Atlas. An *n-dimensional atlas* of a set M is a collection $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha |\ \alpha \in I\}$ of bijective maps from open sets $U_\alpha \subset \mathbb{R}^n$ into subsets $V_\alpha \subset M$ s.t.

(i) $\bigcup_\alpha V_\alpha = M$
(ii) For all $\alpha, \beta \in I$ with $W := V_\alpha \cap V_\beta \neq \emptyset$, the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$.
(iii) $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(W) \to \varphi_\beta^{-1}(W)$ are diffeomorphisms.

Natural topology. If M is a set with an atlas $\mathcal{A}$, then there is a *natural topology* on M s.t. $V \subset M$ open $:\Leftrightarrow$ $\varphi_\alpha^{-1}(V \cap V_\alpha)$ open in $\mathbb{R}^n$ for all $\alpha \in I$ (this is the topology s.t. all V_α are open and all φ_α are continuous).

Manifold. On the set of all atlases, there is the equivalence relation: $\mathcal{A} \sim \mathcal{A}' :\Leftrightarrow \mathcal{A} \cup \mathcal{A}'$ is an atlas. A set M together with an equivalence class of n-dim. atlases is called an *n-dim. (differentiable) manifold*, if M equipped with the natural topology is a second countable Hausdorff space.

Smooth function on a manifold. $f : M \to \mathbb{R}$ is called *smooth*, if $f \circ \varphi_\alpha : U_\alpha \to \mathbb{R}$ is smooth for all $\alpha \in I$. The set of smooth functions on M is denoted by $\mathscr{F}(M)$.

Smooth function btw. manifolds. $f : M \to N$ is called *smooth*, if for all charts φ of M and ψ of N, the map $\psi^{-1} \circ f \circ \varphi$ is smooth.

Derivations, tangent space. A derivation at p is an $\mathbb{R}$-linear map $v : \mathscr{F}(M) \to \mathbb{R}$, that satisfies the Leibniz rule $v(fg) = v(f)g(p) + f(p)v(g)$. $Der_p\mathscr{F}(M) \equiv T_pM$ is the $\mathbb{R}$-vector space of derivations at p, or simply the *tangent space of M at p*.

(i) $\{\partial_1|_0, \ldots, \partial_n|_0\}$ forms a basis of $Der_0\mathscr{F}(\mathbb{R}^n) \cong \mathbb{R}^n$.
(ii) If v is a derivation at p, $v(f)$ depends on f only in an arbitrary small neighbourhood U of p, i.e. the differential $\iota_* : T_pU \to T_pM$ is an isomorphism.
(iii) Let $\varphi : U \to V$ be a chart s.t. $\varphi(0) = p$. From (ii) follows $T_pV \cong T_pM$ and $T_0U \cong T_0\mathbb{R}^n$ and since $\varphi_* : T_0U \to T_pV$ is also an isomorphism, the basis in (i) induces a basis in T_pM: $\{\partial_i|p, i = 1, \ldots, n\}$, where $\partial_i|_p := \varphi_*\partial_i|_0$, i.e. $T_pM \ni v = v^i\partial_i|_p$.

Pull-back of a function. If $\varphi : M \to N$ is a smooth map, then the pullback of functions $f \in \mathscr{F}(N)$ to functions $\varphi^*f \in \mathscr{F}(M)$ is defined by $\varphi^*f := f \circ \varphi$.

Differential. The differential $T_p\varphi$ or push-forward φ_* w.r.t a smooth map $\varphi : M \to N$ is defined by

$$T_p\varphi = \varphi_* : T_pM \to T_{\varphi(p)}M, (\varphi_*v)(f) := v(f \circ \varphi)$$

Immersion, embedding. A smooth map $f : M \to N$ is called *immersion*, if $f_* : T_pM \to T_{f(p)}N$ is injective $\forall p \in M$. f is called *embedding*, if it is an immersion and $f : M \to f(M)$ is a diffeomorphism.

(i) If f is an embedding, then $f(M)$ is a *submanifold* of N.
(ii) If $M \subset N$ is a submanifold, then $\iota : M \to N$ is an embedding.
(iii) *Inverse function theorem:* Given a smooth map $f : M \to N$, if the differential $f_* = T_pf : T_pM \to T_{f(p)}N$ is a linear isomorphism at a point $p \in M$, then there is a neighbourhood U of p s.t. $f : U \to f(U)$ is a diffeomorphism. Thus: *An immersion is locally an embedding.*

Vectorfield. A *vectorfield* $X \in \mathscr{X}(M)$ assigns to every point $p \in M$ a vector $X_p \in T_pM$, s.t. $X_p = X^i(p)\partial_i|_p$ with smooth functions $X^i : M \supset V \to \mathbb{R}$.

□ A vectorfield $X \in \mathscr{X}(M)$ can be viewed as an $\mathbb{R}$-linear map $\mathscr{F}(M) \to \mathscr{F}(M)$, satisfying $X(fg) = X(f)g + fX(g)$. Such maps are called *derivations on M*, thus $\mathscr{X}(M) = Der\mathscr{F}(M)$.

Push-forward/pull-back of vectorfields. For a diffeomorphism $\varphi : M \to N$ the *push-forward* $\varphi_* : \mathscr{X}(M) \to \mathscr{X}(N)$ is given by $(\varphi_*X)_{\varphi(p)}(f) := X_p(f \circ \varphi)\ \forall p \in M$. The *pull-back* φ^* is then defined by $\varphi^* := (\varphi^{-1})_*$.

Tangent vector. A curve $\gamma : (a, b) \to M$ for each t has a tangent vector $\dot\gamma(t) \in T_{\gamma(t)}M$ defined by $\dot\gamma(t) := \gamma_*\frac{\partial}{\partial t}$.

