LECTURE 4-1
REFRACTION, DIFFRACTION, AND DISPERSION

Min Sup Hur

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Outline

- Refraction: Snell's law, Critical Angle, Brewster Angle
- Single-slit Diffraction
- A Gaussian Beam
- Dispersion Relation
- Fourier Decomposition of a Wave Packet
- Phase Velocity & Group Velocity
- Dispersive Wave

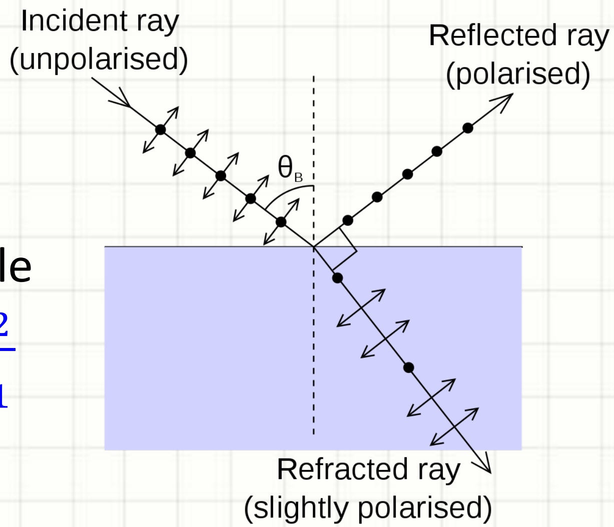
Refraction

- Modify the Maxwell solver to simulate the refraction; $\epsilon_0 \to \epsilon$
  For this purpose, in the unit system of 'c=1', just multiply the square of refractive index to the place of $1/c^2$ in Ampere's law.

$$\nabla\times\vec{B} = \mu_0 \frac{\partial\epsilon\vec{E}}{\partial t} = \frac{n^2}{c^2}\frac{\partial\vec{E}}{\partial t} = n^2\frac{\partial\vec{E}}{\partial t}$$

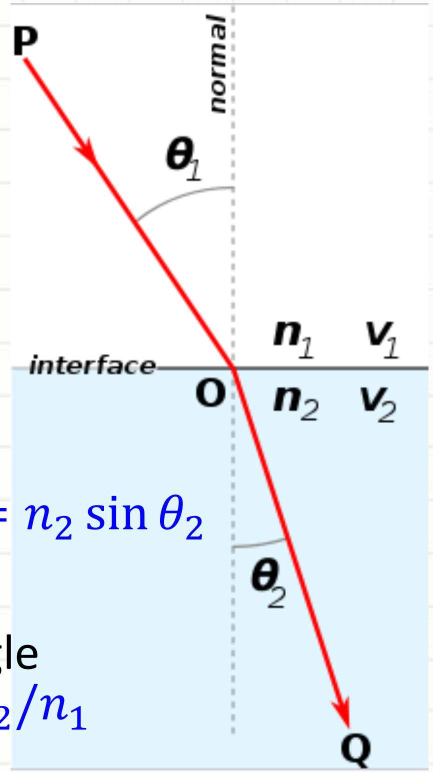
Refractive index: $n = \sqrt{\epsilon/\epsilon_0}$

- Compare the results with the known theory for transmittance and reflectivity. Check the Snell's law and critical angle.
- You can also check the Brewster's angle for p-polarized wave.



Brewster angle

$$\theta_B = \text{atan}\frac{n_2}{n_1}$$



Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Critical angle

$$\sin\theta_c = n_2/n_1$$

What and How to Measure

Example 1. To get the reflectivity and transmittance from the simulation, determine which quantity, e.g. field strength on the ray axis or averaged field strength, you measure. Then determine how you can do that. Once the measurement is completed, compare the results with the theoretical prediction. Check if Snell's law is also satisfied.

Answer: Put an array of virtual probes to grab temporal evolution of the field at different positions.

```
for id in range(nPrbs):
        i1 = pr1[id]; i2=pr2[id]
        e1t = E1[i1,i2];  e2t = E2[i1,i2];  e3t = E3[i1,i2];
        Ep[id].append(sqrt(e1t**2+e2t**2+e3t**2)
```

Single Slit Diffraction

Example 2. Using 2D Maxwell solver, reproduce the single-slit diffraction pattern. Measure the width of the central bright blob and compare it with the theory, i.e. $w = 2D\lambda/d$.

