Distributions I

1 Basic definitions

Let $\Omega \subset \mathbb{R}^n$ nonempty and open (for all sections 1-4).

Test functions. The set of *test functions* $D(\Omega)$ is defined as the space $C_c^\infty(\Omega)$ equipped with the following notion of convergence:

$$\phi_n \to \phi \;:\Leftrightarrow \begin{cases} \exists K \subset \Omega \text{ compact, s.t.} \\ \text{(i) } \operatorname{supp}(\phi_n - \phi) \subset K \ \forall n \\ \text{(ii) } D^\alpha \phi_n \to D^\alpha \phi \text{ unif. on K } \forall \alpha \in \mathbb{N}^n. \end{cases}$$

Distributions. A *distribution* is an element of the dual space $D'(\Omega)$, i.e. a continuous linear functional on $D(\Omega)$. We can introduce the concept of *pointwise convergence* on the space of distributions by

$$T_n \to T \quad :\Leftrightarrow \quad T_n(\phi) \to T(\phi) \quad \forall \phi \in D(\Omega)\,.$$

Locally integrable functions. For $1 \le p \le \infty$ we define the space of *locally integrable functions* by

$$L^p_{\text{loc}}(\Omega) := \left\{ \begin{array}{c} f : \Omega \to \mathbb{C} \\ \text{Borel msrbl.} \end{array} \;\middle|\; \begin{array}{c} \|f\|_{L^p(K)} < \infty \\ \forall \text{ compact } K \subset \Omega \end{array} \right\}.$$

Remarks.
(R1) $L^p_{\text{loc}}(\Omega)$ is a vector space, but not equipped with a simple norm.
(R2) The concepts of *strong and weak convergence* in L^p-spaces can be transferred to $L^p_{\text{loc}}(\Omega)$: $(f_n) \subset L^p_{\text{loc}}(\Omega)$ converges strongly (weakly) to $f \in L^p_{\text{loc}}(\Omega)$ iff f_n converges strongly (weakly) to f in $L^p(K)$ for all compact subsets $K \subset \Omega$.
(R3) We have $L^p(\Omega) \subset L^p_{\text{loc}}(\Omega)$, but $L^p_{\text{loc}}(\Omega) \not\subset L^p(\Omega)$.
(R4) For $q > p$, from Hölder's inequality : $L^q_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$.

2 Functions as distributions

Theorem 1. Let $f \in L^1_{\text{loc}}(\Omega)$, then $T_f : \ D(\Omega) \to \mathbb{C}$ with $T_f(\phi) := \int_\Omega f\phi$ defines a distribution.

Theorem 2. Let $f, g \in L^1_{\text{loc}}(\Omega)$ with $T_f(\phi) = T_g(\phi)$ for all $\phi \in D(\Omega)$, then $f(x) = g(x)$ for a.e. $x \in \Omega$.

Remark.
(R5) From (R4) and theorems 1 and 2, we see that it makes sense to say that functions in $L^p_{\text{loc}}(\Omega)$ (for any $p \ge 1$) *are* distributions and we write $f(\phi) := T_f(\phi)$.

3 Derivatives of distributions

Weak derivative. Let $T \in D'(\Omega)$ and $\alpha \in \mathbb{N}^n$ (a multi-index), then the *weak derivative* (or *distributional derivative*) of T is the distribution defined by

$$(D^\alpha T)(\phi) := (-1)^{|\alpha|} T(D^\alpha \phi) \qquad \forall \phi \in D(\Omega)\,. \quad (1)$$

