LECTURE 3-2
MAXWELL EQUATION

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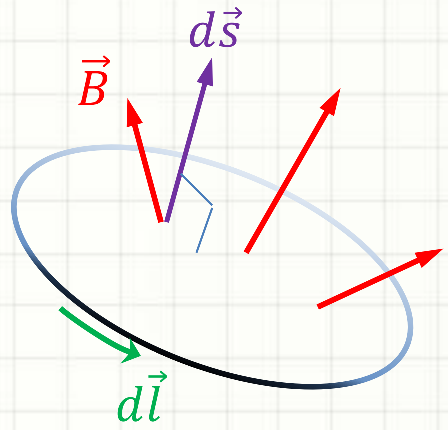
Outline

- Integral Form of Maxwell Equations
- Yee Mesh
- Maxwell Equations on Yee Mesh
- Evolution of the Fields
- Discretized Maxwell Equations
- Error and Stability
- Boundary Conditions

Integral Form of Maxwell Equations



Faraday's Law

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- Take the surface integral at both sides.

$$\int \nabla\times\vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$$

- Use the Stokes theorem to obtain the line integral of $\vec{E}$.

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$$

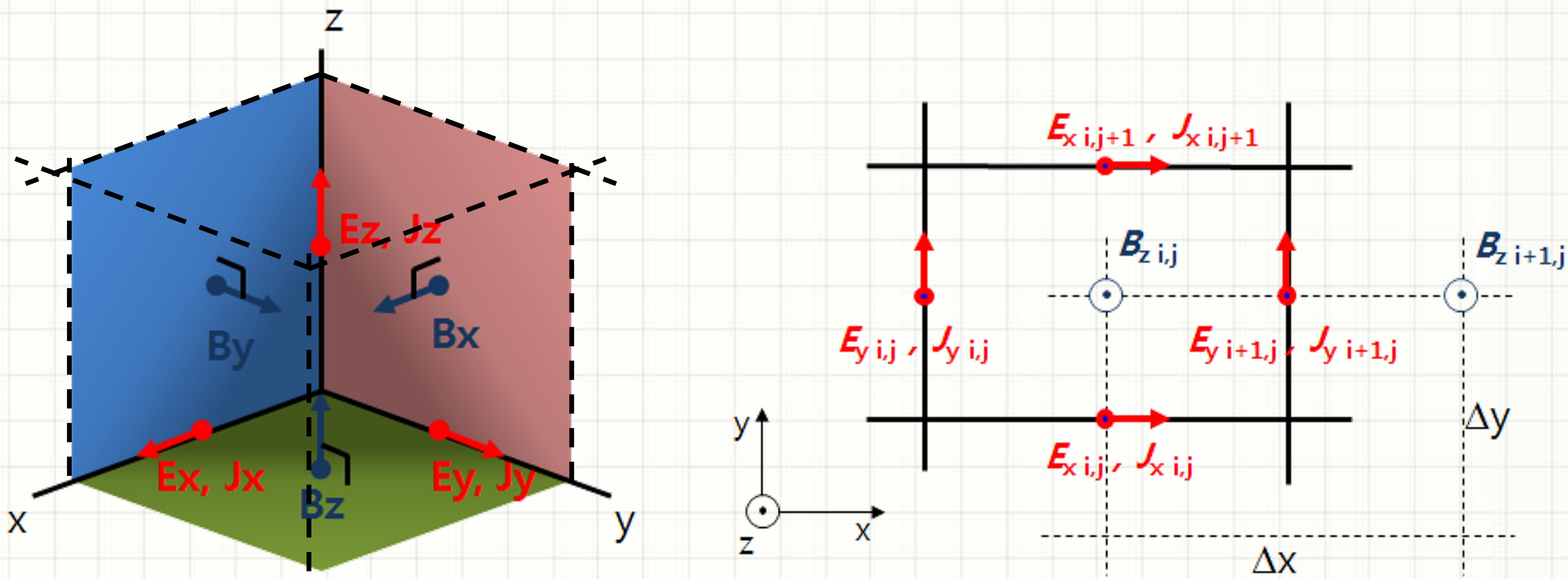
Ampere's Law (with the D-field)

$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

- Taking the same procedure,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\int \vec{J}\cdot d\vec{s} + \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{s}$$

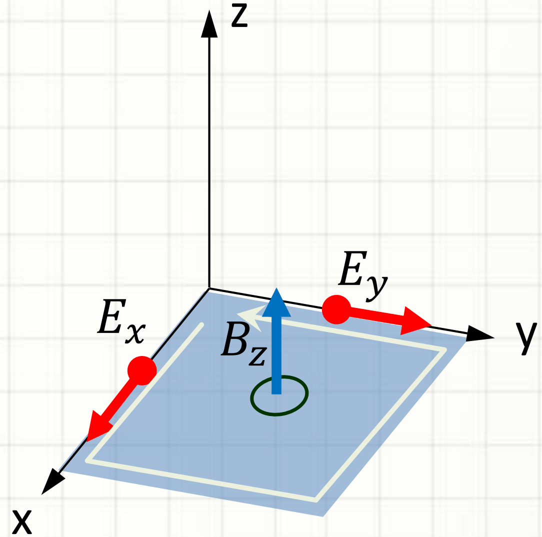
Yee Mesh



z
Ez, Jz
By
Bx
Ex, Jx
Bz
Ey, Jy
x
y
$E_{x\ i,j+1}$, $J_{x\ i,j+1}$
$B_{z\ i,j}$
$B_{z\ i+1,j}$
$E_{y\ i,j}$, $J_{y\ i,j}$
$E_{y\ i+1,j}$, $J_{y\ i+1,j}$
$E_{x\ i,j}$, $J_{x\ i,j}$
Δy
Δx
y
z
x

