Integration

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I think we all started learning integration in 11th class. Many of us may have found it a little hard. So let us sta

**** For example, if I place narrow sheets of rectangular paper side by side, on the ground, I get a bigger rectangular paper. The area of the bigger paper is sum of areas of the smaller papers.**

Contents

1 Introduction to Integration

Let us start by understanding the term **Integration** in English. Simply put, Integration is the act of bringing together smaller components into a single system that functions as one. *****

With this understanding, let us try to

To understand integration in mathematical terms**. Let** let us first recall the concept of **summation**. For example, **sum** of first n natural numbers is

$$\sum_{1}^{n} x = 1 + 2 + 3 + \ldots = n(n+1)/2$$

This is easy to understand, isn't it. Now, then **So what do we mean here?**

Integration is nothing but a continuous version of summation. We know that summation applies to finite, countable sets.***** However, integration occurs over discrete or infinite bounds, but more importantly, over a set of values which is not countably finite.**$** For example, when summation iterates over the discrete set $\{1, 2, \ldots, n\}$, we can sum over all the elements in the set. But, for the

*** That is, we can count the number of elements that we are adding up in a summation.**

$ So here we are talking about two points. The first point pertains to infinte bounds and discrete bounds. Infinite bounds imply integration (section ~\ref{}) over the entire range ??? of the function. On the other hand, discrete bounds implies definite integral (section ~\ref{}) where the interval for the integration is defined. In the second point, we are stating that no matter what the interval for integration is, the number of elements in the set is infinite.

elements of the real closed interval [1, n], integration is defined. For example, sum of square of first n natural numbers is:

$$\sum_{1}^{n} x^2 = 1^2 + 2^2 + 3^2 + \ldots = n(n+1)(2n+1)/6$$

whereas integration is:

$$\int_1^n x^2 dx = 1^2(dx) + (1+dx)^2 dx + (1+2dx)^2 dx + \ldots = (n^3 - 1)/3$$

dx???

1.1 Geometrical Interpretation

We can understand integration as a continuous summation of elements that are infinitesimally small. However, one may ask "What are we summming in integration?". Suppose, we want to integrate a given function $f(x)$. When we plot this curve, the area under the curve $f(x)$ can be divided into infinite rectangles, each with height $f(x)$ and thickness dx, where dx is an infinitesimally small width of x. Thus, integration of the function $f(x)$ is the summation of the area of all these rectangles.

1.2 Indefinite Integration

When we find the area under the curve for all possible values of x, it is called as **indefinite integration**. That is, indefinite integral is the area under the curve for the whole domain.

1.3 Definite Integration

However, what if we want to find the area between certain limits or over a certain region? To do this we introduce the concept of **definite integration**. For a given function $f(x)$ with indefinite integral $g(x)$, its definite integral over the range [a, b] is:

$$\int_a^b f(x)dx = g(b) - g(a),\ where \int f(x)dx = g(x)$$

The range over which we want to find the integral is known as the **limits of the integral**.

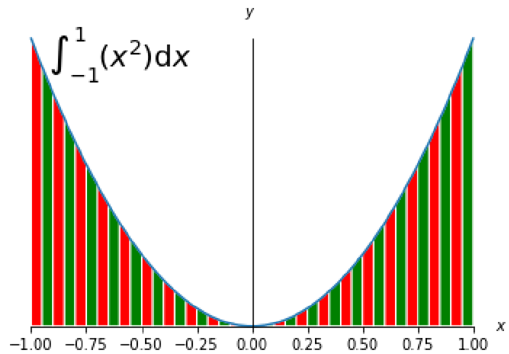

Figure 1: Definite integral of the function x^2 over the range [-1, 1]

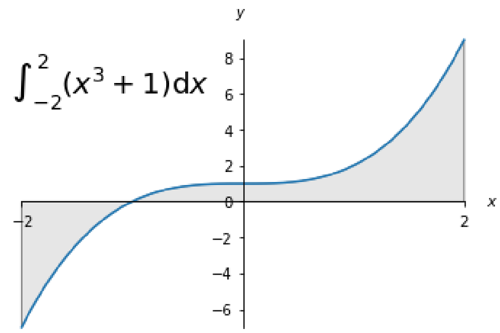

Figure 2: Definite integral of the function $x^3 + 1$ over the range [-2, 2]

Example 1.1. Definite integral of the function x^2 over the range [-1, 1] is:

$$\int_{-1}^{1} x^2 dx = x^3/3|_{-1}^{1} = (1/3) - (-1/3) = 2/3$$

Example 1.2. Definite integral of the function $x^3 + 1$ over the range [-2, 2] is (Figure 2):

$$\int_{-2}^{2} (x^3 + 1)dx = (x^4/4 + x)|_{-2}^{2} = (4 + 2) - (4 - 2) = 4$$

1.4 Multiple integrals

So far, the integrals we have done are **single integrals** where the function $f(x)$ has a single independent variable x. That is, change in variable x only will result in change in the value of the function $f(x)$. But what happens when the function being integrated is dependent on more than one variable? Thus we come to the concept of multiple integrals, specifically double integrals. For the function $f(x, y)$, we first integrate $f(x, y)$ with respect to one of the independent variables, say x, treating the other independent variable y as a constant, to obtain a function, say $g(x, y)$. Then, we integrate $g(x, y)$ with respect to y. That is,

$$\int\int f(x,y)dxdy = \int\left(\int f(x,y)dx\right)dy = \int g(x,y)dy$$

Let us understand through the following examples.

Example 1.3. For f(x,y)=xy, the double integral is:

$$\int\int xydxdy = \int\left(\int xdx\right)ydy = \int(1/2)x^2ydy = (1/4)x^2y^2$$

Example 1.4. For $f(x, y) = (x + y)^2$, the double integral in the range $x \in [0, 1]$, $y \in [0, 1]$ is:

$$\begin{aligned}
&\int_0^1\int_0^1 (x+y)^2dxdy \\
&= \int_0^1\int_0^1 (x^2+y^2+2xy)dxdy \\
&= \int_0^1 ((1/3)x^3 + y^2x + x^2y)|_{x=0}^{x=1}dy \\
&= \int_0^1 ((1/3) + y^2 + y) - 0dy \\
&= ((1/3)y + (1/3)y^3 + (1/2)y^2)|_{y=0}^{y=1} \\
&= (1/3) + (1/3) + (1/2) - 0 = 7/6
\end{aligned}$$

??? [1]

References

[1] J. Stewart. *Essential calculus: Early transcendentals*. Cengage Learning, 2012.