Remarks.
(R6) In the case $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$, we have $(\partial_i T)(\phi) = -T(\partial_i \phi)$.
(R7) The symbol $\nabla T := (\partial_1 T, \ldots, \partial_n T)$ denotes the *distributional gradient* of T.
(R8) For a (classically) differentiable function $f \in C^{|\alpha|}(\Omega) \subset L^1_{\text{loc}}(\Omega)$, we have $(D^\alpha T_f)(\phi) = (-1)^{|\alpha|} \int_\Omega D^\alpha \phi \ f = \int_\Omega \phi \ D^\alpha f = T_{D^\alpha f}(\phi)$, by partial integration. In accordance with $(R5)$, we say: The weak derivative of f is the function $D^\alpha f$, that is the distribution with values $(D^\alpha f)(\phi) = T_{D^\alpha f}(\phi)$ for all $\phi \in D(\Omega)$.
(R9) Since $\phi \in C_c^\infty(\Omega)$, we see from (1) that any distribution is infinitely often differentiable.
(R10) The map of taking the derivative $D^\alpha : D'(\Omega) \to D'(\Omega)$ is continuous, i.e. if $T_m \to T$, then $D^\alpha T_m(\phi) \to D^\alpha T(\phi)$ for all $\phi \in D(\Omega)$, because $D^\alpha \phi \in D(\Omega)$.

Example. It is easy to check, that $\delta_y(\phi) := \phi(y)$ defines a distribution. Then we have $(\partial_i \delta_y)(\phi) = -\partial_i \phi(y)$.

$\boldsymbol{W^{1,p}_{\text{loc}}(\Omega)}$. The class of functions in $L^p_{\text{loc}}(\Omega)$ with first weak derivatives that are also functions in $L^p_{\text{loc}}(\Omega)$ is denoted by $W^{1,p}_{\text{loc}}(\Omega)$.

Remarks.
(R11) Explicitly, for $f \in W^{1,p}_{\text{loc}}(\Omega)$ there is a function $g \in L^p_{\text{loc}}(\Omega)$ s.t. $(\partial_i T_f)(\phi) = -\int_\Omega f \partial_i \phi = \int_\Omega g\phi = T_g(\phi)$ for all $\phi \in D(\Omega)$. Thus we write $g := \partial_i f$.
(R12) $W^{1,p}_{\text{loc}}(\Omega)$ is a vector space.
(R13) If $p < q$, we have for each $f \in W^{1,q}_{\text{loc}}(\Omega)$, that $\partial_i f \in L^q_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$. Thus $W^{1,q}_{\text{loc}}(\Omega) \subset W^{1,p}_{\text{loc}}(\Omega)$.

Sobolev spaces. Since $L^p(\Omega) \subset L^p_{\text{loc}}(\Omega)$, it makes sense to speak of the class of functions in $L^p(\Omega)$ with first weak derivatives that are also in $L^p(\Omega)$. This space is denoted by $W^{1,p}(\Omega)$. For the first m weak derivatives to be in $L^p(\Omega)$, the spaces are called *Sobolev spaces* $W^{m,p}(\Omega)$. They can be endowed with several norms, e.g. $\|f\|_{W^{m,p}} := \sum_{|\alpha| \le m} \|D^\alpha f\|_{L^p(\Omega)}$ in order to form Banach spaces.

4 Calculating with distributions

Lemma. Let $\phi \in D(\Omega)$, $z \in \mathbb{R}^n$, $T \in D'(\Omega)$ and define $\phi_z(x) := \phi(x-z)$, $\mathcal{O}_\phi := \{y \in \mathbb{R}^n | \operatorname{supp}\phi_y \subset \Omega\}$. Then
(i) $\mathcal{O}_\phi$ is open and non-empty.
(ii) $y \mapsto T(\phi_y)$ is in $C^\infty(\mathcal{O}_\phi)$ and $D^\alpha_y T(\phi_y) = (D^\alpha T)(\phi_y)$.
(iii) For $\psi \in L^1(\mathcal{O}_\phi)$ with compact support:

$$\int_{\mathcal{O}_\phi} \psi(y) T(\phi_y) dy = T(\psi * \phi)\,. \quad (2)$$

Theorem. The so called *fundamental theorem of calculus for distributions* states the following:
(i) Let $T \in D'(\Omega), \phi \in D(\Omega)$ and $y \in \mathbb{R}^n$ s.t. $\phi_{ty} \in D(\Omega)$ $\forall t \in [0,1]$, then

$$T(\phi_y) - T(\phi) = \int_0^1 \sum_{j=1}^n y_j (\partial_j T)(\phi_{ty})\, dt\,. \quad (3)$$

(ii) For $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ we get $\forall y \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$:

$$f(x+y) - f(x) = \int_0^1 y \cdot \nabla f(x+ty)\, dt\,. \quad (4)$$