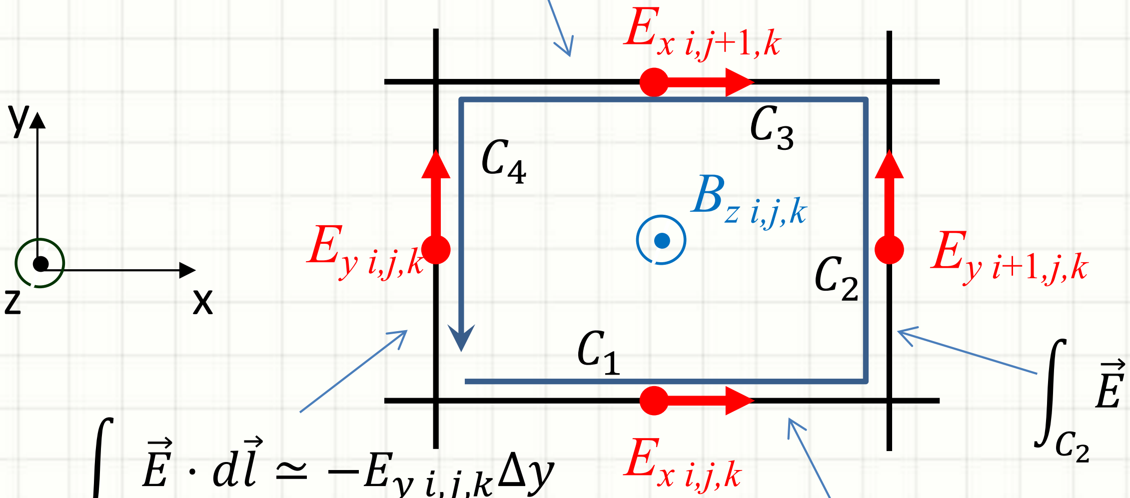
Faraday's Law on Yee Mesh

To use $\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s}$



Mesh (i, j, k)

$$\int_{C_3} \vec{E} \cdot d\vec{l} \simeq -E_{x\,i,j+1,k} \Delta x$$



$$\int_{C_2} \vec{E} \cdot d\vec{l} \simeq E_{y\,i+1,j,k} \Delta y$$

$$\int_{C_4} \vec{E} \cdot d\vec{l} \simeq -E_{y\,i,j,k} \Delta y$$

$$\int_{C_1} \vec{E} \cdot d\vec{l} \simeq E_{x\,i,j,k} \Delta x$$

Faraday's Law on Yee Mesh

- The line integral (left-hand-side) is

$$\oint \vec{E} \cdot d\vec{l} = \int_{C_1} \vec{E} \cdot d\vec{l} + \int_{C_2} \vec{E} \cdot d\vec{l} + \int_{C_3} \vec{E} \cdot d\vec{l} + \int_{C_4} \vec{E} \cdot d\vec{l}$$
$$\simeq \left(E_{y\ i+1,j,k} - E_{y\ i,j,k}\right)\Delta y - \left(E_{x\ i,j+1,k} - E_{x\ i,j,k}\right)\Delta x$$

- The surface integral (right-hand-side) is

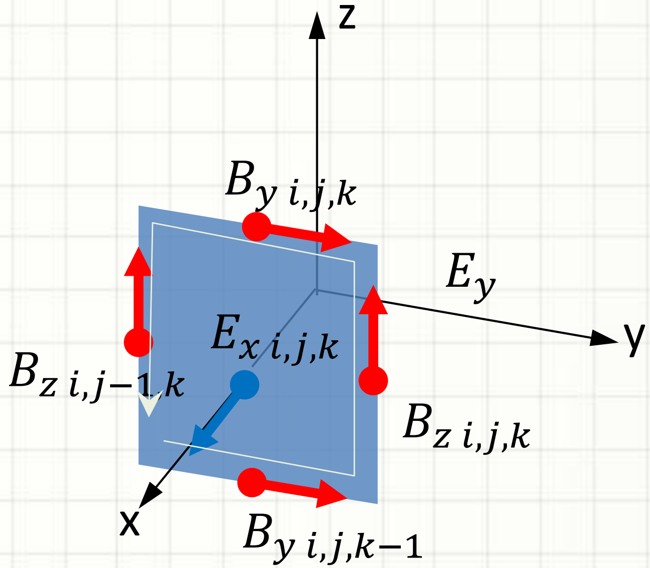
$$\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{s} \simeq \frac{\partial B_{z\ i,j,k}}{\partial t}\Delta x \Delta y$$

