Bounded operators

In the following, $\mathcal{H}$ always denotes a Hilbert space and X a Banach space.

Adjoint operator on $\mathcal{H}$. For any $T \in \mathcal{L}(\mathcal{H})$, there is a unique operator $T^* \in \mathcal{L}(\mathcal{H})$, s.t. $(x, Ty) = (T^*x, y)$ for all $x, y \in \mathcal{H}$ and $\|T\| = \|T^*\|$.

Sketch: Use Riesz representation theorem on the functional $(x, T\cdot)$. For the norm, use $\|T\| = \sup_{\|x\|,\|y\|\leqslant 1} |(x, Ty)|$.

(i) T is called *self-adjoint* if $T^* = T$.
(ii) $(T^*)^* = T$
(iii) If T^{-1} exists, then $(T^{-1})^* = (T^*)^{-1}$.
(iv) $\|T^*T\| = \|T\|^2$

Adjoint operator on Banach spaces. Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$, then there is a unique operator $T' \in \mathcal{L}(Y^*, X^*)$ s.t. $(T'l)(x) = l(Tx)$.

Projection theorem. For any subspace $M \subset \mathcal{H}$ of a Hilbert space, we have $\mathcal{H} = \overline{M} \oplus M^\perp$.

Misc on bounded operators.

(i) A is known, if $(\varphi, A\psi)$ $\forall \varphi, \psi \in \mathcal{H}$ or $(\psi, A\psi)$ $\forall \psi \in \mathcal{H}$ is known (Riesz and polarization).
(ii) $\|A\| = \sup_{\|\varphi\|=\|\psi\|=1} |(\varphi, A\psi)|$ (Riesz on $\|Ax\|$)
(iii) $A = A^* \Leftrightarrow (\psi, A\psi) = (A\psi, \psi) \quad \forall \psi \in \mathcal{H}$
(iv) $\|A\| = \sup_{\|\psi\|=1} |(\psi, A\psi)|$ for $A = A^*$.
(v) $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$
(vi) $\mathcal{H} = \overline{\operatorname{Ran} A} \oplus \operatorname{Ker} A^*$ (using the projection theorem)

Orthogonal Projections. $P \in \mathcal{L}(\mathcal{H})$ is called *orthogonal projection*, if $P^2 = P$ and $P = P^*$.

(i) $P|_{\operatorname{Ran} P} = id|_{\operatorname{Ran} P}$ and $P|_{(\operatorname{Ran} P)^\perp} = 0$.
(ii) $\operatorname{Ran} P$ is closed, hence $\mathcal{H} = \operatorname{Ran} P \oplus \operatorname{Ker} P$.
(iii) $\|P\| = 1$, $P \geqslant 0$.
(iv) For P_1, P_2 orth. projections, $P_1 \leqslant P_2 \Leftrightarrow \operatorname{Im} P_1 \subset \operatorname{Im} P_2$ and in this case they commute and $P_2 - P_1$ is also an orthogonal projection.
(v) For P_1, P_2 orth. projections, $\lim_{n\to\infty}(P_1P_2)^n$ exists and is equal to the orth. projection onto $\operatorname{Ran} P_1 \cap \operatorname{Ran} P_2$.
(vi) $\sigma(P) = \{0, 1\}$
(vii) Resolvent: $(\lambda - P)^{-1} = \frac{1}{\lambda-1}P + \frac{1}{\lambda}(\mathbb{1} - P)$

The resolvent set and spectra. Let $T \in \mathcal{L}(X)$, then

$\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda - T$ bijection with bounded inverse$\}$ is called *resolvent set of* T,

$\sigma(T) := \mathbb{C}\backslash\rho(T)$ is called the *spectrum of* T,

$\sigma_p(T) := \{\lambda$ eigenvalue of $T\}$ is the *point spectrum*,

$\sigma_c(T) := \{\lambda$ is not an e.v. and $\operatorname{Ran}(\lambda - T) \neq \mathcal{H}$ but dense$\}$ is called *continuous spectrum*,

$\sigma_r(T) := \{\lambda$ is not an e.v. and $\operatorname{Ran}(\lambda - T)$ is not dense$\}$ is called *residual spectrum*.

For $\lambda \in \rho(T)$ we call $R_\lambda(T) := (\lambda - T)^{-1}$ the *resolvent*.

$r(T) := \sup_{\sigma(T)} |\lambda|$ denotes the *spectral radius* of T.

Properties of the spectrum.

(i) $\rho(T)$ is open, i.e. $\sigma(T)$ is closed.
(ii) $\sigma(T) \neq \emptyset$.
(iii) $r(T) \leqslant \|T\|$ (i.e. $\sigma(T)$ is compact)
(iv) $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (Hadamard)
(v) If $T = T^*$, then $r(T) = \|T\|$ (use $\|T^*T\|^{1/2} = \|T\|$)
(vi) $\sigma(T^*) = \sigma(T)^*$
(vii) If $T = T^*$, then $\sigma(T) \subset \mathbb{R}$, $\sigma_r(T) = \emptyset$ and eigenvectors to different eigenvalues are orthogonal.
(viii) If $T = T^*$ then $\lambda \in \sigma(T) \Leftrightarrow \exists\ \{\psi_n\}_{n\in\mathbb{N}}$ with $\|\psi_n\| = 1$ s.t. $\|T\psi_n - \lambda\psi_n\| \to 0$.
(ix) If $T = T^*$, then $\inf \sigma(T) = \inf_{\|\psi\|=1}(\psi, T\psi)$
(x) If $A \in \mathcal{L}(\mathcal{H})$ is invertible, then $\sigma(A^{-1})\backslash\{0\} = (\sigma(A)\backslash\{0\})^{-1}$ and $A\psi = \lambda\psi$ iff $A^{-1}\psi = \lambda^{-1}\psi$.
(xi) If A, B commute, then $r(A + B) \leqslant r(A) + r(B)$ and also $r(AB) \leqslant r(A)r(B)$
(xii) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$
(xiii) $\sigma(A^*A)\backslash\{0\} = \sigma(AA^*)\backslash\{0\}$
(xiv) If $A^* = A^{-1}$ (unitary), then $\sigma(A) \subset \{|\lambda| = 1\}$

Results involving resolvents.

