LECTURE 3-1
WAVE EQUATION

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Outline

- Mesh and Discretization
- Stability
- Initial Conditions
- Waves in Python
- Free End Boundary Condition

Discretization of Wave Equation

The wave equation with a source term

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = S(\psi, \vec{r}, t)$$

* The source term originates from scattering or external driver etc.

Discretization

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \simeq \frac{\psi^n_{i+1jk} - 2\psi^n_{ijk} + \psi^n_{i-1jk}}{\Delta x^2} + \frac{\psi^n_{ij+1k} - 2\psi^n_{ijk} + \psi^n_{ij-1k}}{\Delta y^2} + \frac{\psi^n_{ijk+1} - 2\psi^n_{ijk} + \psi^n_{ijk-1}}{\Delta z^2} + O(\Delta x^2, \Delta y^2, \Delta z^2)$$

$$\frac{\partial^2\psi}{\partial t^2} \simeq \frac{\psi^{n+1}_{ijk} - 2\psi^n_{ijk} + \psi^{n-1}_{ijk}}{\Delta t^2} + O(\Delta t^2)$$

$$S \simeq S^n_{ijk}(\psi^n_{ijk})$$

1D Homogeneous Wave Equation

Discrete Wave Eq.

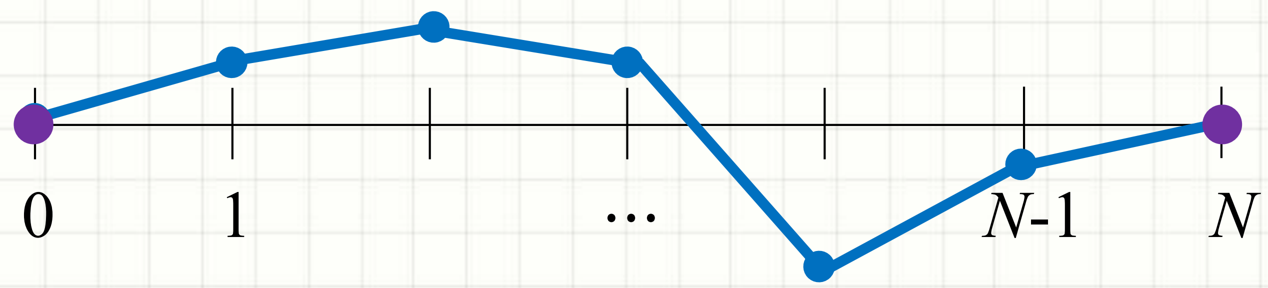
$$\frac{\psi_{i+1}^{n} - 2\psi_{i}^{n} + \psi_{i-1}^{n}}{\Delta x^2} - \frac{1}{v^2}\frac{\psi_{i}^{n+1} - 2\psi_{i}^{n} + \psi_{i}^{n-1}}{\Delta t^2} = 0$$