- Balancing those two,

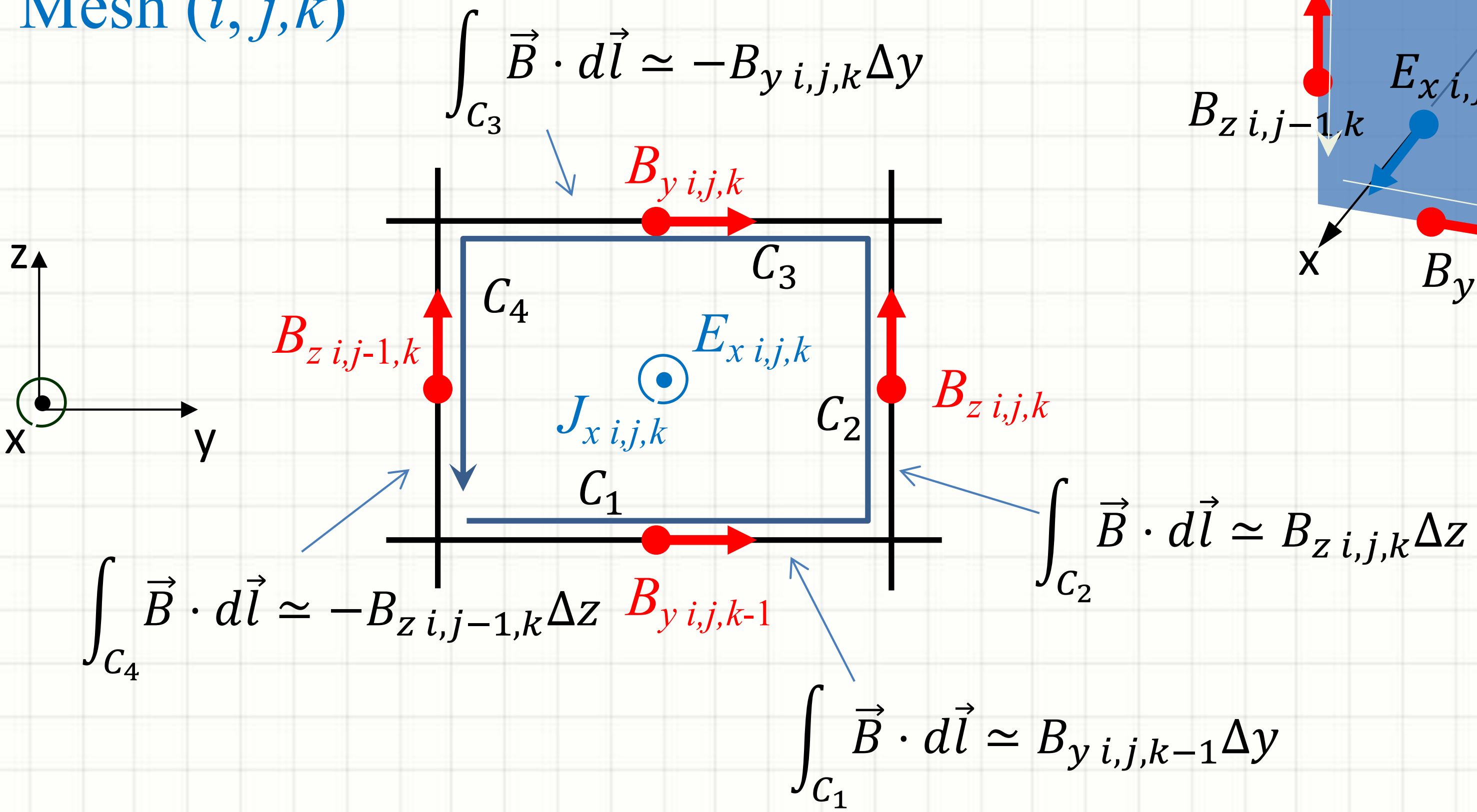
$$\frac{E_{y\ i+1,j,k} - E_{y\ i,j,k}}{\Delta x} - \frac{E_{x\ i,j+1,k} - E_{x\ i,j,k}}{\Delta y} = -\frac{\partial B_{z\ i,j,k}}{\partial t}$$

Ampere's Law on Yee Mesh

To use $$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{s} + \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{s}$$



Mesh (i, j, k)



Ampere's Law on Yee Mesh

- The line integral (left-hand-side) is

$$\oint \vec{B} \cdot d\vec{l} = \int_{C_1} \vec{B} \cdot d\vec{l} + \int_{C_2} \vec{B} \cdot d\vec{l} + \int_{C_3} \vec{B} \cdot d\vec{l} + \int_{C_4} \vec{B} \cdot d\vec{l}$$
$$\simeq \left(B_{z\, i,j,k} - B_{z\, i,j-1,k}\right)\Delta z - \left(B_{y\, i,j,k} - B_{y\, i,j,k-1}\right)\Delta y$$

- The surface integral (right-hand-side) is

$$\int \vec{J} \cdot d\vec{s} \simeq J_{x\, i,j,k}\Delta y \Delta z \qquad \frac{\partial}{\partial t}\int \vec{E} \cdot d\vec{s} \simeq \frac{\partial E_{x\, i,j,k}}{\partial t}\Delta y \Delta z$$

- Balancing those two,

$$\frac{B_{z\, i,j,k} - B_{z\, i,j-1,k}}{\Delta y} - \frac{B_{y\, i,j,k} - B_{y\, i,j,k-1}}{\Delta z} - \mu_0 J_{x\, i,j,k} = \frac{1}{c^2}\frac{\partial E_{x\, i,j,k}}{\partial t}$$

Evolution in Time

Faraday's Law

$$\frac{E_{y\,i+1,j,k} - E_{y\,i,j,k}}{\Delta x} - \frac{E_{x\,i,j+1,k} - E_{x\,i,j,k}}{\Delta y} = \frac{\partial B_{z\,i,j,k}}{\partial t}$$

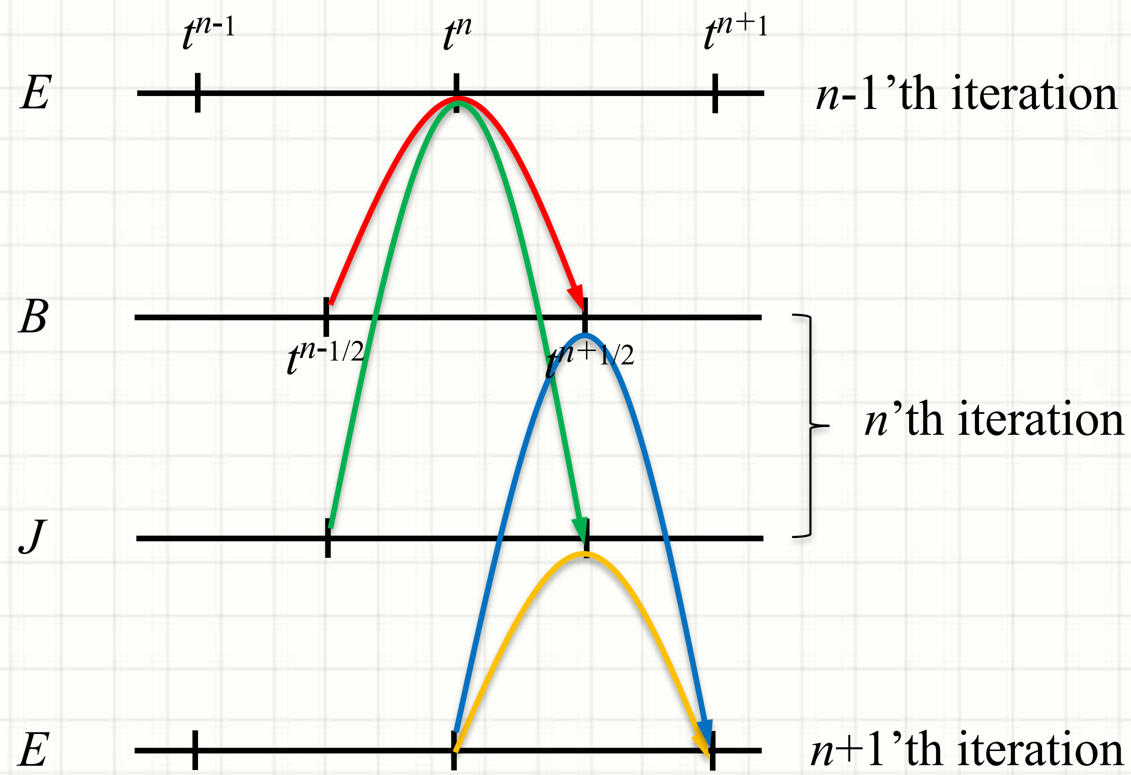
$$\Rightarrow \quad \frac{E^{n}_{y\,i+1,j,k} - E^{n}_{y\,i,j,k}}{\Delta x} - \frac{E^{n}_{x\,i,j+1,k} - E^{n}_{x\,i,j,k}}{\Delta y} = \frac{B^{n+1/2}_{z\,i,j,k} - B^{n-1/2}_{z\,i,j,k}}{\Delta t}$$