(i) $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$ (*resolvent identity*, trivial)
(ii) $R_\lambda R_\mu = R_\mu R_\lambda$
(iii) $R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S - T)R_\lambda(S)$
(iv) $R_\lambda(T) - R_\lambda(S) = R_\lambda(S)(S - T)R_\lambda(T)$
(v) $R_\lambda = \sum_{n=0}^\infty (\lambda - \lambda_0)^n (R_{\lambda_0})^{n+1}$ for $|\lambda - \lambda_0| < \|R_{\lambda_0}\|^{-1}$
(vi) $R_\lambda(T) = \sum_{n=0}^\infty \lambda^{-1-n}\, T^n$ for $|\lambda| > \|T\|$ (Neumann)
(vii) $\|R_\lambda(T)\| \geqslant \operatorname{dist}(\lambda, \sigma(T))^{-1}$
(viii) If $T = T^*$, then $\|R_\lambda(T)\| \leqslant |\mathfrak{Im}(\lambda)|^{-1}$

Compact operators. $T \in \mathcal{L}(X, Y)$ is compact, if and only if one of the following properties hold:

(1) $\overline{T(B)}$ is compact for all bounded $B \subset X$,
(2) Each bounded sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$, s.t. Tx_{n_k} converges.

The following is true:

(i) T compact $\Rightarrow$ if $x_n \rightharpoonup x$, then $Tx_n \to Tx$ in norm.
(ii) If $\{T_n\}$ are compact operators and $\|T_n - T\| \to 0$, then T is compact.
(iii) T compact $\Leftrightarrow$ T' compact.
(iv) The set of compact operators forms an ideal in $\mathcal{L}(X)$, i.e. if T is compact and $S \in \mathcal{L}(X)$, then ST and TS are compact.
(v) Any finite rank operator is compact (in particular any functional $\phi \in X^*$).
(vi) If $T \in \mathcal{L}(\mathcal{H})$ is compact, then the sequence $\{T_N\}$ of finite rank operators defined by $T_N := \sum_{k=1}^N (\varphi_k, \cdot\,)\, T\varphi_k$ converges to T (in operator norm).
(vii) On $C[0, 1]$, the Arzela-Ascoli theorem provides a nice framework to prove compactness: If for any bounded set $A \subset C[0, 1]$, we have that $\{Tf \mid f \in A\}$ is unif. bounded (clear if T is bounded) and unif. equicontinuous, then T is compact. E.g. $Tf = \int k(\cdot, y)f(y)dy$ on $C[0, 1]$ is compact if $k \in C([0, 1] \times [0, 1])$.
(viii) If $T = V(x)U(-i\nabla)$ with U and V bounded and decaying at ∞, then T is compact. (here $U(-i\nabla)$ is def. via FT: $(U(-i\nabla) := \mathcal{F}^{-1}U(\xi)\mathcal{F}$ and $V(x)$ or $U(\xi)$ denote the correspondig multiplication operators)

(ix) Any compact operator $A : X \to Y$ with X being reflexive attains its norm, i.e. $\exists f$ s.t. $\|Af\| = \|A\|$ (follows from $f \mapsto \|Af\|$ being weakly continuous and the unit ball $B \subset X$ being weakly seq. compact applied to an approximating sequence $(x_n) \subset B$ with $\|Ax_n\| \to \|A\|$).

(x) If $T : X \to Y$ is compact with X, Y being infinite dimensional Banach spaces, then T cannot be surjective. (use open mapping theorem and non-compactness of the unit ball)

Multiplication operator. A multiplication operator T with respect to a function V acts as $(T\psi)(x) = V(x)\psi(x)$.

(i) T is bounded iff $V \in L^\infty$.

(ii) T is never compact (unless $V \equiv 0$)

Hilbert-Schmidt integral operator. Let T be an operator on $\mathcal{L}^2(M)$ acting as $Tf(x) = \int_m k(x,y)f(y)\,dy$ with $k \in L^2(M \times M)$, then

(i) T is compact, since it is of Hilbert-Schmidt type.

(ii) $\|T\|_{\mathrm{HS}} = \|k\|_{2\times 2}$

(iii) T^* has kernel $(x,y) \mapsto k(y,x)^*$.

Volterra operator. $V : \mathcal{L}^2[0,1] \to \mathcal{L}^2[0,1]$ with $Vf(x) := \int_0^x f(y)dy$ is called *Volterra integral operator* on $\mathcal{L}^2[0,1]$.

(i) V is compact, since of Hilbert-Schmidt type.

(ii) $(V^n f)(x) = \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy$

(iii) $\|V^n\| \leqslant \frac{1}{(n-1)!}$, hence $r(V) = 0$ and since the spectrum is never empty: $\sigma(V) = \{0\}$.

(iv) V has no eigenvalues (since $\lambda = 0$ is none; if we didn't know the spectrum yet, we could show that $\lambda \neq 0$ cannot be an e.v. by estimating the square of the eigenvalue equation and integrating).

(v) $\|V\| = \|V^*V\|^{1/2} = r(V^*V)^{1/2} = \frac{2}{\pi}$ (V^*V is s.a. and compact; eigenvalue eqn. can be solved by an ODE)

(vi) $\sigma(\mathbb{1} + V) = \{1\}$ and thus also $\sigma((\mathbb{1} + V)^{-1}) = \{1\}$. Moreover $\|(\mathbb{1} + V)^{-1}\| = 1$.

Spectral properties of compact operators.

(i) If $T = T^*$ is compact, then either $-\|T\|$ or $\|T\|$ is an eigenvalue.

(ii) *Fredholm alternative:* If $A \in \mathcal{L}(\mathcal{H})$ is compact, then either $(\mathbb{1} - A)^{-1}$ exists or $A\psi = \psi$ has nontrivial solution.

(iii) *Riesz-Schauder theorem:* If $A \in \mathcal{L}(\mathcal{H})$ is compact, then $\sigma(A)$ is discrete with only possible accumulation at 0 and any $\lambda \in \sigma(A)$ with $\lambda \neq 0$ is an eigenvalue of finite multiplicity.

(iv) *Hilbert-Schmidt theorem:* If $A = A^*$ is compact on $\mathcal{H}$, then there is a complete ONB $\{\phi_n\}$ of $\mathcal{H}$ and $\{\lambda_n\} \subset \mathbb{R}$, s.t. $A\phi_n = \lambda_n \phi_n$. This gives the *spectral decomposition for compact s.a. operators*

$$A = \sum_n \lambda_n \, |\phi_n\rangle\langle\phi_n|$$

(v) *Singular value decomposition for compact operators:* If $A \in \mathcal{L}(\mathcal{H})$ is compact, then there are two orthonormal systems $\{\varphi_n\}$, $\{\psi_n\}$ and $\lambda_n > 0$ with $\lambda_n \to 0$ s.t.

$$A = \sum_n \lambda_n \, |\varphi_n\rangle\langle\psi_n|$$

Continuous functional calculus for bounded s.a. operators. Let $A \in \mathcal{L}(\mathcal{A})$ be self-adjoint.

(i) For a polynomial P, the operator $P(A)$ is well-defined.

(ii) $\sigma(P(A)) = \{P(\lambda) \,|\, \lambda \in \sigma(A)\}$

(iii) *Key observation:* $\|P(A)\| = \sup_{\sigma(A)} |P(\lambda)|$

(iv) *Weierstrass theorem:* Let $f \in C(K)$ with $K \subset \mathbb{C}$ compact, then there is a sequence $\{P_n\}$ of polynomials s.t. $\|f - P_n\|_\infty \to 0$.