No Source Term

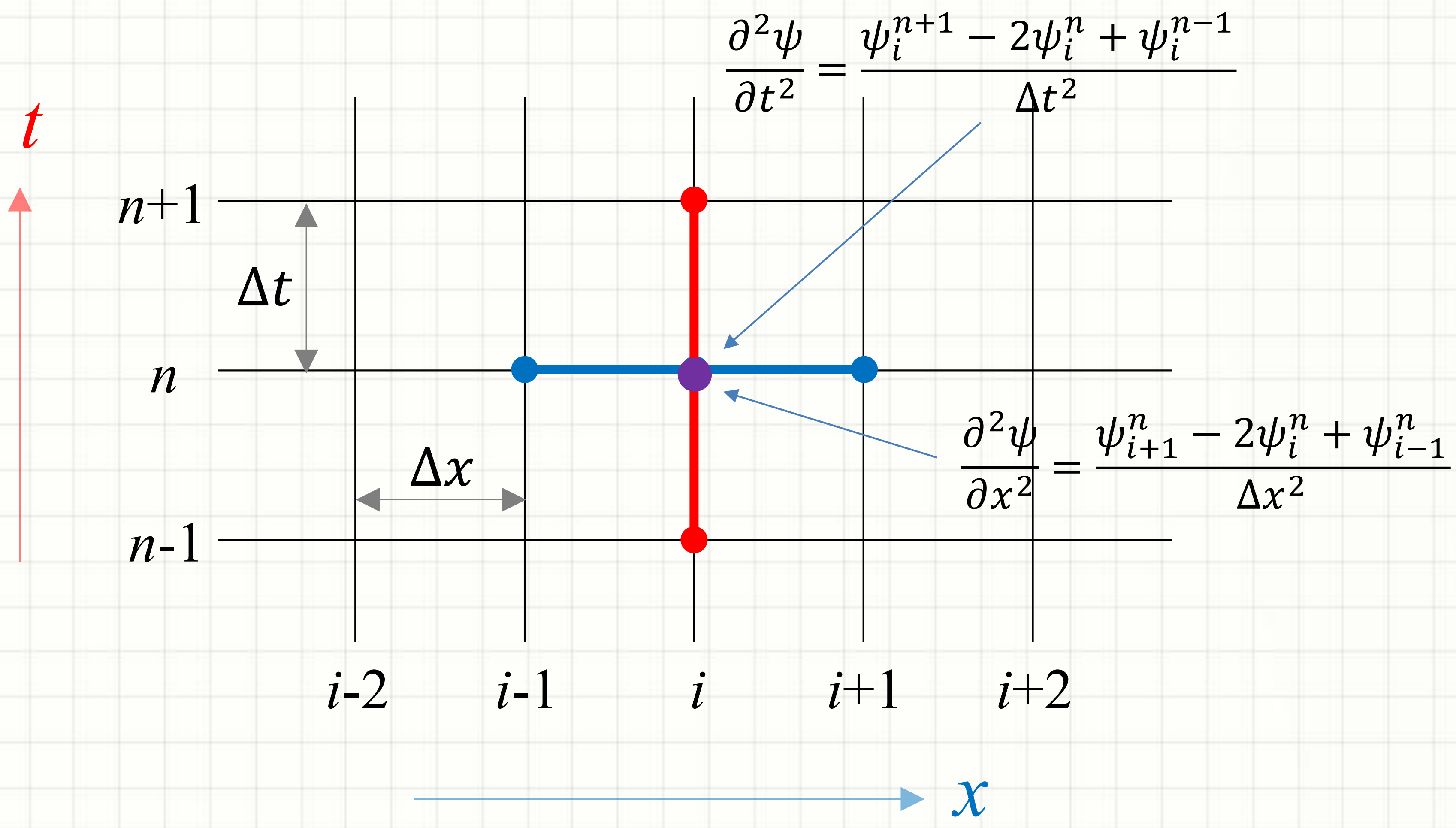
$$\psi_{i}^{n+1} = 2(1-\alpha^2)\psi_{i}^{n} + \alpha^2(\psi_{i+1}^{n} + \psi_{i-1}^{n}) - \psi_{i}^{n-1} \qquad \alpha = \frac{v\Delta t}{\Delta x}$$

Boundary Condition

$$\psi_{0}^{n} = \psi_{N}^{n} = 0$$



Mesh and Time Step for 1D Wave Eq.



Stability Analysis

- Substitute $\psi_j^n = \xi^n e^{ik(j\Delta x)}$ for the discretized wave equation.

$$\xi^{n+1}e^{ik(j\Delta x)} = 2(1-\alpha^2)\xi^n e^{ik(j\Delta x)} + \alpha^2\xi^n\left(e^{ik(j+1)\Delta x} + e^{ik(j-1)\Delta x}\right) - \xi^{n-1}e^{ik(j\Delta x)}$$

- Cancelling out $\xi^n e^{ik(j\Delta x)}$ yields a quadratic equation of ξ.

$$\xi^2 = 2(1-\alpha^2)\xi + \alpha^2\xi\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - 1 \quad \rightarrow \quad \xi^2 - 2[(1-\alpha^2) + \alpha^2\cos k\Delta x]\xi + 1 = 0$$

- The solution of the quadratic equation is

$$\xi = 1 + \alpha^2(\cos k\Delta x - 1) \pm \sqrt{\left(1 + \alpha^2(\cos k\Delta x - 1)\right)^2 - 1}$$