Ampere's Law (with the D-field)

$$\frac{B_{z\,i,j,k} - B_{z\,i,j-1,k}}{\Delta y} - \frac{B_{y\,i,j,k} - B_{y\,i,j,k-1}}{\Delta z} - \mu_0 J_{x\,i,j,k} = \frac{1}{c^2}\frac{\partial E_{x\,i,j,k}}{\partial t}$$

$$\Rightarrow \quad \frac{B^{n+1/2}_{z\,i,j,k} - B^{n+1/2}_{z\,i,j-1,k}}{\Delta y} - \frac{B^{n+1/2}_{y\,i,j,k} - B^{n+1/2}_{y\,ij,k-1}}{\Delta z} - \mu_0 J^{n+1/2}_{x\,i,j,k} = \frac{1}{c^2}\frac{E^{n+1}_{x\,i,j,k} - E^{n}_{x\,i,j,k}}{\Delta t}$$

Evolution in Time



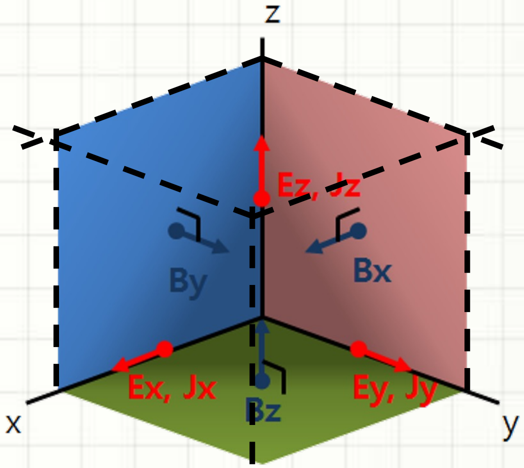
Discretized Maxwell Equations

Faraday's Law

$$\frac{E^n_{z\,i,j+1,k} - E^n_{z\,i,j,k}}{\Delta y} - \frac{E^n_{y\,i,j,k+1} - E^n_{y\,i,j,k}}{\Delta z} = -\frac{B^{n+1/2}_{x\,i,j,k} - B^{n-1/2}_{x\,i,j,k}}{\Delta t}$$

$$\frac{E^n_{x\,i,j,k+1} - E^n_{x\,i,j,k}}{\Delta z} - \frac{E^n_{z\,i+1,j,k} - E^n_{z\,i,j,k}}{\Delta x} = -\frac{B^{n+1/2}_{y\,i,j,k} - B^{n-1/2}_{y\,i,j,k}}{\Delta t}$$

$$\frac{E^n_{y\,i+1,j,k} - E^n_{y\,i,j,k}}{\Delta x} - \frac{E^n_{x\,i,j+1,k} - E^n_{x\,i,j,k}}{\Delta y} = -\frac{B^{n+1/2}_{z\,i,j,k} - B^{n-1/2}_{z\,i,j,k}}{\Delta t}$$



Ampere's Law (with the D-field)

$$\frac{B^{n+1/2}_{z\,i,j,k} - B^{n+1/2}_{z\,i,j-1,k}}{\Delta y} - \frac{B^{n+1/2}_{y\,i,j,k} - B^{n+1/2}_{y\,i,j,k-1}}{\Delta z} - \mu_0 J^{n+1/2}_{x\,i,j,k} = \frac{1}{c^2}\frac{E^{n+1}_{x\,i,j,k} - E^n_{x\,i,j,k}}{\Delta t}$$

$$\frac{B^{n+1/2}_{x\,i,j,k} - B^{n+1/2}_{x\,i,j,k-1}}{\Delta z} - \frac{B^{n+1/2}_{z\,i,j,k} - B^{n+1/2}_{z\,i-1,j,k}}{\Delta x} - \mu_0 J^{n+1/2}_{y\,i,j,k} = \frac{1}{c^2}\frac{E^{n+1}_{y\,i,j,k} - E^n_{y\,i,j,k}}{\Delta t}$$

$$\frac{B^{n+1/2}_{y\,i,j,k} - B^{n+1/2}_{y\,i-1,j,k}}{\Delta x} - \frac{B^{n+1/2}_{x\,i,j,k} - B^{n+1/2}_{x\,i,j-1,k}}{\Delta y} - \mu_0 J^{n+1/2}_{z\,i,j,k} = \frac{1}{c^2}\frac{E^{n+1}_{z\,i,j,k} - E^n_{z\,i,j,k}}{\Delta t}$$

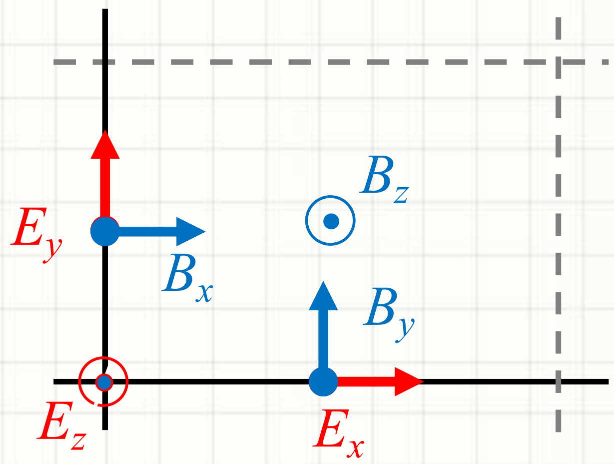
Discretized Maxwell Equations 2D (X-Y)