(v) *Main result:* $\exists!$ map $\phi : C(\sigma(A)) \to \mathcal{L}(A)$, $\phi(f) =: f(A)$,

(a) $fg(A) = f(A)g(A)$, $1(A) = \mathbb{1}$, $\overline{f}(A) = f(A)^*$ (i.e. ϕ is a $*$-*algebraic homomorphism*), $id(A) = A$

(b) ϕ is norm preserving, $\|f(A)\| = \|f\|_\infty$

(c) $A\psi = \lambda\psi \Rightarrow f(A)\psi = f(\lambda)\psi$

(d) $\sigma(f(A)) = \{f(\lambda) \,|\, \lambda \in \sigma(A)\}$

(e) $f \geqslant 0 \Rightarrow f(A) \geqslant 0$

(For the proof one uses that all is true for polynomials and then applies Weierstrass to $f \in C(\sigma(A))$. Then one gets a $\|\cdot\|_\infty$-Cauchy sequence $(P_n)_n$ of polynomials, which gives due to the key observation a Cauchy sequence $(P_n(A))_n \subset \mathcal{L}(\mathcal{H})$ and one calls the limit $f(A)$.)

Bounded functional calculus for bounded s.a. operators. Let $\mathcal{B}(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{C}$ with $\sup |f| < \infty$ and let $A \in \mathcal{L}(\mathcal{A})$ be self-adjoint.

(i) *Example:* $f = \mathbb{1}_\Omega \in \mathcal{B}(\mathbb{R})$ with $\Omega \subset \mathbb{R}$ will give projections, since $f^2 = f$ and $\bar{f} = f$.

(ii) *Riesz-Markov theorem:* Let $K \subset X$ be a compact subset of a locally compact Hausdorff space, then for any functional $\phi \in C(K)^*$ $\exists!$ Borel measure μ on K s.t.

$$\phi(f) = \int_K f \, d\mu$$

(iii) Fix $\psi \in \mathcal{H}$ and consider the (positive; i.e. $\phi(f) \geqslant 0$ for f positive) functional $\phi : C(\sigma(A)) \to \mathbb{C}$, $f \mapsto (\psi, f(A)\psi)$ and its Borel measure μ_ψ given by Riesz-Markov, which then is called *spectral measure*. By construction

$$(\psi, f(A)\,\psi) = \int_{\sigma(A)} f(\lambda)\, d\mu_\psi(\lambda)$$

(iv) *Main result:* $\exists!$ map $\varphi : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(A)$, $\varphi(f) =: f(A)$,

(a) φ extends ϕ from the continuous functional calculus

(b) φ is a $*$-algebraic homomorphism and $id(A) = A$

(c) φ is norm continuous, $\|\varphi(f)\| \leqslant \|f\|_\infty$

(d) If $f_n(x) \to f(x)$ pointwise and $\sup_n \|f_n\|_\infty < \infty$, then $f_n(A)\psi \to f(A)\psi$ in $\mathcal{H}$ for all $\psi \in \mathcal{H}$ (i.e. strongly).

(For the proof, one defines for a given $f \in \mathcal{B}(\mathbb{R})$ and $\psi \in \mathcal{H}$, $(\psi, f(A), \psi) := \int f \, d\mu_\psi$ (integration over $\sigma(A)$). The spectral measure μ_ψ was defined in (iii) using continuous functions. One can show, that this defines a quadratic form and by polarization one gets $(\psi, f(A)\psi')$ for all $\psi, \psi' \in \mathcal{H}$. Then using Riesz theorem, one can extract the action of $f(A)$ and check its properties.)

Unbounded operators

In the following $\mathcal{H}$ denotes a separable Hilbert space and $A : \mathcal{D}(A) \to \mathcal{H}$ a linear map with domain $\mathcal{D}(A) \subset \mathcal{H}$. If nothing else is said, $\mathcal{D}(A)$ is always assumed to be *dense*.

Symmetric operator. A is called *symmetric*, if $(g, Af) = (Ag, f)$ for all $f, g \in \mathcal{D}(A)$ (and $\mathcal{D}(A)$ is dense).

Quadratic form of A**.** The *quadratic form* q_A *associated with* A is defined on $\mathcal{D}(A)$ by $q_A(f) := (f, Af)$.

(i) q_A determines A

Sketch: By polarization one gets (g, Af) for all $f, g \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense, using a corollary of Riesz theorem (any vector is char. by its scalar products with vectors from a dense subset) one finds Af.

(ii) A is symmetric $\Leftrightarrow q_A(f) \in \mathbb{R}$ for all $f \in \mathcal{D}(A)$.

For $\Leftarrow$, taking the imaginary part of $q_A(\psi + i\varphi) = q_A(\psi) + q_A(\varphi) + i((\psi, A\varphi) - (\varphi, A\psi))$ gives $Re(\psi, A\varphi) = Re(\varphi, A\psi)$ and the same with φ replaced by $i\varphi$ gives the rest.

Operator relations. Let A and B be unbounded operators on $\mathcal{H}$, then we write

(i) $A = B \;:\Leftrightarrow \mathcal{D}(A) = \mathcal{D}(B)$ and $A\psi = B\psi \;\forall \psi \in \mathcal{D}(A)$

(ii) $A \subset B \;:\Leftrightarrow \mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$.

Hellinger-Toeplitz theorem. If A is symmetric with $\mathcal{D}(A) = \mathcal{H}$, then A is bounded (i.e. if A is symmetric and unbounded, then it cannot be extended to the whole $\mathcal{H}$)

Adjoint operator. For densely defined $A : \mathcal{D}(A) \to \mathcal{H}$, $\mathcal{D}(A^*) := \{g \in \mathcal{H} \,|\, \exists \tilde{g} \in \mathcal{H} : (g, Af) = (\tilde{g}, f) \;\forall f \in \mathcal{D}(A)\}$ and for $g \in \mathcal{D}(A^*)$, set $A^*g := \tilde{g}$.

(i) *Alt.:* $\mathcal{D}(A^*) = \{g \in \mathcal{H} \,|\, (g, A\,\cdot)$ bdd. functional on $\mathcal{D}(A)\}$ and for $g \in \mathcal{D}(A^*)$, by Riesz $\exists!\, \tilde{g}$ with $(\tilde{g}, \cdot) = (g, A\,\cdot)$, then $A^*g := \tilde{g}$.

(ii) *Upshot*: $(\psi, A\varphi) = (A^*\psi, \varphi) \;\; \forall \varphi \in \mathcal{D}(A), \forall \psi \in \mathcal{D}(A^*)$.

(iii) If A is symmetric, then $A \subset A^*$

(iv) $(\alpha A)^* = \bar{\alpha} A^*$ for all $\alpha \in \mathbb{C}$.