□ If $v \in T_pM$ is tangent to γ at $t = 0$, i.e. $v = \frac{d}{dt}\Big|_0\gamma(t)$, then the push-forward of v w.r.t a map $\phi : M \to N$ is given by $\phi_*v = \frac{d}{dt}\Big|_0\phi(\gamma(t))$.

Integral curve. A curve $\gamma : (a, b) \to M$ is an integral curve of $X \in \mathscr{X}(M)$, if $\dot\gamma(t) = X(\gamma(t))\ \forall t \in (a, b)$.

□ Suppose $X \in \mathscr{X}(M)$ has compact support, then $\forall p \in M$ there is a unique integral curve $\gamma : \mathbb{R} \to M$ of X with $\gamma(0) = p$.

Flow. A *flow* on a manifold M is a smooth map $\varphi : \mathbb{R} \times M \to M$, s.t. $\varphi_0 = id$ and $\varphi_{s+t} = \varphi_s \circ \varphi_t$.

(i) $\varphi_t : M \to M$ is a diffeomorphism $\forall t$.
(ii) φ defines a vectorfield by
$$p \mapsto X_p := \frac{d}{dt}\Big|_0 \varphi_t(p) := \left(\varphi(p)_* \frac{\partial}{\partial t}\right)_p$$
(iii) For any $X \in \mathscr{X}(M)$ with compact support there is a unique flow φ s.t. $X_{\varphi_t(p)} = \frac{d}{dt}\varphi_t(p)$. For v.f. with non-compact support, there is at least a *local flow*.

Lie derivative. Let $X, Y \in \mathscr{X}(M)$. The *Lie derivative* of Y in direction X at $p \in M$ is defined by

$$(\mathcal{L}_XY)_p := \frac{d}{dt}\Big|_0 (\varphi_t)^*Y$$

where φ is the local flow of X in a neighbourhood of p.

(i) For vectorfields X, Y with flows φ, ψ, we have the equivalence: $\mathcal{L}_XY = 0 \Leftrightarrow \psi_s \circ \varphi_t = \varphi_t \circ \psi_s\ \forall s, t$.
(ii) X, Y, φ, ψ as in (i), then $\mathcal{L}_XY = \frac{\partial^2}{\partial s\partial t}\Big|_0 \varphi_{-t} \circ \psi_s \circ \varphi_t$.
(iii) $\mathcal{L}_XY(f) = X(Y(f)) - Y(X(f)) \equiv [X, Y](f)$.

Lie bracket. The *Lie bracket* of two vectorfields X, Y is the vector field def. by $[X, Y](f) := X(Y(f)) - Y(X(f))$.

(i) $[X, Y]$ is $\mathbb{R}$-bilinear in X and Y.
(ii) $[Y, X] = -[X, Y]$.
(iii) $[X, [Y, Z]] + [Z, [X, Y] + [Y, [Z, X]] = 0$.
(iv) $[X, fY] = f[X, Y] + X(f)Y$.
(v) $\varphi_*[X, Y] = [\varphi_*X, \varphi_*Y]$.
(vi) $[X, Y] = (X^j\partial_jY^i - Y^j\partial_jX^i)\partial_i = DY \cdot X - DX \cdot Y$.

Lie group. It is a manifold that is also a group G, s.t. the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

(i) The left-multiplication $\mathcal{L}_g h := gh$ (also the right-mult.) is a diffeomorphism with inverse $\mathcal{L}_{g^{-1}}$.
(ii) There are vectorfields X on a Lie group G that are *left-invariant*, i.e. $\mathcal{L}_g^* X = X \ \forall g \in G$.
(iii) The map $\mathscr{X}(G) \to T_eG, X \mapsto X_e$ defines a 1-1-correspondence btw. left-invariant vector fields on G and T_eG. The inverse map is given by $T_eG \supset \xi \mapsto \underline{\xi}$, where $\underline{\xi}_g := \mathcal{L}_{g*}\xi$.
(iv) If X, Y are left-inv. then $[X, Y]$ is left-inv., too.

Lie algebra. A vector space that is endowed with a pairing $[\cdot,\cdot]$ satisfying the properties (i)-(iii) of the Lie bracket is called *Lie algebra*. The tangent space T_eG of a Lie group G, endowed with $[\xi,\eta] := [\underline{\xi},\underline{\eta}]_e$ is called *Lie algebra of* G and denoted $\mathfrak{g}$.
(i) Let $\xi \in T_eG$, then $\underline{\xi}$ has a (global) flow φ_t^ξ.
(ii) The result (i) allows us to define $\exp : \mathfrak{g} \to G, \xi \mapsto \varphi_t^\xi(e)$.

Relations with antisymmetric combinations. The antisymmetrization of a k-linear map $t \in (V \times \cdots \times V)^*$ is defined by $(\pi^A t)(v,\ldots,w) := \frac{1}{k!}\sum_\sigma \mathrm{sgn}(\sigma) t(\sigma(v,\ldots,w))$ with component notation $(\pi^A t)_{a\ldots b} =: t_{[a\ldots b]}$. Then using the totally antisymmetric Levi-Civita-Symbol $\epsilon_{a\ldots b}$ in n dimensions, we get
(i) $\epsilon^{a\ldots b}\epsilon_{a\ldots b} = n!$
(ii) Any antisymmetric quantity with n indizes has just one degree of freedom: $\omega_{a\ldots b} = \omega_{1\ldots n}\,\epsilon_{a\ldots b}$.
(iii) $\omega_{1\ldots n} = \frac{1}{n!}\omega_{a\ldots b}\epsilon^{a\ldots b}$
(iv) $\det A = \epsilon_{a\ldots b}A_1^a \ldots A_n^b = \frac{1}{n!}\epsilon_{a\ldots b}\epsilon^{c\ldots d}A_c^a \cdots A_d^b$.
(v) $\epsilon_{a\ldots b}\epsilon^{c\ldots d} = n!\,\delta^a_{[b}\cdots\delta^c_{d]} =: n!\,\delta^{a\ldots b}_{c\ldots d}$
(vi) For any antisymmetric quantity α carrying k indices: $\alpha_{c\ldots d} = \delta^a_{[c}\cdots\delta^b_{d]}\,\alpha_{a\ldots b} \equiv \delta^{a\ldots b}_{c\ldots d}\,\alpha_{a\ldots b}$.

