Paley-Wiener-Schwartz theorem

A theorem which relates regularity of a function (or distribution) with the behaviour at infinity of its Fourier transform is often referred to as of Paley-Wiener type. We will study two (in a sense) limiting cases: A version of the classical Paley-Wiener theorem for C_0^∞-functions and a version by L. Schwartz for distributions (the *Paley-Wiener-Schwartz theorem*).

Paley-Wiener theorem. *A function U defined on $\mathbb{R}^n$ is the Fourier transform of some $u \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_A$, if and only if U can be extended to $\mathbb{C}^n$ as an entire function satisfying estimates*

$$|U(\zeta)| \leqslant C_N\,(1+|\zeta|^2)^{-N} e^{A|\operatorname{Im}\zeta|} \quad \forall \zeta \in \mathbb{C}^n$$

for all $N \in \mathbb{N}_0$ and some sequence $(C_N)_{N\in\mathbb{N}_0}$.

(The proof of $\Rightarrow$ consists of extending $\hat{u}$ to $\mathbb{C}^n$ by replacing the n real variables by n complex ones, which is then called the Fourier-Laplace transform and which is holomorphic in $\mathbb{C}^n$, since we can differentiate under the integral and $\zeta \mapsto \exp(-i\langle\xi,\zeta\rangle)$ is entire for all $\xi \in \mathbb{R}^n$. The estimates follow from an estimate on $\zeta^\alpha \hat{u}(\zeta)$ for any multi-index α: $|\zeta^\alpha \hat{u}(\zeta)\,| \leqslant \|\partial^\alpha u\|_1 \exp(A|\operatorname{Im}\zeta|)$ which can be established by replacing $\zeta^\alpha \exp(-i\langle\zeta,\xi\rangle)$ with a derivative w.r.t. ξ of order $|\alpha|$ and performing $|\alpha|$ integrations by parts.

For $\Leftarrow$, we define u to be the inverse Fourier transform of U and show $u \in C_0^\infty$ with $\operatorname{supp} u \subset B_A$, again by differentiating under the integral (using one of the estimates) and by applying Cauchy's integral theorem on the entire function $\exp(i\langle x,\cdot\rangle)U$ to shift the integration in the definition of u into the (n-dim.) complex plane in order to get $|u(x)| \leqslant C\exp(|x|(A-|x|)/\varepsilon)$ for all $\varepsilon > 0$, which then shows $\operatorname{supp} u \subset B_A$.)

Some notions and results from distribution theory.

(1) **Def. (*support of a distribution*).** For $u \in \mathcal{S}'$, we define $\operatorname{supp} u \subset \mathbb{R}^n$ by $x \notin \operatorname{supp} u :\Leftrightarrow \exists$ a neighbourhood V_x of x s.th. $u(\phi) = 0$ for all $\phi \in C_0^\infty(V_x)$. The set of temperate distributions with compact support we will denote by $\mathcal{S}'_0$. Here are some simple properties:
   - (i) *If u is a function, then this definition coincides with its support as a function.*
   - (ii) $u(\phi) = 0$, *if* $\operatorname{supp} u \cap \operatorname{supp}\phi = \emptyset$.
   - (iii) *If $\varphi, \psi \in \mathcal{S}$ s.th. $\varphi(x) = \psi(x)$ for all $x \in \operatorname{supp} u$, then $u(\varphi) = u(\psi)$.*

(2) **Theorem.** *Any distribution $u \in \mathcal{S}'$ can be uniquely extended as a (semi-)linear form to the set of C^∞-functions f with $\operatorname{supp} u \cap \operatorname{supp} f$ being compact. In particular, any $u \in \mathcal{S}'_0$ can be applied to $f \in C^\infty(\mathbb{R}^n)$.* (see [2], thm 2.2.5).