Faraday's Law

$$\frac{E^n_{z\,i,j+1}-E^n_{z\,i,j}}{\Delta y} - \cancel{\frac{E^n_{y\,i,j,k+1}-E^n_{y\,i,j,k}}{\Delta z}} = \frac{B^{n+1/2}_{x\,i,j}-B^{n-1/2}_{x\,i,j}}{\Delta t}$$

$$\cancel{\frac{E^n_{x\,i,j,k+1}-E^n_{x\,i,j,k}}{\Delta z}} - \frac{E^n_{z\,i+1,j}-E^n_{z\,i,j}}{\Delta x} = \frac{B^{n+1/2}_{y\,i,j}-B^{n-1/2}_{y\,i,j}}{\Delta t}$$

$$\frac{E^n_{y\,i+1,j}-E^n_{y\,i,j}}{\Delta x} - \frac{E^n_{x\,i,j+1}-E^n_{x\,i,j}}{\Delta y} = \frac{B^{n+1/2}_{z\,i,j}-B^{n-1/2}_{z\,i,j}}{\Delta t}$$



Ampere's Law (with the D-field)

$$\frac{B^{n+1/2}_{z\,i,j}-B^{n+1/2}_{z\,i,j-1}}{\Delta y} - \cancel{\frac{B^{n+1/2}_{y\,i,j,k}-B^{n+1/2}_{y\,i,j,k-1}}{\Delta z}} - \mu_0 J^{n+1/2}_{x\,i,j} = \frac{1}{c^2}\frac{E^{n+1}_{x\,i,j}-E^n_{x\,i,j}}{\Delta t}$$

$$\cancel{\frac{B^{n+1/2}_{x\,i,j,k}-B^{n+1/2}_{x\,i,j,k-1}}{\Delta z}} - \frac{B^{n+1/2}_{z\,i,j}-B^{n+1/2}_{z\,i-1,j}}{\Delta x} - \mu_0 J^{n+1/2}_{y\,i,j} = \frac{1}{c^2}\frac{E^{n+1}_{y\,i,j}-E^n_{y\,i,j}}{\Delta t}$$

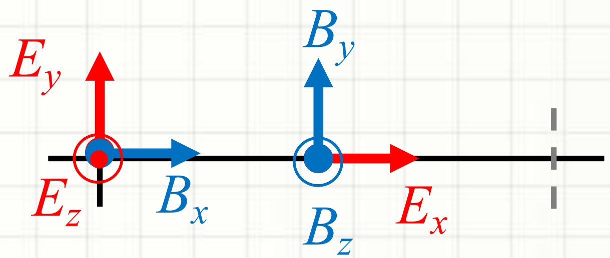
$$\frac{B^{n+1/2}_{y\,i,j}-B^{n+1/2}_{y\,i-1,j}}{\Delta x} - \frac{B^{n+1/2}_{x\,i,j}-B^{n+1/2}_{x\,i,j-1}}{\Delta y} - \mu_0 J^{n+1/2}_{z\,i,j} = \frac{1}{c^2}\frac{E^{n+1}_{z\,i,j}-E^n_{z\,i,j}}{\Delta t}$$

Discretized Maxwell Equations 1D (X)

$$\cancel{\frac{E^{n}_{z\,i,j+1}-E^{n}_{z\,i,j}}{\Delta y}}-\cancel{\frac{E^{n}_{y\,i,j,k+1}-E^{n}_{y\,i,j,k}}{\Delta z}}=\cancel{\frac{B^{n+1/2}_{x\,i,j}-B^{n-1/2}_{x\,i,j}}{\Delta t}}$$

$$\cancel{\frac{E^{n}_{x\,i,j,k+1}-E^{n}_{x\,i,j,k}}{\Delta z}}-\frac{E^{n}_{z\,i+1}-E^{n}_{z\,i}}{\Delta x}=\frac{B^{n+1/2}_{y\,i}-B^{n-1/2}_{y\,i}}{\Delta t}$$

$$\frac{E^{n}_{y\,i+1}-E^{n}_{y\,i}}{\Delta x}-\cancel{\frac{E^{n}_{x\,i,j+1}-E^{n}_{x\,i,j}}{\Delta y}}=\frac{B^{n+1/2}_{z\,i}-B^{n-1/2}_{z\,i}}{\Delta t}$$



$$\cancel{\frac{B^{n+1/2}_{z\,i,j}-B^{n+1/2}_{z\,i,j-1}}{\Delta y}}-\cancel{\frac{B^{n+1/2}_{y\,i,j,k}-B^{n+1/2}_{y\,i,j,k-1}}{\Delta z}}-\mu_0 J^{n+1/2}_{x\,i}=\frac{1}{c^2}\frac{E^{n+1}_{x\,i}-E^{n}_{x\,i}}{\Delta t}$$

$$\cancel{\frac{B^{n+1/2}_{x\,i,j,k}-B^{n+1/2}_{x\,i,j,k-1}}{\Delta z}}-\frac{B^{n+1/2}_{z\,i}-B^{n+1/2}_{z\,i-1}}{\Delta x}-\mu_0 J^{n+1/2}_{y\,i}=\frac{1}{c^2}\frac{E^{n+1}_{y\,i}-E^{n}_{y\,i}}{\Delta t}$$

