Contents

A Study Guide

The book has six chapters: Methods of Proof, Algebra, Real Analysis, Geometry and Trigonometry, Number Theory, Combinatorics and Probability, divided into subchapters such as Linear Algebra, Sequences and Series, Geometry, and Arithmetic. All subchapters are self-contained and independent of each other and can be studied in any order. In most cases they reflect standard undergraduate courses or fields of mathematics. The sections within each subchapter are best followed in the prescribed order.

If you are an *undergraduate student* trying to acquire skills or test your knowledge in a certain field, study first a regular textbook and make sure that you understand it very well. Then choose the appropriate chapter or subchapter of this book and proceed section by section. Read first the theoretical background and the examples from the introductory part; then do the problems. These are listed in increasing order of difficulty, but even the very first can be tricky. Don't get discouraged; put effort and imagination into each problem; and only if all else fails, look at the solution from the back of the book. But even if you are successful, you should read the solution, since many times it gives a new insight and, more important, opens the door toward more advanced mathematics.

Beware! The last few problems of each section can be very hard. It might be a good idea to skip them at the first encounter and return to them as you become more experienced.

If you are a *Putnam competitor*, then as you go on with the study of the book try your hand at the true Putnam problems (which have been published in three excellent volumes). Identify your weaknesses and insist on the related chapters of *Putnam and Beyond*. Every once in a while, for a problem that you have solved, write down the solution in detail, then compare it to the one given at the end of the book. It is very important that your solutions be correct, structured, convincing, and easy to follow.

Mathematical Olympiad competitors can also use this book. Appropriate chapters are Methods of Proof, Number Theory, and Combinatorics, as well as the subchapters 2.1 and 4.2.

An *instructor* can add some of the problems from the book to a regular course in order to stimulate and challenge the better students. Some of the theoretical subjects can also be incorporated in the course to give better insight and a new perspective. *Putnam and*

Beyond can be used as a textbook for problem-solving courses, in which case we recommend beginning with the first chapter. Students should be encouraged to come up with their own original solutions.

If you are a *graduate student* in mathematics, it is important that you know and understand the contents of this book. First, mastering problems and learning how to write down arguments are essential matters for good performance in doctoral examinations. Second, most of the presented facts are building blocks of graduate courses; knowing them will make these courses natural and easy.

It is important to keep in mind that detailed solutions to all problems are given in the second part of the book. After the solution we list the author of the problem and/or the place where it was published. In some cases we also describe how the problem fits in the big picture of mathematics.

"Don't bother to just be better than your contemporaries or predecessors. Try to be better than yourself" (W. Faulkner).

1

Methods of Proof

In this introductory chapter we explain some methods of mathematical proof. They are argument by contradiction, the principle of mathematical induction, the pigeonhole principle, the use of an ordering on a set, and the principle of invariance.

The basic nature of these methods and their universal use throughout mathematics makes this separate treatment necessary. In each case we have selected what we think are the most appropriate examples, solving some of them in detail and asking you to train your skills on the others. And since these are fundamental methods in mathematics, you should try to understand them in depth, for "it is better to understand many things than to know many things" (*Gustave Le Bon*).

1.1 Argument by Contradiction

The method of *argument by contradiction* proves a statement in the following way:

First, the statement is assumed to be false. Then, a sequence of logical deductions yields a conclusion that contradicts either the hypothesis (*indirect method*), *or a fact known to be true* (*reductio ad absurdum*). *This contradiction implies that the original statement must be true.*

This is a method that Euclid loved, and you can find it applied in some of the most beautiful proofs from his *Elements*. Euclid's most famous proof is that of the infinitude of prime numbers.

Euclid's theorem. *There are infinitely many prime numbers.*

Proof. Assume, to the contrary, that only finitely many prime numbers exist. List them as $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots, p_n$. The number $N = p_1 p_2 \ldots p_n + 1$ is divisible by a prime p, yet is coprime to $p_1, p_2, \ldots, p_n$. Therefore, p does not belong to our list of all prime numbers, a contradiction. Hence the initial assumption was false, proving that there are infinitely many primes.

R. Gelca and T. Andreescu, *Putnam and Beyond*, DOI 10.1007/978-3-319-58988-6_1

Here is a variation of this proof using repunits. If there are only finitely many primes, then the terms of the sequence

$$x_1 = 1, x_2 = 11, x_3 = 111, x_4 = 1111, ...$$

have only finitely many prime divisors, so there are finitely many terms of the sequence that exhaust them. Assume that the first n terms exhaust all prime divisors. Then $x_{n!}$ is divisible by $x_1, x_2, \ldots, x_n$, since for each $k \le n$, we can group the digits of $x_{n!}$ in strings of k. Then $x_{n!+1} = 10x_{n!} + 1$ is coprime with all of $x_1, x_2, \ldots, x_n$. This is a contradiction because all prime divisors of terms of the sequence were exhausted by $x_1, x_2, \ldots, x_n$. So there are infinitely many primes. $\square$

We continue our illustration of the method of argument by contradiction with an example of Euler.

Example. Prove that there is no polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2), \ldots$ are all prime numbers.

Solution. Assume the contrary and let $P(0) = p$, p prime. Then $a_0 = p$ and $P(kp)$ is divisible by p for all $k \ge 1$. Because we assumed that all these numbers are prime, it follows that $P(kp) = p$ for $k \ge 1$. Therefore, $P(x)$ takes the same value infinitely many times, a contradiction. Hence the conclusion. $\square$

The last example comes from I. Tomescu's book Problems in Combinatorics (Wiley, 1985).

Example. Let $F = \{E_1, E_2, \ldots, E_s\}$ be a family of subsets with r elements of some set X. Show that if the intersection of any $r+1$ (not necessarily distinct) sets in F is nonempty, then the intersection of all sets in F in nonempty.

Solution. Again we assume the contrary, namely that the intersection of all sets in F is empty. Consider the set $E_1 = \{x_1, x_2, \ldots, x_r\}$ Because none of the x_i, $i = 1, 2, \ldots, r$, lies in the intersection of all the e_j's (this intersection being empty), it follows that for each i we can find some E_{j_i} such that $x_i \notin E_{j_i}$. Then

$$E_1 \cap E_{j_1} \cap E_{j_2} \cap \ldots \cap E_{j_r} = \emptyset,$$

since, at the same time, this intersection is included in E_1 and does not contain any element of E_1. But this contradicts the hypothesis. It follows that our initial assumption was false, and hence the sets from the family F have a nonempty intersection. $\square$

The following problems help you practice this method, which will be used often in the book.

1. Prove that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is an irrational number.

2. Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.

3. Find the least positive integer n such that any set of n pairwise relatively prime integers greater than 1 and less than 2005 contains at least one prime number.

4. Let $\mathcal{F} = \{E_1, E_2, \ldots, E_m\}$ be a family of subsets with $n-2$ elements of a set S with n elements, $n \geq 3$. Show that if the union of any three subsets from $\mathcal{F}$ is not equal to S, then the union of all subsets from $\mathcal{F}$ is different from S.

5. Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

6. The union of nine planar surfaces, each of area equal to 1, has a total area equal to 5. Prove that the overlap of some two of these surfaces has an area greater than or equal to $\frac{1}{9}$.

7. Show that there does not exist a function $f : \mathbb{Z} \to \{1, 2, 3\}$ satisfying $f(x) \neq f(y)$ for all $x, y \in \mathbb{Z}$ such that $|x-y| \in \{2, 3, 5\}$.

8. Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$.

9. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$xf(y) + yf(x) = (x+y)f(x^2+y^2)$$
for all positive integers x and y.

10. Show that the interval $[0, 1]$ cannot be partitioned into two disjoint sets A and B such that $B = A + a$ for some real number a.

11. Let $n > 1$ be an arbitrary real number and let k be the number of positive prime numbers less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k+1$ that is bigger than n.

1.2 Mathematical Induction

The principle of *mathematical induction*, which lies at the very heart of Peano's axiomatic construction of the set of positive integers, is stated as follows.

Induction principle. *Given $P(n)$, a property depending on a positive integer n,*

(i) if $P(n_0)$ is true for some positive integer n_0, and
(ii) if for every $k \geq n_0$, $P(k)$ true implies $P(k+1)$ true,
then $P(n)$ is true for all $n \geq n_0$.

This means that when proving a statement by mathematical induction you should (i) check the base case and (ii) verify the inductive step by showing how to pass from an arbitrary integer to the next. Here is a simple example from combinatorial geometry.

Example. Finitely many lines divide the plane into regions. Show that these regions can be colored by two colors in such a way that neighboring regions have different colors.

Solution. We prove this by induction on the number n of lines. The base case $n = 1$ is straightforward, color one half-plane black, the other white.

For the inductive step, assume that we know how to color any map defined by k lines. Add the $(k+1)$st line to the picture; then keep the color of the regions on one side of this line the same while changing the color of the regions on the other side. The inductive step is illustrated in Figure 1.

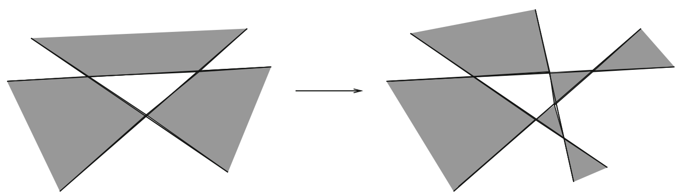

Figure 1

Regions that were adjacent previously still have different colors. Regions that share a segment of the $(k+1)$st line, which were part of the same region previously, now lie on opposite sides of the line. So they have different colors, too. This shows that the new map satisfies the required property and the induction is complete. $\square$

A classical proof by induction is that of Fermat's so-called little theorem.

Fermat's little theorem. *Let p be a prime number, and n a positive integer. Then $n^p - n$ is divisible by p.*

Proof. We prove the theorem by induction on n. The base case $n = 1$ is obvious. Let us assume that the property is true for $n = k$ and prove it for $n = k + 1$. Using the induction hypothesis, we obtain

$$(k+1)^p - (k+1) \equiv k^p + \sum_{j=1}^{p-1} \binom{p}{j} k^j + 1 - k - 1 \equiv \sum_{j=1}^{p-1} \binom{p}{j} k^j \pmod{p}.$$

The key observation is that for $1 \leq j \leq p-1$, $\binom{p}{j}$ is divisible by p. Indeed, examining

$$\binom{p}{j} = \frac{p(p-1)\cdots(p-j+1)}{1 \cdot 2 \cdots j},$$

it is easy to see that when $1 \leq j \leq p-1$, the numerator is divisible by p while the denominator is not. Therefore, $(k+1)^p - (k+1) \equiv 0 \pmod{p}$, which completes the induction. $\square$

The third example is a problem from the 5th W.L. Putnam Mathematical Competition, and it was selected because its solution combines several proofs by induction. If you find it too demanding, think of Vincent van Gogh's words: "The way to succeed is to keep your courage and patience, and to work energetically".

Example. For m a positive integer and n an integer greater than 2, define $f_1(n) = n$, $f_2(n) = n^{f_1(n)}, \ldots, f_{i+1}(n) = n^{f_i(n)}, \ldots$. Prove that

$$f_m(n) < n!! \cdots ! < f_{m+1}(n),$$

where the term in the middle has m factorials.

Solution. For convenience, let us introduce $g_0(n) = n$, and recursively $g_{i+1}(n) = (g_i(n))!$. The double inequality now reads

$$f_m(n) < g_m(n) < f_{m+1}(n).$$

For $m = 1$ this is obviously true, and it is only natural to think of this as the base case. We start by proving the inequality on the left by induction on m. First, note that if $t > 2n^2$ is a positive integer, then

$$t! > (n^2)^{t-n^2} = n^t n^{t-2n^2} > n^t.$$

Now, it is not hard to check that $g_m(n) > 2n^2$ for $m \geq 2$ and $n \geq 3$. With this in mind, let us assume the inequality to be true for $m = k$. Then

$$g_{k+1}(n) = (g_k(n))! > n^{g_k(n)} > n^{f_k(n)} = f_{k+1}(n),$$

which proves the inequality for $m = k + 1$. This verifies the inductive step and solves half of the problem.

Here we pause for a short observation. Sometimes the proof of a mathematical statement becomes simpler if the statement is strengthened. This is the case with the second inequality, which we replace by the much stronger

$$g_0(n)g_1(n) \cdots g_m(n) < f_{m+1}(n),$$

holding true for m and n as above.

As an intermediate step, we establish, by induction on m, that

$$g_0(n)g_1(n) \ldots g_m(n) < n^{g_0(n)g_1(n)\cdots g_{m-1}(n)},$$

for all m and all $n \geq 3$. The base case $m = 1$ is the obvious $n \cdot n! < n^n$. Now assume that the inequality is true for $m = k$, and prove it for $m = k + 1$. We have

$$\begin{aligned} g_0(n)g_1(n) \cdots g_{k+1}(n) &= g_0(n)g_0(n!) \cdots g_k(n!) < g_0(n)(n!)^{g_0(n!)g_1(n!)\cdots g_{k-1}(n!)} \\ &< n(n!)^{g_1(n)\cdots g_k(n)} < (n \cdot n!)^{g_1(n)\cdots g_k(n)} \\ &< (n^n)^{g_1(n)\cdots g_k(n)} = n^{g_0(n)g_1(n)\cdots g_k(n)}, \end{aligned}$$

completing this induction, and proving the claim.

Next, we show, also by induction on m, that $g_0(n)g_1(n)\cdots g_m(n) < f_{m+1}(n)$ for all n. The base case $m = 1$ is $n \cdot n! < n^n$; it follows by multiplying $1 \cdot 2 < n$ and $3 \cdot 4 \cdots n < n^{n-2}$. Let's see the inductive step. Using the inequality for the g_m's proved above and the assumption that the inequality holds for $m = k$, we obtain

$$g_0(n)\cdots g_m(n)g_{m+1}(n) < n^{g_0(n)\cdots g_m(n)} < n^{f_{m+1}(n)} = f_{m+2}(n),$$

which is the inequality for $m = k + 1$. This completes the last induction, and with it the solution to the problem. No fewer than three inductions were combined to solve the problem! □

Listen and you will forget, learn and you will remember, do it yourself and you will understand. Practice induction with the following examples.

12. Prove for all positive integers n the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

13. Prove that $|\sin nx| \le n|\sin x|$ for any real number x and positive integer n.

14. Prove that for any real numbers $x_1, x_2, \ldots, x_n$, $n \ge 1$,

$$|\sin x_1| + |\sin x_2| + \cdots + |\sin x_n| + |\cos(x_1 + x_2 + \cdots + x_n)| \ge 1.$$

15. Prove that $3^n \ge n^3$ for all positive integers n.

16. Let $n \ge 6$ be an integer. Show that

$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$

17. Let n be a positive integer. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

18. Prove that for any positive integer n there exists an n-digit number
(a) divisible by 2^n and containing only the digits 2 and 3;
(b) divisible by 5^n and containing only the digits 5, 6, 7, 8, 9.

19. Prove that for any $n \ge 1$, a $2^n \times 2^n$ checkerboard with a 1×1 corner square removed can be tiled by pieces of the form described in Figure 2.

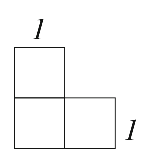

Figure 2

20. Given a sequence of integers $x_1, x_2, \dots, x_n$ whose sum is 1, prove that exactly one of the cyclic shifts

$$x_1, x_2, \dots, x_n; \quad x_2, \dots, x_n, x_1; \dots; \quad x_n, x_1, \dots, x_{n-1}$$

has all of its partial sums positive. (By a partial sum we mean the sum of the first k terms, $k \le n$.)

21. Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m$ be positive integers, $n, m > 1$. Assume that

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m < mn.$$

Prove that in the equality

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_m$$

one can suppress some (but not all) terms in such a way that the equality is still satisfied.

22. Prove that any function defined on the entire real axis can be written as the sum of two functions whose graphs admit centers of symmetry.

23. Prove that for any positive integer $n \ge 2$ there is a positive integer m that can be written simultaneously as a sum of $2, 3, \dots, n$ squares of nonzero integers.

24. Let n be a positive integer, $n \ge 2$, and let $a_1, a_2, \dots, a_{2n+1}$ be positive real numbers such that $a_1 < a_2 < \cdots < a_{2n+1}$. Prove that

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}.$$

25. It is given a finite set A of lines in a plane. It is known that, for some positive integer $k \ge 3$, for every subset B of A consisting of $k^2 + 1$ lines there are k points in the plane such that each line in B passes through at least one of them. Prove that there are k points in the plane such that every line in A passes through at least one of them.

Even more powerful is strong induction.

Induction principle (strong form). *Given $P(n)$ a property that depends on an integer n,*

(i) if $P(n_0), P(n_0+1), \dots, P(n_0+m)$ are true for some positive integer n_0 and nonnegative integer m, and

(ii) if for every $k > n_0 + m$, $P(j)$ true for all $n_0 \le j < k$ implies $P(k)$ true,
then $P(n)$ is true for all $n \ge n_0$.

We use strong induction to solve a problem from the 24th W.L. Putnam Mathematical Competition.

Example. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers m and n. Prove that $f(n) = n$ for every positive integer n.

Solution. The proof is of course by induction on n. Monotonicity implies right away that $f(1) = 1$. However, the base case is not the given $f(2) = 2$, but $f(2) = 2$ and $f(3) = 3$.

So let us find $f(3)$. Because f is strictly increasing, $f(3)f(5) = f(15) < f(18) = f(2)f(9)$. Hence $f(3)f(5) < 2f(9)$ and $f(9) < f(10) = f(2)f(5) = 2f(5)$. Combining these inequalities, we obtain $f(3)f(5) < 4f(5)$, so $f(3) < 4$. But we know that $f(3) > f(2) = 2$, which means that $f(3)$ can only be equal to 3.

The base case was the difficult part of the problem; the induction step is rather straightforward. Let $k > 3$ and assume that $f(j) = j$ for $j < k$. Consider $2^r(2m+1)$ to be the smallest even integer greater than or equal to k that is not a power of 2. This number is equal to either $k, k+1, k+2$, or $k+3$, and since $k > 3$, both 2^r and $2m+1$ are strictly less than k. From the induction hypothesis, we obtain $f(2^r(2m+1)) = f(2^r)f(2m+1) = 2^r(2m+1)$. Monotonicity, combined with the fact that there are at most $2^r(2m+1)$ values that the function can take in the interval $[1, 2^r(2m+1)]$, implies that $f(l) = l$ for $l \leq 2^r(2m+1)$. In particular, $f(k) = k$. We conclude that $f(n) = n$ for all positive integers n. □

A function $f : \mathbb{N} \to \mathbb{C}$ with the property that $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are coprime is called a multiplicative function. Examples include the Euler totient function and the Möbius function. In the case of our problem, the multiplicative function is also strictly increasing. A more general result of P. Erdös shows that any increasing multiplicative function that is not constant is of the form $f(n) = n^\alpha$ for some $\alpha > 0$.