(3) **Theorem.** *If $u, v \in \mathcal{S}'_0$, $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ or $u, v \in \mathcal{S}'$, $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then* (see [2], thm 5.1.1)

$$u\Big(x \mapsto \overline{v(f(x,\cdot))}\Big) = v\Big(y \mapsto \overline{u(f(\cdot,y))}\Big)$$

(4) **Corollary (*Fourier transform on $\mathcal{S}'_0$*).** From (2) and (3) it follows: *The Fourier transform of $u \in \mathcal{S}'_0$ is a function on $\mathbb{R}^n$ given by*

$$\hat{u}(\xi) = u(e^{i\langle\cdot,\xi\rangle}) \quad \forall \xi \in \mathbb{R}^n$$

(5) **Lemma (*differentiate inside u*).** As an application of Taylor's theorem we get: *If $u \in \mathcal{S}'_0$, $\varphi \in C^\infty(\mathbb{R}^{2n})$ or $u \in \mathcal{S}'$, $\varphi \in C_0^\infty(\mathbb{R}^{2n})$, then $x \mapsto u(\varphi(x,\cdot)) \in C^\infty(\mathbb{R}^n)$,*

$$\partial_x^\alpha u(\varphi(x,\cdot)) = u(\partial_x^\alpha \varphi(x,\cdot)) \quad \forall x \in \mathbb{R}^n,\ \alpha \in \mathbb{N}_0^n$$

(6) **Def. (*convolution of distributions with functions*).** For $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$ or $u \in \mathcal{S}'_0$, $\phi \in C^\infty(\mathbb{R}^n)$ we define $u*\phi(x) := u(\overline{\phi}(x-\cdot))\ \forall x \in \mathbb{R}^n$, which implies the following properties:
   - (i) *If u is a function, this agrees with the convolution of functions.*
   - (ii) Using (3), we can show: $(u*\phi)*\psi = u*(\phi*\psi)$, *whenever* $u \in \mathcal{S}', \phi \in C_0^\infty, \psi \in \mathcal{S}$.
   - (iii) From (5) it follows: *For u and ϕ as in (5), $u*\phi \in C^\infty$ and $\partial^\alpha(u*\phi) = u*\partial^\alpha\phi$ for any $\alpha \in \mathbb{N}_0^n$*
   - (iv) From $u*\check{\overline{\varphi}}(0) = u(\phi)$ and (ii) it follows: $\widehat{u*\phi} = \hat{u}\,\hat{\phi}$ *whenever $u \in S'$ and $\phi \in C_0^\infty$.*

Paley-Wiener-Schwartz theorem. *A function U on $\mathbb{R}^n$ is the Fourier transform of a distribution $u \in \mathcal{S}'_0$ with $\operatorname{supp} u \subset B_A$, if and only if U can be extended to $\mathbb{C}^n$ as an entire function satisfying an estimate*

$$|U(\zeta)| \leqslant C\,(1+|\zeta|^2)^N e^{A|\operatorname{Im}\zeta|} \quad \forall \zeta \in \mathbb{C}^n$$

for some constants $C, N \geqslant 0$.