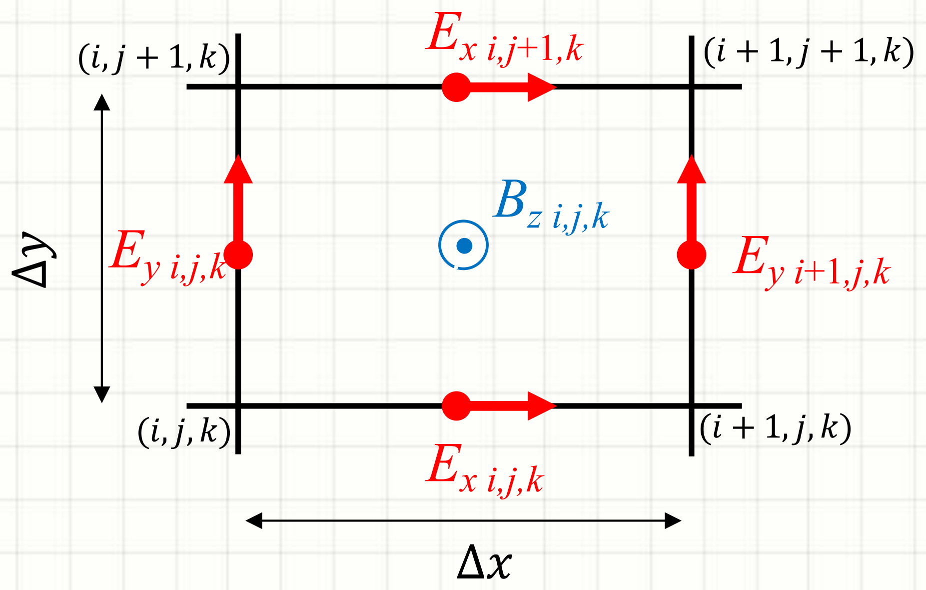
$$\frac{B^{n+1/2}_{y\,i}-B^{n+1/2}_{y\,i-1}}{\Delta x}-\cancel{\frac{B^{n+1/2}_{x\,i,j}-B^{n+1/2}_{x\,i,j-1}}{\Delta y}}-\mu_0 J^{n+1/2}_{z\,i}=\frac{1}{c^2}\frac{E^{n+1}_{z\,i}-E^{n}_{z\,i}}{\Delta t}$$

Error Order

Actual Indices

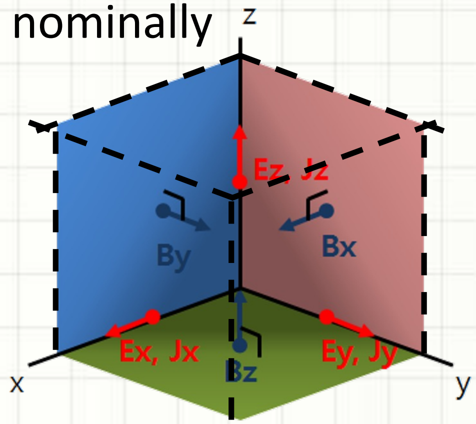
$$\frac{E^n_{y\,i+1,j+1/2,k} - E^n_{y\,i,j+1/2,k}}{\Delta x} - \frac{E^n_{x\,i+1/2,j+1,k} - E^n_{x\,i+1/2,j,k}}{\Delta y} = \frac{B^{n+1/2}_{z\,i+1/2,j+1/2,k} - B^{n-1/2}_{z\,i+1/2,j+1/2,k}}{\Delta t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + O(\Delta x^2, \Delta y^2) = \frac{\partial B_z}{\partial t} + O(\Delta t^2)$$



Stability Analysis

- The electromagnetic fields are of the form $E(B) \sim E_0(B_0)e^{i\vec{k}\cdot\vec{r}-i\omega t} = E_0(B_0)e^{i(k_x i\Delta x+k_y j\Delta y+k_z k\Delta z)-i\omega s\Delta t}$, where s is the time step.
- Substitute this expression for the Faraday's law, which is indexed nominally using the half-index.



$$\frac{E^n_{z\,i,j+1,k+1/2} - E^n_{z\,i,j,k+1/2}}{\Delta y} - \frac{E^n_{y\,i,j+1/2,k+1} - E^n_{y\,i,j+1/2,k}}{\Delta z} = -\frac{B^{n+1/2}_{x\,i,j+1/2,k+1/2} - B^{n-1/2}_{x\,i,j+1/2,k+1/2}}{\Delta t}$$

$$\Rightarrow \quad \frac{1}{\Delta y}E_{z0}\left(e^{ik_y\Delta y}-1\right)e^{i\frac{k_z\Delta z}{2}} - \frac{1}{\Delta z}E_{y0}\left(e^{ik_z\Delta z}-1\right)e^{i\frac{k_y\Delta y}{2}} = -\frac{1}{\Delta t}B_{x0}e^{i\frac{k_y\Delta y}{2}+i\frac{k_z\Delta z}{2}}\left(e^{-i\frac{\omega\Delta t}{2}} - e^{i\frac{\omega\Delta t}{2}}\right)$$

$$\Rightarrow \quad \frac{1}{\Delta y}\sin\frac{k_y\Delta y}{2}E_{z0} - \frac{1}{\Delta z}\sin\frac{k_z\Delta z}{2}E_{y0} = \frac{1}{\Delta t}\sin\frac{\omega\Delta t}{2}B_{x0}$$

- Conducting the same procedure for the other components of Faraday's law,

$$\frac{1}{\Delta z}\sin\frac{k_z\Delta z}{2}E_{x0} - \frac{1}{\Delta x}\sin\frac{k_x\Delta x}{2}E_{z0} = \frac{1}{\Delta t}\sin\frac{\omega\Delta t}{2}B_{y0} \qquad \frac{1}{\Delta x}\sin\frac{k_x\Delta x}{2}E_{y0} - \frac{1}{\Delta y}\sin\frac{k_y\Delta y}{2}E_{x0} = \frac{1}{\Delta t}\sin\frac{\omega\Delta t}{2}B_{z0}$$

- Define a new vector and a scalar quantities ;

$$\vec{\kappa} \equiv \left(\frac{1}{\Delta x}\sin\frac{k_x\Delta x}{2},\quad \frac{1}{\Delta y}\sin\frac{k_y\Delta y}{2},\quad \frac{1}{\Delta z}\sin\frac{k_z\Delta z}{2}\right) \qquad \Omega \equiv \frac{1}{\Delta t}\sin\frac{\omega\Delta t}{2}$$

- The Faraday's law can be then expressed as $\Omega\vec{B} = \vec{\kappa}\times\vec{E}$