The second example is from the 1999 Balkan Mathematical Olympiad, being proposed by B. Enescu.

Example. Let $0 \leq x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ be a sequence of non-negative integers such that for every index k, the number of the terms of the sequence that are less than or equal to k is finite. We denote this number by y_k. Prove that for any two positive integer numbers m and n, the following inequality holds

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j \geq (n+1)(m+1).$$

Solution. We will prove this by strong induction on $s = m + n$.

The base case $s = 0$ is obvious, since either $x_0 > 0$, in which case the first sum is at least 1, or $x_0 = 0$, in which case $y_0 \geq 1$ and the second sum is at least 1. Let us now assume that the inequality holds for all $s \leq N - 1$ and let us prove it for $s = N$.

If $x_n \geq m + 1$, then

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j = \sum_{i=0}^{n-1} x_i + \sum_{j=0}^{m} y_j + x_n,$$

where $\sum_{i=0}^{n-1} x_i$ is taken to be zero if $n = 0$. The induction hypothesis implies that this is greater than or equal to $n(m+1) + (m+1) = (n+1)(m+1)$, and we are done.

If $x_n < m + 1$, then $y_m \geq n + 1$ and so

$$\sum_{i=0}^{n} x_i + \sum_{j=0}^{m} y_j = \sum_{i=0}^{n} x_i + \sum_{j=0}^{m-1} y_j + y_m \geq (n+1)m + (n+1) = (n+1)(m+1),$$

where for the inequality we used again the induction hypothesis. This completes the induction and we are done. □

26. Show that every positive integer can be written as a sum of distinct terms of the Fibonacci sequence. (The Fibonacci sequence $(F_n)_n$ is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$.)

27. Prove that the Fibonacci sequence satisfies the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \text{ for } n \geq 0.$$

28. Prove that the Fibonacci sequence satisfies the identity

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3, \text{ for } n \geq 0.$$

29. Show that an isosceles triangle with one angle of 120° can be dissected into $n \geq 4$ triangles similar to it.

30. Show that for all $n > 3$ there exists an n-gon whose sides are not all equal and such that the sum of the distances from any interior point to each of the sides is constant. (An n-gon is a polygon with n sides.)

31. The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

32. Prove that any polygon (convex or not) can be dissected into triangles by interior diagonals.

33. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \dots \pm n^2$ for some positive integer n and some choice of the signs.

Now we demonstrate a less frequently encountered form of induction that can be traced back to Cauchy's work, where it was used to prove the arithmetic mean-geometric mean inequality. We apply this method to solve a problem from D. Buşneag, I. Maftei, *Themes for Mathematics Circles and Contests* (Scrisul Românesc, Craiova, 1983).

Example. Let $a_1, a_2, \dots, a_n$ be real numbers greater than 1. Prove the inequality

$$\sum_{i=1}^{n} \frac{1}{1+a_i} \geq \frac{n}{1+\sqrt[n]{a_1 a_2 \cdots a_n}}.$$

Solution. As always, we start with the base case:

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} \geq \frac{2}{1+\sqrt{a_1 a_2}}.$$

Multiplying out the denominators yields the equivalent inequality

$$(2 + a_1 + a_2)(1 + \sqrt{a_1 a_2}) \geq 2(1 + a_1 + a_2 + a_1 a_2).$$

After multiplications and cancellations, we obtain

$$2\sqrt{a_1a_2} + (a_1 + a_2)\sqrt{a_1a_2} \geq a_1 + a_2 + 2a_1a_2.$$

This can be rewritten as

$$2\sqrt{a_1a_2}(1 - \sqrt{a_1a_2}) + (a_1 + a_2)(\sqrt{a_1a_2} - 1) \geq 0,$$

or

$$(\sqrt{a_1a_2} - 1)(a_1 + a_2 - 2\sqrt{a_1a_2}) \geq 0.$$

The inequality is now obvious since $a_1a_2 \geq 1$ and $a_1 + a_2 \geq 2\sqrt{a_1a_2}$.

Now instead of exhausting all positive integers n, we downgrade our goal and check just the powers of 2. So we prove that the inequality holds for $n = 2^k$ by induction on k. Assuming it true for k, we can write

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{1+a_i} &= \sum_{i=1}^{2^k} \frac{1}{1+a_i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{1+a_i} \\
&\geq 2^k \left(\frac{1}{1 + \sqrt[2^k]{a_1a_2 \dots a_{2^k}}} + \frac{1}{1 + \sqrt[2^k]{a_{2^k+1}a_{2^k+2} \cdots a_{2^{k+1}}}} \right) \\
&\geq 2^k \frac{2}{1 + \sqrt[2^{k+1}]{a_1a_2 \cdots a_{2^{k+1}}}},
\end{aligned}$$

where the first inequality follows from the induction hypothesis, and the second is just the base case. This completes the induction.

Now we have to cover the cases in which n is not a power of 2. We do the induction backward, namely, we assume that the inequality holds for $n + 1$ numbers and prove it for n. Let $a_1, a_2, \dots, a_n$ be some real numbers greater than 1. Attach to them the number $\sqrt[n]{a_1a_2 \cdots a_n}$. When writing the inequality for these $n + 1$ numbers, we obtain

$$\frac{1}{1+a_1} + \cdots + \frac{1}{1 + \sqrt[n]{a_1a_2 \cdots a_n}} \geq \frac{n+1}{1 + \sqrt[n+1]{a_1 \cdots a_n \sqrt[n]{a_1a_2 \cdots a_n}}}.$$

Recognize the complicated radical on the right to be $\sqrt[n]{a_1a_2 \dots a_n}$. After cancelling the last term on the left, we obtain

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \geq \frac{n}{1 + \sqrt[n]{a_1a_2 \cdots a_n}},$$

as desired. The inequality is now proved, since we can reach any positive integer n by starting with a sufficiently large power of 2 and working backward. □

Try to apply the same technique to the following problems.

34. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2} \text{ for any } x_1, x_2.$$

Prove that

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)=\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$$

for any $x_1, x_2, \ldots, x_n$.

35. Show that if $a_1, a_2, \ldots, a_n$ are nonnegative numbers, then

$$(1+a_1)(1+a_2)\cdots(1+a_n)\geq(1+\sqrt[n]{a_1a_2\cdots a_n})^n.$$

1.3 The Pigeonhole Principle

The *pigeonhole principle* (or *Dirichlet's box principle*) is usually applied to problems in combinatorial set theory, combinatorial geometry, and number theory. In its intuitive form, it can be stated as follows.

Pigeonhole principle. *If $kn+1$ objects ($k \geq 1$ not necessarily finite) are distributed among n boxes, one of the boxes will contain at least $k+1$ objects.*

This is merely an observation, and it was Dirichlet who first used it to prove nontrivial mathematical results. The name comes from the intuitive image of several pigeons entering randomly in some holes. If there are more pigeons than holes, then we know for sure that one hole has more than one pigeon. We begin with an easy problem, which was given at the International Mathematical Olympiad in 1972, proposed by Russia.

Example. Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements.

Solution. Let S be the set of 10 numbers. It has $2^{10}-2=1022$ subsets that differ from both S and the empty set. They are the "pigeons". If $A \subset S$, the sum of elements of A cannot exceed $91+92+\cdots+99=855$. The numbers between 1 and 855, which are all possible sums, are the "holes". Because the number of "pigeons" exceeds the number of "holes", there will be two "pigeons" in the same "hole". Specifically, there will be two subsets with the same sum of elements. Deleting the common elements, we obtain two disjoint sets with the same sum of elements. □

Here is a more difficult problem from the 26th International Mathematical Olympiad, proposed by Mongolia.

Example. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26, prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

Solution. We show more generally that if the prime divisors of elements in M are among the prime numbers $p_1, p_2, \ldots, p_n$ and M has at least $3\cdot 2^n+1$ elements, then it contains a subset of four distinct elements whose product is a fourth power.

To each element m in M we associate an n-tuple $(x_1, x_2, \ldots, x_n)$, where x_i is 0 if the exponent of p_i in the prime factorization of m is even, and 1 otherwise. These n-tuples are the "objects". The "boxes" are the 2^n possible choices of 0's and 1's. Hence, by the pigeonhole principle, every subset of $2^n + 1$ elements of M contains two distinct elements with the same associated n-tuple, and the product of these two elements is then a square.

We can repeatedly take aside such pairs and replace them with two of the remaining numbers. From the set M, which has at least $3 \cdot 2^n + 1$ elements, we can select $2^n + 1$ such pairs or more. Consider the $2^n + 1$ numbers that are products of the two elements of each pair. The argument can be repeated for their square roots, giving four elements a, b, c, d in M such that $\sqrt{ab}\sqrt{cd}$ is a perfect square. Then $abcd$ is a fourth power and we are done. For our problem $n = 9$, while $1985 > 3 \cdot 2^9 + 1 = 1537$. □

The third example comes from the 67th W.L. Putnam Mathematical Competition, 2006.

Example. Prove that for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a nonempty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

Solution. Recall that the fractional part of a real number x is $x - \lfloor x \rfloor$. Let us look at the fractional parts of the numbers $x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_n$. If any of them is either in the interval $\left[0, \frac{1}{n+1}\right]$ or $\left[\frac{n}{n+1}, 1\right]$, then we are done. If not, we consider these n numbers as the "pigeons" and the $n - 1$ intervals $\left[\frac{1}{n+1}, \frac{2}{n+1}\right], \left[\frac{2}{n+1}, \frac{3}{n+1}\right], \ldots, \left[\frac{n-1}{n+1}, \frac{n}{n+1}\right]$ as the "holes". By the pigeonhole principle, two of these sums, say $x_1 + x_2 + \cdots + x_k$ and $x_1 + x_2 + \cdots + x_{k+m}$, belong to the same interval. But then their difference $x_{k+1} + \cdots + x_{k+m}$ lies within a distance of $\frac{1}{n+1}$ of an integer, and we are done. □

More problems are listed below.

36. Given 50 distinct positive integers strictly less than 99, prove that some two of them sum to 99.

37. A sequence of m positive integers contains exactly n distinct terms. Prove that if $2^n \le m$ then there exists a block of consecutive terms whose product is a perfect square.

38. Let $x_1, x_2, \ldots, x_3, \ldots$ be a sequence of integers such that

$$1 = x_1 < x_2 < x_3 < \cdots \text{ and } x_{n+1} \le 2n \text{ for } n = 1, 2, 3, \ldots.$$

Show that every positive integer k is equal to $x_i - x_j$ for some i and j.

39. Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that the equation $ax^2 + by^2 \equiv c \pmod{p}$ has integer solutions.

40. In each of the unit squares of a 10×10 checkerboard, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.

41. Show that there is a positive term of the Fibonacci sequence that is divisible by 1000.

42. Let $x_1 = x_2 = x_3 = 1$ and $x_{n+3} = x_n + x_{n+1}x_{n+2}$ for all positive integers n. Prove that for any positive integer m there is an index k such that m divides x_k.

43. A chess player trains by playing at least one game per day, but, to avoid exhaustion, no more than 12 games a week. Prove that there is a group of consecutive days in which he plays exactly 20 games.

44. Let m be a positive integer. Prove that among any $2m+1$ distinct integers of absolute value less than or equal to $2m-1$ there exist three whose sum is equal to zero.

45. There are n people at a party. Prove that there are two of them such that among the remaining $n-2$ people there are at least $\lfloor \frac{n}{2} \rfloor - 1$, each of whom knows both or else knows neither of the two.

46. Let $x_1, x_2, \ldots, x_k$ be real numbers such that the set

$$A = \{\cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_k) \mid n \geq 1\}$$

is finite. Prove that all the x_i are rational numbers.

Particularly attractive are the problems in which the pigeons and holes are geometric objects. Here is a problem from a Chinese mathematical competition.

Example. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.

Solution. Divide the square into four equal squares, which are the "boxes". From the $9 = 2 \times 4 + 1$ points, at least $3 = 2 + 1$ will lie in the same box. We are left to show that the area of a triangle placed inside a square does not exceed half the area of the square.

Cut the square by the line passing through a vertex of the triangle, as in Figure 3. Since the area of a triangle is $\frac{\text{base} \times \text{height}}{2}$ and the area of a rectangle is base $\times$ height, the inequality holds for the two smaller triangles and their corresponding rectangles. Adding up the two inequalities, we obtain the inequality for the square. This completes the solution. $\square$

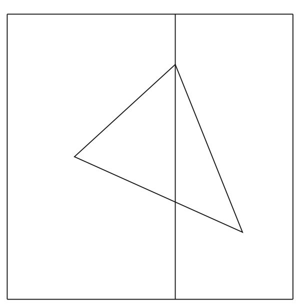

Figure 3

47. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

48. Each of nine straight lines divides a square into two quadrilaterals with the ratio of their areas equal to $r > 0$. Prove that at least three of these lines are concurrent.

49. Show that any convex polyhedron has two faces with the same number of edges.

50. Draw the diagonals of a 21-gon. Prove that at least one angle of less than 1° is formed. (Angles of 0° are allowed in the case that two diagonals are parallel.)

51. Let $P_1, P_2, \ldots, P_{2n}$ be a permutation of the vertices of a regular polygon. Prove that the closed polygonal line $P_1P_2\ldots P_{2n}$ contains a pair of parallel segments.

52. Let S be a convex set in the plane that contains three noncollinear points. Each point of S is colored by one of p colors, $p > 1$. Prove that for any $n \geq 3$ there exist infinitely many congruent n-gons whose vertices are all of the same color.

53. The points of the plane are colored by finitely many colors. Prove that one can find a rectangle with vertices of the same color.

54. Inside the unit square lie several circles the sum of whose circumferences is equal to 10. Prove that there exist infinitely many lines each of which intersects at least four of the circles.

1.4 Ordered Sets and Extremal Elements

An *order* on a set is a relation $\leq$ with three properties: (i) $a \leq a$; (ii) if $a \leq b$ and $b \leq a$, then $a = b$; (iii) $a \leq b$ and $b \leq c$ implies $a \leq c$. The order is called total if any two elements are comparable, that is, if for every a and b, either $a \leq b$ or $b \leq a$. The simplest example of a total order is $\leq$ on the set of real numbers. The existing order on a set can be useful when solving problems. This is the case with the following two examples, the second of which is a problem of G. Galperin published in the Russian journal *Quantum*.

Example. Prove that among any 50 distinct positive integers strictly less than 100 there are two that are coprime.

Solution. Order the numbers: $x_1 < x_2 < \ldots < x_{50}$. If in this sequence there are two consecutive integers, they are coprime and we are done. Otherwise, $x_{50} \geq x_1 + 2 \cdot 49 = 99$. Equality must hold, since $x_{50} < 100$, and in this case the numbers are precisely the 50 odd integers less than 100. Among them 3 is coprime to 7. The problem is solved.

Example. Given finitely many squares whose areas add up to 1, show that they can be arranged without overlaps inside a square of area 2.

Solution. The guess is that a tight way of arranging the small squares inside the big square is by placing the squares in decreasing order of side-lengths.

To prove that this works, denote by x the side length of the first (that is, the largest) square. Arrange the squares inside a square of side $\sqrt{2}$ in the following way. Place the first in the lower-left corner, the next to its right, and so on, until obstructed by the right side of the big square. Then jump to height x, and start building the second horizontal layer of squares by the same rule. Keep going until the squares have been exhausted (see Figure 4).

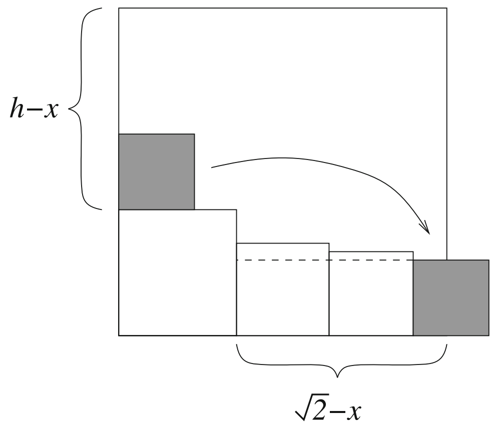

Figure 4

Let h be the total height of the layers. We are to show that $h \leq \sqrt{2}$, which in turn will imply that all the squares lie inside the square of side $\sqrt{2}$. To this end, we will find a lower bound for the total area of the squares in terms of x and h. Let us mentally transfer the first square of each layer to the right side of the previous layer. Now each layer exits the square, as shown in Figure 4.

It follows that the sum of the areas of all squares but the first is greater than or equal to $(\sqrt{2} - x)(h - x)$. This is because each newly obtained layer includes rectangles of base $\sqrt{2} - x$ and with the sum of heights equal to $h - x$. From the fact that the total area of the squares is 1, it follows that

$$x^2 + (\sqrt{2} - x)(h - x) \leq 1.$$

This implies that

$$h \leq \frac{2x^2 - \sqrt{2}x - 1}{x - \sqrt{2}}.$$

That $h \leq \sqrt{2}$ will follow from

$$\frac{2x^2 - \sqrt{2}x - 1}{x - \sqrt{2}} \leq \sqrt{2}.$$

This is equivalent to

$$2x^2 - 2\sqrt{2}x + 1 \geq 0,$$

or $(x\sqrt{2} - 1)^2 \geq 0$, which is obvious and we are done. □

What we particularly like about the shaded square from Figure 4 is that it plays the role of the "largest square" when placed on the left, and of the "smallest square" when placed on the right. Here are more problems.

55. Given $n \geq 3$ points in the plane, prove that some three of them form an angle less than or equal to $\frac{\pi}{n}$.

56. Consider a planar region of area 1, obtained as the union of finitely many disks. Prove that from these disks we can select some that are mutually disjoint and have total area at least $\frac{1}{9}$.

57. Suppose that $n(r)$ denotes the number of points with integer coordinates on a circle of radius $r > 1$. Prove that

$$n(r) < 2\pi \sqrt[3]{r^2}.$$

58. Prove that among any eight positive integers less than 2004 there are four, say a, b, c, and d, such that

$$4 + d \leq a + b + c \leq 4d.$$

59. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of distinct positive integers. Prove that for any positive integer n,

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

60. Let X be a subset of the positive integers with the property that the sum of any two not necessarily distinct elements in X is again in X. Suppose that $\{a_1, a_2, \ldots, a_n\}$ is the set of all positive integers not in X. Prove that $a_1 + a_2 + \cdots + a_n \leq n^2$.

61. Let $P(x)$ be a polynomial with integer coefficients, of degree $n \geq 2$. Prove that the set $A = \{x \in \mathbb{Z} \mid P(P(x)) = x\}$ has at most n elements.

An order on a finite set has *maximal* and *minimal* elements. If the order is total, the maximal (respectively, minimal) element is unique. Quite often it is useful to look at such extremal elements, like in the solution to the following problem.

