1 Cylindrical wave functions

The scalar Helmholtz equation in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \tag{1}$$

Following the method of variable separation, we seek to find solution of the form

$$\psi=R(\rho)\Phi(\phi)Z(z) \tag{2}$$

Substitution of the last Eq. (2) into Eq. (1) yields

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{\rho^2\Phi}\frac{\partial^2\Phi^2}{\partial\phi^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k^2=0 \tag{3}$$

The third term does not depend on variables ρ and ϕ. Hence,

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-k_z^2 \tag{4}$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi^2}{\partial\phi^2}+(k^2-k_z^2)\rho^2=0 \tag{5}$$

Once again, the second term is independent on the variable ρ and z, Thus we may denote

$$\frac{1}{\Phi}\frac{\partial^2\Phi^2}{\partial\phi^2}=-n^2 \tag{6}$$

And finally, we reach equation of one variable that called *Bessel equation*

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)-n^2+(k^2-k_z^2)\rho^2=0 \tag{7}$$

To summarize we define

$$k_\rho^2=k^2-k_z^2 \tag{8}$$

and come to three separated equation, Bessel and two harmonic ones

$$\begin{cases}\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)[(k_\rho\rho)^2-n^2]R=0\\ \frac{\partial^2\Phi^2}{\partial\phi^2}+n^2\Phi=0\\ \frac{\partial^2 Z}{\partial z^2}+k_z^2 Z=0\end{cases} \tag{9}$$

where the second equation according to the geometrical properties are 2π periodical, or

$$\Phi(\phi+2\pi)=\Phi(\phi) \tag{10}$$

so it may be presented via $\cos n\phi$ and $\sin n\phi$. Further let us assume the the propagation constant $k_z=0$. Therefore, according the properties of the Bessel function the field in domain containing the point $\rho=0$ is

$$\psi=J_n(k\rho)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix} \tag{11}$$

and in open domain that does not contain $\rho=0$

$$\psi=H_n(k\rho)\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix} \tag{12}$$

Here one should select the kind of Hankel function according to time dependence, $H_n^{(1)}$ for $e^{-i\omega t}$ and $H_n^{21)}$ for $e^{i\omega t}$.

2 The circular waveguide

Consider a dielectric resonator of radius $\rho = a$ with perfectly conducting borders filled with dielectric of relative permittivity $\epsilon_r = 10$. As the field should be finite inside the waveguide we choose one of form (11). As there is no losses, eigenvalues should be real.

For TM mode the characteristic equation is

$$J_n(\sqrt{\epsilon_r}ka) = 0 \tag{13}$$

The roots of Bessel function

m	0	1	2	3	4
1	2.4048	3.8317	5.1356	6.3802	7.5883
2	5.5201	7.0156	8.4172	9.761	11.0647
3	8.6537	10.1735	11.6198	13.0152	14.3725
4	11.7915	13.3237	14.796	16.2235	17.616

For TE mode the characteristic equation is

$$J_n'(\sqrt{\epsilon_r}ka) = 0 \tag{14}$$

The roots of Bessel function derivative

m	0	1	2	3	4
1	3.8317	1.8412	3.0542	4.2012	5.3176
2	7.0156	5.3314	6.7061	8.0152	9.2824
3	10.1735	8.5363	9.9695	11.3459	12.6819
4	13.3237	11.706	13.1704	14.5858	15.9641

2.1 Open cylindrical resonator

For TM mode the characteristic equation is

$$\sqrt{\epsilon_r}H_n(ka)J_n'(\sqrt{\epsilon_r}ka) - H_n'(ka)J_n(\sqrt{\epsilon_r}ka) = 0 \tag{15}$$

For TE mode the characteristic equation is

$$\frac{1}{\sqrt{\epsilon_r}}H_n(ka)J_n'(\sqrt{\epsilon_r}ka) - H_n'(ka)J_n(\sqrt{\epsilon_r}ka) = 0 \tag{16}$$

The field has the following form

$$\psi = \begin{cases} J_n(k\rho)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix}, & \text{if} \quad \rho \le a \\ H_n(k\rho)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix}, & \text{if} \quad \rho \le a \end{cases} \tag{17}$$