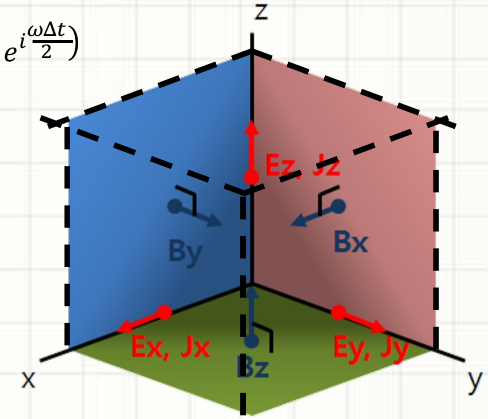
Stability Analysis

- The Ampere's law (current-less) with nominal indices is

$$\frac{B_{z\,i+1/2,j+1/2,k}^{n+1/2}-B_{z\,i+1/2,j-1/2,k}^{n+1/2}}{\Delta y}-\frac{B_{y\,i+1/2,j,k+1/2}^{n+1/2}-B_{y\,i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}=\frac{1}{c^2}\frac{E_{x\,i+1/2,j,k}^{n+1}-E_{x\,i+1/2,j,k}^{n}}{\Delta t}$$

$$\Rightarrow \frac{1}{\Delta y}B_{z0}\left(e^{ik_y\Delta y2}-e^{-ik_y\Delta y2}\right)e^{i\frac{k_x\Delta x}{2}}-\frac{1}{\Delta z}B_{y0}\left(e^{ik_z\Delta z2}-e^{-ik_z\Delta z2}\right)e^{i\frac{k_x\Delta x}{2}}=\frac{1}{c^2\Delta t}E_{x0}e^{i\frac{k_x\Delta x}{2}}\left(e^{-i\frac{\omega\Delta t}{2}}-e^{i\frac{\omega\Delta t}{2}}\right)$$

$$\Rightarrow \frac{1}{\Delta y}\sin\frac{k_y\Delta y}{2}B_{z0}-\frac{1}{\Delta z}\sin\frac{k_z\Delta z}{2}B_{y0}=-\frac{1}{c^2\Delta t}\sin\frac{\omega\Delta t}{2}E_{x0}$$



- For the other components,

$$\frac{1}{\Delta z}\sin\frac{k_z\Delta z}{2}B_{x0}-\frac{1}{\Delta x}\sin\frac{k_x\Delta x}{2}B_{z0}=-\frac{1}{c^2\Delta t}\sin\frac{\omega\Delta t}{2}E_{y0}$$

$$\frac{1}{\Delta x}\sin\frac{k_x\Delta x}{2}B_{y0}-\frac{1}{\Delta y}\sin\frac{k_y\Delta y}{2}B_{x0}=-\frac{1}{c^2\Delta t}\sin\frac{\omega\Delta t}{2}E_{z0}$$

- Then we have

$$-\frac{\Omega}{c^2}\vec{E}=\vec{\kappa}\times\vec{B}$$

Stability Analysis

- Combining two equations, $-\frac{\Omega^2}{c^2}\vec{E} = \vec{\kappa}(\vec{\kappa}\cdot\vec{E}) - \kappa^2\vec{E}$
- $\vec{\kappa}\cdot\vec{E}$ term corresponds to $\nabla\cdot\vec{E}$, which vanishes for space-charge-less systems.
- Although we do not solve $\nabla\cdot\vec{E} = 0$, this is implicitly included in the time-dependent Maxwell equations as long as $\nabla\cdot\vec{J} = 0$, which is the case here.
- Then the dispersion relation is

$$\frac{\Omega^2}{c^2} = \kappa^2$$

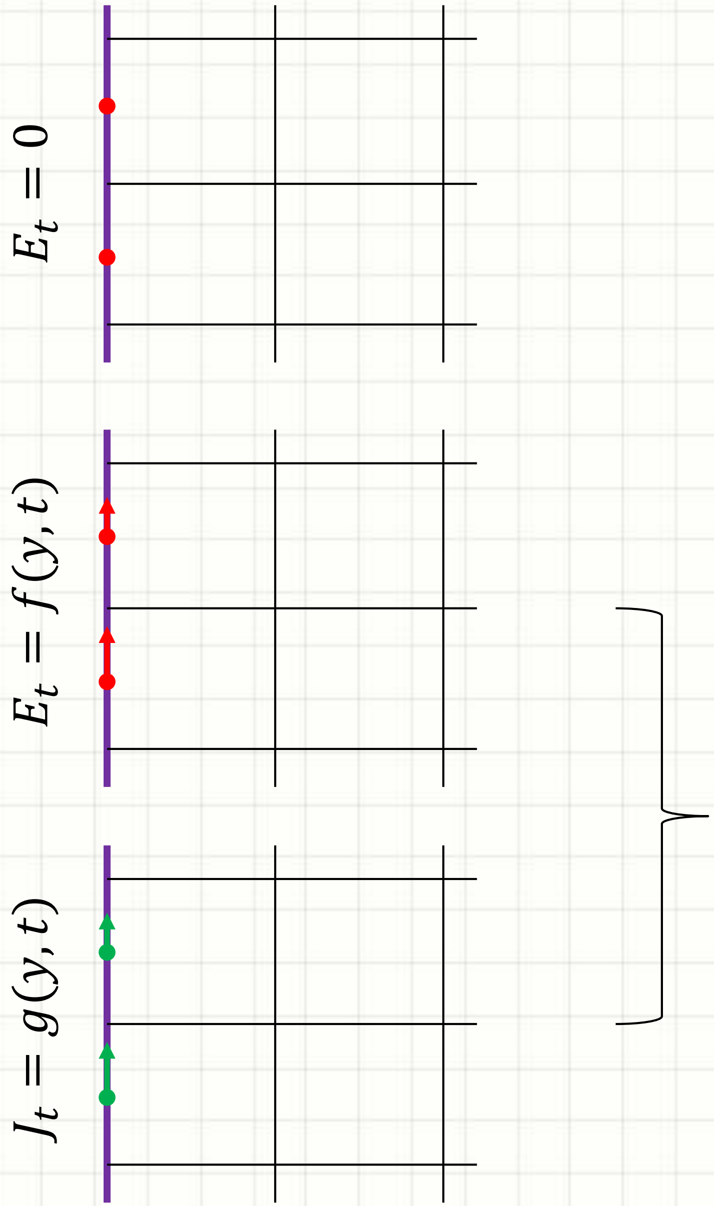
Dispersion relation of the discretized Maxwell equations

$$\Rightarrow \quad \frac{1}{c^2\Delta t^2}\sin^2\frac{\omega\Delta t}{2} = \frac{1}{\Delta x^2}\sin^2\frac{k_x\Delta x}{2} + \frac{1}{\Delta y^2}\sin^2\frac{k_y\Delta y}{2} + \frac{1}{\Delta z^2}\sin^2\frac{k_z\Delta z}{2}$$