(v) If $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(A+B) := \mathcal{D}(A) \cap \mathcal{D}(B)$ are dense, then $A^* + B^* \subset (A + B)^*$.

(vi) $\operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$

(vii) If A is injective and $\operatorname{Ran} A$ dense, then $(A^{-1})^* = (A^*)^{-1}$

(viii) $A \subset B \Rightarrow B^* \subset A^*$.

Self-adjoint operator. A is *self-adjoint*, if $A = A^*$.

(i) If A is s.a., then it is a *maximal symmetric operator* in the sense that if $A \subset B$ with symmetric B, then $A = B$, since $B \subset B^* \subset A^* = A$.

(ii) *Criterium for self-adjointness:* Let A be symmetric. If for some $z \in \mathbb{C}$, $\operatorname{Ran}(A + z) = \operatorname{Ran}(A + \bar{z}) = \mathcal{H}$, then A is self-adjoint.

(iii) If A is injective and self-adjoint, then A^{-1} is self-adjoint ($\{0\} = \operatorname{Ker} A = \operatorname{Ker} A^* = (\operatorname{Ran} A)^\perp$ and (vii) above)

The second adjoint. If $\mathcal{D}(A^*)$ is dense (e.g. if A is symmetric), we may define $A^{**} := (A^*)^*$. It follows

(i) $A \subset A^{**}$ always and for A being symmetric: $A^{**} \subset A^*$, hence in the symmetric case: $A \subset A^{**} \subset A^*$.

(ii) For A symmetric, A^{**} is also symmetric (follows from $A^{**} \subset A^*$), i.e. A^{**} is a *natural sym. extension* of A.

(iii) *Examples show:* A^{**} is an extension of A, which finds the *right* smoothness class (w/o touching b.c.).

Self-adjoint extensions of symmetric operators. Let A be symmetric, then we distinguish the cases

(I) If A^{**} is s.a., then A is called *essentially self-adjoint* and A^{**} is the natural s.a. extension of A.

(II) If A^{**} is not s.a., then $\mathcal{D}(A^*) = \mathcal{D}(A) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$, where $\mathcal{D}_\pm := \operatorname{Ker}(A^* \mp i)$ are called *deficiency spaces* and $n_\pm := \dim \mathcal{D}_\pm$ *deficiency indices.* Furthermore, if $n_+ = n_-$, find all isometries $S : \mathcal{D}_+ \to \mathcal{D}_-$ and set

$$\mathcal{D}_0^S := \{v + Sv \,|\, v \in \mathcal{D}_+\}\,, \;\; \mathcal{D}^S := \mathcal{D}(A) \oplus \mathcal{D}_0^S$$

Then *all self-adjoint extensions of* A are obtained by $A^S := A^*|_{\mathcal{D}^S}$. If $n_+ \neq n_-$, then A has *no* s.a. extension.

Rm *Von Neumann theorem.* If there is a conjugation $C : \mathcal{H} \to \mathcal{H}$ (i.e. C is antilinear, norm-preserving and $C^2 = \mathbb{1}$) with the properties $C(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and $AC = CA$, then A has equal deficiency indices and therefore admits self-adjoint extensions

Graph of an operator. For $A : \mathcal{D}(A) \to \mathcal{H}$, we define its *graph* by

$$\Gamma(A) := \{(\psi, A\psi) \in \mathcal{H} \times \mathcal{H} \,|\, \psi \in \mathcal{D}(A)\}$$

(which is a subspace of the Hilbert space $\mathcal{H} \times \mathcal{H}$ with natural inner product $((\cdot,\cdot),(\cdot,\cdot))_{\mathcal{H}\times\mathcal{H}} := (\cdot,\cdot)_\mathcal{H} + (\cdot,\cdot)_\mathcal{H}$)

Closure of an operator. A is called *closable*, if for all sequences $\{\psi_n\} \subset \mathcal{D}(A)$ with $\psi_n \to 0$ and $A\psi_n \to \varphi$ it follows $\varphi = 0$. Now, if A is closable, we define its *closure* $\bar{A}$ by

$$\Gamma(\bar{A}) := \overline{\Gamma(A)}^{\mathcal{H}\times\mathcal{H}}$$

(the closability of A guarantees, that sequences in $\Gamma(A)$, that converge in $\mathcal{H} \times \mathcal{H}$ converge to points of the graph of an operator)

(i) A is called *closed*, if $\bar{A} = A$ (i.e. if $\Gamma(A)$ is closed).

(ii) $\bar{A}$ is the *smallest closed extension* of A.

(iii) A^* is always closed.

(iv) A closable $\Leftrightarrow \mathcal{D}(A^*)$ dense

(v) $\bar{A} = A^{**}$ and $(\bar{A})^* = A^*$ (if one of the eq. in (iv) hold)

(vi) If A is closable and $\bar{A}$ injective, then $(\bar{A})^{-1} = \overline{A^{-1}}$

(proof of (iii)-(vi) relies heavily on $M^\perp = \overline{M}^\perp$ and $M^{\perp\perp} = \overline{M}$)

Criterum for essential self-adjointness. Let A be symmetric, then $\bar{A}$ is s.a. (i.e. A^{**} s.a., i.e. A ess. s.a.) if and only if one of the two equiv. conditions hold for some $z \in \mathbb{C}\backslash\mathbb{R}$

(1) $\overline{\operatorname{Ran}(A + z)} = \overline{\operatorname{Ran}(A + z^*)} = \mathcal{H}$

(2) $\operatorname{Ker}(A^* + z) = \operatorname{Ker}(A^* + z^*) = \{0\}$

Closed graph theorem. If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is linear, with $\mathcal{D}(A) = \mathcal{H}_1$, then: A bounded $\Leftrightarrow$ $\Gamma(A)$ closed.

Friedrichs extension ($A \Rightarrow q_A$). Let A be symmetric and semibounded from below, i.e. $A \geqslant \gamma$ for some $\gamma \in \mathbb{R}$.

(i) $(\cdot,\cdot)_A := (\,\cdot\,, A\,\cdot\,) + (1-\gamma)\,(\cdot,\cdot)$ defines a new inner product on $\mathcal{D}(A)$ with $\|\cdot\|_A \geqslant \|\cdot\|$.

(ii) Let $\mathcal{Q}(A)$ denote the completion of $\mathcal{D}(A)$ w.r.t. $(\cdot,\cdot)_A$, called the *form domain*. A small argument shows, that $\mathcal{Q}(A) \subset \mathcal{H}$.

(iii) Define $q_A(\psi) := (\psi,\psi)_A - (1-\gamma)\,(\psi,\psi)$ on $\mathcal{Q}(A)$, the *quadratic form of A* ($\Rightarrow q_A(\psi) = (\psi, A\psi)$ on $\mathcal{D}(A)$).