Answer: Use maxwell2d.py. Put a screen line at some point and draw the diffraction pattern along that. a

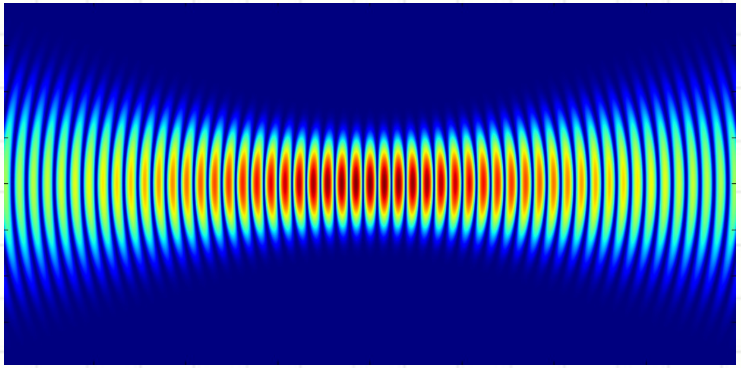
Example 3. Using 3D Maxwell solver, reproduce the diffraction pattern from a circular aperture. Measure the radius of the central bright blob and compare it with the theory, $R = 1.22D\lambda/d$.

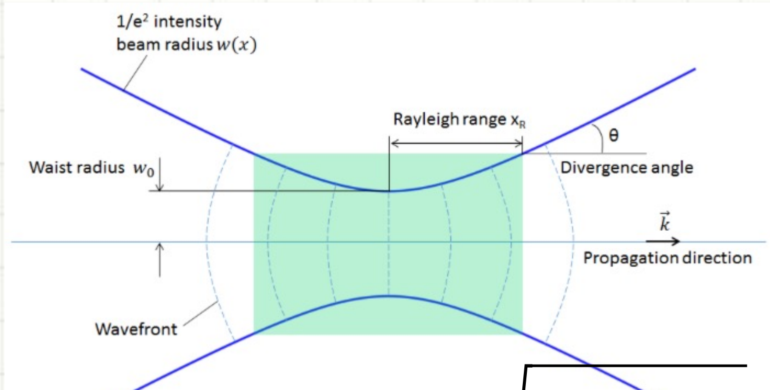
Answer: Use maxwell3d.py. Put a screen plane at some point and draw the diffraction pattern on it.

A Gaussian Beam

- Realistic light beam can be modeled by Gaussian.
- Because of the diffraction, the beam focuses and diffracts.
- Rayleigh length z_R: beam propagation distance over which the beam intensity is lowered by a half from the focus.

$$Z_R = \pi r_0^2/\lambda$$





Spot size evolution $$r = r_0\sqrt{1 + z^2/z_R^2}$$

Example 4. Launch a Gaussian beam from one side of the simulation domain using a 2D Maxwell solver. Measure the focal spot radius and peak intensity as functions of propagation distance. Compare the results with the theory.

Answer: See the appendix for general launcher of a Gaussian beam

Dispersion Relation

- How does the sunlight split into a color spectrum? This is because different color (frequency or wavelength) has different velocity.

- The velocity of the wave is $v = \frac{f}{\lambda}$, where f and λ are frequency and wavelength, respectively. Usually it is represented by angular frequency and wavenumber;

$$v = \frac{\omega}{k}, \qquad \omega \equiv 2\pi f, \quad k \equiv \frac{2\pi}{\lambda}$$

- Generally the wavenumber (k) is determined for a given frequency (ω) depending on the medium property. The relation between ω and k is referred to the dispersion relation.