Exterior form. An *exterior form of degree* k (or k-form) on a vectorspace V is a map $V \times \cdots \times V \to \mathbb{R}$, that is alternating and multilinear, i.e. the set of k-forms $\Lambda^k V^*$ on V is the totally antisymmetric subspace of the dual space $(V \times \cdots \times V)^*$.

Wedge product of 1-forms. Using 1-forms $\varphi^1,\ldots,\varphi^k$, we get a k-form by $\varphi^1 \wedge \cdots \wedge \varphi^k(v_1,\ldots,v_k) := \det(\varphi^i(v_j))$.
(i) We obtain: $\varphi^1 \wedge \cdots \wedge \varphi^k(v_1,\ldots,v_k) = \det(\varphi^l(v_m)) = \epsilon_{i\ldots j}\varphi^i(v_1)\cdots\varphi^j(v_k) = \epsilon_{i\ldots j}\epsilon^{i\ldots j}\varphi^{[1}(v_1)\cdots\varphi^{k]}(v_k)$, i.e. $\varphi^1 \wedge \cdots \wedge \varphi^k = k!\,\varphi^{[1} \otimes \cdots \otimes \varphi^{k]}$.
(ii) Any k-form can be written as $\alpha = \alpha_{a\ldots b}\, e^a \otimes \cdots \otimes e^b = \alpha_{a\ldots b} e^{[a} \otimes \cdots \otimes e^{b]} = \frac{1}{k!}\alpha_{a\ldots b}\, e^a \wedge \cdots \wedge e^b$.
(iii) From (ii): $\alpha = \sum_{a<\cdots<b}\alpha_{a\ldots b}e^a \wedge \cdots \wedge e^b$. Therefore, the set $\{e^{i_1} \wedge \cdots \wedge e^{i_k} | i_1 < \cdots < i_k, i_j = 1,\ldots n\}$ forms a basis of $\Lambda^k V^*$.

Wedge product for k-forms. For $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^p V^*$ let $\alpha \wedge \beta = \sum_{a<\cdots b, c<\cdots d}\alpha_{a\ldots b}\beta_{c\cdots d}e^a \wedge \cdots \wedge e^b \wedge e^c \wedge \cdots \wedge e^d$.
(i) $(\alpha \wedge \beta) \wedge \omega = \alpha \wedge (\beta \wedge \omega)$.
(ii) $\alpha \wedge \beta = (-1)^{k\cdot p}\beta \wedge \alpha$.
(iii) $\alpha \wedge (\beta + \omega) = \alpha \wedge \beta + \alpha \wedge \omega$.

Exterior k-form on a manifold. An exterior k-form ω on a manifold is a choice for each $p \in M$ of an element $\omega_p \in \Lambda^k(T_pM)^*$.

(i) If $f : M \to N$ is differentiable, then the *pull-back* of an exterior k-form ω on N is def. by the action of ω on the pushed vectors: $(f^*\omega)_p(v,\ldots) = \omega_{f(p)}(f_*v,\ldots)$.
(ii) Let $\varphi_\alpha : U_\alpha \to V_\alpha$ be a chart, then the *representation* ω_α of ω is defined by $\omega_\alpha := \varphi_\alpha^*\omega$, i.e. it represents ω as an exterior k-form in $U_\alpha \subset \mathbb{R}^n$, so if $\{\varphi_\alpha\}_{\alpha\in I}$ is a given atlas of M, the set $\{\omega_\alpha\}_{\alpha\in I}$ represents ω in $\mathbb{R}^n$.

Differential forms. The dual basis of $\{\partial_i\}$ in $T_x\mathbb{R}^n$ is denoted by $\{dx^i\}$. Thus an exterior k-form ω on $\mathbb{R}^n$ can be decomposed in each point $x \in \mathbb{R}^n$ as
$$\omega_x = \sum_{i<\cdots<j}(\omega_x)_{i\ldots j}\,dx^i \wedge \cdots \wedge dx^j$$
Now, if the coefficients in this decomposition happen to be smooth functions of x, $(\omega_x)_{i\ldots j} =: \omega_{i\ldots j}(x)$, then ω is called *differential k-form on* $\mathbb{R}^n$. An exterior k-form on a manifold is called *differentiable k-form*, if all its representations $\{\omega_\alpha\}$ in a given atlas $\{\varphi_\alpha\}$ are differentiable k-forms on $\mathbb{R}^n$. Then in $p \in M$, a differentiable k-form can be written as
$$\begin{aligned}\omega_p &= (\varphi_{\alpha*}\omega_\alpha)_p \\ &= \sum_{i<\cdots<j}\omega_{i\ldots j}(\varphi_\alpha^{-1}(p))\,\varphi_{\alpha*}\,(dx^i \wedge \cdots \wedge dx^j)\big|_p \\ &=: \sum_{i<\cdots<j}\omega_{i\ldots j}(p)(dx^i \wedge \cdots \wedge dx^j)_p\end{aligned}$$
Thus a differentiable k-form on a manifold has smooth coefficients $\omega_{i\ldots j}$ with respect to the pushed-forward basis k-forms $\{dx^i \wedge \cdots \wedge dx^j\}_{i<\cdots<j}$. The set of differentiable k-forms is denoted by $\Omega^k(M)$.

Exterior derivative. The *exterior derivative* of a function $f \in \mathcal{F}(M)$ is the 1-form $df := \partial_i f\,dx^i$. For $\omega \in \Omega^k(M)$ we def. $d\omega = \sum_{i<\cdots<j} d\omega_{i\ldots j} \wedge dx^i \wedge \cdots dx^j$. For $\alpha \in \Omega^k(M), \beta \in \Omega^p(M)$:
(i) $d(\alpha + \beta) = d\alpha + d\beta$.
(ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\omega \wedge d\beta$.
(iii) $d^2 = 0$.