Example. Prove that it is impossible to dissect a cube into finitely many cubes, no two of which are the same size.

Solution. For the solution, assume that such a dissection exists, and look at the bottom face. It is cut into squares. Take the smallest of these squares. It is not hard to see that this square lies in the interior of the face, meaning that it does not touch any side of the bottom face. Look at the cube that lies right above this square! This cube is surrounded by bigger cubes, so its upper face must again be dissected into squares by the cubes that lie on top of it. Take the smallest of the cubes and repeat the argument. This process never stops, since the cubes that lie on top of one of these little cubes cannot end up all touching the upper face of the original cube. This contradicts the finiteness of the decomposition. Hence the conclusion. □

By contrast, a square can be dissected into finitely many squares of distinct size. Why does the above argument not apply in this case?

And now an example of a more exotic kind.

Example. Given is a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from any other planet. Prove that the total area of these sets is equal to the surface area of one planet.

Solution. The problem was on the short list of the 22nd International Mathematical Olympiad, proposed by the Soviet Union. The solution below we found in I. Cuculescu's book on the *International Mathematical Olympiads* (Editura Tehnică, Bucharest, 1984).

Choose a preferential direction in space, which defines the north pole of each planet. Next, define an order on the set of planets by saying that planet A is greater than planet B if on removing all other planets from space, the north pole of B is visible from A. Figure 5 shows that for two planets A and B, either $A < B$ or $B < A$, and also that for three planets A, B, C, if $A < B$ and $B < C$ then $A < C$. The only case in which something can go wrong is that in which the preferential direction is perpendicular to the segment joining the centers of two planets. If this is not the case, then $<$ defines a total order on the planets. This order has a unique maximal element M. The north pole of M is the only north pole not visible from another planet.

Now consider a sphere of the same radius as the planets. Remove from it all north poles defined by directions that are perpendicular to the axes of two of the planets. This is a set of area zero. For every other point on this sphere, there exists a direction in space that makes it the north pole, and for that direction, there exists a unique north pole on one of the planets that is not visible from the others. As such, the surface of the newly introduced sphere is covered by patches translated from the other planets. Hence the total area of invisible points is equal to the area of this sphere, which in turn is the area of one of the planets. $\square$

62. Complete the square in Figure 6 with integers between 1 and 9 such that the sum of the numbers in each row, column, and diagonal is as indicated.

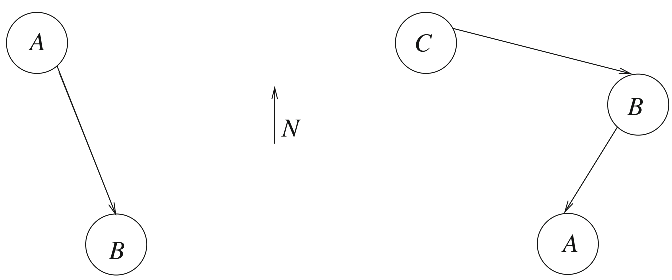

Figure 5

63. Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.

64. Show that any polygon in the plane has a vertex, and a side not containing that vertex, such that the projection of the vertex onto the side lies in the interior of the side or at one of its endpoints.

65. In some country all roads between cities are one-way and such that once you leave a city you cannot return to it again. Prove that there exists a city into which all roads enter and a city from which all roads exit.

66. At a party assume that no boy dances with all the girls, but each girl dances with at least one boy. Prove that there are two girl-boy couples gb and $g'b'$ who dance, whereas b does not dance with g', and g does not dance with b'.

67. In the plane we have marked a set S of points with integer coordinates. We are also given a finite set V of vectors with integer coordinates. Assume that S has the property that for every marked point P, if we place all vectors from V with origin are P, then more of their ends are marked than unmarked. Show that the set of marked points is infinite.

68. The entries of a matrix are real numbers of absolute value less than or equal to 1, and the sum of the elements in each column is 0. Prove that we can permute the elements of each column in such a way that the sum of the elements in each row will have absolute value less than or equal to 2.

69. Find all odd positive integers n greater than 1 such that for any coprime divisors a and b of n, the number $a + b - 1$ is also a divisor of n.

70. The positive integers are colored by two colors. Prove that there exists an infinite sequence of positive integers $k_1 < k_2 < \cdots < k_n < \cdots$ with the property that the terms of the sequence $2k_1 < k_1 + k_2 < 2k_2 < k_2 + k_3 < 2k_3 < \cdots$ are all of the same color.

71. Let $P_1 P_2 \ldots P_n$ be a convex polygon in the plane. Assume that for any pair of vertices P_i and P_j, there exists a vertex P_k of the polygon such that $\angle P_i P_k P_j = \pi/3$. Show that $n = 3$.

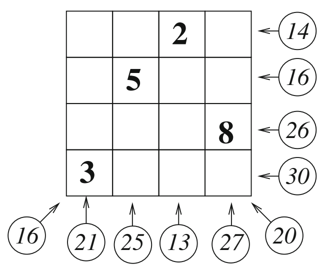

Figure 6

1.5 Invariants and Semi-Invariants

In general, a mathematical object can be studied from many points of view, and it is always desirable to decide whether various constructions produce the same object. One usually distinguishes mathematical objects by some of their properties. An elegant method is to associate to a family of mathematical objects an *invariant*, which can be a number, an algebraic structure, or some property, and then distinguish objects by the different values of the invariant.

The general framework is that of a set of objects or configurations acted on by transformations that identify them (usually called isomorphisms). Invariants then give obstructions to transforming one object into another. Sometimes, although not very often, an invariant is able to tell precisely which objects can be transformed into one another, in which case the invariant is called complete.

An example of an invariant (which arises from more advanced mathematics yet is easy to explain) is the property of a knot to be 3-colorable. Formally, a knot is a simple closed curve in $\mathbb{R}^3$. Intuitively it is a knot on a rope with connected endpoints, such as the right-handed trefoil knot from Figure 7.

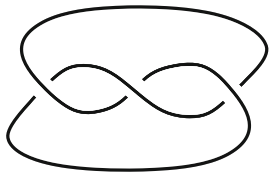

Figure 7

How can one prove *mathematically* that this knot is indeed "knotted"? The answer is, using an invariant. To define this invariant, we need the notion of a knot diagram. Such a diagram is the image of a regular projection (all self-intersections are nontangential and are double points) of the knot onto a plane with crossing information recorded at each double point, just like the one in Figure 7. But a knot can have many diagrams (pull the strands around, letting them pass over each other).

A deep theorem of K. Reidemeister states that two diagrams represent the same knot if they can be transformed into one another by the three types of moves described in Figure 8.

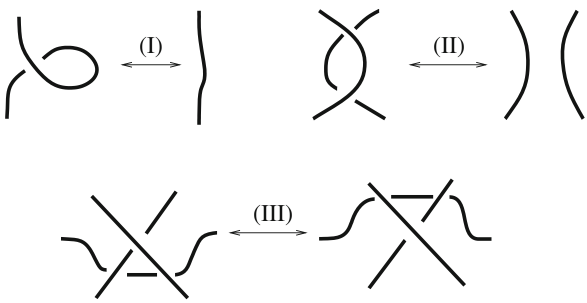

Figure 8

The simplest knot invariant was introduced by the same Reidemeister, and is the property of a knot diagram to be 3-colorable. This means that you can color each strand in the knot diagram by a residue class modulo 3 such that

(i) at least two distinct residue classes modulo 3 are used, and

(ii) at each crossing, $a + c \equiv 2b \pmod 3$, where b is the color of the arc that crosses over, and a and c are the colors of the other two arcs (corresponding to the strand that crosses under).

It is rather easy to prove, by examining the local picture, that this property is invariant under Reidemeister moves. Hence this is an invariant of knots, not just of knot diagrams.

The trefoil knot is 3-colorable, as demonstrated in Figure 9. On the other hand, the unknotted circle is not 3-colorable, because its simplest diagram, the one with no crossings, cannot be 3-colored. Hence the trefoil knot is knotted.

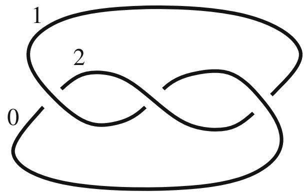

Figure 9

This 3-colorability is, however, not a complete invariant. We now give an example of a complete invariant from geometry. In the early nineteenth century, F. Bolyai and a less well-known mathematician Gerwin proved that given two polygons of equal area, the first can be dissected by finitely many straight cuts and then assembled to produce the second polygon. In his list of 23 problems presented to the International Congress of Mathematicians, D. Hilbert listed as number 3 the question whether the same property remains true for solid polyhedra of the same volume, and if not, what would the obstruction be.

The problem was solved by M. Dehn, a student of Hilbert. Dehn defined an invariant that associates to a finite disjoint union of polyhedra P the sum $I(P)$ of all their dihedral angles reduced modulo rational multiples of π (viewed as an element in $\mathbb{R}/\pi\mathbb{Q}$). He showed that two polyhedra P_1 and P_2 having the same volume can be transformed into one another if and only if $I(P_1) = I(P_2)$, i.e., if and only if the sums of their dihedral angles differ by a rational multiple of π.

It is good to know that the quest for invariants dominated twentieth-century geometry. That being said, let us return to the realm of elementary mathematics with a short list problem from the 46th International Mathematical Olympiad.

Example. There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighboring markers. Prove that one can reach a configuration with only two markers left if and only if $n - 1$ is not divisible by 3.

Solution. We refer to a marker by the color of its visible face. Note that the parity of the number of black markers remains unchanged during the game. Hence if only two markers are left, they must have the same color.

We define an invariant as follows. To a white marker with t black markers to its left we assign the number $(-1)^t$. Only white markers have numbers assigned to them. The invariant S is the residue class modulo 3 of the sum of all numbers assigned to the white markers.

It is easy to check that S is invariant under the operation defined in the statement. For instance, if a white marker with t black markers on the left and whose neighbors are both black is removed, then S increases by $-(-1)^t + (-1)^{t-1} + (-1)^{t-1} = 3(-1)^{t-1}$, which is zero modulo 3. The other three cases are analogous.

If the game ends with two black markers then S is zero; if it ends with two white markers, then S is 2. This proves that $n - 1$ is not divisible by 3.

Conversely, if we start with $n \geq 5$ white markers, $n \equiv 0$ or 2 modulo 3, then by removing in three consecutive moves the leftmost allowed white markers, we obtain a row of $n-3$ white markers. Working backward, we can reach either 2 white markers or 3 white markers. In the latter case, with one more move we reach 2 black markers as desired. □

Now try to find the invariants that lead to the solutions of the following problems.

72. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, 1/\sqrt{2})$ using this operation?

73. There are 2000 white balls in a box. There are also unlimited supplies of white, green, and red balls, initially outside the box. During each turn, we can replace two balls in the box with one or two balls as follows: two whites with a green, two reds with a green, two greens with a white and red, a white and a green with a red, or a green and red with a white.

(a) After finitely many of the above operations there are three balls left in the box. Prove that at least one of them is green.
(b) Is it possible that after finitely many operations only one ball is left in the box?

74. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?

75. Starting with an ordered quadruple of positive integers, a generalized Euclidean algorithm is applied successively as follows: if the numbers are x, y, u, v and $x > y$, then the quadruple is replaced by $x-y, y, u+v, v$. Otherwise, it is replaced by $x, y-x, u, v+u$. The algorithm stops when the numbers in the first pair become equal (in which case they are equal to the greatest common divisor of x and y). Assume that we start with m, n, m, n. Prove that when the algorithm ends, the arithmetic mean of the numbers in the second pair equals the least common multiple of m and n.

76. On an arbitrarily large chessboard consider a generalized knight that can jump p squares in one direction and q in the other, $p, q > 0$. Show that such a knight can return to its initial position only after an *even* number of jumps.

77. Prove that the figure eight knot described in Figure 10 is knotted.

78. In the squares of a 3×3 chessboard are written the signs + and − as described in Figure 11(a). Consider the operations in which one is allowed to simultaneously change all signs in some row or column. Can one change the given configuration to the one in Figure 11(b) by applying such operations finitely many times?

79. The number $99\ldots99$ (having 1997 nines) is written on a blackboard. Each minute, one number written on the blackboard is factored into two factors and erased, each factor is (independently) increased or decreased by 2, and the resulting two numbers are written. Is it possible that at some point all of the numbers on the blackboard are equal to 9?

80. Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

81. For an integer $n \geq 4$, consider an n-gon inscribed in a circle. Dissect the n-gon into $n-2$ triangles by nonintersecting diagonals. Prove that the sum of the radii of the incircles of these $n-2$ triangles does not depend on the dissection.

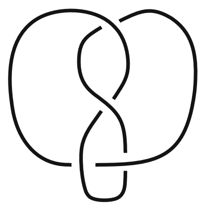

Figure 10

(a)

+	+	−
+	+	−
−	−	+

(b)

−	−	+
+	−	−
−	−	+

Figure 11

In some cases a semi-invariant will do. A *semi-invariant* (also known as monovariant) is a quantity that, although not constant under a specific transformation, keeps increasing (or decreasing). As such it provides a unidirectional obstruction.

For his solution to the following problem from the 27th International Mathematical Olympiad, J. Keane, then a member of the US team, was awarded a special prize.

Example. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all of the five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z, respectively, and $y < 0$, then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution. The answer is yes. The key idea of the proof is to construct an integer-valued semi-invariant whose value decreases when the operation is performed. The existence of such a semi-invariant will guarantee that the operation can be performed only finitely many times.

Notice that the sum of the five numbers on the pentagon is preserved by the operation, so it is natural to look at the sum of the absolute values of the five numbers. When the operation is performed this quantity decreases by $|x|+|z|-|x+y|-|y+z|$. Although this expression is not always positive, it suggests a new choice. The desired semi-invariant should include the absolute values of pairwise sums as well. Upon testing the new expression and continuing this idea, we discover in turn that the desired semi-invariant should also include absolute values of sums of triples and foursomes. At last, with a pentagon numbered v, w, x, y, z and the semi-invariant defined by

$$\begin{aligned} S(v, w, x, y, z) = &|v| + |w| + |x| + |y| + |z| + |v + w| + |w + x| + |x + y| \\ &+ |y + z| + |z + v| + |v + w + x| + |w + x + y| + |x + y + z| \\ &+ |y + z + v| + |z + v + w| + |v + w + x + y| + |w + x + y + z| \\ &+ |x + y + z + v| + |y + z + v + w| + |z + v + w + x|, \end{aligned}$$

we find that the operation reduces the value of S by the simple expression

$$|z + v + w + x| - |z + v + w + x + 2y| = |s - y| - |s + y|,$$

where $s = v+w+x+y+z$. Since $s > 0$ and $y < 0$, we see that $|s-y|-|s+y| > 0$, so S has the required property. It follows that the operation can be performed only finitely many times. $\square$

Using the semi-invariant we produced a proof based on Fermat's infinite descent method. This method will be explained in the Number Theory chapter of this book. Here the emphasis was on the guess of the semi-invariant. And now some problems.

82. A real number is written in each square of an $n \times n$ chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row or column is positive.

83. Starting with an ordered quadruple of integers, perform repeatedly the operation

$$(a, b, c, d) \xrightarrow{T} (|a - b|, |b - c|, |c - d|, |d - a|).$$

Prove that after finitely many steps, the quadruple becomes $(0, 0, 0, 0)$.

84. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

85. Consider the integer lattice in the plane, with one pebble placed at the origin. We play a game in which at each step one pebble is removed from a node of the lattice and two new pebbles are placed at two neighboring nodes, provided that those nodes are unoccupied. Prove that at any time there will be a pebble at distance at most 5 from the origin.

Methods of Proof

1. Assume the contrary, namely that $\sqrt{2} + \sqrt{3} + \sqrt{5} = r$, where r is a rational number. Square the equality $\sqrt{2} + \sqrt{3} = r - \sqrt{5}$ to obtain $5 + 2\sqrt{6} = r^2 + 5 - 2r\sqrt{5}$. It follows that $2\sqrt{6} + 2r\sqrt{5}$ is itself rational. Squaring again, we find that $24 + 20r^2 + 8r\sqrt{30}$ is rational, and hence $\sqrt{30}$ is rational, too. Pythagoras' method for proving that $\sqrt{2}$ is irrational can now be applied to show that this is not true. Write $\sqrt{30} = \frac{m}{n}$ in lowest terms; then transform this into $m^2 = 30n^2$. It follows that m is divisible by 2 and because $2\left(\frac{m}{2}\right)^2 = 15n^2$ it follows that n is divisible by 2 as well. So the fraction was not in lowest terms, a contradiction. We conclude that the initial assumption was false, and therefore $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

2. Assume that such numbers do exist, and let us look at their prime factorizations. For primes p greater than 7, at most one of the numbers can be divisible by p, and the partition cannot exist. Thus the prime factors of the given numbers can be only 2, 3, 5, and 7.

We now look at repeated prime factors. Because the difference between two numbers divisible by 4 is at least 4, at most three of the nine numbers are divisible by 4. Also, at most one is divisible by 9, at most one by 25, and at most one by 49. Eliminating these at most $3 + 1 + 1 + 1 = 6$ numbers, we are left with at least three numbers among the nine that do not contain repeated prime factors. They are among the divisors of $2 \cdot 3 \cdot 5 \cdot 7$, and so among the numbers

$$2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 14,\ 15,\ 21,\ 30,\ 35,\ 42,\ 70,\ 105,\ 210.$$

Because the difference between the largest and the smallest of these three numbers is at most 9, none of them can be greater than 21. We have to look at the sequence $1, 2, 3, \ldots, 29$. Any subsequence of consecutive integers of length 9 that has a term greater than 10 contains a prime number greater than or equal to 11, which is impossible. And from $1, 2, \ldots, 10$ we cannot select nine consecutive numbers with the required property. This contradicts our assumption, and the problem is solved.

Remark. In the argument, the number 29 can be replaced by 27, namely by 21 plus the 6 numbers that can have repeated prime factor.

R. Gelca and T. Andreescu, *Putnam and Beyond*, DOI 10.1007/978-3-319-58988-6

3. The example $2^2, 3^2, 5^2, \ldots, 43^2$, where we considered the squares of the first 14 prime numbers, shows that $n \geq 15$.

Assume that there exist $a_1, a_2, \ldots, a_{15}$, pairwise relatively prime integers greater than 1 and less than 2005, none of which is a prime. Let q_k be the least prime number in the factorization of a_k, $k = 1, 2, \ldots, 15$. Let q_i be the maximum of $q_1, q_2, \ldots, q_{15}$. Then $q_i \geq p_{15} = 47$. Because a_i is not a prime, $\frac{q_i}{q_i}$ is divisible by a prime number greater than or equal to q_i. Hence $a_i \geq q_i^2 = 47^2 > 2005$, a contradiction. We conclude that $n = 15$.

4. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $E_1 = \{x_1, x_2, \ldots, x_{n-2}\}$. Arguing by contradiction, let us assume that $\cup_{k=1}^{m} E_k = S$. Choose E_j and E_k such that $x_{n-1} \in E_j$ and $x_n \in E_k$. Then $E_1 \cup E_j \cup E_k = S$, a contradiction.

(Romanian Mathematical Olympiad, 1986, proposed by I. Tomescu)

5. Arguing by contradiction, we assume that none of the colors has the desired property. Then there exist distances $r \geq g \geq b$ such that r is not attained by red points, g by green points, and b by blue points (for these inequalities to hold we might have to permute the colors).

Consider a sphere of radius r centered at a red point. Its surface has green and blue points only. Since $g, b \leq r$, the surface of the sphere must contain both green and blue points. Choose M a green point on the sphere. There exist two points P and Q on the sphere such that $MP = MQ = g$ and $PQ = b$. So on the one hand, either P or Q is green, or else P and Q are both blue. Then either there exist two green points at distance g, namely M and P, or Q, or there exist two blue points at distance b. This contradicts the initial assumption. The conclusion follows.

(German Mathematical Olympiad, 1985)

6. Arguing by contradiction, let us assume that the area of the overlap of any two surfaces is less than $\frac{1}{9}$. In this case, if $S_1, S_2, \ldots, S_9$ denote the nine surfaces, then the area of $S_1 \cup S_2$ is greater than $1 + \frac{8}{9}$, the area of $S_1 \cup S_2 \cup S_3$ is greater than $1 + \frac{8}{9} + \frac{7}{9}, \ldots$, and the area of $S_1 \cup S_2 \cup \cdots \cup S_9$ is greater than

$$1 + \frac{8}{9} + \frac{7}{9} + \cdots + \frac{1}{9} = \frac{45}{9} = 5$$

a contradiction. Hence the conclusion.

(L. Panaitopol, D. Şerbănescu, *Probleme de Teoria Numerelor şi Combinatorică pentru Juniori* (*Problems in Number Theory and Combinatorics for Juniors*), GIL, 2003)

7. Assume that such an f exists. We focus on some particular values of the variable. Let $f(0) = a$ and $f(5) = b$, $a, b \in \{1, 2, 3\}$, $a \neq b$. Because $|5 - 2| = 3$, $|2 - 0| = 2$, we have $f(2) \neq a, b$, so $f(2)$ is the remaining number, say c. Finally, because $|3 - 0| = 3$, $|3 - 5| = 2$, we must have $f(3) = c$. Therefore, $f(2) = f(3)$. Translating the argument to an arbitrary number x instead of 0, we obtain $f(x + 2) = f(x + 3)$, and so f is constant. But this violates the condition from the definition. It follows that such a function does not exist.

8. Arguing by contradiction, let us assume that such a function exists. Set $f(3) = k$. Using the inequality $2^3 < 3^2$, we obtain

$$3^3 = f(2)^3 = f(2^3) < f(3^2) = f(3)^2 = k^2,$$

hence $k > 5$. Similarly, using $3^3 < 2^5$, we obtain

$$k^3 = f(3)^3 = f(3^3) < f(2^5) = f(2)^5 = 3^5 = 243 < 343 = 7^3.$$

This implies that $k < 7$, and consequently k can be equal only to 6. Thus we should have $f(2) = 3$ and $f(3) = 6$. The monotonicity of f implies that $2^u < 3^v$ if and only if $3^u < 6^v$, u, v being positive integers. Taking logarithms this means that $\frac{v}{u} > \log_2 3$ if and only if $\frac{v}{u} > \log_3 6$. Since rationals are dense, it follows that $\log_2 3 = \log_3 6$. This can be written as $\log_2 3 = \frac{1}{\log_2 3} + 1$, and so $\log_2 3$ is the positive solution of the quadratic equation $x^2 - x - 1 = 0$, which is the golden ratio $\frac{1+\sqrt{5}}{2}$. The equality $2^{\frac{1+\sqrt{5}}{2}} = 3$ translates to $2^{1+\sqrt{5}} = 9$. But this would imply

$$65536 = 2^{5\times 3.2} < 2^{5(1+\sqrt{5})} = 9^5 = 59049.$$

We have reached a contradiction, which proves that the function f cannot exist.

(B.J. Venkatachala, *Functional Equations: A Problem Solving Approach*, Prism Books PVT Ltd., Bangalore, 2002)

9. The constant function $f(x) = k$, where k is a positive integer, is the only possible solution. That any such function satisfies the given condition is easy to check.

Now suppose there exists a nonconstant solution f. There must exist two positive integers a and b such that $f(a) < f(b)$. This implies that $(a+b)f(a) < af(b) + bf(a) < (a+b)f(b)$, which by the given condition is equivalent to $(a+b)f(a) < (a+b)f(a^2+b^2) < (a+b)f(b)$. We can divide by $a + b > 0$ to find that $f(a) < f(a^2 + b^2) < f(b)$. Thus between any two different values of f we can insert another. But this cannot go on forever, since f takes only integer values. The contradiction shows that such a function cannot exist. Thus constant functions are the only solutions.

(Canadian Mathematical Olympiad, 2002)

10. Assume that A, B, and a satisfy $A \cup B = [0, 1]$, $A \cap B = \emptyset$, $B = A + a$. We can assume that a is positive; otherwise, we can exchange A and B. Then $(1 - a, 1] \subset B$; hence $(1 - 2a, 1 - a] \subset A$. An inductive argument shows that for any positive integer n, the interval $(1 - (2n + 1)a, 1 - 2na]$ is in B, the interval $(1 - (2n + 2)a, 1 - (2n + 1)a]$ is in A. However, at some point this sequence of intervals leaves $[0, 1]$. The interval of the form $(1 - na, 1 - (n - 1)a]$ that contains 0 must be contained entirely in either A or B, which is impossible since this interval exits $[0, 1]$. The contradiction shows that the assumption is wrong, and hence the partition does not exist.

(Austrian-Polish Mathematics Competition, 1982)

11. Assume the contrary. Our chosen numbers $a_1, a_2, \ldots, a_{k+1}$ must have a total of at most k distinct prime factors (the primes less than or equal to n). Let $o_p(q)$ denote the highest value of d such that $p^d|q$. Also, let $a = a_1a_2\cdots a_{k+1}$ be the product of the numbers. Then for each prime p,

$$o_p(a) = \sum_{i=1}^{k+1} o_p(a_i),$$

and it follows that there can be at most one *hostile* value of i for which $o_p(a_i) > \frac{o_p(a)}{2}$. Because there are at most k primes that divide a, there is some i that is not hostile for any such prime.

Then $2o_p(a_i) \le o_p(a)$, so $o_p(a_i) \le o_p\left(\frac{a}{a_i}\right)$ for each prime p dividing a. This implies that a_i divides $\frac{a}{a_i}$, which contradicts the fact that the a_i does not divide the product of the other a_j's. Hence our assumption was false, and the conclusion follows.

(Hungarian Mathematical Olympiad, 1999)

12. The base case $n = 1$ is $\frac{1}{2} = 1 - \frac{1}{2}$, true. Now the inductive step. The hypothesis is that

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} = 1 - \frac{1}{2} + \cdots + \frac{1}{2k-1} - \frac{1}{2k}.$$

We are to prove that

$$\frac{1}{k+2} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} = 1 - \frac{1}{2} + \cdots - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2}.$$

Using the induction hypothesis, we can rewrite this as

$$\frac{1}{k+2} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} = \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2},$$

which reduces to

$$\frac{1}{2k+2} = \frac{1}{k+1} - \frac{1}{2k+2},$$

obvious. This completes the induction.

13. The base case is trivial. However, as I.M. Vinogradov once said, "it is the first nontrivial example that matters". And this is $n = 2$, in which case we have

$$|\sin 2x| = 2|\sin x||\cos x| \le 2|\sin x|.$$

This suggests to us to introduce cosines as factors in the proof of the inductive step. Assuming the inequality for $n = k$, we can write

$$\begin{aligned} |\sin(k+1)x| &= |\sin kx \cos x + \sin x \cos kx| \le |\sin kx||\cos x| + |\sin x||\cos kx| \\ &\le |\sin kx| + |\sin x| \le k|\sin x| + |\sin x| = (k+1)|\sin x|. \end{aligned}$$

The induction is complete.

14. As in the solution to the previous problem we argue by induction on n using trigonometric identities. The base case holds because

$$|\sin x_1| + |\cos x_1| \ge \sin^2 x_1 + \cos^2 x_1 = 1.$$

Next, assume that the inequality holds for $n = k$ and let us prove it for $n = k+1$. Using the inductive hypothesis, it suffices to show that

$$|\sin x_{n+1}| + |\cos(x_1 + x_2 + \cdots + x_{n+1})| \ge |\cos(x_1 + x_2 + \cdots + x_n)|.$$

To simplify notation let $x_{n+1} = x$ and $x_1 + x_2 + \cdots + x_n + x_{n+1} = y$, so that the inequality to be proved is $|\sin x| + |\cos y| \ge |\cos(y - x)|$. The subtraction formula gives

$$\begin{aligned} |\cos(y-x)| &= |\cos y \cos x + \sin y \sin x| \le |\cos y||\cos x| + |\sin y||\sin x| \\ &\le |\cos y| + |\sin x|. \end{aligned}$$

This completes the inductive step, and concludes the solution.

(*Revista Mathematică din Timişoara* (*Timişoara Mathematics Gazette*), proposed by T. Andreescu)

15. We expect an inductive argument, with a possible inductive step given by

$$3^{n+1} = 3 \cdot 3^n \geq 3n^3 \geq (n+1)^3.$$

In order for this to work, the inequality $3n^3 \geq (n+1)^3$ needs to be true. This inequality is equivalent to $2n^3 \geq 3n^2 + 3n + 1$, which would, for example, follow from the separate inequalities $n^3 \geq 3n^2$ and $n^3 \geq 3n + 1$. These are both true for $n \geq 3$. Thus we can argue by induction starting with the base case $n = 3$, where equality holds. The cases $n = 0$, $n = 1$, and $n = 2$ can be checked by hand.

16. The base case $2^6 < 6! < 3^6$ reduces to $64 < 720 < 729$, which is true. Assuming the double inequality true for n we are to show that

$$\left(\frac{n+1}{3}\right)^{n+1} < (n+1)! < \left(\frac{n+1}{2}\right)^{n+1}.$$

Using the inductive hypothesis we can reduce the inequality on the left to

$$\left(\frac{n+1}{3}\right)^{n+1} < (n+1)\left(\frac{n}{3}\right)^n,$$

$$\left(1 + \frac{1}{n}\right)^n < 3,$$

while the inequality on the right can be reduced to

$$\left(1 + \frac{1}{n}\right)^n > 2.$$

These are both true for all $n \geq 1$ because the sequence $\left(1 + \frac{1}{n}\right)^n$ is increasing and converges to e, which is less than 3. Hence the conclusion.

17. The left-hand side grows with n, while the right-hand side stays constant, so apparently a proof by induction would fail. It works, however, if we sharpen the inequality to

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2} - \frac{1}{n}, \; n \geq 2.$$

As such, the cases $n = 1$ and $n = 2$ need to be treated separately, and they are easy to check.

The base case is for $n = 3$:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} < 1 + \frac{1}{8} + \frac{1}{27} < \frac{3}{2} - \frac{1}{3}.$$

For the inductive step, note that from

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2} - \frac{1}{n}, \text{ for some } n \geq 3,$$

we obtain

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \frac{1}{(n+1)^3} < \frac{3}{2} - \frac{1}{n} + \frac{1}{(n+1)^3}.$$

All we need to check is

$$\frac{3}{2}-\frac{1}{n}+\frac{1}{(n+1)^3}<\frac{3}{2}-\frac{1}{(n+1)},$$

which is equivalent to

$$\frac{1}{(n+1)^3}<\frac{1}{n}-\frac{1}{(n+1)},$$

or

$$\frac{1}{(n+1)^3}<\frac{1}{n(n+1)}.$$

This is true, completing the inductive step. This proves the inequality.

18. We prove both parts by induction on n. For (a), the case $n = 1$ is straightforward. Assume now that we have found an n-digit number m divisible by 2^n made out of the digits 2 and 3 only. Let $m = 2^n k$ for some integer k. If n is even, then

$$2\times 10^n + m = 2^n(2\cdot 5^n + k)$$

is an $(n + 1)$-digit number written only with 2's and 3's, and divisible by 2^{n+1}. If k is odd, then

$$3\times 10^n + m = 2^n(3\cdot 5^n + k)$$

has this property.

The idea of part (b) is the same. The base case is trivial, $m = 5$. Now if we have found an n-digit number $m = 5^n k$ with this property, then looking modulo 5, one of the $(n + 1)$-digit numbers

$$5\times 10^n + m = 5^n(5\cdot 2^n + k),$$

$$6\times 10^n + m = 5^n(6\cdot 2^n + k),$$

$$7\times 10^n + m = 5^n(7\cdot 2^n + k),$$

$$8\times 10^n + m = 5^n(8\cdot 2^n + k),$$

$$9\times 10^n + m = 5^n(9\cdot 2^n + k)$$

has the required property, and the problem is solved.

(USA Mathematical Olympiad, 2003, proposed by T. Andreescu)

19. We proceed by induction on n. The base case is obvious; the decomposition consists of just one piece. For the induction step, let us assume that the tiling is possible for such a $2^n \times 2^n$ board and consider a $2^{n+1} \times 2^{n+1}$ board. Start by placing a piece in the middle of the board as shown in Figure 49. The remaining surface decomposes into four $2^n \times 2^n$ boards with corner squares removed, each of which can be tiled by the induction hypothesis. Hence we are done.

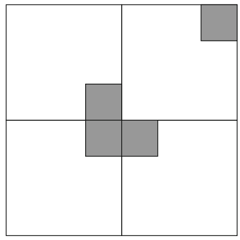

Figure 49

20. The property is clearly true for a single number. Now assume that it is true whenever we have such a sequence of length k and let us prove it for a sequence of length $k+1$: $x_1, x_2, \ldots, x_{k+1}$. Call a cyclic shift with all partial sums positive "good".

With indices taken modulo $k+1$, there exist two terms x_j and x_{j+1} such that $x_j > 0$, $x_{j+1} \leq 0$. Without loss of generality, we may assume that these terms are x_k and x_{k+1}. Define a new sequence by $y_j = x_j$, $j \leq k-1$, $y_k = x_k + x_{k+1}$. By the inductive hypothesis, $y_1, y_2, \ldots, y_k$ has a unique good cyclic shift. Expand y_k into x_k, x_{k+1} to obtain a good cyclic shift of $x_1, x_2, \ldots, x_{k+1}$. This proves the existence. To prove uniqueness, note that a good cyclic shift of $x_1, x_2, \ldots, x_{k+1}$ can start only with one of $x_1, x_2, \ldots, x_k$ (since $x_{k+1} < 0$). It induces a good cyclic shift of $y_1, y_2, \ldots, y_k$ that starts at the same term; hence two good cyclic shifts of the longer sequence would produce two good cyclic shifts of the shorter. This is ruled out by the induction hypothesis, and the uniqueness is proved.

(G. Raney)

21. We induct on $m+n$. The base case $m+n=4$ can be verified by examining the equalities

$$1+1=1+1 \quad \text{and} \quad 1+2=1+2.$$

Now let us assume that the property is true for $m+n=k$ and prove it for $m+n=k+1$. Without loss of generality, we may assume that $x_1 = \max_i x_i$ and $y_1 = \max_i y_i$, $x_1 \geq y_1$. If $m=2$, then

$$y_1 + y_2 = x_1 + x_2 + \cdots + x_n \geq x_1 + n - 1 \geq y_1 + n - 1.$$

It follows that $y_1 = x_1 = n$ or $n-1$, $y_2 = n-1$, $x_2 = x_3 = \cdots = x_n = 1$. Consequently, $y_2 = x_2 + x_3 + \cdots + x_n$, and we are done. If $m > 2$, rewrite the original equality as

$$(x_1 - y_1) + x_2 + \cdots + x_n = y_2 + \cdots + y_m.$$

This is an equality of the same type, with the observation that $x_1 - y_1$ could be zero, in which case x_1 and y_1 are the numbers to be suppressed.

We could apply the inductive hypothesis if $y_1 \geq n$, in which case $y_2 + \cdots + y_m$ were less than $mn - y_1 < (m-1)n$. In this situation just suppress the terms provided by the inductive hypothesis; then move y_1 back to the right-hand side.

Let us analyze the case in which this argument does not work, namely when $y_1 < n$. Then $y_2 + y_3 + \cdots + y_m \leq (m-1)y_1 < (m-1)n$, and again the inductive hypothesis can be applied. This completes the solution.

22. Let f be the function. We will construct g and h such that $f = g + h$, with g an odd function and h a function whose graph is symmetric with respect to the point $(1, 0)$.

Let g be any odd function on the interval $[-1, 1]$ for which $g(1) = f(1)$. Define $h(x) = f(x) - g(x)$, $x \in [-1, 1]$. Now we proceed inductively as follows. For $n \geq 1$, let $h(x) = -h(2-x)$ and $g(x) = f(x)-h(x)$ for $x \in (2n-1, 2n+1]$, and then extend these functions such that $g(x) = -g(-x)$ and $h(x) = f(x) - g(x)$ for $x \in [-2n-1, -2n+1)$. It is straightforward to check that the g and h constructed this way satisfy the required condition.

(*Kvant* (*Quantum*))

23. *First solution.* We prove the property by induction on n. For $n = 2$, any number of the form $n = 2t^2$, t an integer, would work.

Let us assume that for $n = k$ there is a number m with the property from the statement, and let us find a number m' that fulfills the requirement for $n = k + 1$.

We need the fact that every integer $p \geq 2$ can be represented as $a^2 + b^2 - c^2$, where a, b, c are positive integers. Indeed, if p is even, say $p = 2q$, then

$$p = 2q = (3q)^2 + (4q - 1)^2 - (5q - 1)^2,$$

while if p is odd, $p = 2q + 1$, then

$$p = 2q + 1 = (3q - 1)^2 + (4q - 4)^2 - (5q - 4)^2,$$

if $q > 1$, while if $q = 1$, then $p = 3 = 4^2 + 6^2 - 7^2$.

Returning to the inductive argument, let

$$m = a_1^2 + a_2^2 = b_1^2 + b_2^2 + b_3^2 = \cdots = l_1^2 + l_2^2 + \cdots + l_k^2,$$

and also $m = a^2 + b^2 - c^2$. Taking $m' = m + c^2$ we have

$$m' = a^2 + b^2 = a_1^2 + a_2^2 + c^2 = b_1^2 + b_2 + c^2 = \cdots = l_1^2 + l_2^2 + \cdots + l_2^2 + c^2.$$

This completes the induction.