- Define $X \equiv \alpha^2(1-\cos k\Delta x) - 1$. Note that $-1 \le X \le 2\alpha^2 - 1$. The solution is

$$\xi = -X \pm \sqrt{X^2 - 1}$$

- For $\alpha > 1$, there are modes around $k \simeq -\frac{\pi}{\Delta x}$ for which $|\xi| > 1$. Such modes grows, leading to numerical instability.
- For $\alpha \le 1$, $X^2 \le 1$. Then the solution is $\xi = -X \pm i\sqrt{1 - X^2}$
- Its magnitude is $|\xi| = \sqrt{X^2 + (1 - X^2)} = 1$
- Hence

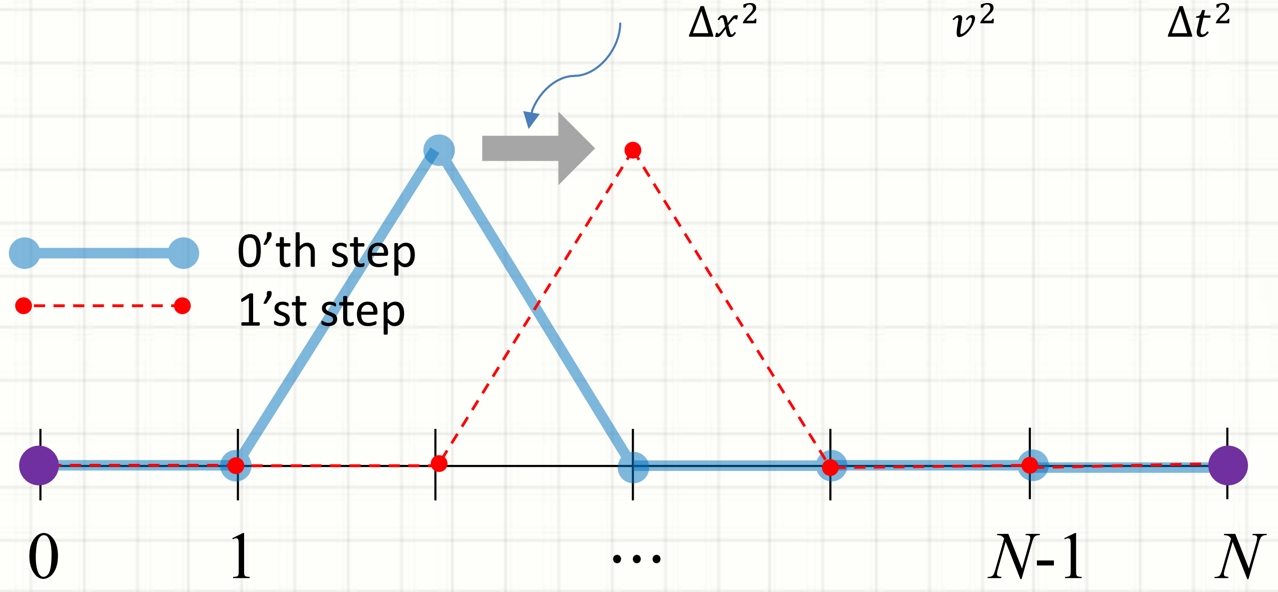
$|\xi| = 1$ when $\alpha \le 1$. Stable and no dissipation.
$|\xi| > 1$ for some modes when $\alpha > 1$. Unstable.

Initial Condition – a Propagating Pulse

A Propagating Pulse

- How can we determine \psi when we know the exact pulse shape at t=0?
- One way is using analytic form (relatively easily obtained by approximating the solution for small dt) can be used for \psi.
- Or a bit less accurate 1st order method can be applied to get psi. The relatively large initial error may not grow as time advances, as long as the solver is stable.

$$\frac{\psi_{i+1}^n - 2\psi_i^n + \psi_{i-1}^n}{\Delta x^2} - \frac{1}{v^2}\frac{\psi_i^{n+1} - 2\psi_i^n + \psi_i^{n-1}}{\Delta t^2} = 0$$



Propagating Pulses in Python

Example 1. Create a python script to simulate the pulse propagation. Start with a Gaussian, half-wavelength pulse, as is in the continuity equation of the previous class. Run the code to look if you obtain expected results.

Answer: Look at the ‘wave-prop.py’.