Interior derivative. For a given vectorfield $Z \in \mathscr{X}(M)$, the *interior derivative* $\iota_Z : \Omega^k(M) \to \Omega^{k-1}(M)$, is def. by $\iota_Z\omega := \omega(Z,\cdots)$. For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^p(M)$:
(i) $\iota_Z(\alpha \wedge \beta) = \iota_Z\alpha \wedge \beta + (-1)^k\alpha \wedge \iota_Z\beta$.
(ii) $\iota_Z^2 = 0$.
(iii) $\iota_{[X,Y]}\omega = X \circ \iota_Y\omega - Y \circ \iota_X\omega$.

Pull-back of forms. For a smooth map $\varphi : M \to N$, the *pull-back* of a differentiable k-form $\omega \in \Omega^k(N)$ is defined by $\varphi^*\omega(Z,\ldots) = \omega(\varphi_*Z,\ldots)$, i.e. $\varphi^*\omega \in \Omega^k(M)$.
(i) $\varphi^*(\alpha + \beta) = \varphi^*\alpha + \varphi^*\beta$.
(ii) For any $f \in \mathcal{F}(M)$: $\varphi^*(f\alpha) = \varphi^*f\,\varphi^*\alpha \equiv (f \circ \varphi)\,\varphi^*\alpha$.
(iii) $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.
(iv) $(\varphi \circ \psi)^* = \psi^*\varphi^*$.
(v) $d\,\varphi^*\omega = \varphi^*d\,\omega$.

Lie derivative of forms. For a given $X \in \mathscr{X}(M)$, we define the *Lie derivative* of a k-form in direction Z by
$$(\mathcal{L}_X\omega)_p := \frac{d}{dt}\bigg|_0 (\varphi_t)^*\,\omega$$
where φ is the local flow of X in a neighbourhood of p.
(i) $\mathcal{L}_Y(\omega(X_1,\ldots,X_p)) = (\mathcal{L}_Y\omega)(\ldots) + \sum_i \omega(\ldots,\mathcal{L}_Y X_i,\ldots)$
(ii) $\mathcal{L}_X\omega = (d\,\iota_X + \iota_X d)\,\omega$

Orientation. If $\{\tilde{e}_a\}$ is a basis of a vector space V, then by a basis transformation $e_a = A_{ab}\tilde{e}_b$, $A \in GL(n,\mathbb{R})$, any other basis $\{e_a\}$ can be constructed. If $\tilde{e}_a$ is the standard basis in $\mathbb{R}^n$, then $\{e_a\}$ is called *right handed*, if $\det A > 0$ and *left handed*, if $\det A < 0$. If $\{\partial_i|_0\}$ is a right handed basis of $\mathbb{R}^n$ and φ a chart around $p \in M$, then the basis elements $\partial_i|_p := \varphi_*\partial_i|_0$ form a *positively oriented* basis of T_pM if $\det\varphi_* > 0$ and a *negatively oriented* basis, if $\det\varphi_* < 0$.

Oriented atlas, orientability. An atlas of a manifold is called *oriented*, if the change of coordinates on any non-empty overlap of coordinate patches preserves orientation, i.e. if it has positive Jacobian determinant. If a manifold admits an oriented atlas, it is called *orientable*.

Partition of unity. On any manifold M, there exists a *partition of unity*, i.e. a set $\{\varphi_\alpha : M \to [0,1]\}$ with $\operatorname{supp}\varphi_\alpha \subset V_\alpha$ and $\sum_\alpha \varphi_\alpha = 1$ (for compact manifolds, there are finitely many φ_i s.t. $\operatorname{supp}\varphi_i \subset V_\alpha$, $\sum_i \varphi_i = 1$).

Manifolds with boundary. Let $\mathbb{H}^n$ be the half space $\{x \in \mathbb{R}^n | x^1 \leqslant 0\}$ equipped with the subspace topology. A *manifold with boundary* is defined by replacing $\mathbb{R}^n$ by $\mathbb{H}^n$ in the definition of a manifold. The boundary ∂M consists of those points $p \in M$ with $p = \varphi(0, x^2, \dots, x^n)$ for some chart of M

(i) The def. of a point on the boundary is indep. of the chosen chart, i.e. it holds either for all charts or for none.
(ii) The boundary ∂M of an n-dim. manifold is an $(n-1)$-dim. manifold.
(iii) An orientation on M induces one on ∂M.

Integration of forms. Let ω be a form of *top degree*, i.e. $\omega \in \Omega^n(M)$. *1st case:* If $\operatorname{supp}\omega \subset f_\alpha(U_\alpha)$ for some α of an atlas $\{f_\alpha\}$ of M, then

$$\begin{aligned}\int_M \omega &:= \int_{U_\alpha} \omega_\alpha = \int_{U_\alpha} \omega_\alpha(x)\, dx^1 \wedge \cdots \wedge dx^n \\ &:= \int_{U_\alpha} \omega_\alpha(x)\, dx^1 \cdots dx^n\end{aligned}$$

2nd case: If $\operatorname{supp}\omega$ is not contained in any $f_\alpha(U_\alpha)$, but if M is compact, there is a partition of unity $\{\varphi_i\}_{i=1,\dots,m}$ s.t. $\forall i = 1,\dots,m$: $\operatorname{supp}\varphi_i \subset f_\alpha(U_\alpha)$ for some α. Then $\forall i = 1,\dots,m$: $\operatorname{supp}\varphi_i\omega \subset f_{\alpha_i}(U_{\alpha_i})$ for some α_i. By setting

$$\int_M \omega := \sum_{i=1}^m \int_M (\varphi_i\omega)_{\alpha_i}$$

the integral reduces to the 1st case.