Second solution. We prove by induction that $m = 25^{n-1}$ can be written as the sum of 1, 2, ..., n nonzero perfect squares. Base case: $1 = 1^2$. Inductive step: Suppose 25^{n-1} can be expressed as the sum of 1, 2, ..., n positive squares. Then 25^n can be written as the sum of p positive squares, for any p in 1, 2, ..., n, by multiplying each addend in the decomposition of 25^{n-1} into p squares by 25. Now let

$$25^{n-1} = (a_1)^2 + \cdots + (a_n)^2.$$

We have

$$25^n = (3a_1)^2 + (4a_1)^2 + (5a_2)^2 + (5a_3)^2 + ... + (5a_n)^2,$$

and we're done (for $n = 1$, we simply have $25 = 9 + 16$).

(*Gazeta Matematică* (*Mathematics Gazette, Bucharest*), 1980, proposed by M. Cavachi, second solution by E. Glazer)

24. We will prove a more general inequality namely that for all $m > 1$,

$$\sqrt[m]{a_1} - \sqrt[m]{a_2} + \sqrt[m]{a_3} - \cdots - \sqrt[m]{a_{2n}} + \sqrt[m]{a_{2n+1}} < \sqrt[m]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}.$$

The inequality from the statement is the particular case $m = n$.

This more general inequality will be proved by induction on n. For $n = 2$, we have to show that if $a_1 < a_2 < a_3$, then

$$\sqrt[m]{a_1} - \sqrt[m]{a_2} + \sqrt[m]{a_3} < \sqrt[m]{a_1 - a_2 + a_3}.$$

Denote $a = a_1$, $b = a_3$, $t = a_2 - a_1 > 0$. The inequality can be written as

$$\sqrt[m]{a + t} - \sqrt[m]{a} > \sqrt[m]{b} - \sqrt[m]{b - t}.$$

Define the function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \sqrt[m]{x + t} - \sqrt[m]{x}$. Its first derivative is $f'(x) = \frac{1}{m}[(x+t)^{(1-m)/m} - x^{(1-m)/m}]$, which is negative. This shows that f is strictly decreasing, which proves the inequality.

For the induction step, let us assume that the inequality holds for $n \leq k - 1$ and prove it for $n = k$. Using the induction hypothesis we deduce that

$$\begin{aligned}&\sqrt[m]{a_1} - \sqrt[m]{a_2} + \sqrt[m]{a_3} - \cdots - \sqrt[m]{a_{2k}} + \sqrt[m]{a_{2k+1}}\\ &< \sqrt[m]{a_1 - a_2 + a_3 - \cdots - a_{2k-2} + a_{2k-1}} - \sqrt[m]{a_{2k}} + \sqrt[m]{a_{2k+1}}.\end{aligned}$$

Using the base case $n = 2$, we deduce that the latter is less than $\sqrt[m]{a_1 - a_2 + a_3 - \cdots - a_{2k} + a_{2k+1}}$, which completes the induction.

(Balkan Mathematical Olympiad, 1998, proposed by B. Enescu)

25. We will say that the lines of the set X pass through k nodes if there are k points in the plane such that each line in X passes through at least one of them. We denote by $S(n, k)$ the statement which says that from the fact that any n lines of set X pass through k nodes it follows that all the lines of X pass through k nodes. We are supposed to prove $S(k^2 + 1, k)$ for $k \geq 1$. We do this by induction.

First note that $S(3, 1)$ is obvious, if any three lines pass through a point, then all lines pass through a point. Next notice that $S(6, 2)$ is a corollary of $S(3, 1)$ by the following argument:

Consider 6 lines, which, by hypothesis pass through 2 points. Then through one of the points, which we call P, pass at least 3 lines. Denote the set of all lines passing through P by M. We will show that any 3 lines in $A \backslash M$ pass through a point. Consider 3 such lines, and add to them 3 lines in M. Then these six lines pass through 2 points. One of these poins must be P, or else the lines passing through P would generate 3 different nodes. Hence the other 3 lines must themselves pass through a point.

The argument can be adapted to prove that $S(6, 2)$ implies $S(10, 3)$. Basically one starts again with 6 lines outside the similar set M, add the 4 lines in M and argue the same.

This argument can be adapted to prove $S((k - 1)^2 + 1, k - 1)$ implies $S(k^2 + 1, k)$ as follows. Consider $k^2 + 1$ points in A and the k points through which they pass. Through one of these points, which we call P, pass at least $k + 1$ lines. Denote by M the set of lines in A that pass through P. We will show that for the lines in $A \backslash M$ any $(k - 1)^2 + 1$ lines pass through $k-1$ nodes. Indeed, to each subset of $(k-1)^2+1$ lines in $A\backslash M$ add lines from M, and some other lines if M is exhausted, until we obtain k^2+1 lines. By hypothesis, these pass through k nodes. One of these nodes is P, for else the more than $k+1$ lines passing through it would pass through at least that many nodes, contradicting the hypothesis. It follows that the lines in $A\backslash M$ pass throgh the remaining $k-1$ nodes, proving the claim. By the induction hypothesis, all lines in $A\backslash M$ pass through $k-1$ nodes. Add P to these nodes to complete the induction step.

(Moscow Mathematical Olympiad, 1995–1996)

26. The property can be checked easily for small integers, which will constitute the base case. Assuming the property true for all integers less than n, let F_k be the largest term of the Fibonacci sequence that does not exceed n. The number $n-F_k$ is strictly less than n, so by the induction hypothesis it can be written as a sum of distinct terms of the Fibonacci sequence, say $n-F_k=\sum_j F_{i_j}$. The assumption on the maximality of F_k implies that $n-F_k<F_k$ (this because $F_{k+1}=F_k+F_{k-1}<2F_k$ for $k\geq 2$). It follows that $F_k\neq F_{i_j}$, for all j. We obtain $n=\sum_j F_{i_j}+F_k$, which gives a way of writing n as a sum of distinct terms of the Fibonacci sequence.

27. We will prove a more general identity, namely,

$$F_{m+n+1}=F_{m+1}F_{n+1}+F_mF_n,\ \text{for } m,n\geq 0.$$

We do so by induction on n. The inductive argument will assume the property to be true for $n=k-1$ and $n=k$, and prove it for $n=k+1$. Thus the base case consists of $n=0$, $F_{m+1}=F_{m+1}$; and $n=1$, $F_{m+2}=F_{m+1}+F_m$ – both of which are true.

Assuming that $F_{m+k}=F_{m+1}F_k+F_mF_{k-1}$ and $F_{m+k+1}=F_{m+1}F_{k+1}+F_mF_k$, we obtain by addition,

$$F_{m+k}+F_{m+k+1}=F_{m+1}(F_k+F_{k+1})+F_m(F_{k-1}+F_k),$$

which is, in fact, the same as $F_{m+k+2}=F_{m+1}F_{k+2}+F_mF_{k+1}$. This completes the induction. For $m=n$, we obtain the identity in the statement.

28. Inspired by the previous problem, we generalize the identity to

$$F_{m+n+p}=F_{m+1}F_{n+1}F_{p+1}+F_mF_nF_p-F_{m-1}F_{n-1}F_{p-1},$$

which should hold for $m,n,p\geq 1$. In fact, we can augment the Fibonacci sequence by $F_{-1}=1$ (so that the recurrence relation still holds), and then the above formula makes sense for $m,n,p\geq 0$. We prove it by induction on p. Again for the base case we consider $p=0$, with the corresponding identity

$$F_{m+n}=F_{m+1}F_{n+1}-F_{m-1}F_{n-1},$$

and $p=1$, with the corresponding identity

$$F_{m+n+1}=F_{m+1}F_{n+1}+F_mF_n.$$

Of the two, the second was proved in the solution to the previous problem. And the first identity is just a consequence of the second, obtained by subtracting $F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}$ from $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$. So the base case is verified. Now we assume that the identity holds for $p = k-1$ and $p = k$, and prove it for $p = k+1$. Indeed, adding

$$F_{m+n+k+1} = F_{m+1}F_{n+1}F_k + F_mF_nF_{k-1} - F_{m-1}F_{n-1}F_{k-2}$$

and

$$F_{m+n+k} = F_{m+1}F_{n+1}F_{k+1} + F_mF_nF_k - F_{m-1}F_{n-1}F_{k-1},$$

we obtain

$$\begin{aligned} F_{m+n+k+1} &= F_{m+n+k-1} + F_{m+n+k} \\ &= F_{m+1}F_{n+1}(F_k + F_{k+1}) + F_mF_n(F_{k-1} + F_k) - F_{m-1}F_{n-1}(F_{k-2} + F_{k-1}) \\ &= F_{m+1}F_{n+1}F_{k+2} + F_mF_nF_{k+1} - F_{m-1}F_{n-1}F_k. \end{aligned}$$

This proves the identity. Setting $m = n = p$, we obtain the identity in the statement.

29. The base case consists of the dissections for $n = 4$, 5, and 6 shown in Figure 50. The induction step jumps from $P(k)$ to $P(k+3)$ by dissecting one of the triangles into four triangles similar to it.

(R. Gelca)

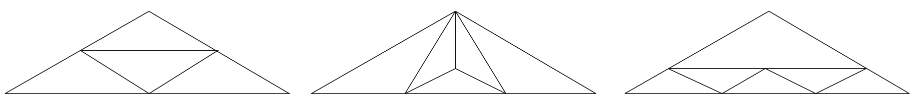

Figure 50

30. First, we explain the inductive step, which is represented schematically in Figure 51. If we assume that such a k-gon exists for all $k < n$, then the n-gon can be obtained by cutting off two vertices of the $(n-2)$-gon by two parallel lines. The sum of the distances from an interior point to the two parallel sides does not change while the point varies, and of course the sum of distances to the remaining sides is constant by the induction hypothesis. Choosing the parallel sides unequal, we can guarantee that the resulting polygon is not regular.

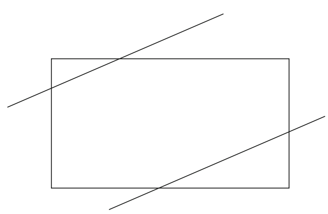

Figure 51

The base case consists of a rectangle ($n = 4$) and an equilateral triangle with two vertices cut off by parallel lines ($n = 5$). Note that to obtain the base case we had to apply the idea behind the inductive step.

31. The property is obviously true for the triangle since there is nothing to dissect. This will be our base case. Let us assume that the property is true for any coloring of a k-gon, for all $k < n$, and let us prove that it is true for an arbitrary coloring of an n-gon. Because at least three colors were used, there is a diagonal whose endpoints have different colors, say red (r) and blue (b). If on both sides of the diagonal a third color appears, then we can apply the induction hypothesis to two polygons and solve the problem.

If this is not the case, then on one side there will be a polygon with an even number of sides and with vertices colored in cyclic order *rbrb* . . . *rb*. Pick a blue point among them that is not an endpoint of the initially chosen diagonal and connect it to a vertex colored by a third color (Figure 52). The new diagonal dissects the polygon into two polygons satisfying the property from the statement, and having fewer sides. The induction hypothesis can be applied again, solving the problem.

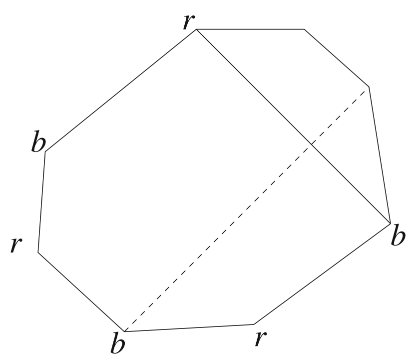

Figure 52

32. We prove the property by induction on the number of vertices. The base case is the triangle, where there is nothing to prove.

Let us assume now that the property holds for polygons with fewer than n vertices and prove it for a polygon with n vertices. The inductive step consists in finding one interior diagonal.

We commence with an interior angle less than π. Such an angle can be found at one of the vertices of the polygon that are also vertices of its convex hull (the convex hull is the smallest convex set in the plane that contains the polygon). Let the polygon be $A_1A_2 \dots A_n$, with $\angle A_nA_1A_2$ the chosen interior angle. Rotate the ray $|A_1A_n$ toward $|A_1A_2$ continuously inside the angle as shown in Figure 53. For each position of the ray, strictly between A_1A_n and A_1A_2, consider the point on the polygon that is the closest to A_1. If for some position of the ray this point is a vertex, then we have obtained a diagonal that divides the polygon into two polygons with fewer sides. Otherwise, A_2A_n is the diagonal.

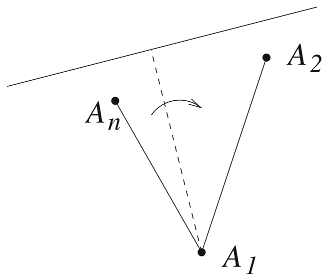

Figure 53

Dividing by the interior diagonal, we obtain two polygons with fewer vertices, which by hypothesis can be divided into triangles. This completes the induction.

33. We induct on the number to be represented. For the base case, we have

$$\begin{aligned}
1 &= 1^2 \\
2 &= -1^2 - 2^2 - 3^2 + 4^2, \\
3 &= -1^2 + 2^2, \\
4 &= -1^2 - 2^2 + 3^2.
\end{aligned}$$

The inductive step is "$P(n)$ implies $P(n+4)$"; it is based on the identity

$$m^2 - (m+1)^2 - (m+2)^2 + (m+3)^2 = 4.$$

Remark. This result has been generalized by J. Mitek, who proved that every integer k can be represented in the form $k = \pm 1^s \pm 2^s \pm \cdots \pm m^s$ for a suitable choice of signs, where s is a given integer ≥ 2. The number of such representations is infinite.

(P. Erdös, J. Surányi)

34. First, we show by induction on k that the identity holds for $n = 2^k$. The base case is contained in the statement of the problem. Assume that the property is true for $n = 2^k$ and let us prove it for $n = 2^{k+1}$. We have

$$f\left(\frac{x_1 + \cdots + x_{2^k} + x_{2^k+1} \cdots + x_{2^{k+1}}}{2^{k+1}}\right) = \frac{f\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + f\left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)}{2}$$

$$= \frac{\frac{f(x_1) + \cdots + f(x_{2^k})}{2^k} + \frac{f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^k}}{2}$$

$$= \frac{f(x_1) + \cdots + f(x_{2^k}) + f(x_{2^k+1}) + \cdots + f(x_{2^{k+1}})}{2^{k+1}},$$

which completes the induction. Now we work backward, showing that if the identity holds for some n, then it holds for $n-1$ as well. Consider the numbers $x_1, x_2, \ldots, x_{n-1}$ and $x_n = \frac{x_1+x_2+\cdots+x_{n-1}}{n-1}$. Using the hypothesis, we have

$$f\left(\frac{x_1+\cdots+x_{n-1}+\dfrac{x_1+\cdots+x_{n-1}}{n-1}}{n}\right)=\frac{f(x_1)+\cdots+f(x_{n-1})+f\left(\dfrac{x_1+\cdots+x_{n-1}}{n-1}\right)}{n},$$

which is the same as

$$f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)=\frac{f(x_1)+\cdots+f(x_{n-1})}{n}+\frac{1}{n}f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right).$$

Moving the last term on the right to the other side gives

$$\frac{n-1}{n}f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)=\frac{f(x_1)+f(x_2)+\cdots+f(x_{n-1})}{n}.$$

This is clearly the same as

$$f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)=\frac{f(x_1)+f(x_2)+\cdots+f(x_{n-1})}{n-1},$$

and the argument is complete.

35. This is a stronger form of the inequality discussed in the beginning, which can be obtained from it by applying the AM-GM inequality.

We first prove that the property holds for n a power of 2. The base case

$$(1+a_1)(1+a_2) \geq (1+\sqrt{a_1a_2})^2$$

reduces to the obvious $a_1 + a_2 \geq 2\sqrt{a_1a_2}$.

If

$$(1+a_1)(1+a_2)+\cdots+(1+a_{2^k}) \geq (1+\sqrt[2^k]{a_1a_2\cdots a_{2^k}})^{2^k}$$

for every choice of nonnegative numbers, then

$$\begin{aligned}(1+a_1)\cdots(1+a_{2^{k+1}}) &= (1+a_1)\cdots(1+a_{2^k})(1+a_{2^k+1})\cdots(1+a_{2^{k+1}})\\ &\geq \left(1+\sqrt[2^k]{a_1\cdots a_{2^k}}\right)^{2^k}(1+\sqrt[2^k]{a_{2^k+1}\cdots a_{2^{k+1}}})^{2^k}\\ &\geq \left[\left(1+\sqrt{\sqrt[2^k]{a_1\cdots a_{2^k}}\sqrt[2^k]{a_{2^k+1}}\cdots a_{2^{k+1}}}\right)^2\right]^{2^k}\\ &= \left(1+\sqrt[2^{k+1}]{a_1\cdots a_{2^{k+1}}}\right)^{2^{k+1}}.\end{aligned}$$

This completes the induction.

Now we work backward. If the inequality holds for $n+1$ numbers, then choosing $a_{n+1} = \sqrt[n]{a_1a_2\cdots a_n}$, we can write

$$(1+a_1)\cdots(1+a_n)(1+\sqrt[n]{a_1\cdots a_n}) \geq \left(1+\sqrt[n+1]{a_1\cdots a_n\sqrt[n]{a_1\cdots a_n}}\right)^{n+1},$$

which is the same as

$$(1+a_1)\cdots(1+a_n)(1+\sqrt[n]{a_1\cdots a_n}) \geq (1+\sqrt[n]{a_1\cdots a_n})^{n+1}.$$

Canceling the common factor, we obtain the inequality for n numbers. The inequality is proved.

36. The "pigeons" are the numbers. The "holes" are the 49 sets

$$\{1, 98\}, \{2, 97\}, \ldots, \{40, 50\}.$$

Two of the numbers fall in the same set; their sum is equal to 99. We are done.

37. As G. Pólya said, "a trick applied twice becomes a technique". Here we repeat the idea of the Mongolian problem from the 26th International Mathematical Olympiad.

Let $b_1, b_2, \ldots, b_n$ be the sequence, where $b_i \in \{a_1, a_2, \ldots, a_n\}$, $1 \leq i \leq m$. For each $j \leq m$ define the n-tuple $K_j = (k_1, k_2, \ldots, k_n)$, where $k_i = 0$ if a_i appears an even number of times in $b_1, b_2, \ldots, b_j$ and $k_i = 1$ otherwise.

If there exists $j \leq m$ such that $K_j = (0, 0, \ldots, 0)$ then $b_1 b_2 \cdots b_j$ is a perfect square and we are done. Otherwise, there exist $j < l$ such that $K_j = K_l$. Then in the sequence $b_{j+1}, b_{j+2}, \ldots, b_l$ each a_i appears an even number of times. The product $b_{j+1}b_{j+2}\cdots b_l$ is a perfect square.