($\Rightarrow$: From (4) we know $\hat{u}(\xi) = u(\exp(i\langle\cdot,\zeta\rangle)$ for all $\xi \in \mathbb{R}^n$ and its obvious extension to $\zeta \in \mathbb{C}^n$ forms an entire function, since we can differentiate inside u due to (5). For the estimate we use a smooth cutoff-function $\psi \in C^\infty(\mathbb{R})$, s.th. $\psi(t) = 1$ for $t \leqslant 1/2$ and $\psi(t) = 0$ for all $t \geqslant 1$. Then set $\psi_\zeta(x) := \psi(|\zeta|(|x| - A))$, which has compact support contained in $B_{A+1/|\zeta|}$ and $\psi_\zeta \equiv 1$ on B_A. Hence by (1), (iii) we have the identity $\hat{u}(\zeta) = u(\psi_\zeta \exp(i\langle\cdot,\zeta\rangle))$, from which we get the estimate: The C^∞-function $\hat{u}$ is bounded on $\{|\zeta| \leqslant 1\}$ and for $|\zeta| \geqslant 1$, it follows $\operatorname{supp}\psi_\zeta \subset B_{A+1}$ and $\|\partial^\alpha(\psi_\zeta \exp(i\langle\cdot,\zeta\rangle))\|_\infty \leqslant C'|\zeta|^{|\alpha|} e^{A|\operatorname{Im}\zeta|} | \leqslant C'(1+|\zeta|^2)^{|\alpha|/2}$ for some constant C'. Hence from $|\hat{u}(\zeta)| \leqslant C''|\psi_\zeta \exp(i\langle\cdot,\zeta\rangle)|_N$ follows the desired estimate, i.e. $|\hat{u}(\zeta)| \leqslant C(1+|\zeta|^2)^{N/2} \exp(A|\operatorname{Im}\zeta|)$.

$\Leftarrow$: The estimate for $\xi \in \mathbb{R}^n$ shows that $U \in \mathcal{S}'$. From the Fourier inversion formula in $\mathcal{S}'$ follows that U is the Fourier transform of some $u \in \mathcal{S}'$. For $\operatorname{supp} u \subset B_A$, we define $u_\varepsilon := u * \varphi_\varepsilon$, where $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for any $\varepsilon > 0$ and φ being a unit test function, i.e. $\varphi \in C_0^\infty$, $\operatorname{supp}\varphi \subset B_1$, $\int\varphi = 1$ and $\varphi \geqslant 0$. It is easy to see that $\operatorname{supp}\varphi_\varepsilon \subset B_\varepsilon$, $\int\varphi_\varepsilon = 1$ and $\hat{\varphi}_\varepsilon(\xi) = \hat{\varphi}(\varepsilon\xi)$. From (6), (iv) it follows $\hat{u}_\varepsilon = \hat{u}\hat{\varphi}_\varepsilon = U\hat{\varphi}_\varepsilon$ and we can apply the Paley-Wiener theorem to φ_ε, i.e. $\hat{\varphi}_\varepsilon$ extends to an entire function on $\mathbb{C}^n$ and we have $|\hat{u}_\varepsilon(\zeta)| \leqslant CC_M(1+|\zeta|^2)^N(1+|\varepsilon\zeta|^2)^{-M}\exp((A+\varepsilon)|\operatorname{Im}\zeta|)$ for some sequence (C_M), $C \geqslant 0$, $N \geqslant 0$ and all $M \in \mathbb{N}_0$. Choosing $M = m + N$ for $m \in \mathbb{N}_0$, we get estimates which allow us to apply the Paley-Wiener theorem again in order to get $u_\varepsilon \in C_0^\infty$ with $\operatorname{supp} u_\varepsilon \subset B_{A+\varepsilon}$. Finally, we can show that $u_\varepsilon \to u$ in $\mathcal{S}'$ as $\varepsilon \to 0$, i.e. $|u_\varepsilon(\phi) - u(\phi)| \to 0$ for any $\phi \in \mathcal{S}'$. By using (6), (ii) this reduces to $|\varphi_\varepsilon * \phi - \phi|_k \to 0$ for some $k \in \mathbb{N}_0$. Due to (6), (iii), this reduces to the case $k = 0$, which follows from a change of variables in the convolution integral. The desired inclusion $\operatorname{supp} u \subset B_A$ then is an easy consequence of $\operatorname{supp} u_\varepsilon \subset B_{A+\varepsilon}$ and the convergence of $u_\varepsilon \to u$ as $\varepsilon \to 0$.)

Sources:

[1] X. Saint Raymond, *Elementary introduction to the theory of pseudodifferential operators*, 1991

[2] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, 1990

[3] F. G. Friedlander, *Introduction to the theory of distributions*, 1998