Stokes theorem. Let M be a compact oriented manifold with boundary and $\omega \in \Omega^{n-1}(M)$, $\iota : \partial M \to M$ the inclusion map, then $\int_{\partial M} \iota^*\omega = \int_M d\omega$.

Exact and closed forms. A k-form $\omega \in \Omega^k(M)$ is called *exact*, if $\exists\, \beta \in \Omega^{k-1}(M)$ s.t. $\omega = d\beta$. ω is called *closed*, if $d\omega = 0$. Thus an exact form is in the image of d and a closed form in the kernel of d.

Poincaré Lemma. If the manifold M is contractible and ω a closed k-form on M, then it is exact.

Cocycle and coboundary group. The set of closed k-forms, i.e. the kernel of d, is called k-th *cocycle group* $Z^k(M,\mathbb{R})$ and the set of exact k-forms, the image set of d, is called k-th *coboundary group* $B^k(M,\mathbb{R})$. They are subgroups (w.r.t addition) of the abelian group $\Omega^k(M,\mathbb{R})$ of linear combinations of k-forms with real coefficients.

Cohomology group. The k-th *(de Rham) cohomology group* $H^k(M,\mathbb{R})$ is the quotient $Z^k(M,\mathbb{R})/B^k(M,\mathbb{R})$.

(i) If M is contractible, then $H^k(M,\mathbb{R}) = 0$.
(ii) If $M = T^2$, then $H^0(T^2,\mathbb{R}) = \mathbb{R}$, $H^2(T^2,\mathbb{R}) = \mathbb{R}$.
(iii) Since for a smooth map $f : M \to N$, $f^*d = d\,f^*$, we have $f^* : H^k(M,\mathbb{R}) \to H^k(N,\mathbb{R})$.

Cohomology ring. Let $H^* := \oplus_k H^k(M,\mathbb{R})$. Then the wedge product $\wedge : H^* \times H^* \to H^*$ gives H^* a ring structure.

De Rham complex. The set $\Omega^* := \oplus_k \Omega^k(M,\mathbb{R})$ together with the sequence

$$\cdots \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \cdots$$

Riemannian metric. It assigns to every point $p \in M$ an inner product, i.e. a bilinear, symmetric, positive definite map $\langle\ ,\ \rangle : T_pM \times T_pM \to \mathbb{R}$, s.t. $g_{ij}(p) := \langle \partial_i|_p, \partial_j|_p\rangle_p$ depends smoothly on p.

(i) Every manifold carries a Riemannian metric.
(ii) Under a change of coordinates: $g'_{ij} = \partial'_i x^k \partial'_j x^l g_{kl}$
(iii) A diffeomorphism/immersion/embedding $f : M \to N$ is called isometry/isometric immersion/isometric embedding, if $\langle f_*v, f_*w\rangle_{f(p)} = \langle v, w\rangle_p$.
(iv) An immersion $f : M \to N$ where N is a Riem. manif. *induces* a metric on M by $\langle v, w\rangle_p := \langle f_*v, f_*w\rangle_{f(p)}$. This is called the *pull-back-metric* $f^*\langle\ ,\ \rangle_p$.

Length of a curve. The length of a curve $\gamma : [a,b] \to M$ is defined by $L(\gamma) := \int_a^b \langle\dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}^{1/2} dt =: \int_a^b |\dot\gamma| dt$.

(i) If $\psi : [a',b'] \to [a,b]$ is a diffeomorphic reparametrization, then $L(\gamma \circ \psi) = L(\gamma)$
(ii) γ with $\dot\gamma(t) \neq 0\ \forall t$ can be reparametrized by *arc length*, i.e. $|\dot\gamma(t)| \equiv 1$.

Volume form. Let M be an oriented n-dim. Riemannian manifold, then the canonical *volume form* is a form of top degree, $vol \in \Omega^n(M)$, s.t. it is nowhere zero and $\forall p \in M$: $vol_p(e_1,\dots,e_n) = 1$, for a positively oriented ONB $\{e_i\}$ of T_pM, i.e. $vol_p = e^1 \wedge \cdots \wedge e^n$.

(i) If $e^i = b_{ij}\tilde{e}^j$, then $vol = (\det b)\,\tilde{e}^1 \wedge \cdots \wedge \tilde{e}^n$.
(ii) For any local coordinate basis $\{\partial_i\}$ of T_pM, there is a transformation matrix (a_{ij}) s.t. $\partial_i = a_{ij}e_j$, where e_j is a positively oriented ONB. Then $dx^i = (a^{-1})^T_{ij}e^j$. Furthermore $g_{ij} = \langle\partial_i, \partial_j\rangle = a_{ik}a_{jk}$, and therefore $\det g = (\det a)^2$. Thus $vol = \sqrt{|\det g|}\,dx^1 \wedge \cdots \wedge dx^n$.
(iii) The quantity $vol(M) := \int_M vol$ is called *total volume* of a manifold.

Affine connection. It is an $\mathbb{R}$-bilinear map

$$\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M), (X,Y) \mapsto \nabla_X Y$$

s.t. it is $\mathscr{F}(M)$-linear in the first argument and a derivation in the second: $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

(i) The $\mathscr{F}(M)$-Linearity of the map $X \mapsto \nabla_X Y$ (fixed Y) is equiv. to the fact, that $(\nabla_X Y)_p$ depends only on X_p.
(ii) $(\nabla_X Y)_p$ depends on Y only in a neighbourhood of p.
(iii) In local coordinates, ∇ is represented by the *Christoffel symbols* def. by $\nabla_{\partial_i}\partial_j =: \Gamma_{ij}^k \partial_k$.
(iv) $\nabla_X Y$ has the coord. rep. $(\nabla_X Y)^k = X(Y^k) + X^i Y^j \Gamma_{ij}^k$.
(v) The space of affine connections is an affine space, i.e. $\sum_i \alpha_i \nabla^i$ is an affine connection if the ∇^i are affine connections and $\sum_i \alpha_i = 1$.
(vi) On each (differentiable) manifold there exists an affine connection (proof by part. of unity and affinity).