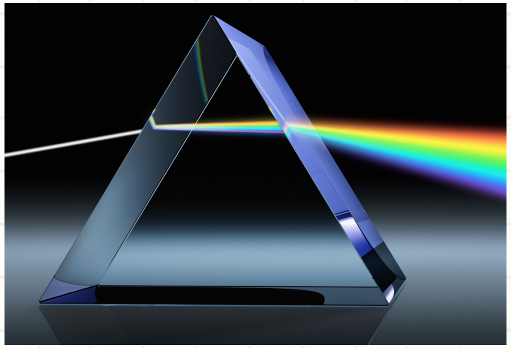


Figure: Britannica

Fourier Decomposition of a Wave Packet

- Fourier Transform

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

Gaussian wave packet

- Fourier Transform of a Gaussian Wave Packet

$$f(x) = e^{-x^2/\sigma^2}\cos x$$

From the Fourier Transform

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{-x^2/\sigma^2}\cos x\right]e^{-ikx}dx = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{-x^2/\sigma^2-i(k-1)x} + e^{-x^2/\sigma^2-i(k+1)x}\right]dx$$

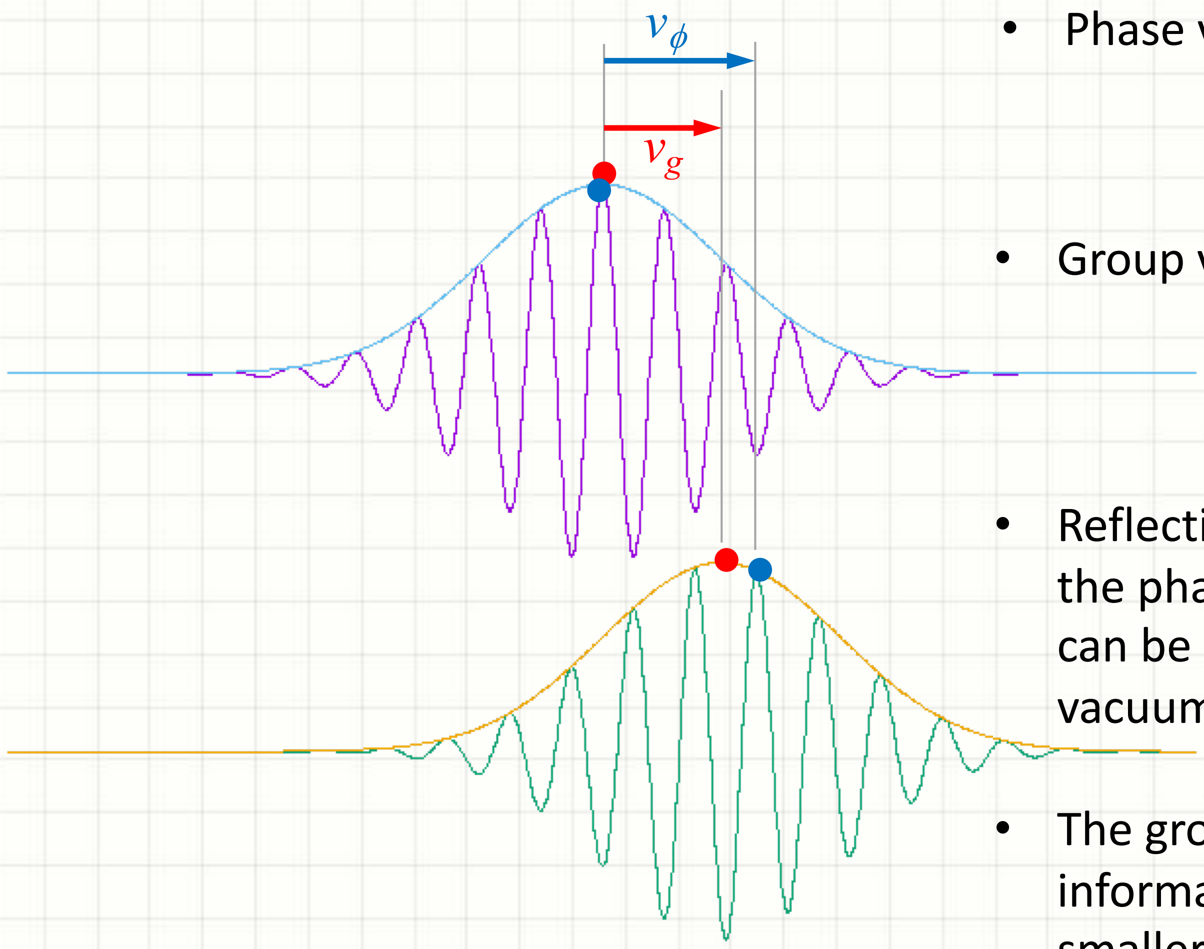
Integration of each term in the integrand is