- ω should be real for a given (k_x, k_y, k_z) for the fields not to grow or decay unstably. From the notion that sine is less than 1, it should be like

$$\frac{1}{c^2\Delta t^2} \geq \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \qquad \mathit{CFL}\ \text{Condition}$$

Boundary Conditions



Conducting Boundary Condition
- Tangential electric field on the conducting surface is zero.

Emitting Boundary Condition
- Time-varying electric field or current emits an electromagnetic wave.

1D Electromagnetic Fields in Python

Example 1. Create a 1D solver of the time-dependent Maxwell equations. Put a line source of oscillating current at the center. Make a plot of the electric field with the same polarization as the current and the magnetic field perpendicular to that.

Answer: Look at the code 'maxwell1d-I.py'. Only the main loop is shown here.

```
…
s=0
while s<smax:
        By[ :-1]  +=    a*(Ez[1: ] – Ez[ :-1])
        Bz[ :-1]  += – a*(Ey[1: ] – Ey[ :-1])
        Ey[1:-1] += – a*(Bz[1:-1] – Bz[0:-2])
        Ey[c]  += –dt*np.sin(w*s*dt)
        Ez[1:-1] +=   a*(By[1:-1] – By[0:-2])
        s+=1
```

$$-\frac{E^n_{z\,i+1}-E^n_{z\,i}}{\Delta x}=\frac{B^{n+1/2}_{y\,i}-B^{n-1/2}_{y\,i}}{\Delta t}$$

$$\frac{E^n_{y\,i+1}-E^n_{y\,i}}{\Delta x}=\frac{B^{n+1/2}_{z\,i}-B^{n-1/2}_{z\,i}}{\Delta t}$$

$$-\frac{B^{n+1/2}_{z\,i}-B^{n+1/2}_{z\,i-1}}{\Delta x}-\mu_0 J^{n+1/2}_{y\,i}=\frac{1}{c^2}\frac{E^{n+1}_{y\,i}-E^n_{y\,i}}{\Delta t}$$

$$\frac{B^{n+1/2}_{y\,i}-B^{n+1/2}_{y\,i-1}}{\Delta x}=\frac{1}{c^2}\frac{E^{n+1}_{z\,i}-E^n_{z\,i}}{\Delta t}$$

2D Electromagnetic Fields in Python

Example 2. Create a 2D solver of the time-dependent Maxwell equations. Enclose the simulation domain by conducting boundaries, but put a small piece of emitting boundary at the left edge, so that it looks like a single-slit diffraction system. Make a plot of the electric field.

Suggested run parameters:

xmax=6, ymax=10, dx=dy=0.01, dt=0.005, f=3, D=1, smax=1000

Answer: Look at the code 'maxwell2d-I.py'

```
while s < smax:
        Ey[lower:upper,0]= np.sin(w*s*dt)   # slit
        Bx[:-1,:-1] +=  -b*(Ez[1:,:-1]-Ez[:-1,:-1])
        By[1:-1,:-1] +=  a*(Ez[1:-1,1:]-Ez[1:-1,:-1])
        Bz[:-1,:-1]+= -a*(Ey[:-1,1:]-Ey[:-1,:-1]) + b*(Ex[1:,:-1]-Ex[:-1,:-1])
        Ex[1:-1,:-1] +=  b*(Bz[1:-1,:-1]-Bz[:-2,:-1])
        Ey[:-1,1:-1] += -a*(Bz[:-1,1:-1]-Bz[:-1,:-2])
        Ez[1:-1,1:-1]+=  a*(By[1:-1,1:-1]-By[1:-1,:-2]) - b*(Bx[1:-1,1:-1]-Bx[:-2,1:-1])

s+=1
```

$$\frac{E^n_{z\,i,j+1} - E^n_{z\,i,j}}{\Delta y} = \frac{B^{n+1/2}_{x\,i,j} - B^{n-1/2}_{x\,i,j}}{\Delta t}$$

$$-\frac{E^n_{z\,i+1,j} - E^n_{z\,i,j}}{\Delta x} = \frac{B^{n+1/2}_{y\,i,j} - B^{n-1/2}_{y\,i,j}}{\Delta t}$$

$$\frac{E^n_{y\,i+1,j} - E^n_{y\,i,j}}{\Delta x} - \frac{E^n_{x\,i,j+1} - E^n_{x\,i,j}}{\Delta y} = \frac{B^{n+1/2}_{z\,i,j} - B^{n-1/2}_{z\,i,j}}{\Delta t}$$

$$\frac{B^{n+1/2}_{z\,i,j} - B^{n+1/2}_{z\,i,j-1}}{\Delta y} = \frac{1}{c^2}\frac{E^{n+1}_{x\,i,j} - E^n_{x\,i,j}}{\Delta t}$$

$$-\frac{B^{n+1/2}_{z\,i,j} - B^{n+1/2}_{z\,i-1,j}}{\Delta x} = \frac{1}{c^2}\frac{E^{n+1}_{y\,i,j} - E^n_{y\,i,j}}{\Delta t}$$

$$\frac{B^{n+1/2}_{y\,i,j} - B^{n+1/2}_{y\,i-1,j}}{\Delta x} - \frac{B^{n+1/2}_{x\,i,j} - B^{n+1/2}_{x\,i,j-1}}{\Delta y} = \frac{1}{c^2}\frac{E^{n+1}_{z\,i,j} - E^n_{z\,i,j}}{\Delta t}$$

END OF LECTURE 3-2