38. The sequence has the property that for any n the first $n+1$ terms are less than or equal to $2n$. The problem would be solved if we showed that given a positive integer n, from any $n+1$ distinct integer numbers between 1 and $2n$ we can choose two whose difference is n. This is true, indeed, since the pigeonhole principle implies that one of the n pairs $(1, n+1)$, $(2, n+2), \ldots, (n, 2n)$ contains two terms of the sequence.

(Austrian-Polish Mathematics Competition, 1980)

39. The "holes" will be the residue classes, and the "pigeons", the numbers ax^2, $c - by^2$, $x, y = 0, 1, \ldots, p-1$. There are $2p$ such numbers. Any residue class, except for 0, can have at most two elements of the form ax^2 and at most two elements of the form $c - by^2$ from the ones listed above. Indeed, $ax_1^2 \equiv ax_2^2$ implies $x_1^2 \equiv x_2^2$, so $(x_1 - x_2)(x_1 + x_2) \equiv 0$. This can happen only if $x_1 = \pm x_2$. Also, $ax^2 \equiv 0$ only when $x = 0$.

We distinguish two cases. If $c - by_0^2 \equiv 0$ for some y_0, then $(0, y_0)$ is a solution. Otherwise, the $2p-1$ numbers ax^2, $c - by^2$, $x = 1, 2, \ldots, p-1$, $y = 0, 1, \ldots, p-1$ are distributed into $p-1$ "holes", namely the residue classes $1, 2, \ldots, p-1$. Three of them must lie in the same residue class, so there exist x_0 and y_0 with $ax_0^2 \equiv c - by_0^2 \pmod{p}$. The pair (x_0, y_0) is a solution to the equation from the statement.

Remark. A more advanced solution can be produced based on the theory of quadratic residues.

40. In any 2×2 square, only one of the four numbers can be divisible by 2, and only one can be divisible by 3. Tiling the board by 2×2 squares, we deduce that at most 25 numbers are divisible by 2 and at most 25 numbers are divisible by 3. There are at least 50 remaining numbers that are not divisible by 2 or 3, and thus must equal one of the numbers 1, 5, or 7. By the pigeonhole principle, one of these numbers appears at least 17 times.

(St. Petersburg City Mathematical Olympiad, 2001)

41. A more general property is true, namely that for any positive integer n there exist infinitely many terms of the Fibonacci sequence divisible by n.

We apply now the pigeonhole principle, letting the "objects" be all pairs of consecutive Fibonacci numbers (F_n, F_{n+1}), $n \geq 1$, and the "boxes" the pairs of residue classes modulo n. There are infinitely many objects, and only n^2 boxes, and so there exist indices $i > j > 1$ such that $F_i \equiv F_j \pmod{n}$ and $F_{i+1} \equiv F_{j+1} \pmod{m}$.

In this case

$$F_{i-1} = F_{i+1} - F_i \equiv F_{j+1} - F_j = F_{j-1} \pmod{n},$$

and hence $F_{i-1} \equiv F_{j-1} \pmod{n}$ as well. An inductive argument proves that $F_{i-k} \equiv F_{j-k} \pmod{n}$, $k = 1, 2, \ldots, j$. In particular, $F_{i-j} \equiv F_0 = 0 \pmod{n}$. This means that F_{i-j} is divisible by n. Moreover, the indices i and j range in an infinite family, so the difference $i - j$ can assume infinitely many values. This proves our claim, and as a particular case, we obtain the conclusion of the problem.

(Irish Mathematical Olympiad, 1999)

42. We are allowed by the recurrence relation to set $x_0 = 0$. We will prove that there is an index $k \leq m^3$ such that x_k divides m. Let r_t be the remainder obtained by dividing x_t by m for $t = 0, 1, \ldots, m^3 + 2$. Consider the triples $(r_0, r_1, r_2), (r_1, r_2, r_3), \ldots, (r_{m^3}, r_{m^3+1}, r_{m^3+2})$. Since r_t can take m values, the pigeonhole principle implies that at least two triples are equal. Let p be the smallest number such that the triple (r_p, r_{p+1}, r_{p+2}) is equal to another triple (r_q, r_{q+1}, r_{q+2}), $p < q \leq m^3$. We claim that $p = 0$.

Assume by way of contradiction that $p \geq 1$. Using the hypothesis, we have

$$r_{p+2} \equiv r_{p-1} + r_p r_{p+1} \pmod{m} \quad \text{and} \quad r_{q+2} \equiv r_{q-1} + r_q r_{q+1} \pmod{m}.$$

Because $r_p = r_q$, $r_{p+1} = r_{q+1}$, and $r_{p+2} = r_{q+2}$, it follows that $r_{p-1} = r_{q-1}$, so $(r_{p-1}, r_p, r_{p+1}) = (r_{q-1}, r_q, r_{q+1})$, contradicting the minimality of p. Hence $p = 0$, so $r_q = r_0 = 0$, and therefore x_q is divisible by m.

(T. Andreescu, D. Miheţ)

43. We focus on 77 consecutive days, starting on a Monday. Denote by a_n the number of games played during the first n days, $n \geq 1$. We consider the sequence of positive integers

$$a_1, a_2, \ldots, a_{77}, a_1 + 20, a_2 + 20, \ldots, a_{77} + 20.$$

Altogether there are $2 \times 77 = 154$ terms not exceeding $11 \times 12 + 20 = 152$ (here we took into account the fact that during each of the 11 weeks there were at most 12 games). The pigeonhole principle implies right away that two of the above numbers are equal. They cannot both be among the first 77, because by hypothesis, the number of games increases by at least 1 each day. For the same reason the numbers cannot both be among the last 77. Hence there are two indices k and m such that $a_m = a_k + 20$. This implies that in the time interval starting with the $(k + 1)$st day and ending with the nth day, exactly 20 games were played, proving the conclusion.

Remark. In general, if a chess player decides to play d consecutive days, playing at least one game a day and a total of no more than m with $d < m < 2d$, then for each $i \leq 2d - n - 1$ there is a succession of days on which, in total, the chess player played exactly i games.

(D.O. Shklyarskyi, N.N. Chentsov, I.M. Yaglom, *Izbrannye Zadachi i Theoremy Elementarnoy Matematiki* (*Selected Problems and Theorems in Elementary Mathematics*), Nauka, Moscow, 1976)

44. The solution combines the induction and pigeonhole principles. We commence with induction. The base case $m = 1$ is an easy check, the numbers can be only $-1, 0, 1$.

Assume now that the property is true for any $2m - 1$ numbers of absolute value not exceeding $2m - 3$. Let A be a set of $2m + 1$ numbers of absolute value at most $2m - 1$. If A contains $2m-1$ numbers of absolute value at most $2m-3$, then we are done by the induction hypothesis. Otherwise, A must contain three of the numbers $\pm(2m - 1)$, $\pm(2m - 2)$. By eventually changing signs we distinguish two cases.

Case I. $2m - 1, -2m + 1 \in A$. Pair the numbers from 1 through $2m - 2$ as $(1, 2m - 2)$, $(2, 2m - 3), \dots, (m - 1, m)$ so that the sum of each pair is equal to $2m - 1$, and the numbers from 0 through $-2m+1$ as $(0, -2m+1), (-1, -2m+2), \dots, (-m+1, -m)$, so that the sum of each pair is $-2m + 1$. There are $2m - 1$ pairs, and $2m$ elements of A lie in them, so by the pigeonhole principle there exists a pair with both elements in A. Those elements combined with either $2m - 1$ or $-2m + 1$ give a triple whose sum is equal to zero.

Case II. $2m-1, 2m-2, -2m+2 \in A$ and $-2m+1 \notin A$. If $0 \in A$, then $0-2m+2+2m-2 = 0$ and we are done. Otherwise, consider the pairs $(1, 2m - 3)$, $(2, 2m - 4), \dots, (m-2, m)$, each summing up to $2m-2$, and the pairs $(1, -2m), \dots, (-m+1, -m)$, each summing up to $-2m + 1$. Altogether there are $2m - 2$ pairs containing $2m - 1$ elements from A, so both elements of some pair must be in A. Those two elements combined with either $-2m + 2$ or $2m - 1$ give a triple with the sum equal to zero.

This concludes the solution.

(*Kvant* (*Quantum*))

45. Denote by Δ the set of ordered triples of people (a, b, c) such that c is either a common acquaintance of both a and b or unknown to both a and b. If c knows exactly k participants, then there exist exactly $2k(n - 1 - k)$ ordered pairs in which c knows exactly one of a and b (the factor 2 shows up because we work with *ordered* pairs). There will be

$$(n-1)(n-2) - 2k(n-1-k) \geq (n-1)(n-2) - 2\left(\frac{n-1}{2}\right)^2 = \frac{(n-1)(n-3)}{2}$$

ordered pairs (a, b) such that c knows either both or neither of a and b. Counting by the c's, we find that the number of elements of Δ satisfies

$$|\Delta| \geq \frac{n(n-1)(n-3)}{2}.$$

To apply the pigeonhole principle, we let the "holes" be the ordered pairs of people (a, b), and the "pigeons" be the triples $(a, b, c) \in \Delta$. Put the pigeon (a, b, c) in the hole (a, b) if c

knows either both or neither of a and b. There are $fn(n-1)(n-3)2$ pigeons distributed in $n(n-1)$ holes. So there will be at least

$$\left\lceil \frac{n(n-1)(n-3)}{2} \Big/ n(n-1) \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor - 1$$

pigeons in one hole, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. To the "hole" corresponds a pair of people satisfying the required condition.

(USA Mathematical Olympiad, 1985)

46. The beautiful observation is that if the sequence

$$a_n = \cos(n\pi x_1) + \cos(n\pi x_2) + \cdots + \cos(n\pi x_k),\ n \geq 1,$$

assumes finitely many distinct values, then so does the sequence of k-tuples $u_n = (a_n, a_{2n}, \ldots, a_{kn})$, $n \geq 1$. By the pigeonhole principle there exist $m < n$ such that $a_n = a_m$, $a_{2n} = a_{2m}, \ldots, a_{kn} = a_{km}$. Let us take a closer look at these relations. We know that $\cos(nx)$ is a polynomial of degree n with integer coefficients in $\cos(x)$, namely the Chebyshev polynomial. If $A_i = \cos(n\pi x_i)$ and $B_i = \cos(m\pi x_i)$, then the previous relations combined with this observation show that $A_1^j + A_2^j + \cdots + A_k^j = B_1^j + B_2^j + \cdots + B_k^j$ for all $j = 1, 2, \ldots, k$. Using Newton's formulas, we deduce that the polynomials having the zeros $A_1, A_2, \ldots, A_k$, respectively, $B_1, B_2, \ldots, B_k$ are equal (they have equal coefficients). Hence there is a permutation σ of $1, 2, \ldots, n$ such that $A_i = B_{\sigma(i)}$. Thus $\cos(n\pi x_i) = \cos(m\pi x_{\sigma(i)})$, which means that $nx_i - mx_{\sigma(i)}$ is a rational number r_i for $1 \leq i \leq k$. We want to show that the x_i's are themselves rational. If $\sigma(i) = i$, this is obvious. On the other hand, if we consider a cycle of σ, $(i_1 i_2 i_3, \ldots, i_s)$, we obtain the linear system

$$\begin{aligned} mx_{i_1} - nx_{i_2} &= r_{i_1}, \\ mx_{i_2} - nx_{i_3} &= r_{i_2}, \\ &\cdots \\ mx_{i_s} - nx_{i_1} &= r_{i_s}. \end{aligned}$$

It is not hard to compute the determinant of the coefficient matrix, which is $n^s - m^s$ (for example, by expanding by the first row, then by the first column, and then noting that the new determinants are triangular). The determinant is nonzero; hence the system has a unique solution. By applying Cramer's rule we determine that this solution consists of rational numbers. We conclude that the x_i's are all rational, and the problem is solved.

(V. Pop)

47. Place the circle at the origin of the coordinate plane and consider the rectangular grid determined by points of integer coordinates, as shown in Figure 54. The circle is inscribed in an 8×8 square decomposed into 64 unit squares. Because $3^2 + 3^2 > 4^2$, the four unit squares at the corners lie outside the circle. The interior of the circle is therefore covered by 60 squares, which are our "holes". The 61 points are the "pigeons", and by the pigeonhole principle two lie inside the same square. The distance between them does not exceed the length of the diagonal, which is $\sqrt{2}$. The problem is solved.

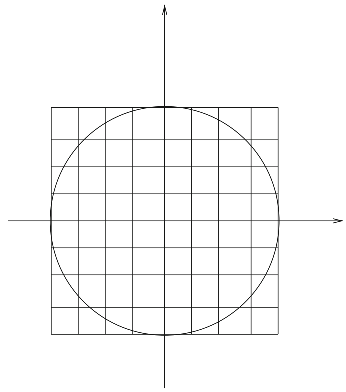

Figure 54

48. If $r = 1$, all lines pass through the center of the square. If $r \neq 1$, a line that divides the square into two quadrilaterals with the ratio of their areas equal to r has to pass through the midpoint of one of the four segments described in Figure 55 (in that figure the endpoints of the segments divide the sides of the square in the ratio r). Since there are four midpoints and nine lines, by the pigeonhole principle three of them have to pass through the same point.

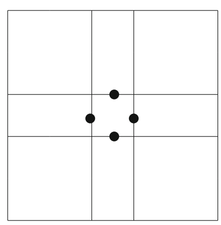

Figure 55

49. Choose a face with maximal number of edges, and let n be this number. The number of edges of each of the n adjacent faces ranges between 3 and n, so by the pigeonhole principle, two of these faces have the same number of edges.

(Moscow Mathematical Olympiad)

50. An n-gon has $\binom{n}{2} - n = \frac{1}{2}n(n-3)$ diagonals. For $n = 21$ this number is equal to 189. If through a point in the plane we draw parallels to these diagonals, $2 \times 189 = 378$ adjacent angles are formed. The angles sum up to $360°$, and thus one of them must be less than $1°$.

51. The geometric aspect of the problem is only apparent. If we number the vertices of the polygon counterclockwise $1, 2, \ldots, 2n$, then $P_1, P_2, \ldots, P_{2n}$ is just a permutation of these numbers. We regard indices modulo $2n$. Then P_iP_{i+1} is parallel to P_jP_{j+1} if and only if $P_i - P_j \equiv P_{j+1} - P_{i+1} \pmod{2n}$, that is, if and only if $P_i + P_{i+1} \equiv P_j + P_{j+1} \pmod{2n}$. Because

$$\sum_{i=1}^{2n}(P_i + P_{i+1}) \equiv 2\sum_{i=1}^{2n} P_i \equiv 2n(2n-1) \equiv 0 \pmod{2n}$$

and

$$\sum_{i=1}^{2n} i = n(2n-1) \equiv n \pmod{2n},$$

it follows that $P_i + P_{i+1}$, $i = 1, 2, \ldots, 2n$, do not exhaust all residues modulo $2n$. By the pigeonhole principle there exist $i \neq j$ such that $P_i+P_{i+1} \equiv P_j+P_{j+1} \pmod{2n}$. Consequently, the sides P_iP_{i+1} and P_jP_{j+1} are parallel, and the problem is solved.

(German Mathematical Olympiad, 1976)

52. Let C be a circle inside the triangle formed by three noncollinear points in S. Then C is contained entirely in S. Set $m = np + 1$ and consider a regular polygon $A_1A_2\ldots A_m$ inscribed in C. By the pigeonhole principle, some n of its vertices are colored by the same color. We have thus found a monochromatic n-gon. Now choose α an irrational multiple of π. The rotations of $A_1A_2\ldots A_m$ by $k\alpha$, $k = 0, 1, 2, \ldots$, are all disjoint. Each of them contains an n-gon with vertices colored by n colors. Only finitely many incongruent n-gons can be formed with the vertices of $A_1A_2\ldots A_m$. So again by the pigeonhole principle, infinitely many of the monochromatic n-gons are congruent. Of course, they might have different colors. But the pigeonhole principle implies that one color occurs infinitely many times. Hence the conclusion.

(Romanian Mathematical Olympiad, 1995)

53. *First solution.* This is an example with the flavor of Ramsey theory (see Section 6.3.3) that applies the pigeonhole principle. Pick two infinite families of lines, $\{A_i, i \geq 1\}$, and $\{B_j, j \geq 1\}$, such that for any i and j, A_i and B_j are orthogonal. Denote by M_{ij} the point of intersection of A_i and B_j. By the pigeonhole principle, infinitely many of the M_{1j}'s, $j \geq 1$, have the same color. Keep only the lines B_j corresponding to these points, and delete all the others. So again we have two families of lines, but such that M_{1j} are all of the same color; call this color c_1.

Next, look at the line A_2. Either there is a rectangle of color c_1, or at most one point M_{2j} is colored by c_1. Again by the pigeonhole principle, there is a color c_2 that occurs infinitely many times among the M_{2j}'s. We repeat the reasoning. Either at some step we encounter a rectangle, or after finitely many steps we exhaust the colors, with infinitely many lines A_i still left to be colored. The impossibility to continue rules out this situation, proving the existence of a rectangle with vertices of the same color.

Second solution. Let there be p colors. Consider a $(p+1) \times \left(\binom{p+1}{2} + 1\right)$ rectangular grid. By the pigeonhole principle, each of the $\binom{p+1}{2} + 1$ horizontal segments contains two points of the same color. There are $\binom{p+1}{2}$ possible configurations of monochromatic pairs, so two must repeat. The repeating pairs are vertices of a monochromatic rectangle.

54. We place the unit square in standard position. The "boxes" are the vertical lines crossing the square, while the "objects" are the horizontal diameters of the circles (Figure 56). Both the boxes and the objects come in an infinite number, but what we use for counting is length on the horizontal. The sum of the diameters is

$$\frac{10}{\pi} = 3 \times 1 + \varepsilon, \ \varepsilon > 0.$$

Consequently, there is a segment on the lower side of the square covered by at least four diameters. Any vertical line passing through this segment intersects the four corresponding circles.

55. If three points are collinear then we are done. Thus we can assume that no three points are collinear. The convex hull of all points is a polygon with at most n sides, which has therefore an angle not exceeding $\frac{(n-2)\pi}{n}$. All other points lie inside this angle. Ordered counterclockwise around the vertex of the angle they determine $n-2$ angles that sum up to at most $\frac{(n-2)\pi}{n}$. It follows that one of these angles is less than or equal to $\frac{(n-2)\pi}{n(n-2)} = \frac{\pi}{n}$. The three points that form this angle have the required property.

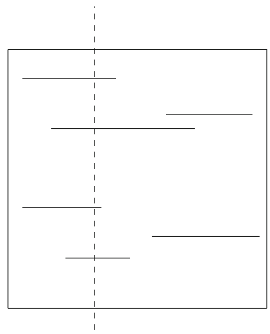

Figure 56

56. Denote by $D(O, r)$ the disk of center O and radius r. Order the disks

$$D(O_1, r_1), D(O_2, r_2), \ldots, D(O_n, r_n),$$

in decreasing order of their radii.