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}-i(k\pm1)x}dx = \int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2}\left(x^2+i\sigma^2(k\pm1)x-\frac{\sigma^4(k\pm1)^2}{4}+\frac{\sigma^4(k\pm1)^2}{4}\right)}dx = \int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2}\left(x+\frac{i\sigma^2(k\pm1)}{2}\right)^2} e^{-\frac{\sigma^2(k\pm1)^2}{4}}dx = \sigma\sqrt{\pi}e^{-\frac{\sigma^2(k\pm1)^2}{4}}$$

- Fourier Modes of a Gaussian Wave Packet

$$A(k) = \frac{\sigma}{2\sqrt{2}}\left(e^{-\sigma^2(k+1)^2/4} + e^{-\sigma^2(k-1)^2/4}\right)$$

Phase Velocity and Group Velocity



- Phase velocity: velocity of a phase front

$$v_\phi = \frac{\omega}{k}$$

- Group velocity: velocity of a wave packet

$$v_g = \frac{d\omega}{dk}$$

- Reflection and refraction are relevant to the phase velocity. The phase velocity can be larger than the speed of light in vacuum.

- The group velocity is relevant to the information transport. It is always smaller than the speed of light in vacuum.

Dispersive Wave

- In vacuum, the light has the same velocity for every frequency. In this case, the dispersion relation is

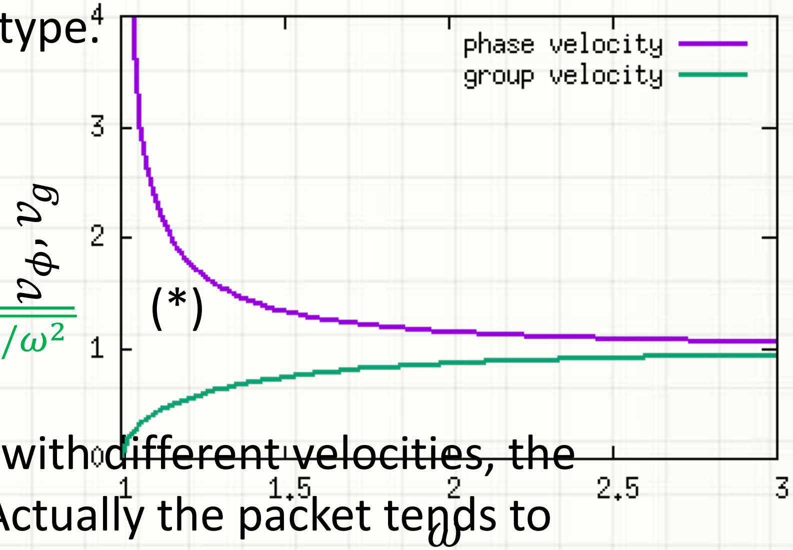
$$\omega = ck \quad : \text{dispersion relation of a light in vacuum}$$

- One typical dispersion relation is the Bohm-Gross type:

$$\omega^2 = A^2k^2 + B$$

- The velocity in this case,

$$v = \frac{\omega}{k} = A\sqrt{1 + \frac{B}{A^2k^2}} = \frac{A}{\sqrt{1 - B/\omega^2}}$$



Phase and group velocities for a Bohm-Gross style dispersion relation

- In the previous example, if the modes propagates with different velocities, the shape of the wave packet may not be preserved. Actually the packet tends to spread as time goes on, i.e. disperses.

- The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{A^2k}{\omega} = A\sqrt{1 - \frac{B}{\omega^2}} = \frac{A}{\sqrt{1 + B/(Ak)^2}}$$

Example: Dispersion of a Pulse

Example 5. Using disp-dispersion.py code, make the modes propagate with different velocities. Look how the shape of the packet changes.
*Suggested parameters: sigma=8, k=[0,3], dk=0.005, time of fly =800 with A=1 and B=0.2 in Eq. (*) in the previous page.