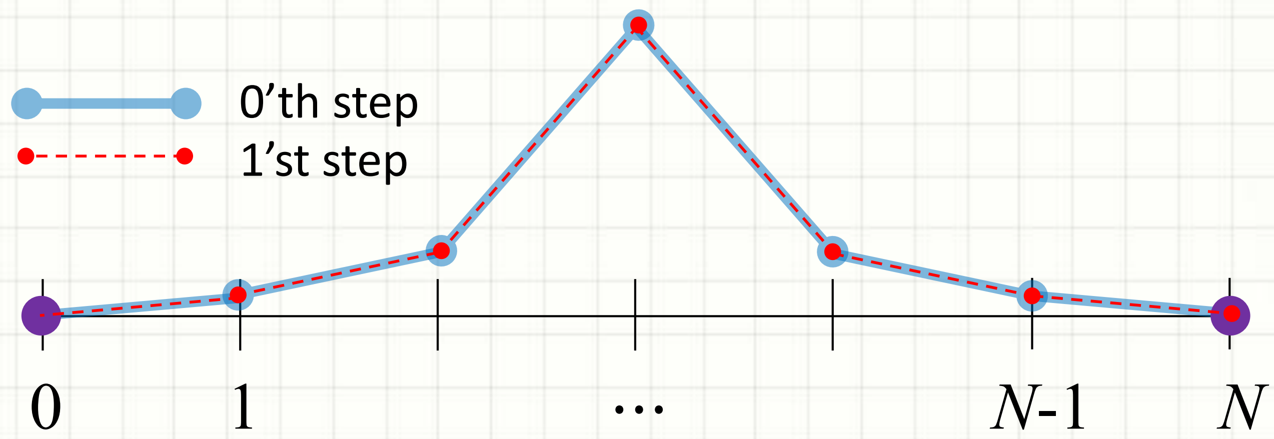
Initial Condition – Plucking

- As the wave equation is 2nd order in time, two initial conditions, $\psi(x, t = 0)$ and $\psi'(x, t = 0)$ should be specified.
- In discrete equations, ψ_i^0 and ψ_i^1 are required (see below).

$$\psi_i^2 = 2(1 - \alpha^2)\psi_i^1 + \alpha^2\left(\psi_{i+1}^1 + \psi_{i-1}^1\right) - \psi_i^0$$

Plucking a String

- Displace some parts of the string for a while, and release.



Plucked String in Python

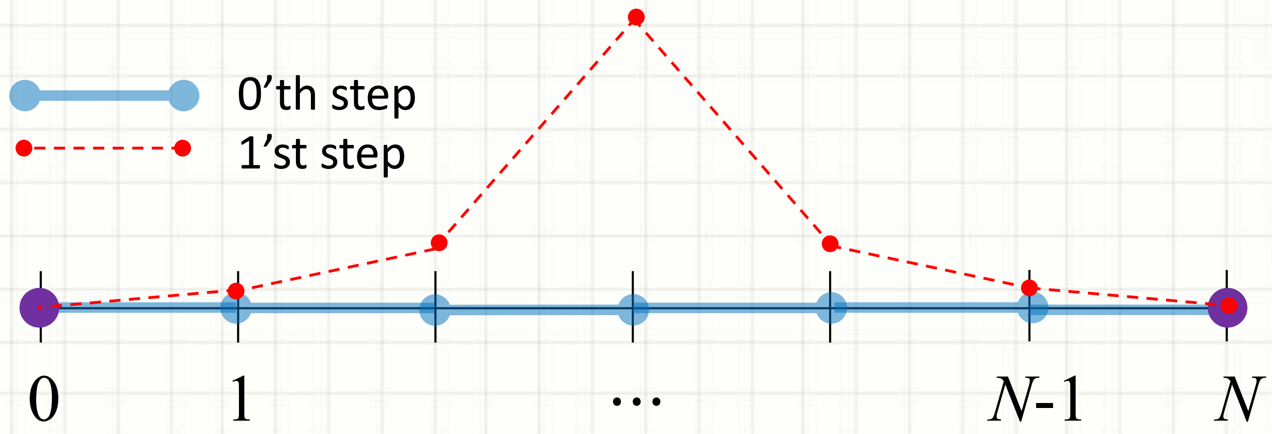
Example 2. Modify the initial condition of the propagating case to reproduce the situation of the plucked string. Run the code to look if you obtain expected results.

Answer: Look at the ‘wave-plucked.py’.