Choose the disk $D(O_1, r_1)$ and then delete all disks that lie entirely inside the disk of center O_1 and radius $3r_1$. The remaining disks are disjoint from $D(O_1, r_1)$. Among them choose the first in line (i.e., the one with maximal radius), and continue the process with the remaining circles.

The process ends after finitely many steps. At each step we deleted less than eight times the area of the chosen circle, so in the end we are left with at least $\frac{1}{9}$ of the initial area. The chosen circles satisfy the desired conditions.

(M. Pimsner, S. Popa, *Probleme de Geometrie Elementară* (*Problems in Elementary Geometry*), Editura Didactică şi Pedagogică, Bucharest, 1979)

57. Given a circle of radius r containing n points of integer coordinates, we must prove that $n < 2\pi\sqrt[3]{r^2}$. Because $r > 1$ and $2\pi > 6$ we may assume $n \geq 7$.

Label the n lattice points counterclockwise $P_1, P_2, \ldots, P_n$. The (counterclockwise) arcs $\overset{\frown}{P_1P_3}, \overset{\frown}{P_2P_4}, \ldots, \overset{\frown}{P_nP_2}$ cover the circle twice, so they sum up to 4π. Therefore, one of them, say $\overset{\frown}{P_1P_3}$, measures at most $\frac{4\pi}{n}$.

Consider the triangle $P_1P_2P_3$, which is inscribed in an arc of measure $\frac{4\pi}{n}$. Because $n \geq 7$, the arc is less than a quarter of the circle. The area of $P_1P_2P_3$ will be maximized if P_1 and P_3 are the endpoints and P_2 is the midpoint of the arc. In that case,

$$\text{Area}(P_1P_2P_3) = \frac{abc}{4r} = \frac{2r\sin\dfrac{\pi}{n}\cdot 2r\sin\dfrac{\pi}{n}\cdot 2r\sin\dfrac{2\pi}{n}}{4r} \leq \frac{2r\dfrac{\pi}{n}\cdot 2r\dfrac{\pi}{n}\cdot 2r\dfrac{2\pi}{n}}{4r} = \frac{4r^2\pi^3}{n^3}.$$

And in general, the area of $P_1P_2P_3$ cannot exceed $\frac{4r^2\pi^3}{n^3}$. On the other hand, if the coordinates of the points P_1, P_2, P_3 are, respectively, (x_1, y_1), (x_2, y_2), and (x_3, y_3), then

$$\begin{aligned}\text{Area}(P_1P_2P_3) &= \pm\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= \frac{1}{2}|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|\end{aligned}$$

Because the coordinates are integers, the area cannot be less than $\frac{1}{2}$. We obtain the inequality $\frac{1}{2} \leq \frac{4r^2\pi^3}{n^3}$, which proves that $2\pi\sqrt[3]{r^2} \geq n$, as desired.

Remark. The weaker inequality $n(r) < 6\sqrt[3]{\pi r^2}$ was given in 1999 at the Iranian Mathematical Olympiad.

58. Order the eight integers $a_1 < a_2 < \cdots < a_8 \leq 2004$. We argue by contradiction. Assume that for any choice of the integers a, b, c, d, either $a + b + c < d + 4$ or $a + b + c > 4d$. Let us look at the situation in which d is a_3 and a, b, and c are a_1, a_2 and a_4. The inequality $a_1 + a_2 + a_4 < 4 + a_3$ is impossible because $a_4 \geq a_3 + 1$ and $a_1 + a_2 \geq 3$. Thus with our assumption, $a_1 + a_2 + a_4 > 4a_3$, or

$$a_4 > 4a_3 - a_2 - a_1.$$

By similar logic,

$$\begin{aligned} a_5 &> 4a_4 - a_2 - a_1 > 16a_3 - 5a_2 - 5a_1, \\ a_6 &> 4a_5 - a_2 - a_1 > 64a_3 - 21a_2 - 21a_1, \\ a_7 &> 4a_6 - a_2 - a_1 > 256a_3 - 85a_2 - 85a_1, \\ a_8 &> 4a_7 - a_2 - a_1 > 1024a_3 - 341a_2 - 341a_1. \end{aligned}$$

We want to show that if this is the case, then a_8 should exceed 2004. The expression $1024a_3 - 341a_2 - 341a_1$ can be written as $683a_3 + 341(a_3 - a_2) + 341(a_3 - a_1)$, so to minimize it we have to choose $a_1 = 1$, $a_2 = 2$, $a_3 = 3$. But then the value of the expression would be 2049, which, as predicted, exceeds 2004. This contradiction shows that our assumption was false, proving the existence of the desired four numbers.

(Mathematical Olympiad Summer Program, 2004, proposed by T. Andreescu)

59. There is no loss of generality in supposing that $a_1 < a_2 < \cdots < a_n < \cdots$. Now proceed by induction on n. For $n = 1$, $a_1^2 \geq \frac{2\times 1+1}{3}a_1$ follows from $a_1 \geq 1$. The inductive step reduces to

$$a_{n+1}^2 \geq \frac{2}{3}(a_1 + a_2 + \cdots + a_n) + \frac{2n+3}{3}a_{n+1}.$$

An equivalent form of this is

$$3a_{n+1}^2 - (2n+3)a_{n+1} \geq 2(a_1 + a_2 + \cdots + a_n).$$

At this point there is an interplay between the indices and the terms of the sequence, namely the observation that $a_1 + a_2 + \cdots + a_n$ does not exceed the sum of integers from 1 to a_n. Therefore,

$$2(a_1 + a_2 + \cdots + a_n) \leq 2(1 + 2 + \cdots + a_n) = a_n(a_n + 1) \leq (a_{n+1} - 1)a_{n+1}.$$

We are left to prove the sharper, yet easier, inequality

$$3a_{n+1}^2 - (2n+3)a_{n+1} \geq (a_{n+1} - 1)a_{n+1}.$$

This is equivalent to $a_{n+1} \geq n + 1$, which follows from the fact that a_{n+1} is the largest of the numbers.

(Romanian Team Selection Test for the International Mathematical Olympiad, proposed by L. Panaitopol)

60. Again, there will be an interplay between the indices and the values of the terms. We start by ordering the a_i's increasingly $a_1 < a_2 < \cdots < a_n$. Because the sum of two elements of X is in X, given a_i in the complement of X, for each $1 \leq m \leq \frac{a_i}{2}$, either m or $a_i - m$ is not in X. There are $\lceil \frac{a_i}{2} \rceil$ such pairs and only $i - 1$ integers less than a_i and not in X, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. Hence $a_i \leq 2i - 1$. Summing over i gives $a_1 + a_2 + \cdots + a_n \leq n^2$ as desired.

(Proposed by R. Stong for the USAMO, 2000)

61. Because $P(P(x)) - x$ is a polynomial of degree n, A is finite. If $a, b \in A$, $a \neq b$, then $a - b$ divides $P(a) - P(b)$ and $P(a) - P(b)$ divides $P(P(a)) - P(P(b) = a - b$. It follows that $|a - b| = |P(a) - P(b)|$. Let the elements of A be $x_1 < x_2 < \cdots < x_k$. We have

$$\begin{aligned} x_k - x_1 &= \sum_{i=1}^{k-1}(x_{i+1} - x_i) = \sum_{i=1}^{k-1} |P(x_{i+1}) - P(x_i)| \\ &\geq \left| \sum_{i=1}^{k-1}(P(x_{i+1}) - P(x_i)) \right| = |P(x_k) - P(x_1)| = x_k - x_1. \end{aligned}$$

It follows that the inequality in this relation is an equality, and so all the numbers $P(x_{i+1} - P(x_i)$ have the same sign. So either $P(x_{i+1}) - P(x_i) = x_{i+1} - x_i$, for all i, or $P(x_{i+1}) - P(x_i) = x_i - x_{i+1}$ for all i. It follows that the numbers $x_1, x_2, \ldots, x_k$ are the roots of a polynomial equation of the form $P(x) \pm x = a$. And such an equation has at most n real roots. The conclusion follows.

(Romanian Mathematics Competition, 1986, proposed by Gh. Eckstein)

62. Call the elements of the 4×4 tableau a_{ij}, $i, j = 1, 2, 3, 4$, according to their location. As such, $a_{13} = 2$, $a_{22} = 5$, $a_{34} = 8$ and $a_{41} = 3$. Look first at the row with the *largest* sum, namely, the fourth. The unknown entries sum up to 27; hence all three of them, a_{42}, a_{43}, and a_{44}, must equal 9. Now we consider the column with *smallest* sum. It is the third, with

$$a_{13} + a_{23} + a_{33} + a_{43} = 2 + a_{23} + a_3 + 9 = 13.$$

We see that $a_{23} + a_{33} = 2$; therefore $a_{23} = a_{33} = 1$. We than have

$$a_{31} + a_{32} + a_{33} + a_{34} = a_{31} + a_{32} + 1 + 8 = 26.$$

Therefore, $a_{31} + a_{32} = 17$, which can happen only if one of them is 8 and the other is 9. Checking the two cases separately, we see that only $a_{31} = 8$, $a_{32} = 9$ yields a solution, which is described in Figure 57.

	7	**2**	**2**	**3**	*14*
	3	**5**	**1**	**7**	*16*
	8	**9**	**1**	**8**	*26*
	3	**9**	**9**	**9**	*30*
16	*21*	*25*	*13*	*27*	*20*

Figure 57

Remark. Such puzzles would appear in the Sunday edition of the *San Francisco Chronicle* at the time of publication of this book.

63. There are only finitely many polygonal lines with these points as vertices. Choose the one of minimal length $P_1P_2 \dots P_n$. If two sides, say P_iP_{i+1} and P_jP_{j+1}, intersect at some point M, replace them by P_iP_j and $P_{i+1}P_{j+1}$ to obtain the closed polygonal line $P_1 \dots P_iP_jP_{j-1} \dots P_{i+1}P_{j+1} \dots P_n$ (Figure 58). The triangle inequality in triangles MP_iP_j and $MP_{i+1}P_{j+1}$ shows that this polygonal line has shorter length, a contradiction. It follows that $P_1P_2 \dots P_n$ has no self-intersections, as desired.

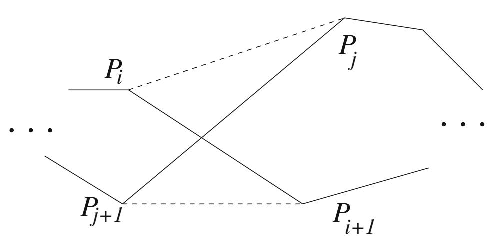

Figure 58

64. Let A_iA_{i+1} be the longest side of the polygon (or one of them if more such sides exist). Perpendicular to it and at the endpoints A_i and A_{i+1} take the lines L and L', respectively. We argue on the configuration from Figure 59.

If all other vertices of the polygon lie to the right of L', then $A_{i-1}A_i > A_iA_{i+1}$, because the distance from A_i to a point in the half-plane determined by L' and opposite to A_i is greater

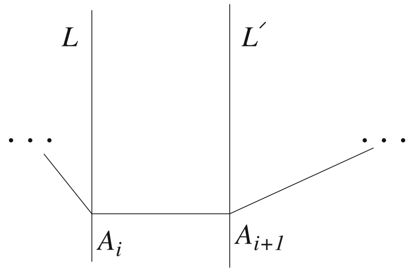

Figure 59

than the distance from A_i to L'. This contradicts the maximality, so it cannot happen. The same argument shows than no vertex lies to the left of L. So there exists a vertex that either lies on one of L and L', or is between them. That vertex projects onto the (closed) side A_iA_{i+1}, and the problem is solved.

Remark. It is possible that no vertex projects in the interior of a side, as is the case with rectangles or with the regular hexagon.

(M. Pimsner, S. Popa, *Probleme de Geometrie Elementară* (*Problems in Elementary Geometry*), Editura Didactică şi Pedagogică, Bucharest, 1979)

65. *First solution*: Consider the oriented graph of roads and cities. By hypothesis, the graph has no cycles. Define a partial order of the cities, saying that $A < B$ if one can travel from A to B. A partial order on a finite set has maximal and minimal elements. In a maximal city all roads enter, and from a minimal city all roads exit.

Second solution: Pick an itinerary that travels through a maximal number of cities (more than one such itinerary may exist). No roads enter the starting point of the itinerary, while no roads exit the endpoint.

(*Kvant* (*Quantum*))

66. Let b be a boy dancing with the maximal number of girls. There is a girl g' he does not dance with. Choose as b' a boy who dances with g'. Let g be a girl who dances with b but not with b'. Such a girl exists because of the maximality of b, since b' already dances with a girl who does not dance with b. Then the pairs (b, g), (b', g') satisfy the requirement.

(26th W.L. Putnam Mathematical Competition, 1965)

67. Arguing by contradiction, assume that we can have a set of finitely many points with this property. Let $V_1 \subset V$ be the vectors whose x-coordinate is positive or whose x-coordinate is 0 and the y-coordinate is positive. Let $V_2 = V \backslash V_1$.

Order the points marked points lexicographically by their coordinates (x, y). Then examining the largest point we obtain $|V_1| < |V_2|$ and examining the smallest point we obtain $|V_2| < |V_1|$. This impossible. The conclusion follows.

(*Kvant (Quantum)*, proposed by D. Rumynin)

68. Let $(a_{ij})_{ij}$, $1 \le i \le m$, $1 \le j \le n$, be the matrix. Denote the sum of the elements in the ith row by s_i, $i = 1, 2, \ldots, m$. We will show that among all matrices obtained by permuting the elements of each column, the one for which the sum $|s_1| + |s_2| + \cdots + |s_m|$ is minimal has the desired property.

If this is not the case, then $|s_k| \geq 2$ for some k. Without loss of generality, we can assume that $s_k \geq 2$. Since $s_1 + s_2 + \cdots + s_m = 0$, there exists j such that $s_j < 0$. Also, there exists an i such that $a_{ik} > a_{ij}$ for otherwise s_j would be larger than s_k. When exchanging a_{ik} and a_{ij} the sum $|s_1| + |s_2| + \cdots + |s_m|$ decreases. Indeed,

$$\begin{aligned}|s_k - a_{ik} + a_{ij}| + |s_j + a_{ik} - a_{ij}| &= s_k - a_{ik} + a_{ij} + |s_j + a_{ik} - a_{ij}| \\ &< s_k - a_{ik} + a_{ij} + |s_j| + a_{ik} - a_{ij},\end{aligned}$$

where the equality follows from the fact that $s_k \geq 2 \geq a_{ik} - a_{ij}$, while the *strict* inequality follows from the triangle inequality and the fact that s_j and $a_{ik} - a_{ij}$ have opposite signs. This shows that any minimal configuration must satisfy the condition from the statement. Note that a minimal configuration always exists, since the number of possible permutations is finite.

(Austrian-Polish Mathematics Competition, 1984)

69. We call a number *good* if it satisfies the given condition. It is not difficult to see that all powers of primes are good. Suppose n is a good number that has at least two distinct prime factors. Let $n = p^r s$, where p is the smallest prime dividing n and s is not divisible by p. Because n is good, $p+s-1$ must divide n. For any prime q dividing s, $s < p+s-1 < s+q$, so q does not divide $p+s-1$. Therefore, the only prime factor of $p+s-1$ is p. Then $s = p^c - p + 1$ for some integer $c > 1$. Because p^c must also divide n, $p^c + s - 1 = 2p^c - p$ divides n. Because $2p^{c-1} - 1$ has no factors of p, it must divide s. But

$$\begin{aligned}\frac{p-1}{2}(2p^{c-1}-1) = p^c - p^{c-1} - \frac{p-1}{2} < p^c - p + 1 &< \frac{p+1}{2}(2p^{c-1}-1) \\ &= p^c + p^{c-1} - \frac{p+1}{2},\end{aligned}$$

a contradiction. It follows that the only good integers are the powers of primes.

(Russian Mathematical Olympiad, 2001)

70. Let us assume that no infinite monochromatic sequence exists with the desired property, and consider a maximal white sequence $2k_1 < k_1 + k_2 < \cdots < 2k_n$ and a maximal black sequence $2l_1 < l_1 + l_2 < \cdots < 2l_m$. By maximal we mean that these sequences cannot be extended any further. Without loss of generality, we may assume that $k_n < l_m$.

We look at all white even numbers between $2k_n + 1$ and some arbitrary $2x$; let W be their number. If for one of these white even numbers $2k$ the number $k + k_n$ were white as well, then the sequence of whites could be extended, contradicting maximality. Hence $k + k_n$ must be black. Therefore, the number b of blacks between $2k_n + 1$ and $x + k_n$ is at least W.

Similarly, if B is the number of black evens between $l_m + 1$ and $2x$, the number w of whites between $2l_m + 1$ and $x + l_m$ is at least B. We have $B + W \geq x - l_m$, the latter being the number of even integers between $2l_m + 1$ and $2x$, while $b + w \leq x - k_n$, since $x - k_n$ is the number of integers between $2k_n + 1$ and $x + k_n$. Subtracting, we obtain

$$0 \leq (b - W) + (w - B) \leq l_m - k_n,$$

and this inequality holds for all x. This means that as x varies there is an upper bound for $b - W$ and $w - B$. Hence there can be only a finite number of black squares that cannot be written as $k_n + k$ for some white $2k$ and there can only be a finite number of white squares which cannot be written as $l_m + 1$ for some black $2l$. Consequently, from a point onward all white squares are of the form $l_m + l$ for some black $2l$ and from a point onward all black squares are of the form $k_n + k$ for some white $2k$.

We see that for k sufficiently large, k is black if and only if $2k - 2k_n$ is white, while k is white if and only if $2k - 2l_m$ is black. In particular, for each such k, $2k - 2k_n$ and $2k - 2l_m$ have the same color, opposite to the color of k. So if we let $l_m - k_n = a > 0$, then from some point onward $2x$ and $2x + 2a$ are of the same color. The arithmetic sequence $2x + 2na$, $n \geq 0$, is thus monochromatic. It is not hard to see that it also satisfies the condition from the statement, a contradiction. Hence our assumption was false, and sequences with the desired property do exist.

(Communicated by A. Neguţ)

71. We begin with an observation that will play an essential role in the solution. Given a triangle XYZ, if $\angle XYZ \leq \frac{\pi}{3}$, then either the triangle is equilateral or else $\max\{XY, YZ\} > XZ$, and if $\angle XYZ \leq \frac{\pi}{3}$, then either the triangle is equilateral or else $\min\{YX, YZ\} < XZ$.