Covariant derivative. Given an affine connection ∇ and a curve γ on M, the covariant derivative of V along γ is def. as $\frac{DV}{dt} := \nabla_{\dot\gamma} V$.

(i) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$ for all smooth $f : I \to \mathbb{R}$.
(ii) In local coordinates: $\frac{DV}{dt}(t) = (\dot V^k + \dot x^i V^j \Gamma_{ij}^k)\partial_k$, where $V(t) := V(\gamma(t))$.

Parallel vector field along a curve. A vectorfield V is called *parallel* along γ if $\frac{DV}{dt} \equiv 0$. Given ∇, a curve γ and a vector $v_0 \in T_{\gamma(t_0)}M$, there is a unique parallel vectorfield V along γ with $V(\gamma(t_0)) = v_0$.

Parallel transport. The parallel transport along γ w.r.t ∇ is the linear map $T_pM \to T_qM, v_0 = V(t_0) \mapsto V(t_1)$.

Remark. One can recover ∇ from parallel transport: Let $X, Y \in \mathscr{X}(M)$, $p \in M$. Pick $\gamma : (-\varepsilon, \varepsilon) \to M$ s.t. $\dot\gamma(0) = X_p$. Let $e_1(t), \dots, e_n(t)$ be a basis of parallel vector fields along γ and $Y(\gamma(t)) =: V(t) = V^i(t)e_i(t)$. Then we can recover $(\nabla_X Y)_p = \frac{DV}{dt}(0) = \dot V^i(0)e_i(0) + 0$, since the e^i are parallel.

Compatibility with the metric. On a Riemannian manifold, a connection ∇ is said to be *compatible with the metric*, if parallel transport along any curve is a linear isometry $T_pM \to T_qM$. This is equivalent to

(i) For all parallel vectorfields V, W along γ: $\langle (V(t), W(t)\rangle_{\gamma(t)}$ is independent of t.
(ii) $\forall$ v.f. V, W along γ: $\frac{d}{dt}\langle V(t), W(t)\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$.
(iii) $\forall$ v.f. X, Y, Z: $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Symmetric connection. We say, an affine connection ∇ is *symmetric* (or torsion-free), if $\nabla_X Y - \nabla_Y X = [X,Y]$, for all $X, Y \in \mathscr{X}(M)$. This is equivalent to say $\Gamma_{ij}^k = \Gamma_{ji}^k$.

☐ If ∇ is symmetric, then for $\Gamma : \mathbb{R}\times\mathbb{R} \to M$, $\frac{D}{ds}\frac{\partial\Gamma}{\partial t} = \frac{D}{dt}\frac{\partial\Gamma}{\partial s}$.

Levi-Civita-Connection. On a Riemannian manifold, there exists a unique connection, which is symmetric and compatible with the metric, the *Levi-Civita-Connection*.

(i) The LC-connection is given by $\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y,Z]\rangle - \langle Y, [X,Z]\rangle + \langle Z, [X,Y]\rangle)$. In coordinates: $\Gamma_{ij}^m = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})g^{km}$.
(ii) If $\overline\nabla$ is the LC-connection on $\overline M$, then the LC-connection ∇ on a submanifold $M \subset \overline M$ is given by $\nabla_X Y(p) = \pi(\overline\nabla_{\overline X}\overline Y(p))$, where $\pi : T_p\overline M \to T_pM$ is the orthogonal projection, $\overline X$ and $\overline Y$ are extensions of $X, Y \in \mathscr{X}(M)$, s.t. they are equal to X, Y on a small neighbourhood.

1st variation of length. Let M be a Riemannian manifold and ∇ the LC-connection. Let $\gamma : [a,b] \to M$ be a curve from p to q, $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ a smooth map with $\Gamma(0,t) = \gamma(t)$, $\Gamma(s,a) = p$, $\Gamma(s,b) = q$. The *first variation of length* at γ in direction $V(t) := \frac{\partial\Gamma}{\partial s}(0,t)$ with fixed endpoints is defined by $dL(\gamma)V := \frac{d}{ds}L(\Gamma(s,\cdot))|_0$.

(i) Assuming $|\dot\gamma(t)| \equiv 1$, we obtain

$$dL(\gamma)V = -\int_a^b \langle V, \frac{D\dot\gamma}{dt}\rangle dt$$

(ii) $dL(\gamma)V = 0\ \forall V$ is a necessary condition for γ to minimize length. By (i), this is equivalent to $\frac{D\dot\gamma}{dt} \equiv 0$.

Geodesic. A curve γ is called *geodesic*, if $\frac{D\dot\gamma}{dt} \equiv 0$.

(i) If γ is a geodesic, then $\frac{d}{dt}|\dot\gamma|^2 = 2\langle \frac{D\dot\gamma}{dt}, \dot\gamma\rangle = 0$. Thus $|\dot\gamma|$ is constant, so geodesics are parametrized proportional to arc length.
(ii) In local coordinates, the equations that determine a geodesic read $\ddot x^k + \Gamma_{ij}^k \dot x^i \dot x^j = 0$.
(iii) $\forall p \in M$ and $v \in T_pM$, $\exists \varepsilon > 0$ and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, $\dot\gamma(0) = v$.
(iv) $\forall p \in M$, $\exists \varepsilon > 0$ s.t. $\forall v \in T_pM$ with $|v| \leqslant \varepsilon$ $\exists$ unique geodesic $\gamma_v : (-2,2) \to M$ with $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$.
(v) For $a > 0$ and γ_v as above, the curve $t \mapsto \gamma_v(at)$ is a geodesic with initial position p and initial velocity av for $t \in (-2/a, 2/a)$.