(iv) *Friedrichs extension:* For A as above, there is a *unique* extension $\tilde{A}$, with $\tilde{A} \geqslant \gamma$ and $\mathcal{D}(\tilde{A}) \subset \mathcal{Q}(A)$. (Moreover, $\mathcal{D}(\tilde{A}) = \{g \in \mathcal{Q}(A) \,|\, \exists \tilde{g} : (f,g)_A = (f,\tilde{g})\ \forall f \in \mathcal{Q}(A)\}$ and for $g \in \mathcal{D}(\tilde{A})$, $\tilde{A}g = \tilde{g} - (1-\gamma)g$)

Friedrichs extension via quadratic forms ($q \Rightarrow A$).

(i) Let $\mathcal{Q} \subset \mathcal{H}$ be a dense subspace. A map $q : \mathcal{Q} \to \mathbb{C}$ with is called *quadratic form*, if $q(\psi) = s(\psi,\psi)$ for some sesquilinear form s.

(ii) Given q, we can recover s by polarization.

(iii) If $s(f,g) = \overline{s(g,f)}$, then s and q are called *hermitian*, which is equivalent to q being real.

(iv) If q is hermitian, then it is called *semibounded*, if $\exists\,\gamma \in \mathbb{R}$ s.t. $q(\psi) \geqslant \gamma\,\|\psi\|^2\ \forall \psi \in \mathcal{Q}$.

(v) Given $q \geqslant \gamma$, we call $\|\cdot\|_q := q(\,\cdot\,) + (1-\gamma)\|\cdot\|^2$ *the norm associated with q* (which is stronger than $\|\cdot\|$) and s_q its sequilinear form.

(vi) Let $\mathcal{H}_q$ denote the completion of $\mathcal{Q}$ w.r.t. to $\|\cdot\|_q$.

(vii) q is called *closable*, if $\mathcal{H}_q \subset \mathcal{H}$ ($\Leftrightarrow$ for any $\|\cdot\|_q$-C.seq. $(\psi_n) \subset \mathcal{Q}$ with $\|\psi_n\| \to 0$, we need $\|\psi_n\|_q \to 0$). q is *closed*, if $\mathcal{Q}$ is complete w.r.t. $\|\cdot\|_q$.

(viii) *Theorem:* Let $q \geqslant \gamma$ be closed (or closable and work with the closure), then there is a *unique* s.a. operator $A \geqslant \gamma$, s.t. $\mathcal{Q} = \mathcal{Q}(A)$, $q = q_A$. Moreover $\mathcal{D}(A) = \{g \in \mathcal{H}_q \,|\, \exists \tilde{g} : s_q(f,g) = (f,\tilde{g})\ \forall f \in \mathcal{H}_q\}$ and for $g \in \mathcal{D}(A)$, $Ag = \tilde{g} - (1-\gamma)g$.

Remarks on the two methods.

(i) A symmetric operator is always closable, but it may have no s.a. extensions. In contrast, q may not be closable, but if it is, then there is a s.a. operator A, if additionally q is semibounded.

(ii) Clearly, if we start from a symmetric operator $A \geqslant \gamma$, then q_A defined on $\mathcal{Q}(A)$ from the first method can be used in the second method. By construction: $\|\cdot\|_{q_A} = \|\cdot\|_A$, $\mathcal{H}_{q_A} = \mathcal{Q}(A)$ and the s.a. extension $\tilde{A}$ coincides with the operator A in (viii) of the second method.

Resolvents of unbounded operators. Let A be densely defined and closed.

(i) $\rho(A) := \{z \in \mathbb{C} \,|\, A - z \text{ bijection on } \mathcal{D}(A), (A-z)^{-1} \text{ bdd}\}$ is the *resolvent set of A*, where the second condition is superfluous by the closed graph theorem.

(ii) $\sigma(A) := \mathbb{C} \backslash \rho(A)$ is the *spectrum of A*.

(iii) $R_A : \rho(A) \to \mathcal{L}(\mathcal{H}), z \mapsto (A-z)^{-1}$ is the *resolvent of A*.

(iv) $((A-z)^{-1})^* = (A^* - \bar{z})^{-1}$

(v) $R_A(z) = R_A(z') + (z-z')R_A(z)R_A(z')$ (*resolvent ident.*)

(vi) $R(z) = \sum_{j=0}^{\infty} (z-z_0)^j R(z_0)^{j+1}$ for $|z - z_0| < \frac{1}{\|R(z_0)\|}$

(vii) From (vi): $\rho(A)$ is open and R_A is analytic.

(viii) $\|R(z)\| \geqslant (\operatorname{dist}(z, \sigma(A)))^{-1}$

(ix) *Weyl sequences:* $z \in \sigma(A)$ if there is $(\psi_n) \subset \mathcal{D}(A)$ with $\|\psi_n\| = 1$ and $\|(A-z)\psi_n\| \to 0$. (*e.g.:* For $-\Delta$ the functions $x \to e^{ipx}$ fulfill the eigenvalue equation with eigenvalues p^2 but are not in $\mathcal{L}^2$, i.e. no eigenfunctions. But the functions $\psi_n := e^{ipx}\varphi(\frac{x}{n})n^{-d/2}$ with $\varphi \in C_0^\infty$ form a Weyl sequence for p^2.)

(x) The converse of (viii) holds if $z \in \partial\sigma(A)$.

(xi) If A is injective, then $\sigma(A^{-1})\backslash\{0\} = (\sigma(A)\backslash\{0\})^{-1}$ and for $z \neq 0$: $A\psi = z\psi \Leftrightarrow A^{-1}\psi = z^{-1}\psi$.

(xii) Let A be symmetric, then $A = A^* \Leftrightarrow \sigma(A) \subset \mathbb{R}$. Also, $A \geqslant \gamma \Leftrightarrow \sigma(A) \subset [\gamma, \infty)$. Moreover $\|R(z)\| \leqslant |\operatorname{Im} z|^{-1}$ and if $A \geqslant \gamma$, then $\|R(\lambda)\| \leqslant \frac{1}{|\lambda - \gamma|}$ for any $\lambda < \gamma$

(xiii) If A is symmetric, then all its eigenvalues are real and eigenfunctions belonging to different eigenvalues are orthogonal.

(xiv) If A is symmetric and has an ONB of eigenvectors, then A is essentially self-adjoint.

Multiplication operator. Let $a : \mathbb{R} \to [0,\infty)$ be a positive function and $A : \mathcal{D}(A) \to \mathcal{L}^2(\mathbb{R}), Af(x) := a(x)f(x)$ the associated *multiplication operator*, then its natural domain is $\mathcal{D}(A) = \{f \in \mathcal{L}^2 \,|\, af \in \mathcal{L}^2\}$.