Choose vertices A and B that minimize the distance between vertices. If C is a vertex such that $\angle ACB = \frac{\pi}{3}$, then $\max\{CA, CB\} \leq AB$, so by our observation the triangle ABC is equilateral. So there exists an equilateral triangle ABC formed by vertices of the polygon and whose side length is the minimal distance between two vertices of the polygon. By a similar argument there exists a triangle $A_1B_1C_1$ formed by vertices whose side length is the maximal distance between two vertices of the polygon. We will prove that the two triangles are congruent.

The lines AB, BC, CA divide the plane into seven open regions. Denote by R_A the region distinct from the interior of ABC and bounded by side BC, plus the boundaries of this region except for the vertices B and C. Define R_B and R_C analogously. These regions are illustrated in Figure 60. Because the given polygon is convex, each of A_1, B_1, and C_1 lies in one of these regions or coincides with one of A, B, and C.

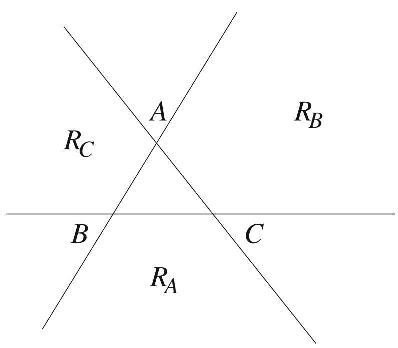

Figure 60

If two of A_1, B_1, C_1, say A_1 and B_1, are in the same region R_X, then $\angle A_1XB_1 < \frac{\pi}{3}$. Hence $\max\{XA_1, XB_1\} > A_1B_1$, contradicting the maximality of the length A_1B_1. Therefore, no two of A_1, B_1, C_1 are in the same region.

Suppose now that one of A_1, B_1, C_1 (say A_1) lies in one of the regions (say R_A). Because $\min\{A_1B, A_1C\} \geq BC$, we have that $\angle BA_1C \leq \frac{\pi}{3}$. We know that B_1 does not lie in R_A. Also, because the polygon is convex, B does not lie in the interior of the triangle AA_1B_1, and C does not lie in the interior of triangle AA_1B_1. It follows that B_1 lies in the closed region bounded by the rays $|A_1B$ and $|A_1C$. So does C_1. Therefore, $\frac{\pi}{3} = \angle B_1A_1C_1 \le \angle BA_1C \le \frac{\pi}{3}$, with equalities if B_1 and C_1 lie on rays $|A_1B$ and $|A_1C$. Because the given polygon is convex, this is possible only if B_1 and C_1 equal B and C in some order, in which case $BC = B_1C_1$. This would imply that triangles ABC and $A_1B_1C_1$ are congruent.

The remaining situation occurs when none of A_1, B_1, C_1 are in $R_A \cup R_B \cup R_C$, in which case they coincide with A, B, C in some order. Again we conclude that the two triangles are congruent.

We have proved that the distance between any two vertices of the given polygon is the same. Therefore, given a vertex, all other vertices are on a circle centered at that vertex. Two such circles have at most two points in common, showing that the polygon has at most four vertices. If it had four vertices, it would be a rhombus, whose longer diagonal would be longer than the side, a contradiction. Hence the polygon can only be the equilateral triangle, the desired conclusion.

(Romanian Mathematical Olympiad, 2000)

72. Because

$$a^2 + b^2 = \left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2,$$

the sum of the squares of the numbers in a triple is invariant under the operation. The sum of squares of the first triple is $\frac{13}{2}$ and that of the second is $6 + 2\sqrt{2}$, so the first triple cannot be transformed into the second.

(D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles*, AMS, 1996)

73. Assign the value i to each white ball, $-i$ to each red ball, and -1 to each green ball. A quick check shows that the given operations preserve the product of the values of the balls in the box. This product is initially $i^{2000} = 1$. If three balls were left in the box, none of them green, then the product of their values would be $\pm i$, a contradiction. Hence, if three balls remain, at least one is green, proving the claim in part (a). Furthermore, because no ball has value 1, the box must contain at least two balls at any time. This shows that the answer to the question in part (b) is *no*.

(Bulgarian Mathematical Olympiad, 2000)

74. Let I be the sum of the number of stones and heaps. An easy check shows that the operation leaves I invariant. The initial value is 1002. But a configuration with k heaps, each containing 3 stones, has $I = k + 3k = 4k$. This number cannot equal 1002, since 1002 is not divisible by 4.

(D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles*, AMS, 1996)

75. The quantity $I = xv + yu$ does not change under the operation, so it remains equal to $2mn$ throughout the algorithm. When the first two numbers are both equal to $\gcd(m, n)$, the sum of the latter two is $\frac{2mn}{\gcd(m,n)} = 2\operatorname{lcm}(m, n)$.

(St. Petersburg City Mathematical Olympiad, 1996)

76. We can assume that p and q are coprime; otherwise, shrink the size of the chessboard by their greatest common divisor. Place the chessboard on the two-dimensional integer lattice such that the initial square is centered at the origin, and the other squares, assumed to have side length 1, are centered at lattice points. We color the chessboard by the Klein four group

$$K = \{a, b, c, e \mid a^2 = b^2 = c^2 = e,\ ab = c,\ ac = b,\ bc = a\}$$

as follows: if (x, y) are the coordinates of the center of a square, then the square is colored by e if both x and y are even, by c if both are odd, by a if x is even and y is odd, and by b if x is odd and y is even (see Figure 61). If p and q are both odd, then at each jump the color of the location of the knight is multiplied by c. Thus after n jumps the knight is on a square colored by c^n. The initial square was colored by e, and the equality $c^n = e$ is possible only if n is even.

If one of p and q is even and the other is odd, then at each jump the color of the square is multiplied by a or b. After n jumps the color will be $a^k b^{n-k}$. The equality $a^k b^{n-k} = e$ implies $a^k = b^{n-k}$, so both k and $n - k$ have to be even. Therefore, n itself has to be even. This completes the solution.

(German Mathematical Olympiad)

77. The invariant is the 5-colorability of the knot, i.e., the property of a knot to admit a coloring by the residue classes modulo 5 such that

(i) at least two residue classes are used;

(ii) at each crossing, $a + c \equiv 2b \pmod 5$,

where b is the residue class assigned to the overcrossing, and a and c are the residue classes assigned to the other two arcs.

c	b	c	b	c	b
a	e	a	e	a	e
c	b	c	b	c	b
a	e	a	e	a	e
c	b	c	b	c	b
a	e	a	e	a	e

Figure 61

A coloring of the figure eight knot is given in Figure 62, while the trivial knot does not admit 5-colorings since its simplest diagram does not. This proves that the figure eight knot is knotted.

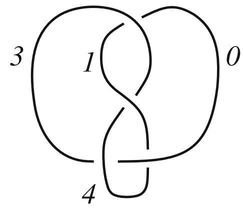

Figure 62

78. The answer is no. The idea of the proof is to associate to the configuration (a) an encoding defined by a pair of vectors $(v, w) \in \mathbb{Z}_2^2$ square contains a $+$ if the ith coordinate of v is equal to the jth coordinate of w, and a $-$ otherwise. A possible encoding for our configuration is $v = w = (1, 1, 0)$. Any other configuration that can be obtained from it admits such an encoding. Thus we choose as the invariant the *possibility* of encoding a configuration in such a manner.

It is not hard to see that the configuration in (b) cannot be encoded this way. A slick proof of this fact is that the configuration in which all signs are negative except for the one in the center can be obtained from this by the specified move, and this latter one cannot be encoded. Hence it is impossible to transform the first configuration into the second.

(Russian Mathematical Olympiad 1983–1984, solution by A. Badev)

79. The answer is no. The essential observation is that

$$99\ldots 99 \equiv 99 \equiv 3 \pmod 4.$$

When we write this number as a product of two factors, one of the factors is congruent to 1 and the other is congruent to 3 modulo 4. Adding or subtracting a 2 from each factor produces numbers congruent to 3, respectively, 1 modulo 4. We deduce that what stays invariant in this process is the parity of the number of numbers on the blackboard that are congruent to 3 modulo 4. Since initially this number is equal to 1, there will always be at least one number that is congruent to 3 modulo 4 written on the blackboard. And this is not the case with the sequence of nines. This proves our claim.

(St. Petersburg City Mathematical Olympiad, 1997)

80. Without loss of generality, we may assume that the length of the hypotenuse is 1 and those of the legs are p and q. In the process, we obtain homothetic triangles that are in the ratio $p^m q^n$ to the original ones, for some nonnegative integers m and n. Let us focus on the pairs (m, n).

Each time we cut a triangle, we replace the pair (m, n) with the pairs $(m + 1, n)$ and $(m, n + 1)$. This shows that if to the triangle corresponding to the pair (m, n) we associate the weight $\frac{1}{2^{m+n}}$, then the sum I of all the weights is invariant under cuts. The initial value of I is 4. If at some stage the triangles were pairwise incongruent, then the value of I would be strictly less than

$$\sum_{m,n=0}^{\infty} \frac{1}{2^{m+n}} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{2^n} = 4,$$

a contradiction. Hence a configuration with all triangles of distinct sizes cannot be achieved.

(Russian Mathematical Olympiad, 1995)

81. *First solution*: Here the invariant is given; we just have to prove its invariance. We first examine the simpler case of a cyclic quadrilateral $ABCD$ inscribed in a circle of radius R. Recall that for a triangle XYZ the radii of the incircle and the circumcircle are related by

$$r = 4R\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}.$$

Let $\angle CAD = \alpha_1$, $\angle BAC = \alpha_2$, $\angle ABD = \beta$. Then $\angle DBC = \alpha_1$, and $\angle ACD = \beta$, $\angle BDC = \alpha_2$, and $\angle ACB = \angle ADB = 180^\circ - \alpha_1 - \alpha_2 - \beta$. The independence of the sum of the inradii in the two possible dissections translates, after dividing by $4R$, into the identity

$$\sin\frac{\alpha_1+\alpha_2}{2}\sin\frac{\beta}{2}\sin\left(90^\circ - \frac{\alpha_1+\alpha_2+\beta}{2}\right) + \sin\left(90^\circ - \frac{\alpha_1+\alpha_2}{2}\right)\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}$$
$$= \sin\frac{\alpha_1+\beta_1}{2}\sin\frac{\alpha_2}{2}\sin\left(90^\circ - \frac{\alpha_1+\alpha_2+\beta}{2}\right) + \sin\left(90^\circ - \frac{\alpha_1+\beta_1}{2}\right)\sin\frac{\alpha_1}{2}\sin\frac{\beta}{2}.$$

This is equivalent to

$$\cos\frac{\alpha_1+\beta_1+\alpha_2}{2}\left(\sin\frac{\alpha_1+\alpha_2}{2}\sin\frac{\beta}{2} - \sin\frac{\alpha_1+\beta}{2}\sin\frac{\alpha_2}{2}\right)$$
$$= \sin\frac{\alpha_1}{2}\left(\sin\frac{\beta}{2}\cos\frac{\alpha_1+\beta_1}{2} - \sin\frac{\alpha_2}{2}\cos\frac{\alpha_1+\alpha_2}{2}\right),$$

or

$$\cos\frac{\alpha_1+\alpha_2+\beta}{2}\left(\cos\frac{\alpha_1+\alpha_2-\beta}{2} - \cos\frac{\alpha_1-\alpha_2+\beta}{2}\right)$$
$$= \sin\frac{\alpha_1}{2}\left(\sin\left(\beta_1+\frac{\alpha_1}{2}\right) - \sin\left(\alpha_2+\frac{\alpha_1}{2}\right)\right).$$

Using product-to-sum formulas, both sides can be transformed into

$$\cos(\alpha_1+\alpha_2) + \cos\beta_1 - \cos(\alpha_1+\beta_1) - \cos\alpha_2.$$

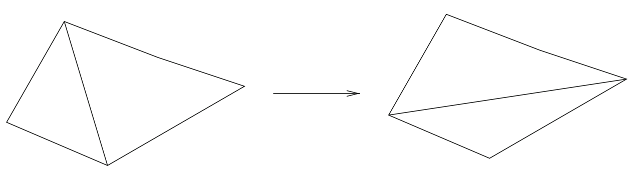

Figure 63

The case of a general polygon follows from the particular case of the quadrilateral. This is a consequence of the fact that any two dissections can be transformed into one another by a sequence of *quadrilateral moves* (Figure 63). Indeed, any dissection can be transformed into a dissection in which all diagonals start at a given vertex, by moving the endpoints of diagonals one by one to that vertex. So one can go from any dissection to any other dissection using this particular type as an intermediate step. Since the sum of the inradii is invariant under quadrilateral moves, it is independent of the dissection.

Second solution: This time we use the trigonometric identity

$$1 + \frac{r}{R} = \cos X + \cos Y + \cos Z.$$

We will check therefore that the sum of $1 + \frac{r_i}{R}$ is invariant, where r_i are the inradii of the triangles of the decomposition. Again we prove the property for a cyclic quadrilateral and then obtain the general case using the quadrilateral move. Using the fact that the sum of cosines of supplementary angles is zero and chasing angles in the cyclic quadrilateral $ABCD$, we obtain

$$\begin{aligned}\cos\ & \angle DBA + \cos\angle BDA + \cos\angle DAB + \cos\angle BDC + \cos\angle CBD + \cos\angle CDB \\ &= \cos\angle DBA + \cos\angle BDA + \cos\angle CBD + \cos\angle CDB \\ &= \cos\angle DCA + \cos\angle BCA + \cos\angle CAD + \cos\angle CAB \\ &= \cos\angle DCA + \cos\angle CAD + \cos\angle ADC + \cos\angle BCA + \cos\angle CAB + \cos\angle ABC,\end{aligned}$$

and we are done.

Remark. A more general theorem states that two triangulations of a polygonal surface (not necessarily by diagonals) are related by the move from Figure 63 and the move from Figure 64 or its inverse. These are called Pachner moves.

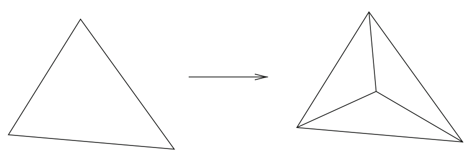

Figure 64

(Indian Team Selection Test for the International Mathematical Olympiad, 2005, second solution by A. Tripathy)

82. Let S be the sum of the elements of the table. By performing moves on the rows or columns with negative sum, we obtain a strictly increasing sequence $S_1 < S_2 < \cdots$. Because S can take at most 2^{n^2} values (all possible sign choices for the entries of the table), the sequence becomes stationary. At that time no row or column will have negative sum.

83. Skipping the first step, we may assume that the integers are nonnegative. The semi-invariant is $S(a, b, c, d) = \max(a, b, c, d)$. Because for nonnegative numbers x, y, we have $|x - y| \leq \max(x, y)$, S does not increase under T. If S decreases at every step, then it eventually becomes 0, in which case the quadruple is $(0, 0, 0, 0)$. Let us see in what situation S is preserved by T. If

$$S(a, b, c, d) = S(T(a, b, c, d)) = S(|a - b|, |b - c|, |c - d|, |d - a|),$$

then next to some maximal entry there must be a zero. Without loss of generality, we may assume $a = S(a, b, c, d)$ and $b = 0$. Then

$$\begin{aligned}(a, 0, c, d) &\xrightarrow{T} (a, c, |c-d|, |d-a|)\\ &\xrightarrow{T} (|a-c|, |c-|c-d|, ||c-d|-|d-a||, |a-|d-a||).\end{aligned}$$

Can S stay invariant in both these steps? If $|a-c| = a$, then $c = 0$. If $|c-|c-d|| = a$, then since a is the largest of the four numbers, either $c = d = a$ or else $c = 0$, $d = a$. The equality $||c-d|-|d-a|| = a$ can hold only if $c = 0$, $d = a$, or $d = 0$, $c = a$. Finally, $|a-|d-a|| = a$ if $d = a$. So S remains invariant in two consecutive steps only for quadruples of the form

$$(a, 0, 0, d), (a, 0, 0, a), (a, 0, a, 0), (a, 0, c, a),$$

and their cyclic permutations.

At the third step these quadruples become

$$(a, 0, d, |d-a|), (a, 0, a, 0), (a, a, a, a), (a, c, |c-a|, 0).$$

The second and the third quadruples become $(0, 0, 0, 0)$ in one and two steps, respectively. Now let us look at the first and the last. By our discussion, unless they are of the form $(a, 0, a, 0)$ or $(a, a, 0, 0)$, respectively, the semi-invariant will decrease at the next step. So unless it is equal to zero, S can stay unchanged for at most five consecutive steps. If initially $S = m$, after $5m$ steps it will be equal to zero and the quadruple will then be $(0, 0, 0, 0)$.

84. If a, b are erased and $c < d$ are written instead, we have $c \le \min(a, b)$ and $d \ge \max(a, b)$. Moreover, $ab = cd$. Using derivatives we can show that the function $f(c) = c + \frac{ab}{c}$ is strictly decreasing on $\left(0, \frac{a+b}{2}\right)$, which implies $a + b \le c + d$. Thus the sum of the numbers is nondecreasing. It is obviously bounded, for example by n times the product of the numbers, where n is the number of numbers on the board. Hence the sum of the numbers eventually stops changing. At that moment the newly introduced c and d should satisfy $c + d = a + b$ and $cd = ab$, which means that they should equal a and b. Hence the numbers themselves stop changing.

(St. Petersburg City Mathematical Olympiad, 1996)

85. To a configuration of pebbles we associate the number

$$S = \sum \frac{1}{2^{|i|+|j|}},$$

where the sum is taken over the coordinates of all nodes that contain pebbles. At one move of the game, a node (i, j) loses its pebble, while two nodes (i_1, j_1) and (i_2, j_2) gain pebbles. Since either the first coordinate or the second changes by one unit, $|i_k| + |j_k| \le |i| + |j| + 1$, $k = 1, 2$. Hence

$$\frac{1}{2^{|i|+|j|}} = \frac{1}{2^{|i|+|j|+1}} + \frac{1}{2^{|i|+|j|+1}} \le \frac{1}{2^{|i_1|+|j_1|}} + \frac{1}{2^{|i_2|+|j_2|}},$$

which shows that S is a nondecreasing semi-invariant. We will now show that at least one pebble is inside or on the boundary of the square R determined by the lines $x \pm y = \pm 5$. Otherwise, the total value of S would be less than

$$\begin{aligned}\sum_{|i|+|j|>5} \frac{1}{2^{|i|+|j|}} &= 1 + 4\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2^{i+j}} - \sum_{|i|+|j|\leq 5}\frac{1}{2^{|i|+|j|}} \\ &= 1 + 4\sum_{i=1}^{\infty}\frac{1}{2^i}\sum_{j=0}^{\infty}\frac{1}{2^j} - 1 - 4\left(1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32}\right).\end{aligned}$$

This equals $9 - \frac{65}{8} = \frac{7}{8}$ which is impossible, since the original value of S was 1. So there is always a pebble inside R, which is at distance at most 5 from the origin.