Answer: Look at the code 'disp-dispersion.py'. Main part of the code is,

```
…
z=0*x
for var in k:
        vel=A*np.sqrt(1+B/A**2/(var**2+0.1))
        z += sgm*(np.exp(-sgm**2*(var+1)**2/4)+np.exp(-sgm**2*(var-1)**2/4))/np.sqrt(8) *np.cos(var*(x-vel*propT))
```

END OF
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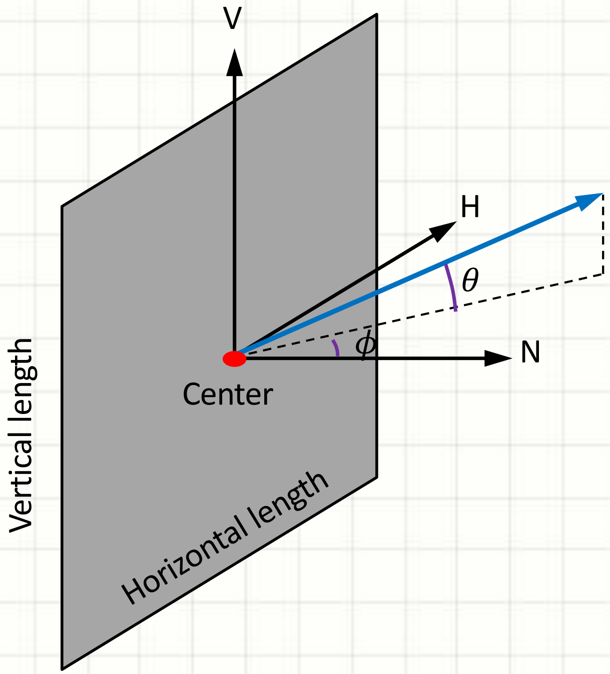
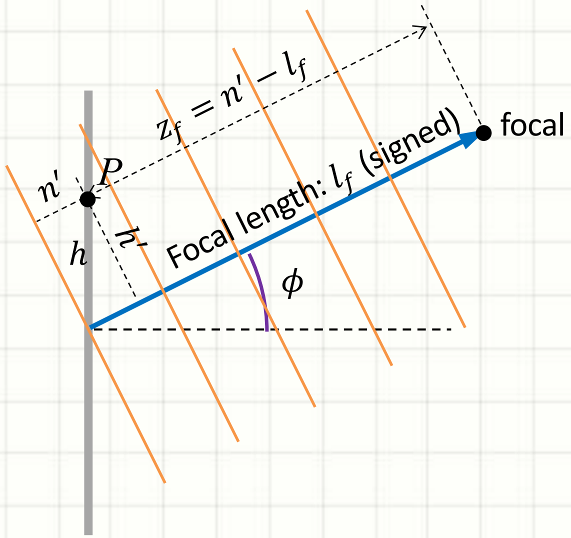
General EM Incidence Boundary





Paraxial wave

$$E = A(t)\left(\sqrt{\frac{r_{0h}}{r_h}}\exp\left[-\frac{h^2}{r_h^2}(1+i\alpha_h)+i\beta_h\right]\right)\left(\sqrt{\frac{r_{0v}}{r_v}}\exp\left[-\frac{v^2}{r_v^2}(1+i\alpha_v)+i\beta_v\right]\right)$$

$z_{h,v} = \pi r_{0h,v}^2/\lambda$	Rayleigh length
$\alpha_{h,v} = -z_f/z_{h,v}$	Radial phase factor
$\beta_{h,v} = (1/2)\ \mathrm{atan}\,\alpha_{h,v}$	Gouy phase
$r_{h,v} = r_{0h,v}\sqrt{1+\alpha_{h,v}^2}$	Beam radius

Fields at point P

$$h' = h\cos\phi \qquad v' = -h\sin\phi\sin\theta + v\cos\theta$$
$$n' = h\sin\phi\cos\theta + v\sin\theta$$

Rotation to wave frame

$$\psi = -kn' + 2\pi ft + \frac{h'^2}{r_h^2}\alpha_h + \frac{v'^2}{r_v^2}\alpha_v - \beta_h - \beta_v, \qquad \psi_t = \psi + \psi_0$$

Phase

$$E'_{h,v}(P) = A(t)\sqrt{\frac{r_{0h}r_{0v}}{r_h r_v}}\exp\left(-\frac{h'^2}{r_h^2}-\frac{v'^2}{r_v^2}\right)\cos(\psi_t)$$

Transverse field in wave frame

$$E'_n(P) = \frac{\lambda}{\pi}\left[E'_h(P)\frac{h'}{r_h^2}(\alpha_h\cos\psi_t + \sin\psi_t)\right],$$

$h \to v$ for s-polarization.