Exponential map. The exponential map at $p \in M$ is the map $\exp_p : T_pM \supset B_\varepsilon(0) \to M, v \mapsto \gamma_v(1)$.

(i) For $\exp_{p*} = T_0 \exp_p : T_0(T_pM) \to T_pM$ we get $\exp_{p*} v = \exp_{p*} \frac{d}{dt}\Big|_0 tv = \frac{d}{dt}\Big|_0 \exp_p(tv) = \frac{d}{dt}\Big|_0 \gamma_{tv}(1) = \frac{d}{dt}\Big|_0 \gamma_v(t) = v$, i.e. $T_0 \exp_p = id : T_pM \to T_pM$.
(ii) $\forall p \in M\ \exists \varepsilon > 0$ and $V \ni p$ open s.t. $\exp_p : B_\varepsilon(0) \to V$ is a diffeomorphism (by (i) and the inverse function theorem, since id is an isomorphism).
(iii) *Gauss lemma:* Let $p \in M$, $v \in B_\varepsilon(0) \subset T_pM$ ($\varepsilon > 0$ as in (iv) above), $w \in T_v(T_pM)$, then
$$\langle T_v \exp_p(v), T_v \exp_p(w)\rangle_{\exp_p(v)} = \langle v, w\rangle_p$$

Geodesic normal ball/sphere. By $B_r(p)$, we denote the image of $B_r(0)$ under $\exp_p$, where r is chosen so small such that $\exp_p : B_r(0) \to B_r(p)$ is a diffeomorphism. The boundary of the *geodesic normal ball* $B_r(p)$ is denoted by $S_r(p) := \partial B_r(p)$ and is called *geodesic normal sphere.*

(i) Let $\gamma : [0,1] \to B_\varepsilon(p)$ be a geodesic with $\gamma(0) = p$, $\gamma(1) = q \neq p$. Then $L(\gamma) \leqslant L(c)$ for any other curve c from p to q and equality if c is a monotone repar. of γ.
(ii) For a curve $\gamma : [a,b] \to M$ being a geodesic is equivalent to say that γ is *locally length minimizing* ($\exists \varepsilon > 0$ s.t. $\forall t \in [a, b-\varepsilon]$ $\gamma|_{[t,t+\varepsilon]}$ minimizes length among all curves with the same endpoints).

Geodesic normal coordinates. Let $p \in M$, $B_\varepsilon(p)$ a geodesic normal ball and $\{e_i\}$ a ONB of T_pM. The coordinates $(x_1, \dots x_n)$ of $v \in B_\varepsilon(0)$ with respect to $\{e_i\}$ can be used for the point $\exp_p(v) \in B_\varepsilon(p)$, since $\exp_p$ is a diffeomorphism btw. $B_\varepsilon(0)$ and $B_\varepsilon(p)$. These coordinates are called *(geodesic) normal coordinates*.

(i) In normal coordinates: $g_{ij}(0) = \delta_{ij}$.
(ii) $x^i(t) := tv^i$ is a geodesic $\forall v \in T_pM$. Then $0 = \ddot{x}^k + \Gamma^k_{ij}(0)\dot{x}^i\dot{x}^j$, i.e. $\Gamma_{ij}v^iv^j = 0$. Since Γ^k_{ij} is symmetric in (i,j), it follows $\Gamma^k_{ij}(0) = 0$.
(iii) $g_{ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$.

Curvature tensor. The *curvature tensor* of M is the map $R : \mathscr{X}(M)\times\mathscr{X}(M)\times\mathscr{X}(M) \to \mathscr{X}(M)$, $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. In local coordinates, we can define coeffients R^l_{ijk} by $R(\partial_i,\partial_j)\partial_k =: R^l_{ijk}\partial_l$.

(i) R is $\mathscr{F}(M)$-linear in each variable X, Y, Z, so $[R(X,Y)Z]_p$ depends only on $X(p), Y(p), Z(p)$.
(ii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, the *1st Bianchi identity*.
(iii) $R(X,Y)Z = -R(Y,X)Z$.
(iv) $\langle R(X,Y)Z, T\rangle = -\langle R(X,Y)T, Z\rangle$.
(v) $\langle R(X,Y)Z, T\rangle = \langle R(Z,T)X, Y\rangle$.
(vi) By $\mathscr{F}(M)$-linearity: $R(X,Y)Z = X^iY^jZ^kR^l_{ijk}\partial_l$.
(vii) $R^m_{ijk} = \partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl}$.
(viii) Notation: $R(X,Y,Z,T) := \langle R(X,Y)Z, T\rangle$. Then in local coordinates $R(\partial_i,\partial_j,\partial_k,\partial_m) =: R_{ijkm} = R^l_{ijk}g_{lm}$. Then the symmetries read $R_{ijkl} + R_{jkil} + R_{kijl} = 0$, $R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$, $R_{ijkl} = R_{klij}$.
(ix) *Cartan's theorem:* A Riemannian manifold M is locally isometric to Euclidean space (i.e. near each $p \in M$ $\exists$ local coordinates in which $g_{ij}(x) \equiv \delta_{ij}$, i.e. $R \equiv 0$)

Sectional curvature. Let M be a Riemannian manifold, $p \in M$, $X, Y \in T_pM$ linearly independent, then the *area of the parallelogram spanned by the vectors* X, Y is defined by $|X \wedge Y| := \sqrt{|X|^2|Y|^2 - \langle X,Y\rangle^2}$. The quantity $K(\sigma) := K(X,Y) := R(X,Y,Y,X)/|X\wedge Y|^2$ only depends on the plane $\sigma := span\{X,Y\} \subset T_pM$ and is called *sectional curvature of the plane* σ.

(i) $K(X,Y)$ is invariant under $(X,Y) \to (Y,X)$, $(X,Y) \to (\lambda X, Y)$, $(X,Y) \to (X+\lambda Y, Y)$.
(ii) Using (i), one can show: The sectional curvature K only depends on σ.

