy$$

(The claim (i) follows from aux. lemma 2 after applying aux. lemma 1.ii to $\gamma_j(y) := I[\partial_y^\alpha a(\cdot,y)f_j]$, where once more the key lemma shows the required inequality for γ_j. The first statement in (ii) then follows from aux. lemma 1.i and the equation follows by smuggling in the ordinary integral $\int e^{ir(y)} I[a(\cdot,y)f_j] f_j(y) dy$ in the def. of the left-hand side and then using Fubini's theorem together with the estimate for γ_j established in the proof of (i).)

Source: X. Saint Raymond, *Elementary introduction to the theory of pseudodifferential operators*, 1991