(i) $\mathcal{D}(A) = \mathcal{L}^2 \quad \Leftrightarrow \quad \|a\|_\infty < \infty$

(ii) $\mathcal{Q}(A) = \{f \in \mathcal{L}^2 \,|\, \sqrt{a}f \in \mathcal{L}^2\}$

More generally, $a : \mathbb{R} \to \mathbb{C}$, then

(i) $R_A(z) = (A-z)^{-1}$ is multiplication by $x \mapsto \frac{1}{a(x)-z}$

(ii) Thus, if for some z, $|\frac{1}{a(x)-z}| \leqslant C$ for μ-a.e. $x \in \mathbb{R}$, then $z \in \rho(A)$ and $\|R_A(z)\| \leqslant C$.

(iii) $\rho(A) = \{z \in \mathbb{C} \,|\, \exists\,\epsilon > 0 \text{ s.t. } \mu(\{x : |a(x) - z| \leqslant \epsilon\}) = 0\}$

(iv) z is an eigenvalue of A, if $\mu(a^{-1}(\{z\})) > 0$ with eigenfunction $\mathbb{1}_{a^{-1}(\{z\})}$

Normal operator. We say, that a densely defined operator A is *normal*, if $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $\|A\psi\| = \|A^*\psi\|$.

Spectral Theorem

Projection valued measure. A map $P : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ on the Borel-σ-algebra $\mathcal{B} = \mathcal{B}(\mathbb{R})$ with the properties

(1) $P(\Omega)$ orth. projection $\forall \Omega \in \mathcal{B}$,
(2) $P(\mathbb{R}) = \mathbb{1}$,
(3) $\Omega = \cup_{n=1}^{\infty} \Omega_n \Rightarrow \sum_{n=1}^{\infty} P(\Omega_n)\psi = P(\Omega)\psi \ \forall \psi \in \mathcal{H}$, for disjoint Borel sets Ω_n (i.e. strong convergence; actually, here weak and strong convergence are equivalent)

is called *projection valued measure* (pvm). Consequences of the definition are

(i) $P(\emptyset) = 0$, $P(\Omega^c) = \mathbb{1} - P(\Omega)$
(ii) $P(\Omega_1 \cap \Omega_2) + P(\Omega_1 \cup \Omega_2) = P(\Omega_1) + P(\Omega_2)$
(iii) $P(\Omega_1)P(\Omega_2) = P(\Omega_2)P(\Omega_1) = P(\Omega_1 \cap \Omega_2)$
(iv) $\Omega_1 \subset \Omega_2 \Rightarrow P(\Omega_1) \leqslant P(\Omega_2)$.

Resolution of the identity. Let P be a pvm, then for $\lambda \in \mathbb{R}$, the orth. projection $P_\lambda := P((-\infty, \lambda])$ is called *resolution of identity.*

(i) $P_{\lambda_1} \leqslant P_{\lambda_2}$ for $\lambda_1 \leqslant \lambda_2$.
(ii) $\lim\limits_{\lambda_n \downarrow \lambda} P_{\lambda_n} = P_\lambda$ (strong limit)
(iii) $\lim\limits_{\lambda \to \infty} P_\lambda = \mathbb{1}$

Spectral measure. We define for any $\psi, \varphi \in \mathcal{H}$ a complex measure $\mu_{\psi,\varphi}$ on $\mathbb{R}$ by $\mu_{\psi,\varphi}(\Omega) := (\psi, P(\Omega)\varphi)$. Then the positive finite measure $\mu_\psi := \mu_{\psi,\psi}$ is called *spectral measure.*

Integral w.r.t. pvm for simple functions. Let f be a simple function, i.e. $f = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{\Omega_j}$ with disjoint sets Ω_j, then define

$$P : f \mapsto P(f) := \sum_{j=1}^{n} \alpha_j P(\Omega_j) =: \int f(\lambda)\, dP(\lambda)$$

(it is easy to check, that $P(f)$ does not depend on the choice of (Ω_j) for a given simple function f)

(i) $P(\mathbb{1}_{\Omega_j}) = P(\Omega_j)$
(ii) $(\varphi, P(f)\psi) = \sum_j \alpha_j \, \mu_{\varphi,\psi}(\Omega_j) = \int f \, d\mu_{\varphi,\psi}$
(iii) $\|P(f)\psi\|^2 = \sum_j |\alpha_j|^2 \mu_\psi(\Omega_j) = \int |f|^2 \, d\mu_\psi$
(iv) From (iii) we find: $\|P(f)\| \leqslant \|f\|_\infty$, i.e. P is a bounded linear map from simple functions to bounded operators.

Integral w.r.t. pvm for bounded Borel functions. Since simple functions are dense in $\mathcal{B}(\mathbb{R})$ (the space of bounded Borel functions on $\mathbb{R}$) w.r.t. $\|\cdot\|_\infty$, by the bounded extension theorem, we can uniquely extend P to $\mathcal{B}(\mathbb{R})$, i.e. we get a bounded linear map

$$P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}), f \mapsto P(f) =: \int f \, dP$$

with properties

(i) $P(\bar{f}) = P(f)^*$
(ii) $P(fg) = P(f)P(g)$
(iii) $(P(g)\varphi, P(f)\psi) = \int \bar{g} f \, d\mu_{\varphi,\psi}$ for $f, g \in \mathcal{B}(\mathbb{R})$, $\varphi, \psi \in \mathcal{H}$
(iv) If $f_n \to f$ converges pointwise and $\sup_n \|f_n\|_\infty < \infty$, then $P(f_n) \to P(f)$ strongly.

Integral for unbounded functions. Let $f : \mathbb{R} \to \mathbb{C}$ be a (probably unbounded) Borel function.

(1) $D_f := \{\psi \in \mathcal{H} \,|\, \int |f|^2 d\mu_\psi < \infty\} \subset \mathcal{H}$ is a dense subspace and serves as domain for $P(f)$.
(2) Define $\Omega_n := \{\lambda \in \mathbb{R} \,|\, |f(\lambda)| \leqslant n\}$ and $f_n := \mathbb{1}_{\Omega_n} f$, then $f_n \in \mathcal{B}(\mathbb{R})$ and by dominated convergence, $f_n \to f$ in $\mathcal{L}^2(\mu_\psi)$ for $\psi \in D_f$.
(3) The action of $P(f)$ on D_f is constructed using (ii): Let $\psi \in D_f$, then for (f_n) as in (ii), $\|P(f_n)\psi - P(f_m)\psi\|^2 = \int |f_n - f_m|^2 d\mu_\psi$ shows, that $(P(f_n)\psi)_n$ is a Cauchy sequence in $\mathcal{H}$ and (as can be verified) we may denote its limit by $P(f)\psi$.