Isometric immersions. Let $\overline{M}$ a Riemannian manifold. If $f : M \to \overline{M}$ is an immersion, then M is locally a submanifold of $\overline{M}$ with the induced pull-back-metric $f^*\langle\ ,\ \rangle_p$. $\forall p \in M$ we can decompose $T_p\overline{M} = T_pM \oplus (T_pM)^\perp$. If $\overline{\nabla}$ is the LC-connection on $\overline{M}$, then using the orth. projection $\pi : T_p\overline{M} \to T_pM$, the LC-connection on M is defined by $\nabla_XY(p) = \pi(\overline{\nabla}_{\overline{X}}\overline{Y}(p))$. We define a map $B(X,Y) := (\overline{\nabla}_{\overline{X}}\overline{Y})^\perp = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_XY$.

(i) $B(X,Y) \in \mathscr{X}(M)^\perp$, i.e. it is a v.f. along M perp. to M.
(ii) B is $\mathscr{F}(M)$-bilinear and symmetric in X, Y.

Second fundamental form. For $N \in \mathscr{X}(M)^\perp$ define a sym. bilinear form $B_N(X,Y) := \langle B(X,Y), N\rangle$. Then $\mathit{II}_N(X) := B_N(X,X)$ is called the *second fundamental form* of $M \subset \overline{M}$ at $p \in M$ in direction N. The associated operator $S_N : T_pM \to T_pM$ defined by $\langle S_NX, Y\rangle := B_N(X,Y)$ is self-adjoint.

(i) $S_NX = -\pi(\overline{\nabla}_{\overline{X}}N)$.
(ii) *Gauss theorem:* Let $f : M \to \overline{M}$ be an isometric immersion. Then the sectional curvatures $K, \overline{K}$ of $M, \overline{M}$ are related by $K(X,Y) - \overline{K}(X,Y) = \langle B(X,X), B(Y,Y)\rangle - |B(X,Y)|^2$ where $X, Y \in T_pM$ are orthonormal.

Gauss curvature. Let $M \subset R^{n+1}$ be a hypersurface, i.e. $\dim M = n$. The *Gauss map* $N : M \to S^n, p \mapsto N_p$ assigns each point p to the unit normal vector N_p of the hypersurface at p. The differential/push-forward of N at p is the map $T_pN : T_pM \to T_{N_p}S^n = T_pM$. The *Gauss curvature* of $M \subset \mathbb{R}^{n+1}$ at p is $K(p) := \det(-T_pN)$.

(i) $T_pN(X) = \overline{\nabla}_XN$ (cov. deriv. on $\mathbb{R}^{n+1}$ = ord. deriv.)
(ii) From (i) of the par. on second fund. form: $S_N = -T_pN$.
(iii) $K(p) = \det S_N = \det(B_N(e_i,e_j))$
(iv) For a surface $M \in \mathbb{R}^3$ by the Gauss theorem and since $\overline{K} \equiv 0$: $K(X,Y) = \langle B(X,X), B(Y,Y)\rangle - |B(X,Y)|^2$. Thus $K(X,Y) = K(p)$, i.e. the sectional curvature (intrinsic) of a surface in $\mathbb{R}^3$ equals the Gauss curvature (extrinsic).
(v) Let $\overline{M}$ be a Riemannian manifold, $\sigma \subset T_p\overline{M}$ a plane and $M = \exp_p(\sigma)$. Then for $X \in \sigma$, $\overline{\nabla}_{\overline{X}}\overline{X} = 0$, $B(X,X) = (\overline{\nabla}_{\overline{X}}\overline{X})^\perp = 0$ and since B is symmetric $B(X,Y) = 0$ $\forall X, Y$. Then by Gauss theorem: $K(\sigma) - \overline{K}(\sigma) = 0$, i.e. $\overline{K}(\sigma) = K(\sigma) =$ Gauss curv. of M. Thus the sect. curvature of a plane $\sigma \subset T_pM$ equals the Gauss curvature at p of the surface $\exp_p(\sigma)$.

Ricci tensor. The *Ricci tensor* is def. by $Ric(X,Y) := tr(Z \mapsto R(X,Z)Y)$ i.e. $R_{ik} = R^j_{ijk}$.

(i) $Ric(e_n, e_n) = -\sum_{i=1}^{n-1} K(e_i, e_n)$.
(ii) $scal(p) := tr(Ric : T_pM \to T_pM) = R^j_{ijm}g^{mi}$.
(iii) In $\dim M = 2$: $scal = -2K$.

Covariant derivative of tensors. The covariant derivative of 1-forms along a vector field X is the 1-form defined by $(\nabla_X\omega)(Y) := X(\omega(Y)) - \omega(\nabla_XY)$. For a general tensor T, set $\nabla_XT(\omega_1,\dots;X_1,\dots) := X(T(\omega_1,\dots;X_1,\dots)) - \sum_i T(\dots,\nabla_X\omega_i,\dots;X_1,\dots) - \sum_j T(\omega_1,\dots;\dots\nabla_XX_j,\dots)$.

(i) We can also view this as $\nabla T(\omega_1,\dots;X_1,\dots,Z) := \nabla_ZT(\dots)$.
(ii) $\nabla g(X,Y,Z) = Z\langle X,Y\rangle - \langle\nabla_ZX,Y\rangle - \langle X,\nabla_ZY\rangle$, thus $\nabla g = 0 \Leftrightarrow \nabla$ is comp. with g.
(iii) *2nd Bianchi identity:* $(\nabla_XR)(Y,Z)W + (\nabla_YR)(Z,X)W + (\nabla_ZR)(X,Y)W = 0$.
(iv) With $\mathrm{div}Ric(X) := tr(Z \mapsto (\nabla_ZRic)(X))$ one can show $dscal = 2\,\mathrm{div}Ric$.