Initial Condition – Impulse

Hammering a String

- When you hit a piano key, the hammer hit the string.
- This procedure corresponds to the case that $\psi_i^0 = 0$ and $\psi_i^1 \neq 0$ at particular i's.

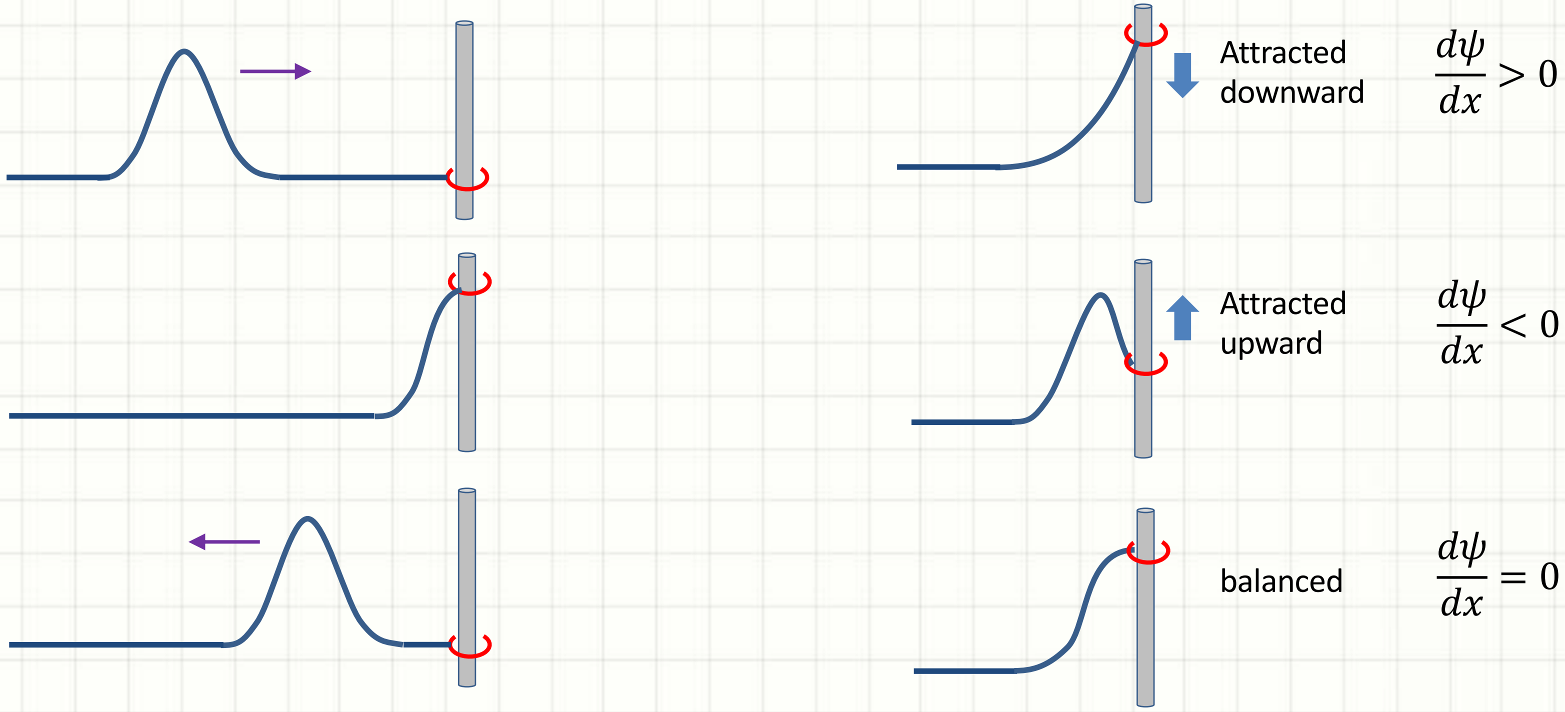


Impulse in Python

Example 3. Modify the initial condition of the previous example to make it the impulse problem.

Answer: Look at the 'wave-impulse.py'.

Boundary Condition for Free Ends



- When the free end has no friction, the acceleration is infinite, moving the node to the balanced position instantaneously.
- Hence the free end boundary condition is ‘normal derivative=0’.

$$\frac{\partial \psi}{\partial n} = 0$$

Free end boundary condition

END OF LECTURE 3-1