Important properties:

(i) $P(f)^* = P(\bar{f})$
(ii) $\|P(f)\psi\|^2 = \int |f|^2 \, d\mu_\psi$
(iii) $(\varphi, P(f)\psi) = \int f \, d\mu_{\varphi,\psi}$

Cyclic subspaces. Let P be a pvm and $\psi \in \mathcal{H}$, then its *cyclic subspace* is given by $H_\psi := \{P(g)\psi \,|\, g \in \mathcal{L}^2(\mu_\psi)\}$

(i) H_ψ is a closed subspace.
(ii) H_ψ is invariant under $P(f)$ for any $f \in \mathcal{L}^2(\mu_\psi)$, i.e. if P_ψ is the projection onto H_ψ, then $P_\psi P(f) \subset P(f) P_\psi$ (and equality in the case when f is bounded). Thus we can decompose
$$P(f) = P(f)|_{H_\psi} + P(f)|_{H_\psi^\perp}$$
(iii) The isometry $U_\psi : H_\psi \to \mathcal{L}^2(\mu_\psi)$, $P(g)\psi \mapsto g$ satisfies
$$U_\psi P(f)|_{H_\psi} = f \, U_\psi$$
i.e. under U_ψ we can identify $P(f)$ acting on H_ψ as multiplication operator by f
(iv) $\psi \in \mathcal{H}$ is called *cyclic vector*, if $H_\psi = \mathcal{H}$
(v) $\{\psi_j\}$ is called a *spectral set of vectors*, if $H_{\psi_i} \perp H_{\psi_j}$ for $i \neq j$ and it is called *spectral basis*, if $\mathcal{H} = \oplus_j H_{\psi_j}$
(vi) *Theorem:* Let P be a pvm, then $\exists$ a (not unique) spectral basis $\{\psi_j\}$ s.t. $U : \mathcal{H} = \oplus_j H_{\psi_j} \to \oplus_j L^2(\mu_{\psi_j})$ satisfies
$$U \, P(f) = f \, U$$
(vii) The cardinality of a spectral basis is *not* unique, but there can be found a minimal one, which is called *spectral multiplicity of P*.

Borel transform. The *Borel transform* of a (pos.) measure μ on $\mathbb{R}$ is given by $F_\mu(z) := \int \frac{1}{\lambda - z} \, d\mu(\lambda)$ for any $z \in \mathbb{C} \backslash \mathbb{R}$.

(i) $\operatorname{Im} F_\mu(z) = \int \frac{\operatorname{Im} z}{|\lambda - z|^2} \, d\mu(\lambda)$
(ii) F_μ is a *Herglotz function*, i.e. $F_\mu : \mathbb{C}_+ \to \mathbb{C}_+$ is analytic, if $\mu(\mathbb{R}) < \infty$.
(iii) *Stieltjes inversion formula:* If F_μ is the Borel tranform of a measure μ, then
$$\mu((-\infty, \lambda]) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{-\infty}^{\lambda + \delta} \operatorname{Im} F_\mu(t + i\varepsilon) \, dt$$
(iv) If F is a Herglotz function with $|F(z)| \leqslant \frac{M}{\operatorname{Im} z}$ for some $M > 0$ then there is a pos. measure μ on $\mathbb{R}$ with $F = F_\mu$ and $\mu(\mathbb{R}) \leqslant M$.

Assigning a pvm to any s.a. operator. Let $A^* = A$.

(i) For $\psi \in \mathcal{H}$ set $F_\psi(z) := (\psi, R_A(z)\psi)$ for any $z \in \mathbb{C}_+$, then $\operatorname{Im} F_\psi(z) = \operatorname{Im} z \, \|R_A(z)\|^2 > 0$ (here we use that A is s.a.) so $F_\psi : \mathbb{C}_+ \to \mathbb{C}_+$; and it is analytic, since R_A is analytic on $\rho(A) \supset \mathbb{C}_+$. Hence we found: F_ψ is a Herglotz function. Moreover, from $\|R_A(z)\| \leqslant (\operatorname{Im} z)^{-1}$, it follows
$$|F_\psi(z)| \leqslant \frac{\|\psi\|^2}{\operatorname{Im} z}$$

(ii) From above: $\exists\, \mu_\psi$ s.t. $\mu_\psi(\mathbb{R}) \leqslant \|\psi\|^2$ and $F_{\mu_\psi} = F_\psi$, i.e.
$$(\psi, R_A(z)\psi) = \int \frac{1}{\lambda - z} \, d\mu_\psi(\lambda)$$
Also, by polarization, we can construct a complex measure $\mu_{\varphi,\psi}$ out of μ_ψ, s.t.
$$(\varphi, R_A(z)\psi) = \int \frac{1}{\lambda - z} \, d\mu_{\varphi,\psi}(\lambda)$$

(iii) For $\Omega \subset \mathbb{R}$, define $s_\Omega(\varphi, \psi) = \int \mathbb{1}_\Omega \, d\mu_{\varphi,\psi}$, which is sesquilinear and bounded (linearity from the identity above, and boundedness from the construction of $\mu_{\varphi,\psi}$ out of the finite measure μ_ψ). Thus, a corollary to Riesz theorem shows that s_Ω is the sesquilinear form of a unique bounded operator $P(\Omega)$, i.e.
$$\mu_{\varphi,\psi}(\Omega) = \int \mathbb{1}_\Omega \, d\mu_{\varphi,\psi} = (\varphi, P(\Omega)\psi)$$

(iv) $P_A := P$ *defined above is a pvm.*
(The proof boils down to $P(\Omega_1)P(\Omega_2) = P(\Omega_1 \cap \Omega_2)$, which is shown by first identifying $d\mu_{R_A(\bar{z})\varphi,\psi} = (\lambda - z)^{-1} d\mu_{\varphi,\psi}$ and $d\mu_{\varphi,P(\Omega)\psi} = \mathbb{1}_\Omega \, d\mu_{\varphi,\psi}$ using uniqueness under Borel transform and then $\mu_{\varphi,P(\Omega_1)P(\Omega_2)\psi} = \mu_{\varphi,P(\Omega_1\cap\Omega_2)}$)

(v) From $(\varphi, P_A(f)\psi) = \int f \, d\mu_{\varphi,\psi}$ together with (ii), the boundedness of $R_A(z)$ and Riesz theorem follows
$$P_A((\lambda - z)^{-1}) = R_A(z)$$
Now, from $P(\lambda) = P((\lambda - z)^{-1})^{-1} + z = A$ and uniqueness of each of the above steps, it follows the main goal of our lecture:

Spectral theorem for s.a. unbounded operators. *Let A be a self-adjoint operator on $\mathcal{H}$, then there exists a unique projection valued measure P_A s.t.*
$$A = \int \lambda \, dP_A(\lambda)$$

Consequences of the spectral theorem.