Longitudinal field

$$E_h(P) = E'_n\cos\theta\sin\phi + E'_h\cos\phi$$
$$E_v(P) = E'_n\sin\theta$$

p-polariation.

$$E_h(P) = E'_n\cos\theta\sin\phi - E'_v\sin\theta\sin\phi$$
$$E_v(P) = E'_n\sin\theta + E'_v\cos\theta$$

s-polariation.

Field at P

General EM Incidence Boundary

```
def gaussianEMemitter(t,E, dh, dv, mth, pol, phi, theta, frq, psi0, chrp, fl, r0, dur, Apk) :
        _pi=np.pi;  _clght= 2.99792458e8
        _dm = E.ndim
        if  _dm > 1:   (_szh, _szv) = E.shape
        else:  (_szh, _szv) = (len(E), 1)
        _ch = dh* (_szh-1)/2;   _cv = dv*(_szv-1)/2
        if len(r0) >1 :  _r0h = r0[0];  _r0v = r0[1]
        else: _r0h = _r0v = r0[0]
        _sf = np.sin(phi);  _sq = np.sin(theta);  _cf = np.cos(phi);  _cq = np.cos(theta)
        lmda = _clght/frq;   _w = 2.0*_pi*frq;   _k = _w/clght
        _zh = _pi*r0[0]**2/lmda;   _zv = _pi*r0[1]**2/lmda

        for _i in range(_szh):   for _j in range(_szv):
                _h = _i* dh – _ch;   _v = _j*dv – _cv
                _np = _h*_sf*_cq + _v*_sq;  _hp = _h*_cf; _vp = – _h*_sf*_sq + _v*_cq
                _zf = _np – fl
                _ah = – _zf / _zh;   _av = –_zf/_zv
                _bh = 0.5*np.atan(_ah);   _bv = 0.5*np.atan(_av)
                _rh = _r0h*np.sqrt(1+ _ah**2);  _rv = _r0v*np.sqrt(1+_av**2)
                _Dh = (_hp/_rh)**2;  _Dv = (_vp/_rv)**2
                _psi = –_k*_np + _w*t +_Dh*_ah +_Dv*_av – _bh – _bv+psi0
                _S = np.sqrt((_r0h/_rh)*(_r0v/_rv))*np.exp(–_Dh – _Dv )
                _Ep =Apk*_S*np.cos(_psi)
                if pol ==‘p’:
                        _Enp = (lmda/_pi)*_Ep*(_hp/_rh**2)*(_ah*np.cos(_psi)+np.sin(_psi))
                        if mth=‘sum’:   E[_i,_j] += _Enp*_cq*_sf + _Ep*_cf
                        else:  E[
                else:
                        _Enp = (lmda/_pi)*_Ep*(_vp/_rv**2)*(_av*np.cos(_psi)+np.sin(_psi))
                        if mth=‘sum’:  E[_i,_j] = _Enp*_cq + _Ep*_cq
        return
```

Rayleigh length: $z_{h,v} = \pi r_{0h,v}^2/\lambda$

Coordinate transform: $h' = h\cos\phi,\ v' = -h\sin\phi\sin\theta + v\cos\theta,\ n' = h\sin\phi\cos\theta + v\sin\theta$

Radial phase factor: $\alpha_{h,v} = -z_f/z_{h,v}$

Gouy phase: $\beta_{h,v} = (\operatorname{atan}\alpha_{h,v})/2$

Beam radius: $r_{h,v} = r_{0h,v}\left(1+\alpha_{h,v}^2\right)^{1/2}$

Phase: $\psi = -kn' + 2\pi f t + \frac{h'^2}{r_h^2}\alpha_h + \frac{v'^2}{r_v^2}\alpha_v - \beta_h - \beta_v + \psi_0$

Envelope

Transverse field in wave frame: $E'_{h,v}(P) = A(t)S\cos\psi$