(i) $\mathcal{D}(A) = \{\psi \in \mathcal{H} : \int \lambda^2 \, d\mu_\psi < \infty\}$

(ii) $\sigma(A) = \{\lambda \in \mathbb{R} \,|\, P_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \ \forall \varepsilon > 0\}$

(iii) $P_A(\sigma(A)) = \mathbb{1}$ and equivalently $P_A(\rho(A) \cap \mathbb{R}) = 0$

(iv) $P_A(f)$ is determined by $f|_{\sigma(A)}$.

Spectral types and quantum dynamics

Absolutely continuous and singular measures. A measure μ on $\mathbb{R}$ is called *absolutely continuous*, if $\mu(A) = 0$ for every $A \subset \mathbb{R}$ with Lebesgue measure 0 and *singular*, if $\mu(\mathbb{R} \backslash B) = 0$ for some set B of Lebesgue measure 0.

Radon-Nikodym theorem. Any measure μ on $\mathbb{R}$ can be decomposed in $\mu = \mu_{ac} + \mu_s$ with μ_{ac} being absolutely continuous, μ_s being singular and $\exists f \in \mathcal{L}^1$ s.t. $\mu_{ac}(\Omega) = \int_\Omega f \, dx$. Furthermore $\mu_s = \mu_{sc} + \mu_{pp}$ where μ_{pp} is a *pure-point measure*, i.e. $\mu_{pp} = \sum_j c_j \delta_{\lambda_j}$ with the property $\mu_s\{\lambda_j\} = \mu_{pp}\{\lambda_j\}$, i.e. $c_j = \mu_s\{\lambda_j\}$ and $\mu_{sc} := \mu_s - \mu_{pp}$.

Riemann-Lebesgue Lemma. If μ is absolutely continuous, then $\int e^{-it\lambda} \, d\mu(\lambda) \to 0$.

Decomposition of $\mathcal{H}$. For a given $\mu = \mu_a c + \mu_{sc} + \mu_{pp}$, we can decompose $\mathbb{R} = M^{ac} \cup M^{sc} \cup M^{pp}$ with $M^\sharp$ being the support of $\mu^\sharp$, i.e. $\mu^\sharp = \mu|_{M^\sharp}$. Now, if $A = A^*$, then $\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^{sc} \oplus \mathcal{H}^{pp}$ where $\mathcal{H}^\sharp := \{\psi \in \mathcal{H} \,|\, \mu_\psi \text{ is } \sharp\}$ and A leaves $H^\sharp$ invariant.

Strongly continuous unitary group. A group of unitary operators, $\{U(t)\}_{t \in \mathbb{R}}$, is called *strongly continuous unitary group on $\mathcal{H}$*, if $\lim_{t \to t_0} U(t)\psi = U(t_0)\psi$ for any $\psi \in \mathcal{H}$ and $t_0 \in \mathbb{R}$. Its *generator* A is defined by
$$A\psi := \lim_{t \to 0} \frac{i}{t}(U(t)\psi - \psi)$$
with $\mathcal{D}(A) := \{\psi \in \mathcal{H} \,|\, \text{the above limit exists}\}$.

Self-adjoint generator. For any $A = A^*$, $U(t) = e^{-itA}$ form a strongly continuous unitary group with generator A (i.e. the limit above exists if and only if $\psi \in \mathcal{D}(A)$ and coincides with $A\psi$). Moreover $U(t)\mathcal{D}(A) = \mathcal{D}(A)$ and $U(t)A = AU(t)$.

Stone theorem. If $U(t)$ is a strongly continuous unitary group, then its generator A is self-adjoint and
$$U(t) = e^{-itA}$$

Wiener theorem. If μ is a finite (complex) measure on $\mathbb{R}$ and $\hat{\mu}(t) := \int e^{-it\lambda} \, d\mu(\lambda)$, then
$$\frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 \, dt \ \overset{T\to\infty}{\longrightarrow} \ \sum_{\lambda \in \mathbb{R}} |\mu\{\lambda\}|^2$$
(this is called *convergence in Cesaro-average sense*)

Application of Wiener's theorem. For $A = A^*$, let $\psi \in \mathcal{H}^{ac} \oplus \mathcal{H}^{sc}$, i.e. μ_ψ is continuous, then $\mu_{\varphi,\psi}$ is continuous for any φ (since $\|P_A\{\lambda\}\psi\|^2 = \mu_\psi\{\lambda\} = 0$ and therefore $\mu_{\varphi,\psi}\{\lambda\} = (\varphi, P_A\{\lambda\}\psi) = 0$). Moreover, we have $(\varphi, U(t)\psi) = (\varphi, P_A(e^{-it\lambda})\psi) = \hat{\mu}_{\varphi,\psi}(t)$ and therefore by Wiener's theorem, $(\varphi, U(t)\psi)$ converges to 0 in Cesaro-average sense.

Kato-Rellich type theorems

A-boundedness. If A, B are operators on $\mathcal{H}$, then we say that B *is A-bounded*, if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $\exists\, a, b > 0$ s.t. $\|B\psi\| \leqslant a\|A\psi\| + b\|\psi\|$ for all $\psi \in \mathcal{D}(A)$. The smallest possible a is called *A-bound of B*.

Kato-Rellich theorem. If A is (ess.) self-adjoint, B is symmetric and B is A-bounded with a bound $a < 1$, then $A + B$ is (ess.) self-adjoint on $\mathcal{D}(A + B) = \mathcal{D}(A)$.

Relatively compact operators. B is called *relatively compact* with respect to A, if $BR_A(z)$ is compact for some $z \in \rho(A)$ ($\Leftrightarrow \forall\, z \in \rho(A)$ for connected $\rho(A)$).

Lemma. Let $A = A^*$ and B be relatively compact w.r.t. A, then B is A-bounded with an arbitrarily small bound, i.e. $\forall \varepsilon > 0\, \exists b_\varepsilon$ with $\|B\psi\| \leqslant \varepsilon\|A\psi\| + b_\varepsilon\|\psi\|$.

Discrete and essential spectrum. Let $A = A^*$, then we define the *discrete spectrum* by

$$\sigma_d(A) := \{\lambda \in \sigma_p(A) \mid \exists \varepsilon : \operatorname{rank} P(\lambda - \varepsilon, \lambda + \varepsilon) < \infty\}$$

i.e. the discrete spectrum consists of σ_p without eigenvalues of infinite multiplicity and accumulation points. The *essential spectrum* is defined as

$$\sigma_{\text{ess}}(A) := \sigma(A) \backslash \sigma_d(A)$$

It immediately follows for compact A, that $\sigma(A) \subset \{0\}$.

Theorem. If A, K are self-adjoint and K is relatively compact w.r.t. A, then $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$.

Lemma. Let $A = A^*$, then $\lambda \in \sigma_{\text{ess}}(A)$ if and only if there is a *singular Weyl sequence* ψ_n for λ, i.e. $\|\psi_n\| = 1$, $\psi_n \rightharpoonup 0$ and $\|(A - \lambda)\psi_n\| \to 0$.